A glimpse of 2D and 3D quantum field theories through number theory

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Publications

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[2] F. Ferrari, V. Reys,
"Mixed Rademacher and BPS black holes",
JHEP 1707 (2017) 094, `arXiv:1702.02755 [hep-th]`.

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[3] M. C. N. Cheng, S. Chun, F. Ferrari, S. Gukov, S. M. Harrison,
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Other publications by the author:

[4] M. C. N. Cheng, F. Ferrari, S. M. Harrison, N. M. Paquette,
"Landau-Ginzburg Orbifolds and Symmetries of K3 CFTs",
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“[...] les quelques pages de démonstration qui suivent tirent toute leur force du fait que l’histoire est entièrement vraie, puisque je l’ai imaginée d’un bout à l’autre. Sa réalisation matérielle proprement dite consiste essentiellement en une projection de la réalité, en atmosphère biaise et chauffée, sur un plan de référence irrégulièrement ondulé et présentant de la distorsion. On le voit, c’est un procédé avouable, s’il en fut.”

Boris Vian, *L’écume des jours*

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Chapter 1

Introduction

At the beginning of the 20th century, an Indian mathematician, S. Ramanujan, revolutionized the field of number theory with mysterious results on elliptic functions, hypergeometric series, arithmetic functions, and congruences. Back then, the lack of proofs and demonstrations in Ramanujan's propositions caused skepticism among many mathematicians. Around the same time Ramanujan wrote his famous manuscripts, A. Einstein formulated the theory of special (1905) and general relativity (1915). None of these physical theories received immediate recognition but, fortunately for Einstein, it did not take too long before experimental evidence supporting the new theoretical formulation of gravity to the detriment of alternative theories, such as the ether theory. The relativistic discovery was soon accompanied by the "quantum revolution".

The first half of the 20th century witnessed the advent of quantum mechanics, while the second half of the century was marked by the development of quantum field theory. The latter unifies the relativistic theory of gravity with the notions of quantum mechanics and provides a formalism to accurately describe particle physics through the so-called Standard Model. After the developments of quantum field theory, unsolved issues emerged and difficulties arose in reconciling the macroscopic view of general relativity with the microscopic perspective depicted by quantum field theory. These two frameworks not only represent physics at different energy scales but also use different mathematical underpinnings and formalisms. Although they appear as mutually independent descriptions of Nature, there exist physical systems where both of them are essential. They are both required, for instance, in the analysis of certain astrophysical objects (such as black holes) or to give an account of the early stages of the Universe. Therefore, soon the need to produce a quantum gravity theory, unifying and harmonizing two radically different formalisms and views of nature into a single theory, arose.

String theory is an attempt[1] to unveil the unifying theory and to comprehend

[1]Other attempts come, for instance, from loop quantum gravity. Part of the discussion that

quantum gravity. This theory does not only quantize the geometry of spacetime by averaging over all possible spacetime geometry weighted by the exponential of the action of the theory but, in doing so, it also provides a way to regulate ultra-violet divergences, which are notorious for being present in quantum field theories.

The value of string theory comes from the way it critically reconsiders the concept of elementary particles. The one-dimensional elementary entity is substituted by a two-dimensional entity, the string. Strings sweep out a surface as they move through spacetime; more explicitly we can imagine a closed string to be given by a loop, or a circle, which forms a cylinder as time elapses. Alternatively, a string can be thought as a map from a Riemann surface, the so-called worldsheet in the string theory jargon, to the spacetime. The theory on the worldsheet of the string is a conformal field theory, a quantum field theory invariant under coordinate transformations that do not alter the angles but can rescale the lengths of the theory. As in usual quantum field theories, string theory has a perturbative expansion where multi-loops correspond to conformal field theories on different Riemann surfaces with higher genus (and possible boundaries). For instance, the one-loop vacuum amplitude corresponds to a genus one amplitude.

Despite the lack of experimental evidence and of a guidance provided by observations, string theory furnishes new tools to investigate quantum field theories. Moreover, it provides new ideas via the AdS/CFT correspondence to describe black hole thermodynamics, probe condensed matter systems, as well as to tackle the long-standing problem of confinement in quantum chromodynamics. Another important role played by string theory is in the field of mathematics. Indeed string theory appears inextricably intertwined with mathematics as evidenced by, among others, the discovery of mirror symmetry and its connection to enumerative geometry and algebraic topology, and the advent of the geometric Langlands program in the physics realm.

Recently a renewed connection between string theory and number theory emerged from the development of a coherent mathematical framework in which to understand mock modular forms. These objects were first described in 1920 by Ramanujan. Just a few days before his death he wrote a letter to Hardy [5], where he investigated the asymptotic properties of certain q-hypergeometric series. Many years later, in 2008, a precise definition of mock modular forms was formulated by Zwegers [6], providing a consistent mathematical framework that describes the intrinsic features of these almost modular objects. The definition of mock modular forms extends the one of modular forms in the following way: mock modular

follows justifies our choice of dealing with string theory instead of alternative views of quantum gravity.

forms have the peculiarity of transforming as modular forms only after the addition of a non-holomorphic term. Moreover, they appeared in seemingly unrelated fields of research, and they found applications which go far beyond the original observations that Ramanujan reported to Hardy. For instance, a wide range of applications of these objects to string theory has been identified in systems which exhibit wall-crossing phenomena or string theories and conformal field theories with non-compact target space [7, 8].

Broadly speaking, the principal subjects investigated in this thesis belong to the string theory, the conformal field theory or the number theory realm. Before diving into this work, we introduce the reader to the basic concepts and the main topics discussed in the rest of the thesis. The Ariadne thread that connects the different topics in this introduction is the generating function of integer partitions.

1.1 Integer partitions of a free boson

Part of the work by Rogers, Ramanujan, Hardy, MacMahon, and Rademacher was driven by their interest in the partition of integers[2]. To understand what this is and what it led to, consider the sequence of integers

$$\{1,\, 2,\, 3,\, 5,\, 7,\, 11,\, 15,\dots\}\,. \tag{1.1.1}$$

Each number in this sequence represents the number of integer partitions of a given integer[3]. The fifth entry, for instance, counts the number of ways 5 can be decomposed into a sum of integers (i.e. $1+1+1+1+1$, $2+1+1+1$, $2+2+1$, $3+1+1$, $3+2$, $4+1$, 5). The information encrypted in this sequence, or any sequence, can be packaged into a single function, more precisely a formal power series of q, called the generating function of the sequence. To the above sequence, one can associate the generating function of unrestricted partitions, also known as Euler series,

$$\mathcal{P}(q) := 1 + \sum_{n=1}^{\infty} \alpha(n) q^n\,. \tag{1.1.2}$$

The coefficients $\alpha(n)$ coincides with the elements of the sequence, while n stands for the number that is decomposed, or equivalently the position of $\alpha(n)$ in the sequence. The properties of the list (1.1.1) turns out to be much harder to examine

[2] Leibniz seems to be the first one who wrote about partition of integers, followed great contributions by Euler and J. Sylvester. Reviews of their seminal works as “partition theorists” is summarized in [9].

[3] Eliding the last number from the sequence generates a list of prime numbers, whose generating function is still mysterious.

than the function $\mathcal{P}(q)$ itself. The analysis of generating functions of sequences has been a central topic in different branches of mathematics, chiefly in enumerative combinatorics.

Non-trivial identities and congruences can be inferred by comparing the series (1.1.2) with its product representation

$$\mathcal{P}(q) = \prod_{n=1}^{\infty}(1-q^n)^{-1} . \tag{1.1.3}$$

If we take the variable q to be equal to $e^{2\pi i\tau}$ with $\tau \in \mathcal{H}$, we obtain a holomorphic function of τ convergent in the upper half-plane $\mathcal{H}$ (or inside the unit disc $|q| < 1$). Here we deviate from combinatorics to enter the realm of number theory. This is in fact related to one of the most important modular form: the Dedekind eta function

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n) . \tag{1.1.4}$$

A fundamental feature of modular forms is that the spaces they comprise are finite dimensional, and in addition, many of them are spanned by forms with rational (Fourier) coefficients [10]. These coefficients turn out to be important from many perspectives: they might encode numbers of partitions, multiplicities of lattice vectors, representations of integers by quadratic forms, characters of affine Lie algebras, and quite remarkably characters of representations of sporadic finite groups.

Besides their importance in mathematics, they have a direct application to physics: they provide information about the spectrum or the dynamics of various physical systems. In this thesis, we investigate their appearance (and the appearance of various other number theoretical objects) as partition functions of (super)conformal field theories and other quantum field theories.

Partition functions

Given a statistical (thermodynamical) system with a discrete spectrum of states, the associated partition function can be viewed as the generating function of the numbers of states (the dimension of the Hilbert space $\mathscr{H}$) at the different energy levels,

$$Z(\tau) = \mathrm{Tr}_{\mathscr{H}}\left(e^{-2\pi\tau H}\right) , \tag{1.1.5}$$

the parameter τ is proportional to the inverse of the temperature and it is a purely imaginary quantity[4], the energy level is the eigenvalue of the Euclidean

[4] In general $\tau = \tau_1 + i\tau_2$. Here we restrict to a rectangular torus with size 1 and τ_2.

time evolution operator H (the Hamiltonian) on the state.

Moreover, for a more general system we can grade the different states comprising the theory not only by their eigenvalues under H but under any mutually commuting operator that can be simultaneously diagonalized with the Hamiltonian of the system. This is exactly what occurs in the computation of characters of affine Kac-Moody Lie algebras, which are defined as traces over formal exponentials of elements of the maximal torus (Cartan subalgebra §3.1). In other words, the affine Lie algebra $\mathfrak{g}$ reflects the local symmetries of the system. Indeed, denoting by J_i the elements of the Cartan subalgebra (generators of the maximal abelian subalgebra of $\mathfrak{g}$) with $i = 1, \ldots, r-1$ where r is the rank of the Cartan subalgebra, we can consider more generally

$$Z(\tau, z_1, z_2, \ldots, z_{r-1}) = \mathrm{Tr}_{\mathscr{H}}\left(e^{-2\pi\tau H} e^{2\pi i z_1 J_1} \ldots e^{2\pi i z_{r-1} J_{r-1}}\right). \tag{1.1.6}$$

Unfortunately, it is usually difficult to compute the partition function for general theories. Nevertheless, we might be able to access a subsector of the Hilbert space or to compute generating functions, which can be recovered from the above expression by setting certain z_i's to zero. When one deals with supersymmetric theory these specialized partition functions[5] might reproduce indices of the space-time manifold. In a physics language, these are protected quantities under change in the couplings of the system, that might be even be accessible for computation thanks to supersymmetry.

The free boson and the Heisenberg algebra

An example of counting function that relates to the study of unrestricted partitions of integers is the partition function of a chiral conformal field theory composed of 24 free bosons[6]. A theory is said to be conformal if it is invariant under reparametrization (general coordinate transformation) and scaling. The bosonic fields $X(\Sigma)$ are maps from a two-dimensional surface $\Sigma = S^1 \times \mathbb{R}$, an infinite elongated cylinder, and the 24-dimensional space $\mathbb{R}^{24}$. In string theory the two-dimensional space corresponds to the worldsheet of a closed string.

The trace in the definition of the partition function in (1.1.5) is responsible for the identification of the two boundaries of the cylinder at the initial and final time, thus giving rise to a theory defined on a torus $S^1 \times S^1$. More generally, we might allow for possible rotations of the final spatial S^1, which corresponds to the

[5]Introducing fermioinic fields require the insertion of $(-1)^F$.

[6]This is the chiral-half of the CFT at the base of the worldsheet description of bosonic string theory, or similarly the left-moving sector of the heterotic string theory.

insertion of another formal exponential with operator P representing translation in the Euclidean space S^1. Defining $(H+P)/2 = (\ell_0 - c/24)$, we obtain

$$Z(\tau) = \mathrm{Tr}_{\mathscr{H}}\left(e^{-2\pi\tau_2 H} e^{2\pi i \tau_1 P}\right) = \mathrm{Tr}_{\mathscr{H}}\left(q^{\ell_0 - c/24}\right), \tag{1.1.7}$$

where $\tau = (\tau_1 + i\tau_2)$, and as usual $q = e^{2\pi i\tau}$. Here we restricted to the left-moving degrees of freedom, and thus took $(\bar{\ell}_0 - \bar{c}/24) = (H-P)/2 = 0$, the generalization to non-chiral CFT can be easily derived. The constant c has been added to shift the vacuum energy density such that it vanishes in the limit that the radius of the circle is taken to be infinite[7].

Due to the equations of motion[8] describing the dynamics of the bosonic field, they can be decomposed into modes which satisfy the Heisenberg algebra

$$[a_m^\mu, a_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}\,, \tag{1.1.8}$$

where $\eta^{\mu\nu}$ represents the spacetime metric. By rescaling $a_m^\mu \to 1/\sqrt{m} a_m^\mu$ we get the commutation relations for the harmonic oscillators $[a_m^\mu, a_n^\nu] = \eta^{\mu\nu}\delta_{m+n,0}$. The ground states are defined as the states annihilated by the lowering operators (a_n, $n > 0$), and in this case, they are specified by a one-parameter family of vacua, labeled by the eigenvalue of the momentum operator, which is proportional to the mode a_0. The Fock space of this theory is constructed by acting with the creation operators a_{-n}, $n > 0$, on the ground states. In terms of this description $(\ell_0 - c/24) = (H+P)/2 = \frac{1}{2}{a_0}^2 + \sum_{n=1}^\infty a_{-n}\cdot a_n - c/24$ where the product corresponds to the inner product with respect to the spacetime metric. By neglecting the term arising form the zero modes (a_0), the partition function for a single boson is

$$\eta(q)^{-1} = q^{-1/24}\prod_{n=1}^\infty (1-q^n)^{-1}\,, \tag{1.1.9}$$

and it coincides (up to an overall q-power) with the generating function of integer partitions $\mathcal{P}(q)$. Clearly, for 24 independent bosons we have

$$\mathrm{Tr}_{\mathscr{H}}\left(q^{\sum_{n=1}^\infty a_{-n}\cdot a_n - c/24}\right) = q^{-1}\prod_{n=1}^\infty \frac{1}{(1-q^n)^{24}}\,. \tag{1.1.10}$$

The coefficients in the Fourier expansion of the above expression represents the number of colored partitions of a positive integer n through integers of 24 colors. The same result is recovered by solely looking at the symmetries of the system and representations thereof. The Fock space of the theory carries (projective)

[7] The constant shift can be seen to arise from normal ordering of the operators and can be computed through zeta function regularization. It corresponds to the normalization used for characters of affine Lie algebras. And, finally, it reflects the central extension of the Witt algebra.

[8] They are derived from a variation principle of the action characterizing the theory.

unitary representations of the associated groups of symmetries, which in this case correspond to the Virasoro algebra. Counting the number of states with fixed ℓ_0-eigenvalue, that is to say, given energy in a chiral CFT, reduces to a computation of the number of states in a given irreducible representation (or Verma module). In particular, it enters the same combinatorics as the one entering the calculation of unrestricted integer partitions[9].

Based on the same reasoning, we can construct partition functions of more general chiral conformal field theories. An example is provided by CFTs built out of a finite number of bosonic fields which are periodic functions of a lattice[10]. Particular attention is given in chapter 3 to chiral (super)CFTs where the bosonic and/or fermionic fields take values on an even unimodular lattice Λ. The target-space coincides with the compact space $\mathbb{R}^n/\Lambda$ or $\mathbb{Z}_k$-orbifolds of this geometry. The partition functions (or elliptic genus) of these theories are Jacobi modular forms, whose decomposition into characters of the superconformal algebra makes their mock-character apparent. The reader who does not know the technical jargon is encouraged to go to chapter 3 for explanations and details.

1.2 Asymptotics and black holes

Asymptotic of q-series

There have always been several motivations to study the asymptotic of generating functions of sequences (holomorphic q-series), primarily because of the wide range of applications of these q-series. Initial results on the asymptotic of the Eulerian series $\mathcal{P}(q)$ came from the study of Ramanujan and Hardy in [11]. The difficulty in deriving an asymptotic formula for the function $\mathcal{P}(q)$ lies in the presence of an infinite number of poles located on the unit circle. By cleverly turning this obstacle into an advantage, Hardy and Ramanujan examined the asymptotic of the function and managed to gather information about it directly from its pole structure. This led them to formulate the theory of the circle method.

From the result in [11], the asymptotic formula for the Fourier coefficients of $\mathcal{P}(q)$

[9]Repeated applications of the creation operators a_{-n} (or equivalently ℓ_{-n}) with $n > 0$ on the vacuum give rise to integer partitions of n.

[10]If the lattice is a root lattice of a Lie algebra then the theory has a so-called Kac-Moody symmetry.

reads

$$\alpha(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k \leq c\sqrt{n}} \mathfrak{K}_n^0(k, \rho_\eta^{-1}) \sqrt{k} \frac{d}{dn} \left(\frac{e^{\frac{C}{k}\sqrt{n-1/24}}}{\sqrt{n-1/24}} \right) + O\big(n^{-1/4}\big) , \qquad (1.2.1)$$

where c is an arbitrary constant, $C = \pi\sqrt{2/3}$, ρ_η is the multiplier system of the η-function defined in Appendix A, equation (A.2.2), and $\mathfrak{K}$ is the Kloosterman sum defined in (2.5.6). Rademacher derived an explicit formula for the Fourier coefficient as an absolutely convergent series [12]

$$\alpha(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \mathfrak{K}_n^0(k, \rho_\eta^{-1}) \sqrt{k} \frac{d}{dn} \left(\frac{\sinh\left(\frac{C}{k}\sqrt{n-1/24}\right)}{\sqrt{n-1/24}} \right) , \qquad (1.2.2)$$

where the Bessel function $I_{3/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left(\frac{\sinh(z)}{z}\right)$ and the rest is as above. This technique was later improved by Rademacher and led to the definition of what is referred to as Rademacher series.

The expression of the Fourier coefficients of certain automorphic forms were independently constructed by other authors (see §2.5 for references) through the so-called averaging method. This method efficiently constructs a modular function (and in particular can be applied to build the partition function of integer sequences) directly from the knowledge of its symmetries (and a few more details). This idea goes back at least as far as Poincaré [13].

The Rademacher sum, independently of the method used to construct it, is a powerful tool which allows to completely reconstruct a (mock) modular form once its modular transformations and q-polar terms at the different cusps of the modular group are known. It is intimately tied to the sum over the different geometries arising from a quantum gravity path integral, the so-called Farey tail [14], as briefly reviewed in chapter 3 and chapter 4.

Black hole entropy

From a physical viewpoint, the estimate of the asymptotic of partition functions (or related objects) describes the rate of growth of states in $\mathscr{H}$ (or subsectors thereof) which in a thermodynamical system is related to the entropy of the system. The the statistical description of the partition function and the Euclidean version of a gravity path integral provide the starting point for the study of the thermodynamics of quantum gravity objects such as black holes. In addition, thanks to the AdS/CFT correspondence[11], the entropy of a class of black holes was

[11] This is a correspondence between a supergravity theory in Anti de Sitter space in $d+1$ dimensions and a conformal field theory in d dimensions [15].

shown to be encoded in the index of a supersymmetric conformal field theory.

An example of microscopic counting of degeneracies of a black hole system, that will appear again in connection with the physical configuration analyzed in chapter 4, is provided below for a class of so-called "small black holes". These are black holes with vanishing horizon area in the supergravity theory (which can be viewed as low-energy effective theory of a string theory), but non-zero area in the corresponding string theory. For this class of examples the degeneracies of states comprising the black hole system can be computed exactly via string theory.

The physical system we are interested in is Type IIB string theory on K3 $\times T^2$ or equivalently heterotic string theory on T^6. The small black hole solutions (also called Dabholkar-Harvey states) are 1/2-BPS states solutions of the Type IIB string theory. The associated microscopic function can be more easily computed in the heterotic frame and was shown to be given by

$$Z(\tau) = 16q^{-1} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{24}} = \sum_n d_n q^n \,, \tag{1.2.3}$$

where the constant term accounts for the degeneracy of the right-moving supersymmetric ground states and $d_n/16$ counts the number of excited left-moving transverse bosons at each energy level graded by n. The reader is referred to [16] and references therein for more details.

The leading contribution to the black hole entropy S_{BH}, that is the large-n degeneracy of $Z(\tau)$, can be extracted from the high temperature limit $\tau \to 0$ of the microscopic counting function. This can be computed by first approximating $Z(\tau)$ with $(-i\tau)^{12} e^{-2\pi i/\tau}$, where we exploited the modular transformation of $\eta(\tau)$

$$\eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau) \,, \tag{1.2.4}$$

and then by evaluating the integral

$$d_n = \frac{1}{2\pi i} \oint Z(\tau) \frac{dq}{q^n} \tag{1.2.5}$$

by saddle point approximation. This yields

$$S_{BH} = \log(d_n) \sim_{n\to\infty} 4\pi \sqrt{n} \,. \tag{1.2.6}$$

The same result can be recovered from the Cardy formula [17]. A close cousin of the Rademacher sum given in (1.2.2) reconstructs all higher order (quantum gravity) corrections to this approximation.

Another interesting phenomenon is that these 1/2-BPS states can form bound states and these are responsible for the wall-crossing phenomenon for 1/4-BPS states described in chapter 4. In particular, in this chapter we will examine Rademacher sums for 1/4-BPS states that are not bound states of two Dabholkar-Harvey states, and that are also called single center solutions.

1.3 Mock, false and 3-manifolds

Mock modular forms and false theta functions

The generating function of unrestricted partition defined in (1.1.2) converges inside the unit circle and its singularities fill out the entire unit circle creating a natural boundary. At the same time, $\mathcal{P}(q)$ converges outside the disc, and, in fact, the problem of convergence is restricted to the unit circle. Therefore, one may wonder if it is possible to modify the Rademacher expression in (1.2.2) so to recover the function on the other side of the disc. The first to address this question was Rademacher in [18].

Following [18, 19], we formally write the coefficients of $\mathcal{P}(q)$ for $|q| > 1$ and thus $\tau \in \mathcal{H}^-$ (the lower half-plane) as $\widetilde{\alpha}(-n) = -\alpha(n)$ for $n > 0$. This yields the formula[12]

$$\widetilde{\alpha}(m) := \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \mathfrak{K}^{0}_{-m}(k, \rho_\eta^{-1}) \sqrt{k} \frac{d}{dm} \left(\frac{\sin\left(\frac{C}{k}\sqrt{m+1/24}\right)}{\sqrt{m+1/24}} \right) \tag{1.3.1}$$

The function $\widetilde{\mathcal{P}}(q) = 1 + \sum_{m=0}^{\infty} \widetilde{\alpha}(m) q^{-m}$ was proven by Petersson to vanish identically [20]. Similar considerations hold for other modular forms of negative weights. Due to these results, Rademacher called the above expression the “expansion of zero”.

The “expansion of zero” principle was later extended to the context of mock modular forms, references are given in chapter 5. Similarly to the above, mock theta functions[13] present infinitely many singularities on the unit circle, however, it is possible to extend their definition from the upper to the lower half-plane. This time the function is a mock modular form on one side of the plane and a non-vanishing function on the other side. Surprisingly, the expansion of zero principle produces completely different functions on the other side of the plane: from a mock theta function one obtains a false theta function on the other side of the plane. The flow between one side and the other of the plane is dictated by an elusive

[12] Note the appearance of the sine instead of the hyperbolic sine in (1.2.2).

[13] We restrict here to mock modular forms of weight 1/2 whose shadow is a cusp form, which are examples of harmonic Maass forms and include the instances brought to light by Ramanujan [5].

number-theoretical object, the so-called quantum modular form, which will be one of the players in chapter 5.

3-manifolds invariants

From the start, quantum modular forms have been introduced to investigate physical[14] and often topological quantities from a number theory perspective. Many instances of quantum modular forms were detected in relation to 3-manifold and knot invariants. An example of such invariants are the so-called WRT invariants. These were described in terms of a three-dimensional Chern-Simons path integral in [22].

Certain WRT invariants provide examples of the quantum modular forms described in the previous section in relation to false theta functions. The physics of the connection between quantum modular forms and false theta functions came into view only recently, thanks to the discovery of the 3d-3d correspondence [23]. The latter provided a bridge between Chern-Simons theory on M_3 and a 3d supersymmetric gauge theory $T[M_3]$. Moreover, it gave a new way to describe topological phases of matter in 2+1 dimensions, e.g. quantum Hall effect, topological insulators and superconductors (see e.g. [24]). The theory on which we focus for the rest of the discussion corresponds to a supersymmetric version of the topological phase of matter where the 3d theory has a mass gap and leads to gapless 2d excitations on its boundary, which makes the combined system non-anomalous.

The 3d theory has $\mathcal{N} = 2$ supersymmetry and 2d "edge modes" on the boundary, which preserve 2d $\mathcal{N} = (0, 2)$ supersymmetry. The supersymmetric partition function for the BPS sector (preserving two out of four supercharges) of this supersymmetric gauge theory living on a cigar ($D^2 \times_q S^1$) and with specific boundary condition $\mathcal{B}_a$ is the so-called "half-index", $\widehat{Z}_a(q)$. Half-indices were conjectured to be building blocks for the supersymmetric index $\mathcal{I}(q)$, which is the index of $T[M_3]$ on $S^2 \times_q S^1$ [25],

$$\mathcal{I}(q) = \sum_a |\mathcal{W}_a| \widehat{Z}_a(q) \widehat{Z}_a(q^{-1}) \,, \tag{1.3.2}$$

where the sum is over boundary conditions $\mathcal{B}_a$, which corresponds to equivalence classes of abelian flat connections of M_3 and $\mathcal{W}_a$ is the stabilizer group of the element a under the action of the Weyl group $a \mapsto -a$. The half-index was recently argued to define a homological block [25]. The latter is an invariant of M_3 and it corresponds to the graded Euler characteristic of the Hilbert space of the theory living on $D^2 \times_q S^1$ with boundary condition $\mathcal{B}_a$ on the boundary of

[14]The physical origin accounts for the vague name and definition provided in [191].

the disc D^2. Homological blocks turn out to be given by false theta functions for specific classes of 3-manifolds. In chapter 5, we investigate this point further. The limit of q to a point on the unit circle of the form $e^{2\pi i/k}$, with $k \in \mathbb{Z}$ recovers the quantum modular form associated to the value of the Chern-Simons path integral on M_3 with integer level k.

Contrary to Ramanujan's expectation, false theta functions turn out to have a dominant role in the definition of these new homological invariants, and furthermore, they appear to be related to mock modular forms on the other side of the plane. Indeed one may wonder what is the homological block associated to $\overline{M}_3$, the closed oriented 3-manifolds with opposite orientation with respect to M_3 or, similarly, what is the "other half-index" entering the computation of the supersymmetric index: $\widehat{Z}_a(q^{-1})$ is supposed to be an appropriate extension of $\widehat{Z}_a(q)$ to the region $|q| > 1$ (or, equivalently, to $\mathrm{Im}(\tau) < 0$). This is exactly the problem that was addressed in the previous subsection and that will be discussed in more details in chapter 5.

1.4 A note to the reader

This thesis presents part of the interplay between number theory, conformal field theory and string theory in the past and present years. Even though the presentation of the subjects is primarily adapted for physicists, part of the text might be more accessible to number theorists. In fact, not only do the topics differ from chapter to chapter but also the language used to investigate these topics varies in accordance with the tradition of each field. We hope that the attempt at merging the various fields presented in this text will not be viewed as an unusual and unwieldy hybrid, but rather as en enrichment of the discussion.

The mathematical review presented in chapter 2 is intended as an introduction to specific topics in number theory. As such, it might allow the reader to learn more about this subject. On the other hand, it might be beneficial to restrict to particular sections which serve as the basis for the rest of the thesis. Indeed, since the purpose is to give a self-contained introduction, certain items are not strictly necessary to the understanding of subsequent chapters. However, they provide a complement to the discussion which strives for a certain elegance in our exposition. Broadening this exposition was not only meant to achieve a complete treatment but it also serves as a starting point to connect the topics analyzed here to other works. In this respect, the analysis of Jacobi theta series for general lattices does not directly apply to any of the later sections except for the simple A_1-lattice, to which we could have restricted ourselves from the start. Jacobi theta series of generic lattices, however, enter the study of different gauge theories and naturally appear in the context of additive and multiplicative lifts.

The remaining three chapters are based on my research during the last four years. Similarly to the Introduction, the common root of these chapters lies in the first chapter, or in other words stems from the underlying number theory. At the beginning of each chapter, we briefly review some physics background and provide the motivations that led to the work presented therein. We conclude each chapter with a summary of the main results and expand on possible future directions. Apart from the physical results the presentation includes new mathematical results, see for instance chapters 4 and 5. Finally, in 5.9 we give a summary of the results presented in this thesis, as well as a discussion of open problems and future directions.

1.4.1 Outline

We begin with a mathematical introduction to the thesis in chapter 2. The prominent (even if sometimes not manifest) role of lattices and theta functions makes it natural to open this chapter with a presentation of these notions in the first section §2.1. Follows a whole section §2.2 dedicated to Siegel and Jacobi modular forms. Mock modular forms are reviewed in section §2.3. Section §2.4 deals with an important representation of the metaplectic group (double cover of $SL_2(\mathbb{Z})$). The last part of this chapter, specifically section §2.5, surveys two techniques for the explicit construction of modular objects. A geometrical review on modular forms along with basic concepts on modular groups is reported in Appendix A.

The second chapter is dedicated to the study of extremal conformal field theories. After a brief introduction to the topic and its relation to moonshine and sporadic groups, we review the basic concepts for the definition of a conformal field theory §3.1. In §3.2 we depict its relation to three-dimensional quantum gravity and summarize the known examples of (super) ECFT. Later we review the important features of the monster CFT §3.3, and examples of extremal theory with central charge 12 and 24 in section §3.4 and §3.5 respectively. The Rademacher expansion of the twining functions is analyzed in §3.6 and concluding remarks are conveyed in the last section §3.7 together with open questions. Relevant definition of superconformal characters, twining functions and character tables are reported in Appendix B.

In chapter 4 we derive the Rademacher expansion for mixed mock Jacobi forms. The connection with the physics of supersymmetric black holes and their exact entropy is provided at the beginning of the chapter. In section §4.2 the microscopic derivation of the counting function is reviewed. The main theorem and its proof are reported in section §4.3, few details have been postponed to the Appendix C. Section §4.4 describes the macroscopic derivation performed via the quantum entropy function in the literature and compares these results with the one we obtained. To conclude, we summarize our results and provide possible resolution to the mismatch of the macroscopic and the microscopic state counting in §4.5.

In the last chapter, we investigate invariants of 3-manifolds through quantum modular forms, false theta functions, and mock modular forms. In section §5.1 we outline the known connection between three-manifolds invariants, three-dimensional Chern-Simons theory, and quantum modular forms. The description of these type of quantum modular forms naturally leads to the definition of false theta functions §5.2. Superconformal 3d $\mathcal{N} = 2$ theories and homological blocks are introduced in section §5.3. The explicit formula for homological blocks of certain type of 3-manifolds are derived in §5.4. Later, in section §5.5 a description of these objects in terms of analytically continued CS theory and the resurgence analysis of compact CS is reported and an interpretation in terms of false theta functions and Weil representation of the modular group is given. Finally, we analyze the extension of the homological blocks outside the unit disc in section §5.6 and section §5.8.

Chapter 2

Number theory prelude

In this chapter, we review most of the mathematical background relevant to the thesis.

The first section, primarily based on [26], describes some of the features of lattice theory. The theory of lattices permeates the whole thesis, either simply through the lattice of integers (that is the A_1-lattice) or more explicitly in chapter 3 in relation to sporadic groups, error-correcting codes, and Kac-Moody algebras. Quadratic forms of integral lattices and associated theta series connect lattice theory to another branch of mathematics: number theory.

The rest of the chapter is an introduction to some of the concepts of number theory. In section §2.2.1, we briefly summarize the concept of Siegel upper-half plane and Siegel modular form, for further reading see for instance [27]. In recent years, this type of automorphic forms has acquired particular relevance in physics in relation to BPS indices corresponding to enumerative invariants such as Gromov-Witten and Gopakumar-Vafa invariants [28, 29], or BPS indices accounting for a microscopic description of the Bekenstein-Hawking entropy of supersymmetric black holes [8]. In chapter 4 we investigate further this topic in the specific example of extremal dyonic black holes in four dimensions.

Follows the theory of Jacobi forms for integer lattices [30]; particular emphasis is given to the theory of Jacobi forms associated to the lattice of integers, as main reference for this we refer to [31]. Due to the leading role played by the Jacobi theta series for the A_1-lattice both in the classical theory of Jacobi modular forms and in the more recent theory of mock Jacobi forms, this turns out to be one of the central subjects of this thesis. Nonetheless, it is worthwhile pointing out the appearance of generic lattice Jacobi forms in the construction of Siegel modular forms [32, 33] and the study of BPS invariants of six-dimensional superconformal field theories [34, 35].

Later in section §2.3, the concept of Jacobi modular form is extended to the notion of mock Jacobi forms, described via mock modular forms [6,37]. A wide range of applications of these nearly modular objects to string theory has been identified, for instance, in systems which exhibit wall-crossing phenomena or configurations with a continuous spectrum of states [7]. The properties of mock theta functions are further analyzed in chapter 5 in relation with quantum modular forms, false theta functions and the physics of 3-manifolds invariants.

Both mock Jacobi forms and Jacobi modular forms belong to the space of elliptic functions, the analysis of the latter furnishes important clues on the structure of the spaces of Jacobi forms. In section §2.4, a specific method based on the study of irreducible projective representations of the modular group $SL_2(\mathbb{Z})$ is described to obtain the space of optimal mock Jacobi forms of weight 1. Ingredients appearing in this construction beautifully emerge in the study of umbral moonshine [36] and homological blocks of three-manifolds invariants [25].

The explicit construction of (mock) Jacobi forms is accomplished via the tool of Rademacher sums, introduced in section §2.5. These methods provide new insights into the theory of (super)conformal field theory, Euclidean quantum gravity in Anti de Sitter space, and $\mathcal{N} = 2$ three-dimensional theories. The relation between Rademacher series and elliptic genus of supersymmetric conformal field theories is investigated in chapter 3; an extension of this technique is constructed in chapter 4 to compute the quantum entropy function of quarter-BPS in $\mathcal{N} = 4$ theories in four dimensions. Lastly, in chapter 5 this method is implemented as a mean to leak from the upper half-plane, where false theta functions are defined, to the lower half, where mock modular forms appear.

2.1 Lattices and Quadratic forms

We begin by reviewing the basic notions of the classical theory of lattices.

Definition 2.1.1. A lattice is a discrete additive group Λ over a real vector space V endowed with a symmetric bilinear form $(\cdot, \cdot) : \Lambda \times \Lambda \to \mathbb{Q}$.

In the following we take V to be $\mathbb{R}^n$ and the bilinear form to be the standard scalar product over $\mathbb{R}^n$. Let $\{e_i\}_{i=1,\dots,n}$ denote a basis of Λ, and let n be the rank of the lattice[1]; the scalar products between basis vectors is encoded in the so-called Gram matrix

$$A := M^t M \,, \tag{2.1.1}$$

[1]Throughout this section we restrict to positive definite lattices.

where the column vectors of M consist of the basis vectors of Λ. The quadratic form associated to Λ is given by

$$(x,x) := \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j (e_i, e_j) = c^t A c \tag{2.1.2}$$

for any vector $x = \sum_{i=1}^{n} c_i e_i \in \Lambda$, $c_i \in \mathbb{Z}$. Two lattices are said to be equivalent if the corresponding Gram matrices are related by a unitary transformation. We denote by $\Lambda(a)$ the lattice Λ rescaled by a constant a.[2] A quantity characterizing the lattice independently of the chosen basis is the determinant of Λ

$$\det(\Lambda) := \det(A)\,, \tag{2.1.3}$$

that is the square of the volume of a fundamental region. A lattice is said to be *integral* if its quadratic form has value in $\mathbb{Z}$; moreover, an integral lattice is *even* if the norm of each vector $x \in \Lambda$ has value in $2\mathbb{Z}$. For an even lattice Λ, we define the level of Q to be the smallest positive integer N such that NA^{-1} is again an even matrix (it has even number on the diagonal and integers in the off-diagonal entries). A notable example of integral lattice is the root lattice, which is defined as the lattice generated by vectors with norm $(x,x) = 2$ (the roots). As long as we take as reference lattice the minimal integral one, the Gram matrix of the root lattice coincides with the Cartan matrix of the corresponding Lie algebra.

The dual of a lattice Λ is defined as

$$\Lambda^* := \{y \in \mathbb{R}^n \,|\, (y,x) \in \mathbb{Z},\ \forall x \in \Lambda\}\,. \tag{2.1.4}$$

Its Gram matrix is the inverse of the one associated to Λ, and therefore the determinant of Λ^* is the inverse of $\det(\Lambda)$. According to the definition of the dual lattice, a necessary and sufficient condition for a lattice to be integral is that $\Lambda \subseteq \Lambda^*$. A fundamental object for our later discussion is the quotient group (or discriminant group) Λ^*/Λ. The order of Λ^*/Λ is given by the determinant of Λ. When the absolute value of the determinant of Λ is equal to one, the original lattice coincides with the dual lattice and Λ is said to be a *unimodular* (or *self-dual*) lattice.

Another way to describe the dual lattice Λ^* is through the "gluing theory" [26]. The latter provides a technique to build an n-dimensional lattice starting from an n-dimensional sublattice. Here we review the main ingredients of this theory. An integral lattice Λ, embedded in $\mathbb{R}^n$, can be constructed from an n-dimensional

[2] In [26] two lattices are considered to be equivalent if they are related by a rescale of the basis vectors. We prefer to adopt a different convention, and define an equivalence transformation as a map that leaves the absolute value of the determinant of Λ invariant.

sublattice $\widetilde{\Lambda}$ of the form $\Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_k$ by gluing the different components Λ_i with a finite number of vectors. The latter are representatives of the cosets Λ_i^*/Λ_i for $i = 1, \ldots, k$; therefore the maximum number of "glue" vectors associated to the i-th component Λ_i is given by $|\det(\Lambda_i)|$. To understand the gluing procedure, consider a generic vector $x \in \Lambda$. This can be expressed as a sum of k-vectors, each belonging to the subspace spanned by Λ_i but not necessarily to Λ_i itself. Due to the integrality of the lattice Λ, the inner-product of x with any vector $x_i \in \Lambda_i$ is an integer. Since the above description is unaltered if we add to x a vector of Λ_i, we can think of x as a representative of the coset Λ_i^*/Λ_i. Lastly, the glue vectors are well-defined if and only if they have integral inner-products and are closed under addition modulo $\Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_k$. Clearly, the dual lattice Λ^* can be described through this procedure, and it is given by

$$\Lambda^* = \cup_{i=1}^{\det\Lambda} (\Lambda + [i]) \tag{2.1.5}$$

where $[i]$ stands for the i-th glue vector. Generally the automorphism group of a lattice determined through the gluing procedure is the group of permutations of the different components of the sublattice $\widetilde{\Lambda}$. The automorphism group that leaves the different components fixed consists of (inner) automorphisms, which translate a glue vector $[i]$ by an element of Λ_i, and (outer) automorphisms, which permute the glue vectors of each component of the sublattice. If Λ_i is a root lattice, the translation of a glue vector by an element of Λ_i corresponds to the action of the Weyl group. To elucidate this statement, denote by r_i a root of Λ_i and by y a glue vector for Λ, then the action of the Weyl group on y corresponds to a reflection in the plane orthogonal to r_i

$$y \mapsto y - 2\frac{(y, r_i)}{(r_i, r_i)} r_i = y - (y, r_i) r_i = y - n r_i, \qquad n \in \mathbb{Z}\,. \tag{2.1.6}$$

On the other hand, a permutation of the glue vectors associated to Λ_i coincides with an automorphism of the Dynkin diagram of Λ_i.

The quotient group Λ^*/Λ can be viewed as a finite quadratic module so long as Λ is an even lattice [38]. This statement is clarified by the following definition [39,40].
Definition 2.1.2. A finite quadratic module (G, Q) is a finite abelian group G equipped with a non-degenerate quadratic form $Q : G \to \mathbb{Q}/\mathbb{Z}$, such that

$$Q(nx) = n^2 Q(x) \qquad x \in G\,, n \in \mathbb{Z} \tag{2.1.7}$$

and for any x, $y \in G$ the map $B(x, y) := Q(x + y) - Q(x) - Q(y)$ is a bilinear, symmetric form which vanishes for all x if and only if $y = 0$.

An automorphism of a finite quadratic module is an isomorphism $\varphi : G \to G$ such that $Q \circ \varphi = Q$; we refer to the automorphism group of G as $O(G)$, the orthogonal group of G. The level of a quadratic module is the smallest positive integer N such that $NQ(x) \equiv 0$ for any $x \in G$.

The finite quadratic module (or discriminant module) associated to the even lattice Λ with bilinear form $(\cdot,\cdot)$ is defined by

$$D_{\Lambda,Q} := \left(\Lambda^*/\Lambda\,,\; x+\Lambda \;\mapsto\; \frac{1}{2}(x,x)+\mathbb{Z}\right) \tag{2.1.8}$$

where $x \in \Lambda^*/\Lambda$ and the quadratic form $Q(x) := \frac{1}{2}(x,x) \bmod \mathbb{Z}$. Any discriminant form can be constructed from the quotient group of an even lattice by a suitable rescaling of the quadratic form.

To each lattice we can associate a formal power series $\theta_\Lambda \in \mathbb{Z}[[q]]$ whose coefficients encode the multiplicity of the vectors with a fixed norm

$$\theta_\Lambda(q) := \sum_{x\in\Lambda} q^{(x,x)/2}\,. \tag{2.1.9}$$

In other words, the theta function of an even lattice Λ is the generating function of the number of solutions of the quadratic equation

$$Q(x) = c^t A\, c = m\,, \qquad m \in \mathbb{Z} \tag{2.1.10}$$

where m denotes the power of q in the theta series. If two lattices have different theta series they are certainly inequivalent, however the converse does not always hold: two inequivalent lattices might have the same theta series. This happens for instance in the case of the D_{16}^+ lattice and the $E_8 \oplus E_8$ lattice[3].

Here we list a number of examples of integral lattices that will be useful in the sequel, for more details the reader is referred to [26].

$\mathbb{Z}^n$ **lattice.** The n-dimensional integer lattice

$$\mathbb{Z}^n = \{(x_1,\ldots,x_n) \in \mathbb{Z}^n \,|\, x_i \in \mathbb{Z}\,,\, i=1,\ldots,n\} \tag{2.1.11}$$

is a unimodular lattice with Gram matrix equal to the identity matrix in n dimensions. Restricting to the one-dimensional case, the associated theta function is

$$\theta_{\mathbb{Z}}(\tau) = \sum_{m\in\mathbb{Z}} q^{\frac{m^2}{2}} = 1 + 2q^{1/2} + 2q^2 + 2q^{9/2} + 2q^4 \ldots\,, \tag{2.1.12}$$

The shifted lattice $\mathbb{Z}+1/2$ has theta series

$$\theta_{\mathbb{Z}+1/2}(\tau) = \sum_{m\in\mathbb{Z}} q^{\frac{(m+1/2)^2}{2}} = 2q^{1/8} + 2q^{9/8} + 2q^{25/8} + \ldots \tag{2.1.13}$$

[3]This reflects the equality between the partition functions of heterotic string theory with gauge group $SO(32) \sim SU(32)/\mathbb{Z}_2$ and $E_8 \oplus E_8$ respectively.

In the following we denote by $\theta_3(\tau)$ the theta series of the one-dimensional integer lattice, and $\theta_2(\tau)$ the theta series of the shifted lattice.

A_n **lattice.** The A_n root lattice is defined by

$$A_n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{Z}^{n+1} |\, x_0 + \cdots + x_n = 0\}\,. \tag{2.1.14}$$

The A_1 lattice has a one-dimensional Gram matrix ($A = 2$), the elements of the quotient group are the vectors $[0] = (0,0)$ and $[1] = \left(\frac{1}{2}, -\frac{1}{2}\right)$ and the corresponding theta series are given by

$$\theta_{A_1}(\tau) = \theta_{\mathbb{Z}}(2\tau) = \theta_3(2\tau), \quad \theta_{A_1+[1]}(\tau) = \theta_{\mathbb{Z}+1/2}(2\tau) = \theta_2(2\tau)\,. \tag{2.1.15}$$

The A_2 root lattice (also known as planar hexagonal lattice) is represented by the Gram matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \det(A_2) = 3 \tag{2.1.16}$$

and has quadratic form $Q(x) = 2(c_1^2 - c_1 c_2 + c_2^2)$. The dual lattice consists of points (x_1, x_2, x_3) with $x_1 \equiv x_2 \equiv x_3 \pmod 1$ and $\sum_i x_i = 0$. The determinant of the lattice being equal to 3, there are 3 generators of the group A_2^*/A_2. According to the dual lattice condition, they can be chosen to be

$$[0] = (0,0,0), \qquad [1] = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right), \qquad [2] = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{2.1.17}$$

The theta series associated to the vector [0] is

$$\begin{aligned} \theta_{A_2}(\tau) = \sum_{\ell,m\in\mathbb{Z}} q^{\ell^2-\ell m+m^2} &= \sum_{\ell,m\in\mathbb{Z}} q^{(\ell-m/2)^2+3m^2/4} \\ &= \theta_3(2\tau)\theta_3(6\tau) + \theta_2(2\tau)\theta_2(6\tau) \end{aligned} \tag{2.1.18}$$

where the last equality is achieved by separating even and odd m. The theta series corresponding to the vectors [1] and [2] are related by the order two automorphism, which acts by changing the signs of all the coordinates. This coincides with the symmetry of the A_2-Dynkin diagram and leads to the identity $\theta_{A_2+[1]}(\tau) = \theta_{A_2+[2]}(\tau)$. The explicit expression of this theta series is

$$\theta_{A_2+[1]}(\tau) = \frac{1}{2}\theta_2(2\tau)\big(\theta_2(2\tau/3) - \theta_2(6\tau)\big) + \frac{1}{2}\theta_3(2\tau)\big(\theta_3(2\tau/3) - \theta_3(6\tau)\big) \tag{2.1.19}$$

More generally, for any A_n with $n \geq 2$ this order two automorphism interchanges the vector $[i]$ with $[n-i]$ belonging to A_n^*/A_n.

D_n **lattice.** The D_n root lattice is defined by

$$D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n |\, x_1 + \cdots + x_n = 0 \,(\mathrm{mod}\, 2)\} \tag{2.1.20}$$

The Gram matrix is given by the associated Cartan matrix and it has determinant equal to 4 for any n. The quotient group is defined by the four vectors

$$[0] = (0, 0, \dots, 0)\,,\;\; [1] = (1/2, 1/2, \dots, 1/2)\,, \qquad (2.1.21)$$
$$[2] = (0, 0, \dots, 1)\,,\;\; [3] = (1/2, 1/2, \dots, -1/2)\,.$$

The theta series for this integral lattice is given by

$$\theta_{D_n}(\tau) = \frac{1}{2}\big(\theta_3(\tau)^n + \theta_4(\tau)^n\big) \qquad (2.1.22)$$

where the first term on the right hand side is the theta series for the integer lattice $\mathbb{Z}^n$, while the second series is defined as

$$\theta_4(\tau) := \sum_{m\in\mathbb{Z}} (-q)^{\frac{m^2}{2}} = 1 - 2q^{1/2} + 2q^2 - 2q^{9/2} + \dots \qquad (2.1.23)$$

This series again enumerates the number of integers with fixed norm but with an additional grading, which differentiates between odd and even integers. The remaining vectors give rise to the theta series

$$\theta_{D_n+[1]}(\tau) = \theta_{D_n+[3]}(\tau) = \frac{1}{2}\theta_2(\tau)^n\,, \qquad \theta_{D_n+[2]}(\tau) = \frac{1}{2}\big(\theta_3(\tau)^n - \theta_4(\tau)^n\big). \qquad (2.1.24)$$

From (2.1.5) the theta series of the dual lattice is

$$\theta_{D_n^*}(\tau) = \theta_2(\tau)^n + \theta_3(\tau)^n\,. \qquad (2.1.25)$$

Another important lattice directly constructed from the D_n-lattice is

$${D_n}^+ := D_n \cup (D_n + [1])\,, \qquad (2.1.26)$$

whose theta series can be readily computed,

$$\theta_{D_n^+}(\tau) = \frac{1}{2}\big(\theta_2(\tau)^n + \theta_3(\tau)^n + \theta_4(\tau)^n\big)\,. \qquad (2.1.27)$$

E_8 lattice. The E_8 lattice is an even unimodular 8-dimensional lattice, defined by $E_8 := D_8^+$. There are two coordinate systems usually used to describe this lattice: the even coordinate system, which consists of points of the form

$$\{(x_1, \dots, x_8) \in \mathbb{Z}^8 \vee (x_1, \dots, x_8) \in \mathbb{Z}^8 + (1/2, \dots, 1/2) \mid x_1 + \dots + x_8 \equiv 0 \pmod 2\}$$

and the odd coordinate system, which consists of points of the form

$$\{(x_1, \dots, x_8) \in \mathbb{Z}^8 \vee (x_1, \dots, x_8) \in \mathbb{Z}^8 + (1/2, \dots, 1/2) \mid x_1 + \dots + x_8 \equiv 2x_8 \pmod 2\}$$

The theta series can be derived from (2.1.27) and it is explicitly given by

$$\begin{aligned}\theta_{E_8}(\tau) &= \frac{1}{2}(\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8)\\ &= 1 + 240q^2 + 2160q^4 + 6720q^6 + \dots\end{aligned} \tag{2.1.28}$$

In 8 dimension the unique instance of unimodular lattice (up to equivalence) is the E_8-lattice. Moreover, one can prove that there exist even unimodular lattices only for $n \equiv 0\,(8)$.[4] If the dimension of the lattice is smaller or equal to 24 the number of even unimodular lattices is finite: in 8 dimensions there is the aforementioned E_8 lattice, in 16 dimensions there are the D_{16}^+ and the $E_8 \oplus E_8$ lattices [42], and lastly in 24 dimensions the only examples are the Niemeier lattices [43], to which we now turn to.

Niemeier lattices. Niemeier lattices are the unique even unimodular 24-dimensional lattices. There are in total 24 Niemeier lattices. Among these, there is the so-called Leech lattice, which has vectors of minimal norm 4, and other 23 lattices which are uniquely determined by their root systems, X. In order to differentiate between them, we denote the Leech lattice by Λ_L, and the remaining 23 lattices by Λ_N. The latter are a union of simply-laced root systems with the same Coxeter number and of total rank 24.

The formal power series $\theta_\Lambda(q)$ provides a bridge between lattice theory and number theory. To describe this connection, we take $q = e^{2\pi i\tau}$ with $\tau \in \mathcal{H}$. The theta series is a holomorphic function of τ as proven in [44]. In addition, if Λ is a positive definite lattice of rank r, $\theta_\Lambda(\tau)$ is a modular form of weight $r/2$ with some character for a congruence subgroup of $SL_2(\mathbb{Z})$ of level N [31]. If Λ is even and unimodular the theta series turns out to be a modular form with respect to the full modular group. For instance, take $\Lambda = E_8$: the theta series associated to this root lattice is a weight 4 modular form for $SL_2(\mathbb{Z})$. Since the space of modular form of this weight with respect to the full modular group is one-dimensional and it is spanned by the Eisenstein series of weight 4, $\theta_\Lambda(\tau)$ is completely fixed by its first Fourier coefficient and it is in fact given by

$$E_4(\tau) := 1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^{2m} \tag{2.1.29}$$

where $\sigma_k(m)$ is the sum of the k-th powers of the divisors of m.

Lattice theta series, named by Jacobi “Thetanullwerte”, arise from the restriction of the Jacobi modular forms $\theta_\Lambda(\tau, z)$ to $z = 0$. In the next section, we describe in

[4] A proof is given in [26]. From the perspective of number theory, this fact can be easily proven using properties of modular forms, as explained in [41].

more details these modular objects starting from their relation to Siegel modular forms.

2.2 From Siegel to Jacobi modular forms

This section is devoted to the theory of Siegel and Jacobi modular forms. Siegel modular forms can be viewed as functions defined on a genus g-surface. By restricting to the genus-2 case, we define Jacobi modular forms and the Jacobi modular group [31], whose properties acquire an interesting perspective from the viewpoint of Siegel modular forms [30]. We continue with the description of the isomorphism between spaces of Jacobi modular forms and spaces of modular forms.

2.2.1 Siegel modular forms

Let the space of $g \times g$ matrices over the real field be $M_g(\mathbb{R})$. Denote by $\mathcal{H}_g$ the Siegel upper half-space, which is defined as follows

$$\mathcal{H}_g := \{Z = X + iY \,|\, X, Y \in M_g(\mathbb{R}),\ X = X^t,\ Y = Y^t,\ Y > 0\}\,. \tag{2.2.1}$$

$\mathcal{H}_g$ consists of all symmetric $g \times g$ complex matrices whose imaginary part is positive definite, that is to say for any non-zero vector $v \in \mathbb{R}^g$ the term $v^t\,\mathrm{Im}(Z)\,v$ is strictly positive. If we take g to be one we recover the definition of the upper half-plane

$$\mathcal{H} := \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\big(\tau\big) > 0\}\,. \tag{2.2.2}$$

On the other hand, when $g = 2$ the Siegel upper half-plane is defined by

$$\mathcal{H}_2 = \{Z = \begin{pmatrix} \tau & z \\ z & w \end{pmatrix} \,|\, Z \in M_2(\mathbb{C}),\ \mathrm{Im}(Z) > 0\}\,, \tag{2.2.3}$$

where the condition $\mathrm{Im}(Z) > 0$ implies $\tau \in \mathcal{H}, w \in \mathcal{H}$ and $\mathrm{Im}(\tau)\mathrm{Im}(\omega) - (\mathrm{Im}(z))^2 > 0$. Notice that for any $(\tau, z) \in \mathcal{H} \times \mathbb{C}$ there is a $w \in \mathcal{H}$ such that $Z \in \mathcal{H}_2$.
Similarly to the case of the upper-half plane $\mathcal{H}$, where the action of a generic element γ of the special linear group $SL_2(\mathbb{R})$ on $\mathcal{H}$ is given by fractional linear transformations

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}\,, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})\,,\ \tau \in \mathcal{H}\,, \tag{2.2.4}$$

the symplectic group $Sp_g(\mathbb{R})$ acts on $\mathcal{H}_g$ by linear transformations

$$\Xi\langle Z\rangle = (AZ + B)(CZ + D)^{-1}, \qquad \Xi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_g(\mathbb{R}),\ Z \in \mathcal{H}_g\,, \tag{2.2.5}$$

where

$$Sp_g(\mathbb{R}) = \left\{ \Xi \in M_{2g}(\mathbb{R}) \;\middle|\; \Xi^t \begin{pmatrix} 0 & -\mathbb{I}_g \\ \mathbb{I}_g & 0 \end{pmatrix} \Xi = \begin{pmatrix} 0 & -\mathbb{I}_g \\ \mathbb{I}_g & 0 \end{pmatrix} \right\}. \tag{2.2.6}$$

Given the action of the symplectic group on $\mathcal{H}_g$, to each $Sp_g(\mathbb{Z})$-orbit in $\mathcal{H}_g$ one can associate a genus g surface.

Definition 2.2.1. Let g be greater than or equal to two. A Siegel modular form $f : \mathcal{H}_g \to \mathbb{C}$ of integer weight k is a holomorphic function satisfying the functional equation

$$f(\Xi\langle Z\rangle) = \det(CZ + D)^k\, f(Z)\,, \qquad \forall\, \Xi \in Sp_g(\mathbb{Z})\,. \tag{2.2.7}$$

Let $|_k$ indicate the slash-operator, which acts on functions of the Siegel upper half-plane according to the rule

$$(f|_k\, \Xi)(Z) := \det(CZ + D)^{-k} f(\Xi\langle Z\rangle)\,. \tag{2.2.8}$$

Moreover, this action turns out to be transitive, i.e. it satisfies

$$f|_k(\Xi_1 \cdot \Xi_2) = (f|_k\Xi_1)|_k\Xi_2\,. \tag{2.2.9}$$

The transitivity of the slash-operator is ensured by the cocycle condition satisfied by the automorphic factor $J(\Xi, Z) := \det(CZ + D)$

$$J(\Xi_1 \cdot \Xi_2, Z) = J(\Xi_1, \Xi_2\langle Z\rangle)\, J(\Xi_2, Z) \tag{2.2.10}$$

and by the fact that $(\Xi_1 \cdot \Xi_2)\langle Z\rangle = \Xi_1\langle\Xi_2\langle Z\rangle\rangle$. In terms of the slash-operator the functional equation (2.2.7) satisfied by Siegel modular forms reads

$$f|_k\, \Xi = f\,, \qquad \forall\, \Xi \in Sp_g(\mathbb{Z})\,. \tag{2.2.11}$$

For a genus-2 Siegel modular form the above equation encodes the periodicity properties of $f(Z)$. In particular, it implies that there exists a Fourier expansion for $f(Z)$ with respect to w

$$f(Z) = \sum_{m=0}^{\infty} \varphi_m(\tau, z) e^{2\pi i m w}\,, \tag{2.2.12}$$

where $\varphi_m(\tau, z)$ are holomorphic Jacobi modular forms[5] [31]. The definition and properties of Jacobi modular forms are reported in the next section. In chapter 4 we examine in details an example of meromorphic Siegel modular form: the inverse of the weight 10 Igusa cusp form. This emerges as the counting function

[5]From the holomorphicity of $f(Z)$ derives a positivity condition on the coefficients of the Jacobi modular form $\varphi_m(\tau, z)$. The definition of holomorphic Jacobi modular forms is given in definition 2.2.2.

of quarter-BPS states in Type IIB string theory compactified on K3$\times T^2$. The Fourier expansion of meromorphic Siegel modular forms gives rise to meromorphic Jacobi modular forms with negative and positive indexes. One of the reason why meromorphic modular objects appear as counting functions of black holes degeneracies can be traced back to the wall-crossing phenomenon.

2.2.2 Jacobi modular forms

The Jacobi modular group[6], $\Gamma^J(\mathbb{Z})$, is the parabolic modular subgroup of $Sp_2(\mathbb{Z})$ defined by matrices of the following form,

$$\Gamma^J(\mathbb{Z}) := \left\{ \begin{pmatrix} * & 0 & * & * \\ * & 1 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp_2(\mathbb{Z}) \right\}. \tag{2.2.13}$$

There exists an embedding $SL_2(\mathbb{Z}) \hookrightarrow \Gamma^J(\mathbb{Z})$ given by the elements of $\Gamma^J(\mathbb{Z})$ of the form

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] := \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp_2(\mathbb{Z}) \right\}. \tag{2.2.14}$$

The modular group $SL_2(\mathbb{Z})$ is a normal subgroup of $\Gamma^J(\mathbb{Z})$. Moreover, the Jacobi modular group turns out to be equal to $SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$, where $H(\mathbb{Z})$ denotes the Heisenberg group[7]. The latter consists of the unipotent elements of $\Gamma^J(\mathbb{Z})$

$$H(\mathbb{Z}) := \left\{ [\,(q,p),\, r\,] := \begin{pmatrix} 1 & 0 & 0 & p \\ q & 1 & p & r \\ 0 & 0 & 1 & -q \\ 0 & 0 & 0 & 1 \end{pmatrix},\ p,q,r \in \mathbb{Z} \right\} \tag{2.2.15}$$

with group operation

$$[\,(q,p),\, r\,].[\,(q',p'),\, r\,] = [\,(q+q',p+p'),\, r+r' - \det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}\,]. \tag{2.2.16}$$

[6]Here we restrict for simplicity to the Jacobi modular group $\Gamma^J(\mathbb{Z})$, instead of more general discrete subgroups of $\Gamma^J(\mathbb{R})$.

[7]Sometimes this is called the polarized Heisenberg group to differentiate from the classical Heisenberg group.

The embedding of $SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$ into $Sp_2(\mathbb{Z})$ is described by

$$[\,(q,p),\, r\,]\,.\,\gamma = \begin{pmatrix} a & 0 & b & p \\ aq+pc & 1 & bq+dp & r \\ c & 0 & d & -q \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.2.17}$$

The Heisenberg group is the central extension of the abelian group $\mathbb{Z}\times\mathbb{Z}$, and its center is given by

$$C_{H(\mathbb{Z})} = \{[\,(0,0),\, r\,], r \in \mathbb{Z}\}\,. \tag{2.2.18}$$

The non-commutativity of $H(\mathbb{Z})$ is reflected in the fact that given two element $h, h' \in H(\mathbb{Z})$ we have

$$h.h'.(h)^{-1}.(h')^{-1} = [\,(0,0),\, -2\det\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}]\,, \tag{2.2.19}$$

$$\text{where} \quad h = [\,(q,p),\, r\,]\,, \qquad h' = [\,(q',p'),\, r'\,]\,. \tag{2.2.20}$$

The action of $SL_2(\mathbb{Z})$ on $H(\mathbb{Z})$ is by conjugation

$$\gamma^{-1}.\,[\,(q,p)\,,\, r\,]\,.\,\gamma = [(q,p)\,.\,\gamma, r] = [\,(aq+pc, bq+dp)\,,\, r\,]\,, \tag{2.2.21}$$

and it corresponds to the right-action of $SL_2(\mathbb{Z})$ on the complex torus defined by $(q\tau+p)|_\gamma = (aq+cp)\tau + (bq+dp)\,.$

The explicit action of the parabolic group $\Gamma^J(\mathbb{Z})$ on $\mathcal{H}_2$ is specified by (2.2.21)

$$\gamma\langle Z\rangle = \begin{pmatrix} \frac{a\tau+b}{c\tau+d} & \frac{z}{c\tau+d} \\ \frac{z}{c\tau+d} & w - \frac{cz^2}{c\tau+d} \end{pmatrix} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \tag{2.2.22}$$

$$h\langle Z\rangle = \begin{pmatrix} \tau & z+q\tau+p \\ z+q\tau+p & w+q^2\tau+2qz+qp+r \end{pmatrix} \qquad h = [\,(p,q),\, r\,] \in H(\mathbb{Z}) \tag{2.2.23}$$

Similarly to the case of Siegel modular forms, we define the slash-operator $(|_{k,m})$ of weight k and index m by its action on functions of two variables $(\tau, z) \in \mathcal{H}\times\mathbb{C}$ as

$$\varphi|_{k,m}\, g \;:=\; \left(\varphi(\tau,z)e^{2\pi i m w}|_k\, g\right) e^{-2\pi i m w}\,, \qquad g \in \Gamma^J(\mathbb{Z}) \tag{2.2.24}$$

where the slash operator on the right-hand side represents the slash operator (2.2.8), defined on Siegel modular forms.

Definition 2.2.2. A Jacobi modular form of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}_{>0}$ with respect to $\Gamma^J(\mathbb{Z})$ is a holomorphic function $\varphi : \mathcal{H}\times\mathbb{C} \to \mathbb{C}$ which obeys

$$\varphi|_{k,m}\, g \;=\; \varphi\,, \qquad \forall g \in \Gamma^J(\mathbb{Z}) \tag{2.2.25}$$

and whose Fourier coefficients, defined by the expansion

$$\varphi(\tau, z) = \sum_{n,\ell \in \mathbb{Z}} c(n, \ell)\, q^n\, y^\ell\,, \tag{2.2.26}$$

satisfy certain positivity conditions.
Depending on the asymptotic growth of the coefficients, a Jacobi form is called *holomorphic* if $c(n, \ell) = 0$ unless $4mn \geq \ell^2$, *weak* if $c(n, \ell) = 0$ unless $n \geq 0$ and it is said to be a *cusp* Jacobi form if $c(n, \ell) = 0$ unless $4mn > \ell^2$. Lastly, if the coefficients satisfy the weaker condition that $c(n, \ell) = 0$ unless $n \geq n_0$ for some possibly negative integer n_0, the associated Jacobi form is a *weakly holomorphic* Jacobi form.

The transformations properties of the Jacobi form $\varphi(\tau, z)$ under the generators of the Jacobi modular group comprise the modular transformation of $\varphi(\tau, z)$

$$\varphi\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^w\, e^{\frac{2\pi i m c z^2}{c\tau+d}}\, \varphi(\tau, z) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})\,, \tag{2.2.27}$$

and the elliptic transformation of $\varphi(\tau, z)$

$$\varphi(\tau, z + p\tau + q) = e^{-2\pi i m(p^2\tau + 2pz)}\varphi(\tau, z) \qquad\qquad \forall (p,\, q) \in \mathbb{Z} \times \mathbb{Z}\,, \tag{2.2.28}$$

which encodes the action of the elements of the Heisenberg group of the form $[\,(p, q),\, 0]$. More generally a Jacobi modular form transforms with a non-trivial character $\chi : H(\mathbb{Z}) \to \{\pm 1\}$ under the Heisenberg group, and a multiplier system $\rho : SL_2(\mathbb{Z}) \to \mathbb{C}^*$ under the modular group, see [30] for more details. The functional equation given in definition 2.2.2 thus becomes

$$\varphi|_{k,m}\, h = \chi(h)\varphi, \qquad \forall h \in H(\mathbb{Z})\,, \qquad \varphi|_{k,m}\, \gamma = \rho(\gamma)\varphi, \qquad \forall \gamma \in SL_2(\mathbb{Z})\,. \tag{2.2.29}$$

The definition of Jacobi modular forms can be extended to Jacobi modular forms of positive definite lattices, also called Jacobi forms of lattice index. These represent Jacobi forms in many variables; to be more precise, the elliptic variable $z \in \mathbb{C}$ turns into a lattice variable $v \in (\Lambda \otimes \mathbb{C})$. Jacobi forms of this type were introduced in [45,46], further references for Jacobi lattice index and their relation to finite quadratic modules are [38,39,47]. In the sequel we follow the treatment given in [30], note however that our notation sometimes differs from the one in the original literature.

Define the Heisenberg group associated to the even lattice Λ as the central extension of the group $\Lambda \times \Lambda$, that is to say

$$H(\Lambda) := \{[\,(q, p)\,,\, r\,]\,|\, q,\, p \in \Lambda,\, r \in \frac{1}{2}\mathbb{Z}\}\,. \tag{2.2.30}$$

The group structure is defined by the operation

$$[\,(q,p),\,r\,].[\,(q',p'),\,r\,] = [\,(q+q',p+p'),\,r+r'-(p,q')/2+(p',q)/2\,]\,, \quad (2.2.31)$$

where $(\cdot,\cdot)$ represents the bilinear form of the lattice. The modular group $SL_2(\mathbb{Z})$ acts on $H(\Lambda)$ as in equation (2.2.21). In addition, the Jacobi modular group associated with a lattice Λ can be taken to be $\Gamma^J(\Lambda) := SL_2(\mathbb{Z}) \ltimes H(\Lambda)$.

To define a Jacobi modular form associated to a lattice Λ, we consider the action of the Jacobi modular group $\Gamma^J(\Lambda)$ on $\mathcal{H}(\Lambda)$, where $\mathcal{H}(\Lambda)$ is defined as

$$\mathcal{H}_2(\Lambda) := \left\{ Z = \begin{pmatrix} \tau \\ v \\ w \end{pmatrix} \middle|\, \tau,\, w \in \mathcal{H},\, v \in \Lambda\otimes\mathbb{C}\,,\, \big(2\mathrm{Im}(\tau)\mathrm{Im}(w) - (\mathrm{Im}(v),\mathrm{Im}(v))\big) > 0 \right\}. \quad (2.2.32)$$

A Jacobi modular form with respect to $\Gamma^J(\Lambda)$ can be defined similarly to above. In the following definition to ease the notation we denote by the same symbol the non-trivial character of $\Gamma^J(\Lambda)$ and its restriction to the characters of the generators of the Jacobi group.

Definition 2.2.3. Let χ be a character of finite order of the Jacobi modular group $\Gamma^J(\Lambda)$. Given a positive definite even lattice Λ, a Jacobi form of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Q}_{>0}$ associated to the lattice Λ is a holomorphic function $\varphi : \mathcal{H} \times (\Lambda\otimes\mathbb{C}) \to \mathbb{C}$ which satisfies the following functional equations with respect to the generators of the Jacobi modular group $\Gamma^J(\Lambda)$

$$\varphi\Big(\frac{a\tau+b}{c\tau+d}, \frac{v}{c\tau+d}\Big) = (c\tau+d)^k\, e^{\frac{\pi i m c(v,v)}{c\tau+d}}\, \chi(\gamma)\, \varphi(\tau,v) \qquad \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,,$$

$$\varphi(\tau, v+p\tau+q) = e^{-\pi i m((p,p)\tau+2(p,v))}\chi(h)\, \varphi(\tau,v) \qquad \forall\, h = [\,(q,p)\,,\, 0\,] \in H(\Lambda)\,.$$

The bilinear form $(\cdot,\cdot)$ is the quadratic form associated to the lattice. Due to the periodicity in the variables τ and v, the Jacobi modular form has Fourier expansion

$$\varphi(\tau,v) = \sum_{\substack{n\in\mathbb{Z}\,,\, x\in\Lambda^* \\ 2n\geq(x,x)}} c(n,x)e^{2\pi i(n\tau+(x,v))}\,. \quad (2.2.33)$$

This definition can be similarly written in terms of an appropriately defined slash operator. We refrain from giving all the details to avoid cluttered notations. Note that a rescaling of the lattice accompanied by a rescaling of its bilinear form amounts to a change in the index of the Jacobi form [30]. Moreover, we get back to the definition of holomorphic Jacobi form stated in 2.2.2 by choosing the lattice $\Lambda = A_1(m)$ with standard bilinear form rescaled by a factor of m^{-1}.

An example of a Jacobi form for a positive definite unimodular lattice Λ is

$$\theta_\Lambda(\tau,v) = \sum_{x\in\Lambda} e^{\pi i(x,x)\tau+2\pi i(x,v)}\,. \quad (2.2.34)$$

This provides a refinement of the lattice theta function introduced in (2.1.9) for unimodular lattices; in fact, it does not only count the number of vectors with given norm, but it also grades the vectors in Λ by the value of their scalar product with a fixed vector $v \in (\Lambda \otimes \mathbb{C})$.

Let Λ be as above, and write the scalar product as $(x, v) = \ell\, z$, where $\ell \in \mathbb{Z}$ and $z \in \mathbb{C}$, then we have the expansion

$$\theta_\Lambda(\tau, v) = \sum_{n,\ell\in\mathbb{Z}} a_{\Lambda,v}(n,\ell) e^{2\pi i(n\tau+\ell z)} . \tag{2.2.35}$$

where $a_{\Lambda,v}(n,\ell) = \mathrm{Card}(\{x \in \Lambda | (x,x) = 2n,\ (x,v) = \ell z\})$.

The refined lattice theta series (2.2.34) have a prominent role in the theory of Jacobi modular forms. Part of the reason why this happens is disclosed in the next section through the Theorem 2.2.4.

2.2.3 Theta series and Vector-valued modular forms

Jacobi modular forms of lattice index are intimately tied to modular forms of half-integral weight. This relation is based on the theta decomposition of Jacobi modular forms displayed in theorem 2.2.4 and further explained by theorem 2.2.7. We refer to [31] for an account of the standard Jacobi forms and to [40, 48] for the generalization to a generic even lattice. The modular forms arising under decomposition of a Jacobi form for Λ transform under projective representations of the modular group. These are analyzed at the end of the section in terms of Weil representations associated to the finite quadratic module of Λ.

The following theorem describes the decomposition of a Jacobi modular form in terms of vector-valued modular forms and lattice theta series. The analogue theorem for the A_1-lattice is reported below.

Theorem 2.2.4. *Let Λ be an even positive definite lattice. A Jacobi form indexed by Λ (see §2.2.2) decomposes according to*

$$\varphi(\tau, v) = \sum_{x\in\Lambda^*/\Lambda} h_x(\tau)\theta_{\Lambda,x}(\tau, v) \tag{2.2.36}$$

where $\theta_{\Lambda,x}(\tau, v)$ is the theta series

$$\theta_{\Lambda,x}(\tau, v) := \sum_{\substack{\xi\in\Lambda^* \\ \xi\equiv x\ (mod\,\Lambda)}} e^{\pi i(\tau(\xi,\xi)+2(\xi,v))} , \tag{2.2.37}$$

and $\{h_x(\tau)\}_{x\in\Lambda^*/\Lambda}$ *is a d-dimensional vector-valued modular form defined as*

$$h_x(\tau) := \sum_{\substack{N\in\mathbb{Q} \\ N\equiv -(x,x)/2\,(mod\,\mathbb{Z})}} c\Big(N + \frac{(x,x)}{2}, x\Big) e^{2\pi i N\tau} \tag{2.2.38}$$

where $d = |\Lambda^*/\Lambda|$ *is the order of the quotient group.*

Proof. To simplify the notation denote by x, instead of $[x]$, a representative of an equivalence class of the discriminant group Λ^*/Λ. Through the action of the Heisenberg group on the Jacobi modular form $\varphi(\tau, v)$, we obtain

$$\varphi(\tau, v) = \sum_{\substack{n\in\mathbb{Z}\,,\,x\in\Lambda^* \\ 2n\geq(x,x)}} c(n,x) e^{2\pi i(n\tau+(x,v))} \tag{2.2.39}$$

$$= e^{\pi i((p,p)\tau+2(p,v))}\varphi(\tau, v + p\tau + q) \qquad [(p,q),0] \in H(\Lambda) \tag{2.2.40}$$

where $e^{2\pi i(x,q)} = 1$ for $x \in \Lambda^*$ and $q \in \Lambda$. Comparing the Fourier expansion of $\varphi(\tau, v)$ with the last line, we observe that the Fourier coefficients of $\varphi(\tau, v)$ depend only on x modulo Λ and on the combination $N = n - (x,x)/2$.

Therefore, we can express the Fourier expansion of $\varphi(\tau, v)$ in terms of x and N as follows

$$\varphi(\tau, v) = \sum_{\substack{N\in\mathbb{Q},\,x\in\Lambda^* \\ N\equiv -(x,x)/2\,(\mathrm{mod}\,\mathbb{Z})}} c\Big(N + \frac{(x,x)}{2}, x\Big) e^{2\pi i(N\tau+(x,x)\tau/2+(x,v))}$$

$$= \sum_{x\in\Lambda^*/\Lambda} \sum_{\substack{N\in\mathbb{Q},, \\ N\equiv -(x,x)/2\,(\mathrm{mod}\,\mathbb{Z})}} c\Big(N + \frac{(x,x)}{2}, x\Big) e^{2\pi i N\tau} \sum_{\substack{\xi\in\Lambda^* \\ \xi\equiv x\,(\mathrm{mod}\,\Lambda)}} e^{2\pi i((\xi,\xi)\tau/2+(\xi,v))}.$$

Lastly, substituting the definition of $\theta_{\Lambda,x}(\tau, v)$ and $h_x(\tau)$ we prove the theorem. ■

The transformation properties of the theta series $\theta_{\Lambda,x}(\tau, v)$ and the vector-valued modular form $h_x(\tau)$ are dictated by projective representations of the modular group. Projective representations of $SL_2(\mathbb{R})$ correspond to representation of its double cover, the so-called metaplectic group $Mp_2(\mathbb{R})$. The elements of $Mp_2(\mathbb{R})$ are pairs $(\gamma, j(\gamma,\tau))$ where $\gamma \in SL_2(\mathbb{R})$ and $j(\gamma,\tau)$ is a holomorphic function $j : \mathcal{H} \to \mathbb{C}$ given by $j(\gamma,\tau)^2 = c\tau + d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The group multiplication is defined by

$$(\gamma, j(\gamma,\tau))(\gamma', j(\gamma',\tau)) = (\gamma\gamma', j(\gamma,\gamma'\tau)j(\gamma',\tau))\,. \tag{2.2.41}$$

Alternatively, we can define the metaplectic group through its generators

$$\widetilde{T} = (T, 1)\,, \qquad \widetilde{S} = (S, \sqrt{\tau}) \tag{2.2.42}$$

and relations $\widetilde{S}^2 = (\widetilde{ST})^3$, $\widetilde{S}^8 = \mathbb{I}_{Mp_2(\mathbb{Z})}$, where $\widetilde{S}^2$ is the generator of the center of the metaplectic cover. Note that representations of the metaplectic group allow to keep track of the choice of branch cut appearing in the automorphic factor $j(\gamma, \tau)$. The group $Mp_2(\mathbb{R})$ acts on $\mathcal{H}$ via the natural homomorphism $Mp_2(\mathbb{R}) \to SL_2(\mathbb{R})$ which projects onto the first term in the pair. Therefore we define the double-cover of the Jacobi group as $\widetilde{\Gamma}^J(\mathbb{R}) := Mp_2(\mathbb{R}) \ltimes H(\mathbb{Z})$.

The unitary representations of the metaplectic group relevant to us are defined on $\mathbb{C}[\Lambda^*/\Lambda]$, the group algebra of the discriminant group Λ^*/Λ. $\mathbb{C}[\Lambda^*/\Lambda]$ can be viewed as a vector space of dimension $|\Lambda^*/\Lambda|$ over $\mathbb{C}$, with basis vectors $\mathfrak{e}_x$, labelled by $x \in \Lambda^*/\Lambda$, and group law defined by $\mathfrak{e}_x \mathfrak{e}_y = \mathfrak{e}_{x+y}$. The unitary representations of $Mp_2(\mathbb{R})$ on $\mathbb{C}[\Lambda^*/\Lambda]$ are therefore the representation associated to the finite quadratic module $D_{\Lambda,Q}$. Recall that to each quotient group of an even positive definite lattice we can associated a finite quadratic module $D_{\Lambda,Q}$ through the map defined in equation (2.1.8).

Definition 2.2.5. The Weil representation[8] associated to $D_{\Lambda,Q}$ is the left $Mp_2(\mathbb{R})$-module structure defined by its action on the generators of the metaplectic group as

$$\rho_{\Lambda^*/\Lambda}\big(\widetilde{T}\big)\, \mathfrak{e}_x = e^{\pi i Q(x)} \mathfrak{e}_x\,, \qquad \rho_{\Lambda^*/\Lambda}\big(\widetilde{S}\big)\, \mathfrak{e}_x = \frac{i^{-r/2}}{\sqrt{|\Lambda^*/\Lambda|}} \sum_{y \in \Lambda^*/\Lambda} e^{-\pi i B(x,y)} \mathfrak{e}_y\,, \tag{2.2.43}$$

where r is the rank of the lattice Λ, and $\mathfrak{e}_x$ is a basis vector for $\mathbb{C}[\Lambda^*/\Lambda]$. We assume the representation to be trivial on some congruence subgroup of sufficiently large level.

The set of theta series $\{\theta_{\Lambda,x}(\tau, v)\}_{x \in \Lambda^*/\Lambda}$ defines a module for the Weil representation of $D_{\Lambda,Q}$ as we now illustrate.

Theorem 2.2.6. *Let Λ be an even positive definite lattice of finite rank r, and let $D_{\Lambda,Q}$ be its associated discriminant form. The vector-valued theta series $\Theta_\Lambda(\tau, v)$ associated to $D_{\Lambda,Q}$, defined by*

$$\Theta_\Lambda(\tau, v) := \sum_{x \in \Lambda^*/\Lambda} \theta_{\Lambda,x}(\tau, v)\, \mathfrak{e}_x\,, \tag{2.2.44}$$

is a vector-valued Jacobi form for Λ for $\widetilde{\Gamma}^J(\Lambda) := Mp_2(\mathbb{Z}) \ltimes H(\Lambda)$ of weight $r/2$ and multiplier system for $Mp_2(\mathbb{Z})$ given by $\rho_{\Lambda^*/\Lambda}$.

[8] The name of these representations was attributed to Weil, because he used similar constructions in [49].

The transformation properties of $\Theta_\Lambda(\tau, v)$ under an element $\widetilde{\gamma} \in Mp_2(\mathbb{Z}) < \Gamma^J(\Lambda)$ are thus given by

$$\Theta_\Lambda\left(\frac{a\tau+b}{c\tau+d}, \frac{v}{c\tau+d}\right) = j(\gamma,\tau)^r\, e^{\frac{\pi i c(v,v)}{c\tau+d}}\, \rho_{\Lambda^*/\Lambda}(\widetilde{\gamma})\, \Theta_\Lambda(\tau, v)\,, \tag{2.2.45}$$

$$\forall\, \widetilde{\gamma} = \big(\gamma, j(\gamma,\tau)\big) \in Mp_2(\mathbb{Z})\,, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{2.2.46}$$

This functional equation can be proven by means of the Poisson summation formula (see e.g. [31]).

Theorem 2.2.7. *Let Λ be a positive definite even lattice of rank r, and let ρ be the associated metaplectic representation defined in 2.2.5. The application*

$$\varphi(\tau, v) = \sum_{x\in\Lambda^*/\Lambda} h_x(\tau)\theta_{\Lambda,x}(\tau, v) \;\mapsto\; h(\tau) := \sum_{x\in\Lambda^*/\Lambda} h_x(\tau)\mathfrak{e}_x \tag{2.2.47}$$

defines an isomorphism $J_{k,\Lambda} \to M_{k-r/2}(\rho^*)$ from the space of weight k Jacobi forms for Λ and the space of modular forms of weight $k - r/2$ for the dual representation $\rho^*(\gamma)$ of $Mp_2(\mathbb{Z})$. Here ρ^* denotes the representation

$$\rho^*(\gamma)\, \mathfrak{e}_i = \sum_{j=1}^{n} \overline{\rho(\gamma)_{ji}}\, \mathfrak{e}_j \tag{2.2.48}$$

An important example: $A_1(m)$-lattice .
Consider the lattice $A_1(m)$, that is the A_1-lattice rescaled by m, and let $Q(x)$ be its associated $\mathbb{Q}/\mathbb{Z}$-valued quadratic form. From the definition 2.1.2, we denote the finite quadratic module of $\big(A_1(m), Q\big)$ as

$$D_{A_1(m),Q} = \left(\mathbb{Z}/2m\mathbb{Z}\,,\; x \;\mapsto\; \frac{x^2}{2m} + \mathbb{Z}\right), \tag{2.2.49}$$

where $Q(x)$ is congruent to $\frac{x^2}{2m}$ modulo integers. The Weil representation associated to this finite quadratic module is defined by its action on the generators of the metaplectic group as [31, 38, 39]

$$\rho_{\mathbb{Z}/2m\mathbb{Z}}\big(\widetilde{T}\big)\, \mathfrak{e}_x = e^{2\pi i \frac{x^2}{4m}}\, \mathfrak{e}_x\,, \qquad \rho_{\mathbb{Z}/2m\mathbb{Z}}\big(\widetilde{S}\big)\, \mathfrak{e}_x = \frac{\sqrt{-i}}{\sqrt{2m}} \sum_{y\in\mathbb{Z}/2m\mathbb{Z}} e^{-2\pi i \frac{xy}{2m}}\, \mathfrak{e}_y\,. \tag{2.2.50}$$

The vector-valued theta series $\theta_m(\tau, z)$ of weight $1/2$, index m, whose components are given by

$$\theta_{m,\ell}(\tau, z) := \sum_{\substack{r\in\mathbb{Z} \\ r\equiv \ell \bmod 2m}} q^{r^2/4m}\, y^r\,, \tag{2.2.51}$$

define a module for the representation of the metaplectic group defined in (2.2.50).

Owing to the theorem 2.2.4, a Jacobi form of weight k and index m can be decomposed as

$$\varphi(\tau,z) = \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\,\theta_{m,\ell}(\tau,z)\,, \tag{2.2.52}$$

where

$$h_\ell(\tau) = \sum_{\substack{N\geq N_0\\ N\equiv-\ell^2\,(\mathrm{mod}\,4m)}} C_\ell\,(N)\,q^{N/4m}\,, \tag{2.2.53}$$

is a cusp form, holomorphic modular form or weakly holomorphic modular form depending on the positivity condition satisfied by the Fourier coefficients of $\varphi(\tau,z)$, described in definition 2.2.2. Notice that we denoted by $C_\ell\,(N)$ the Fourier coefficients of $h_\ell(\tau)$ (instead of $c(n,\ell)$) in order to stress the dependence on the discriminant $N = 4mn - \ell^2$. The functions $h_\ell(\tau)$ are the components of the vector-valued modular form $\underline{h}(\tau)$ of weight $k-1/2$ and dual multiplier system with respect to the one of the theta function. The above decomposition leads to the isomorphism between the space of weight k and index m Jacobi forms and a space of vector-valued modular forms, as shown in [31].

2.3 Mock Jacobi and mock modular forms

In this section we introduce two of the central objects in our subsequent discussion: vector-valued mock modular forms and mock Jacobi forms. We primarily follow the following reviews [37,50–53].

Mock modular forms were brought to light at the beginning of the 20th century by Ramanujan [5]. In his last manuscript to Hardy, he examines the mysterious nature of certain hypergeometric series[9], which he named mock theta functions. The distinctive features of these functions are the infinite number of exponential singularities at roots of unity and the fact that, as Ramanujan said, "*it is inconceivable to construct a theta-function* (weakly holomorphic modular form) *to cut out the singularities of the original function*". That is to say, there is no weakly holomorphic modular form that can compensate all the divergences of a mock theta function at rationals ($\tau\in\mathbb{Q}$). In fact, it was later proven in [54] that a single modular form is not sufficient to cancel all the singularities. Nevertheless, there exists a collection of weakly holomorphic modular forms that correctly subtracted

[9]A q-hypergeometric series is a sum of the form $\sum_{n=0}^{\infty} f_n(q)$, where $f_n(q)\in\mathbb{Q}[q]$ for all $n\geq 0$ and $f_{n+1}(q)/f_n(q)\in\mathbb{Q}[q,q^n]$, $n\geq 1$.

to the mock theta function gives rise to a finite limit at any rational point [55]. Among the 17 examples listed by Ramanujan [56], there is

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 + \dots \tag{2.3.1}$$

where $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ is the Pochhammer symbol. This q-hypergeometric series has exponential singularities at all even order roots of unity while it is convergent at odd roots of unity. In the case of $f(q)$, Ramanujan observed that any singularity at even primitive k-th roots of unity disappears by subtracting a term of the form $(-1)^{k/2}b(q)$. Or in other words

$$f(q) - (-1)^{k/2}b(q) = O(1) \tag{2.3.2}$$

where $q^{-1/24}b(q)$ is a modular form defined by

$$b(q) := (1-q)(1-q^3)(1-q^5)\dots(1-2q+2q^4-\dots)\,. \tag{2.3.3}$$

Some references for details on this mock theta function are [54, 57–59], to just name a few. The extension of this result to other mock theta functions [54,59] and in particular to the universal mock theta functions was proven in [60, 61], while the relation between Ramanujan's mock theta functions and the universal mock theta functions is described in [62].

Although Ramanujan's last letter to Hardy dates back to 1920, it was not until the work of Zwegers [6] that a consistent mathematical framework for mock theta functions was developed. Mock theta functions are now considered to be examples of mock modular forms [37]: holomorphic functions which transforms as proper modular forms only after the addition of a non-holomorphic term.

Definition 2.3.1. A vector-valued mock modular form of weight w, multiplier system ρ and shadow $g(\tau)$ with respect to the modular group Γ is a holomorphic vector-valued function $\underline{\varphi}(\tau)$ with at most exponential growth at the cusps and such that there exists a non-holomorphic function

$$\widehat{\underline{\varphi}}(\tau) = \underline{\varphi}(\tau) + \underline{g}^*(\tau), \tag{2.3.4}$$

called the completion of $\underline{\varphi}$, which transforms as a modular form of weight w and multiplier system ρ with respect to Γ. The obstruction for $\underline{\varphi}(\tau)$ to be a modular form is given by $\underline{g}^*(\tau)$, the non-holomorphic Eichler integral of the modular form $\underline{g}(\tau)$, which is defined as

$$g_\ell^*(\tau) = \frac{(4\pi\tau_2)^{1-w}}{w-1}\,\overline{c(0)} + \sum_{n>0} \overline{c(n)}\, n^{w-1}\, \Gamma(1-w; 4\pi n\tau_2)\, q^{-n}\,. \tag{2.3.5}$$

where the first term on the right-hand side has to be substituted by $-\overline{c(0)}\log(4\pi\tau_2)$ if $w = 1$, $\tau_2 = \mathrm{Im}(\tau)$ and $\Gamma(1-w;z) = \int_z^\infty t^{-w}e^{-t}dt$ is the upper incomplete Gamma function.

A convenient representation for the non-holomorphic Eichler integral of the shadow is

$$\underline{g}^*(\tau) = \left(\frac{i}{2\pi}\right)^{w-1} \int_{-\overline{\tau}}^{i\infty} (z+\tau)^{-w}\, \overline{\underline{g}(-\overline{z})}\, \mathrm{d}z\,, \tag{2.3.6}$$

which holds if $w > 1$ or if $\underline{g}(\tau)$ is a cusp form (i.e. $c(0) = 0$). The shadow $\underline{g}(\tau)$ is a modular form of weight $2-w$ and multiplier system conjugate to the one of $\underline{\varphi}(\tau)$. The definition of the non-holomorphic Eichler integral in (2.3.5) can be recovered as a solution of the differential equation

$$(4\pi\tau_2)^k\, \frac{\partial \underline{g}^*(\tau)}{\partial\overline{\tau}} = -2\mathrm{i}\pi\overline{\underline{g}(\tau)}\,. \tag{2.3.7}$$

Clearly, a modular form is simply a special case of a mock modular form with trivial shadow. Mock modular forms are often studied as holomorphic parts of (weak) harmonic Maass forms[10]. Despite the fact that this definition does not include all the instances of mock modular forms, the examples considered in this thesis can be viewed as the holomorphic parts of (weak) harmonic Maass forms.

Special examples of mock modular forms are the so-called mock theta functions: mock modular forms whose shadow is a unary theta series. Theta series associated to a quadratic form in one variable only appear for lattices of rank one and three and are thus "Thetanullwerte" of weight 1/2 and 3/2 respectively. A unary theta series of weight 1/2 is a modular form with Fourier expansion

$$\sum_{n\in\mathbb{Z}} \epsilon(n) q^{an^2}\,, \qquad a \in \mathbb{Q}_{>0}\,, \tag{2.3.8}$$

where $\epsilon(n)$ is an even periodic function. A weight 3/2 unary theta series is a cusp form with Fourier expansion

$$\sum_{n\in\mathbb{Z}} n\varepsilon(n) q^{an^2}\,, \qquad a \in \mathbb{Q}_{>0}\,, \tag{2.3.9}$$

where $\varepsilon(n)$ is an odd periodic function. The former can as well be expressed in terms of the index m theta function as follows

$$\vartheta^0_{m,r}(\tau) = \theta_{m,r}(\tau,z)\Big|_{z=0}\,. \tag{2.3.10}$$

[10]Harmonic Maass forms are real analytic functions annihilated by the weight k hyperbolic Laplacian, which satisfies the modular functional equation proper of a modular form (see Appendix A and a growth condition at the cusps, similarly to the completion of a mock modular form defined above. See [50] for details.

The latter can also be written in terms of the index m theta function as

$$\mathcal{D}\vartheta_{m,r}(\tau) = \frac{1}{2\pi i}\frac{\partial}{\partial z}\theta_{m,r}(\tau,z)\Big|_{z=0}. \tag{2.3.11}$$

Many examples of mock modular forms, including the celebrated Ramanujan's mock theta functions, were revealed as counting functions in (super)conformal field theories, and string theories with non-compact target space, see for instance [7, 8, 63]. In chapter 3 we examine the properties of mock modular forms which arise in connection to extremal conformal field theories with small central charge. Quite surprisingly mock theta functions can be viewed as (lower half-plane) companions of Roger's false theta functions. This is explained in details in chapter 5, where we also provide a physical interpretation to this fact by relating false theta functions and mock theta functions to homological blocks of Seifert 3-manifolds.

An extension of the concept of mock modular form, that will be important for our discussion in chapter 4, is given by mixed mock modular forms. The space of weakly holomorphic mixed mock modular forms lies in the tensor product of the space of modular forms and the space of weakly holomorphic mock modular forms. Strikingly, these objects arise in the counting problem of immortal dyons in string theory on K3 $\times T^2$.
Definition 2.3.2. A mixed mock modular form, $\underline{h}(\tau)$, is a mock modular form whose completion transforms as a modular form of weight w and takes the form

$$\widehat{\underline{h}}(\tau) = \underline{h}(\tau) + \sum_{\ell} f_\ell(\tau)\, g^*_\ell(\tau)\,, \tag{2.3.12}$$

where $\underline{f}(\tau)$ is a (vector-valued) holomorphic modular form of weight s and $\underline{g}(\tau)$ is a (vector-valued) holomorphic modular form of weight $2-w+s$.

It turns out that most of the classical examples of mock theta functions occur as components of vector-valued mock modular forms, arising from the decomposition of mock Jacobi forms. The mock Jacobi form $\psi(\tau,z)$ transforms under the elliptic transformation according to 2.2.28 but it satisfies a "weaker" modular transformation with respect to a Jacobi modular form. This definition generalizes the concept of Jacobi modular forms which is recovered when taking a vanishing shadow. Due to the fact that the application $z\mapsto\psi(\tau,z)$ defines an elliptic function we can still apply the theta-decomposition theorem 2.2.4, and obtain a mock modular form of weight $k-1/2$ associated to the original mock Jacobi form. However, in this case there is no analogue of theorem 2.2.7, in other words there is no isomorphism between the space of mock modular forms and the space of Jacobi forms [8].
Definition 2.3.3. A mock Jacobi form of weight k and index m is a holomorphic function $\psi:\mathcal{H}\times\mathbb{C}\to\mathbb{C}$ whose completion $\widehat{\psi}(\tau,z)$ transforms as a Jacobi modular

form under $SL_2(\mathbb{Z})$. The completion is defined by

$$\widehat{\psi}(\tau, z) = \psi(\tau, z) + \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} g_\ell^*(\tau)\theta_{m,\ell}(\tau, z)\,. \tag{2.3.13}$$

where $\underline{g}^*(\tau)$ represents the (non-holomorphic) Eichler integral of the shadow $\underline{g}(\tau)$.

Optimal mock Jacobi forms of weight 2, associated to weight 3/2 mock modular forms, were first discussed in [8], while weight 1 optimal mock Jacobi forms were classified in [64]. The analysis in [64] brought to light many interesting relations between Ramanujan's mock theta functions, umbral moonshine, genus zero groups, and Weil representations of the double cover of $SL_2(\mathbb{Z})$, which are examined in the next section.

2.4 Elliptic functions and the Weil representation

All the instances of Jacobi modular forms analyzed in previous sections are elliptic functions. In this section we describe some general features of elliptic functions, and briefly review different methods to examine the structure of spaces of elliptic functions. More details can be found in [31]. An explicit description of the space of mock Jacobi forms of critical weight is derived from an analysis of the Weil representations of the metaplecitc group. This reviews the works [38–40] on the Weil representation associated to finite quadratic module and [64] for the result on mock Jacobi forms of weight 1.

Both Jacobi modular forms and mock Jacobi modular forms are elliptic functions with respect to the z variable.[11] This means that for fixed $\tau \in \mathcal{H}$, the assignment $z \mapsto \varphi(\tau, z)$ defines a holomorphic function $\varphi : \mathbb{C} \to \mathbb{C}$ satisfying the functional equation

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)}\varphi(\tau, z)\,, \qquad \forall\,(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z} \tag{2.4.1}$$

and therefore it obeys the following theorem.

Theorem 2.4.1. *Take $m \in \mathbb{Z}_{\geq 0}$. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be an holomorphic function. If φ is not identically zero and satisfies*

$$\varphi(z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)}\varphi(z)\,, \qquad \forall \lambda, \mu \in \mathbb{Z} \tag{2.4.2}$$

then it has exactly $2m$ zeroes (counting multiplicity) in any fundamental domain for the action of $(\mathbb{Z}\tau + \mathbb{Z})$ on $\mathbb{C}$.

[11] In this section we restrict to the A_1 lattice.

A proof of the theorem is reported in [31] for a Jacobi form of index m, and naturally extended to meromorphic Jacobi forms. In this case $2m$ corresponds to the number of zeroes minus the number of poles and therefore m can be any integer. Similarly, the theorem applies to mock Jacobi forms thanks to their elliptic properties.

Theorem 2.4.1 leads to several important proofs about the space of elliptic functions, $\mathcal{E}_m$, and specifically about the spaces of Jacobi forms [31]. For instance it provides the basis to derive the finiteness of the space of Jacobi forms. In addition, it directly derives from theorem 2.4.1 that an holomorphic (mock) Jacobi forms $\varphi(\tau, z)$ is determined up to a constant by its zeroes. That is to say, expanding $\varphi(\tau, z)$ around $z = 0$ we obtain

$$\varphi(\tau, z) = \sum_{d \geq d_0} \chi_d(\tau) z^d \tag{2.4.3}$$

where d_0 is the order of $\varphi(\tau, z)$ at $z = 0$, and the first $2m + 1$ coefficients $\chi_d(\tau)$ uniquely determine $\varphi(\tau, z)$. If $\varphi(\tau, z)$ is purely modular then the functional equation it satisfies provides further constraints on these coefficients [31]. This reasoning allows to quickly determine the dimension of the spaces of Jacobi forms with small index, while a basis can be constructed using theta-eta blocks [65].

In the rest of the section we provide another procedure to concretely build certain spaces of (mock) Jacobi forms in terms of the Weil representation of the finite quadratic module $D_{A_1(m),Q}$. This method turns out to be an efficient mechanism for the construction of the space of optimal mock Jacobi forms of critical weight 1 with rational coefficients [64] to which we will turn to at the end of the section.

First, notice that the Weil representation of the finite quadratic module $D_{A_1(m),Q}$ is a reducible representation when $m > 1$. The reducibility of the representation is reflected into symmetries of the theta function, or in other words in a non-trivial automorphism group for $D_{A_1(m),Q}$. For instance, consider the transformation $r \mapsto -r$ applied to the components of the theta function $\theta_{m,-r}(\tau, z) = \theta_{m,r}(\tau, -z)$. This property together with the modular transformation of the weight k Jacobi forms $\varphi(\tau, z)$ (or its completion $\widehat{\varphi}(\tau, z)$ in case of mock Jacobi forms) under $-\mathbb{I} \in SL_2(\mathbb{Z})$ leads to the following identity

$$h_{-r}(\tau) = (-1)^k h_r(\tau)\,, \qquad r \in (\mathbb{Z}/2m\mathbb{Z})^* \tag{2.4.4}$$

where $h_r(\tau)$ is a component of a vector-valued (mock) modular form. Therefore, depending on the weight of $\varphi(\tau, z)$, we can reduce the $2m$ vector $h(\tau)$ to a vector of the form $\{h_r + h_{-r}\}_{0 \leq r \leq m}$ if k is an even integer, and to a vector of the form $\{h_r - h_{-r}\}_{0<r<m}$ if k is an odd integer. These new combinations provide vector-valued (mock) modular forms invariant under the transformation $r \mapsto -r$,

which corresponds to a specific element of the automorphism group of the finite quadratic module, that is $-1 \in O_m$ as we will see in a moment. The same reasoning can be extended to other elements of the orthogonal group O_m. Moreover, since the action of the orthogonal group O_m commutes with $Mp_2(\mathbb{Z})$, we can define simultaneous eigenspaces for the action of O_m on the space spanned by the theta functions $\Theta_{A_1(m)}(\tau, z)$[12]. This leads to a characterization of the irreducible metaplecitc representations [39, 40] through the action of O_m on the space generated by $\Theta_{A_1(m)}(\tau, z)$.

To completely reduce the Weil representation to an irreducible representation a deeper analysis of the automorphism group of the finite quadratic module and a few more definitions are required. Owing to the definition of a finite quadratic module, we examine the automorphism group of the finite quadratic module $D_{A_1(m),Q}$, defined in (2.2.49). The automorphism group of this discriminant form is the orthogonal group O_m, which is given by

$$O_m := \{a \bmod 2m \mid a^2 = 1 \bmod 4m\} . \tag{2.4.5}$$

Denote by Ex_m the set of exact divisors of m

$$\mathrm{Ex}_m := \{n \in \mathbb{Z}_{>0} \mid n|m \wedge (n, m/n) = 1\} \tag{2.4.6}$$

endowed with the operation $n * n' := nn'/(n, n')^2$. The latter defines a group structure on Ex_m. The elements of O_m turn out to be in one to one correspondence with the elements of Ex_m. The isomorphism between Ex_m and O_m is realized by the map

$$n \mapsto a(n) , \text{ where } \quad a(n) = \begin{cases} a = -1 & \bmod 2n \\ a = 1 & \bmod 2m/n \end{cases} \quad n \in \mathrm{Ex}_m , \, a \in O_m . \tag{2.4.7}$$

The action of the orthogonal group on $D_{A_1(m),Q}$ can be translated into an action on the space of elliptic functions $\mathcal{E}_m$: if we chose the elements of O_m to belong to $(\mathbb{Z}/2m\mathbb{Z})^*$, then $a \in O_m$ permutes the components of the theta functions according to

$$\varphi \cdot a(n) := \sum_{r \bmod 2m} h_r(\tau)\theta_{m,ra}(\tau, z) . \tag{2.4.8}$$

To translate this statement in terms of the group of exact divisors we define the so-called Eichler-Zagier operator $\mathcal{W}_m(n)$, described in [31, 66].

Definition 2.4.2. Let n be an exact divisor of m. Define the Eichler-Zagier operator (EZ) $\mathcal{W}_m(n)$ on $\mathcal{E}_m$ as

$$\varphi|\mathcal{W}_m(n) := \frac{1}{n} \sum_{a,b=0}^{n-1} e\Big(m\Big(\frac{a^2}{n^2}\tau + 2\frac{a}{n}z + \frac{ab}{n^2}\Big)\Big)\varphi\Big(\tau, z + \frac{a}{n}\tau + \frac{b}{n}\Big) . \tag{2.4.9}$$

[12]This is an orthonormal basis. The action of the orthogonal group of this space turns out to be unitary with respect to the Petersson inner product [31]

This operator defines an involution on $\mathcal{E}_m$.

The isomorphism between O_m and Ex_m is then reflected into an isomorphic action on $\mathcal{E}_m$,

$$\varphi \cdot a(n) = \varphi|\mathcal{W}_m(n)\,. \tag{2.4.10}$$

Moreover, $\mathcal{W}_m$ form a group of involutions on $\mathcal{E}_m$[13]. If n divides m but it is not an exact divisor, we can still define the $\mathcal{W}_m(n)$ operator which is not an involution on $\mathcal{E}_m$. The action on $\mathcal{E}_m$ can be seen as an action on the space of theta functions of index m. This explains why an involution corresponds to permutations of the components.

Another way of expressing the action of the Eichler-Zagier operator is via the $\boldsymbol{\Omega}$-matrices, first defined by Capelli-Itzykson-Zuber in relation to modular invariant $\widehat{su}(2)$-characters[14]. The $2m \times 2m$ $\boldsymbol{\Omega}$-matrix is defined by

$$\boldsymbol{\Omega}_m(n)_{r,r'} := \begin{cases} 1 & \text{if } r = -r' \bmod 2n \ \wedge \ r = r' \bmod 2m/n\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.11}$$

Note that this matrix simply implements the operation given in equation (2.4.8). The $\boldsymbol{\Omega}$-matrix is an invertible matrix if $n \in \mathrm{Ex}_m$. Moreover, it preserves the product in Ex_m: $\boldsymbol{\Omega}_m(n)\boldsymbol{\Omega}_m(n') = \boldsymbol{\Omega}_m(n * n')$. The action of the EZ operator on $\mathcal{E}_m$ can be analyzed in terms of the $\boldsymbol{\Omega}$-matrices thanks to the following equality

$$\varphi|\mathcal{W}_m(n) = h^t.\,\boldsymbol{\Omega}_m(n).\,\theta_m\,. \tag{2.4.12}$$

Note that for fixed a and b the action of the EZ operator on $\varphi(\tau, z)$ reads

$$\begin{aligned} e\Big(m\Big(\frac{a^2}{n^2}\tau + 2\frac{a}{n}z + \frac{ab}{n^2}\Big)\Big)\varphi\Big(\tau, z + \frac{a}{n}\tau + \frac{b}{n}\Big) &= \\ \sum_{r \bmod 2m} e\Big(m\frac{ab}{n^2} + r\frac{b}{n}\Big)\, h_r(\tau)\theta_{m,r+2m\frac{a}{n}}(\tau, z) \end{aligned} \tag{2.4.13}$$

where we used the theta decomposition of the elliptic function $\varphi(\tau, z)$ defined in (2.2.52). This displays precisely the isomorphism in (2.4.12) once the sum over a and b is reinserted.

Examples. Let us compute explicit the action of O_m and Ex_m on $\mathcal{E}_m$ for $m = 6$. The groups are specified by $O_6 = \{1, 5, 7, 11\}$, and $\mathrm{Ex}_6 = \{1, 2, 3, 6\}$. The identity elements give rise to a trivial action on the space of elliptic functions. Consider the action of $7 \in O_6$, this element acts by $\varphi \cdot a = \sum_{r\,(\mathrm{mod}\,12)} h_r\theta_{6,7r}$

[13]This follows from the group structure of O_m and Ex_m, and it can be explicitly see via the representation in terms of the Ω-matrices defined below.

[14]The relation between affine $\widehat{su}(2)$-characters and the ADE-classification is beautifully reflected in the umbral moonshine conjecture [36].

and it is associated to the element $2 \in \mathrm{Ex}_6$ via the map $a(2) = 7$. On the other hand, the action of $\varphi|\mathcal{W}_6(2)$ can be computed explicitly: for even r the only terms that contribute to the sum over a and b are the ones with vanishing a and give back the original function, thus the even components are fixed by this action; while for odd r only the terms with $a = 1$ contribute and lead to the permutation $(1,7),(3,9),(5,11)$. Therefore we obtain the equality $\varphi|\mathcal{W}_6(2) = \varphi \cdot a(2) = \sum_{r(12)} h_r \theta_{6,7r}$, for $a(2) = 7$. Similarly, $\mathcal{W}_6(3)$ fixes the components that are multiple of 3, and exchanges $(1,5),(1,10),(7,11),(4,8)$. Correspondly, $\varphi|\mathcal{W}_6(3) = \varphi \cdot a(3) = \sum_{r(12)} h_r \theta_{6,5r}$, where $a(3) = 5$. Lastly, the element $-1 \equiv 11 \in O_6$, which corresponds to the element $6 \in \mathrm{Ex}_6$, permutes the components according to $r \mapsto -r \pmod{12}$.

The space $\mathcal{E}_m$ can therefore be decomposed into eigenspaces $\mathcal{E}_m^\alpha$ for the action of O_m as follows

$$\mathcal{E}_m^\alpha := \{\varphi \in \mathcal{E}_m \,|\, \varphi \cdot a = \alpha(a)\varphi \,, \forall a \in O_m\}\,, \tag{2.4.14}$$

where $\alpha \in \mathrm{Hom}(O_m, \mathbb{C}^*)$. As a result, for a given m the different representations can be labelled by subgroups $K < \mathrm{Ex}_m$, with K defined as the kernel of the map $n \mapsto \alpha(a(n))$ or such that $\alpha = 1$. Simultaneous eigenspaces for the action of O_m corresponds to irreducible representations for the finite quadratic module $D(A_1(m), Q)$. As long as m is square-free, restricting to eigenvectors of O_m produces irreducible representations. When m is not square-free, the irreducible representation is given by taking the orthogonal complement of the images of operators U_d given by $U_d(\phi(\tau, z)) = \phi(\tau, dz)$ with respect to the Petersson metric [31]. Irreducible representations are labelled by subgroups $K < \mathrm{Ex}_m$ with the property that K together with m generate the whole group of exact divisors. Following a tradition initiated in [67], we denote the pair (m, K) by $m + K = m + n, n', \ldots$ for $K = \{1, n, n', \ldots\}$. For instance, we denote by $6+2$ the pair $m = 6$ and $K = \{1, 2\} \subset \mathrm{Ex}_6$.

The eigenspaces $\mathcal{E}_m^\alpha$ for the action of O_m on $\mathcal{E}_m$ can be explicitly constructed by means of the EZ-operators. Since $\mathcal{W}_m(n)$ are involutions for $n||m$, we can define the projection matrices

$$P_m^\pm(n) := (\mathbb{I} \pm \Omega_m(n))/2\,. \tag{2.4.15}$$

In this way the projection operator associated to $m + K$, for m square-free it is given by

$$P^{m+K} := \prod_{n \in K} P_m^+(n) \prod_{n \in \mathrm{Ex}_m \setminus K} P_m^-(n) \tag{2.4.16}$$

while for $m = p^2 m'$ where m' is square-free and p is prime, we have

$$P^{m+K} := \prod_{n \in K} P_m^+(n) \prod_{n \in \mathrm{Ex}_m \setminus K} P_m^-(n) \Big(\mathbb{I} - \Omega_m(p)/p\Big). \tag{2.4.17}$$

The basis vector for the irreducible Weil representation is then given by

$$\theta_r^{m+K}(\tau, z) = 2^{|K|} \sum_{r' \in \mathbb{Z}/2m} P_{rr'}^{m+K} \theta_{m,r'}(\tau, z) \tag{2.4.18}$$

where $|K|$ is the order of the group. In case m is not square-free, these vectors sit in the orthogonal complement (with respect to Petersson scalar product) of the images of operators U_d.

Note that due to the projection operator P^{m+K}, we have $\theta_r^{m+K} = \pm\theta_{r'}^{m+K}$ among various pairs (r, r'). Denote by σ^{m+K} the set of linearly independent vectors θ_r^{m+K} and by d^{m+K} the cardinality of this set. Explicitly, σ^{m+K} is the set of orbits $\mathrm{Ex}_m r$ of Ex_m in $\mathbb{Z}/2m$ with the property $P_{rr} \neq 0$. Therefore, the vector-valued theta functions Θ^{m+K}, whose components are θ_r^{m+K} with $r \in \sigma^{m+K}$, are basis vectors of the irreducible Weil representation.

Many of the ingredients presented in the construction of irreducible Weil representations emerge unexpectedly from the resurgence analysis of the homological blocks of 3-manifolds invariant [68]. This topic is investigated in chapter 5.

2.5 Rademacher sums

The main content of this section is the determination of explicit expressions for the Fourier coefficients of modular-type of objects. This topic is part of a central problem in mathematics: the estimations of the asymptotics of q-series. Several techniques are available to compute the asymptotic growth of the Fourier coefficients of modular objects. Among these is the well-known method of steepest descent, which provides an estimate of the asymptotic growth via a saddle-point approximation. In this section we describe two other powerful techniques that allow to exactly reconstruct the Fourier coefficients of the modular object via the so-called Rademacher sum, going beyond the estimates on its asymptotics. Despite the fact that the latter two methods share many features, they rely on different approaches. Roughly speaking, one is based on the analysis of the pole structure of the function, while the other one is based on the symmetries of the function.

Apart from furnishing an efficient method to reconstruct (mock) modular forms, the Rademacher sum is intimately tied to (monstrous and umbral) moonshine, conformal field theories and quantum gravity. The role of the Rademchaer sum in the different physical contexts constitutes one of the central theme of the subsequent Chapters.

2.5.1 The circle method

The natural approach to a complex integral is to deform the contour of integration in order to obtain the largest contribution from the function in a relatively small path/region. Examples are given by the residues method and the method of steepest descent; yet another method was discovered by Hardy and Ramanujan around 1918 [11]. They considered the generating function of unrestricted partitions, $\mathcal{P}(q) = 1 + \sum_{n>0} \alpha(n) q^n$, whose Fourier coefficients are given by Cauchy's theorem as

$$\alpha(n) = \frac{1}{2\pi i} \oint_\gamma \frac{\mathcal{P}(q)}{q^{n+1}} \, \mathrm{d}q \,, \tag{2.5.1}$$

where the contour γ encircles the origin of the q-plane and lies inside the unit disc. For a complete review of the circle method the reader is referred to [18]. $\mathcal{P}(q)$ is the Euler series introduced in the Introduction

$$\mathcal{P}(q) = q^{1/24} \eta(\tau)^{-1} = \prod_{n=1}^{\infty} (1 - q^n)^{-1} \,, \tag{2.5.2}$$

Hardy and Ramanujan noticed that the Fourier coefficients $\alpha(n)$ could be reconstructed following the pole structure of the function $\mathcal{P}(q)$. In other words, the poles of this function indicate the path of integration to follow in order to have the biggest contribution to the integral along a relatively small portion of the path. In the case at hand, the heaviest contribution is given by the pole at $q = 1$, where all the different roots of unity appear with decreasing weight for increasing denominator. This led to the decomposition of the circle γ into a sum of Farey arcs,

$$\oint_\gamma \quad \longrightarrow \quad \sum_{\substack{0 \le h < k \le N \\ (h,k)=1}} \int_{\xi_{h,k}} \tag{2.5.3}$$

Here $\xi_{h,k}$ denotes the Farey arc centered in h/k and bounded by $\frac{h+h_1}{k+k_1}$ and $\frac{h+h_2}{k+k_2}$, with $\frac{h_1}{k_1}$ and $\frac{h_2}{k_2}$ indicating the preceding and the consecutive Farey fractions of h/k respectively. Indeed, the Farey series of order N, $\mathcal{F}_N$ [69], is constituted by irreducible fractions h/k in ascending order such that $0 \le h < k \le N$, and $(h, k) = 1$, i.e. h and k are coprime.
Each center of the Farey arc represents a root of unity and the mediants delimitate the neighborhood of the corresponding root of unity.

This strategy was refined by Hardy and Littlewood [70] and adopted by Rademacher to derive an exact expression for the Fourier coefficients of the J-function [71] and, together with Zuckerman, to prove the form of the Fourier coefficients of certain modular forms with negative weights [72].

Following their discussion, for a generic modular form $f(\tau) = \sum_{n>n_0} \alpha(n)q^n$ the modular variable q is substituted by $e^{-\frac{2\pi}{N^2}+2\pi i\phi+2\pi i\frac{h}{k}}$, such that in the limit $N \to \infty$ the closed curve γ tends to the unit circle. After having transformed the contour of integration into a sum of Farey arcs and implemented the above substitution, the integral takes the form

$$\alpha(n) = \lim_{N\to\infty} \sum_{\substack{0\leq h<k\leq N \\ (h,k)=1}} e^{-2i\pi n\frac{h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} f\left(e^{-\frac{2\pi}{N^2}+2i\pi\frac{h}{k}+2i\pi\phi}\right) e^{\frac{2\pi n}{N^2}-2i\pi n\phi}\,\mathrm{d}\phi\,, \tag{2.5.4}$$

where $\vartheta'_{h,k}$ and $\vartheta''_{h,k}$ are the mediants of the Farey arc after the change of variables. The function f represents the J-function or a more general modular form; an explanation regarding the type of object one can consider will follow shortly.
Consider the variable $z = \frac{k}{N^2} - ik\phi$ and denote an element of $SL(2,\mathbb{Z})$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix} \qquad hh' \equiv -1\,(\mathrm{mod}\,k) \tag{2.5.5}$$

such that $\tau = \frac{1}{k}(h+iz)$ and $\gamma\tau = \frac{1}{k}(h' + \frac{i}{z})$. Note that $z \in \mathbb{C}$, $Re(z) > 0$. Furthermore, k may be restricted to take positive values, thanks to the symmetry under $-\mathbb{I}$ of f.
Since we focus on the behavior of the function close to the roots of unity, performing a modular transformation on f allows one to obtain an estimate of the function which, in the neighborhoods of roots of unity, is dominated by the polar terms in the q-expansion. Further refinement of the estimate invokes the limit for $N \to \infty$, and leads to an expression in terms of Bessel functions and Kloosterman sums. The latter is defined by

$$\mathfrak{K}_m^{(n)}(k,\rho) := \sum_{\substack{0\leq h<k \\ (h,k)=1}} e^{2\pi i\left(-\frac{h}{k}m+\frac{h'}{k}n\right)} \rho(\gamma)^{-1}\,, \tag{2.5.6}$$

where $\rho(\gamma)$ is the multiplier system of the modular form under consideration. Following this procedure Rademacher obtained an explicit expression for $\mathcal{P}(q^{24})$ [73]. More subtle analysis on the convergence of the series, involving a deeper examination of the Kloosterman sum, enabled him to reproduce the J-function [71]. This is the unique weakly holomorphic modular function with respect to $SL_2(\mathbb{Z})$ with expansion

$$q^{-1} + O(q) \qquad \text{as } \tau \to i\infty\,,$$

that is

$$J(\tau) = q^{-1} + 196884q + \ldots\,. \tag{2.5.7}$$

The Rademacher sum is given by the following expression, which up to the constant term $2\mathfrak{r}$ equals $J(\tau)$,

$$q^{-1} + 2\mathfrak{r} + \sum_{m=1}^{\infty} \frac{2\pi}{\sqrt{m}} \sum_{k=1}^{\infty} \frac{\mathfrak{K}_m^{(-1)}(k)}{k} I_1\left(\frac{4\pi\sqrt{m}}{k}\right) q^n . \tag{2.5.8}$$

In the expression above $\mathfrak{K}_m^{(-1)}(k)$ is the the Kloosterman sum defined in (2.5.6) with trivial multiplier system ($\rho(\gamma) = 1$ for any $\gamma \in SL_2(\mathbb{Z})$), and I_1 denotes the I-Bessel function defined in (C.2.10). Although the constant term $\mathfrak{r}$ might need more care for the proof of convergence, it can be recovered by analyzing the behavior of $\mathfrak{K}_m^{(-1)}(k)$ at $m = 0$ [71] and it is given explicitly below.

Progress has been made in the past years to extend the method to modular objects with positive weights and with weaker modular properties, such as mock modular forms.

As explained in section 2.3, the modular transformation of a mock modular form involves an extra term: the non-holomorphic Eichler integral of a modular form. Nonetheless, the growth of this term is negligible with respect to the growth of the rest of the function, and therefore the asymptotic is unaltered by this term. The asymptotics for one of Ramanujan's mock theta function were first proven via the circle method in [74] and later refined in [75]. Further results include the explicit expression for the coefficients of a (quasi-)mock modular form in [76] and for coefficients of a mixed-mock modular form in [77]. The complication that might arise in the case of a mixed-mock modular form is that the additional term in the modular transformation might contribute to the final asymptotics. We derive an explicit form for the Fourier coefficients of a mixed mock modular through the circle method in chapter 4; this proof allows to then compare the microscopic counting function of immortal dyons in 4d to the macroscopic macroscopic counting function, computed via localization technique in [78].

2.5.2 The averaging method

The origin of the Poincaré sum traces back to a basic idea that a function can be made invariant under the action of a group Γ by averaging over its images

$$\sum_{\gamma\in\Gamma} f(\gamma \cdot \tau).$$

The shortcoming of this method lies in the fact that the above expression is bound to diverge for a generic complex function $f(\tau)$ and group Γ. The convergence is improved if we take the seed function to be invariant under a subgroup of Γ. In this way the the sum reduces to a sum over the elements of the quotient group. Starting from this basic principle, Poincaré was led to the expression [13]

$$\mathcal{P}_w^{(n)}(\tau) := \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} j(\gamma, \tau)^{-2w} e(n\,\gamma\tau) \tag{2.5.9}$$

where $j(\gamma, \tau)$ is the automorphic factor defined above equation (2.2.41), and Γ_∞ is the subgroup of $SL_2(\mathbb{Z})$ generated by $\langle T, -\mathbb{I}\rangle$, i.e. the (parabolic) subgroup fixing the infinite cusp. As long as the weight w is an even integer number strictly greater than two and $n \geq 0$, $\mathcal{P}_w^{(n)}(\tau)$ converges absolutely and it reproduces a modular form with respect to the full modular group; if instead n is strictly positive, $\mathcal{P}_w^{(n)}(\tau)$ turns out to be a cusp form for $SL_2(\mathbb{Z})$.

The analysis of Poincaré [13] was restricted to modular forms of even weight greater than two and trivial multiplier system with respect to the full modular group. Later Petersson [79, 80] generalized this discussion to different groups and multiplier systems and generic weights bigger than two. Generalized Poincaré sums turned out to be a powerful tool in the construction of a basis for the spaces of cusp forms with weight $w > 2$, further developed in the spectral theory of automorphic forms, see [81] and references therein.

An extension of (2.5.9) to a modular form of weight w, multiplier system $\rho : \Gamma \to S^1$ with respect to the group Γ is [15]

$$\mathcal{P}_{\Gamma,w,\rho}^{(n)}(\tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, \tau)^{-2w} \rho(\gamma)^{-1} e(n\,\gamma\tau). \tag{2.5.10}$$

where the multiplier system is such that $\rho(T^h) = e(\mu)$, $0 \leq \mu < 1$. Here Γ is a subgroup of $SL_2(\mathbb{R})$ commensurable[16] with $SL_2(\mathbb{Z})$ and containing $-\mathbb{I}$. We denote by $h \in \mathbb{Z}_{>0}$ the width of Γ at infinity, that is to say the minimal positive integer such that $T^h = \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$. The subgroup of Γ fixing the infinite cusp is then generated by $\Gamma_\infty := \langle T^h, -\mathbb{I}\rangle$.

The sum is well-defined as a sum over elements of the right-coset $\Gamma_\infty\backslash\Gamma$ so long as the summands are invariant under the action of Γ_∞. To check if this holds, we examine what happens to the sum if we change the representative of the coset from an element $\gamma = \delta\widetilde{\gamma}$, to $\widetilde{\gamma}$ where $\delta = T^h$. For an integer weight Poincaré sum,

[15] We restrict to integer weight such that no phase for the choice of branch cut of the logarithm needs to be included.

[16] The group Γ_1 is said to be commensurable with Γ_2 when the index of $\Gamma_1 \cup \Gamma_2$ in Γ_1 and Γ_2 is finite.

we get

$$\mathcal{P}^{(n)}_{\Gamma,w,\rho}(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,\tau)^{-2w}\rho(\gamma)^{-1}e(n\gamma\tau) \tag{2.5.11}$$

$$= \sum_{\widetilde{\gamma}\in\Gamma_\infty\backslash\Gamma} j(\delta\widetilde{\gamma},\tau)^{-2w}\rho(\delta\widetilde{\gamma})^{-1}e(n\delta\widetilde{\gamma}\tau) \tag{2.5.12}$$

$$= \sum_{\widetilde{\gamma}\in\Gamma_\infty\backslash\Gamma} j(\widetilde{\gamma},\tau)^{-2w}\rho(\widetilde{\gamma})^{-1}\rho(\delta)^{-1}e(n(\widetilde{\gamma}\tau+h)) \tag{2.5.13}$$

$$= \sum_{\widetilde{\gamma}\in\Gamma_\infty\backslash\Gamma} j(\widetilde{\gamma},\tau)^{-2w}\rho(\widetilde{\gamma})^{-1}e(-\mu)e(n(\widetilde{\gamma}\tau+h))\,. \tag{2.5.14}$$

To get to the third line, we used the following identity

$$j^{2w}(\alpha\beta,\tau)\rho(\alpha\beta) = j^{2w}(\alpha,\beta\tau)j^{2w}(\beta,\tau)\rho(\alpha)\rho(\beta)$$

which holds for any α, $\beta\in\Gamma$. Therefore, the invariance of the sum under a change of coset representative requires $nh-\mu\in\mathbb{Z}$. Note that no extra constraints arise from the other generator of Γ_∞.

Relying on the absolute convergence of the sum in (2.5.10) for weights strictly bigger than 2, we can in addition prove its modular transformation properties. Again for an integer weight Poincaré sum, we get

$$P_{\Gamma,w,\rho,n}(\alpha\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,\alpha\tau)^{-2w}\rho(\gamma)^{-1}e(n\gamma\,\alpha\tau) \tag{2.5.15}$$

$$= j(\alpha,\tau)^{2w}\rho(\alpha)\sum_{\gamma'\in\Gamma_\infty\backslash\Gamma} j(\gamma',\tau)^{-2w}\rho(\gamma')^{-1}e(n\gamma'\tau) \tag{2.5.16}$$

where $\gamma'=\gamma\alpha$. This equation corresponds to the functional equation satisfied by a modular form of weight w and multiplier system ρ with respect to Γ given in (A.1.21). Changing the sum into a sum over γ' we have reshuffled the order in which we sum over cosets, however no complication arises thanks to the absolute convergence of the sum. The absolute convergence of the series is lost for smaller weights. Already at $w=2$ the sum requires a regularization procedure to be conditionally convergent.

Under the influence of the works by Rademacher on the J-function, and the analysis of Knopp on (period) abelian integrals [82,83] Niebur designed a new type of Poincare sum which provides an efficient machinery to construct modular objects of vanishing and negative weights [84]. Throughout this thesis we refer to this object as Rademacher sum. The Rademacher sum of weight w and multiplier system ρ with respect to the modular group Γ is given by

$$\mathcal{R}^{(n)}_{\Gamma,w,\rho}(\tau) := \mathfrak{r} + \lim_{K\to\infty}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_{K,K^2}} \Re^n_w(\gamma,\tau)\,j(\gamma,\tau)^{-2w}\rho(\gamma)^{-1}e(n\,\gamma\tau)\,, \tag{2.5.17}$$

where n stands for the negative q-power in its Fourier expansion and $\mathfrak{r}$ corresponds to the coefficient defined in (2.5.22). The sum is taken over representatives of the right coset of $\Gamma_{K,K^2} = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \,|\, 0 \leq c < K, -K^2 < d < K^2\}$ by Γ_∞. Due to the conditional convergence of the series, the sum has to be taken in a particular order: specifically the summands are chosen with increasing c. Lastly, the regularization factor is

$$\mathfrak{R}^n_w(\gamma, \tau) = \frac{\overline{\gamma}(1-w, 2\pi i n(\gamma\tau - \gamma\infty))}{\Gamma(1-w)},$$

where $\overline{\gamma}$ denotes the lower incomplete gamma function $\overline{\gamma}(1-w, x) = \int_0^x t^{-w} e^{-t} dt$. Specializing this compact formula to the case of vanishing weight, single simple pole at infinity ($n = -1$), and trivial multiplier system we recover the Rademacher expression for the J-function (2.5.7) up to the constant $\mathfrak{r}$. The addition of a constant in this case does not modify the modular properties of the function at hand; however for general weights it is a necessary ingredient to simplify the modular transformation of the object.[17] In contrast to (2.5.7), here the sum over coset representatives includes a term with vanishing c and a constant $\mathfrak{r}$.

Remarkably, Niebur proved that the Rademacher construction defined by the above regularization gives rise to a conditionally convergent series, that he referred to as *automorphic integral*. The latter is defined as a holomorphic map $\varphi : \mathbb{H} \to \mathbb{C}$ obeying

$$\varphi(\gamma\tau) = (c\tau + d)^w \rho(\gamma)\Big(\varphi(\tau) - p(w, \gamma^{-1}\infty, g)\Big) \tag{2.5.18}$$

where

$$p(w, \tau, g) := \frac{1}{\Gamma(1-w)} \int_{-\overline{\tau}}^{i\infty} (z+\tau)^{-w} \overline{g(-\overline{z})} dz\,,$$

and g is a cusp form of weight $(2-w)$ and conjugate multiplier system, $\overline{\rho}(\gamma)$. The regularization procedure was thus proven to lead to what is now known as a mock modular form. Consequently, if the space of cusp forms of dual weight is empty the automorphic integral reduces to a modular form. This happens, for instance, in the case of the J-function and for all the McKay-Thompson series arising in monstrous moonshine[18], as reviewed in 3.3. In addition, Niebur showed that the Rademacher sum gives a basis for the vector space of automorphic integrals of negative weight w and multiplier system ρ for a generic modular group Γ.

This technique was generalized to weight 1/2 mock modular forms in [87, 88]. In [89] a collection of references together with the history of the Rademacher sum

[17] A detailed analysis on the role of the constant term is presented in [86].

[18] The dimension of the space of holomorphic 1-forms on the Riemann surface (isomorphic to the space of cusp forms of weight 2) is equal to its genus. Therefore if we take the Riemann surface to be given by compactification of $\Gamma\backslash\mathcal{H}$ and restrict to those groups Γ that give rise to a Riemann sphere, then the weight 0 modular object with respect to this group has to be a modular form since there is no shadow available to form a mock modular form.

from the perspective of Poincaré sums is presented. Further developments in the context of harmonic Maass forms are reported in [90, 91].

Although until now we focused on scalar-valued Rademacher sums, the main objects of the next sections are vector-valued Rademacher sums, recently constructed in [86, 92–96]. Following these results, the definition (5.8.2) can readily be generalized to the vector-valued case

$$\mathcal{R}^{(n_i)}_{\Gamma,w,\rho}(\tau)_j = \mathfrak{r}_j + \lim_{K\to\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_{K,K^2}} \mathfrak{R}^{n_i}_w(\gamma,\tau) j(\gamma,\tau)^{-2w} \rho(\gamma)^{-1}_{ji} e(n_i\gamma\tau)\,. \tag{2.5.19}$$

This corresponds to the contribution of the pole of order $(-n_i)$ at the infinite cusp[19] to the j-th component of the Rademacher sum. If multiple polar terms are present in the Fourier expansion of the mock modular form then all polar contributions must be taken into account.

The above defined Rademacher sum leads to series expressions for its coefficients. In the following we consider suitable Rademacher sums to reproduce vector-valued mock modular forms of weight smaller or equal to 1/2 and with a single pole at the infinite cusp (more general pole structures are easily obtained extending the following derivation). Denote by $\mathcal{R}^{(n_i)}_{\Gamma,w,\rho}(K,\tau)_j$ the expression on the right hand side of (2.5.19) before taking the limit for $K\to\infty$, In order to extract the Fourier coefficients of the modular object defined by (2.5.19) we split the series into a double coset decomposition as follows [93]

$$\begin{aligned}
\mathcal{R}^{(n_i)}_{\Gamma,w,\rho}(K,\tau)_j &= \mathfrak{r}_j + \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_{K,K^2}} \mathfrak{R}^{n_i}_w(\gamma,\tau) j(\gamma,\tau)^{-2w} \rho(\gamma)^{-1}_{ji} e(n_i\gamma\tau) \\
&= \mathfrak{r}_j + \delta_{ij} q^{n_i} + \\
&+ \sum_{\gamma\in\Gamma_\infty\backslash\Gamma^*_{K,K^2}} e^{-2\pi i n_i(\gamma\tau-\gamma\infty)} \sum_{m=0}^{\infty} \frac{(2\pi i n_i(\gamma\tau-\gamma\infty))^{m+1-w}}{\Gamma(m+2-w)} j(\gamma,\tau)^{-2w}\rho(\gamma)^{-1}_{ji} e(n_i\gamma\tau) \\
&= \mathfrak{r}_j + \delta_{ij} q^{n_i} + \\
&+ \sum_{\gamma\in\Gamma_\infty\backslash\Gamma^*_{K,K^2}/\Gamma_\infty} \sum_{\substack{\ell\in\mathbb{Z}\\ |\ell|<K^2/ch}} e^{2\pi i n_i \frac{a}{c} - 2\pi i \ell\mu_j} \rho(\gamma)^{-1}_{ji} \sum_{m=0}^{\infty} \left(\frac{2\pi i n_i}{c}\right)^{m+1-w} \frac{(c\tau+d+\ell ch)^{-m-1}}{\Gamma(m+2-w)} \\
&= \mathfrak{r}_j + \delta_{ij} q^{n_i} + \\
&+ \sum_{0<c<K} \sum_{0<d<ch} \rho(\gamma)^{-1}_{ji} e^{2\pi i n_i \frac{a}{c}} \sum_{m=0}^{\infty} \left(\frac{2\pi i n_i}{c}\right)^{m+1-w} \sum_{\substack{\ell\in\mathbb{Z}\\ |\ell|<K^2/ch}} e^{-2\pi i \ell\mu_j} \frac{(c\tau+d+\ell ch)^{-m-1}}{\Gamma(m+2-w)}
\end{aligned}$$

[19] The definition of the Rademacher sum at different cusps of Γ can be found in [86, 93].

In the second line we removed from the summation the identity element, and denoted the reduced sum by Γ^*_{K,K^2}. In the third line we wrote the summation in terms of a double coset $\gamma \in \Gamma_\infty \backslash \Gamma^*_{K,K^2}/\Gamma_\infty$, this amounts to replace γ by $\gamma T^{h\ell}$ and include a new sum over $\ell \in \mathbb{Z}$, and we took an element of Γ to be given by $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$. In the fourth line we interchanged the sums over m and ℓ since they converge absolutely. Before proceeding, we introduce the Lipschitz summation formula. Take $\mathrm{Im}(w) > 0$, $p \geq 1$, $0 \leq \alpha < 1$, then we have

$$\sum_{\substack{|n|<N \\ n\in\mathbb{Z}}} \frac{e(-n\alpha)}{(w+n)^p} - \frac{(-2\pi i)^p}{\Gamma(p)} \sum_{\substack{m+\alpha>0 \\ m\in\mathbb{Z}}} (m+\alpha)^{p-1} e((m+\alpha)w) =$$

$$= \begin{cases} -\pi i + O(1/N) & \text{If}\, \alpha = 0 \wedge p = 1 \\ O(1/N^2) & \text{otherwise.} \end{cases}$$

The convergence is locally uniform in τ as N approaches infinity. By means of this expression, we obtain

$$\sum_{\substack{\ell\in\mathbb{Z} \\ |\ell|<K^2/ch}} e^{-2\pi i \ell \mu_j} \frac{(c\tau + d + \ell ch)^{-m-1}}{\Gamma(m+2-w)} = -\frac{\pi i}{ch}\delta_{m,0}\delta_{\mu_j 0} +$$

$$+ \sum_{\substack{k'+\mu_j>0 \\ k'\in\mathbb{Z}}} (k'+\mu_j)^m \frac{(-2\pi i)^{m+1}}{(ch)^{m+1}\Gamma(m+1)} e\left((k'+\mu_j)\frac{c\tau+d}{ch}\right) + O\left(\frac{ch}{K^2}\right).$$

Substituting the right hand side back into $\mathcal{R}^{(n_i)}_{\Gamma,w,\rho}(\tau)_j$, we finally arrive at the Fourier expansion of $\mathcal{R}^{(n_i)}_{\Gamma,w,\rho}(\tau)_j$

$$\mathcal{R}^{(n_i)}_{\Gamma,w,\rho}(\tau)_j =$$

$$= \delta_{ij} q^{n_i} + 2\mathfrak{r}_j + \sum_{\substack{k_j>0 \\ hk_j\in\mathbb{Z}+\mu_j}} q^{k_j} \sum_{c>0} \mathfrak{K}^{(n_i)}_{k_j}(c,\rho)_{ji} \frac{-2\pi i}{ch} \left(-\frac{k_j}{n_i}\right)^{\frac{w-1}{2}} J_{1-w}\left(\frac{4\pi i}{c}\sqrt{-k_j n_i}\right) \tag{2.5.20}$$

where $J_s(x)$ is the J-Bessel function, $\mathfrak{K}^{(n_i)}_{k_j}(c,\rho)$ denotes the matrix-valued Kloosterman sum

$$\mathfrak{K}^{(n)}_k(c,\rho) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty} e(n\gamma\infty - k\gamma^{-1}\infty)\rho(\gamma)^{-1} = \sum_{0<d<c} e\left(n\frac{a}{c} + k\frac{d}{c}\right)\rho(\gamma)^{-1}, \tag{2.5.21}$$

which coincides with (2.5.6) in the case of scalar-multiplier, and the constant $\mathfrak{r}_j$ is zero unless $\mu_j = 0$, in which case it is equal to

$$\mathfrak{r}_j = -\frac{(2\pi i)^{2-w}\,(-n_i)^{1-w}}{2h\,\Gamma(2-w)}\mathfrak{s}_0^{(n_i)}(1-w/2)\,, \tag{2.5.22}$$

where $\mathfrak{s}_{k_j}^{(n_i)}$ is the Kloosterman Selberg Zeta function defined by

$$\mathfrak{s}_k^{(n)}(1-w/2) = \sum_{c>0} \frac{\mathfrak{K}_k^{(n)}(c,\rho)}{c^{2(1-w/2)}}\,. \tag{2.5.23}$$

Equation (2.5.20) expresses once again the contribution of the i-th component, which has a pole of order $(-n_i)$ at the infinite cusp, to the j-th component. Negative weight Rademacher sums with poles at different cusps can be found in [93].

Chapter 3

Extremal conformal field theories

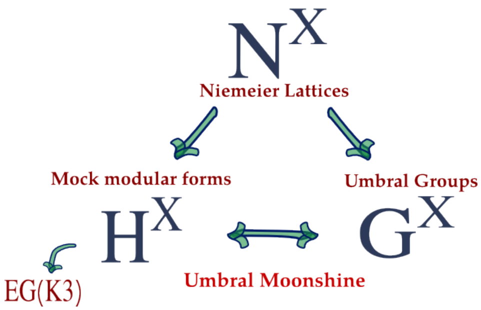
Lattice theory, number theory and sporadic finite groups find a physical realization in this chapter in the form of extremal conformal field theories [97, 98].

One of the most impressive mathematical results of the 20th century is the classification of finite simple groups. The result is that there are 18 infinite families of simple groups as well as the so-called 26 sporadic simple groups, which do not arise as part of any infinite family. Though they are known to exist, there is not yet a deep understanding of the role of sporadic groups in physics.

Several sporadic groups admit a realization as symmetry group of a lattice. This, for instance, occurs in the case of the Conway group Co_0, the automorphism group of the Leech lattice. Additionally, many sporadic groups can be represented as automorphism groups of error-correcting codes. Examples are given by the binary and ternary Golay codes whose automorphism groups are respectively the Mathieu groups M_{24} and M_{12}.

Even more striking than the relation between lattices and the theory of error correcting codes is the connection between sporadic groups and number theory. This subject was first brought to light around 1978 by a group of mathematicians: Mckay, Thompson, Conway and Norton. They observed a surprising relation between the J-function (2.5.7) and the monster group $\mathbb{M}$, the largest sporadic simple group. The Fourier expansion of this modular function was shown to encode the dimensions of the irreducible representations of the monster group. Because of the mystery surrounding this discovery, they named it “moonshine”.

A particularly interesting moonshine was discovered in 2011 [99] from the decomposition of the elliptic genus of a K3 surface into irreducible characters of the $\mathcal{N} = 4$ superconformal algebra. This phenomenon was later shown to be part of the so-called umbral moonshine [36, 94]. The latter comprises 23 instances of moonshine, each labeled by a Niemeier lattice with non-trivial root system, described in section §2.1. This time the relation between number theory and group theory manifests itself in the coefficients of the q-series of mock modular forms, which encode the characters of certain finite groups, the umbral groups.



A fundamental ingredient in the characterization and the construction of the mock modular forms appearing in umbral moonshine is provided by Rademacher sums [95]. Besides its power and effectiveness in constructing modular objects from few ingredients §2.5, the Rademacher sum is also thought to have a physical meaning in the context of the AdS/CFT correspondence: when the (mock) modular form has an interpretation as a partition function in 2d CFT, each individual term appearing in the Rademacher sum resembles the contribution from a classical saddle point to the path integral of the dual AdS_3 gravity theory.

In recent years, the possibility has been raised that the presence of sporadic groups as global symmetry groups is related to the fact that the twining functions (partition functions with the insertion of an element of the symmetry group) of CFTs are expressible as Rademacher sums at a single cusp.

In this chapter, we review the work [1] where this question was addressed for the extant examples of 2d extremal chiral (super)conformal field theories with small central charge, i.e. $c \leq 24$. These are conformal field theories characterized by a minimal spectrum of primary operators consistent with both the (super)Virasoro algebra and modular invariance. The Hilbert space of these theories admits the action of a global symmetry group which is either a sporadic simple group or is very closely related to one [63, 98, 100, 101].

The prototypical example of extremal CFT is the monster CFT, whose famous genus zero property is intimately tied to the Rademacher summability at the infinite cusp of its twined partition functions. However, there are now several additional known examples of extremal CFTs, all of which have at least $\mathcal{N} = 1$ supersymmetry. We study the $\mathcal{N} = 1$ ECFT with Conway symmetry, and a number of ECFTs with extended superconformal algebras. We find that in most cases, the special Rademacher summability property characterizing monstrous and umbral moonshine[1] does not hold for the other extremal CFTs, with the exception of the Conway module and two $c = 12$, $\mathcal{N} = 4$ superconformal theories with M_{11} and M_{22} symmetry. Therefore, it appears that the action of a sporadic symmetry group is only compatible when the singularities at other cusps are incorporated. The fact that twining functions display singular behavior at multiple cusps leads to various interesting questions that we report in the conclusion of this chapter.

The outline of the rest of the chapter is as follows. In §3.1, we discuss aspects of the representation theory of the Virasoro and the affine algebra to introduce the notion of conformal field theory. Section §3.2 explains the relation between extremal CFTs and pure gravity in 3d Anti de Sitter space and summarizes the examples of conformal field theories that are considered in the chapter. In §3.3, we review the construction of the monster CFT, the genus zero property, and its connection with the Rademacher sum. In §3.4 and 3.5 we review the other known ECFTs, in central charge 12 and 24 respectively. We present our results in §3.6 on the Rademacher summability of the twined partition functions of these other ECFTs. Finally, we conclude with a summary and discussion of open questions in §3.7. Appendices A and B contain additional details which complement the main text.

3.1 Conformal field theories

Conformal field theories (CFTs) in two dimensions are defined by a left and a right chiral algebra and a set of fusion rules. The former specifies the spectrum of local operators, and the latter determines the algebra of the operators, that is to say the operator product between the different fields. One of the simplest examples of chiral algebra is the Virasoro algebra. Further examples are provided by the loop algebra and the affine (Kac-Moody) algebra, which can be thought as extensions of the Virasoro algebra by a set of currents (integer weights fields). Important features of chiral algebras arise from the analysis of their representation theory and can be encoded in the characters of the algebra.

The representation theory of chiral algebras is the main focus of the following section, see [102–104] for more details. We refrain from giving a review of super-Virasoro and affine superalgebra here, the basic definitions are reviewed in [105] or references therein. In this section, we use the language of conformal field theories, even though a more rigorous formulation can be given in terms of vertex operator algebra (VOA).

[1]Notice that umbral moonshine is a non-chiral theory.

Any chiral algebra is an infinite dimensional Lie algebra. An instance is provided by the Witt algebra, a $\mathbb{Z}$-graded Lie algebra of polynomial growth. The commutation relations in terms of its generators ℓ_n are

$$[\ell_n, \ell_m] = (m-n)\ell_{m+n}\,, \qquad n, m \in \mathbb{Z}\,. \tag{3.1.1}$$

Through the realization $\ell_n = -z^{n+1}\partial_z$, the Witt algebra can be viewed as algebra of infinitesimal analytic (holomorphic) coordinate transformations $z \to f(z)$, $z \in \mathbb{C}$. The Virasoro algebra is the central extension of the Witt algebra by an operator C, and its commutators are restricted to take the form[2]

$$[\ell_n, \ell_m] = (m-n)\ell_{m+n} + \delta_{n,-m}\frac{m(m^2-1)}{12}C\,, \qquad [\ell_n, C] = 0 \tag{3.1.2}$$

A central role in conformal field theories is played by the representation theory of the Virasoro algebra. Due to the triangular decomposition

$$\oplus_{n>0}\,\mathbb{C}\ell_n \oplus \mathbb{C}\ell_0 \oplus \mathbb{C}C \oplus_{n>0}\mathbb{C}\ell_n\,, \tag{3.1.3}$$

Virasoro representations can be split into highest-weight representations, the so-called Verma modules[3]. Each Verma module $V_{c,h}$ is generated by a vector satisfying

$$\ell_0|h\rangle = h|h\rangle\,, \quad C|h\rangle = c|h\rangle\,, \quad \ell_n|h\rangle = 0\,, \quad \forall n > 0 \tag{3.1.4}$$

where c represents the central charge and h the conformal dimension (or conformal weight) of the vector $|h\rangle$. The action of ℓ_n with $n < 0$ on $|h\rangle$, the highest-weight state, generates the infinite-dimensional module $V_{c,h}$. In CFT language, $|h\rangle$ represents a primary field, and its associated conformal family $V_{c,h}$ is composed by $|h\rangle$ and its descendants $\ell_{i_j} \ldots \ell_{i_k}|h\rangle$, $i \in \mathbb{Z}_{<0}$, $k \in \mathbb{Z}_{>0}$. The Hilbert space of a chiral CFT is therefore defined as $\mathscr{H} = \oplus_h V_{c,h}$. In addition, the tensor product between $V_{c,h}$ and its anti-holomorphic counterpart $\overline{V}_{\overline{h},\overline{c}}$ gives rise to the Hilbert space of a generic CFT upon summing over the conformal dimensions $(h, \overline{h})$. In this chapter we restrict to unitary chiral CFTs, whose Hilbert space can be written as $\mathscr{H} = \oplus_{h\geq 0} V_{c,h}$.

Information about the Verma modules comprising the theory are encoded in the characters of the module

$$\chi_{c,h}(\tau) := \mathrm{Tr}_{\mathscr{H}_h}\, q^{\ell_0 - c/24} = \sum_{n=0}^{\infty} \dim(h+n)\, q^{n+h-c/24}\,, \tag{3.1.5}$$

[2]Notice that only a rescaling of the term $\frac{m(m^2-1)}{12}C$ by field redefinition is allowed by the structure of the Witt algebra.

[3]Despite the slight difference between the definition of a module and a representation, throughout the text we use both terms interchangeably.

where $\dim(h+n)$ is the number of linearly independent states at level n. In view of the construction of the Verma modules, each coefficient $\dim(h+n)$ is bounded by the number of integer partitions of n. Verma modules defined this way might be reducible; to obtain an irreducible representation the null submodule, i.e. the module of zero-norm states, has to be quotiented out.

The formal q-series defining $\chi_{c,h}(\tau)$ turns out to be a holomorphic function of τ, which nicely transforms under modular transformations. If the conformal field theory admits a discrete symmetry, the action of this symmetry on the Hilbert space of the theory can be exploited to construct twining functions, corresponding to characters of the finite group. For a chiral conformal field theory with discrete symmetry group G, and such that G does not permute the modules of the chiral algebra, its twining functions are defined as

$$\chi_g(\tau) := \mathrm{Tr}_{\mathscr{H}_h}\, g q^{\ell_0 - c/24}, \qquad \forall\, g \in G\,. \tag{3.1.6}$$

where we suppressed the indices c, h for ease of notation. This is a class function, as it only depends on the conjugacy class $[g]$ of the element g, and reduces to (3.1.5) if g is the identity element of the group. Furthermore, the functions $\chi_g(\tau)$ are highly constrained as they must transform under the subgroup of the modular group which preserves the corresponding g-twisted boundary condition on the torus, as reviewed in §B.1. Thus they naturally provide a link between the representation theory of G and a distinguished set of modular forms.

Another important instance of infinite dimensional Lie algebra is the affine (Kac-Moody) Lie algebra. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra of rank r endowed with a non-degenerate symmetric invariant bilinear form $(\cdot|\cdot) : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$, where invariant means that $([a,b]|c) = (a|[b,c])$. The associated affine (Kac-Moody) algebra is an extension of the Lie algebra of polynomial maps $S^1 \to \mathfrak{g}$, also called the loop algebra. Affine algebras are usually denoted in the physics literature as current algebras. We prefer to refer to them as affine algebras, to emphasize the role played by the Weyl group, see later.

The process of affinization of a finite dimensional Lie algebra $\mathfrak{g}$ consists of taking a one-dimensional central extension of the loop algebra, denoted as $\mathbb{C}K$, together with a one-dimensional non-central extension, $\mathbb{C}d$,

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d\,, \tag{3.1.7}$$

and endowing the algebra with the brackets

$$[at^m, bt^n] = [a,b]t^{m+n} + m\delta_{m,-n}(a|b)K\,, \tag{3.1.8}$$

where the symmetric bilinear form of $\mathfrak{g}$ has been extended to $\widehat{\mathfrak{g}}$ such that $(at^m|bt^n) = \delta_{m,-n}(a|b)$, $(\mathfrak{g}[t,t^{-1}]|\mathbb{C}k \oplus \mathbb{C}d) = 0\,,\, (k|d) = 1,\ (k|k) = (d|d) = 0$.

The non-central extension d represents the differential operator $t\frac{d}{dt}$, which corresponds to the generator of the Virasoro algebra ℓ_0 up to a sign. This operator has been added to the affine construction such that there exists a maximal abelian subalgebra which consists of the Cartan subalgebra of $\mathfrak{g}$, i.e. the zero-modes $\{H_0^1, \dots, H_0^r\}$, together with K and $-\ell_0$. Note that the algebra of derivation can itself be central extended to recover the Virasoro algebra. A generic weight λ of $\widehat{\mathfrak{g}}$ can be written as $(\overline{\lambda}; k_\lambda; n_\lambda)$, where $\overline{\lambda}$ is a weight of $\mathfrak{g}$, and the remaining two components are the eigenvectors of the Cartan generators K and $-\ell_0$. The scalar product between two weights of $\widehat{\mathfrak{g}}$ is given by $(\lambda,\mu) = (\overline{\lambda},\overline{\mu}) + k_\lambda n_\mu + k_\mu n_\lambda$.

The extension of finite simple Lie algebras to the realm of affine algebras brings about many new elements. An instance is provided by the addition of imaginary roots[4], to be differentiated from the real roots appearing in the context of finite Lie algebra. Another important feature of affine algebras is the structure of the Weyl group, which leads to the appearance of theta functions in the characters of affine (Kac-Moody) algebras.

In analogy to finite Lie algebras, the Weyl group W is generated by reflections with respect to the real roots. A reflection with respect to a real root α corresponds to a reflection with respect to the hyperplane perpendicular to the root $\alpha = (\overline{\alpha}; 0; m)$,

$$s_\alpha \lambda := \lambda - (\lambda, \alpha^\vee)\alpha\,, \tag{3.1.9}$$

where the co-root is defined as $\alpha^\vee := \frac{2}{|\overline{\alpha}|^2}(\overline{\alpha}; 0; m)$. In addition, imaginary roots are unaffected by Weyl reflections. To describe the structure of the Weyl group we split the reflection s_α with respect to the root α into $s_{\overline{\alpha}}(t_{\overline{\alpha}^\vee})^m\lambda$, where $t_{\overline{\alpha}^\vee}\lambda = s_{\overline{\alpha}} s_{\overline{\alpha}+\overline{\delta}}\lambda = (\overline{\lambda}+k\overline{\alpha}^\vee; k; n+(|\overline{\lambda}|^2 - |\overline{\lambda}+k\overline{\alpha}^\vee|^2)/2k)$, for $\lambda = (\overline{\lambda}; k; n)$. The action of $t_{\overline{\alpha}^\vee}$ on the finite part of the weight λ equals a translation by a coroot $\overline{\alpha}^\vee$. Thanks to this new operation, the Weyl group of $\widehat{\mathfrak{g}}$ can be expressed as $W = \overline{W} \ltimes T$, where $\overline{W}$ is the Weyl group of the finite Lie algebra and T is the group of translations [102, 104].

Affine characters associated to integrable highest-weight representations[5], irreducible representations of $\widehat{\mathfrak{g}}$, are defined as follows. Denote by λ the weight corresponding to an integrable highest-weight representation of $\widehat{\mathfrak{g}}$, then the associated character reads

$$\widetilde{\chi}_\lambda := \mathrm{Tr}_{\mathscr{H}_\lambda} e^h = \sum_{\lambda' \in \mathcal{W}_\lambda} \mathrm{mult}_\lambda(\lambda') e^{\lambda'}\,, \tag{3.1.10}$$

where $\mathcal{W}_\lambda$ is the set of weights of the representation and h is a vector whose components are the elements of the Cartan subalgebra of $\widehat{\mathfrak{g}}$. The character can be

[4] Note that imaginary roots have vanishing squared-norm. Imaginary roots with negative squared-norm only appear in the context of Borcherds-Kac-Moody algebras.

[5] The integrable highest-weight representations are representations whose projections onto the $\mathfrak{su}(2)$ algebra associated with any real root are finite.

expressed in terms of sums over elements of the Weyl group W as

$$\widetilde{\chi}_\lambda = \frac{\sum_{w\in W} \epsilon(w) e^{w(\lambda+\rho)}}{\sum_{w\in W} \epsilon(w) e^{w(\rho)}}\,, \tag{3.1.11}$$

where $\epsilon(w)$ is the signature of w, that is $(-1)^{l(w)}$ where $l(w)$ is the minimum number of reflections in the decomposition of w, ρ is the affine Weyl vector and w_i are the elements of the basis dual to the simple coroots. Moreover, exploiting the explicit structure of the Weyl group, a sum over W can be split into a sum over the finite Weyl group $\overline{W}$ and a sum over the space of coroots $\Lambda^\vee$, through the identity

$$\sum_{w\in W} \epsilon(w) e^{w\lambda} = \sum_{w\in \overline{W}} \epsilon(w) e^{\frac{1}{2k}(\lambda,\lambda)\delta} \Theta_{w\lambda}\,. \tag{3.1.12}$$

where δ is the imaginary root $(0;0;1)$ and the theta function is

$$\Theta_\lambda = e^{-\frac{1}{2k}(\lambda,\lambda)\delta} \sum_{\overline{\alpha}^\vee \in \Lambda^\vee} e^{t_{\overline{\alpha}^\vee}\lambda}\,. \tag{3.1.13}$$

Explicitly the affine character reads

$$\chi_\lambda = \frac{\sum_{w\in \overline{W}} \epsilon(w) \Theta_{w(\lambda+\rho)}}{\sum_{w\in \overline{W}} \epsilon(w) \Theta_{w(\rho)}} \tag{3.1.14}$$

where we removed the tilde to denote a different normalization. This links the theory of affine algebras to theta functions, one of the main players in the previous chapter.

It is convenient to specialize the above character to a particular point $\xi = -2\pi i(\zeta;\tau;0)$, with $\zeta = \sum_{i=1}^r z_i \alpha_i^\vee$, and $(\lambda,\zeta) = \sum_{i=1}^r z_i\lambda_i$, so that

$$\chi_\lambda(\tau,z) = \mathrm{Tr}_{\mathcal{H}_\lambda} e^{2\pi i\tau \ell_0} e^{-2\pi i \sum_i z_i h^i} \tag{3.1.15}$$

where h^i are the Cartan generators[6] of the finite Lie algebra. Evaluating Θ_λ at this specific point reduces the theta function to

$$\Theta_\lambda(\tau,z) = \sum_{\gamma\in\,\Lambda^\vee + \overline{\lambda}/k} q^{\frac{k}{2}(\gamma,\gamma)} e^{-2\pi i k(\gamma,z)}\,. \tag{3.1.16}$$

As stated in [103] by Gannon, “the affine algebras supply classic examples of moonshine, in that the characters of their integrable modules are vector-valued Jacobi functions for $SL_2(\mathbb{Z})$”.

Heisenberg algebra. An example of affine algebra is the Heisenberg algebra (or oscillator algebra). The seed algebra, in this case, is a one-dimensional commuting algebra with brackets $[a_m, a_n] = m\delta_{m-n}$. As such it is regarded as the

[6]These are usually expressed in the Chevalley basis.

extension of the $\mathfrak{u}(1)$ algebra, or as the algebra of a free boson ($S^1 \times \mathbb{R} \to M$, where M is the target space). The Virasoro generators are given in terms of oscillators a_n as

$$\ell_n = \frac{1}{2} \sum_{k\in\mathbb{Z}} : a_{n-k} a_k : \, . \tag{3.1.17}$$

where the notation : ... : stands for normal ordering (see e.g. [102]). The character of a highest-weight state with zero conformal dimension can be obtained through the combinatorics of the action of the Virasoro generators, and it is given by

$$\chi_0(\tau) = \eta(\tau)^{-1} \, , \tag{3.1.18}$$

where $\eta(\tau)$ is the Dedekind eta function defined in (A.2.1).

Lattice theory. Let Λ be an integral lattice with bilinear form $(\cdot,\cdot)$, and consider the complexified lattice $\mathbb{C} \otimes_{\mathbb{Z}} \Lambda$ with extended bilinear form. We can associate to it a conformal field theory, given by rank(Λ)-boson, which are periodic functions of the internal space-coordinates and with period defined by the lattice Λ. Due to the periodicity of the bosonic field, its momentum is forced to lie in the dual lattice. The lattice CFT partition function is therefore

$$\chi_{\Lambda+\gamma}(\tau) = \eta(\tau)^{-\mathrm{rank}(\Lambda)} \sum_{\substack{\mu\in\Lambda^* \\ \mu\equiv\gamma \mod \Lambda}} q^{(\mu,\mu)/2} \tag{3.1.19}$$

where $\gamma \in \Lambda^*/\Lambda$, and $\mu \in \Lambda^*$ represents the sum over allowed momenta. The summation of q on the r.h.s coincides with the lattice partition functions $\theta_{\Lambda+\gamma}(\tau)$ defined in §2.1. On the other hand, the eta functions reflect the action of the Virasoro generators and ensure $\chi_{\Lambda+\gamma}(\tau)$ to be a weight zero modular form with respect to a congruence group of the full modular group $SL_2(\mathbb{Z})$. Generically, the set of characters $\chi_{\Lambda+\gamma}(\tau)$ with γ in the quotient group furnishes a representation of the full modular group. This statement can be explicitly proven through the Poisson summation formula. The reader is referred to [106] for the definition and modular properties of characters associated to a lattice Λ.

If the lattice Λ is unimodular, there exists only one character and it is therefore a modular function with possibly non-trivial multiplier system. Only for $c \equiv 0 \mod 24$ the multiplier system reduces to the trivial one.

In the next sections we deal with different lattice CFTs, and thus characters of the form (3.1.19), and their supersymmetric analogue.

3.2 Extremal CFTs

Extremal conformal field theories (ECFTs) are theories whose only operators with dimension smaller or equal to $c/24$ are the vacuum and its (super)Virasoro descendants. The most famous example of ECFT is the conformal field theory with central charge $c = 24$ that appears in monstrous moonshine [107]. The definition of extremal field theories in the physics literature[7] first appeared in [98] in relation to pure three-dimensional gravity with a negative cosmological constant.

In the first part of this section, we briefly review the potential connection of 2d ECFTs with 3d quantum gravity in AdS_3. After the seminal paper by [98], there have been numerous works investigating this conjecture further (see, e.g., [108–118]); however as of now it has neither been proven nor disproven. In the second part, we delineate the ECFTs that are examined in the rest of the chapter and display some of their properties.

Pure gravity in AdS_3 is described by the action

$$S = \frac{1}{16G_N\pi}\int d^3x\,\sqrt{g}\Big(R+\frac{2}{\ell^2}\Big)+\frac{k'}{4\pi}\int \mathrm{Tr}\Big(\omega\wedge d\omega+\frac{2}{3}\omega\wedge\omega\wedge\omega\Big) \tag{3.2.1}$$

where G_N denotes the Newton constant, the first term is the Einstein-Hilbert action with cosmological constant given by $-\frac{2}{\ell^2}$ and Ricci scalar R, and the second term represents the Chern-Simons gravitational term in terms of the spin connection ω. Even though there is neither matter nor physical gravitons, this theory admits black hole solutions as long as the cosmological constant is negative. Another interesting feature of three-dimensional gravity is that the action takes a gauge invariant form if written in terms of a gauge field A, defined by combining the spin connection with the vielbein such that infinitesimal local Lorentz transformations and diffeomorphisms translate into gauge transformations of A. The spin connection ω is an $SO(2,1)$ gauge field ($SO(3)$ for Euclidean signature), which combined with the drei-bein e gives rise to a connection A of $SO(2,2)$[8]. Therefore, up to a redefinition of the Chern-Simons gravitational coupling the 3d gravity action (3.2.1) can be recast in the following form

$$S = \frac{k_L}{4\pi}\int_M \mathrm{Tr}(A_L\wedge dA_L+\frac{2}{3}A_L\wedge A_L\wedge A_L)-\frac{k_R}{4\pi}\int_M \mathrm{Tr}(A_R\wedge dA_R+\frac{2}{3}A_R\wedge A_R\wedge A_R) \tag{3.2.2}$$

The minus sign between the first term and the second is just to ensure the levels k_L, k_R to be positive. Notice that the gauge description of 3d gravity might break

[7] The first reference where ECFT were introduced is [97].

[8] Here the Chern-Simons theory is presented in Lorentzian signature, a presentation in Euclidean signature is reported in chapter 5 in connection to knot theory.

down if we try to go beyond perturbation theory around a classical solution of the gravity theory. By comparison of (3.2.1) and (3.2.2) we obtain the relation

$$k_L + k_R = \frac{\ell}{8G_N} . \tag{3.2.3}$$

In order to have a consistent quantum field theory, further constraints have to be imposed on the level of the Chern-Simons theories. Assuming as in [98] that there is no contribution from the gravitational Chern-Simons term leads to $k_L = k_R$ with $k_L, k_R \in \mathbb{Z}$, and therefore $k := k_L = k_R = \ell/16G_N$.

The quantization condition on the level of the Chern-Simons theory translates to constraints on the central charge defining the asymptotic Virasoro algebra. The latter was discovered by Brown-Henneaux to act on the physical space with $c_L = c_R = 3\ell/2G_N$. From the above assumptions we immediately obtain $c_L = c_R = 24k$, that is to say the dual CFT can be homomorphically factorized. A hint to this holomorphic factorization comes already from the factorized form of the action in (3.2.2). The dual conformal field theory for pure gravity was conjectured to be a double copy (corresponding to the holomorphic and the anti-holomorphic sector) of the monster CFT at $c = 24$ [98]. This is the only conformal field theory with central charge 24 and with no affine (Kac-Moody) symmetry[9]. In fact, any extension of the chiral algebra corresponds to the addition of operators of conformal dimensions one, that by definition cannot belong to the spectrum of an extremal CFT. The operators with dimension strictly bigger than $c/24$, were interpreted in [98] as operators responsible for the creation of BTZ black holes.

After the analysis of [112], which unveiled certain discrepancies in the semiclassical analysis of pure gravity, Witten's proposal was revisited in [114]. Here deformations of the EH action by the addition of the gravitational Chern-Simons term were considered. At a critical value of the Chern-Simons coupling this theory, also called topological massive gravity, is argued to generate a stable theory: chiral gravity. The dual conformal field theory was proposed to be a chiral ECFT, more precisely a single copy of the monster CFT. Further evidence for this proposal have been reported in [119].

Other examples of ECFTs at higher central charges and with $\mathcal{N} = 1$ supersymmetry were studied in [98], in particular Witten derived partition functions for putative extremal (S)CFTs. Modular invariance and holomorphicity constrain the allowed values of the central charge to be $c = 24k$ or $c = 12k^*$, $k, k^* \in \mathbb{N}$, for bosonic and $\mathcal{N} = 1$ CFTs, respectively. The authors of [120] similarly derived elliptic genera for putative extremal SCFTs with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal symmetry and conjectured that theories with such elliptic genera, if they exist,

[9] Here we assume the uniqueness of the monster moonshine module, as conjectured in [107].

would be dual to pure ($\mathcal{N} = 2$ and $\mathcal{N} = 4$) supergravity in AdS_3. Furthermore, they found that such theories can only exist for a finite set of small central charges due to constraints coming from the modular and elliptic properties of the elliptic genus. In particular, parameterizing the central charge for these theories as $c = 6m$, $m \in \mathbb{N}$,[10] such Jacobi forms exist only for $m \leq 13$, $m \neq 6, 9, 10, 12$, and $m \leq 5$, in the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ cases, respectively.[11]

Given the minimal mathematical input arising from physical reasoning via the AdS/CFT, one surprise is that there are a number of known (chiral) CFTs with small central charge and extremal spectrum. Furthermore, they each have global symmetry groups related to sporadic finite simple groups. We summarize the existing extremal CFTs here and introduce notation we will use throughout the text. In the bosonic case, there is one known ECFT at $k = 1$ [98], usually denoted as $\mathcal{V}^\natural$; this is the famous CFT with global symmetry group the monster group ($\mathbb{M}$), which was constructed by Frenkel, Lepowsky, and Meurman (FLM) in [107]. In the case of $\mathcal{N} = 1$ chiral CFTs, it was pointed out in [98] that there are ECFTs with $k^* = 1, 2$ and symmetry related to the sporadic group Co_0 ("Conway zero") first built in [101, 107] and [100], respectively. We refer to these as $\mathcal{E}^{\mathcal{N}=1}_{k^*=1} := \mathcal{V}^{s\natural}$ and $\mathcal{E}^{\mathcal{N}=1}_{k^*=2}$.

Moreover, a number of additional extremal SCFTs with extended supersymmetry were constructed recently: $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SCFTs with $m = 2$ [63] ($\mathcal{E}^{\mathcal{N}=2}_{m=2}(G)$ and $\mathcal{E}^{\mathcal{N}=4}_{m=2}(G)$), SCFTs with $c = 12$ and $SW(3/2, 2)$ superconformal symmetry [121] ($\mathcal{E}^{\mathrm{Spin}(7)}(G)$), an $m = 4$, $\mathcal{N} = 2$ SCFT with M_{23} symmetry [122] ($\mathcal{E}^{\mathcal{N}=2}_{m=4}$), and an $m = 4$, $\mathcal{N} = 4$ SCFT with M_{11} symmetry [123] ($\mathcal{E}^{\mathcal{N}=4}_{m=4}$). Because there exist multiple extremal SCFTs with central charge 12 and extended superconformal symmetry, we distinguish them by specifying their global symmetry group G. We will describe these theories in much greater detail in §3.4 and §3.5.

Finally, we would like to point out that the K3 non-linear sigma model is also an extremal $\mathcal{N} = 4$ SCFT with $m = 1$ ($\mathcal{E}^{\mathrm{K3}}(G)$) according to the definition of [120]. However, unlike the other known examples of ECFTs, it is not chiral. Symmetry groups of K3 non-linear sigma models have since been classified [124] and are in one-to-one correspondence with subgroups $G \subset Co_0$ such that G preserves a 4-plane in the non-trivial 24-dimensional irreducible representation of Co_0, denoted as **24**. In Table 3.1 we present the list of known extremal CFTs, including their central charges, chiral algebras, and global symmetry groups.

The FLM monster module, $\mathcal{V}^\natural$, described in details in §3.3, enjoys a number of striking properties, including the fact that its twining functions as defined

[10] It would be interesting to consider a generalization to half-integer m in the $\mathcal{N} = 2$ case.

[11] One interesting question is whether one can define a notion of "near-extremal" CFT which extends to arbitrarily high central charge. See [117, 120] for attempts in this direction.

ECFT	c	$\mathcal{A}$	Symmetry Group (G)
$\mathcal{V}^\natural$	24	Virasoro	$\mathbb{M}$
$\mathcal{E}^{\mathcal{N}=1}_{k^*=2}$		$\mathcal{N}=1$	$\sim Co_0$
$\mathcal{E}^{\mathcal{N}=2}_{m=4}$		$\mathcal{N}=2$	M_{23}
$\mathcal{E}^{\mathcal{N}=4}_{m=4}$		$\mathcal{N}=4$	M_{11}
$\mathcal{V}^{s\natural}$	12	$\mathcal{N}=1$	Co_0
$\mathcal{E}^{\mathrm{Spin}(7)}$		$\mathcal{SW}(3/2,2)$	$\{G \subset Co_0 \vert G$ fixes a 1-plane$\}$
$\mathcal{E}^{\mathcal{N}=2}_{m=2}$		$\mathcal{N}=2$	$\{G \subset Co_0 \vert G$ fixes a 2-plane$\}$
$\mathcal{E}^{\mathcal{N}=4}_{m=2}$		$\mathcal{N}=4$	$\{G \subset Co_0 \vert G$ fixes a 3-plane$\}$
$\mathcal{E}^{\mathrm{K3}}$	(6,6)	$\mathcal{N}=4$	$\{G \subset Co_0 \vert G$ fixes a 4-plane$\}$

Table 3.1: Known extremal CFTs with central charge c, chiral algebra $\mathcal{A}$, and global symmetry group G. An n-plane corresponds to an n-dimensional subspace in the representation **24** of Co_0.

in (3.1.6) (and known as "McKay-Thompson series") furnish Hauptmoduln for genus zero groups. This is the famous "genus zero" property of monstrous moonshine [67], which was shown in [86] to be equivalent to a particular feature of their Rademacher sums: each of these functions can be expressed as a Rademacher sum with only a simple pole at the infinite cusp[12]. That is to say, one can represent these functions as a sum over representatives of $\Gamma_\infty\backslash\Gamma$ about the pole $\left(q^{-1}\right)$, where Γ_∞ is the subgroup of Γ that fixes the $i\infty$-cusp. A similar property is crucial in the formulation of umbral moonshine [36, 88, 94], where again the polar structure at the infinite cusp is sufficient to recover almost all the functions[13]. Thus a natural question is: can the twining functions of the other examples of ECFTs be expressed as Rademacher sums at the infinite cusp? This question is particularly compelling given the proposed connection between the Rademacher sum and the path integral of quantum gravity in AdS$_3$, beginning with [14].

Via the AdS$_3$/CFT$_2$ correspondence, one associates the partition function of the 2d CFT on a torus with the Euclidean quantum gravity path integral in three dimensions with asymptotically AdS boundary conditions. The bulk path integral is evaluated on a solid torus whose boundary is the torus of the 2d CFT; its semi-classical saddle points correspond to representatives of equivalence classes of contractible cycles of the solid torus and are thus labeled by elements of the coset $\Gamma_\infty\backslash\Gamma$ for $\Gamma = SL_2(\mathbb{Z})$ and Γ_∞ the subgroup which stabilizes the contractible

[12]For a recent account of the relation between Rademacher sums, Faber polynomials, and traces of singular moduli see [125].

[13]Another construction based on theta-eta blocks is reported in [95].

cycle. The sum over saddle points precisely appears in the Rademacher expansion of the CFT partition function, as noted above in the case of the monster CFT, suggesting a physical interpretation of this expression via holography. An explicit connection between the monster CFT and a family of 3d chiral gravities [114] was proposed in [85, 86]. One caveat to a holographic interpretation of Rademacher sums appearing in monstrous moonshine, however, is that the AdS radius in three dimensions is proportional to the central charge of the CFT. Thus only for very large c does one have reason to trust the semi-classical bulk path integral, which is decidedly not the case for the monster CFT, which has $c = 24$. Nevertheless, it is striking that such an interpretation seems to remain valid in this context.

3.3 Properties of the monster CFT

In this section we review the construction of the monster CFT and discuss some of its defining properties, which include the genus zero property, the Rademacher summability of its twining functions and the connection between these properties and holomorphic orbifolds of the theory. For a short review on holomorphic orbifolds the reader is referred to the Appendix B.1 and references therein.

Given a positive-definite even unimodular lattice Λ of rank $24k$ one can construct a bosonic chiral conformal field theory with modular invariant partition function by compactifying the theory of $24k$ chiral bosons on the torus $\mathbb{R}^{24k}/\Lambda$. In the case of $k = 1$, there are 24 such lattices: the Leech lattice, Λ_L, which has no roots, and the 23 Niemeier lattices, Λ_N, which can be uniquely specified by their root systems. These are a union of simply-laced root systems with the same Coxeter number and of total rank 24. We will call the 23 chiral bosonic CFTs on $\mathbb{R}^{24}/\Lambda_N$ the *Niemeier CFTs*, and the theory on $\mathbb{R}^{24}/\Lambda_L$ the *Leech CFT*. We label their associated modules as $\mathcal{V}^N$ and $\mathcal{V}^L$, respectively. The partition function of each of these theories is simply given by[14]

$$\mathcal{Z}^{\Lambda_N}(\tau) := \mathrm{Tr}_{\mathcal{V}^N}\, q^{L_0 - c/24} = \frac{\theta_{\Lambda_N}(\tau)}{\eta^{24}(\tau)} = J(\tau) + 24(h+1), \tag{3.3.1}$$

where θ_{Λ_N} is the lattice theta function, h is the Coxeter number associated to Λ_N and $J(\tau)$ is defined in (2.5.7). The constant comprises the contribution of the length-squared two vectors (roots) and the level-one bosonic states. In the case of the Leech lattice, we define $h = 0$ for Λ_L so that the partition function of the

[14] more general lattices with Lorentzian signature yields Borcherds-Kac-Moody algebras (imaginary roots).

Leech CFT is simply

$$\mathcal{Z}^{\Lambda_L}(\tau) := \mathrm{Tr}_{\mathcal{V}^L}\, q^{L_0-c/24} = \frac{\theta_{\Lambda_L}(\tau)}{\eta^{24}(\tau)} = J(\tau) + 24. \tag{3.3.2}$$

The monster CFT [107] is constructed from a $\mathbb{Z}_2$ orbifold of the Leech CFT. The $\mathbb{Z}_2$ acts on the 24 coordinates as

$$h : x_i \mapsto -x_i, \quad \forall i = 1, \ldots, 24,$$

and the Hilbert space $\mathscr{H}$ of the Leech CFT splits into two Hilbert spaces $\mathscr{H}_\pm$ consisting of states which are either invariant or anti-invariant under the orbifold action:

$$\mathscr{H}_\pm := \{\psi \in \mathscr{H} | h\psi = \pm\psi\}. \tag{3.3.3}$$

Furthermore, there is a twisted sector Hilbert space $\mathscr{H}^{tw}$ arising from the fixed points of the orbifold action; this is once again the direct sum of two Hilbert spaces comprised of twisted sector states which are invariant or anti-invariant under the orbifold action:

$$\mathscr{H}^{tw}_\pm := \{\psi^{tw} \in \mathscr{H}^{tw} | h\psi^{tw} = \pm\psi^{tw}\}. \tag{3.3.4}$$

The resulting Hilbert space of $\mathcal{V}^\natural$ is

$$\mathscr{H}_{\mathcal{V}^\natural} := \mathscr{H}_+ \oplus \mathscr{H}^{tw}_+. \tag{3.3.5}$$

The partition function of the theory is given by

$$\mathcal{Z}_{\mathcal{V}^\natural}(\tau) = \mathrm{Tr}_{\mathcal{V}^\natural}\, q^{L_0-c/24} = J(\tau). \tag{3.3.6}$$

Following [98], this is the partition function of a bosonic ECFT with smallest possible central charge.

The action of the monster group on $\mathcal{V}^\natural$ allows one to define, for each conjugacy class $g \in \mathbb{M}$, the so-called "McKay-Thompson" series $T_g(\tau)$, by

$$T_g(\tau) = \mathrm{Tr}_{\mathcal{V}^\natural}\, g q^{L_0-c/24}\,, \tag{3.3.7}$$

which is a modular function for Γ_g, an Atkin-Lehner type subgroup of $SL_2(\mathbb{R})$. See Appendix A for the precise definition of Γ_g. Therefore, it follows that one can interpret $\mathcal{V}^\natural$ as an infinite-dimensional $\mathbb{Z}$-graded $\mathbb{M}$-module,

$$\mathcal{V}^\natural = \bigoplus_{n=-1}^{\infty} \mathcal{V}^\natural_n$$

whose graded trace reproduces the McKay-Thompson series via

$$T_g(\tau) = \sum_{n=-1}^{\infty} (\mathrm{Tr}_{\mathcal{V}^\natural_n}\, g) q^n$$

and where one gets $J(\tau)$ by taking $g = e$, the identity element of $\mathbb{M}$. Moreover, each $T_g(\tau)$ is a Hauptmodul for Γ_g; i.e. it defines an isomorphism between the compactified fundamental domain $\overline{\mathcal{F}} = \Gamma_g \backslash \overline{\mathcal{H}}$ (see Appendix A.1.1 for the relevant definitions) and the Riemann sphere with finitely many points removed as

$$T_g : \overline{\mathcal{F}} \to \mathbb{C} \cup \{i\infty\}, \tag{3.3.8}$$

such that any meromorphic Γ_g-invariant function can be expressed as a rational function of $T_g(\tau)$. Due to the isomorphism in (3.3.8), Γ_g is called a *genus zero group*. This distinguishing feature of the McKay-Thompson series (genus zero property) was first conjectured by Conway and Norton in [67], later confirmed via an explicit construction of the module [107] and finally proved in [126].

The genus zero property reflects the pole structure of $T_g(\tau)$ in the following way. For any element $g \in \mathbb{M}$, $T_g(\tau)$ has a unique simple pole at the infinite cusp and is bounded at all the other cusps of Γ_g, or in other words has exponential growth in $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup i\infty$ only at the images of $i\infty$ under Γ_g. This extends the defining property of the J-function (2.5.7) to non-trivial conjugacy classes of the monster group. There are many more beautiful properties which distinguish the monster CFT from, say, the Leech and Niemeier CFTs or other bosonic chiral CFTs with central charge 24. We focus specifically on the following results, which elucidate the connection between the genus zero property of the McKay-Thompson series, their Rademacher summability, and the nature of g-orbifolds of $\mathcal{V}^\natural$.

Firstly, in [86], Duncan and Frenkel showed that the Hauptmodul property could be rephrased in terms of the Rademacher summability of T_g around the infinite cusp as long as the modular group has width one at the infinite cusp.

Theorem 3.3.1 (Duncan-Frenkel). *For all $g \in G$, there is a (Atkin-Lehner type) $\Gamma_g < SL_2(\mathbb{R})$ and a multiplier system $\epsilon_g : \Gamma_g \to \mathbb{C}^\times$ such that*

$$T_g(\tau) = \mathcal{R}^{(-1)}_{\Gamma_g,0,\epsilon_g}(\tau) - 2\mathfrak{r}(g)\,, \tag{3.3.9}$$

is the normalized Hauptmodul for Γ_g.

In the above, the constant $\mathfrak{r}(g)$ is given by the formula in equation (2.5.22). We slightly changed notation with respect to section §2.5.2 to stress its dependence on the conjugacy class of g. In addition, they proved that Γ_g has genus zero if and only if the Rademacher sum $\mathcal{R}^{(-1)}_{\Gamma_g,0,\epsilon_g}(\tau)$ is a function invariant under Γ_g. Therefore, for each Γ_g the associated Rademacher sum reduces to a modular function, specifically the Hauptmodul.[15] As discussed in §2.5, this must be due to the absence of a cusp form for dual weight (weight two) and conjugate multiplier system. In fact,

[15] A Hauptmodul is said to be normalized when no constant term appears in its Fourier expansion.

the space of cusp forms of weight two is isomorphic to the space of holomorphic differentials on $\overline{\mathcal{F}}$, which is empty when $\overline{\mathcal{F}}$ is a Riemann sphere.

Secondly, besides constraining the Fourier expansion of $T_g(\tau)$, the genus zero property was shown to correspond to a condition on the vacuum structure of sectors twisted by elements of the monster group, [127]. This is a consequence of the modular properties of a twisted-twined function, which connect the expansion of a twining function at a particular cusp to the ground state energy of a twisted sector. That is to say, depending on the group Γ_g, the g-twisted sector is related to the g-twined function by either an element of Γ_g or an element belonging to the normalizer group of $\Gamma_0(N)$ in $SL_2(\mathbb{R})$, defined in Appendix A.1.1. In [127] it was proved that the former case corresponds to a twisted sector with one negative energy state, while in the latter case no negative energy state appears. Therefore, for all $g \in \mathbb{M}$ the g-twisted sector of the $\langle g \rangle$-orbifold theory is either completely determined by the untwisted sector and the cusp corresponding to the g-twisted sector is equivalent to the infinite cusp (under the action of Γ_g) or the g-twisted sector spectrum has no negative energy states and the cusp corresponding to the g-twisted sector is inequivalent to the cusp at ∞. This condition together with a closure condition (c.f. Appendix A.1.1) which relates the different cusps is sufficient to ensure that T_g is a Hauptmodul for Γ_g. Again this property is directly encoded in the Rademacher expression for T_g.

Finally, another result by Tuite [128] relates the orbifold partition function for several conjugacy classes in $\mathbb{M}$ to the (conjectured) uniqueness of the module.

Theorem 3.3.2 (Tuite). *Assuming the uniqueness of $\mathcal{V}^\natural$, then the genus zero property holds if and only if orbifolding $\mathcal{V}^\natural$ with respect to a monster element reproduces the monster module itself or the Leech theory.*

We would like to understand the extent to which (suitable generalizations of) the above-mentioned properties hold in other cases of extremal CFTs. Specifically, one can ask if there is an analogue of Theorem 3.1 for the twining functions of other extremal CFTs. We investigate this question in §3.6 and comment on a possible extension of Theorem 3.2 in §3.7.

3.4 Central charge 12

In this section we discuss a family of extremal superconformal field theories with central charge 12. Each of the theories discussed in this section arises from the same underlying chiral SCFT whose Neveu-Schwarz (NS) and Ramond (R) sectors are vertex operator algebras which, following [129], we will refer to as $\mathcal{V}^{s\natural}$ and $\mathcal{V}^{s\natural}_{tw}$, respectively. As we will see, $\mathcal{V}^{s\natural}$ is in many senses the supersymmetric analogue of the monster CFT, $\mathcal{V}^{\natural}$. In §3.4.1 we review the construction of $\mathcal{V}^{s\natural}$. In §3.4.2 we describe a number of extremal SCFTs which arise upon viewing $\mathcal{V}^{s\natural}$ as a module for a $c = 12$ superconformal algebra (SCA) with extended supersymmetry. We discuss the symmetry groups of each of these theories in §3.4.3, as well as the mock modular forms whose coefficients encode the graded character of the corresponding G-module for each of these extremal CFTs. The material reviewed in this section primarily arises from the papers [63, 101, 121, 129, 130].

3.4.1 The Conway ECFT

Before defining $\mathcal{V}^{s\natural}$, we begin by describing a closely related theory, $\mathcal{V}^{sE_8}$, which we call the *super–E_8 CFT*. The latter is the $\mathcal{N} = 1$ SCFT obtained by compactifying eight chiral bosons on the eight-dimensional torus $\mathbb{R}^8/\Lambda_{E_8}$ with their eight chiral fermionic superpartners, where Λ_{E_8} is the E_8 root lattice. The theory has two sectors corresponding to whether the fermions have 1/2-integer (NS) or integer (R) grading along the spatial direction. From this description, the partition functions are easily determined to be

$$\mathcal{Z}^{sE_8}_{\mathrm{NS}}(\tau) = \mathrm{Tr}_{\mathrm{NS}}\, q^{L_0 - c/24} = \frac{E_4(\tau)\theta_3(\tau,0)^4}{\eta^{12}(\tau)} = \frac{1}{\sqrt{q}} + 8 + 276q^{1/2} + 2048q + \dots \tag{3.4.1}$$

in the NS sector, and

$$\mathcal{Z}^{sE_8}_{\mathrm{R}}(\tau) = \mathrm{Tr}_{\mathrm{R}}\, q^{L_0 - c/24} = \frac{E_4(\tau)\theta_2(\tau,0)^4}{\eta^{12}(\tau)} = 16 + 4096q + 98304q^2 + \dots \tag{3.4.2}$$

in the R sector. The functions (3.4.1) and (3.4.2) are invariant under the modular groups Γ_θ and $\Gamma_0(2)$, respectively, and they are part of a vector-valued representation of $SL_2(\mathbb{Z})$.[16]

The super–E_8 CFT is not extremal because of the eight fermions of dimension 1/2 in the NS sector. However, by taking a $\mathbb{Z}_2$ orbifold of the theory, one can remove these states and construct an extremal $\mathcal{N} = 1$ theory, $\mathcal{V}^{s\natural}$. This is analogous to the $\mathbb{Z}_2$ orbifold which removes the 24 dimension one currents of the Leech CFT in the construction of $\mathcal{V}^{\natural}$. In fact, $\mathcal{V}^{s\natural}$ has two distinct but equivalent constructions:

(A) A $\mathbb{Z}_2$ orbifold of the theory on the eight-torus $\mathbb{R}^8/\Lambda_{E_8}$ which acts as $X_i \to -X_i$ on the eight chiral bosons and as $\psi_i \to -\psi_i$ on their eight fermionic superpartners.

(B) A $\mathbb{Z}_2$ orbifold of 24 free chiral fermions, λ_α, which acts as $\lambda_\alpha \to -\lambda_\alpha$.

[16] See Appendix A.1.1 for the definitions of the relevant modular groups.

Construction (A) was first discussed in [107]. Construction (B) was first discussed in [101], where the two constructions were shown to be equivalent as vertex operator superalgebras, and further in [129], where it was shown that certain graded traces in this theory furnish normalized Hauptmoduln analogous to the McKay-Thompson series in monstrous moonshine. It is apparent that (A) is an $\mathcal{N} = 1$ supersymmetric extension of the E_8 current algebra. Furthermore, (B) enjoys a hidden $\mathcal{N} = 1$ superconformal symmetry as well. In particular, there are $2^{12} = 4096$ dimension–$\frac{3}{2}$ twist fields arising from zero modes of the λ_i acting on the twisted sector ground state. In [101] it is shown that there exists a linear combination of these twist fields that satisfies the OPEs for a supercurrent of an $\mathcal{N} = 1$ SCA with central charge 12. Moreover, the subgroup of Spin(24) which preserves this choice of supercurrent is the discrete subgroup Co_0 [129].

The partition function of $\mathcal{V}^{s\natural}$ can be computed using either construction. In the NS sector, the result is

$$\begin{aligned} \mathcal{Z}^{s\natural}_{\text{NS}}(\tau) = \text{Tr}_{\text{NS}}\, q^{L_0 - c/24} &= \frac{1}{2}\left(\frac{E_4(\tau)\theta_3(\tau,0)^4}{\eta^{12}(\tau)} + 16\frac{\theta_4^4(\tau,0)}{\theta_2^4(\tau,0)} + 16\frac{\theta_2^4(\tau,0)}{\theta_4^4(\tau,0)}\right) \\ &= \frac{1}{2}\sum_{i=2}^{4}\frac{\theta_i^{12}(\tau,0)}{\eta^{12}(\tau)} \\ &= \frac{1}{\sqrt{q}} + 276q^{1/2} + 2048q + 11202q^{3/2} + \dots \qquad (3.4.3) \\ &= K(\tau) - 24, \end{aligned}$$

where the formula in the first line arises from construction (A) and that in the second from construction (B). In the last line we have introduced an expression in terms of $K(\tau)$, a Hauptmodul for the modular subgroup Γ_θ (c.f. Appendix A.1.1). The lack of constant term in this partition function indicates that $\mathcal{V}^{s\natural}$ furnishes an example of an extremal $\mathcal{N} = 1$ SCFT with $k^* = 1$, according to [98]; i.e., there are no primary fields of dimension smaller than or equal to $c/24 = 1/2$. For more details on the moonshine properties of $\mathcal{V}^{s\natural}$, see the papers [101, 129].

3.4.2 More extremal theories

Focusing on construction (B) of $\mathcal{V}^{s\natural}$, it is straightforward to construct a number of additional extremal SCFTs where the chiral algebra is an extension of the $\mathcal{N} = 1$ superconformal algebra. In [121] theories with an $\mathcal{SW}(3/2, 2)$ SCA are discussed, whereas in [63], theories with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SCAs are discussed. In each case the approach is the same: given a choice of supercurrent $\mathcal{W}$ which generates an $\mathcal{N} = 1$ SCA with $c = 12$, one can pick an additional one, two, or three fermions to generate a chiral algebra which enhances the $\mathcal{N} = 1$ SCA to an extended version.

That each of these theories furnishes an example of an ECFT is straightforward to see from the character decomposition of their (flavored) partition functions. At $c = 12$, the extremal constraint forces the states of conformal dimension smaller than 1 in the NS sector to be superconformal descendants of the identity. We review this for each of these cases in turn.

1. If one chooses one of the 24 fermions, say λ_1, one can generate a chiral $c = 1/2$ Ising model. This enhances the $\mathcal{N} = 1$ SCA to a $c = 12$ $\mathcal{S}W(3/2, 2)$, i.e. the SCA which arises on the worldsheet theory of a non-linear sigma model with target space a manifold of Spin(7) holonomy [131]. See Appendix B.3.1 for a summary of the representation theory and characters of this algebra. It follows from the discussion in [121] and the Appendix that the partition function of $\mathcal{V}^{s\natural}$ has the following decomposition into Spin(7) characters,

$$\mathcal{Z}^{s\natural}_{\rm NS}(\tau) = \widetilde{\chi}^{\rm NS}_0(\tau) + 0\widetilde{\chi}^{\rm NS}_{\frac{1}{16}}(\tau) + 23\widetilde{\chi}^{\rm NS}_{\frac{1}{2}}(\tau) + \sum_{n=1}^{\infty} b_n \chi^{\rm NS}_{0,n}(\tau) + \sum_{n=1}^{\infty} c_n \chi^{\rm NS}_{\frac{1}{16},n}(\tau), \tag{3.4.4}$$

   where the constraint of extremality is satisfied by the fact that the coefficient in front of $\chi^{\rm NS}_{\frac{1}{16}}(\tau)$ is zero. We will denote this theory by $\mathcal{E}^{\rm Spin(7)}(G)$, where the group G is the symmetry group of the theory, and depends on the choice of fermion λ_1.

2. If one chooses two of the 24 fermions, one can generate a $\widehat{u(1)}_2$ current algebra which, together with the $\mathcal{N} = 1$ supercurrent, satisfies the OPEs of a $c = 12$ $\mathcal{N} = 2$ SCA [63]. The partition function of $\mathcal{V}^{s\natural}$ graded by this additional $U(1)$ is a weak Jacobi form for $SL_2(\mathbb{Z})$ of weight zero and index two, which takes the form

$$\mathcal{Z}^{s\natural}_{\rm R}(\tau, z) = {\rm Tr}_{\rm R}(-1)^F q^{L_0 - c/24} y^{J_0} = \frac{1}{2}\frac{1}{\eta^{12}(\tau)} \sum_{i=2}^{4} (-1)^{i+1} \theta_i(\tau, 2z) \theta_i^{11}(\tau, 0)\,, \tag{3.4.5}$$

   and admits the following decomposition into $c = 12$, $\mathcal{N} = 2$ characters

$$\begin{aligned}\mathcal{Z}^{s\natural}_{\rm R}(\tau, z) = {}& 23\,{\rm ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{1}{2},0} + {\rm ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{1}{2},2} + \Big(770\,({\rm ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},1} + {\rm ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},-1}) + \\ & + 13915\,({\rm ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},1} + {\rm ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},-1}) + \ldots\Big) + \big(231\,{\rm ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{3}{2},2} + 5796\,{\rm ch}^{\mathcal{N}=2}_{\frac{3}{2};\frac{5}{2},2} + \ldots\big).\end{aligned} \tag{3.4.6}$$

   From the discussion of the representation theory of the $\mathcal{N} = 2$ superconformal algebra in Appendix B.3.2, one sees from this character decomposition that the theory is an extremal $\mathcal{N} = 2$ theory. We will denote this theory by $\mathcal{E}^{\mathcal{N}=2}_{m=2}(G)$ where G is the global symmetry group of the theory and depends on the choice of $\mathcal{N} = 2$ superconformal algebra.

3. Finally, by choosing three fermions one can generate an $\widehat{su(2)}_2$ current algebra which becomes part of a $c = 12$ $\mathcal{N} = 4$ SCA when combined with the $\mathcal{N} = 1$ supercurrent [63]. The partition function of the theory with an additional grading by the Cartan of the $SU(2)$ coincides with the expression in (3.4.5). Furthermore, it admits the following decomposition into $c = 12$, $\mathcal{N} = 4$ superconformal characters

$$\mathcal{Z}_{\mathrm{R}}^{s\natural}(\tau, z) = 21\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{1}{2},0} + \mathrm{ch}^{\mathcal{N}=4}_{2;\frac{1}{2},1} + \left(560\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{3}{2},\frac{1}{2}} + 8470\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{5}{2},\frac{1}{2}} + 70576\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{7}{2},\frac{1}{2}} + \dots\right) + \left(210\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{3}{2},1} + 4444\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{5}{2},1} + 42560\,\mathrm{ch}^{\mathcal{N}=4}_{2;\frac{7}{2},1} + \dots\right). \tag{3.4.7}$$

The representation theory of the $\mathcal{N} = 4$ SCA is reviewed in Appendix B.3.3; with this information one can check that the above theory furnishes an extremal $\mathcal{N} = 4$ superconformal theory. We will denote this theory by $\mathcal{E}^{\mathcal{N}=4}_{m=2}(G)$, where again G is the global symmetry group of the theory and depends on the choice of $\mathcal{N} = 4$ superconformal algebra.

See table 3.2 for a summary of the relation between $\mathcal{V}^{s\natural}$ and these different superconformal algebras.

ESCFT	Fermions	Chiral algebra	$\mathcal{A}$
$\mathcal{E}^{\mathrm{Spin}(7)}$	1	Ising	$\mathcal{SW}(3/2, 2)$
$\mathcal{E}^{\mathcal{N}=2}_{m=2}$	2	$\widehat{u(1)}_2$	$\mathcal{N} = 2$
$\mathcal{E}^{\mathcal{N}=4}_{m=2}$	3	$\widehat{su(2)}_2$	$\mathcal{N} = 4$

Table 3.2: Superconformal algebras (with central charge 12) generated by a subset of fermions using construction (B) of $\mathcal{V}^{s\natural}$.

3.4.3 Symmetry groups and twining functions

In this section we consider the above-mentioned ECFTs in more detail, beginning with an analysis of their global discrete symmetry groups. In order to do this, we restrict to construction (B), where the discrete symmetries are most transparent.

Viewed as a theory with no supersymmetry, the continuous group Spin(24) has a natural action on the 24 fermions as signed permutations. In [101] it was shown that the choice of $\mathcal{N} = 1$ supercurrent in $\mathcal{V}^{s\natural}$ breaks the Spin(24) symmetry of the 24 fermions to the discrete group Co_0, the group of automorphisms of the Leech lattice. Likewise, for each choice of superconformal algebra $\mathcal{A}$ introduced in the previous section, there is a distinct ECFT whose global symmetry group G

is the subgroup of Co_0 which preserves the choice of fermions used to construct $\mathcal{A}$. There is moreover a geometrical interpretation of these symmetry groups: the distinct choices of superconformal algebra constructed from n fermions are in one-to-one correspondence with subgroups $G < Co_0$ which preserve an n-dimensional subspace in the unique non-trivial irreducible 24-dimensional representation of Co_0 (**24**) [63, 121]. We refer to such a group as n-plane preserving if it preserves an n-dimensional subspace in the representation **24**. In the following table we have listed examples of G which arise as subgroups of Co_0 preserving a choice of relevant superconformal algebra.

ESCFT	$\mathcal{A}$	Symmetry group (G)
$\mathcal{V}^{s\natural}$	$\mathcal{N}=1$	Co_0
$\mathcal{E}^{\mathrm{Spin}(7)}(G)$	$\mathcal{SW}(3/2,2)$	M_{24}, Co_2, Co_3
$\mathcal{E}^{\mathcal{N}=2}_{m=2}(G)$	$\mathcal{N}=2$	M_{23}, M_{12}, McL, HS
$\mathcal{E}^{\mathcal{N}=4}_{m=2}(G)$	$\mathcal{N}=4$	$M_{22}, M_{11}, U_4(3)$

For each of these theories one can construct character-valued twined partition functions for each conjugacy class $[g] \in G$. The twined functions are completely characterized by the action of g on the 24 fermions, and thus by the eigenvalues of g in the irreducible representation **24**.

Firstly, for every $[g] \in G$, where G is either Co_0 or a subgroup of Co_0 preserving a vector in the **24**, the corresponding g-twined partition function in the NS sector is

$$\mathcal{Z}^{s\natural}_{\mathrm{NS},g}(\tau) = \mathrm{Tr}_{\mathrm{NS}}\, g q^{L_0 - c/24} = \frac{1}{2}\sum_{i=1}^{4} \epsilon_i(g) \prod_{k=1}^{12} \frac{\theta_i(\tau, \rho_{g,k})}{\eta(\tau)} \tag{3.4.8}$$

where the definition of the $\epsilon_i(g)$ can be found in [129]. Also, we have defined $e(\rho_{g,k}) = \lambda_{g,k}$, where $k = 1, \ldots, 24$, $\rho_{g,k} \in [0, 1/2]$ and $\lambda_{g,k}$ is an eigenvalue of g. The latter corresponds to one of the 24 roots of the rational polynomial

$$\prod_{\ell|n} (t^\ell - 1)^{k_\ell}\,, \tag{3.4.9}$$

where $n = o(g)$ is the order of g, ℓ's are the positive divisors of n, and k_ℓ's are integers defined by the 24-dimensional irreducible representation of g. The data encoded in (3.4.9) can be succinctly written in terms of a formal product: the Frame shape of $[g]$,

$$\pi_g := \prod_{\ell|n} \ell^{k_\ell}\,. \tag{3.4.10}$$

In [129] it was proved that, similar to the case of $\mathcal{V}^\natural$, $\mathcal{V}^{s\natural}$ furnishes a $\frac{1}{2}\mathbb{Z}$-graded Co_0-module, whose graded characters are encoded in the coefficients of the twined functions $\mathcal{Z}^{s\natural}_{\mathrm{NS},g}(\tau)$. Furthermore, for all $g \in Co_1$, the functions (3.4.8) together with $\mathcal{Z}^{s\natural}_{\mathrm{R},g}(\tau)$ and $\mathcal{Z}^{s\natural,-}_{\mathrm{NS},g}(\tau)$,[17] form a vector-valued representation of a modular group $\Gamma_g < SL_2(\mathbb{R})$ with $\Gamma_0(o(g)) \subseteq \Gamma_g$.[18]

For every $[g] \in G$ where G is a subgroup of Co_0 preserving (at least) a 2-plane in **24**, the corresponding $U(1)$-graded g-twined function in the R Hilbert space reads

$$\begin{aligned}\mathcal{Z}^{s\natural}_{\mathrm{R},g}(\tau,z) &= \mathrm{Tr}_{\mathrm{R}}(-1)^F q^{L_0-c/24} y^{J_0} \\ &= \frac{1}{2}\frac{1}{\eta(\tau)^{12}}\sum_{i=1}^{4}(-1)^{i+1}\epsilon_{g,i}\,\theta_i(\tau,2z)\prod_{k=2}^{12}\theta_i(\tau,\rho_{g,k})\,.\end{aligned} \tag{3.4.11}$$

Moreover, it was shown in [63, 121] that $\mathcal{V}^{s\natural}$, equipped with a choice of extended superconformal algebra $\mathcal{A}$ (either $\mathcal{SW}(3/2,2)$, $\mathcal{N}=2$, or $\mathcal{N}=4$) furnishes a G-module for the discrete group G which preserves $\mathcal{A}$ and whose graded characters are encoded in the coefficients of a set of vector-valued mock modular forms whose corresponding shadows are (vector-valued) unary theta series. We summarize these results here and report the necessary definitions in Appendix B.2 and B.3.

1. $\mathcal{E}^{\mathrm{Spin}(7)}(G)$: Let $\mathcal{A}$ be the choice of Spin(7) algebra, and G the symmetry group preserving $\mathcal{A}$. G is a subgroup of Co_0 which fixes a one-plane. From the discussion in Appendix B.3.1 on the representation theory of the Spin(7) algebra, it is apparent that one can rewrite the graded partition function of equation (3.4.4) as

$$\mathcal{Z}^{s\natural}_{\mathrm{NS}}(\tau) = \mathcal{P}(\tau)\left(24\mu^{NS}(\tau) + h_1^{\mathrm{Spin}(7)}(\tau)\Theta^{NS}_{\frac{1}{16}}(\tau) + h_7^{\mathrm{Spin}(7)}(\tau)\Theta^{NS}_0(\tau)\right), \tag{3.4.12}$$

where $h^{\mathrm{Spin}(7)}$ is a weight 1/2 vector-valued mock modular form for $SL_2(\mathbb{Z})$ with shadow given by $24\underline{\widetilde{S}}$, multiplier system given by the inverse of $\underline{\widetilde{S}}$. The definition of $\underline{\widetilde{S}}$ is given in (B.3.14). The definition of $\Theta^{NS}_{\frac{1}{16}}$ and the one of Θ^{NS}_0 are given in (B.3.6), (B.3.5) respectively. Moreover, the g-twined functions for all conjugacy classes $g \in G$ have a similar expansion given by

$$\mathcal{Z}^{s\natural}_{\mathrm{NS},g}(\tau) = \mathcal{P}(\tau)\left(\chi_g\mu^{NS}(\tau) + h_{g,1}^{\mathrm{Spin}(7)}(\tau)\Theta^{NS}_{\frac{1}{16}}(\tau) + h_{g,7}^{\mathrm{Spin}(7)}(\tau)\Theta^{NS}_0(\tau)\right), \tag{3.4.13}$$

where $\chi_g = \mathrm{Tr}_{\mathbf{24}}g$, and $h_g^{\mathrm{Spin}(7)}$ is a weight 1/2 vector-valued mock modular form for $\Gamma_0(n)$, $n = o(g)$ with shadow $\chi_g\underline{\widetilde{S}}$ and, whenever $\chi_g \neq 0$, multiplier system given by the inverse multiplier system of $\underline{\widetilde{S}}$ restricted to $\Gamma_0(n)$.[19]

[17] The upper index "$-$" stands for the insertion of $(-1)^F$ in the trace over the NS Hilbert space.

[18] For $g \in Co_0$ but $g \notin Co_1$, a slightly different set of functions forms a vector-valued representation of Γ_g.

[19] When $\chi_g = 0$, the multiplier system is more complicated. This case is described in [121].

2. $\mathcal{E}^{\mathcal{N}=2}_{m=2}(G)$: Now we let $\mathcal{A}$ be a choice of $\mathcal{N} = 2$ superconformal algebra, and G the two-plane preserving subgroup of Co_0 which preserves $\mathcal{A}$. In [63] it was shown that one can rewrite equation (3.4.6) as

$$\mathcal{Z}^{\mathcal{N}=2}_{m=2}(\tau, z) = \frac{e\left(\frac{3}{4}\right)}{\Psi_{1,-\frac{1}{2}}(\tau, z)}\left(24\,\widetilde{\mu}_{\frac{3}{2};0}(\tau, z) + \sum_{j-\frac{3}{2}\in\mathbb{Z}/3\mathbb{Z}} h^{\mathcal{N}=2}_j(\tau)\theta_{\frac{3}{2},j}(\tau, z)\right), \tag{3.4.14}$$

where $h^{\mathcal{N}=2}$ is a weight $1/2$ vector-valued mock modular form. Furthermore, $h^{\mathcal{N}=2}$ has shadow given by $24S_{3/2}$ and inverse multiplier system to that of $S_{3/2}$, where $S_{3/2}$ is defined in (B.2.4). For all conjugacy classes $g \in G$, one can also write

$$\mathcal{Z}^{\mathcal{N}=2}_{m=2,g}(\tau, z) = \frac{e\left(\frac{3}{4}\right)}{\Psi_{1,-\frac{1}{2}}(\tau, z)}\left(\chi_g\,\widetilde{\mu}_{\frac{3}{2};0}(\tau, z) + \sum_{j-\frac{3}{2}\in\mathbb{Z}/3\mathbb{Z}} h^{\mathcal{N}=2}_{g,j}(\tau)\theta_{\frac{3}{2},j}(\tau, z)\right), \tag{3.4.15}$$

where $h^{\mathcal{N}=2}_g$ is a weight $1/2$ vector-valued mock modular form for $\Gamma_0(n)$ with shadow $\chi_g S_{3/2}$ and multiplier given by the inverse multiplier of $S_{3/2}$ restricted to $\Gamma_0(n)$ whenever $\chi_g \neq 0$. When $\chi_g = 0$, $h^{\mathcal{N}=2}_g$ is modular and has a more complicated multiplier system (c.f. [63]). See equation (A.2.12) for the definition of half-integer indices theta functions.

3. $\mathcal{E}^{\mathcal{N}=4}_{m=2}(G)$: Finally, let $\mathcal{A}$ be a choice of $\mathcal{N} = 4$ superconformal algebra, and G the three-plane preserving subgroup of Co_0 which preserves $\mathcal{A}$. It follows from [63] that equation (3.4.7) can be rewritten as

$$\mathcal{Z}^{\mathcal{N}=4}_{m=2}(\tau, z) = (\Psi_{1,1}(\tau, z))^{-1}\left(24\,\mu_{3;0}(\tau, z) + \sum_{j\in\mathbb{Z}/6\mathbb{Z}} h^{\mathcal{N}=4}_j(\tau)\theta_{3,j}(\tau, z)\right), \tag{3.4.16}$$

where $h^{\mathcal{N}=4}$ is a weight $1/2$ vector-valued mock modular form and with shadow given by $24S_3$ and inverse multiplier system to that of S_3, (B.2.4). For all conjugacy classes $g \in G$, one can also write

$$\mathcal{Z}^{\mathcal{N}=4}_{m=2,g}(\tau, z) = (\Psi_{1,1}(\tau, z))^{-1}\left(\chi_g\,\mu_{3;0}(\tau, z) + \sum_{j\in\mathbb{Z}/6\mathbb{Z}} h^{\mathcal{N}=4}_{g,j}(\tau)\theta_{3,j}(\tau, z)\right), \tag{3.4.17}$$

where $h^{\mathcal{N}=4}_g$ is a weight $1/2$ vector-valued mock modular form for $\Gamma_0(n)$ with shadow $\chi_g S_3$ and multiplier given by the inverse multiplier of S_3 restricted to $\Gamma_0(n)$ whenever $\chi_g \neq 0$. When $\chi_g = 0$, $h^{\mathcal{N}=4}_g$ is modular and again has a more complicated multiplier system (c.f. [63].)

3.5 Central charge 24

In this section we discuss three extremal superconformal field theories with central charge 24. Each of these SCFTs can be constructed as a nonlocal $\mathbb{Z}_2$ orbifold of bosons on a 24-dimensional torus given by $\mathbb{R}^{24}/\Lambda$ where Λ is either the Leech lattice or one of two other Niemeier lattices (c.f. §3.3).

3.5.1 Extremal theories

In [100] it was discussed how to construct an $\mathcal{N}=1$ SCFT from a $\mathbb{Z}_2$ orbifold of the Leech (or a Niemeier) CFT, where the $\mathbb{Z}_2$ acts on the 24 coordinates x_i as $h : x_i \to -x_i$, $\forall i$. As discussed in §3.3, the original Hilbert space $\mathscr{H}$ and the twisted Hilbert space $\mathscr{H}^{tw}$ split respectively into subspaces $\mathscr{H}_\pm, \mathscr{H}_\pm^{tw}$ of invariant and anti-invariant states under the h action (c.f. equations (3.3.3), (3.3.4)).

The key observation in [100] is that in the twisted sector Hilbert space $\mathscr{H}_-^{tw}$ there are 2^{12} ground states of dimension 3/2; this is precisely the dimension of an $\mathcal{N}=1$ supercurrent. In fact, the authors show that one can construct a consistent chiral $\mathcal{N}=1$ SCFT by choosing a linear combination of dimension-3/2 twist fields as a supercurrent. Furthermore, the Hilbert space naturally splits into R and NS sector depending on whether the OPE with the supercurrent has a branch-cut (R) or not (NS). The NS sector Hilbert space is given by

$$\mathscr{H}_{NS} = \mathscr{H}_+ \oplus \mathscr{H}_-^{tw},$$

and Ramond sector Hilbert space is given by

$$\mathscr{H}_R = \mathscr{H}_- \oplus \mathscr{H}_+^{tw}.$$

The partition function in the NS sector then reads

$$\begin{aligned} \mathcal{Z}_{\mathrm{NS}}^{\mathcal{N}=1}(\Lambda;\tau) &= \mathrm{Tr}_{\mathrm{NS}}\, q^{L_0-c/24} = \frac{1}{2}\frac{\theta_\Lambda(\tau)}{\eta^{24}(\tau)} + 2^{11}\left(\frac{\eta^{12}(\tau)}{\theta_2^{12}(\tau)} - \frac{\eta^{12}(\tau)}{\theta_3^{12}(\tau)} + \frac{\eta^{12}(\tau)}{\theta_4^{12}(\tau)}\right) \\ &= K(\tau)^2 - 48K(\tau) + 12(h+2) \\ &= \frac{1}{q} + 12h + 4096q^{\frac{1}{2}} + 98580q + 1228800q^{\frac{3}{2}} + \dots \end{aligned} \tag{3.5.1}$$

where $\theta_\Lambda(\tau)$ is the lattice theta function, h is the Coxeter number of the root system and the function $K(\tau)$ is defined in equation (A.1.14). Again, together with the characters $\mathrm{Tr}_{\mathrm{R}}\, q^{L_0-c/24}$ and $\mathrm{Tr}_{\mathrm{NS}}(-1)^F q^{L_0-c/24}$, the partition function of equation (3.5.1) transforms in a three-dimensional representation of $SL_2(\mathbb{Z})$. Furthermore, the function $\mathrm{Tr}_{\mathrm{R}}(-1)^F q^{L_0-c/24} = 12(h+2)$ computes the Witten index of the corresponding $\mathcal{N}=1$ SCFT.

In the case where $\Lambda = \Lambda_L$, the Leech lattice, this is precisely the partition function of an extremal $\mathcal{N} = 1$ SCFT with $k^* = 2$ (which we call $\mathcal{E}^{\mathcal{N}=2}_{k^*=2}$) as defined in [98]. The orbifold removes all dimension 1 currents in the NS sector; this fact both ensures extremality and precludes the possibility of constructing a superconformal algebra with extended supersymmetry. However, in the case where Λ is any of the other Niemeier lattices, a nontrivial current algebra survives the orbifold. The authors of [122] show that for $N = A_1^{24}$, one can construct a $\widehat{u(1)}_4$ current algebra, which, together with the supercurrent, satisfies the OPEs of an $\mathcal{N} = 2$ superconformal algebra with central charge 24. Furthermore, they show that the graded partition function in the Ramond sector is precisely the weak Jacobi form which captures the spectrum of an extremal $\mathcal{N} = 2$ SCFT with $m = 4$ (which we call $\mathcal{E}^{\mathcal{N}=2}_{m=4}$) according to [120]:

$$\begin{aligned}
\mathcal{Z}^{\mathcal{N}=2}_{m=4}(\Lambda_{A_1^{24}};\tau,z) &= \mathrm{Tr}_{\mathrm{R}}(-1)^F y^{J_0} q^{L_0-c/24} \\
&= \frac{1}{y^4} + 46 + y^4 + \ldots \\
&= 47\mathrm{ch}_{\frac{7}{2};1,0}(\tau,z) + \mathrm{ch}_{\frac{7}{2};1,4}(\tau,z) \\
&+ (32890 + 2969208q + \ldots)(\mathrm{ch}_{\frac{7}{2};2,1}(\tau,z) + \mathrm{ch}_{\frac{7}{2};2,-1}(\tau,z)) \\
&+ (14168 + 1659174q + \ldots)(\mathrm{ch}_{\frac{7}{2};2,2}(\tau,z) + \mathrm{ch}_{\frac{7}{2};2,-2}(\tau,z)) \\
&+ (2024 + 485001q + \ldots)(\mathrm{ch}_{\frac{7}{2};2,3}(\tau,z) + \mathrm{ch}_{\frac{7}{2};2,-3}(\tau,z)) \\
&+ (23 + 61984q + \ldots)\mathrm{ch}_{\frac{7}{2};2,4}(\tau,z),
\end{aligned} \tag{3.5.2}$$

where $\mathrm{ch}_{\ell;h,Q}$ denotes the $\mathcal{N} = 4$ character of central charge $c = 3(2\ell+1)$, dimension h, and charge Q in the Ramond sector (c.f. Appendix B.3.2), and we use the fact that the $\mathrm{ch}_{\ell;h+1,Q} = q\mathrm{ch}_{\ell;h,Q}$ for the non-BPS characters.

Similarly, when $N = A_2^{12}$, in [123] it is shown that one can construct an $\widehat{su(2)}_4$ current algebra which, along with the supercurrent, generates an $\mathcal{N} = 4$ superconformal algebra with $c = 24$. A straightforward computation of the graded partition function illustrates that this theory furnishes an example of an extremal $\mathcal{N} = 4$ SCFT with $c = 24$ (which we call $\mathcal{E}^{\mathcal{N}=4}_{m=4}$:

$$\begin{aligned}
\mathcal{Z}^{\mathcal{N}=4}_{m=4}(\Lambda_{A_2^{12}};\tau,z) &= \mathrm{Tr}_{\mathrm{R}}(-1)^F y^{J_3} q^{L_0-c/24} \\
&= \frac{1}{y^4} + \frac{1}{y^2} + 56 + y^2 + y^4 + \ldots \\
&= 55\mathrm{ch}_{4;1,0}(\tau,z) + \mathrm{ch}_{4;1,2}(\tau,z) \\
&+ (18876 + 1315512q + \ldots)(\mathrm{ch}_{4;2,\frac{1}{2}}(\tau,z) + \mathrm{ch}_{4;2,-\frac{1}{2}}(\tau,z)) \\
&+ (12045 + 1152943q + \ldots)(\mathrm{ch}_{4;2,1}(\tau,z) + \mathrm{ch}_{4;2,-1}(\tau,z)) \\
&+ (1980 + 391974q + \ldots)(\mathrm{ch}_{4;2,\frac{3}{2}}(\tau,z) + \mathrm{ch}_{4;2,-\frac{3}{2}}(\tau,z)) \\
&+ (33 + 45990q + \ldots)\mathrm{ch}_{4;2,2}(\tau,z),
\end{aligned} \tag{3.5.3}$$

where the details of the characters can be found in Appendix B.3.3.

3.5.2 Symmetry groups and twining functions

Like the extremal theories with central charge 12 discussed in §3.4, the theories described in the previous subsection furnish modules for a number of sporadic groups. We first consider $\mathcal{E}^{\mathcal{N}=1}_{k^*=2}$. The symmetry group of this theory arises from the automorphism group of the Leech lattice and a quantum symmetry coming from the $\mathbb{Z}_2$ orbifold. As discussed in [100], this is an extension of the group Co_0 by a finite abelian group. We do not discuss this theory in more detail here, though it would be interesting to investigate the properties of its twining functions.

Similarly, the discrete symmetry groups of the other two extremal theories we consider in this section arise from the automorphism group of the underlying Niemeier lattice Λ_N. The Niemeier lattices contain vectors generated by the root systems and additional so-called glue vectors. For the lattices with root systems A_1^{24} and A_2^{12}, the glue vectors are specified by elements of the extended binary Golay code and extended ternary Golay code, respectively. See e.g. [26] for a detailed description.

The automorphism group of the A_1^{24} Niemeier lattice is the Mathieu group M_{24}. It acts naturally on the 24 copies of the A_1 root system in its 24-dimensional (reducible) permutation representation. Furthermore, its action on the glue vectors is inherited from its natural action on the binary Golay code as automorphisms. Note that we must choose a particular A_1 root system to construct the affine $\widehat{u(1)}_4$ current algebra which becomes part of the $\mathcal{N} = 2$ SCA with $c = 24$. The choice of this root system breaks the M_{24} symmetry of the theory to an M_{23} subgroup, where the M_{23} fixes the distinguished coordinate direction, say x_1, associated with this A_1, and acts as a subgroup of S_{23} on the remaining coordinates. In the next section, we discuss the derivation of the twining functions

$$\mathcal{Z}^{\mathcal{N}=2}_{m=4,g}(\tau, z) = \mathrm{Tr}_{\mathrm{R}}\, g\, (-1)^F y^{J_0} q^{L_0 - c/24} \tag{3.5.4}$$

for conjugacy classes $g \in M_{23}$. These functions are weak Jacobi forms of weight zero and index 4 for the group $\Gamma_0(n)$ where $n = o(g)$, and they have the expansion

$$\mathcal{Z}^{\mathcal{N}=2}_{m=4,g}(\tau, z) = \frac{1}{y^4} + 2\mathrm{Tr}_{\mathbf{23}} g + y^4 + O(q)\,, \tag{3.5.5}$$

where $\mathbf{23} = \mathbf{1} + \mathbf{22}$ is the 23-dimensional permutation representation of M_{23}.

On the other hand, the automorphism group of the A_2^{12} Niemeier lattice is $2.M_{12}$, an extension of the Mathieu group M_{12}, where the M_{12} acts as a subgroup of S_{12}

on the 12 root systems, and the extension includes the order two automorphism of the A_2 Dynkin diagram. The action of $2.M_{12}$ on the glue vectors of the lattice follows from its action on the ternary Golay code, which specifies the glue vectors. In order to construct an affine $\widehat{su(2)}_4$ current algebra which becomes part of an $\mathcal{N}=4$ SCA with $c=24$, one chooses a distinguished A_2 root system corresponding to two directions, say x_1, x_2. The subgroup of $2.M_{12}$ which preserves the $\mathcal{N}=4$ SCA is then a copy of M_{11} which fixes x_1, x_2 and permutes the other 11 root systems. We discuss the twining functions

$$\mathcal{Z}^{\mathcal{N}=4}_{m=4,g}(\tau,z) = \mathrm{Tr}_{\mathrm{R}} g(-1)^F y^{J_0} q^{L_0-c/24} \tag{3.5.6}$$

for certain conjugacy classes $g \in M_{11}$, in Appendix B.5. These functions are weak Jacobi forms of weight zero and index 4 for $\Gamma_0(n)$, $n=o(g)$, and they have the expansion

$$\mathcal{Z}^{\mathcal{N}=4}_{m=4,g}(\tau,z) = \frac{1}{y^4} + \frac{1}{y^2} + (5\mathrm{Tr}_{\mathbf{11}}g+1) + y^2 + y^4 + O(q), \tag{3.5.7}$$

where $\mathbf{11} = \mathbf{1} + \mathbf{10}$ is the 10-dimensional permutation representation of M_{11}.

Just as discussed in the previous section for central charge 12, the two ECFTs $\mathcal{E}^{\mathcal{N}=2}_{m=4}$, $\mathcal{E}^{\mathcal{N}=4}_{m=4}$ with central charge 24 furnish G-modules whose graded characters are encoded in the coefficients of certain vector-valued modular forms, where G is the global symmetry group of the theory. We discuss the properties of these mock modular forms for each case below.

1. $\mathcal{E}^{\mathcal{N}=2}_{m=4}$: From the discussion in Appendix B.3.2, it is clear that we can rewrite the graded partition function of equation (3.5.2) as

$$\mathcal{Z}^{\mathcal{N}=2}_{m=4}(\tau,z) = \frac{e\left(\frac{3}{4}\right)}{\Psi_{1,-\frac{1}{2}}(\tau,z)}\left(48\,\widetilde{\mu}_{\frac{7}{2};0}(\tau,z) + \sum_{j-\frac{7}{2}\in\mathbb{Z}/7\mathbb{Z}} \widetilde{h}^{\mathcal{N}=2}_j(\tau)\theta_{\frac{7}{2},j}(\tau,z)\right), \tag{3.5.8}$$

where $\widetilde{h}^{\mathcal{N}=2}$ is a weight $\frac{1}{2}$ vector-valued mock modular form for $SL_2(\mathbb{Z})$ with shadow $48S_{\frac{7}{2}}$, defined in (B.2.4), and multiplier system inverse to that of $S_{\frac{7}{2}}$. Similarly, the g-twined functions of equation (3.5.4) have an expansion

$$\mathcal{Z}^{\mathcal{N}=2}_{m=4,g}(\tau,z) = \frac{e\left(\frac{3}{4}\right)}{\Psi_{1,-\frac{1}{2}}(\tau,z)}\left(2\chi_g\,\widetilde{\mu}_{\frac{7}{2};0}(\tau,z) + \sum_{j-\frac{7}{2}\in\mathbb{Z}/7\mathbb{Z}} \widetilde{h}^{\mathcal{N}=2}_{g,j}(\tau)\theta_{\frac{7}{2},j}(\tau,z)\right), \tag{3.5.9}$$

where $\chi_g = \mathrm{Tr}_{\mathbf{24}}g$ and $\widetilde{h}^{\mathcal{N}=2}_g$ is a weight $\frac{1}{2}$ vector-valued mock modular form for $\Gamma_0(n)$, $n=o(g)$, shadow $2\chi_g S_{\frac{7}{2}}$, and multiplier system inverse to that of $S_{\frac{7}{2}}$. In Appendix B.6.2, we present tables of the first several coefficients of the $\widetilde{h}_j^{\mathcal{N}=2}$ for all $g \in M_{23}$, as well as their decompositions into irreducible M_{23} representations.

2. $\mathcal{E}_{m=4}^{\mathcal{N}=4}$: Similarly, the discussion in Appendix B.3.3 indicates that we can write the graded partition function in (3.5.3) as

$$\mathcal{Z}_{m=4}^{\mathcal{N}=4}(\tau,z) = (\Psi_{1,1}(\tau,z))^{-1}\left(60\,\mu_{5;0}(\tau,z) + \sum_{j\in\mathbb{Z}/10\mathbb{Z}} \widetilde{h}_j^{\mathcal{N}=4}(\tau)\theta_{5,j}(\tau,z)\right), \tag{3.5.10}$$

where $\widetilde{h}^{\mathcal{N}=4}$ is a weight $\frac{1}{2}$ vector-valued mock modular form for $SL_2(\mathbb{Z})$ with shadow $60S_5$ and multiplier system the inverse to that of S_5. The g-twined functions (3.5.6) for conjugacy classes $g \in M_{11}$ similarly give rise to vector-valued mock modular forms through the expansion

$$\mathcal{Z}_{m=4,g}^{\mathcal{N}=4}(\tau,z) = (\Psi_{1,1}(\tau,z))^{-1}\left(5(\mathrm{Tr}_{\mathbf{12}}g)\,\mu_{5;0}(\tau,z) + \sum_{j\in\mathbb{Z}/10\mathbb{Z}} \widetilde{h}_{g,j}^{\mathcal{N}=4}(\tau)\theta_{5,j}(\tau,z)\right), \tag{3.5.11}$$

where $\widetilde{h}_g^{\mathcal{N}=4}$ is a weight $\frac{1}{2}$ vector-valued mock modular form for $\Gamma_0(n)$, $n = o(g)$, with shadow $5(\mathrm{Tr}_{\mathbf{12}}g)S_5$ where $\mathbf{12} = \mathbf{1} + \mathbf{11}$ and multiplier system the inverse of that of S_5.

3.5.3 Twining functions

In this section, we outline the basic ingredients necessary to compute the twined partition functions of $\mathcal{E}_{m=4}^{\mathcal{N}=2}$ for $[g] \in M_{23}$ and the twined partition functions of $\mathcal{E}_{m=4}^{\mathcal{N}=4}$ with $[g] \in M_{11}$. The approach is similar, so we discuss the two cases in parallel, pointing out distinctions when they occur. More details can be found in Appendix B.5.

The starting point for each is the Niemeier CFT with target $\mathbb{R}^{24}/\Lambda_N$, for $N = A_1^{24}$ and A_2^{12} in the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ cases, respectively. The partition function of the CFT consists of primary states coming from lattice vectors, primary states coming from the 24 currents $i\partial x_i$, and the Virasoro descendants; i.e. it is just given by equation (3.3.1). For $\Lambda = A_1^{24}$, it will be useful to think of this CFT in the following way. The ith copy of the A_1 root system furnishes an affine $\widehat{su(2)}_1$ current algebra, generated by the vertex operators

$$e^{\pm i\sqrt{2}x_i}, \quad i\partial x_i, \tag{3.5.12}$$

and therefore the partition function $\mathcal{Z}^{\Lambda_N}(\tau)$ of the theory has a natural decomposition into characters of $\left(\widehat{su(2)}_1\right)^{24}$.

There are two irreducible modules of $\widehat{su(2)}_1$—one arising from the vacuum representation, which has a ground state of conformal weight zero, and a second from a highest-weight state of conformal weight $\frac{1}{4}$. We will denote these representations as [0] and [1], respectively. The characters of these irreducible modules are given by

$$\chi_0(\tau) = \mathrm{Tr}_{[0]} q^{L_0 - c/24} = \frac{\theta_3(2\tau)}{\eta(\tau)}, \tag{3.5.13}$$

$$\chi_1(\tau) = \mathrm{Tr}_{[1]} q^{L_0 - c/24} = \frac{\theta_2(2\tau)}{\eta(\tau)}, \tag{3.5.14}$$

where $c = 1$ is the Sugawara central charge of the current algebra. The full lattice theta function for Λ_N with $N = A_1^{24}$ consists of all lattice vectors which are linear combinations of root vectors and glue vectors. The glue vectors can be specified in terms of elements of the extended binary Golay code. This is a length-24 binary code with weight enumerator

$$x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}, \tag{3.5.15}$$

where the coefficient of the term $x^n y^m$ gives the number of vectors in the code with n zeros and m ones. Thus we can rewrite the partition function in terms of $\widehat{su(2)}_1$ characters as

$$\mathcal{Z}^{\Lambda_N}(\tau) = \chi_0^{24}(\tau) + 759(\chi_0^{16}(\tau)\chi_1^8(\tau) + \chi_0^8(\tau)\chi_1^{16}(\tau)) + 2576\chi_0^{12}(\tau)\chi_1^{12}(\tau) + \chi_1^{24}(\tau) \tag{3.5.16}$$

for $N = A_1^{24}$.

Similarly, in the case of $N = A_2^{12}$, the partition function can naturally be written in terms of characters of $\left(\widehat{su(3)}_1\right)^{12}$, as each of the 12 A_2 root systems furnishes a copy of $\widehat{su(3)}_1$. There are three irreducible $\widehat{su(3)}_1$ modules we will call [i] and with characters we refer to as $\chi_i(\tau)$, $i = 0, 1, 2$. The vacuum module [0] has conformal dimension $h = 0$ and the two nontrivial primaries both have conformal dimension $h = \frac{1}{3}$. Furthermore, the glue vectors are now specified in terms of elements of the extended ternary Golay code, which is a length-12 ternary code with weight enumerator

$$x^{12} + y^{12} + z^{12} + 22(x^6y^6 + y^6z^6 + z^6x^6) + 220(x^6y^3z^3 + y^6z^3x^3 + z^6x^3y^3). \tag{3.5.17}$$

Therefore, for $N = A_2^{12}$ we can write the partition function in terms of $\widehat{su(3)}_1$ characters by replacing x, y, z in the weight enumerator with χ_0, χ_1, χ_2, respectively. Furthermore, the formulas for these characters are given by

$$\chi_0(\tau) = \mathrm{Tr}_{[0]} q^{L_0 - c/24} = \frac{\theta_3(2\tau)\theta_3(6\tau) + \theta_2(2\tau)\theta_2(6\tau)}{\eta^2(\tau)} \tag{3.5.18}$$

for the vacuum character, and

$$\chi_i(\tau) = \mathrm{Tr}_{[i]} q^{L_0 - c/24} = \frac{\theta_3(2\tau)\theta_3\left(\frac{2\tau}{3}\right) + \theta_2(2\tau)\theta_2\left(\frac{2\tau}{3}\right)}{2\eta^2(\tau)} - \frac{\chi_0(\tau)}{2} \tag{3.5.19}$$

for the nontrivial primaries with $i = 1, 2$, where in this case $c = 2$. Thus the partition function of the theory can be written as

$$\mathcal{Z}^{\Lambda_N}(\tau) = \chi_0^{12}(\tau) + 264\chi_0^6(\tau)\chi_1^6(\tau) + 440\chi_0^3(\tau)\chi_1^9(\tau) + 24\chi_1^{12}(\tau) \tag{3.5.20}$$

for $N = A_2^{12}$.

Now we consider a $\mathbb{Z}_2$ orbifold of the above theories. We will call the $\mathbb{Z}_2$ symmetry h, which acts with a minus sign on the 24 coordinates of the torus:

$$h : x_i \to -x_i. \tag{3.5.21}$$

In the case of $N = A_1^{24}$, the orbifold preserves a $\left(\widehat{u(1)}_4\right)^{24}$ current algebra out of the $\left(\widehat{su(2)}_1\right)^{24}$ and, similarly, for $N = A_2^{12}$, the orbifold preserves an $\left(\widehat{su(2)}_4\right)^{12}$ within the $\left(\widehat{su(3)}_1\right)^{12}$. We choose one copy of $\widehat{u(1)}_4$ and $\widehat{su(2)}_4$ to generate the R-symmetry of the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ algebras, respectively. In the NS sector the corresponding level-4 current is given by the h-invariant linear combination

$$J_0 = 2\left(e^{i\sqrt{2}x_1} + e^{-i\sqrt{2}x_1}\right) \tag{3.5.22}$$

for the $\mathcal{N} = 2$ case and by

$$J_3 = \sqrt{2}\left(e^{i\sqrt{2}x_1} + e^{-i\sqrt{2}x_1}\right) \tag{3.5.23}$$

for the $\mathcal{N} = 4$ case [122, 123].

We consider the Ramond sector partition function graded by these currents and by $(-1)^F$. The Hilbert space is composed of the anti-invariant states in the untwisted sector and the invariant states in the twisted sector. Thus we have to compute the trace

$$\begin{aligned} \mathcal{Z}_{m=4}^{\mathcal{N}=2(4)}(\tau, z) &= \mathrm{Tr}_{\mathscr{H}_R}(-1)^F q^{L_0 - \frac{c}{24}} y^{J_0(J_3)} \\ &= \mathrm{Tr}_{\mathscr{H}}\left(\frac{1-h}{2}\right)(-1)^F q^{L_0 - \frac{c}{24}} y^{J_0(J_3)} + \mathrm{Tr}_{\mathscr{H}^{tw}}\left(\frac{1+h}{2}\right)(-1)^F q^{L_0 - \frac{c}{24}} y^{J_0(J_3)} \end{aligned} \tag{3.5.24}$$

where the first term in the second line implements a projection onto the anti-invariant states in the untwisted sector Hilbert space, $\mathscr{H}$, and the second term a projection onto the invariant states in the twisted sector Hilbert space, $\mathscr{H}^{tw}$. Furthermore, we will also consider the twining functions

$$\mathcal{Z}_{m=4,g}^{\mathcal{N}=2(4)}(\tau, z) = \mathrm{Tr}_{\mathscr{H}_R} g(-1)^F q^{L_0 - \frac{c}{24}} y^{J_0(J_3)} \tag{3.5.25}$$

defined for $g \in M_{23}, M_{11}$, respectively. These functions are weak Jacobi forms of weight zero and index four for $\Gamma_0(n)$ where $n = o(g)$. We discuss the explicit computation of each of these terms in Appendix B.5.

3.6 Rademacher summability

Inspired by the relation between the genus zero property of monstrous moonshine and the Rademacher sum construction of the McKay-Thompson series underlined in [86, 88], we examine the Rademacher expansion at the infinite cusp for the twined functions of the ECFTs introduced in §3.4 and §3.5. We begin in section 3.6.1 by discussing the Co_0-module $\mathcal{V}^{s\natural}$, and analyze the other $c = 12$ and $c = 24$ theories in §3.6.2 and §3.6.3, respectively. The results presented in §3.6.2 and §3.6.3 are obtained by numerically computing the coefficients in equation (2.5.20) to high accuracy and comparing with the known twining functions described in §3.4 and §3.5, respectively.

Finally, in §B.4 we discuss the following curious property of the twining functions for the theory $\mathcal{E}^{\mathcal{N}=2}_{m=2}(M_{23})$. The functions which cannot be expressed as Rademacher sums at the infinite cusp precisely correspond to conjugacy classes $g \in M_{23}$ such that $3|o(g)$. In this case, however, the expansion of these functions about cusps inequivalent to $i\infty$ either has no pole, or the coefficients in such an expansion can be directly related to the coefficients which appear in the expansion of the function at $i\infty$. This property might be interpreted as a generalization of the results of Tuite [127] reviewed in §3.3 for McKay-Thompson series for genus zero groups with Atkin-Lehner involutions.

3.6.1 The Conway module

The Conway theory $\mathcal{V}^{s\natural}$ [129] furnishes a $\frac{1}{2}\mathbb{Z}$-graded Co_0-module, i.e.,

$$\mathcal{V}^{s\natural} = \bigoplus_{n=-1}^{\infty} \mathcal{V}^{s\natural}_{\frac{n}{2}}. \tag{3.6.1}$$

In [101] it was shown that the action of Co_0 is entirely dictated by the eigenvalues of g in its 24-dimensional irreducible representation (**24**) and therefore only depends on the conjugacy class $[g]$. In the following, we refer to the latter via the Frame shape π_g defined in (3.4.10). For each conjugacy class $[g] \in Co_0$, one can define a set of character-valued twined partition functions by taking traces in the NS Hilbert space with and without insertion of $(-1)^F$. These correspond to the previously defined $Z^{s\natural,-}_{\mathrm{NS},g}(\tau)$, $Z^{s\natural}_{\mathrm{NS},g}(\tau)$ and take the form

$$\mathcal{Z}^{s\natural,-}_{\mathrm{NS},g}(\tau) = \sum_{n=-1}^{\infty} (\mathrm{Tr}_{\mathcal{V}^{s\natural}_{\frac{n}{2}}} (-1)^n g) q^{\frac{n}{2}} \, . \tag{3.6.2}$$

$$\mathcal{Z}^{s\natural}_{\mathrm{NS},g}(\tau) = \sum_{n=-1}^{\infty} (\mathrm{Tr}_{\mathcal{V}^{s\natural}_{\frac{n}{2}}} g) q^{\frac{n}{2}} \, . \tag{3.6.3}$$

As it was previously mentioned, the Conway module $\mathcal{V}^{s\natural}$ and the monster module $\mathcal{V}^{\natural}$ have many common features. First of all, in both of these theories the graded traces associated to the particular module are simply related to the Hauptmoduln of the corresponding modular group. However, the fermionic nature of the Conway module is reflected in its half-integer grading, in contrast to the $\mathbb{Z}$-graded monster module. As a result, normalized Hauptmoduln arise only after rescaling the twining functions $Z^{s\natural,-}_{\mathrm{NS},g}(2\tau)$ and $Z^{s\natural}_{\mathrm{NS},g}(2\tau)$. This genus zero property of $\mathcal{V}^{s\natural}$ was shown to hold in [129]; from this the analogue of Theorem 3.3.1 for $\mathcal{V}^{\natural}$ directly follows (c.f. Theorem 4.9 in [129].)

Specifically, all twining functions can be expressed as Rademacher sums at the infinite cusp both (i) as scalar-valued Rademacher sums with respect to appropriate subgroups of genus zero groups appearing in monstrous moonshine, and (ii) as vector-valued Rademacher sums where the vector includes (3.6.2), (3.6.3) and $\mathcal{Z}^{s\natural}_{\mathrm{R},g}(\tau)$ and transforms under a modular group containing $\Gamma_0(N)$ and contained in $\mathcal{N}(N)$. The explicit modular properties of the twining functions can be found in [129].

3.6.2 Modules with $c = 12$

Following §3.4.3 and [63], it is clear that the extremal SCFTs $\mathcal{E}^{\mathrm{Spin}(7)}(G)$, $\mathcal{E}^{\mathcal{N}=2}_{m=2}(G)$, $\mathcal{E}^{\mathcal{N}=4}_{m=2}(G)$ with central charge 12 furnish G-modules for the global symmetry group G of the theory. We will denote these G-modules by $\mathcal{V}^{\mathcal{A},G}$, where $\mathcal{A}$ denotes the extended superconformal algebra and

$$\mathcal{V}^{\mathcal{A},G} = \bigoplus_{r\in\{\alpha\}_{\mathcal{A}}} \bigoplus_{n=1}^{\infty} V^{\mathcal{A},G}_{r,n} \, , \qquad \mathcal{A} \in \{\mathrm{Spin}(7), \mathcal{N}=2, \mathcal{N}=4\}. \tag{3.6.4}$$

The corresponding graded characters are the coefficients of certain vector-valued mock modular forms $h^{\mathcal{A}}_g$, defined by

$$h^{\mathcal{A}}_{g,r}(\tau) = a_r q^{-\frac{r^2}{b_{\mathcal{A}}}} + \sum_{n=1}^{\infty} (\mathrm{Tr}_{\mathcal{V}^{\mathcal{A},G}_{r,n}} g) q^{n-\frac{r^2}{b_{\mathcal{A}}}} \, . \tag{3.6.5}$$

The constants $a_r, b_{\mathcal{A}}, \{\alpha\}_{\mathcal{A}}$ appearing in the expansion are displayed below together with the symmetry groups G on which we focus in this section. In the case of the $\mathcal{N} = 4$ theories, we consider two different embeddings of M_{11} into Co_0, which we refer to as $M_{11}^{(1)}$ and $M_{11}^{(2)}$. $M_{11}^{(1)}$ can be described as the subgroup of M_{12} which fixes a point in its 12-dimensional permutation representation; on the other hand, $M_{11}^{(2)}$ is the subgroup of M_{12} which fixes a certain length-12 vector in its 12-dimensional permutation representation.

$\mathcal{A}$	$\{\alpha\}_{\mathcal{A}}$	$b_{\mathcal{A}}$	$\{a_r\}$	G
$\mathcal{SW}(3/2,2)$	$\{1,7\}$	120	$\{-1,1\}$	M_{24}
$\mathcal{N}=2$	$\{\pm\frac{1}{2},\frac{3}{2}\}$	6	$\{-1,1\}$	M_{23}, M_{12}
$\mathcal{N}=4$	$\{1,2\}$	12	$\{-2,-1\}$	$M_{22}, M_{11}^{(1)}, M_{11}^{(2)}$

The functions $h_{g,r}^{\mathcal{A}}$ comprise a vector-valued mock modular form of weight 1/2 and multiplier system ρ_g with respect to Γ_g a congruence subgroup of $SL_2(\mathbb{Z})$. In particular, if we denote by n the order of g, Γ_g equals $\Gamma_0(n)$. Moreover, given the Frame shape of $[g]$ in (3.4.10), we denote by ξ the smallest cycle appearing in the Frame shape. Note that we choose to focus on the Mathieu groups, which are distinguished as all of their twined Jacobi forms have particularly nice behavior at other cusps according to [63]. However, as reviewed in §3.4.3, there exists an extremal $\mathcal{SW}(3/2,2)$, $\mathcal{N}=2$, and $\mathcal{N}=4$ CFTs for each subgroup of Co_0 which preserves a one-, two-, or three-plane, respectively. We leave a general analysis of such cases to future work.

The components of the functions $h_g^{\mathcal{A}}$ can be written in terms of a Rademacher sum as

$$h_{g,s}^{\mathcal{A}}(\tau) = \mathcal{R}_{\Gamma_g,\,1/2,\,\rho_g}^{(\{-\frac{r^2}{b_{\mathcal{A}}}\})}(\tau)_s\,, \tag{3.6.6}$$

where $\{-\frac{r^2}{b_{\mathcal{A}}}\}$ denotes the set of exponents of the q-polar terms which appear in the vector-valued mock modular form $h_g^{\mathcal{A}}$ and $\mathcal{R}_{\Gamma_g,\,1/2,\,\rho_g}^{(\{-\frac{r^2}{b_{\mathcal{A}}}\})}(\tau)_s$ corresponds to a sum of Rademacher sums for each polar q-term (see section §2.5.2). More explicitly, $\mathcal{R}_{\Gamma_g,\,1/2,\,\rho_g}^{(\{-\frac{r^2}{b_{\mathcal{A}}}\})}(\tau)_s = \sum_r a_r \mathcal{R}_{\Gamma_g,\,1/2,\,\rho_g}^{(-\frac{r^2}{b_{\mathcal{A}}})}(\tau)_s$, where the terms on the right-hand side are defined in equation (2.5.20).

In the following, we report the necessary data for the construction of the functions $h_g^{\mathcal{A}}$ via a Rademacher sum and the conjugacy classes of G whose twining function cannot be reproduced by a Rademacher expansion at the infinite cusp.

1. $\mathcal{E}^{\mathrm{Spin}(7)}(G)$: The weight 1/2 vector-valued mock modular form $h^{\mathrm{Spin}(7)}$ for $SL_2(\mathbb{Z})$ is derived from equation (3.4.12). We report here the first few Fourier

coefficients

$$\begin{aligned} h_1^{\mathrm{Spin}(7)}(\tau) &= q^{-\frac{1}{120}}(-1 + 1771q + 35650q^2 + 374141q^3 + \ldots)\,, \\ h_7^{\mathrm{Spin}(7)}(\tau) &= q^{-\frac{49}{120}}(1 + 253q + 7359q^2 + 95128q^3 + \ldots)\,. \end{aligned} \tag{3.6.7}$$

These expressions fix the coefficients of the negative q-power terms for all the twined versions $h_g^{\mathrm{Spin}(7)}$; these polar terms arise from the G-invariant NS ground state. The multiplier system of these mock modular forms is the inverse multiplier of the vector-valued unary theta series $\widetilde{\underline{S}}$, defined in (B.3.14), as long as $\chi_g \neq 0$. The latter is completely specified by its representation on the generators of $SL_2(\mathbb{Z})$

$$\rho(T) = \begin{pmatrix} e(\frac{1}{120}) & 0 \\ 0 & e(\frac{49}{120}) \end{pmatrix}\,, \qquad \rho(S) = \begin{pmatrix} -\sqrt{\frac{2}{5+\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}} \\ \sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}} \end{pmatrix}\,. \tag{3.6.8}$$

When the element g has no fixed points (i.e. $\chi_g = 0$) the multiplier system is not constrained by that of the shadow and it is given by the inverse of (3.6.8) times a Frame shape-dependent phase

$$\nu_g = e\left(-\frac{c\,d}{n\,\xi}\right)\,. \tag{3.6.9}$$

The Rademacher series defined in (2.5.20) reproduces $h_g^{\mathrm{Spin}(7)}$ for all conjugacy classes in M_{24} except for the conjugacy classes reported in Table 3.3 and the one with Frame shape 12^2 for which it has not been found the correct multipler system.

2. $\mathcal{E}_{m=2}^{\mathcal{N}=2}(G)$: From equation (3.4.14) $h^{\mathcal{N}=2}$ is a weight 1/2 vector-valued mock modular form whose first few coefficients are given by

$$\begin{aligned} h_{\frac{1}{2}}^{\mathcal{N}=2}(\tau) &= h_{-\frac{1}{2}}^{\mathcal{N}=2}(\tau) = -q^{-1/24} + 770q^{23/24} + 13915q^{47/24} + \ldots \\ h_{\frac{3}{2}}^{\mathcal{N}=2}(\tau) &= q^{-9/24} + 231q^{15/24} + 5796q^{37/24} + \ldots\,. \end{aligned} \tag{3.6.10}$$

The multiplier system is given by the inverse of the half-index theta function multiplier, defined in equation (A.2.16).

In the case $G = M_{23}$, the Rademacher expression (2.5.20) coincides with the vector-valued mock modular form $h_g^{\mathcal{N}=2}$ for all the conjugacy classes except those for which $3|o(g)$. However, see §B.4 for an analysis of the structure of these functions at the other poles of Γ_g.

Similarly, in the case $G = M_{12}$ the functions $h_g^{\mathcal{N}=2}$ corresponding to the conjugacy classes for which $3|g$ cannot be reproduced by the Rademacher series at the infinite cusp, except for the Frame shape 3^8. Additionally, the Rademacher expansion of $h_g^{\mathcal{N}=2}$ at the infinite cusp fails when g has Frame shape $4^2 8^2$. The multiplier system of $h_g^{\mathcal{N}=2}$ for the conjugacy classes in π_g with no fixed points and which can be reproduced using the Rademacher expansion is given by the inverse of (A.2.16) times the phase

$$\nu_g = e\left(-\frac{c\,d}{n\,\xi}\right). \tag{3.6.11}$$

3. $\mathcal{E}_{m=2}^{\mathcal{N}=4}(G)$: Equation (3.4.16) defines $h^{\mathcal{N}=4}$, a vector-valued mock modular form whose first few coefficients are given by

$$\begin{aligned} h_1^{\mathcal{N}=4}(\tau) &= -h_{-1}^{\mathcal{N}=4}(\tau) = -2q^{-1/12} + 560q^{11/12} + 8470q^{23/12} + \dots \\ h_2^{\mathcal{N}=4}(\tau) &= -h_{-2}^{\mathcal{N}=4}(\tau) = -q^{-4/12} - 210q^{8/12} - 4444q^{16/12} + \dots \end{aligned} \tag{3.6.12}$$

The multiplier system of these weight 1/2 mock modular forms with respect to $\Gamma_0(n)$ is the conjugate of the shadow $\chi_g S_3$, (c.f. [63]). Due to the symmetry of the theta function and the modular properties of the Jacobi form, it follows that $h_{m,r}^{\mathcal{N}=4} = -h_{m,-r}^{\mathcal{N}=4}$. Thus, among the 6 components of the mock modular form only three of them are linearly independent.

In this case, we find that the Rademacher sum (2.5.20) coincides with $h_g^{\mathcal{N}=4}$ for all conjugacy classes $g \in G$ where $G = M_{22}, M_{11}^{(1)}$. For $G = M_{11}^{(2)}$, the conjugacy class labeled by the Frame shape $2^4\,4^4$ has multiplier system given by the inverse of the theta multiplier (A.2.15) times (3.6.11) whereas for $4^2\,8^2$ there is no match between the Rademacher sum and the twining function.

To summarize, we report below the conjugacy classes corresponding to the mock modular forms that cannot be reconstructed using solely the information at the infinite cusp for these $c = 12$ theories.

ECFT	Frame shapes with additional poles
$\mathcal{E}^{\mathrm{Spin}(7)}\,(M_{24})$	$1^6\,3^6 - 1^4\,5^4 - 1^2\,2^2\,3^2\,6^2 - 2^2\,10^2 - 2.4.6.12 - 1.3.5.15$
$\mathcal{E}_{m=2}^{\mathcal{N}=2}\,(M_{23})$	$1^6\,3^6 - 1^2\,2^2\,3^2\,6^2 - 1.3.5.15$
$\mathcal{E}_{m=2}^{\mathcal{N}=2}\,(M_{12})$	$1^6\,3^6 - 1^2\,2^2\,3^2\,6^2 - 6^4 - 4^2\,8^2$
$\mathcal{E}_{m=2}^{\mathcal{N}=4}\,(M_{22})$	None
$\mathcal{E}_{m=2}^{\mathcal{N}=4}\,(M_{11}^{(1)})$	None
$\mathcal{E}_{m=2}^{\mathcal{N}=4}\,(M_{11}^{(2)})$	$4^2 8^2$

Table 3.3: Pole structure of $h_g^{\mathcal{A}}(\tau)$ for certain extremal theories with central charge 12.

3.6.3 Modules with $c = 24$

In analogy to the theories with central charge 12, the two extremal CFTs $\mathcal{E}^{\mathcal{N}=2}_{m=4}$, $\mathcal{E}^{\mathcal{N}=4}_{m=4}$ with central charge 24 furnish G-modules whose graded characters are encoded in the coefficients of certain vector-valued modular forms. We will denote these modules by $\widetilde{\mathcal{V}}^{\mathcal{A},G}$,

$$\widetilde{\mathcal{V}}^{\mathcal{A},G} = \bigoplus_{j\in\{\widetilde{\alpha}\}_{\mathcal{A}}} \bigoplus_{n=1}^{\infty} \widetilde{V}^{\mathcal{A},G}_{j,n} , \qquad \mathcal{A} \in \{\mathcal{N}=2, \mathcal{N}=4\}, \tag{3.6.13}$$

where $G = M_{23}$ and M_{11}, respectively. From these considerations and the description given in §3.4.3, we see that the mock modular forms can be written as

$$\widetilde{h}^{\mathcal{A}}_{g,j}(\tau) = \widetilde{a}_j q^{-\frac{j^2}{\beta_{\mathcal{A}}}} + \sum_{n=1}^{\infty} (\mathrm{Tr}_{\widetilde{\mathcal{V}}^{\mathcal{A},G}_{j,n}} g) q^{n-\frac{j^2}{\beta_{\mathcal{A}}}} , \tag{3.6.14}$$

where the data appearing above can be summarized succinctly in the following table:

$\mathcal{A}$	$\{\widetilde{\alpha}\}_{\mathcal{A}}$	$\beta_{\mathcal{A}}$	$\{a_r\}$	G
$\mathcal{N}=2$	$\{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \frac{7}{2}\}$	14	$\{-1,1,-1,1\}$	M_{23}
$\mathcal{N}=4$	$\{1,2,3,4\}$	20	$\{-4,-3,-2,-1\}$	M_{11}

Furthermore, for all $g \in G$, there is a modular group $\Gamma_g < SL_2(\mathbb{Z})$ with $\Gamma_g = \Gamma_0(n)$, where $n = o(g)$, such that $\widetilde{h}^{\mathcal{A}}_g(\tau)$ is a vector-valued mock modular form of weight $1/2$ and multiplier system ρ_g with respect to Γ_g. In Appendix B.5 we explicitly compute the functions $\widetilde{h}^{\mathcal{N}=2}_g$. Furthermore, we discuss the computation of $\widetilde{h}^{\mathcal{N}=4}_g$ for three conjugacy classes in M_{11}.[20]

Similarly to above, the components of the functions $\widetilde{h}^{\mathcal{A}}_g$ can be written in terms of a Rademacher sum as

$$\widetilde{h}^{\mathcal{A}}_{g,s}(\tau) = \mathcal{R}^{(\{-\frac{j^2}{\beta_{\mathcal{A}}}\})}_{\Gamma_g,\,1/2,\,\rho_g}(\tau)_s , \tag{3.6.15}$$

where $\{-\frac{j^2}{\beta_{\mathcal{A}}}\}$ denotes the set of exponents of the q-polar terms which appear in the vector-valued mock modular form $\widetilde{h}^{\mathcal{A}}_g$. The components of the Rademacher sum $\mathcal{R}^{(\{-\frac{j^2}{\beta_{\mathcal{A}}}\})}_{\Gamma_g,\,1/2,\,\rho_g}(\tau)$ are given by $\mathcal{R}^{(\{-\frac{j^2}{\beta_{\mathcal{A}}}\})}_{\Gamma_g,\,1/2,\,\rho_g}(\tau)_s = \sum_j \widetilde{a}_j \mathcal{R}^{(-\frac{j^2}{\beta_{\mathcal{A}}})}_{\Gamma_g,\,1/2,\,\rho_g}(\tau)_s$.

[20]For the other conjugacy classes in M_{11}, we compare $\widetilde{h}^{\mathcal{N}=4}_g$ with the Rademacher formula by computing the first couple coefficients of the twined function and seeing already that they do not match.

1. $\mathcal{E}^{\mathcal{N}=2}_{m=4}$: The first few Fourier coefficients of the mock-modular form $\widetilde{h}^{\mathcal{N}=2}$ are

$$\begin{aligned}
\widetilde{h}^{\mathcal{N}=2}_{\frac{1}{2}}(\tau) = \widetilde{h}^{\mathcal{N}=2}_{-\frac{1}{2}}(\tau) &= -q^{-1/56} + 32890q^{55/56} + 2969208q^{111/56} + \dots \\
\widetilde{h}^{\mathcal{N}=2}_{\frac{3}{2}}(\tau) = \widetilde{h}^{\mathcal{N}=2}_{-\frac{3}{2}}(\tau) &= q^{-9/56} + 14168q^{47/56} + 1659174q^{103/56} + \dots \\
\widetilde{h}^{\mathcal{N}=2}_{\frac{5}{2}}(\tau) = \widetilde{h}^{\mathcal{N}=2}_{-\frac{5}{2}}(\tau) &= -q^{-25/56} + 2024q^{31/56} + 485001q^{87/56} + \dots \\
\widetilde{h}^{\mathcal{N}=2}_{\frac{7}{2}}(\tau) &= q^{-49/56} + 23q^{7/56} + 61894q^{63/56} + \dots . \qquad (3.6.16)
\end{aligned}$$

As before, the coefficients multiplying the polar q-terms are singlets under the action of g. The multiplier system is constrained by the multiplier system of the unary theta series $S_{\frac{7}{2}}$ and corresponds to the inverse of the half-integral index theta function (A.2.16). We find that the functions can be reproduced by a Rademacher sum at the infinite cusp only for $\pi_g \in \{1^{24},\ 1^2\,11^2,\ 1.23\}$.

2. $\mathcal{E}^{\mathcal{N}=4}_{m=4}$: The vector-valued mock modular form $\widetilde{h}^{\mathcal{N}=4}$ has the first few coefficients,

$$\begin{aligned}
\widetilde{h}^{\mathcal{N}=4}_{1}(\tau) = -\widetilde{h}^{\mathcal{N}=4}_{-1}(\tau) &= -4q^{-1/20} + 18876q^{19/20} + 1315512q^{39/20} + \dots \\
\widetilde{h}^{\mathcal{N}=4}_{2}(\tau) = -\widetilde{h}^{\mathcal{N}=4}_{-2}(\tau) &= -3q^{-4/20} - 12045q^{16/20} - 1152943q^{36/20} + \dots \\
\widetilde{h}^{\mathcal{N}=4}_{3}(\tau) = -\widetilde{h}^{\mathcal{N}=4}_{-3}(\tau) &= -2q^{-9/20} + 1980q^{11/20} + 391974q^{31/20} + \dots \\
\widetilde{h}^{\mathcal{N}=4}_{4}(\tau) = -\widetilde{h}^{\mathcal{N}=4}_{-4}(\tau) &= -q^{-16/20} - 33q^{4/20} - 45990q^{24/20} + \dots , \qquad (3.6.17)
\end{aligned}$$

where $\widetilde{h}^{\mathcal{N}=4}_{0}(\tau) = \widetilde{h}^{\mathcal{N}=4}_{5}(\tau) = 0$. The multiplier system is given by the conjugate multiplier system of $5(\mathrm{Tr}_{\mathbf{12}}g)S_5$ and therefore equals the inverse of the theta function multiplier system (A.2.15). We find that the Rademacher sum (2.5.20) correctly reproduces the twining function $\widetilde{h}^{\mathcal{N}=4}_{g}$ only for $\pi_g \in \{1^{24},\ 1^2\,11^2\}$.

To summarize, we report in Table 3.4 the conjugacy classes which can be reconstructed from solely the information of the infinite cusp and thus do not have poles at any additional cusps.

ECFT	Frame shapes with no additional poles
$\mathcal{E}^{\mathcal{N}=2}_{m=4}$	$1^{24} - 1^2\,11^2 - 1.23$
$\mathcal{E}^{\mathcal{N}=4}_{m=4}$	$1^{24} - 1^2\,11^2$

Table 3.4: Pole structure of $\widetilde{h}^{\mathcal{A}}_{g}(\tau)$ for the two extremal theories with central charge 24. In contrast to Table 3.3, we report the conjugacy classes where the only pole is at the infinite cusp.

3.7 Conclusion

In this work we have investigated the Rademacher summability properties of the twining functions of known extremal CFTs. Inspired by the genus zero property of monstrous moonshine, and its connection to the Rademacher summability of the monstrous McKay-Thompson series at the infinite cusp, we consider a similar expansion for the twined graded characters associated with the other extremal CFTs. Similarly to $\mathcal{V}^\natural$ and $\mathcal{V}^{s\natural}$, we find that $\mathcal{E}^{\mathcal{N}=4}_{m=2}(G)$ for $G = M_{22}$ and $M_{11}^{(1)}$ have the property that all associated twining functions can be written as Rademacher sums at the infinite cusp. However, all of the other cases we consider have at least one conjugacy class whose graded character has pole at additional cusps of the corresponding modular group which are inequivalent to infinity.

In studying the Rademacher properties of $h_g^{\mathcal{A}}$ and $\widetilde{h}_g^{\mathcal{A}}$, in §3.6 we examined the Rademacher sum of the corresponding polar term for the group $\Gamma_g = \Gamma_0(n)$. However, in the case of $\mathcal{V}^\natural$ and $\mathcal{V}^{s\natural}$ it is the case that many of the McKay-Thompson series are Hauptmoduln for subgroups of $SL_2(\mathbb{R})$ with additional Atkin-Lehner involutions (c.f. Appendix A). One obvious question is whether those functions which, according to the results of §3.6.2 and §3.6.3, have poles at cusps inequivalent to $i\infty$ under Γ_g are nevertheless Rademacher sums at infinity for a different modular group with more general Atkin-Lehner involutions. However, in the case of half-integral index theta functions it does not seem possible to extend the multiplier system to these groups.

More generally, in the case of $\mathcal{V}^\natural$ and $\mathcal{V}^{s\natural}$, the Rademacher summability property applies directly to the twined partition functions. In the cases of the other ECFTs we consider, we study the mock modular forms which arise only after decomposing the partition function into superconformal characters. Though these objects are natural to consider both from a physical and algebraic point of view, one could also consider the Rademacher properties of the twined (flavored) partition function itself. In the Spin(7) case, the (twined) partition function is simply the (twined) partition function of $\mathcal{V}^{s\natural}$; from this point of view the functions of the Spin(7) theory are naturally related to Rademacher summable functions (though for different modular groups and with different polar structure.)

On the other hand, in the case of the theories with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetry, the flavored partition function is a Jacobi form of index m and has a natural theta expansion into length $2m$ vector-valued modular forms of weight $w = -1/2$. However, these functions in general have at least as many poles as the mock modular forms we studied.

In table 3.5 we summarize the Rademacher summability of the different ECFTs.

ESCFT	Twining function	Rademacher summable $[g]$
$\mathcal{V}^\natural$	$T_g(\tau)$	**All** $[\mathbf{g}] \in \mathbb{M}$ [86]
$\mathcal{V}^{s\natural}$	$\mathcal{Z}^{s\natural}_{\mathrm{NS},g}(\tau)$	**All** $[\mathbf{g}] \in \mathbf{Co_0}$ [129]
$\mathcal{E}^{\mathrm{Spin}(7)}(M_{24})$	$h_g^{\mathrm{Spin}(7)}(\tau)$	All but 6 $[g] \in M_{24}$
$\mathcal{E}^{\mathcal{N}=2}_{m=2}(M_{23})$	$h_g^{\mathcal{N}=2}(\tau)$	All but 3 $[g] \in M_{23}$
$\mathcal{E}^{\mathcal{N}=2}_{m=2}(M_{12})$	$h_g^{\mathcal{N}=2}(\tau)$	All but 4 $[g] \in M_{12}$
$\mathcal{E}^{\mathcal{N}=4}_{m=2}(M_{22})$	$h_g^{\mathcal{N}=4}(\tau)$	**All** $[\mathbf{g}] \in \mathbf{M_{22}}$
$\mathcal{E}^{\mathcal{N}=4}_{m=2}(M_{11}^{(1)})$	$h_g^{\mathcal{N}=4}(\tau)$	**All** $[\mathbf{g}] \in \mathbf{M_{11}^{(1)}}$
$\mathcal{E}^{\mathcal{N}=4}_{m=2}(M_{11}^{(2)})$	$h_g^{\mathcal{N}=4}(\tau)$	All but 1 $[g] \in M_{11}^{(2)}$
$\mathcal{E}^{\mathcal{N}=2}_{m=4}$	$\widetilde{h}_g^{\mathcal{N}=2}(\tau)$	3 classes $[g] \in M_{23}$
$\mathcal{E}^{\mathcal{N}=4}_{m=4}$	$\widetilde{h}_g^{\mathcal{N}=4}(\tau)$	2 classes $[g] \in M_{11}$

Table 3.5: Extremal CFTs whose Rademacher summability properties have been proven already ($\mathcal{V}^\natural$ and $\mathcal{V}^{s\natural}$) or are considered in §3.6. In the second column we list the twining functions and in the third column we report our main results.

In particular, in the third column we report the number of conjugacy classes in the global symmetry group of the theory whose corresponding twining function is a Rademacher sum at the infinite cusp. We have indicated in bold those theories for which all twining functions are Rademacher summable in this sense.

Our work suggests that the surprising connection between ECFTs and sporadic groups is, in fact, more general than the connection between ECFTs and Rademacher summability. More insights might come from a deeper examination of the holomorphic orbifolds of the ECFTs. Indeed, the form of the partition functions for holomorphic orbifolds of the monster CFT turn out to be highly constrained by the Hauptmodul property and the uniqueness conjecture of the vacuum. Given a generic element $g \in \mathbb{M}$, $\langle g \rangle$-orbifold is either $\mathcal{V}^\natural$ or $\mathcal{V}^L$, as proved in [128]. We expect a similar reasoning to hold for the Conway module, whose uniqueness was proven in [132], and its relation to the two other $c = 12, \mathcal{N} = 1$ SCFTs, the super-E_8 theory and the theory of 24 free fermions. Most of the other examples of $c = 12$ ECFTs analyzed here are different from $\mathcal{V}^\natural$ and $\mathcal{V}^{s\natural}$ in that not all the mock modular forms appearing in the decomposition of the twining functions are Rademacher summable. However, after a preliminary analysis in the case of $\mathcal{E}^{\mathcal{N}=2}_{m=2}(M_{23})$ we found that the holomorphic $\langle g \rangle$-orbifolds for which $g \in M_{23}$ and $\pi_g \in \{1^8\, 2^8,\ 1^6\, 3^6,\ 1^4\, 5^4,\ 1^3\, 7^3\}$ reproduce the original partition function (3.4.5). It would be interesting to explore this further for the other theories considered in this chapter.

In this work we did not analyze the case of the $\mathcal{E}_{k^*=2}^{\mathcal{N}=1}$, the ECFT with Co_0 symmetry first constructed in [100]. One question for the future is to derive the corresponding twining functions of this theory and consider whether they have any special Rademacher summability properties at the infinite cusp of the appropriate modular groups.

Another example one could consider is K3 non-linear sigma models (NLSM). These theories are extremal according to the definition of [120]; however, they are not chiral CFTs. Their symmetry groups and possible twined elliptic genera have been classified; they are related to four-plane-preserving subgroups of the group Co_0 [124, 133]. It would be interesting to consider in general the Rademacher summability properties of all possible twining functions which can arise for K3 NLSMs. In the case where the symmetry element belongs to the Mathieu group M_{24}, it follows from [88] that these twining functions[21] are Rademacher sums about the infinite cusp. However, a general analysis has yet to be performed. One interesting point to note is that in [133] it was conjectured[22] that all possible twining functions of K3 CFTs arise from either umbral moonshine or Conway moonshine in a precise way proposed in [135] and [136], respectively. So in this (roundabout) sense, they arise from functions which are Rademacher summable at the infinite cusp due to this property of umbral and Conway moonshine. It is also the case that the elliptic genus of a $\langle g\rangle$-orbifold of a K3 CFT either reproduces the K3 elliptic genus or the elliptic genus of T^4.[23] One could investigate in this case whether there is a connection between the Rademacher summability of the g-twined functions and whether the $\langle g\rangle$-orbifold yields a K3 or a torus theory.

It would be interesting to understand more deeply the origin of the curious connection described in Appendix B.4, which relates the mock modular forms arising from $\mathcal{E}_{m=2}^{\mathcal{N}=2}(M_{23})$ and the twining functions of Mathieu moonshine. More specifically, we have observed a precise relation between the coefficients in the expansion of $h_g^{\mathcal{N}=2}$ at two inequivalent cusps of $\Gamma_0(n)$ where it has poles in the case of $g \in \{3A, 6A, 15AB\}$. Is this indicative of a larger symmetry of these functions?

We end with the following comments and open questions inspired by our work:

- The connection between the Rademacher expansion of a CFT partition function and the path integral of 3d quantum gravity in AdS first suggested in [14] primarily served as a source of motivation for our analysis. However, in the case of a g-twined partition function of a CFT with discrete symmetry group G, a physical interpretation of its Rademacher summability at the infinite

[21]By twining functions here we mean mock modular forms which arise from a decomposition of the form (B.3.28).
[22]In [134] this conjecture was proved in a physical sense via a derivation from string theory.
[23]This is just zero due to fermionic zero modes.

cusp is lacking. It would be interesting to find a physical interpretation in instances where such a property holds.

- The authors of [137] considered a certain compactification of heterotic string theory to two dimensions to provide a physical derivation of the genus zero property of monstrous moonshine. The Hauptmodul property of the monstrous McKay-Thompson series was shown to arise from T-duality symmetries which arise upon considering CHL orbifolds of this string compactification. An interesting question is whether the additional ECFTs we consider in this work have any connection with 2d string compactifications, and if this point of view can shed any light on the properties considered in this chapter.

- We can certainly construct infinite families of 2d CFTs with sporadic symmetry groups and arbitrarily high central charge by considering symmetric products of the theories studied in this chapter; however, they will no longer be extremal [120]. Do these symmetric product theories have any special properties? Are there other theories (extremal or not) with large sporadic group symmetry and $c > 24$ which don't arise from this symmetric product construction? Assuming a connection between Rademacher sums and sporadic groups, can one use Rademacher summability techniques to search for such CFTs at higher central charge?

- Finally, how ubiquitous are 2d CFTs with sporadic group symmetries? Do such theories play a special role in physics, i.e. in 3d gravity and/or string theory? How can one interpret the presence of polar terms at different cusps from the point of view of the gravity dual theory? Does the Rademacher sum around multiple poles also admit a natural physics interpretation in terms of a path integral formulation of quantum gravity?

Answering these questions would give us new insight into the AdS/CFT correspondence and also shed light on the physics behind the extremality (optimality) condition in umbral moonshine and related contexts.

Chapter 4

A mixed Farey tail

In this chapter Siegel modular forms, mock Jacobi forms and the Rademacher sum find an application in the realm of black hole physics. We omit here a thorough description of black holes and simply review the aspects relevant to this thesis.

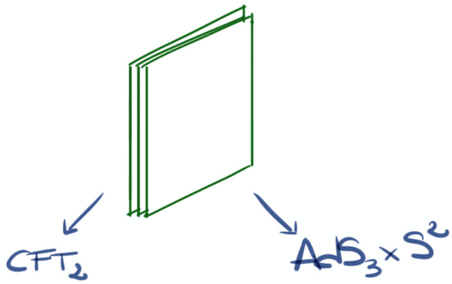
To a first approximation, a black hole is a gravitational object that can be described as a black body system characterized by certain thermodynamical quantities [138]. As first asserted by Bekenstein and Hawking, the entropy of a black hole in a semi-classical regime is proportional to the area of its event horizon [139,140]. The fact that the number of degrees of freedom contained in a space-time region is proportional to the area instead of the volume of this region hinted at a holographic description of black holes. This immediately led to wonder what is the the statistical interpretation of this thermodynamic quantity, and in particular how to describe the internal constituents responsible for the thermodynamic properties of black holes.

String theory was argued to offer through the AdS/CFT correspondence a natural framework in which to examine the thermodynamical properties of black holes or more general extended black objects[1]. A black hole from the string theory perspective is realized as a stack of D-branes. These may be taken to wrap various cycles of the manifold describing the target-space geometry and become a bound state upon compactification on this internal space. By sending the gravitational coupling to infinity this bound state starts gravitating and eventually forms a black hole.

The AdS/CFT correspondence provides a realization of holography by relating a gravitational theory on an asymptotically Anti de Sitter space (AdS) to a conformal field theory (CFT) living in a space with one dimension less. An instance of AdS/CFT was mentioned in the previous chapter in relation to extremal CFTs, which are putative duals to pure gravity in AdS_3. In this chapter the corre-

[1]The reader is referred to the original work [15] and the review [141] for further reading.

spondence between a gravity theory and a conformal field theory emerges from different descriptions of a D-brane system in string theory. In particular, it was observed in [15] that the string theory may be examined either from the viewpoint of the field theory describing the low-energy dynamic on the worldvolume of the D-branes or from the effective supergravity theory which accounts for the near-horizon geometry.



The two theories are conjectured to be the same[2], however they have a valid perturbative description in different regions of the space of coupling constants. Specifically, when one side is weakly coupled the other is strongly coupled and vice-versa. Therefore, one way to make use of the correspondence is to compute protected quantities whose values in the strongly coupled quantum field theory can be easily extrapolated from the weakly coupled one. An important class of black holes that allows to define such protected quantities via supersymmetry is the class of extremal black holes[3]; these are zero temperature black holes which saturate certain BPS bounds and are thus usually referred to as BPS black holes.

In striving to describe the microstates of BPS black holes in supergravity theories, valuable insights have been gained from the partition functions of BPS states in the corresponding string theory compactifications. These partition functions can often be related to an index, whose computation at weak string coupling is able to describe, when going to strong coupling, the exact degeneracies of BPS black holes. Thus, at least for the class of BPS extremal black holes, string theory provides an answer to the question posed at the beginning of this introduction regarding the statistical interpretation of their entropy.

[2]In this introduction we are considering the decoupling limit between bulk and internal theory as $l_s \to 0\,(\alpha' \to 0)$.

[3]One of the important feature of extremal black holes is an enhancement of the symmetries of the system close to the horizon, which is a feature of the attractor mechanism [142].

4.1 Beyond the classical entropy

The first match between the macroscopic entropy of a class of five-dimensional supersymmetric black holes and the microscopic counting of degeneracies associated to a brane system was achieved in the limit of large charges in [143]. An agreement between the macroscopic and a microscopic computation for large charges has been later obtained for different black hole systems.

Nevertheless, to obtain an exact match, and thus to go beyond the limit of large charges on the supergravity side requires a definition for the exact entropy of black holes beyond the semi-classical approximation. The quantum entropy function [144], which is precisely such a proposal to go beyond perturbation theory, gives a definition of the exact entropy in the form of a path integral in the near-horizon region of the black holes. Owing to the supersymmetry preserved by these solutions, localization techniques can then be used to compute the functional integral exactly. In certain situations, localization provides a powerful tool to look for an exact supergravity result to be compared to the string theory expectation [145].

On the microscopic side, microscopic counting functions for the degeneracy of D-brane systems were observed to be related to certain types of automorphic forms, specifically meromorphic (holomorphic) Siegel modular forms in the context of $\mathcal{N} = 4$ ($\mathcal{N} = 8$) string theory compactifications down to four dimensions. The most famous example is the D1-D5 systems describing extremal dyonic black holes, which are the supersymmetric analogues of Reissner-Nordstrom black holes. The relation between automorphic forms and microstate counting in these systems comes from the so-called second quantization, as was first fully addressed in [146, 147]. The relation between number theory, in particular meromorphic Siegel and Jacobi forms, and black hole microstate counting was developed and expounded in [8]. More recent reviews on the role of Siegel modular forms in relation to black hole physics are [134,148] and yet the relation between black holes and class groups in [149]. Finally, the relevant automorphic forms also appear in the Weyl-Kac-Borcherds denominator formulas for generalized Kac-Moody algebras, which is suggestive of a role played by such algebra in black hole physics. The link to the physics of black holes and the property known as wall-crossing for BPS states was made in [150, 151]. Note that these are the Borcherds-Kac-Moody algebras introduced by Borcherds in the proof of monstrous moonshine. Several clues about a possible black hole realization of other instances of moonshine, such as Mathieu and umbral moonshine, were also displayed in [134, 152].

Coming back to the comparison between the exact entropy of supergravity black hole solutions and BPS state counting in string theory, it has been shown that the degeneracies of immortal black holes in four-dimensional $\mathcal{N} = 4$ string theory (which are stable throughout the moduli space of the compactification) are counted by the Fourier coefficients of mixed mock modular forms, as reviewed in the next section. The physical origin of the mock character for the generating function of these degeneracies can ultimately be related to the non-compactness of the target-space of the brane system.

The precise form of the Fourier coefficients for the mixed mock modular counting functions, which is an asymptotic series expansion, has a natural interpretation in the dual macroscopic picture, namely as a sum over semi-classical contributions of saddle-point configurations to the functional integral in the near-horizon geometry of the corresponding black holes. This interpretation has been coined the Black Hole Farey Tail [14, 153], motivating the title of this chapter. Despite the fact that the Farey Tail was first introduced in the context of the D1-D5 system, it applies to any system that has a microscopic description in terms of a 2d CFT and has a dual description as a supergravity theory on a space-time containing an asymptotically AdS_3 factor.

In the toroidal compactification of Type IIB to four dimensions [154], the asymptotic of the counting function of 1/8-BPS states reproduces, at leading order, the celebrated Bekenstein-Hawking entropy of the corresponding dyonic 1/8-BPS black holes [155]. But it has also been shown, using supersymmetric localization techniques directly in the low-energy supergravity theory, that all perturbative and non-perturbative corrections to this entropy can be captured and studied analytically [145,156,157]. Such a study revealed an exact match between the Rademacher expansion, reflecting the microscopic counting, and supergravity degeneracies of 1/8-BPS states.

It is of course interesting to ask whether a similar situation occurs in other types of compactifications. For instance, a more intricate case to examine is the one of Type IIB string theory compactified on $K3 \times T^2$. This black hole system provides a rare opportunity to compare the sub-leading terms in the entropy of the brane system with the macroscopic entropy provided by the supergravity theory. The degeneracy of 1/4-BPS black holes in Type IIB on $K3\times T^2$ displays certain interesting features [8, 146, 150, 151, 158–162]: both single-centered and multi-centered configurations are present, walls of marginal stability appear in the moduli space of the theory and finally the counting functions are given by mixed mock modular objects. The mixed-mock character of these functions is a consequence of the wall-crossing phenomenon and implies that the 1/4-BPS states counting problem can be translated into the question of recovering the exact Fourier coefficients of certain mixed mock Jacobi forms [8].

In general, mock Jacobi forms can be decomposed into vector-valued mock modular forms. Mock modular forms are characterized by a pair of functions, the form itself and its shadow, as reviewed in §2.3. When the shadow is a modular form, the Rademacher series can be applied to recover the Fourier coefficients of mock Jacobi forms. However, the mock Jacobi forms arising in the counting problem of 1/4-BPS states in Type IIB string theory on $K3 \times T^2$ have mixed mock components, and their shadows are therefore more complicated objects. Such mixed mock modular forms have been much less studied in the literature, with the notable exceptions of [77,163]. In the following, we employ the method introduced by Bringmann and Manschot [77] to obtain an exact asymptotic expansion for the Fourier coefficients of the mixed mock Jacobi forms relevant to the string theory states counting problem. In particular, we derive an exact formula for the Fourier coefficients of these modular objects, providing a concrete result one should be able to recover from the supergravity theory, following the above general discussion.

Our result can be viewed as an exact formula in inverse powers of the charges for the degeneracy of states making up 1/4-BPS black holes in an $\mathcal{N} = 4$ theory of supergravity. Similarly to the 1/8-BPS black holes in $\mathcal{N} = 8$ supergravity mentioned previously, a localization computation can also be conducted to determine the exact quantum entropy of such black holes directly in the low-energy effective theory. Progress in this direction has been reported in [78]. It is therefore of interest to compare the macroscopic results obtained using this method to the exact microscopic result derived in this chapter. Some aspects of the supergravity computation, however, are still lacking to conduct a precise comparison on par with the toroidal case. This points to interesting questions regarding localization computations in the context of supergravity which should be examined and answered in order to obtain a complete understanding of the microstates of 1/4-BPS black holes in line with their description as bound states of D-branes.

The rest of the chapter is organized as follows: in section 4.2, we review 1/4-BPS states in string theory compactified on $K3\times T^2$ and some properties of their Siegel counting functions. In section 4.3, we introduce the relevant mock Jacobi forms for the string theory counting problem and explain how a generalization of the circle method allows for an exact derivation of their Fourier coefficients. The main result is given in (4.3.33). In section 4.4.2, we examine the macroscopic computation of the exact entropy of 1/4-BPS black holes in 4d $\mathcal{N} = 4$ supergravity and explain which aspects, in our opinion, remain to be understood to obtain a complete match with the microscopic results. We close with some comments in section 4.5. Appendix C contains some technical facts of our derivation. Relevant aspects and definitions of the general theory of (mock) Jacobi forms are collected in chapter 2, Appendix A.2 and Appendix C.1.

4.2 Type IIB string theory on K3 $\times T^2$

In this section, we summarize the microscopic derivation of the counting functions for 1/4-BPS states in Type IIB string theory compactified on K3 $\times T^2$.

4.2.1 Microscopic derivation

We consider Type IIB string theory compactified on K3 $\times T^2$, which leads to an $\mathcal{N} = 4$ string theory in four dimensions. Writing $T^2 = S^1 \times \widetilde{S}^1$, we focus on the following duality frame [158]: Q_5 D5-branes wrapping K3 $\times S^1$, Q_1 D1-branes wrapping S^1, momentum n along S^1, momentum ℓ along $\widetilde{S}^1$ and one unit of KK monopole charge on $\widetilde{S}^1$. This set-up can be equivalently described by placing the above D1-D5 system at the center of a Taub-NUT space (see [159] for a review of the different string theory frames). The asymptotic region of the Taub-NUT takes the form $\mathbb{R}^3 \times \widetilde{S}^1$, while at the tip the transverse space is $\mathbb{R}^4$. In this configuration the four-dimensional black hole has momentum ℓ along the asymptotic circle $\widetilde{S}^1$ [164].

The resulting four-dimensional $\mathcal{N} = 4$ (preserving 16 real space-time supercharges) string theory has a T-duality group $O(22, 6, \mathbb{Z})$ and an S-duality group $SL(2, \mathbb{Z})$, while the U-duality group is the product $O(22, 6, \mathbb{Z}) \times SL(2, \mathbb{Z})$. The unique[4] quartic (in the charges) invariant of this U-duality group is $\Delta := Q_e^2 Q_m^2 - (Q_e \cdot Q_m)^2 = 4mn - \ell^2$, where the electric and magnetic charge vectors are given by the following T-duality invariants

$$Q_e^2 = 2n\,, \quad Q_m^2 = 2\,Q_5(Q_1 - Q_5) = 2m\,, \quad Q_e \cdot Q_m = \ell\,. \tag{4.2.1}$$

In this theory, we are interested in the degeneracies of 1/4-BPS states when the S^1 circle is large. The worldvolume theory on the D-branes reduces to a two-dimensional supersymmetric sigma model with target space a deformation of the symmetric product $\mathrm{Sym}^{m+1}(\mathrm{K3})$ [147, 166]. The contribution of this (4, 4) two-dimensional SCFT with target-space $\mathrm{Sym}^{m+1}(\mathrm{K3})$ to the counting function of 1/4 BPS states is given by its elliptic genus [146, 147]

$$Z_{\mathrm{Sym}^{m+1}(\mathrm{K3})}(\tau, z, \sigma) = \frac{1}{p} \prod_{\substack{m>0,\, n\geq 0,\\ l\in\mathbb{Z}}} \frac{1}{\left(1 - p^m q^n y^l\right)^{c(mn,l)}} \tag{4.2.2}$$

[4]In the rest of this chapter we focus on dyons with charge vectors Q and P such that $\mathcal{I} := \gcd(Q \wedge P) = 1$. The number $\mathcal{I}$ is also a quartic invariant of the discrete U-duality group and the dyon degeneracies for other values of invariant have been analyzed in e.g. [165].

where $p = e^{2\pi i\sigma}$ and $c(mn, l)$ are the Fourier coefficients in the expansion of the elliptic genus of a single K3-surface

$$Z_{\text{K3}}(\tau, z) = 8\Big(\frac{\theta_2^2(\tau, z)}{\theta_2^2(\tau)} + \frac{\theta_3^2(\tau, z)}{\theta_3^2(\tau)} + \frac{\theta_3^2(\tau, z)}{\theta_3^2(\tau)}\Big)\,.$$

To recover the complete counting function of 1/4 BPS in 4d an extra factor has to be included. The latter corresponds to the contribution of the center-of-mass degrees of freedom of the D-brane system and the motion of the Taub-NUT space itself [159]. This additional term was demonstrated in [158] to be equal to the inverse of the Jacobi form $\eta^{18}(\tau)\,\theta_1^2(\tau, z)$.

This term is responsible for the meromorphicity in the counting function of black hole microstates[5] in 4d,

$$Z_{1/4\text{-BPS}}(\tau, z, \sigma) = \frac{1}{qyp} \prod_{\substack{(m,n,l)>0,\\ m,n,l\in\mathbb{Z}}} \frac{1}{\big(1 - p^m q^n y^l\big)^{c(mn,l)}} \tag{4.2.3}$$

where $(m, n, l) > 0$ stands for either $m > 0$ or $m = 0$, $n > 0$, or $m = n = 0$, $l > 0$. This expression coincides with the multiplicative lift (or Borcherds lift [48]) of the Jacobi form $Z_{\text{K3}}(\tau, z)$, and it turns out to be a meromorphic Siegel modular form (see (2.2.7) for the definition)

$$Z_{1/4\text{-BPS}}(\tau, z, \sigma) = \frac{1}{\Phi_{10}(\tau, z, \sigma)}\,, \tag{4.2.4}$$

specifically the inverse of the Igusa cusp form. Siegel modular forms usually admit two different representation, one is the product representation given above, and the other one is a sum representation which corresponds to an additive lift of a certain Jacobi form. Details can be found in [30]. Restricting to the case of the Igusa cusp form, it can be expressed as additive lift of the automorphic correction $\eta^{18}(\tau)\,\theta_1^2(\tau, z)$.

Notice that the Fourier coefficients of the K3 elliptic genus in the limit for $y \to 1$ (i.e. $z = 0$) exhibit a particular property

$$Z_{\text{K3}}(\tau, z)\big|_{z=0} = \Big(20 + 2(y + y^{-1}) + \sum_{n>0, l\in\mathbb{Z}} c(4n - l^2) q^n y^l\Big)\Big|_{y\to 1} = 24 \tag{4.2.5}$$

$$\Rightarrow \sum_{l\in\mathbb{Z}} c(4n - l^2) = 0 \qquad \text{for all } n > 0. \tag{4.2.6}$$

[5]Notice that the meromorphicity of the counting function is an intrinsic characteristic of the 4d theory that is absent from the 5d picture, since in the latter case there is no contribution from the center-of-mass motion and the Taub-Nut dynamics.

As analyzed in [150] this feature is responsible for the factorization of the automorphic form Φ_{10} when it approaches the double pole at $z = 0$,

$$\frac{\Phi_{10}(\tau,z,\sigma)}{(1-y^{-1})^2}\bigg|_{y\to 1} = \frac{pqy}{(1-y^{-1})^2} \prod_{\substack{r,s\geq 0, t\in\mathbb{Z}\\ r=s=0,t<0}} \left(1-q^s y^t p^r\right)^{c(4rs-t^2)}\bigg|_{y\to 1} = pq\,\eta^{24}(\tau)\eta^{24}(\sigma) \tag{4.2.7}$$

The interpretation of this factorization comes from the analysis of the wall-crossing phenomenon: crossing a wall of marginal stability a 1/4-BPS state may decay into two 1/2-BPS states.

In order to extract the relevant information for the dyonic degeneracy, a precise procedure has to be implemented [8]. Consider first the Fourier expansion[6] of $Z_{1/4\text{-BPS}}(\tau,z,\sigma)$ in the σ variable:

$$\frac{1}{\Phi_{10}(\tau,z,\sigma)} = \sum_{m\geq -1} \psi_m(\tau,z)\, e^{2\pi i m\sigma}\,. \tag{4.2.8}$$

The functions ψ_m are meromorphic Jacobi forms of weight -10 and index m with a double pole at $z = 0$ (together with all the points differing by a translation of the period lattice). The meromorphic Jacobi forms ψ_m can then be canonically split into two terms,

$$\psi_m(\tau,z) = \psi_m^F(\tau,z) + \psi_m^P(\tau,z)\,, \tag{4.2.9}$$

where ψ_m^F are mixed mock Jacobi forms and ψ_m^P are meromorphic mock Jacobi forms, which account for the polar part of ψ_m

$$\psi_m^P(\tau,z) = \frac{p_{24}(m+1)}{\eta(\tau)^{24}} \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s}y^{2ms+1}}{(1-yq^s)^2}\,. \tag{4.2.10}$$

Above, $p_{24}(m+1)$ denotes the coefficient of q^m in $\eta(\tau)^{-24}$ and the second factor is the Appell-Lerch sum of weight 2 and index m. Throughout the text, we will often simply write mock Jacobi forms to refer to forms with generic shadow and explicitly use the attribute mixed when the distinction is necessary. Physically, the splitting (4.2.9) elucidates the interpretation of the wall-crossing phenomenon [150, 151, 159–162, 167, 168] in the low-energy supergravity theory. Contrary to the toroidal case, the K3 $\times\, T^2$ compactification allows for multi-centered configurations of bound states of black holes [169, 170]. These configurations are stable in certain regions of the moduli space of the supergravity theory, and unstable in others. The splitting (4.2.9) takes this into account: the piece ψ_m^F contains the degeneracies of the single-centered immortal gravitational configurations,

[6] Recall that the grading (n,m,ℓ), defined in equation (4.2.1), corresponds to the T-duality invariant integers associated to the chemical potentials (τ,σ,z). Since we deal with meromorphic Siegel modular form the expansion in equation (2.2.12) still holds but this time the Fourier coefficients are meromorphic Jacobi forms.

which are stable throughout the moduli space and hence devoid of poles, while ψ_m^P encodes the physics related to multi-centered gravitational configurations decaying or appearing when tuning the moduli. The ability of the latter to capture all the walls of marginal stability is built-in by the averaging over s, the spectral flow variable, in (4.2.10). Along the lines of [7], one can interpret the appearance of mock modular forms as due to the non-compactness of the target-space of the sigma model associated with the Taub-NUT space.

In this chapter, we focus on the single-centered contributions, whose degeneracy is captured by the Fourier coefficients of the mock Jacobi forms ψ_m^F. As we will see in the next section, the mock character of the counting functions has a significant impact on the evaluation of these coefficients, and in particular it implies that the standard Rademacher formula (2.5.19) needs to be generalized in order to obtain an analogous convergent asymptotic expansion.

4.3 A mixed Rademacher sum

In this section, we extend the circle method described in §2.5.1 for (mock) modular forms to the case of mixed mock modular forms, along the lines of [77]. In the second part, a description of the mock modular objects of interest to us will be followed by the derivation of the Rademacher expansion for their Fourier coefficients.

4.3.1 Mixed mock modular forms

The aim of the following analysis is to derive an analytic expression for the Fourier coefficients of the mixed mock modular forms arising in the theta-decomposition of the mock Jacobi forms ψ_m^F for different values of m, answering the question raised in section 4.2 about the construction of a Rademacher expansion for vector-valued mixed mock modular forms.

Before applying the circle method to these mixed mock modular forms, we examine the precise structure of the mock Jacobi forms under consideration. A detailed treatment of these functions is provided in [8]. There it was shown that, after multiplication by the discriminant function $\eta(\tau)^{24}$, the mixed mock Jacobi form ψ_m^F can be split into two terms,

$$\eta(\tau)^{24}\,\psi_m^F(\tau,z) = \varphi_{2,m}^{\text{mock}}(\tau,z) + \varphi_{2,m}^{\text{true}}(\tau,z)\,, \tag{4.3.1}$$

whose defining characteristics are as follows. The first term reflects the mock character of the counting function, and as such encodes the failure of ψ_m^F to be modular, while the second is a weakly holomorphic Jacobi form. Adding any (weak) Jacobi form of weight 2 and index m to $\varphi^{\text{mock}}_{2,m}$ and subtracting it from $\varphi^{\text{true}}_{2,m}$ does not modify the defining properties of these functions. Therefore, the explicit form of $\varphi^{\text{mock}}_{2,m}$ is obtained only after imposing another condition. Interestingly, the splitting ambiguity is (almost) removed[7] by demanding an *optimal* growth condition on the Fourier coefficients of $\varphi^{\text{mock}}_{2,m}$, which translates into a growth of at most $\exp(\frac{\pi}{m}\sqrt{4mn-\ell^2})$.

Based on the above considerations, we define $\Phi^{\text{opt}}_{2,m}$ to be the mock Jacobi form obtained from $\varphi^{\text{mock}}_{2,m}$ after imposing the optimal growth condition. So long as m is a prime power, the optimal mock Jacobi form $\Phi^{\text{opt}}_{2,m}$ is unique [8], and it can be expressed in terms of the Hurwitz-Kronecker class numbers[8]

$$\Phi^{\text{opt}}_{2,m}(\tau,z) = \mathcal{H}(\tau,z)|\mathcal{V}^{(1)}_{2,m}\,, \quad \text{for } m \text{ a prime power}\,. \tag{4.3.2}$$

The optimality, in this case, translates into the choice of a holomorphic mock Jacobi form. The action of the Hecke-like operator $\mathcal{V}^{(1)}_{k,m}$ is defined in (A.2.20), while $\mathcal{H}(\tau,z)$ is the mock Jacobi form of weight 2 and index 1 whose Fourier coefficients are given by Hurwitz-Kronecker numbers,

$$\mathcal{H}(\tau,z) = \mathcal{H}_0(\tau)\,\theta_{1,0}(\tau,z) + \mathcal{H}_1(\tau)\,\theta_{1,1}(\tau,z)\,, \tag{4.3.3}$$

with

$$\mathcal{H}_\ell(\tau) = \sum_{n=0}^{\infty} H(4n+3\ell)\,q^{n+\frac{3\ell}{4}}\,, \qquad \ell = \{0,1\}\,, \tag{4.3.4}$$

and $\theta_{m,\ell}(\tau,z)$ is defined in Appendix A, as well as in equation (2.2.51). The first few Hurwitz-Kronecker numbers are given in Table 4.1. By convention we have the value $H(0) = -1/12$, and $H(n) = 0$ for $n < 0$.

n	3	4	7	8	11	12	15	16
$H(n)$	1/3	1/2	1	1	1	4/3	2	3/2

n	19	20	23	24	27	28	31	32
$H(n)$	1	2	3	2	4/3	2	3	2

Table 4.1: Some Hurwitz-Kronecker numbers.

[7]The splitting becomes more subtle and the choice ceases to be unique when the index m is not a prime power. See below for an explicit example.

[8]Hurwitz-Kronecker class numbers are defined as the number of $PSL(2,\mathbb{Z})$-equivalent classes of quadratic forms of discriminant $-n$, weighted by the inverse of the order of their stabilizer in $PSL(2,\mathbb{Z})$.

When m is not a prime power, the choice of an optimal function is not unique [8]. For instance, there are three choices of optimal functions for $m = 6$, the first non-prime power:

$$\Phi^{\text{opt, I}}_{2,6}(\tau, z) = \frac{1}{2}\mathcal{H}|\mathcal{V}^{(1)}_{2,6}(\tau, z) + \frac{1}{24}\,\mathcal{F}_6(\tau, z)\,, \tag{4.3.5}$$

$$\Phi^{\text{opt, II}}_{2,6}(\tau, z) = \Phi^{\text{opt, I}}_{2,6}(\tau, z) - \frac{1}{24}\,\mathcal{K}_6(\tau, z)\,, \tag{4.3.6}$$

$$\Phi^{\text{opt, III}}_{2,6}(\tau, z) = \Phi^{\text{opt, I}}_{2,6}(\tau, z) + \frac{1}{4}\,\mathcal{K}_6(\tau, z)\,, \tag{4.3.7}$$

where the mock Jacobi forms of weight 2 and index 6 $\mathcal{F}_6(\tau, z)$ and $\mathcal{K}_6(\tau, z)$ are given in equation (C.1.1) and (C.1.5) respectively. In the following, we will deal explicitly (see footnote 12) with the cases $m = 1 \ldots 7$, to incorporate cases where the index is prime (1, 2, 3, 5 and 7), a prime power (4), or neither (6). However, the extension to other non-prime power cases can also be obtained using results contained in [8].

For any index, the completion of $\Phi^{\text{opt}}_{2,\,m}$, which is a Jacobi form of weight 2 and index m, satisfies [8]

$$\widehat{\Phi}^{\text{opt}}_{2,\,m}(\tau, z) = \Phi^{\text{opt}}_{2,\,m}(\tau, z) - \sqrt{\frac{m}{4\pi}} \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} (\vartheta^0_{m,\ell})^*(\tau)\theta_{m,\ell}(\tau, z)\,, \tag{4.3.8}$$

where $\vartheta^0_{m,\ell}(\tau) := \theta_{m,\ell}(\tau, z)|_{z=0}$ as defined in (2.3.10) and * denotes the Eichler integral (2.3.6). Therefore, the theta-decomposition of $\Phi^{\text{opt}}_{2,\,m}$,

$$\Phi^{\text{opt}}_{2,\,m}(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h^{\text{opt}}_\ell(\tau)\,\theta_{m,\ell}(\tau, z)\,, \tag{4.3.9}$$

specifies the optimal mock modular forms[9] h^{opt}_ℓ of weight 3/2 and shadow given by the unary theta series of weight 1/2, $\vartheta^0_{m,\ell}(\tau)$. This suffices to show that, for any m, the completion of h^{opt}_ℓ, $\widehat{h}^{\text{opt}}_\ell$, is a modular form of weight 3/2 with a multiplier system dual to that of $\vartheta^0_{m,\ell}(\tau)$. This multiplier system is discussed in Appendix A.

Eventually, the Fourier coefficients of the mock Jacobi forms ψ^F_m can be determined provided one can compute the coefficients of the vector-valued mixed mock

[9]The components of the vector-valued mock modular form $\vec{h}^{\text{opt}}(\tau)$ satisfy $h^{\text{opt}}_\ell(\tau) = h^{\text{opt}}_{-\ell}(\tau)$, due to the symmetries of the theta function together with the modular properties of the Jacobi form $\widehat{\Phi}^{\text{opt}}_{2,\,m}$.

modular forms entering their theta-decomposition,

$$\psi^F_m(\tau,z) = \frac{\varphi^{\text{true}}_{2,m}(\tau,z)}{\eta^{24}(\tau)} + \frac{\Phi^{\text{opt}}_{2,m}(\tau,z)}{\eta^{24}(\tau)} \tag{4.3.10}$$

$$= \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} \left[\frac{h^{\text{true}}_\ell(\tau)}{\eta^{24}(\tau)} + \frac{h^{\text{opt}}_\ell(\tau)}{\eta^{24}(\tau)}\right]\theta_{m,\ell}(\tau,z)\,. \tag{4.3.11}$$

In particular, this structure shows that the standard Rademacher expansion can be applied to the first term in the theta-decomposition, which is nothing but a modular form, and that obtaining the coefficients of ψ^F_m reduces to finding an expression for the coefficients of $h^{\text{opt}}_\ell(\tau)/\eta^{24}(\tau)$. The latter are vector-valued mixed mock modular forms of weight $-21/2$ and dimension $2m$.

4.3.2 Mixed Rademacher sum via the circle method

Following the treatment of Bringmann and Manschot [77], we can generalize the circle method to recover the Fourier coefficients of such forms. Focusing on the mixed mock modular part, we define

$$f_{m,\ell}(\tau) := \frac{h^{\text{opt}}_\ell(\tau)}{\eta^{24}(\tau)} = \sum_{n\geq n_0} \alpha_m(n,\ell)\, q^{n-\frac{\ell^2}{4m}}\,, \quad \ell\in\mathbb{Z}/2m\mathbb{Z}\,, \tag{4.3.12}$$

where $n_0 = 0$ for $\ell \neq 0$ and $n_0 = -1$ for $\ell = 0$, and

$$\widetilde{f}_{m,\ell}(\tau) := q^{\frac{\ell^2}{4m}}\, f_{m,\ell}(\tau)\,. \tag{4.3.13}$$

Applying Cauchy's theorem to $\widetilde{f}_{m,\ell}$ and decomposing the contour integral via the Farey sequence as explained in section 2.5.1, we arrive at an equation similar to (2.5.4), the major difference being that $\widetilde{f}_{m,\ell}$ is now a component of a vector-valued form,

$$\alpha_m(n,\ell) = \sum_{\substack{0\leq h<k\leq N\\(h,k)=1}} e^{-2i\pi n\frac{h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \mathrm{d}\phi\, \widetilde{f}_{m,\ell}\Big(e^{-\frac{2\pi}{N^2}+2i\pi\frac{h}{k}+2i\pi\phi}\Big)\, e^{\frac{2\pi n}{N^2}-2i\pi n\phi}\,. \tag{4.3.14}$$

For the time being, we restrict to the Fourier coefficients with $4mn-\ell^2>0$. We introduce the variable $z = (\frac{k}{N^2} - ik\phi)$ and use the modular property of $f_{m,\ell}$. The form of the latter is dictated by the functional equation

$$\begin{aligned} f_{m,\ell}\Big(\frac{1}{k}(h+iz)\Big) =\; & z^{21/2}\,\psi(\gamma)_{\ell j}\, f_{m,j}\Big(\frac{1}{k}\Big(h'+\frac{i}{z}\Big)\Big) \\ & - \sqrt{\frac{m}{8\pi^2}}\, z^{21/2}\,\psi(\gamma)_{\ell j}\,\eta^{-24}\Big(\frac{1}{k}\Big(h'+\frac{i}{z}\Big)\Big)\,\mathcal{I}_{m,j}\Big(\frac{1}{kz}\Big)\,, \end{aligned} \tag{4.3.15}$$

where a sum over $j \in \mathbb{Z}/2m\mathbb{Z}$ is implied, the multiplier system $\psi(\gamma)$ is given in (4.3.27) and

$$\mathcal{I}_{m,j}(x) = \int_0^\infty \frac{\vartheta^0_{m,j}(iw - \frac{h'}{k})}{(w+x)^{3/2}}\,\mathrm{d}w\,. \tag{4.3.16}$$

Due to the mixed-mock character of $f_{m,\ell}$, the modular transformation is contaminated by the shadow of $\Phi^{\mathrm{opt}}_{2,m}$ (the Eichler integral of the unary theta series as in (4.3.8)) divided by the discriminant function. Therefore, using this transformation rule,

$$\alpha_m(n,\ell) = \sum_{\substack{0\le h<k\le N\\(h,k)=1}} z^{21/2}\, e^{-2i\pi(n-\frac{\ell^2}{4m})\frac{h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \mathrm{d}\phi\, e^{\frac{2\pi}{k}(n-\frac{\ell^2}{4m})z}\, \psi(\gamma)_{\ell j} \times$$
$$\times \left[f_{m,j}\Big(\frac{1}{k}\Big(h'+\frac{i}{z}\Big)\Big) - \sqrt{\frac{m}{8\pi^2}}\,\eta^{-24}\Big(\frac{1}{k}\Big(h'+\frac{i}{z}\Big)\Big)\,\mathcal{I}_{m,j}\Big(\frac{1}{kz}\Big)\right], \tag{4.3.17}$$

where again a sum over $j \in \mathbb{Z}/2m\mathbb{Z}$ is implicit. Two distinct terms appear in the integrand: one reflects modularity while the other is generated solely by the shadow of h^{opt}_ℓ through the $\mathcal{I}_{m,j}$ integral. We denote these terms by Σ_1 and Σ_2, respectively. Note that up to now, we haven't made use of the explicit form of the function $f_{m,\ell}$ but only of its transformation properties.

To complete the circle method, one has to estimate the behavior of the different terms in the limit for $N \to \infty$ and thus prove the convergence of the above series. The analysis yielding the exact expression of the Fourier coefficients $\alpha_m(n,\ell)$ is performed in Appendix C.2, and here we only mention the main steps of the proof. The first term Σ_1 in (4.3.17) is dominated (in the limit $N \to \infty$) by the polar terms of $f_{m,\ell}$. We denote their contribution by Σ_1^* and refer to them using tilde variables. Introducing the usual combinations

$$\Delta := 4mn - \ell^2\,, \qquad \widetilde{\Delta} := 4m\widetilde{n} - \widetilde{\ell}^2\,, \tag{4.3.18}$$

as well as $\widetilde{n}_0$ such that

$$\widetilde{n}_0 = \begin{cases} 0 & \text{for } \widetilde{\ell} \neq 0 \\ -1 & \text{for } \widetilde{\ell} = 0\,, \end{cases} \tag{4.3.19}$$

we have

$$\Sigma_1^* = \sum_{\substack{\widetilde{n}\ge\widetilde{n}_0\\ \widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}\\ \widetilde{\Delta}<0}} \alpha_m(\widetilde{n},\widetilde{\ell}) \sum_{\substack{0\le h<k\le N\\(h,k)=1}} e^{2\pi i\left(-\frac{h}{k}\frac{\Delta}{4m}+\frac{h'}{k}\frac{\widetilde{\Delta}}{4m}\right)}\psi(\gamma)_{\ell\widetilde{\ell}} \times$$
$$\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{21/2} e^{\frac{2\pi}{k}\left(z\frac{\Delta}{4m}-\frac{\widetilde{\Delta}}{4mz}\right)}\,\mathrm{d}\phi\,. \tag{4.3.20}$$

Here, $\alpha_m(\widetilde{n},\widetilde{\ell})$ are the polar coefficients of $f_{m,\ell}$. Using (4.3.2) and taking into account the discriminant function in the denominator, we can straightforwardly obtain an analytic expression in terms of the Hurwitz-Kronecker numbers[10] displayed in Table 4.1:

Lemma 4.3.1. *For m prime, the polar coefficients of $f_{m,\ell}$ are given by*

$$\alpha_m(\widetilde{n},\widetilde{\ell}) = \sum_{d|(\widetilde{n}+1,\widetilde{\ell},m)} d\, H\Big(\frac{4m(\widetilde{n}+1)-\widetilde{\ell}^2}{d^2}\Big)\,. \tag{4.3.21}$$

Proof. The identity immediately follows from the action of the Hecke operator $\mathcal{V}^{(1)}_{2,m}$ for m prime defined in (A.2.20) on the mock Jacobi form $\mathcal{H}$ defined in (4.3.3). ∎

When m is not prime but a prime power, we can still make use of (4.3.2) and work out the action of the Hecke-like operator on the Hurwitz-Kronecker generating function. For instance, for the lowest prime power $m=4$, we have

$$\begin{aligned}\alpha_4(\widetilde{n},\widetilde{\ell}) = &\sum_{d|(\widetilde{n}+1,\widetilde{\ell},4)} d\, H\Big(\frac{16(\widetilde{n}+1)-\widetilde{\ell}^2}{d^2}\Big)\\ &-2\begin{cases} H\big(4(\widetilde{n}+1)-\big(\frac{\widetilde{\ell}}{2}\big)^2\big) & \text{if } \ 2\,|\,\widetilde{\ell}\,,\\ 0 & \text{otherwise}\,.\end{cases}\end{aligned} \tag{4.3.22}$$

The case for m neither prime or a prime power needs to be treated separately since, as we mentioned above, the choice of an optimal function is not unique. For instance, when $m=6$ and if we make the choice (I) in (4.3.5), then

$$\alpha_6(\widetilde{n},\widetilde{\ell}) = \frac{1}{2}\sum_{d|(\widetilde{n}+1,\widetilde{\ell},6)} d\, H\Big(\frac{24(\widetilde{n}+1)-\widetilde{\ell}^2}{d^2}\Big) + \frac{1}{24}\, c_{\mathcal{F}_6}(\widetilde{n},\widetilde{\ell})\,, \tag{4.3.23}$$

where $c_{\mathcal{F}_6}$ are the polar coefficients of the mock Jacobi form $\mathcal{F}_6$ (C.1.1), given by

$$c_{\mathcal{F}_6}(-1,1) = c_{\mathcal{F}_6}(4,11) = -c_{\mathcal{F}_6}(0,5) = -c_{\mathcal{F}_6}(1,7) = -1\,, \tag{4.3.24}$$

$$c_{\mathcal{F}_6}(0,1) = c_{\mathcal{F}_6}(5,11) = -c_{\mathcal{F}_6}(1,5) = -c_{\mathcal{F}_6}(2,7) = 11\,. \tag{4.3.25}$$

Similar formulae for the polar coefficients in the case where m is a prime power greater than 4 or m not a prime power can also be obtained by applying the definitions of the optimal mock modular forms (4.3.2) and the results of [8].

Following [77], the boundaries of the integral in (4.3.20) can be written in a symmetric form up to an error term (see Appendix C.2) which vanishes in the $N\to\infty$

[10] Recall that by convention $H(n)=0$ for $n<0$.

limit. We are then left with the integral representation of the standard I-Bessel function of weight 23/2 (C.2.10). If we now take an extension of the definition of the Kloosterman sum given in (2.5.6) for vector-valued mock modular forms,

$$\mathfrak{K}_{\mu}^{(\nu)}(k,\psi^{-1})_{ij} = \sum_{\substack{0\leq h<k\\(h,k)=1}} e^{2\pi i\left(-\frac{h}{k}\mu+\frac{h'}{k}\nu\right)}\,\psi(\gamma)_{ij}\,, \tag{4.3.26}$$

where

$$\psi(\gamma)_{\ell j} := -e^{i\pi/4}\,\rho(\gamma)_{\ell j}^{-1} \tag{4.3.27}$$

and $\rho(\gamma)$ is defined in (A.2.15), we obtain a first Rademacher-type contribution to the coefficients $\alpha_m(n,\ell)$,

$$\Sigma_1 = \sum_{\substack{\widetilde{n}\geq\widetilde{n}_0\\ \widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}\\ \widetilde{\Delta}<0}} \alpha_m(\widetilde{n},\widetilde{\ell}) \sum_{k=1}^{\infty} \mathfrak{K}\Big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m};k,\psi^{-1}\Big)_{\ell\widetilde{\ell}}\, \frac{2\pi}{k}\left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\Big(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\Big)\,. \tag{4.3.28}$$

Above and in what follows, we take $\mathfrak{K}(\mu,\nu;k,\psi^{-1}) = \mathfrak{K}_{\mu}^{(\nu)}(k,\psi^{-1})$ to simplify the notation. We now deal with the shadow contribution Σ_2 in (4.3.17). Implementing the results of Appendix C.2 regarding the representation of the Eichler integral of $\vartheta^0_{m,j}(\tau)$, we obtain a more suitable form for the estimation of the asymptotic of the latter via the following Lemma:

Lemma 4.3.2. *For $x\in\mathbb{C}$ and $Re(x)>0$, $\mathcal{I}_{m,j}(x)$ takes the form*[11]

$$\int_0^{\infty}\frac{\vartheta^0_{m,j}(iw-\frac{h'}{k})}{(w+x)^{3/2}}\,\mathrm{d}w = \sum_{\substack{g\,(2mk)\\ g\equiv j\,(2m)}} e^{-\pi i\frac{g^2h'}{2mk}}\left(\frac{2\delta_{0,g}}{\sqrt{x}} - \frac{1}{\sqrt{2m}\,\pi k^2 x}\int_{-\infty}^{+\infty} e^{-2\pi x m u^2} f_{k,g,m}(u)\,\mathrm{d}u\right), \tag{4.3.29}$$

$$f_{k,g,m}(u) := \begin{cases} \dfrac{\pi^2}{sinh^2(\frac{\pi u}{k}-\frac{\pi i g}{2mk})} & if g\not\equiv 0\,(2mk)\,,\\[2mm] \dfrac{\pi^2}{sinh^2(\frac{\pi u}{k})}-\dfrac{k^2}{u^2} & if g\equiv 0\,(2mk)\,, \end{cases} \tag{4.3.30}$$

where $\delta_{0,g}=0$ for $g\not\equiv 0$ $(2mk)$ and $\delta_{0,g}=1$ for $g\equiv 0$ $(2mk)$.

Proof. The result is readily derived from Mittag-Leffler theory following the method outlined in [171]. ■

[11] To shorten the notation from now on, we will adopt the convention of writing only brackets when the variable is defined over a finite field. For instance, $g\equiv j\,(\mathrm{mod}\,2m)$ becomes $g\equiv j\,(2m)$ and $g\,(2mk)$ stands for $g\in\mathbb{Z}/2mk\mathbb{Z}$.

This shows that there are two contributions to Σ_2, one coming from $g \equiv 0\,(2mk)$ and the other from $g \not\equiv 0\,(2mk)$. For both of them, only the polar coefficient of $\eta(\tau)^{-24}$ contributes to the $N \to \infty$ limit.
After evaluating the integrals over the Farey sequences in the same way as for Σ_1, we are thus left with two contributions

$$\Sigma_{2,\, g\equiv 0(2mk)} = \\ = \frac{\sqrt{2m}}{\kappa} \sum_{k=1}^{\infty} \Re\left(\frac{\Delta}{4m}, -1\,; k, \psi^{-1}\right)_{\ell 0} \frac{1}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right), \qquad (4.3.31)$$

and

$$\Sigma_{2,\, g\not\equiv 0(2mk)} = \\ = -\frac{1}{2\pi\kappa} \sum_{k=1}^{\infty} \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z} \\ g(2mk) \\ g\equiv j(2m)}} \Re\left(\frac{\Delta}{4m}, -1-\frac{g^2}{4m}\,; k, \psi^{-1}\right)_{\ell j} \frac{1}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times \\ \times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} \mathrm{d}u\, f_{k,g,m}(u)\,(1-mu^2)^{25/4}\, I_{25/2}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right). \qquad (4.3.32)$$

The normalization factor κ in the two expressions above is 2 for $m = 1$ and 1 otherwise, and is related to the normalization of the Eichler integral of the shadow in (4.3.8). It has been chosen to be consistent with the $m = 1$ case discussed in [77].

Putting all three contributions together we arrive at our main result, stated in the following theorem[12].

Theorem 4.3.3. *The Fourier coefficients of $f_{m,\ell}$ defined in (4.3.12) for $\Delta = 4mn - \ell^2 > 0$ are given by the exact formula*

[12]Numerically this formula has been tested up to $m = 7$, in order to have a direct comparison with the supergravity computation presented in [78] and to include the cases where m is prime, a prime power or neither. However, the proof applies for any index.

$$
\begin{aligned}
&\alpha_m(n,\ell) = \\
&= 2\pi \sum_{\substack{\widetilde{n}\geq \widetilde{n}_0 \\ \widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z} \\ \widetilde{\Delta}<0}} \alpha_m(\widetilde{n},\widetilde{\ell}) \sum_{k=1}^{\infty} \frac{\mathfrak{K}\big(\frac{\Delta}{4m},\frac{\widetilde{\Delta}}{4m}\,;k,\psi^{-1}\big)_{\ell\widetilde{\ell}}}{k} \left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\!\left(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\right) \\
&+ \frac{\sqrt{2m}}{\kappa} \sum_{k=1}^{\infty} \frac{\mathfrak{K}\big(\frac{\Delta}{4m},-1\,;k,\psi^{-1}\big)_{\ell 0}}{\sqrt{k}} \left(\frac{4m}{\Delta}\right)^{6} I_{12}\!\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right) \\
&- \frac{1}{2\pi\kappa} \sum_{k=1}^{\infty} \sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z} \\ g(2mk) \\ g\equiv j(2m)}} \frac{\mathfrak{K}\big(\frac{\Delta}{4m},-1-\frac{g^2}{4m}\,;k,\psi^{-1}\big)_{\ell j}}{k^2} \left(\frac{4m}{\Delta}\right)^{25/4} \times \\
&\times \int_{-1/\sqrt{m}}^{+1/\sqrt{m}} \mathrm{d}u\, f_{k,g,m}(u)\, I_{25/2}\!\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right)(1-mu^2)^{25/4}\,. \qquad (4.3.33)
\end{aligned}
$$

The detailed proof is provided in Appendix C.2. Despite the fact that we derived the above expression for $\Delta > 0$, the limit for $\Delta \to 0$ exists and it correctly reproduces the constant terms of the vector-valued mock modular forms.

It is interesting to examine the $n \to \infty$ asymptotic behavior of the coefficients $\alpha_m(n,\ell)$. Using the asymptotic behavior of the I-Bessel function in this regime (C.2.11), we find

Corollary. *For m prime or a prime power, the leading asymptotic terms for $n \to \infty$ are:*

$$
\alpha_m(n,\ell) = \left(-\frac{\alpha_m(-1,0)}{2\sqrt{m}}\, n^{-6} - \frac{1}{2\sqrt{2}\,\kappa\,\pi}\, n^{-25/4} + \mathcal{O}\big(n^{-13/2}\big)\right) e^{4\pi\sqrt{n}}\,. \qquad (4.3.34)
$$

As an illustration, we also have for the first non prime or prime power index $m = 6$, in the limit $n \to \infty$,

$$
\alpha_6(n,\ell) = \left(-\frac{\beta_\ell}{2\sqrt{6}}\, n^{-6} - \frac{1}{2\sqrt{2}\,\pi}\, n^{-25/4} + \mathcal{O}\big(n^{-13/2}\big)\right) e^{4\pi\sqrt{n}}\,, \qquad (4.3.35)
$$

with $\beta_\ell = \mathfrak{K}(\infty, -25/24;\, 1, \psi^{-1})_{\ell 1}$, which is finite and smaller than 1 for $\ell \in \mathbb{Z}/12\mathbb{Z}$. Note that the exponential behavior of the Fourier coefficients is compatible with the result of Theorem 9.3 in [8] after taking into account the fact that we have divided the optimal mock Jacobi form of weight 2 and index m, $\Phi^{\text{opt}}_{2,m}$ in (4.3.2) and (4.3.5), by the discriminant function.

4.4 The quantum entropy function

It is in principle possible to recover the degeneracies of BPS states in string theory by computing the entropy of BPS black holes in supergravity. In the large charge (thermodynamic) limit, this has been thoroughly investigated, starting with the celebrated result of [143]. However, it is also well-known that the Bekenstein-Hawking entropy of BPS black holes receives quantum corrections which should be examined carefully. Thanks in part to supersymmetry, such corrections can be computed exactly by means of the quantum entropy function [144], as briefly alluded to in the introduction of this chapter. This formalism goes beyond the leading entropy contribution by re-summing all quantum corrections, and in certain cases turns out to be able to reproduce the exact degeneracies of BPS states described by the above modular objects.

In the following, we introduce the concept of quantum entropy function. The discussion is followed by a survey of the macroscopic computation of the exact entropy of 1/4-BPS black holes in the 4d in the low-energy effective supergravity theory.

4.4.1 The quantum entropy function

Sen motivated a definition of the full quantum-corrected entropy of extremal dyonic black holes in theories of supergravity on the basis of the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence [144], where the AdS_2 factor is a universal factor in the near-horizon region of extremal black holes in any dimension. This quantum entropy of course receives its main contribution from the classical Bekenstein-Hawking entropy, but also encodes the corrections to the area-law coming from higher derivative corrections and quantum fluctuations of the supergravity fields in the near-horizon region. As such, this quantity has a chance of completely reproducing the string-theoretic result for the degeneracy of BPS states which correspond to BPS black holes in the low-energy effective theory.
According to Sen's definition, the exact entropy of such black holes is formally defined as a path-integral (the expectation value of a Wilson line) on the near-horizon Euclidean AdS_2 region,

$$\exp[\mathcal{S}_{\mathrm{macro}}](q,p) := W(q,p) = \left\langle \exp[-\mathrm{i}\, q_I \oint \mathrm{d}\tau\, A^I_\tau] \right\rangle^{\mathrm{finite}}_{AdS_2} , \tag{4.4.1}$$

where τ is the (Euclidean) time direction, A^I_μ are the gauge fields of the vector multiplets under which the black hole is charged with electric charges q_I and magnetic charges p^I, and the superscript 'finite' denotes a regularization scheme to take care of the infinite volume of AdS_2 [144]. According to the expectations borne out of the attractor mechanism [142], this exact entropy only depends on the charges of the black hole.

4.4.2 Macroscopic derivation

For 1/4-BPS black holes in 4d $\mathcal{N} = 4$ supergravity the quantum entropy function $W(q,p)$ is expected to match the microscopic result (4.3.33) but as we will explain in this section, some discrepancies (exponentially subleading in the charges) still remain, pointing to interesting physics in the supergravity picture which is not yet fully understood.

In order to be able to explicitly evaluate the quantum entropy function for these black holes, one regards them as 1/2-BPS solutions of a truncated $\mathcal{N} = 2$ supergravity theory [172]. Their near-horizon region is then full-BPS and is an attractor $\mathrm{AdS}_2 \times S^2$ geometry [173] with an enhanced $SL(2) \times SU(2)$ bosonic symmetry. In $\mathcal{N} = 2$ supergravity, one can also make use of the superconformal formalism [174, 175] and work with an off-shell formulation of the theory, which turns out to be extremely convenient to apply supersymmetric localization techniques to compute the path-integral (4.4.1) [145, 156].

A large class of $\mathcal{N} = 2$ superconformal gravity actions (F-terms) are entirely specified by a holomorphic function $F(X^I, \widehat{A})$ called the prepotential, which is a homogenous function of degree 2. Here, X^I are the scalar fields sitting in the vector multiplets, and $\widehat{A}$ is the lowest component of the square of the Weyl multiplet (which contains the graviton), $\widehat{A} = (T^-_{\mu\nu})^2$. Another class of actions (D-terms) have been showed not to contribute to the quantum entropy function in [176], so one can safely focus on the F-terms. For such actions, it was shown in [156, 177] using supersymmetric localization that the leading contribution to $W(q,p)$, denoted by a hat in the formula below and corresponding to the leading saddle-point of the black hole partition function where the Weyl multiplet is at the attractor value $\mathrm{AdS}_2 \times S^2$, takes the form

$$\widehat{W}(q,p) = \int_{\mathcal{M}_Q} \prod_{I=0}^{n_v} [\mathrm{d}\phi^I] \exp\Big[-\pi\, q_I \phi^I + 4\pi\, \mathrm{Im} F\Big(\frac{\phi^I + ip^I}{2}\Big)\Big] Z_1(\phi^I)\,, \qquad (4.4.2)$$

where

- $\mathcal{M}_Q$ is the localizing manifold, which is specified by all field configurations satisfying $Q\Psi = 0$ for all fermions Ψ in the theory and for the supercharge Q

with the algebra $Q^2 = L_0 - J_0$, with L_0 the Cartan generator of $SL(2)$ and J_0 the Cartan generator of $SU(2)$.

- ϕ^I are the coordinates on $\mathcal{M}_Q$, parameterizing the fluctuations of the supergravity fields in the near-horizon AdS_2 region,
- $[d\phi^I]$ is a measure[13] taking into account the curvature of $\mathcal{M}_Q$,
- Z_1 is a one-loop determinant factor arising from quadratic field fluctuations orthogonal to $\mathcal{M}_Q$ and which depends on the prepotential and the field content of the theory [179, 180],
- the prepotential is evaluated at the attractor value $\widehat{A} = -64$ [173].

We stress that this localized form of the quantum entropy function only depends on the prepotential $F(X^I, \widehat{A})$ and the field content (one Weyl multiplet, n_v vector multiplets and n_h hypermultiplets) of the truncated $\mathcal{N} = 2$ theory, as well as on the measure $[d\phi^I]$. This is the only data which must be specified in order to obtain the exact quantum entropy of the 1/4-BPS black holes considered above and is an upshot of using the superconformal off-shell formalism.

For the K3 $\times$ T^2 case, the field content of the truncated $\mathcal{N} = 2$ theory is that of $n_v + 1 = 24$ vector multiplets (including the conformal compensator) and $n_h = 0$ hypermultiplets, and the prepotential is given by the exact expression [172, 181]

$$F(X^I, \widehat{A}) = -\frac{X^1 X^a C_{ab} X^b}{X^0} - \frac{\widehat{A}}{128\, i\pi} \log \eta^{24}\left(\frac{X^1}{X^0}\right), \quad a, b = 2 \ldots 23\,, \tag{4.4.3}$$

where C_{ab} is the intersection matrix of the 2-cycles on K3 and $\eta(\tau)$ is the Dedekind eta function (A.2.1). Observe that in this case, there are brane instanton corrections proportional to the chiral background field $\widehat{A}$ and that depend on the vector multiplet scalars through the ratio X^1/X^0 (which is the axion-dilaton in the gauge-fixed, Poincare supergravity theory).

Moreover, the computation of the quantum entropy function becomes more troublesome since one has to deal with the instanton contributions to the prepotential (4.4.3). For the leading saddle-point to the quantum entropy function (unorbifolded near-horizon geometry), this was examined in [78]. One should also examine potential other geometries corresponding to sub-leading saddle-points and understand what is the effect of summing over these in the full $W(q, p)$. Investigations in this direction have been conducted in [182, 183], where it was shown that orbifolded near-horizon geometries give exponentially suppressed contributions to

[13] We believe that this measure is subtle and not yet very well understood. We will comment on this below. For an attempt at deriving this measure, see [178].

the entropy, thus fitting the Farey Tail picture presented in the introduction to this chapter.

The question we would like to address now, harking back to the previous section, is whether an $\mathcal{N}=4$ supergravity computation can reproduce the Rademacher expansion of ψ_m^F, which encodes the degeneracies of single-center (immortal) 1/4-BPS black holes.

As explained below (4.3.10), the mock modular forms entering the theta-decomposition of ψ_m^F consist of a truly modular part and a mixed mock modular part. Equation (4.3.33) provides an exact expression for the Fourier coefficients of the latter. To reconstruct the coefficients of the former it is enough to apply the Rademacher expansion (2.5.20) to the modular form $h_\ell^{\text{true}}/\eta^{24}$ of weight $-21/2$.
As a result, the complete formula for the Fourier coefficients of ψ_m^F contains three types of terms: an m-dependent number of I-Bessel functions of weight 23/2 (coming from both the true and the mock parts), an I-Bessel function of weight 12 and the integral of an I-Bessel function of weight 25/2 times a hyperbolic function.

It was shown in [78] that localizing the quantum entropy function with the prepotential (4.4.3) yields a sum of I-Bessel functions of weight 23/2, along with additional contributions that were dubbed "edge-effects". A comparison in the I-Bessel 23/2 sector was conducted for the cases $m = 1\ldots 7$, and although there seemed to be a good agreement between the two calculations, some discrepancies remained (see Tables in [78]).
The measure of the localizing manifold $[\mathrm{d}\phi^I]$ used in [78] was inspired by a saddle-point approximation of the Igusa cusp form derived in [159]. Using such a set-up, the localized quantum entropy function takes the form [78],

$$\widehat{W}(n,\ell,m) = 2^{-12}\sum_{\mathfrak{n},\bar{\mathfrak{n}}\geq -1}(m-\mathfrak{n}-\bar{\mathfrak{n}})\,p_{24}(\mathfrak{n}+1)\,p_{24}(\bar{\mathfrak{n}}+1)\,e^{i\pi(\mathfrak{n}-\bar{\mathfrak{n}})\frac{\ell}{m}}\times$$
$$\times\int_{\gamma_2}\frac{\mathrm{d}\tau_2}{\tau_2^{13}}\exp\Big[-\pi\tau_2\frac{\Delta(\mathfrak{n},\bar{\mathfrak{n}})}{m}+\frac{\pi}{\tau_2}\Big(n-\frac{\ell^2}{4m}\Big)\Big]\times$$
$$\times\int_{\gamma_1}\mathrm{d}\tau_1\exp\left[\frac{\pi m}{\tau_2}\Big(\tau_1+i(\mathfrak{n}-\bar{\mathfrak{n}})\frac{\tau_2}{m}-\frac{\ell}{2m}\Big)^2\right], \tag{4.4.4}$$

with $\Delta(\mathfrak{n},\bar{\mathfrak{n}}) = 4\,m\,\bar{\mathfrak{n}}-(m-\mathfrak{n}+\bar{\mathfrak{n}})^2$, $p_{24}(\mathfrak{n}+1)$ is the $\mathfrak{n}^{\text{th}}$ Fourier coefficient of $\eta(\tau)^{-24}$ ($\mathfrak{n}$ denotes the instanton number).

Above, we have used (4.2.1) to express the macroscopic charges (p,q) entering the quantum entropy function (4.4.1) in terms of the microscopic data (n,ℓ,m) to facilitate the comparison.

It is important to stress that this localization computation does not include subleading saddle-points of the quantum entropy function (4.4.1) corresponding to orbifolded near-horizon geometries [183], and it should therefore only be compared to the $k = 1$ term in the asymptotic expansion for the Fourier coefficients of the single-center counting function ψ_m^F.

The next step to compute the integrals deals with the choice of contours γ_1, γ_2 (see also [178] for a discussion of such contours). The choice made in [78] led to the correct number of I-Bessel functions of weight 23/2 for any m by limiting the sum in (4.4.4) over a finite, m-dependent range for $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, but introduced the edge-effects. A close look at their contribution reveals that they take the form of I-Bessel functions of weight 12 whose prefactors and arguments are functions of $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ but for which the sum is not truncated, thus leading to a mismatch in the analytic formula as explained in details in [2]. In addition, the integral of the Bessel functions of weight 25/2 multiplied by a hyperbolic function (the $f_{k,g,m}(u)$ defined in (4.3.30)) in the Fourier coefficients of ψ_m^F seems absent from the supergravity calculation.
At present, a contour prescription which, while keeping the finite sum over I-Bessel functions of weight 23/2, would also lead to a single I-Bessel function of weight 12 along with an integral of an I-Bessel of weight 25/2 is still missing. It would be illuminating to find motivations based on low-energy physics for this contour and explore whether such considerations could lead to an agreement between the supergravity answer and the Fourier coefficients of ψ_m^F obtained in the previous section.

The results obtained in this chapter can be used as a guiding principle. In fact, one can take the point of view that the precise structure provided by the Rademacher expansion offers a steady guide to applying localization techniques to the quantum entropy function. A priori, such a computation is not straightforward and numerous aspects of the supergravity theory under consideration need to be examined in details before one can trust the results. In the conclusion, we will outline what we believe are necessary modifications to the supergravity calculation which could lead to an agreement between the microscopic and the macroscopic result.

4.5 Conclusion

In this chapter, we proved an exact formula for the Fourier coefficients of the mixed mock modular forms entering the theta-decomposition of the counting function of single-centered 1/4 BPS black holes in $\mathcal{N} = 4$ supergravity. This result was obtained using an extension of the circle method first implemented in the context of moduli spaces of stable coherent sheaves on $\mathbb{P}^2$ [77]. The (mock) modular properties of the functions highly constrain the form of the Fourier coefficients, allowing

to predict not only the growth of the black hole degeneracies for large charges but also the precise value of the degeneracy at fixed charges, which receives both perturbative and non-perturbative corrections.
Motivated by the precise match between the microstate counting function of 1/8-BPS states in Type IIB compactified on T^6 and the quantum entropy function computed by means of localization in the corresponding supergravity theory, we are led to a similar discussion for the case of immortal dyons in the $\mathcal{N} = 4$ theories. However, as explained in section 4.4.2, the low-energy effective field theory computation needs further corrections in order to reproduce the first ($k = 1$) term in the Rademacher expansion. Here we mention various subtleties arising in the calculation of the localized quantum entropy function in supergravity.
The supergravity localization result relies on the form of the measure $[\mathrm{d}\phi^I]$ along the localizing manifold. Such a measure is difficult to obtain from first principles. By definition, it is the induced measure from the full field configuration space of $\mathcal{N} = 2$ superconformal gravity to the localizing manifold (which is a particular slice in the configuration space specified by BPS solutions). However, the former measure is not known at present. The form used in [78] was borrowed from a saddle-point approximation of the microscopic degeneracies; in order to obtain the exact measure, one might need to include corrections to this approximation which would lead to a modification of the final supergravity result.
The choice of contour for the complex integrals (4.4.4) could also be modified to exhibit the same structure as the Fourier coefficients of the single-center counting functions ψ^F_m. Lastly, by inverting the approach, we believe that the exact microscopic results obtained herein could be used to infer what the localizing measure is and what the contour of integration in the localized quantum entropy function must be in order to guarantee a matching between string-theoretic and supergravity counting of 1/4-BPS states in $\mathcal{N} = 4$ theories. Moreover, the exact expression of the coefficients might shed some light on the type of geometries needed to reproduce the correct sum over saddle-points (near-horizon geometries) in the full $W(q, p)$.
In conclusion, our result suggests that some non-perturbative aspects of the supergravity result would benefit from a systematic analysis. Such an analysis should be conducted under the guidance of the exact microscopic results obtained in section 4.3, which provides a precise goal to aim for in the low-energy theory.

The results of section 4.3 may also find wider applications to other types of mixed mock modular forms arising in different physical contexts. For instance, our results might be extended to the mixed mock modular forms arising in compactifications of string theory on CY_3–folds leading to four-dimensional black holes with $\mathcal{N} = 2$ supersymmetry [184], or the related five-dimensional spinning black holes [185].

Chapter 5

From false to mock via 3d $\mathcal{N} = 2$ theories

The study of knots is one of the central themes in topology. The main tools to categorize a knot, and thus understanding its structure, are knot invariants. The purpose of an invariant of a knot is to discriminate between inequivalent knots. Unfortunately, knot invariants do not always discern the inequivalent structure of two knots. This resulted in a constant endeavor to construct more efficient or suitable knot invariants. Similarly, it is convenient to describe 3-manifolds via the corresponding invariants. An important link between 3-manifolds and knots is provided by the operation of integral surgery[1] on links (disconnected union of knots). Thanks to this technique every closed oriented 3-manifold can be realized from a 3-sphere S^3 [186].

Since the early stage of knot theory both in mathematics and physics the perspective on these entities has been purely two-dimensional. They have been, in fact, analyzed via two-dimensional projections such as braid group representations and quantum groups, or considered in relation to 2d conformal field theories and statistical mechanics. In 1989 knot theory and the topology of 3-manifold were proven to be related to a three-dimensional topological quantum field theory (TQFT) [22]. Although this gave a clear 3-dimensional and topological definition for knot and 3-manifold invariants, it did not explain why these invariants are either Laurent polynomials or formal q-series. The structure of an invariant as Laurent polynomial or integral q-series is apparent if we regard it as a homological invariant, that is to say, a generating function of dimensions (or other quantities) of homological spaces. Physically, these can be interpreted as (refined) topological indices of the Hilbert spaces of topological field theories (or more general theories). By the same token, knots and 3-manifold invariants can be viewed as the objects responsible for the "decategorification" of specific homological spaces[2], for a review of this aspect

[1]This is sometimes referred to as Dehn surgery.

[2]The process of associating numbers (e.g. dimensions) to vector spaces is here described as

see [187].

What is described in this chapter gives evidence to the fact that the categorification of 3-manifold invariants does not only appear to be natural in the realm of topological quantum field theory but it also finds a natural home in the realm of number theory. A number-theoretical structure has been already revealed in the perturbative expansion of the Chern-Simons path integral over Seifert fibered 3-manifolds in [190]. In this work, we analyze the deep inter-relation between the non-perturbative structure of this topological quantum field theory and number theory. A dominant role is played by false theta functions (objects similar to theta functions but with a sum over lattice vectors graded by a plus or a minus sign depending on the element of the lattice) and Weil representations of $Mp_2(\mathbb{Z})$, the double cover of $SL_2(\mathbb{Z})$. In addition, the study of certain topological invariants hints at a connection with the space of optimal mock Jacobi forms[3] of weight one with rational coefficients [64]. This space is spanned by mock Jacobi forms whose associated[4] mock modular forms are mock theta functions, i.e. mock modular forms whose shadow are unary theta series. These half-weight mock theta functions are interesting (up to an overall constant) integral q-series. They comprise examples of the mock theta functions that appear in the famous letter to Hardy by Ramanujan, and strikingly in the umbral moonshine conjecture [36]. Here we discover a topological framework in which these objects are expected to emerge as invariants of 3-manifolds.

The idea behind this work is rooted in the physics of the 3d-3d correspondence [23]. This correspondence relates a 3d supersymmetric gauge theory to a Chern-Simons theory (CS) on a 3-manifold M_3, to which we can naturally associate topological invariants. Roughly speaking, the origin of the 3d-3d correspondence can be traced back to the existence of $\mathcal{N} = (2, 0)$ 6d theories. Indeed, one way to describe this correspondence is from a 6d perspective. Six-dimensional superconformal field theories (SCFT) form a finite discrete set labeled by simply-laced Lie algebras $\mathfrak{g}$. These theories may be viewed as the IR fixed points of more general six-dimensional theories. Although these theories are quite mysterious objects, their existence allows drawing conclusions, correspondences, and dualities in lower dimensional theories by compactification of the original 6d theory on different manifolds[5]. Indeed, the topology of the manifold on which one compactifies the 6d theory controls a variety of protected quantities.

decategorification. The opposite procedure is named categorification.

[3]Optimal mock Jacobi forms are Jacobi forms that have minimal possible growth in their coefficients.

[4]They are associated via the theta decomposition; see Theorem 2.2.4.

[5]Other examples of correspondences where the 6d perspective has shown to be useful are the 4d-2d correspondence [188] and the AGT correspondence [189].

In this work, we focus on 3d $\mathcal{N} = 2$ theory $T[M_3]$, which originates from the parent 6d theory as twisted compactification on a closed oriented 3-manifold M_3. The main player in this chapter is the so-called "half-index", supersymmetric partition function of $T[M_3]$ on $D^2 \times \mathbb{R}$ with boundary condition $\mathcal{B}_a$ along the boundary of the disc, as illustrated in Figure 5.1. Half-indices are invariant of the homological

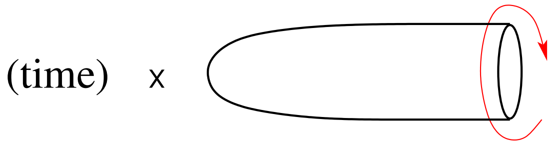


Figure 5.1: A homological block (a.k.a. half-index) counts BPS states of 3d $\mathcal{N} = 2$ theory on (time) $\times$ (cigar).

spaces, corresponding to BPS sectors of the Hilbert space of the 3d theory $T[M_3]$ and because of this, they are sometimes referred to as "homological blocks". From the Chern-Simons perspective, a specific linear combination of homological blocks represents the resummation of the perturbative series of the CS path integral around a flat connection a. Homological blocks constitute our starting point to investigate the role of number theory in the study of 3-manifold topology.

The outline of the chapter is as follows. In the first section, we review compact $SU(2)$ CS theory in relation to 3-manifold invariants and quantum modular forms. Section §5.2 defines false theta functions and describes their connection with non-holomorphic Eichler integrals of unary theta series. To conclude the section, we describe how to fold a false theta function through the Weil representations. In section §5.3 we briefly review some features of the supersymmetric 3d theory $T[M_3]$ and describe in more detail the half-index of $T[M_3]$. In section §5.4 we explicit compute the homological blocks for a subset of indefinite and the negative-definite Seifert fibered manifolds M_3. At the end of the section, we explain for which type of M_3 this calculation is bound to fail. In section §5.5 we first review the resurgence analysis of CS theory and then describe it by means of false theta functions and the Weil representations of $Mp_2(\mathbb{Z})$. The description of half-indices for positive definite and a subset of indefinite graphs is given in section §5.6 in terms of mock theta functions. We provide an example where we analyze both the false and the mock theta function perspective in section §5.7. Finally, in section §5.8 we examine a map from an optimal mock theta function to a false theta function via Rademacher sums. We close with a summary of the results and some open questions in section §5.9.

5.1 The WRT invariants and quantum modular forms

As aforementioned, the WRT invariants have a central role in the analysis of the topology of 3-manifolds. This is reviewed is section §5.1.1 in connection to $SU(2)$ Chern-Simons theory. For more details on the WRT invariants, we refer the reader to the original work by Witten [22] and the book by Turaev [186]. Besides their importance in topology, WRT invariants happen to be interesting objects in number theory. This is described in section §5.1.2. A detailed analysis of the quantum modular form associated to the Poincare homology sphere, $\Sigma(2,3,5)$, and other Brieskorn homology spheres was first provided in [190], where the concept of quantum modular form was invented. This result has been extended to some other Seifert manifolds in [192, 193].

5.1.1 WRT invariants

The Witten-Reshetikhin-Turaev invariant of a closed (connected) oriented 3-manifold M_3 was first introduced in [22] in relation to a topological quantum field theory: the three-dimensional Chern-Simons theory, denoted briefly by CS, with compact gauge group G and partition function

$$Z[k;M_3] = \int \mathcal{D}A\, e^{i(k-2)\mathcal{S}^{cs}(A)}\,. \tag{5.1.1}$$

The path integral is taken over equivalence classes of connections A on a principal G-bundle modulo the action of the gauge group; the parameter k denotes the renormalized level of the CS partition function, and the Action of the theory is

$$\mathcal{S}^{cs}(A) = \frac{1}{4\pi}\int_{M_3} \mathrm{Tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big)\,. \tag{5.1.2}$$

The critical points of the path integral (5.1.1) are flat connections, i.e. solutions of the equation

$$dA + A\wedge A = 0\,. \tag{5.1.3}$$

These are determined by the holonomies of M_3, that is by the homomorphisms $\rho : \pi_1(M_3) \to G$ modulo gauge transformations. To guarantee the invariance of $Z[k;M_3]$ under any gauge transformation, the level k is required to take integer values.

The topological nature of this QFT is expected and in part ensured by the independence of the partition function and the observables from the metric. The observables of this theory are non-local functionals of the gauge connection, called Wilson lines. In the following, we restrict to the compact gauge group $G = SU(2)$.

Denote by R an irreducible representation of $SU(2)$, and by C an oriented closed curve in M_3. A Wilson line (or loop) is defined by the trace in R of the element of the group specified by the holonomy of A around C, that is to say

$$W_R(C) := \mathrm{Tr}_R\left\{\mathrm{P}\, e^{\int_C A_i dx^i}\right\}. \tag{5.1.4}$$

Take M_3 to be the 3-sphere S^3, we define a *knot* to be a smooth embedding $K : S^1 \hookrightarrow S^3$, and a *link* L to be the union of non-intersecting knots. The expectation value of the Wilson line for a knot K in the n-dimensional representation R of $SU(2)$,

$$Z[k; M_3, K] = \langle W_R(K)\rangle = \int \mathcal{D}\mathcal{A}\, e^{i(k-2)\int_{M_3}\mathcal{L}_{cs}(A)}\, W_R(K)\,, \tag{5.1.5}$$

gives rise to a knot invariant: the Jones polynomial or WRT invariant. $Z[k; M_3, K]$ turns out to be a Laurent polynomial[6] of $\mathsf{q} = e^{2\pi i/k}$, which corresponds to the graded Euler characteristic of the Khovanov homology. As mentioned in the introduction to the chapter, the relation between knot invariants and CS partition functions was first brought to light by Witten's derivation of the principles of knot theory through the axioms of topological quantum field theory [22]. In addition, a TQFT interpretation of "surgery" on S^3 was provided in [22].

In [194, 195] Lickorish and Wallace proved that any closed compact oriented 3-manifold M_3 can be constructed by surgery on S^3 along a framed link $\mathcal{L}$. This statement is a reformulation of the fact that each closed oriented 3-manifold bounds a compact oriented 4-manifold (c.f. [196]). Thanks to this observation, any computation of an invariant of M_3 reduces to the computation of invariants of links in S^3. To perform surgery on S^3, we scribble a circle C inside the 3-manifold and thicken it to a tubular neighborhood, so to produce a solid torus with center outlined by K. Removing the solid torus from S^3 splits the manifold into two pieces: the solid torus itself and the complement of K. Any 3-manifold can be constructed by performing an appropriate diffeomorphism on the boundary of the complement of the knot and then gluing the solid torus back. After having performed a diffeomorphism on its boundary we obtain a new manifold M_3 by gluing the solid torus back together with the complement.

Since multiple links can reproduce the same M_3, any invariant of M_3 should be invariant under this redundancy in the description which is captured by the so-called Kirby moves. Reshetikhin and Turaev determined a completely combinatorial definition of $Z[k; M_3]$ for integer level k [198], by means of the representation theory of the quantum groups $U_{\mathsf{q}}(\mathfrak{sl}_2)$. This further led to the definition of WRT invariants through modular tensor categories [186].

[6]In knot theory it is practice to work with a complex variable q, the analytic continuation q.

Throughout the chapter, we adopt the following normalization of the WRT invariant

$$Z[k; S^2 \times S^1] = 1\,, \qquad Z[k; S^3] = \sqrt{\frac{2}{k}} \sin\left(\frac{\pi}{k}\right). \tag{5.1.6}$$

The mathematical definition is normalized such that $\tau[S^3] = 1$, and therefore the two definitions are related by

$$Z[k; M_3] = \frac{1}{i\sqrt{2k}} (q^{1/2} - q^{-1/2})\, \tau[M_3]\,. \tag{5.1.7}$$

5.1.2 Quantum Modular Forms

As briefly described in §5.1.1, the WRT-invariant of a 3-manifold acquires the meaning of Chern-Simons partition function over M_3 so long as the level $k \in \mathbb{Z}$. By extending the definition of WRT invariants to rational values of k we obtain examples of quantum modular forms [190, 191]. The definition of a quantum modular form, formulated in [191], was made as vague as possible to include many examples that resemble modular objects but do not fall in this category for different reasons, for instance they are neither analytic nor Γ-covariant functions. As opposed to modular forms, they are defined at the boundary of the upper-half plane (and only asymptotically), or in other words on a subset of $\mathbb{P}^1(\mathbb{Q})$. The discrete topology of this space does not allow to define analytic functions on it. Even the Γ-covariance cannot be accomplished because of the transitivity of the action (of Γ on $\mathbb{P}^1(\mathbb{Q})$), which would lead to a trivial definition.

Although we can demand neither (continuity or) analyticity nor modularity, we require that the failure of one precisely compensates the failure of the other one. In particular, we assume the modular obstruction to obey some property of analyticity or continuity[7].

Definition 5.1.1. A quantum modular form of weight w and multiplier χ on Γ is a function $Q : \mathbb{Q} \to \mathbb{C}$ such that for every $\gamma \in \Gamma$ the period function $p_\gamma(x) : \mathbb{Q}\backslash\{\gamma^{-1}(\infty)\} \to \mathbb{C}$ defined by

$$p_\gamma(x) := Q(x) - Q|_{w,\chi}\gamma(x) \tag{5.1.8}$$

has some analyticity or continuity property. Moreover, the function $\gamma \mapsto p_\gamma$ is a cocycle on Γ.

The definition of the slash-operator is reported at the end of Appendix A.1.2. The above definition can be extended to an identity between formal power series by

[7]This holds for all the example considered in this chapter, notice however that this is not true in general. The modular obstructions associated to Jones polynomials in the context of the Volume Conjecture are neither analytic nor continuous.

associating to $Q(x)$ a formal power series $Q(x+i\epsilon) \in \mathbb{C}[[\epsilon]]$ where $x \in \mathbb{Q}$. In doing so, the period function extends to a real-analytic function of $\mathbb{R}$ minus finitely many points, and we might be able to extend $Q(x)$ to a function $Q : (\mathbb{C}\backslash\mathbb{R}) \cup \mathbb{Q} \to \mathbb{C}$ analytic on $(\mathbb{C}\backslash\mathbb{R})$, whose asymptotic expansion at rational points coincides to all orders to the formal power series $Q(x+i\epsilon)$. $Q(x)$ is then called a "strong quantum modular form". Concluding, one defines a function on $\mathcal{H}^+ \cup \mathbb{Q} \cup \mathcal{H}^-$ which is $\mathbb{C}^\infty$ and which is defined close to rational points only orthogonally to the real line. It is worth pointing out that the extension of $Q(x)$ to an actual function of $\mathbb{C}$ is not canonical, since adding an analytic function (in $\mathcal{H}^\pm$) which vanishes at all orders as one approaches vertically the real line does not modify the original definition of this quantum modular form. This feature is further analyzed in subsequent sections §5.6.2.

On account of the above, the WRT invariants are quantum modular forms defined on a subset of rationals of the form $1/k$ with $k \in \mathbb{Z}$, at these values the topological quantum field theory defined by three-dimensional Chern-Simons is a consistent quantum field theory. Furthermore, the WRT invariants were argued to furnish examples of strong quantum modular forms [190]. In the following section, we analyze the extension of this type of quantum modular forms to the upper half-plane[8], and leave the extension to the lower half-plane to section §5.6.2.

5.2 False theta functions

This section deals with false theta functions and their connection to holomorphic and non-holomorphic Eichler integrals of a unary theta series. First, we introduce the concept of holomorphic Eichler integral in section §5.2.1. A holomorphic Eichler integral of unary theta series corresponds to the extension of a quantum modular form in the upper half-plane. On the other hand, in the lower half-plane non-holomorphic Eichler integrals of a unary theta series appear. They are described and related to the concept of quantum modular form in section §5.2.2. Concluding we identify holomorphic Eichler integral of unary theta series of weight $3/2$ with false theta functions and represent the latter through Weil representations of $Mp_2(\mathbb{Z})$.

[8]Notice that what is defined as upper or lower half-plane is a simple matter of convention at this point.

5.2.1 Holomorphic Eichler integrals

Eichler integrals were first constructed in [199] to describe the $(w-1)$-fold primitive of a weight $w \in \mathbb{Z}$ cusp form $f(\tau)$. The integral representation of the Eichler integral of $f(\tau)$ is

$$\widetilde{f}(\tau) := \int_{\tau}^{i\infty} dz \, \frac{f(z)}{(z-\tau)^{2-w}} \,, \qquad \tau \in \mathcal{H} \tag{5.2.1}$$

which coincides with the expression

$$\widetilde{f}(\tau) = \sum_{n \geq 1} c(n) n^{1-w} q^n \,, \tag{5.2.2}$$

where the coefficients $c(n)$ are the Fourier coefficients of $f(\tau)$. Despite the fact that $\widetilde{f}(\tau)$ is not a modular form, the modular properties of this object are envisaged in the difference $(\widetilde{f}|_{2-w}\gamma - \widetilde{f})$. For integer weights one can show that $(\widetilde{f}|_{2-w}\gamma - \widetilde{f})$ defines a polynomial of degree at most $w-2$, called the period polynomial of $f(\tau)$ [10].

For the purposes of this chapter, the previous definition of periods has to be extended to include cusp forms of half-integral weights. We denote a half-integral weight cusp forms by $g(\tau)$. The treatment here primarily follows [190]. When the weight of the cusp form $g(\tau)$ is half-integral the above definition of $\widetilde{g}(\tau)$ as integral representation has a branch cut, moreover the period p_g is not a polynomial anymore. However, we can define the holomorphic Eichler integral via the sum representation (5.2.2). Take $g(\tau)$ to be the weight $3/2$ unary theta series $\mathcal{D}\vartheta_{m,r}(\tau)$, defined in section §2.3. Its holomorphic Eichler integral takes the form[9]

$$\widetilde{g}(\tau) = \sum_{n \geq 1} \varepsilon(n) q^{an^2} \,, \qquad a \in \mathbb{Q}_{>0} \,. \tag{5.2.3}$$

No irrationality is introduced in this Eichler integral thanks to the definition of a unary theta series. Moreover, $\widetilde{g}(\tau)$ can be proven to be bounded as q tends to a root of unity radially. We report the main steps that lead to the proof of this statement for the limit $q \to 1$, which corresponds to the perturbative limit of the TQFT.

Take $q^a = e^{2\pi i x - t}$, where $t \in \mathbb{R}_{>0}$ and $\xi = e^{2\pi i x}$ is a root on unity. In addition, to explore the perturbative limit, we take $x = 0$. Following [190, 200], as long as $\varepsilon : \mathbb{Z} \to \mathbb{C}$ is a periodic function with vanishing mean value, the L-series $L(s, \varepsilon)$ defined for $\mathrm{Re}(s) > 1$ extends holomorphically to $\mathbb{C}$ and it takes the form

$$L(-r, \varepsilon) = -\frac{P^r}{r+1} \sum_{n=1}^{M} \varepsilon(n) B_{r+1}\left(\frac{n}{P}\right) \quad (r = 0, 1, \dots) \,, \tag{5.2.4}$$

[9]Note that we normalized this seres so to remove the constant $a^{-1/2}$.

where $B_k(x)$ denotes the k-th Bernoulli polynomial and P is any period of ε.[10] By considering the Mellin transform integral

$$\int_0^\infty \left(\sum_{n=1}^\infty \varepsilon(n) e^{-n^2 t} \right) t^{s-1} dt = \Gamma(s) L(2s, \varepsilon), \quad (\mathrm{Re}(s) > 1/2) \tag{5.2.5}$$

and its analytic continuation to negative values of s, it can be proven that the function $\sum_{n\geq 1} \varepsilon(n) e^{-n^2 t}$ has asymptotic expansion

$$\sum_{n\geq 1} \varepsilon(n) e^{-n^2 t} \sim L(-2r, \varepsilon) \frac{(-t)^r}{r!} , \tag{5.2.6}$$

as t approaches 0 from above. Therefore, the asymptotic expansion of the Eichler integral (5.2.3) is given by the right-hand side of equation (5.2.6).

The above can be used to show that there exists a quantum modular form $Q(x+it')$ which coincides as formal power series to the radial limit of the holomorphic Eichler integral of the cusp form $g(\tau)$

$$Q(x + it') = \lim_{t \to 0^+} \widetilde{g}(x + it') , \tag{5.2.7}$$

where $t' = t/2\pi$. However, the modular properties of this object are not as simple as in the case of integer weights and they turn out to be related to the lower half-plane extension of the quantum modular form.

5.2.2 Non-holomorphic Eichler integrals

From now on assume that $g(\tau)$ is a cusp form of weight $3/2$ with real coefficients (i.e. $g(\tau) = \overline{g(-\overline{\tau})}$). The non-holomorphic Eichler integral of the cusp form $g(\tau)$ in the lower-half plane, is defined for $z \in \mathcal{H}^-$ as

$$\widetilde{g}^*(z) := g^*(-z) = C \int_{\overline{z}}^{i\infty} g(w)(w - z)^{-1/2} dw. \tag{5.2.8}$$

The definition of the *-operator was given in §2.3.6 in relation to mock modular forms, where the cusp form $g(\tau)$ represented the shadow of the mock modular form. Since there is no canonical normalization of the shadow, we took an overall constant C in front. Note that by taking the difference between $g^*(\tau)$ and the modular transformed non-holomorphic Eichler integral we arrive at the definition of the period integral[11]

$$p_\gamma(\tau) = C \int_{-d/c}^{i\infty} g(w)(w-\tau)^{-1/2} dw \qquad \tau \in \mathcal{H}^-, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.2.9}$$

[10]Since we are mainly interested in the limit $q \to 1$, we will not introduce the twisted L-series.

[11]This was defined already in section §2.5 in the context of spectral theory and Rademacher sums.

This reflects the mock modular properties of the non-holomorphic Eichler integral of half-integral weight cusp forms [6], and it is the form of the non-holomorphic Eichler integral $\widetilde{g}^*$ when it approaches rational points.

As proved in [61, 190], the asymptotic expansion at all $x \in \mathbb{Q}$ of $\widetilde{g}(\tau)$ is up to an overall minus sign equal to the one of $\widetilde{g}^*(z)$, namely we have that

$$\lim_{t' \to 0+} \widetilde{g}^*(x - it') = - \lim_{t' \to 0+} \widetilde{g}(x + it') \,. \tag{5.2.10}$$

where $t' = t/2\pi$. The asymptotic expansions of $\widetilde{g}^*(\tau)$ $(\tau \in \mathcal{H}^-)$ and $\widetilde{g}(\tau)$ $(\tau \in \mathcal{H}^+)$ are said to agree at a rational number x if there exist $\alpha(n)$ such that as $t' \to 0^+$

$$\widetilde{g}\big(x + it'\big) \sim \sum_{n \geq 0} \alpha(n) t'^n \,, \qquad \widetilde{g}^*\big(x - it'\big) \sim - \sum_{n \geq 0} (-1)^n \alpha(n) t'^n \,. \tag{5.2.11}$$

It then follows from the work in [55] and from equation (5.2.10) that the left-hand side of equation (5.2.10) defines a strong quantum modular form. In Figure 5.2 we illustrate pictorially what we have just described.

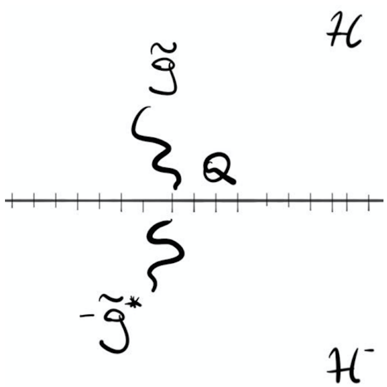


Figure 5.2: The holomorphic Eichler integral of $g(\tau)$ in the upper half-plane defines a quantum modular form $Q(x + it)$ when it approaches the real line. This coincides with the quantum modular form defined by the non-holomorphic Eichler integral of $-g(\tau)$ on the other side of the plane.

Note that $\widetilde{g}^*(z)$ satisfies a (mock) modular transformation (5.2.9), $\widetilde{g}(\tau)$ instead has a nice Fourier expansion (5.2.12). The fact that the modular properties of the function $\widetilde{g}(\tau)$ only appear near rational points is due to the fact that close to the real line $\widetilde{g}(\tau)$ inherits the mock modular properties of $\widetilde{g}^*(z)$. This property of $\widetilde{g}(\tau)$ will be widely used in section §5.5 in relation to the resurgence analysis of the analytically continued Chern-Simons theory.

5.2.3 Weil folding

In this section, we describe Eichler integrals of a unary theta series of weight 3/2 as false theta functions and we analyze the associated Weil representations of $Mp_2(\mathbb{Z})$.

Concretely, consider the Weil representation corresponding to the finite abelian group $\mathbb{Z}/2m$ equipped with a quadratic form $\mathbb{Z}/2m \to \mathbb{Q}/\mathbb{Z}$, defined in equation (2.2.49). As described in §2.4, we label a representation by $m+K = m+n, n', \ldots$ for $K = 1, n, n', \ldots$, with $K \subset \mathrm{Ex}_m$ and $m \notin K$. As in section §2.4, we use the notation σ^{m+K} to denote the set of linearly independent vectors in the vector space Θ_m, the $2m$ dimensional representation of $Mp_2(\mathbb{Z})$, and $d^{m+K} := |\sigma^{m+K}|$.

Eichler integral of unary theta series, as given in (5.2.3), can be written as false theta functions, whose defining sums run over all integers but this time positive and negative signs are associated with different elements of the lattice. False theta functions appeared first in the work by Rogers in [201] (and later by Fine in [202]). This can be as written as a false theta function

$$\Psi_{m,r}(\tau) = \sum_{n>0} (\delta^{[2m]}_{n,r} - \delta^{[2m]}_{n,-r}) q^{n^2/4m} \tag{5.2.12}$$

where $\delta^{[c]}_{a,b} = 1$ if $a \equiv b \bmod c$ and 0 otherwise. Note that $\Psi_{m,r} = -\Psi_{m,-r}$, therefore we focus on irreducible representations corresponding to $\alpha \in \mathrm{hom}(\mathrm{O_m}, \mathbb{C}^\times)$ with $\alpha(-1) = -1$, or equivalently $K \subset \mathrm{Ex}_m$ with $m \notin K$.

From the definition of unary theta function (2.3.10), (2.3.11) and the reduced thetas (2.4.18) we can readily define the corresponding reduced (or folded) unary theta series. This corresponds to

$$\mathcal{D}\vartheta^{m+K}_r(\tau) = 2^{|K|} \sum_{\substack{n\in\mathbb{Z} \\ n\equiv r \bmod 2m}} n\, P^{m+K}_{rr'} q^{n^2/4m}\,, \tag{5.2.13}$$

and the associated false theta function is

$$\Psi^{m+K}_r(\tau) := \widetilde{\mathcal{D}\vartheta^{m+K}_r}(\tau) = 2^{|K|} \sum_{n\geq 0} P^{m+K}_{rn}\, q^{n^2/4m}\,. \tag{5.2.14}$$

where $|K|$ is the order of the group K and we denoted by P^{m+K}_{rn} the entry of the projection corresponding to the r and $n \pmod{2m}$. The projection P^{m+K} for m square-free is given by

$$P^{m+K} := \prod_{n\in K} P^+_m(n) P^-_m(m)\,, \tag{5.2.15}$$

while for $m = p^2 m'$ where m' is square-free and p is prime reads

$$P^{m+K} := \prod_{n \in K} P^+_m(n) P^-_m(m) \Big(\mathbb{I} - \Omega_m(p)/p\Big). \tag{5.2.16}$$

These expressions derive from equation (2.4.16) and (2.4.17) respectively upon taking $K \subset \mathrm{Ex}_m$ with $m \notin K$. Note that by definition Ψ^{m+K}_r can be expressed as linear combination of false thetas of the form $\Psi_{m,r}(\tau)$. In particular, equation (5.2.12) can be written as

$$\Psi_{m,r}(\tau) := \sum_{n \geq 0} P^-_m(m)_{rn}\, q^{n^2/4m}\,. \tag{5.2.17}$$

which corresponds to the case when $K = \{1\}$. We denote the S-matrix of the representation Θ^{m+K} by $\mathcal{S}^{m+K}$. Explicitly, from

$$P^{m+K}_{\ell\ell'} = \sum_{r \in \sigma^{m+K}} \frac{P^{m+K}_{\ell r} P^{m+K}_{r\ell'}}{P^{m+K}_{rr}}$$

we see that the entries are given by

$$\mathcal{S}^{m+K}_{rr'} = \sum_{\ell \in \mathbb{Z}/2m} \frac{\mathcal{S}_{r\ell} P^{m+K}_{\ell r'}}{P^{m+K}_{r'r'}}, \quad r, r' \in \sigma^{m+K}, \tag{5.2.18}$$

where $\mathcal{S}_{\ell\ell'}$ corresponds to $\rho_{\mathbb{Z}/2m\mathbb{Z}}(\widetilde{S})$ in (2.2.50) and $\mathcal{S}_{r\ell}$ is a rectangular matrix whose rows are labels by elements of σ^{m+K} instead of elements of $\mathbb{Z}/2m$. Similarly, the T-matrix is given by

$$\mathcal{T}^{m+K}_{rr'} = \sum_{\ell \in \mathbb{Z}/2m} \frac{\mathcal{T}_{r\ell} P^{m+K}_{\ell r'}}{P^{m+K}_{r'r'}}, \quad r, r' \in \sigma^{m+K}, \tag{5.2.19}$$

where $\mathcal{T}_{\ell\ell'}$ corresponds to $\rho_{\mathbb{Z}/2m\mathbb{Z}}(\widetilde{T})$ in (2.2.50).

An example is given by $m + K = 30 + 6, 10, 15$, the unary theta series in this case are

$$\mathcal{D}\vartheta^{30+6,10,15}_1(\tau) = q^{1/120}(1 + 11q + 19q^3 + 29q^7 - 31q^8 + \dots) \tag{5.2.20}$$

$$\mathcal{D}\vartheta^{30+6,10,15}_7(\tau) = q^{49/120}(7 + 13q + 17q^2 + 23q^4 - 37q^{11} + \dots) \tag{5.2.21}$$

and the corresponding holomorphic Eichler integrals are

$$\Psi^{30+6,10,15}_1(\tau) = q^{1/120}(1 + q + q^3 + q^7 - q^8 + \dots) \tag{5.2.22}$$

$$\Psi^{30+6,10,15}_7(\tau) = q^{49/120}(1 + q + q^2 + q^4 - q^{11} + \dots) \tag{5.2.23}$$

These objects will appear later in section §5.5.3 in relation to the Poincaré homological sphere $\Sigma(2,3,5)$.

5.3 3d $\mathcal{N} = 2$ supersymmetric theories

The 3d-3d correspondence is a correspondence between a topological CS theory on M_3 and a 3d $\mathcal{N} = 2$ gauge theory. It was first developed in [23], and later explored in various works, e.g. [203–206]. This correspondence can be described in terms of a six-dimensional theory, from which these three-dimensional theories should originate. The 6d theory, from the M-theory perspective, represents the dynamics of a stack of five-branes in $\mathbb{R}^5 \times CY_3$, the M5-branes live on $\mathbb{R}^3 \times M_3$ and the 3-manifold M_3 is embedded in the Calabi-Yau as special Lagrangian sub-manifold. The IR limit (fixed point) of this six-dimensional theory is an $\mathcal{N} = (2, 0)$ superconformal field theory labeled by a simply-laced Lie algebra $\mathfrak{g}$. We focus on a six-dimensional theory of A_1-type, which determines the dynamics of two coincident M5-branes. The two sides of the correspondence are described by compactification of the parent 6d theory on different 3-manifolds.

Denote by $T[M_3]$ the 3d $\mathcal{N} = 2$ supersymmetric field theory. This theory is recovered from the 6d theory by performing a topological twist along M_3, that preserves four supercharges, and then by compactification on this 3-manifold. This supersymmetric gauge theory depends solely on the topology of M_3 and the Lie algebra labeling the parent theory [203]. Depending on which type of partition function (or index) of 3d $\mathcal{N} = 2$ theory $T[M_3]$ we may want to compute, we place the supersymmetric theory on a certain three-manifold[12]. Specifically, we take the 3-manifold to be $D^2 \times \mathbb{R}$ to compute the half-index, $S^2 \times \mathbb{R}$ for the superconformal index and $\mathbb{R}^2_q \times \mathbb{R}$ for the vortex partition function. The topological side (or Chern-Simons side) is recovered by compactification of the 6d theory on the three-manifold on which the gauge theory lives. For instance, compactification[13] on on $S^2 \times_q S^1$ leads to a complex Chern-Simons theory with specific coupling (see [205] for more details).

In the original paper [23], the connection between these apparently different 3d theories has been based upon the matching of different quantities. First, the moduli space of (complex) $G_{\mathbb{C}}$ flat connections on M_3 and the moduli space of supersymmetric vacua of $T[M_3]$ were required to be homeomorphic [207]

$$\mathcal{M}_{susy}(T[M_3]) \cong \mathcal{M}_{flat}(M_3, G_{\mathbb{C}})\,. \tag{5.3.1}$$

This requirement derives from the fact that the equations of motion of complex

[12] Instead of the non-compact space $\mathbb{R}^3$, we consider the supersymmetric gauge theory on a different space. The remaining non-compact dimension can be viewed as an S^1 upon taking the trace over the Hilbert space of states.

[13] The notation $\times_q$ stands for a fibered space, where we rotate the disk D^2 by $\arg(q)$ when we go around the S^1.

Chern-Simons, which define $SL_2(\mathbb{C})$ flat connections, appear as boundary conditions for partial differential equations in the Vafa-Witten theory in 4d and as vacua for the effective twisted superpotential arising from the reduction of the 3d theory $T[M_3]$ on a circle. To relate further to $T[M_3]$ we consider the 5d theory (that is the 6d compactified on S^1) on $M_3 \times D^2$; the compactification on the disc yields a direct connection to the choice of boundary conditions on the asymptotic[14] S^1. Later, we will draw attention to a subset of boundary conditions which corresponds to gauge equivalent classes of abelian flat connections.

The original prescription suggests reconstructing a supersymmetric 3d theory from an ideal triangulation of M_3 (or vice-versa). For the type of manifolds considered in this chapter, it is more convenient to reconstruct M_3 by surgery of knots and from this construction build the corresponding gauge theory [188]. Surgery of knots or links provides an effective and concrete method to compute topological invariants of generic 3-manifolds M_3, as briefly described in section §5.1.1. Moreover, thanks to the analysis by Rohklin this technique can be interpreted from a four-dimensional perspective. Indeed, surgery of a closed oriented 3-manifold M_3 can be recovered from the construction via handles of a four-manifold M_4, whose boundary is M_3 [196]; see [188] for the construction of 3d $\mathcal{N} = 2$ theory via 4-manifolds.

5.3.1 Half-index

In this section, we focus on the definition of the half-index of $T[M_3]$, and the corresponding object in Chern-Simons theory for closed oriented manifold M_3, which are Seifert fibrations of a genus zero surface. One of the important features of the half-index is that it serves as building block for various partition functions of 3d $\mathcal{N} = 2$ theories such as the superconformal index and the topologically twisted index. Later in section §5.6.1 we examine in more details the construction of superconformal indices.

The half-index of $T[M_3]$ was shown to count BPS states (preserving two out of four supercharges) of the Hilbert space of the 3d theory on $D^2 \times_q S^1$ [25, 208]. Due to the non-compactness of the spacetime, the vacuum of the theory $T[M_3]$ on $D^2 \times_q S^1$ is specified by a boundary condition $\mathcal{B}_a$ on the region $\partial D^2 \sim S^1$. We, therefore, denote the half-index as $Z(D^2 \times_q S^1; \mathcal{B}_a)$. The boundary condition was shown to correspond to an $\mathcal{N} = (0, 2)$ 2d theory $T[M_4]$ on the boundary of the 3d space[15]. We can thus think of our system as a 2d-3d coupled system [188].

[14]The disc D^2 can be depicted as an elongated cigar with geometry $\mathbb{R} \times \mathbb{R}$ at the tip of the cigar and $S^1 \times \mathbb{R}$ asymptotically.

[15]The 2d theory $T[M_4]$ can be constructed from the parent 6d theory by compactification on

Moreover, the presence of a 2d theory makes the combined system non-anomalous.

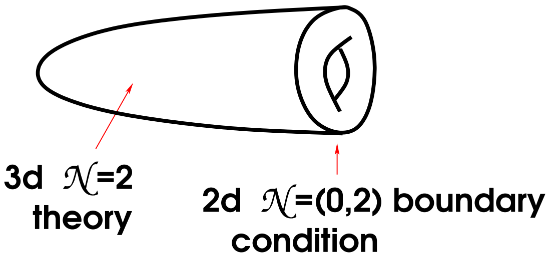


Figure 5.3: A 3d $\mathcal{N} = 2$ theory with a 2d $\mathcal{N} = (0, 2)$ boundary condition $\mathcal{B}_a$.

From the previous section, the choice of boundary condition labeling the half-index corresponds to supersymmetric vacua on $M_3 \times S^1$ and to complex flat connections of the Chern-Simons theory on M_3, labeled by a in Figure 5.3. However, we restrict to boundary conditions labeled by abelian flat connections, this will turn out to be essential for the interpretation of the half-indices as homological invariants (see (5.3.2)). Different justifications to this fact were given in [25]; another important argument was provided by the resurgence analysis in [68]. Through the resurgence analysis, it was shown that the perturbative expansion around abelian flat connections suffices to reconstruct the half-index completely. This is further analyzed in §5.5.

The 3d-2d coupled system describing the theory $T[M_3]$ on $D^2 \times_q S^1$ can be constructed by gluing together two 4-manifolds with an equal boundary, the reader is referred to [188] for details on this construction. To summarize, the 3d bulk theory can be determined from the gauge theory description of the surgery of M_3. Specifically, the theory is obtained by gluing a link complement theory to the theory of the solid torus defined by the link itself. A knot-complement theory is defined by compactification of the 6d $(2, 0)$ theory on S^3 with a codimension-two defect wrapping the knot K in S^3. Similarly, the translation of the gluing of the knot in the 3d field theory is described in [188]. A simplification occurs when the bulk 3d theory admits a UV description as abelian gauge theory, in this case, the bulk theory is described by a (sort of) quiver Chern-Simons theory, whose Lagrangian is completely specified by the plumbing graph of the knot on which the surgery of M_3 is based [188]. Complications arise if one wants to determine the boundary

the partially twisted 4-manifold M_4.

2d theory $T[M_4]$. This theory arises from compactification of the parent 6d $(2,0)$ theory on the partially twisted (via Vafa-Witten topological twist) 4-manifold M_4. Hints for the definition of this theory in our examples are provided in [209].

Even though we do not know the explicit form of this 3d-3d coupled system, we can use the topological side of the 3d-3d correspondence to compute the half-index (or homological block) of $T[M_3]$ with boundary condition $\mathcal{B}_a$. The half-index corresponds to the so-called homological block, a homological invariant recently introduced [25,208] in relation to categorification of WRT invariants. Through the 3d-3d correspondence, the homological block $\widehat{Z}_a(q)$ is defined as the graded Euler characteristic of $\mathscr{H}_a$, that is to say

$$\widehat{Z}_a(q) := \sum_{\substack{i\in\mathbb{Z}+\Delta_a \\ j\in\mathbb{Z}}} q^i(-1)^j \dim\mathscr{H}_a^{i,j} \tag{5.3.2}$$

where the i grading corresponds to the charge under the $U(1)_q$ rotation of D^2, j grades the charge under the $U(1)_R$, the R symmetry group[16], and $\mathscr{H}_a$ is the BPS sector Hilbert space annihilated by two of the four supercharges of the 3d theory with boundary condition determined by a,

$$\mathscr{H}_a[M_3] := \oplus_{\substack{i\in\mathbb{Z}+\Delta_a \\ j\in\mathbb{Z}}} \mathscr{H}_a^{i,j}\,. \tag{5.3.3}$$

Finally, we can view the homological block $\widehat{Z}_a(q)$ as

$$\widehat{Z}_a(q) \;=\; Z(D^2\times_q S^1;\mathcal{B}_a)\,, \tag{5.3.4}$$

the supersymmetric index of the entire 3d theory with a 2d supersymmetric boundary condition $\mathcal{B}_a$ indexed by an abelian flat connection a.

Note, the integrality of the coefficients is crucial for the interpretation as graded Euler characteristic (5.3.2). As claimed in [25,208], one of the important characteristics of the homological block $\widehat{Z}_a(q)$, in fact, is that it is more suitable for categorification than the WRT invariant. Strikingly, it knows about many topological features of the 3-manifold that are not directly accessible from the WRT invariant. We analyze in details these features in section §5.5. In section §5.5.2 and §5.6.2 we relate the explicit form of these integral q-series with specific number theory objects: false theta function and mock modular forms respectively.

In the regime of weak coupling, the 3d-2d system flows to a non-trivial 3d $\mathcal{N} = 2$ SCFT, whose half-index was argued to be expressible as contour integral

$$\widehat{Z}_a \;=\; \prod_i \int \frac{dz_i}{2\pi i z_i}\, F_{3d}(z_i)\,\Theta^{(a)}_{2d}(q,z_i) \tag{5.3.5}$$

[16] For Seifert manifolds a third grading appears, this gives rise to the cyclotomic expansion of $\widehat{Z}_a(q)$. This grading is not considered in the rest of the chapter, however, it would be interesting to explore this aspect further.

where the two factors in the integrand, $F_{3d}(z)$ and $\Theta^{(a)}_{2d}(q,z)$ correspond to the contributions of 3d theory and 2d boundary degrees of freedom[17], respectively, and z_i represents the elements in the complexified Cartan of the gauge group. The contribution from the boundary theory corresponds to the elliptic genus of the 2d theory $T[M_4]$, which was argued to coincide with the partition function of the 4d Vafa-Witten TQFT [188]. The choice of boundary condition $\mathcal{B}_a$ can be visualized as a choice of contour of integration. Unfortunately, in our examples it is difficult to explicitly compute the index through localization since the precise details of the 3d-2d theory are not known. The path we take, following [25], involves computing the homological blocks through the Chern-Simons partition instead of the gauge theory side. This is the topic of the next section.

5.4 Homological blocks for plumbed manifolds

The purpose of this section is to review the technique implemented in [25] to compute the homological blocks for negative definite Seifert 3-manifolds and to extend this definition to certain classes of indefinite 3-manifolds[18]. Specifically, in the first section we derive the explicit expression for the homological blocks of negative definite manifolds and "half" of the indefinite manifolds. These q-series turn out to be related to false theta functions (examples are provided in later sections, see for instance §5.5.3 and §5.7). The reason why the result obtained in the first section holds is outlined in section 5.4.2. Notice that our theorem demonstrates rather strikingly that a Seifert 3-manifold belongs to one side of the complex plane. Moreover, our derivation provides a prescription to indicate on which side of the plane the homological blocks associated to M_3 live. The companion manifolds, obtained by flipping the orientation of M_3, are naturally defined for $|q| > 1$, or equivalently $\tau \in \mathcal{H}^-$ in the lower half-plane. The associated homological blocks are constructed through a different procedure described in section §5.6, §5.8.

[17]The 2d-3d coupled system of interest for the following discussion is when the 3d $\mathcal{N} = 2$ theory is gapped but nevertheless is in a non-trivial topological phase, since it is coupled to a gapless 2d theory. The "dominant" contribution to the half-index comes from 2d massless degrees of freedom, but the nice modular behavior of the 2d elliptic genus is "spoiled" by the non-trivial contribution of the bulk theory.

[18]We define a 3-manifold to be negative definite, indefinite or positive definite depending on the eigenvalues of its linking matrix M. Throughout we assume that M does not have vanishing eigenvalues.

5.4.1 Homological blocks and WRT invariants

Throughout this chapter, we restrict to Seifert manifolds whose base spaces are genus zero surfaces. Aside from its applications in low-dimensional topology, the advantage of working with this class of 3d $\mathcal{N} = 2$ theories $T[M_3]$ is that the homological blocks (5.3.4) can be explicitly computed for many non-trivial examples.

Seifert manifolds are completely described by a connected plumbing graph $\mathcal{G}$, displayed in Figure 5.4 (see e.g. [210]). Note that the graph is characterized by a single high-valency vertex ($\deg(v) \geq 3$), labeled in the figure by b. The graph defines a link $\mathcal{L}(\mathcal{G})$ in S^3, along which we can perform a Dehn surgery to obtain M_3. As we already mentioned in §5.1.1, different surgeries (and thus different plumbing graphs) may produce homeomorphic manifolds if they are connected by Kirby moves, therefore any invariant of M_3 defined through surgery must be invariant under Kirby moves [196].

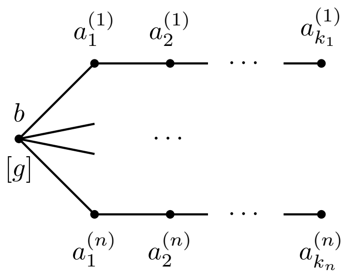


Figure 5.4: Plumbing graph for a Seifert manifold $M_3\big(b, g; \{\frac{q_i}{p_i}\}_{i=1}^n\big)$.

Let $\mathcal{G}$ be a star-shaped graph presenting M_3. Denote by $\mathbb{V}$ the set of vertices and by L its cardinality. The linking matrix of $\mathcal{L}(\mathcal{G})$ is the integral symmetric matrix M, whose (i,j)-entry is the linking coefficient between the vertices v_i and v_j, and whose diagonal represents the framing numbers[19] associated to v_i for $i \in \{1, \dots L\}$. The plumbed manifolds $M_3(\mathcal{G})$ have $\mathrm{Tor} H_1(M_3, \mathbb{Z}) = \mathbb{Z}^L / M\mathbb{Z}^L$. Moreover, in our examples $b_1(M_3) = 0$.

The $L \times L$ matrix M induces a non-degenerate symmetric bilinear pairing on the $\mathrm{Tor} H_1(M_3, \mathbb{Z})$. We refer to [211] for further details.

[19]The framing coefficients $a_j^{(i)}$ are related to surgery coefficients $\frac{q_i}{p_i}$ via continued fraction:

$$\frac{q_i}{p_i} = -\cfrac{1}{a_1^{(i)} - \cfrac{1}{a_2^{(i)} - \cfrac{1}{a_3^{(i)} - \cdots}}}$$

Definition 5.4.1. Define the finite quadratic module of the finite abelian group $\mathrm{Tor} H_1(M_3, \mathbb{Z})$, endowed with quadratic form $Q_M(x)$, as

$$\mathcal{LK}_{H_1(M_3)} := \left(\mathrm{Tor} H_1(M_3, \mathbb{Z}) \,,\; x \mapsto -(x, M^{-1}x) + \mathbb{Z} \right) \tag{5.4.1}$$

The associated symmetric bilinear form $\lambda_M(x,y) := Q_M(x+y) - Q_M(x) - Q_M(y) = -2(x, M^{-1}y)$ defines the linking pairing.

Not all the homological blocks of Seifert 3-manifolds can be computed with the formula derived in this section. Anticipating the result presented in the next section, in theorem 5.4.2 we single out the 3-manifolds whose homological blocks are defined by the expressions given in this section.

Theorem 5.4.2. *Take $M_3(\mathcal{G})$ to be a plumbed 3-manifold, whose plumbing graph $\mathcal{G}$ is a tree. Denote by M the linking matrix of Γ and by M^{-1} its inverse. Assume there is only one high-valency vertex and let i denote the position in the linking matrix corresponding to the high-valency vertex. Then for $(M^{-1})_{ii} < 0$, the homological blocks associated to $T[M_3]$ converges for $|q| < 1$. The opposite orientated manifold $\overline{M_3}(\overline{\mathcal{G}})$ converges for $|q| > 1$.*

An explanation to theorem 5.4.2 is reported in the next section, the reader is referred to the original article [3] for more details. A proposal for the homological block of the opposite orientated manifold $\overline{M_3}(\overline{\mathcal{G}})$ is displayed in section §5.6.2.

The connection between the homological block and the WRT invariant was conjectured in [25] to be given by the following.

Conjecture

The WRT invariant $\tau[M_3(\mathcal{G})]$ can be decomposed as follow

$$\tau[M_3(\mathcal{G})] = \frac{1}{(q^{1/2} - q^{-1/2})} \sum_{\substack{a \in \mathbb{Z}^L/M\mathbb{Z}^L \\ b \in (2\mathbb{Z}^L + \delta)/2M\mathbb{Z}^L}} e^{2\pi i Q_M(a)}\, \widetilde{X}_{ab} \widehat{Z}_b(q) \Big|_{q \to \mathsf{q}} \tag{5.4.2}$$

where $q = e^{2\pi i \tau}$, $\tau \in \mathcal{H}$, and $\mathsf{q} = e^{2\pi i/k}$ with $k \in \mathbb{Z}_{>0}$. Moreover, $\widehat{Z}_b(q) \in q^{\Delta_b}\mathbb{Z}[[q]]$ converges for $|q| < 1$, $\Delta_b \in \mathbb{Q}$, $c \in \mathbb{Z}_+$ and

$$\widetilde{X}_{ab} = \frac{e^{\pi i \lambda_M(a,b)}}{|\det M|^{1/2}} \tag{5.4.3}$$

where $\Delta_b = -\frac{3L - \sum_v a_v}{4} + \max_{\ell \in 2M\mathbb{Z}^L + b} \frac{Q_M(\ell)}{4} \in \mathbb{Q}$.

The label a represents abelian flat connections of the CS theory on M_3, which indeed are elements of the quotient group $\mathbb{Z}^L/M\mathbb{Z}^L$. Moreover, we can associate to each b a choice of 2d $\mathcal{N} = (0,2)$ boundary condition $\mathcal{B}_b$, which corresponds to elements of the quotient group $(2\mathbb{Z}^L + \delta)/2M\mathbb{Z}^L$, where $\delta \in \mathbb{Z}^L/2\mathbb{Z}^L$ and its components are $\delta_v \equiv \deg(v) \mod 2$. Notice that these two spaces are isomorphic, and the map between these two spaces is provided by the matrix $\widetilde{X}_{ab}$[20]. For simplicity, we have here neglected the symmetry between homological blocks induced by the action of the Weil group of $SU(2)$, which transforms $a \to -a$. We return to it at the end of this section.

The definition of homological invariants categorizing WRT invariants was proposed in [25]. The derivation here follows the result in [25, 211, 212]. Expressing M_3 as surgery on a link $\mathcal{L}(\mathcal{G})$, where $\mathcal{G}$ is the resolution graph of the singularity defined by the link $\mathcal{L}(\mathcal{G})$, we obtain the WRT invariant in terms of the Jones polynomial of this link

$$\tau[M_3(\mathcal{G})] = \frac{F[\mathcal{L}(\mathcal{G})]}{F[\mathcal{L}(+1\bullet)]^{b_+}F[\mathcal{L}(-1\bullet)]^{b_-}} \tag{5.4.4}$$

where $b_\pm$ are the number of positive and negative eigenvalues of the linking matrix M, $\pm 1\bullet$ stands for a plumbing graph with one vertex corresponding to an unknot with ± 1 framing and F is a function of the Jones polynomial of $\mathcal{L}(\mathcal{G})$.

The WRT invariant of $M_3(\mathcal{G})$ in its simplified form [25] is given by

$$\tau[M_3(\mathcal{G})] = \frac{e^{-\pi i\sigma/4}\mathsf{q}^{3\sigma/4-\sum_v a_v/4}}{2(2k)^{L/2}(\mathsf{q}^{1/2}-\mathsf{q}^{-1/2})} \times \\ \times \sum_{n\in\mathbb{Z}^L/2k\mathbb{Z}^L}{}' \mathsf{q}^{(n,Mn)/4} \prod_{v\in\mathbb{V}} \left(\frac{1}{\mathsf{q}^{n_v/2}-\mathsf{q}^{-n_v/2}}\right)^{\deg(v)-2}, \tag{5.4.5}$$

where $\mathsf{q} = e^{2\pi i/k}$, $k \in \mathbb{Z}_{>}0$, σ is the signature of M. Note that throughout the Chapter we denote vectors by simple letters and add a subscript to it when we consider each component separately, e.g. n is an L-dimensional vector which belongs to $\mathbb{Z}^L/2k\mathbb{Z}^L$ and its components are denoted by n_v where $v \in \mathbb{V} = \{1, \ldots, L\}$. The sum Σ' excludes the terms $n_v \equiv 0 \pmod k$, for which the summands diverge. In order to obtain (5.4.2) we have to remove the k-dependence from the above summands, this can be achieved through the Gauss sum reciprocity formula, as long as n takes values in the whole lattice $\mathbb{Z}^L/2k\mathbb{Z}^L$. Therefore, we first regularize the sum including the divergent terms. The only piece that needs to be regularized is

$$\sum_{n\in\mathbb{Z}^L/2k\mathbb{Z}^L}{}' \mathsf{q}^{(n,Mn)/4} \prod_{v\in\mathbb{V}} \left(\frac{1}{\mathsf{q}^{n_v/2}-\mathsf{q}^{-n_v/2}}\right)^{\deg(v)-2} \tag{5.4.6}$$

[20] In [208], $\widetilde{X}_{ab}$ with $a, b \in \mathbb{Z}^L/M\mathbb{Z}^L$ was related to the S-transform of the affine algebra $\widehat{u}(1)^2_p$ for $M_3 = L(p, 1)$ a Lens space.

following [25] we substitute this expression by

$$\lim_{w\to 1}\sum_{n\in\mathbb{Z}^L/2k\mathbb{Z}^L} \mathsf{q}^{(n,Mn)/4}F_w(x)\Big|_{x=\mathsf{q}^{n/2}} \tag{5.4.7}$$

where

$$F_w(x) := 2^{-L}\prod_{v\in\mathbb{V}}\frac{(x_v-1/x_v)^{\Delta_v}}{(x_v-w/x_v)^{\deg(v)-2+\Delta_v}+(wx_v-1/x_v)^{\deg(v)-2+\Delta_v}} \tag{5.4.8}$$

where $\Delta_v \equiv \deg(v)-1 \,(\mathrm{mod}\, 2)$ for any $v\in\mathbb{V}$ and $w\in\mathbb{C}$ such that $0<|w|<1$. Moreover, we can "expand" the function F as follows

$$F_w(\mathsf{q}^{n/2}) = \sum_{\ell\in 2k\mathbb{Z}^L} F_w^{\ell}\,\mathsf{q}^{(n,\ell)/2}\,, \qquad F_w^{\ell} = \sum_m N_{m,\ell}w^m \in \mathbb{Z}[w] \tag{5.4.9}$$

where ℓ is such that $(n,\ell)\in\mathbb{Z}$. Since F_1^{ℓ} is independent of Δ_v, for any $v\in\mathbb{V}$, we can take $\Delta_v=0$ for any v, and therefore define

$$\sum_{\ell\in 2k\mathbb{Z}^L} F_1^{\ell}\,q^{(n,\ell)/2} = \tag{5.4.10}$$

$$= 2^{-L}\prod_{v\in\mathbb{V}}\left\{\underset{x\to 0}{\text{Expansion}}\frac{1}{(x_v-1/x_v)^{\deg(v)-2}}+\underset{x\to\infty}{\text{Expansion}}\frac{1}{(x_v-1/x_v)^{\deg(v)-2}}\right\}\Bigg|_{x=\mathsf{q}^{n/2}}$$

It is worth pointing out that this expression is responsible for the form of the q-series, in particular the growth of the series is dictated by the elements of the lattice $2k\mathbb{Z}^L$ for which F_1^{ℓ} is different from zero and the $\deg(v)\geq 3$. Given the above regularization, we implement the Gauss sum reciprocity formula to the regularized sum (5.4.7). For a detailed analysis of Gauss sum in this context the reader is referred to [211–213].

Let Λ be the integral lattice of rank $L<\infty$, and inner product $(\cdot,\cdot)$. Denote by M a self-adjoint automorphism and defined a symmetric bilinear form $g:\Lambda\times\Lambda\to\mathbb{Z}$ by $g(x,y) := (x,My)$ for $x,y\in\Lambda$. Assume that $M(\Lambda^*)\subset\Lambda^*$. The Gauss sum reciprocity formula is given by [212]

$$\sum_{n\in\mathbb{Z}^L/2k\mathbb{Z}^L}\exp\left(\frac{\pi i}{2k}(n,Mn)+2\pi i(\ell,n)\right) =$$
$$\frac{e^{\pi i\sigma/4}(2k)^{L/2}}{|\det M|^{1/2}}\sum_{a\in\mathbb{Z}^L/M\mathbb{Z}^L}\exp\Big(-2\pi i k\big(a+\ell, M^{-1}(a+\ell)\big)\Big). \tag{5.4.11}$$

where $x\in\Lambda$, $y\in\Lambda^*$ and $\ell\in\Lambda(2k)=2k\mathbb{Z}^L$ and $k\in\mathbb{Z}_{>0}$. For $\mathsf{q}=e^{2\pi i/k}$ we apply this identity to equation (5.4.7), and thanks to the definition of F_w in (5.4.9), we obtain

$$\sum_{\ell\in 2k\mathbb{Z}^L} F_w^\ell \sum_{n\in\mathbb{Z}^L/2k\mathbb{Z}^L} \mathsf{q}^{(n,Mn)/4}\mathsf{q}^{(n,\ell)/2} =$$
$$= \frac{e^{\pi i\sigma/4}(2k)^{L/2}}{|\det M|^{1/2}} \sum_{\ell'\in\mathbb{Z}^L} F_w^{\ell'} \sum_{a\in\mathbb{Z}^L/M\mathbb{Z}^L} e^{-2\pi i k(a,M^{-1}a)} e^{-2\pi i(\ell',M^{-1}a)} e^{-2\pi i(\ell',M^{-1}\ell')/4k}$$
$$= \frac{e^{\pi i\sigma/4}(2k)^{L/2}}{|\det M|^{1/2}} \sum_{\substack{a\in\mathbb{Z}^L/M\mathbb{Z}^L\\ b\in(2\mathbb{Z}^L+\delta)/2M\mathbb{Z}^L}} e^{-2\pi i k(a,M^{-1}a)} e^{-2\pi i(b,M^{-1}a)} \sum_{\ell'\in 2M\mathbb{Z}^L+b} F_w^{\ell'}\mathsf{q}^{-\frac{(\ell',M^{-1}\ell')}{4}}$$

where we took $\ell'=\ell/2k$, $\delta\in\mathbb{Z}^L/2\mathbb{Z}^L$, whose components are $\delta_v\equiv\deg(v) \mod 2$. Note that rescaling ℓ by $2k$ does not change the integers F_1^ℓ. In fact, we can express the limit in (5.4.7) as an "analytic continuation" of the right-hand side of the expression

$$\lim_{w\to 1}\sum_{\ell\in 2M\mathbb{Z}^L+b} F_w^\ell\mathsf{q}^{-\frac{(\ell,M^{-1}\ell)}{4}} = \lim_{q\to\mathsf{q}}\sum_{\ell\in 2M\mathbb{Z}^L+b} F_1^\ell q^{-\frac{(\ell,M^{-1}\ell)}{4}} \tag{5.4.12}$$

where $q=e^{2\pi i\tau}$, $\tau\in\mathcal{H}$, and F_1^ℓ as in (5.4.10).

Eventually, from the expression of the WRT invariant given in (5.4.2) we obtain the explicit expression for the homological block $\widehat{Z}_b(q)$

$$\widehat{Z}_b(q) = q^{3\sigma/4-\sum_v a_v/4}\sum_{\ell\in 2M\mathbb{Z}^L+b} F_1^\ell q^{-\frac{(\ell,M^{-1}\ell)}{4}}\,. \tag{5.4.13}$$

This integral (up to an overall constant term) q-series $\widehat{Z}_b(q)$ was conjectured to be equal to the half-index of the 3d-2d coupled system, as described in §5.3.1.

The above expression can be packaged into a convenient form by grouping the terms with equal CS functional $e^{-2\pi i k(a,M^{-1}a)}$ and by making explicit the symmetry under the action of the Weyl group. Since we restricted to the case with gauge group $SU(2)$, the Weyl group is the abelian group $\mathbb{Z}_2$ and its action on the elements $a\in\mathbb{Z}^L/M\mathbb{Z}^L$ is $W: a\mapsto -a$. The fact that the Weyl group is a symmetry of the system amounts to say that $a\equiv -a\,(\mathrm{mod}\,M\mathbb{Z}^L)$. Denote the stabilizer group of the element a as $\mathcal{W}_a$ which equals the abelian group $\mathbb{Z}_2$ if a is a fixed point of the Weyl group action and it is trivial otherwise. The WRT invariant (5.4.2) can thus be expressed as[21]

$$\tau[M_3(\mathcal{G})] = \frac{1}{(q^{1/2}-q^{-1/2})}\sum_{a\in(\mathbb{Z}^L/M\mathbb{Z}^L)/\mathbb{Z}_2} e^{\pi i Q_M(a)}\, Z_a(q)\Big|_{q\to\mathsf{q}} \tag{5.4.14}$$

[21]This decomposition of the WRT invariant was already envisaged in [192, 193].

where

$$Z_a(q) := \sum_{b\in(2\mathbb{Z}^L+\delta)/2M\mathbb{Z}^L/\mathbb{Z}_2} \frac{2}{|\mathcal{W}_b|} X_{ab}\widehat{Z}_b(q) \tag{5.4.15}$$

is a combination of homological blocks and might not have rational coefficients. The relation between $\widehat{Z}_b(q)$ and $Z_a(q)$ can be read from the form of X_{ab}

$$X_{ab} := \frac{e^{\pi i\lambda_M(a,b)} + e^{-\pi i\lambda_M(a,b)}}{|\mathcal{W}_a||\det M|^{1/2}} \, . \tag{5.4.16}$$

The role of X_{ab} in the description of the type of flat $SL_2(\mathbb{C})$ connections of M_3 is investigated in [3].

5.4.2 Growth of q-series

Given the expression for the homological blocks of Seifert manifolds, one may wonder whether it is still possible to associate a convergent q-series and if it is still given by equation (5.4.13) for more general M_3. The theorem below provides an intrinsic characterization of the type of manifolds for which a convergent q-series for $|q| < 1$ can be constructed.

Theorem 5.4.3. *Take $M_3(\mathcal{G})$ to be a plumbed 3-manifold, whose plumbing graph $\mathcal{G}$ is a tree. Denote by M the linking matrix of $\mathcal{G}$ and by M^{-1} its inverse. For multiple high-valency vertices the growth of the q-series defining the homological blocks is dictated by the signs of $(M^{-1})_{ii}$, where i labels the positions in the linking matrix of the high-valency vertices. If $(M^{-1})_{ii} < 0$ for all i then the homological block converges inside the unit disc. Conversely, if there exists i such that $(M^{-1})_{ii} > 0$, the series is unbounded from below. Lastly, when both positive and negative entries appear the homological blocks growth in different directions.*

Equation (5.4.10) displays the preferred role played by the vertices with degree strictly bigger than two. As it was already pointed out above, if we denote by Λ_M the subset of the lattice $2\mathbb{Z}^L + \delta$ where F_1^ℓ does not vanishes, then the number of directions in the lattice where the series growth equals the number of high-valency vertices. Moreover, it is always possible to represent a Seifert fibered manifold with a single high-valency vertex. Notice that when there is only one high-valency vertex, there is only one direction in which the series can grow. Therefore, the q-series either converges in the upper half-plane or it is badly unbounded from below[22].

On the other hand, when there are multiple high-valency vertices, there may be

[22]Note that a homological block cannot be given by a series unbounded from below because of its interpretation as half-index.

multiple, independent unbounded directions. An example of this behavior is provided below.

Consider a plumbing graph with two high-valency vertices, to single out the direction of growths of the q-series we should consider the sign of the elements $(M^{-1})_{ii}$ where i labels the high-valency vertices. More general examples can be deduced from this one. The series again can be unbounded from below or can converge inside the unit disc, however in this case, there are more possibilities to produce an ill-defined series. In fact, it is sufficient to have a single element $(M^{-1})_{ii} < 0$ to produce a series unbounded from below. Consider for instance a 3-manifold with plumbing graph displayed in Figure 5.5. The adjacency matrix M and its inverse

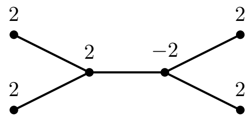


Figure 5.5: An example of plumbing graph, which yields unbounded homological blocks.

are

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad M^{-1} = \frac{1}{16} \begin{pmatrix} 12 & 4 & -6 & -6 & -2 & -2 \\ 4 & -4 & -2 & -2 & 2 & 2 \\ -6 & -2 & 11 & 3 & 1 & 1 \\ -6 & -2 & 3 & 11 & 1 & 1 \\ -2 & 2 & 1 & 1 & 7 & -1 \\ -2 & 2 & 1 & 1 & -1 & 7 \end{pmatrix}.$$

In this case $i = 1$ indexes the left-high-valency vertex while $i = 2$ labels the right-high-valency vertex. The inverse linking matrix has diagonal entries 12 and -4 (up to an overall constant term) in $i = 1, 2$ respectively, therefore we expect from the theorem to have an unbounded growth from below due to the first vertex and a natural growth from the second one. This is what is also witnessed from the explicit expression of a homological block

$$\widehat{Z} = \frac{1}{4}\left(\cdots + q^{-19} + 2q^{-11} + 2q^{-8} - 6q^{-7} - 2 - 2q^{4} - 2q^{5} - 2q^{8} + q^{9} + \cdots\right).$$

In the above derivation, we have restricted ourselves to a particular type of plumbing graph, however, if there exist two (or more) plumbing graphs $\mathcal{G}$, $\mathcal{G}'$ whose resulting manifolds are homeomorphic the final result should coincide. The result is proven to be invariant under 3d Kirby moves in [3].

5.5 The modular side of resurgence

We review the resurgence analysis described in [68] under a new light provided by the Weil representation of the metaplectic group[23] $Mp_2(\mathbb{Z})$. Our viewpoint does not only explain the relation between analytically continued Chern-Simons theory and number theory but it allows to efficiently extract information about the non-abelian $SL_2(\mathbb{C})$ flat connections of M_3. We refer to [3] for an account on this topic and focus in this section on the description of the resurgence analysis. The treatment is restricted to three singular fibered Seifert manifolds.

In section §5.5.1 we briefly introduce the resurgence analysis of analytically continued Chern-Simons and some aspects of the CS theory. We continue in section §5.2.3 with a description of the resurgence analysis and its relation to false theta functions and the Weil representations of $Mp_2(\mathbb{Z})$. This provides a complete treatment for the resurgence analysis of Brieskorn homology spheres (Seifert manifolds with trivial $H_1(M_3, \mathbb{Z})$) and a partial treatment for the other Seifert manifolds. For a complete discussion on Seifert manifolds with non-trivial first homology group, the reader is referred to [3]. Finally, in section §5.5.3 we illustrate a concrete example of Seifert manifold where the previous results can be applied.

5.5.1 Analytically continued Chern-Simons

The resurgence analysis of Chern-Simons for Seifert fibered 3-manifolds was performed for a class of Seifert fibered 3-manifolds in [68][24]. This technique, primarily based on the resurgent analysis by Jean Écalle and the Picard-Lefschetz theory, allows reproducing the non-perturbative contributions to the Chern-Simons path integral of $SU(2)$ Chern-Simons starting from the knowledge of its perturbative expansion. In order to perform this analysis the behavior of the Chern-Simons path integral for generic values of k has to be examined. This forces us to consider analytically continued CS theory[25], that is the holomorphic sector of $SL_2(\mathbb{C})$ complex CS theory. For a complete treatment of complex Chern-Simons theory via Morse's theory the reader is referred to [215].

The CS action defined in equation (5.1.2) becomes a holomorphic function on the space of complex connections $\mathcal{A}$. Given the critical points α of the CS action, one

[23]Recall from chapter 2 that faithful representations of the metaplectic group are projective representations of $SL_2(\mathbb{Z})$.

[24]The resurgent technique was first conjectured to hold for any Seifert fibered 3-manifolds and proved to hold in few examples in [214].

[25]Recall that in the case of compact Chern-Simons §5.1.1 k has to be an integer to satisfy the quantization condition.

derives by steepest descent a suitable basis of integration cycles Γ_α, the so-called Lefschetz thimbles, on which the path integral converges. These are determined by the flow equations, ordinary differential equations corresponding to the Yang-Mills instanton equations of a 4-manifold. For further details and for an interpretation from a 2d perspective see [22,68].

Denote by $\mathcal{C}$ a convergent cycle for the Chern-Simons path integral. This cycle can be decomposed into the basis of Lefschetz thimbles as $\mathcal{C}=\sum_\alpha n_\alpha \Gamma_\alpha$, where n_α are the so-called trans-series parameters. The convergent cycle, along which we take the Chern-Simons path integral, has to be chosen such that, when we evaluate the integral for integer values of the level k, we recover the $SU(2)$ CS partition function[26] $Z[k;M_3]$ (5.1.1). Therefore, the Chern-Simons path integral takes the form

$$Z_{SL_2(\mathbb{C})}[M_3]=\int_{\mathcal{C}\subset\widetilde{\mathcal{A}}}\mathcal{D}\mathcal{A}\, e^{i(k-2)\mathcal{S}^{cs}(\mathcal{A})}\,, \tag{5.5.1}$$

where $\widetilde{\mathcal{A}}$ represents the (universal) cover space of $SL_2(\mathbb{C})$ gauge connections modulo based gauge transformations. The reason why the integral is taken over this space of gauge connections is that the action has to be single-valued on the Lefschetz thimbles and since the level is not restricted to be an integer anymore, the partition function is invariant only under the subgroup of gauge transformations connected to the identity.

The critical submanifolds of the complex CS action are connected components of the moduli space of flat connections

$$\mathcal{M}_{\mathfrak{a}}\subset\mathcal{M}_{\text{flat}}(M_3,SL_2(\mathbb{C}))\times\mathbb{Z} \tag{5.5.2}$$

where $\mathfrak{a}\in\pi_0(\mathcal{M}_{\text{flat}}(M_3,SL_2(\mathbb{C})))\times\mathbb{Z}$. This label takes into account the lift of the complex connection into the universal cover space of $SL_2(\mathbb{C})$ flat connections. If we forget about this lift, the connected components of the moduli space of flat connections are denoted by

$$\mathcal{M}_\alpha\subset\mathcal{M}_{\text{flat}}(M_3,SL_2(\mathbb{C}))=\text{Hom}(\pi_1(M_3),SL_2(\mathbb{C}))/SL_2(\mathbb{C}) \tag{5.5.3}$$

where $\alpha\in\pi_0(\mathcal{M}_{\text{flat}}(M_3,SL_2(\mathbb{C})))$.

The resurgence analysis produces a trans-series expansion around each critical point of the CS partition function. This can be written from the perspective of compact CS as

$$Z[k;M_3]\sim\sum_\alpha n_\alpha e^{2\pi i k S_\alpha} Z_\alpha^{\text{pert}}(k) \tag{5.5.4}$$

[26]Although the contribution of complex flat connections is not directly visible in $SU(2)$ Chern-Simons, it is possible to gather data regarding complex flat connections through the decomposition of the $SU(2)$ CS path integral into homological blocks and a closer look at the relevant $SL_2(\mathbb{Z})$ representation [3].

where k is an integer and the perturbative expansion is

$$Z_\alpha^{\rm pert}(k) = \begin{cases} \sum_{n\geq 0} a_n^\alpha k^{-n-1/2} & \alpha \in \text{Afc} \quad (a_0^\alpha = 0\,, \alpha \in \text{central}) \\ \sum_{n\geq 0} a_n^\alpha k^{-n} & \alpha \in \text{nAfc} \end{cases} \tag{5.5.5}$$

As written above, there are different types of flat connections which can be classified by their stabilizers of the $SU(2)$ action on $\text{Hom}(\pi_1(M_3), SU(2))$ as follows:

- An abelian flat connection[27] (Afc) has a non-trivial stabilizer. Among the abelian flat connections, there are the so-called central flat connections which have stabilizer equal to $SU(2)$.
- A non-abelian flat connection (nAfc) has trivial stabilizer. Moreover, we say that an $SL(2,\mathbb{C})$ flat connection is real if it is conjugate to an $SU(2)$ flat connection and complex otherwise.

Recall that the stabilizer of a flat $G_{\mathbb{C}}$-connection $\mathcal{A}$ is defined as the group of gauge transformations that leave $\mathcal{A}$ invariant.

A central role is played by abelian flat connections. The connected components of abelian flat connections for $SU(2)$ Chern-Simons are defined by the embedding $U(1) \hookrightarrow SU(2)$; that is to say, they are determined by a homomorphism $\mathcal{M}_{\text{flat}}(M_3; U(1)) \to \mathcal{M}_{\text{flat}}(M_3; SU(2))$. If we further take equivalence classes under the action of the Weyl group, which acts on an abelian flat connection a as $a \mapsto -a$, we arrive at the conclusion that they are elements[28] of $H_1(M_3,\mathbb{Z})/\mathbb{Z}_2$. If not stated otherwise, in the rest of the text when we consider abelian flat connections we refer to the equivalence classes under the Weyl group action.

As mentioned in section §5.3.1, we can reduce the sum over critical points to a sum over (solely) abelian flat connections. Physical motivations for this fact were proposed in [25] and further justifications were provided from the resurgence analysis [68], where the contribution from non-abelian connections was shown to be encoded in the non-perturbative piece of the trans-series. Moreover, to each abelian flat connection $a \in \pi_0 \mathcal{M}^{\rm ab}_{\rm flat}(M_3, SL_2(\mathbb{C}))$ we associate a boundary condition $\mathcal{B}_a$ for the supersymmetric theory on $D^2 \times_q S^1$. As reviewed earlier, this is possible due to the isomorphism between the two spaces. The $SL_2(\mathbb{Z})$ action on the boundary torus $\partial D^2 \times S^1 \sim T^2$ acts on the boundary conditions through a representation of the form

$$X_{ab} = \frac{\sum_{a' \in \{\mathbb{Z}_2\text{-orbit of } a\}} e^{\pi i \lambda_M(a',b)}}{\sqrt{|\text{Tor} H_1(M_3,\mathbb{Z})|}}\,, \qquad T_{ab} = \delta_{a,b} e^{\pi i \lambda_M(a,b)}, \tag{5.5.6}$$

[27] Since their holonomies lie in the maximal torus $T_{\mathbb{C}} \subset G_{\mathbb{C}}$, abelian flat connections forms a representation of $H_1(M_3,\mathbb{Z})$.

[28] We can consider the first homology group and do not restrict to its torsion part here, because in all our examples they coincide.

where $a, b \in H_1(M_3, \mathbb{Z})/\mathbb{Z}_2$, the $\mathbb{Z}_2$ acts as the Weyl group on $H_1(M_3, \mathbb{Z})$, and $\lambda_M(\cdot,\cdot)$ is the bilinear form associated to the abelian group $\mathrm{Tor} H_1(M_3, \mathbb{Z})$. This corresponds to (5.4.1) upon changing the range of the b index.

5.5.2 Resurgence and Weil representations

In this section we investigate the relation between the number theory objects introduced earlier in this chapter and the resurgence analysis of analytically continued CS on Seifert manifolds with three singular fibers.

Thanks to theorem 5.4.2 and equation (5.4.13), we can associate to specific plumbed 3-manifold $M_3(\mathcal{G})$ a set of q-series labeled by abelian fat connections. These q-series correspond to the homological blocks of the 3d-2d couple system and are closely related to the expressions of false theta functions[29] (5.2.14). More precisely, for the class of Seifert 3-manifolds considered here, we have

$$\widehat{Z}_a(\tau) = \mathfrak{c}\, q^{\Delta}(\Psi_r^{m+K}(\tau) + \mathfrak{d}(\tau))\,, \qquad a \in H_1(M_3, \mathbb{Z})/\mathbb{Z}_2 \tag{5.5.7}$$

where $\Delta \in \mathbb{Q}$, $\mathfrak{d}(\tau)$ is a finite q-series (which typically vanishes or is given by a number) and $\mathfrak{c}$ is a constant. If we look at different abelian flat connections $\mathfrak{c}$, Δ, $\mathfrak{d}(\tau)$ and r might vary, however the representation $m+K$ is fixed by the 3-manifold $M_3(\mathcal{G})$. This representation can be irreducible or reducible. Therefore, we can associate to each manifold $M_3(\mathcal{G})$ a specific Weil representation. Explicit rules to associate a Weil representation to a given plumbing graph $\mathcal{G}$ are known for Seifert manifolds with trivial first homology group $H_1(M_3, \mathbb{Z})$, for other examples the representation can be inferred from (5.5.7) but no general rule is known at the moment. Some more details are provided in section §5.5.3 where we consider an example of Brieskorn homology spheres (Seifert manifolds with $|H_1(M_3, \mathbb{Z})|=1$). An example with non-trivial first homology group can also be found later in section §5.7.1.

As explained in section §5.2.2, close to the real line the false theta function[30] $\Psi_{m,r}$ (see equation (5.2.12)) inherits the transformation properties of the associated non-holomorphic Eichler integral and thus of the quantum modular form (5.1.8). Explicitly, we have the following identity [216]

$$\frac{1}{\sqrt{k}}\Psi_{m,r}\left(\frac{1}{k}\right) = \sum_{n\geq 0} \frac{c_n}{k^{n+1/2}}\left(\frac{\pi i}{2m}\right)^n + \sum_{r' \in \sigma^{m+K}} (\mathcal{S}P_m^-(m))_{rr'}\Psi_{m,r'}(-k)\,, \tag{5.5.8}$$

[29] The connection between false theta functions and 3-manifold invariants was already observed in many works, such as [68, 192, 193, 208, 216], the novelty here is the used of metaplectic representation.

[30] For simplicity of exposition we consider $\Psi_{m,r}$ instead of Ψ_r^{m+K}. Notice, however, that the relation is simply given by the projection operators, as explained in section §5.2.3.

where we have taken S^{m+K} to be given by $(\mathcal{S}P_m^-(m))$, since as described in section §5.2.3 $\Psi_{m,r}$ can be thought of as minimal folding with $K = 1$. The r component corresponds to the abelian flat connection a from equation (5.5.7), and $\Psi_r^{m+K}(-k)$ is explicitly given by

$$\Psi_{m,r}(-k) = \left(1 - \frac{r}{m}\right) e^{-2\pi i k \frac{r^2}{4m}}, \tag{5.5.9}$$

where the exponential is given by the entry of the matrix $\mathcal{T}^{m+K}$ defined in (5.2.19) and the coefficients are expressed through the zeta-function regularisation as

$$\sum_{n>0} (P_m^-(m))_{rn} = \lim_{t\to 1} \frac{t^r - t^{2m-r}}{1 - t^{2m}} = 1 - \frac{r}{m}, \tag{5.5.10}$$

Assuming that (5.5.7) holds, we can express the homological block $\widehat{Z}_a(q)$ as linear combinations of different $\Psi_{m,r}(\tau)$. Therefore, the left-hand side of equation (5.5.8) corresponds to a specific term which contributes to $\widehat{Z}_a(q)$ and which is evaluated at $q = e^{2\pi i/k}$, $k \in \mathbb{Z}_{>0}$. Furthermore, the first term on the right-hand side captures (part of) the perturbative expansion around the abelian flat connection, while the second term corresponds to (part of) the non-perturbative contribution from other irreducible flat connections. This interpretation follows from the trans-series expansion provided in (5.5.4) and (5.5.5). Note that the perturbative expansion corresponds to the asymptotic expansion of the false theta in (5.2.6) with $t = 2\pi/ik$.

The Borel transform of the perturbative contribution of $\Psi_{m,r}$ (see equation (5.2.12)) is [68]

$$B\Psi_{m,r}^{\text{pert}}(\xi) = \sum_{n\geq 0} \frac{c_n}{\Gamma(n+1/2)} \left(\frac{\pi i}{2m}\right)^n \xi^{n-1/2} = \frac{1}{\sqrt{\pi\xi}} \frac{\sinh[(m-r)\sqrt{\frac{2\pi i\xi}{m}}]}{\sinh[m\sqrt{\frac{2\pi i\xi}{m}}]}, \tag{5.5.11}$$

where the gamma function $\Gamma(n+1/2)$ is equal to $\frac{\sqrt{\pi}}{4^n}\frac{(2n)!}{n!}$. From the Borel resummation in [68] it was obtained the following identity

$$\frac{1}{\sqrt{k}}\Psi_{m,r}\left(\frac{1}{k}\right) = \frac{1}{2}\left(\int_{ie^{i\delta}\mathbb{R}_+} - \int_{ie^{-i\delta}\mathbb{R}_+}\right) \frac{d\xi}{\sqrt{\pi\xi}} \frac{\sinh[(m-r)\sqrt{\frac{2\pi i\xi}{m}}]}{\sinh[m\sqrt{\frac{2\pi i\xi}{m}}]} e^{-k\xi}, \tag{5.5.12}$$

which gives the exact expression for (5.5.8) The poles of the integrand are located at $\xi = 2\pi\frac{n^2}{4m}$ for $n \in \mathbb{Z}_{\geq 0}$. Moreover, the residues are given by

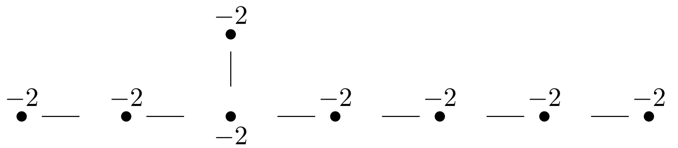
$$\text{Res}_{\xi=2\pi\frac{n^2}{4m}}\left(B\Psi_{m,r}^{\text{pert}}(\xi)\right) = -\frac{1}{\pi\sqrt{2m}}\sin\left(\frac{rn\pi}{m}\right) e^{-\pi i k\frac{n^2}{2m}} \tag{5.5.13}$$

where the factor $\frac{1}{\sqrt{2m}}\sin\left(\frac{rn\pi}{m}\right)$ corresponds to $\sqrt{-i}(\mathcal{S}P_m^-(m))_{rn}$. Instead of considering the infinite set of poles labeled by n, it is convenient to group them together by the form of the CS action $e^{-\pi i k \frac{n^2}{2m}}$ and thus label them by elements of the set σ^{m+K}. Thanks to equation (5.5.10), then we could equivalently take the residue to be given by $\sqrt{-i}(\mathcal{S}P_m^-(m))_{rr'}(1 - \frac{r'}{m})$ with $r' \in \sigma^{m+K}$. We postpone the interpretation of the integral expression (5.5.12) in terms of number theory objects to section §5.7.

To summarize, we consider the expression of the homological blocks of M_3 in terms of false theta functions. This allows us to extract the values of the CS invariants of non-abelian flat connections directly from (a subset of) the diagonal entries of $\mathcal{T}^{m+K}$, and specify the residues of the poles via $\mathcal{S}^{m+K}$ and the projection operators P^{m+K}. For Brieskorn homology spheres, this analysis allow to predict the number and type of non-abelian (real and complex) flat connections, as described in details in the next section. However, when $H_1(M_3, \mathbb{Z})$ is non-trivial it is necessary to consider the expression of the WRT invariant (5.4.2), or equivalently the $SU(2)$ Chern-Simons partition function, to make predictions on the non-abelian flat connections. The missing ingredient in the above treatment is, in fact, the sum over homological blocks that enter in the definition of $\tau[M_3]$ thorough $\widetilde{X}_{ab}$ or in other words the information about the $SL_2(\mathbb{Z})$ representation acting on abelian flat connections. A complete treatment of this more general case can be found in [3].

5.5.3 The Poincaré homology sphere

Consider the Poincaré homology sphere $\Sigma(2, 3, 5)$, specified by the Seifert data $(-1; 1/2, 1/3, 1/5)$ and plumbing graph



(5.5.14)

This corresponds to the E_8 Dynkin diagram with vertices labeled by the framing parameters. This example goes back to [190] and was later revisited by many authors such as [216], and [68]. The relevant (projective) $SL_2(\mathbb{Z})$ representation is given by $m + K = 30 + 6, 10, 15$. This is directly derived from the presentation $\Sigma(a, b, c)$ and the rule[31] $m = abc$ and $K = \{1, ab, bc, ac\}$. This irreducible representation has $\sigma^{m+K} = \{1, 7\}$ and thus dimension $d^{m+K} = 2$. The corresponding false

[31] This rule holds for any Brieskorn sphere, i.e. three singular fibers Seifert manifold with trivial integer first homology group. Further details are provided in [3].

theta functions are

$$\Psi_1^{30+6,10,15} = \Psi_{30,(1)+(11)+(19)+(29)}\,, \tag{5.5.15}$$

$$\Psi_7^{30+6,10,15} = \Psi_{30,(7)+(13)+(17)+(23)}\,. \tag{5.5.16}$$

where the right-hand side corresponds to a sum over false theta functions of the form (5.2.12). The T- and S-matrix are

$$\mathcal{T}^{30+6,10,15} = \begin{pmatrix} e\left(-\frac{1}{120}\right) & 0 \\ 0 & e\left(-\frac{7^2}{120}\right) \end{pmatrix}, \tag{5.5.17}$$

$$\mathcal{S}^{30+6,10,15} = \sqrt{i}\frac{2}{\sqrt{5}} \begin{pmatrix} \sin(\frac{\pi}{5}) & \sin(\frac{2\pi}{5}) \\ \sin(\frac{2\pi}{5}) & -\sin(\frac{\pi}{5}) \end{pmatrix}. \tag{5.5.18}$$

Since the integer first homology group $H_1(M_3, \mathbb{Z})$ is trivial, there is only one abelian flat connection and a corresponding homological block $\widehat{Z}_0(q)$, which in terms of the above false theta functions reads

$$\widehat{Z}_0 = 2q^{\frac{1}{120}} - \Psi_1^{30+6,10,15}(\tau) = 1 - q - q^3 - q^7 + q^8 + q^{14} + q^{20} + \ldots. \tag{5.5.19}$$

To extract the number and type of non-abelian flat connections, we have to check which are the terms that appear in the WRT invariant (5.4.2). The sum over a reduces in this case to a single term, and thus, the WRT invariant is directly proportional to $\widehat{Z}_0$. From equation (5.5.8), we see that there are $d^{m+K} = 2$ nAfcs which are labeled by $r \in \{1, 7\}$ and whose CS invariants is given by the exponent in the diagonal entries of the T-matrix. Specifically, we have two nAfcs with CS invariant $-r^2/4m$, that is to say $-1/120$ and $-7^2/120$. Among these nAfcs there might be complex connections. These should not be visible in the WRT invariant, and thus they have vanishing residue $(\mathcal{S}P_m^-(m))_{1r}(1-\frac{r}{m})$. The Poincaré homology sphere has no complex connection.

To describe the perturbative contribution originating from the false theta function $\Psi_1^{30+6,10,15}(q)$, we write

$$\Psi_1^{30+6,10,15}(q) = \sum_{n\geq 1} \chi_+(n) q^{n^2/120}\,, \qquad \chi_+(n) = \begin{cases} \left(\frac{12}{n}\right) & n \equiv 1(5) \\ -\left(\frac{12}{n}\right) & n \equiv 4(5) \\ 0 & \text{otherwise} \end{cases} \tag{5.5.20}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol.

Take $q = e^{2\pi i \tau}$ with $\tau = x + it$ and $x = 0$. The asymptotic expansion of this false theta function as t approaches 0 from above (or equivalently as k tends to infinity) is given by (5.2.6)

$$\sum_{n\geq 1} \chi_+(n) e^{-n^2 \frac{t}{120}} \sim L(-2r, \chi_+) \frac{1}{r!} \left(\frac{-t}{120}\right)^r, \tag{5.5.21}$$

where

$$L(-r,\chi_+) = -\frac{(60)^r}{r+1}\sum_{n=1}^{60}\chi_+(n)B_{r+1}\left(\frac{n}{60}\right) \quad (r=0,1,\dots)\,. \tag{5.5.22}$$

Therefore the asymptotic expansion of $\Psi_1^{30+6,10,15}(q)$ as q approaches e^{-t} is

$$\Psi_1^{30+6,10,15}(e^{-t}) \sim \sum_{r\geq 0}\frac{-(60)^r}{(2r+1)r!}\sum_{n=1}^{60}\chi_+(n)B_{2r+1}\left(\frac{n}{60}\right)\left(\frac{-t}{120}\right)^r \tag{5.5.23}$$

The coefficients in this expansion agree with the perturbative expansion around the abelian flat connection given in [68]. Moreover, if we define

$$c_n = \frac{-(60)^r}{(2n+1)n!}\sum_{\ell=1}^{60}\chi_+(\ell)B_{2n+1}\left(\frac{\ell}{60}\right) \tag{5.5.24}$$

and make use of the identity

$$\sum_{n\geq 0} c_n\frac{n!}{(2n)!}z^{2n} = \frac{\sinh(z)+\sinh(11z)+\sinh(19z)+\sinh(29z)}{\sinh(30z)} = 2\frac{\cosh(5z)\cosh(9z)}{\cosh(15z)} \tag{5.5.25}$$

we obtain the integrand expression for the resurgence integral (5.5.12).

The asymptotic expansion of $\Psi_1^{30+6,10,15}$ around $k\to\infty$ corresponds to a quantum modular form in the neighborhood of this root of unity,

$$Q(t) = \left(2+\frac{119}{60}t+\frac{129361}{14400}t^2+\frac{353851559}{5184000}t^3+\frac{1806970377121}{2488320000}t^4+\dots\right). \tag{5.5.26}$$

This shows an example of what we have described in section §5.2: The formal power series defining a quantum modular form at a root of unity can be obtained from the radial limit of the false theta function towards that root of unity. If we consider the limit $q\to 1$ (i.e. $k\to\infty$) the expansion describes the perturbative expansion around an abelian flat connection, which in this case corresponds to $a=0$.

5.6 Exploring the other side

In this section, we focus on the homological blocks $\widehat{Z}_a(q^{-1})$, $a\in H_1(M_3,\mathbb{Z})/\mathbb{Z}_2$, which correspond to the homological blocks for the 3-manifold $\overline{M}_3$. First, in section §5.6.1, we review what is their role in the construction of the superconformal index of $T[M_3]$. Later, in section §5.6.2, we conjecture that they can be expressed in terms of mock theta functions.

5.6.1 Superconformal index

In this section we take a closer look at the refined Witten index for the 3d $\mathcal{N} = 2$ theory $T[M_3]$ on $S^2 \times_q S^1$, the so-called superconformal index. Concretely, the superconformal index is defined as

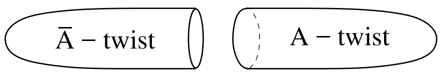
$$\mathcal{I}(q) := \mathrm{Tr}_{\mathscr{H}_{S^2}} (-1)^F q^{R/2+J_3} = Z_{T[M_3]}(S^2 \times_q S^1) \tag{5.6.1}$$

where J_3 is the generator of the $SO(3)$ isometry of S^2, R is the 3d R-symmetry, and $\mathscr{H}_{S^2}$ is the space of BPS states of the 3d $\mathcal{N} = 2$ SCFT. This index can be interpreted in terms of the $SL_2(\mathbb{C})$ Chern-Simons theory on M_3. Indeed, one of the statements of 3d-3d correspondence is that the partition function $Z_{T[M3]}(S^2 \times_q S^1)$ coincides with the partition function of complex Chern-Simons on M_3 with purely imaginary level [203]. As mentioned in the introduction to the thesis, the superconformal index $\mathcal{I}(q)$ was conjectured [25, 208] to be factorizable into two half-indices,

$$\mathcal{I}(q) = \sum_{a \in \mathrm{Tor} H_1(M_3,\mathbb{Z})/\mathbb{Z}_2} |\mathcal{W}_a| \widehat{Z}_a(q) \widehat{Z}_a(q^{-1}) \qquad \in \mathbb{Z}[[q]] \tag{5.6.2}$$

where $\widehat{Z}_a(q^{-1})$ is an appropriate extension of $\widehat{Z}_a(q)$ to the region $|q| > 1$. The conjectured factorization of this index into half-indices was inspired by similar results for holomorphic blocks [217, 218] and by the topological/anti-topological fusion [219]. Moreover, it resembles the factorization of the complex Chern-Simons theory on M_3 into a holomorphic and an anti-holomorphic piece [215, 218]. Note that the gluing defined by the above factorization can be interpreted as performing a topological twist along a $D^2 \times S^1$ and an anti-topological twist along the other $D^2 \times S^1$ [25]:

$$\mathcal{I}(q) = $$



(5.6.3)

Mathematically, the existence of the extension $\widehat{Z}_a(q^{-1})$ across the border $\mathrm{Im}(\tau) = 0$ is completely non-obvious, but from the physics perspective can be understood as a result of orientation reversal (parity) transformation on one of the hemispheres D^2 that upon gluing produce a 2-sphere S^2.

On the Chern-Simons side, it seems natural to take $\widehat{Z}_a(q^{-1})$ to be the homological block associated to the manifold $\overline{M}_3$ with opposite orientation, i.e.

$$\widehat{Z}_a(q^{-1}) \equiv \widehat{Z}_a(q)\Big|_{M_3 \to \overline{M}_3} \tag{5.6.4}$$

An intuitive reason why this should be the case is given by the CP-invariance[32] of the CS partition function.

Of particular interest is the case when the label a takes only one value (which we choose to be $a=0$), or in other words $H_1(M_3,\mathbb{Z})$ is trivial. This case relates to the Turaev-Viro invariant of M_3 as reviewed in [25]. Since there is only one boundary condition $\mathcal{B}_0$ and only one homological block $\widehat{Z}_0(q)$, the superconformal index becomes

$$\mathcal{I}(q) \;=\; |\mathcal{W}_0|\widehat{Z}_0(q)\,\widehat{Z}_0(q^{-1})\,. \tag{5.6.5}$$

Thanks to the results of the next section we can give a candidate q-series for $\mathcal{I}(q)$.

As we shall see in the next sections, the question about the extension of $\widehat{Z}_a(q)$ across the unit circle and, thus, the search for $\widehat{Z}_a(q^{-1})$ are deeply inter-related to number-theoretical properties of the original q-series (5.4.13). The latter, in turn, is determined by the physical properties of the combined 2d-3d system.

5.6.2 Mock modular forms

This story binds once more to the last letter Ramanujan wrote to Hardy [5]. As reviewed in chapter 2, he got attracted by certain q-series due to their unexpected asymptotic behavior towards roots of unity. The letter comprised several examples of q-series which converge inside the unit disc, these were called by the author mock theta functions. One might wonder what happens to the series if q is substituted by $1/q$. It turns out that in most of the examples the series, in the form of a q-hypergeometric series, still converges outside the unit circle and its extension relates to a false theta function. Problems of convergence are restricted to the unit circle and, in fact, poles are usually present at an infinite number of roots of unity.

Given a mock theta function, $h(q)$, it is therefore often possible to define its extension outside the unit circle as $h(1/q)$. As depicted below, for certain mock thetas the interesting extension does not arise only via the replacement of q by $1/q$ but also shifting by a constant or multiplying the transformed function by a certain q-power. Consider, for instance, the order 5 Ramanujan's mock theta function $\chi_0(q)$. This can be extended outside the unit circle by substituting $1/q$ in place of q

$$\chi_0(q)=\sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1})_n} \qquad \xrightarrow{q\to 1/q} \qquad \sum_{n=0}^{\infty}\frac{q^{-n}}{(q^{-n-1};q^{-1})_n}=\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{3}{2}n^2-\frac{1}{2}n}}{(q^{n+1})_n}$$

[32] The charge operator C acts on $\widehat{Z}_a(q)$ as $q\mapsto q^{-1}$, the parity P, instead, change the orientation of the 3-manifold.

where $(a)_n$ is a short-hand notation for the Pochhammer symbol $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$. The right-hand side of the above equation defines up to a constant $\widetilde{\chi}_0(q)$

$$\widetilde{\chi}_0(q) = 2 - \chi_0(1/q) = 2 - \sum_{n=0}^{\infty}(-1)^n \frac{q^{\frac{3}{2}n^2-\frac{1}{2}n}}{(q^{n+1})_n}. \tag{5.6.6}$$

This series can be written as [216]

$$\widetilde{\chi}_0(q) = 1 + q + q^3 + q^7 - q^8 - q^{14} - q^{20} + \dots \tag{5.6.7}$$

The series $\widetilde{\chi}_0(q)$ coincides with the false theta function $\Psi_1^{30+6,10,15}(q)$, defined in equation (5.7.4) in relation with the homological block for the Poincaré homology sphere. In [216] Hikami relates several Ramanujan's mock theta functions to false thetas that appear in the context of 3-manifold invariants: Starting from the q-hypergeometric series expression of the mock theta functions, he performs the inversion map $q \mapsto 1/q$ and rewrite the result in terms of a false theta function. Despite the simplicity of the operation $q \mapsto 1/q$, in certain cases this map fails to reproduce the expected object on the other side of the plane. In particular, from a mock modular form one might obtain a false theta function plus the quotient of a partial theta function by a modular form; whereas from different hypergeometric series expressions of the same false theta function one might get different mock modular forms which differ by a modular form. Possible ambiguities produced by this inversion map are explored in detail in [58]. This apparent ambiguity from one side to the other of the plane is partially resolved by the Rademacher sum approach (see section §5.8). Through this technique one can associate to a mock modular form a unique false theta function. Explicit q-series examples are reported in section §5.8.2. Before showing the explicit map from mock theta functions to false theta functions, we investigate the asymptotic expansions of mock theta functions close to rational points.

Denote by Γ the modular group of a mock modular form $h(\tau)$ of weight k and multiplier system ρ. Take $M^!_{k,\chi}(\Gamma)$, $M_{k,\chi}(\Gamma)$, $S_{k,\chi}(\Gamma)$ to be respectively the space of weakly holomorphic modular forms, holomorphic modular form and cusp forms of weight k and multiplier system χ on Γ. Denote the space of mock modular forms of weight k and multiplier system χ on Γ by $\mathbb{M}_{k,\chi}(\Gamma)$. If $h(\tau)$ belongs to $\mathbb{M}_{1/2,\chi}(\Gamma)$ and its shadow is a cusp form, then the completion $\widehat{h}(\tau)$ is a weight $1/2$ harmonic Maass form. All Ramanujan examples of mock theta functions fall into this category [6, 200].

Suppose Γ has $t > 1$ inequivalent cusps, $\{q_1, \dots, q_t\}$. Then $h(\tau)$ has exponential singularities at infinitely many rationals, [5]. Moreover, for every weight $1/2$ weakly holomorphic modular form $G(\tau)$, $h(\tau) - G(\tau)$ has exponential singularities at infinitely many rationals. The proof of this fact is presented in [54], based on the results of Bruinier and Funke [220]. Despite this fact, there is a collection $\{G_j(\tau)\}_{j=1}^t$ of weakly holomorphic modular forms such that $h(\tau)-G_j(\tau)$ is bounded towards all cusps equivalent to q_j, [55] (for half-integral weights and $\Gamma=\Gamma_0(N)$). The limiting value of the modified mock modular form $h(\tau)-G_j(\tau)$ defines a quantum modular forms $Q:\mathbb{Q}\to\mathbb{C}$ given by [55]

$$Q(x):=\lim_{t\to0^+}(h(\tau)-G_j(\tau))(x+it)$$

for each x equivalent to q_j. A linear injective map is given from the space of mock modular forms to the space of quantum modular forms $(Q_{k,\chi}(\Gamma))$. Then we have a linear injective map $\mu:\mathbb{M}_{k,\chi}(\Gamma)/M^!_{k,\chi}(\Gamma)\to Q_{k,\chi}(\Gamma)$ given by $\mu(f)=Q$.

On the other hand, if we focus on the shadow $g(\tau)$ of the mock modular form $h(\tau)$, we have the injective map $\nu:\mathbb{M}_{k,\chi}(\Gamma)/M^!_{k,\chi}(\Gamma)\to S_{2-k,\overline{\chi}}(\Gamma)$ given by $\nu(h)=g$. Notice that (independently of $G_j(\tau)$) we can associate the shadow to a mock theta function. In fact, for a given q_j we can choose $G_j(\tau)$ such that Q is determined by the shadow:

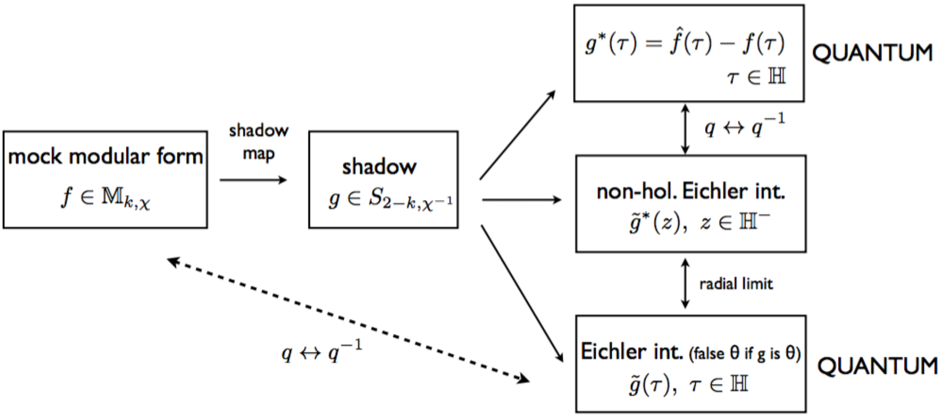
$$Q(x)=-\lim_{t\to0^+}g^*(x+it),$$

and moreover $Q(x+it)$ and $-g^*(x+it)$ has the same perturbative expansion for all cusps $x\in\Gamma q_j$, [3,55].

From the discussion in section §5.2.2 and in particular equation (5.2.10), we conclude that the holomorphic Eichler integral of $g(\tau)$, i.e. $\widetilde{g}(\tau)$, has the same asymptotic expansion as $h(\tau)-G_j(\tau)$, and therefore they define the same quantum modular form. The link between $\widetilde{g}(\tau)$ and $h(\tau)-G_j(\tau)$ is provided by the cusp form $g(\tau)$, which is nothing but the shadow of the mock modular form $h(\tau)$. The unary theta series of weight $3/2$ initially described in relation to false theta functions, in this section acquires the meaning of shadow of the mock modular form $h(\tau)$. We summarise the relation between these objects in Figure 5.6. Note that the $q\leftrightarrow q^{-1}$ line between mock and Eichler integral of its shadow is in dashed line, since the $q\leftrightarrow q^{-1}$ map is non-unique in both directions. See also [58] for more details.

The mock theta function $h(\tau)$ provides a natural candidate for the homological block $\widehat{Z}_a(q^{-1})$ for different reasons. First of all, it is given by a rational q-series (this has been proven for the class of examples called optimal mock theta function [64] to which we restrict in the next section), which up to an overall factor can always be written as an integral q-series. In addition, the quantum modular form defined from the limit of a false theta function can be reproduced as the limit of a mock theta function corrected by an appropriate modular form $G_j(\tau)$. In the following section we examine the limit of a mock theta function at a specific

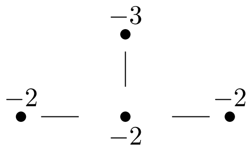
Figure 5.6: The relation between the different number-theoretic objects involved.



rational, where the mock theta function is bounded. This provides an example of the asymptotic expansion of a mock theta function whose associated modular form $G_j(\tau)$ vanishes.

5.7 An example

Consider the Seifert manifold $M_3 = M\left(-2; \frac{1}{2}, \frac{1}{3}, \frac{1}{2}\right)$, with surgery presentation $S^3_{8/1}(\mathbf{3}_1^r)$. This manifold is characterized by the following plumbing graph



(5.7.1)

or equivalently by the linking matrix

$$M = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix} \tag{5.7.2}$$

The finite abelian group $H_1(M_3, \mathbb{Z})/\mathbb{Z}_2$ has five independent elements and therefore 5 abelian flat connections. The Chern-Simons actions of abelian flat connections are:

$$CS[a] = 2\lambda(a,a) = \begin{cases} 0 \quad \text{mod} \quad \mathbb{Z} \quad \text{for} \quad a = (0,0,0,0),(1,-1,0,-1) \\ \frac{7}{8} \quad \text{mod} \quad \mathbb{Z} \quad \text{for} \quad a = (0,-1,0,0),(0,0,0,-1) \\ \frac{1}{2} \quad \text{mod} \quad \mathbb{Z} \quad \text{for} \quad a = (0,0,-1,0) \end{cases} .$$

where $a \in \mathbb{Z}_L/M\mathbb{Z}_L$. A general rule relating the pair (m,K) with the above plumbing graph is still under construction, more details are given in [3]. We observe that the homological blocks are given by

$$\widehat{Z}_{(1,-1,-1,-1)}(q) = q^{-5/12}(2q^{1/24} - \Psi_1^{6+2}) = q^{-3/8}(1 + q - q^2 + q^5 - q^7 + q^{12} + \ldots)$$
$$\widehat{Z}_{(3,-3,-1,-3)}(q) = -q^{-5/12}\Psi_1^{6+2} = q^{-3/8}(-1 + q - q^2 + q^5 - q^7 + q^{12} + \ldots)$$
$$\widehat{Z}_{(3,-1,-5,-3)}(q) = \widehat{Z}_{(3,-3,-5,-1)}(q) = -\frac{1}{2}q^{-5/12}\Psi_2^{6+2} = q^{-1/4}(-1 + q^4 - q^8 + q^{20} - q^{28} + q^{48} + \ldots)$$
$$\widehat{Z}_{(1,-1,-1,-3)}(q) = q^{-5/12}\Psi_4^{6+2} = 2q^{1/4}(1 - q^2 + q^{10} - q^{16} + q^{32} - q^{42} + \ldots)$$

and therefore the associated representation is $m + K = 6 + 2$. The labels of the homological block in the above equation correspond to elements of $(2\mathbb{Z}^L + \delta)/2M\mathbb{Z}^L$. In this case, the group $H_1(M_3,\mathbb{Z})$ is non-trivial and thus we have a non-trivial X-matrix, which in this basis reads

$$X_{ab} = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 2 & 2 & -2 \\ -2 & -2 & 2 & 2 & 2 \end{pmatrix} \tag{5.7.3}$$

Moreover, the trans-series contributions[33] of each flat connections is summarized in the table below. Note that there are two central flat connections with trivial CS invariant and two abelian flat connections which CS invariant 7/8.

CS action	type	trans-series
0	Afc (central)	$e^{2\pi i k\cdot 0}\left(\frac{\pi i}{4\sqrt{2}}k^{-3/2} + \frac{7\pi^2}{96\sqrt{2}}k^{-5/2} + \mathcal{O}(k^{-7/2})\right)$
$\frac{7}{8}$	Afc	$e^{2\pi i k\frac{7}{8}}\left(-\sqrt{2}k^{-1/2} + \frac{2\sqrt{2}\pi i}{3}k^{-3/2} + \mathcal{O}(k^{-5/2})\right)$
$\frac{1}{2}$	Afc	$e^{2\pi i k\frac{1}{2}}\left(-\frac{2\sqrt{2}}{3}k^{-1/2} - \frac{11\pi i}{54\sqrt{2}}k^{-3/2} + \mathcal{O}(k^{-5/2})\right)$
$-\frac{4}{24}$	nAfc (real)	$e^{-2\pi i k\frac{4}{24}}e^{\frac{3\pi i}{4}}2\sqrt{2}$

[33] We report the the trans-series contributions up to an overall factor of $\frac{-iq^{-5/12}}{2\sqrt{2}|\det M|^{1/2}}$.

5.7.1 On the false side

In the following we derive the perturbative expansion, and the quantum modular form related to

$$\Psi_1^{6+2}(q) = \sum_{n\geq 1} \left(\frac{-12}{n}\right) q^{n^2/24}\,, \tag{5.7.4}$$

where the symbol $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol. Similarly to the previous example, and following section §5.2, the asymptotic expansion of $\Psi_1^{6+2}(q)$ as q approaches e^{-t} and $t \searrow 0$ is

$$\Psi_1^{6+2}(e^{-t}) \sim \sum_{r\geq 0} L(-2r, C)\frac{(-t/4m)^r}{r!} = \sum_{r\geq 0} \frac{-(12)^r}{(2r+1)r!} \sum_{n=1}^{12} \left(\frac{-12}{n}\right) B_{2r+1}\left(\frac{n}{12}\right)\left(\frac{-t}{24}\right)^r \tag{5.7.5}$$

As depicted in section 5.1.2, we expect the asymptotic expansion of $\psi(q)$ to define a quantum modular form, that for $k \to \infty$ is given by

$$Q(t) = \frac{1}{2}\left(\frac{4}{3} + \frac{5}{54}t + \frac{17}{864}t^2 + \frac{455}{62208}t^3 + \frac{207913}{53747712}t^4 + \frac{378635}{143327232}t^5 + \dots\right). \tag{5.7.6}$$

We use the expansion in (5.7.5) to define [68]

$$c_n = \frac{-(12)^n}{(2n+1)n!} \sum_{\ell=1}^{12} \left(\frac{-12}{\ell}\right) B_{2n+1}\left(\frac{\ell}{12}\right), \tag{5.7.7}$$

the generating function of these coefficients is given by

$$\sum_{n\geq 0} c_n \frac{n!}{(2n)!} z^{2n} = \frac{\sinh(5z) - \sinh(z)}{\sinh(6z)} = \frac{\sinh(2z)}{\sinh(3z)}\,. \tag{5.7.8}$$

If we now consider the integral expression (5.5.12) for the false theta function $\Psi_1^{6+2}(q)$, we obtain

$$\Psi_1^{6+2}(q) = \frac{\sqrt{k}}{2}\left(\int_{ie^{i\delta}\mathbb{R}_+} - \int_{ie^{-i\delta}\mathbb{R}_+}\right)\frac{d\xi}{\sqrt{\pi\xi}}\frac{\sinh[2\sqrt{\frac{2\pi i\xi}{6}}]}{\sinh[3\sqrt{\frac{2\pi i\xi}{6}}]}e^{-k\xi} \tag{5.7.9}$$

$$= \frac{\sqrt{k}}{2}\left(\int_{e^{i\delta}\mathbb{R}_+} - \int_{e^{-i\delta}\mathbb{R}_+}\right)dy \frac{\sqrt{12i}}{\pi i}\frac{\sinh(2y)}{\sinh(3y)}e^{-\frac{6ky^2}{2\pi i}} \tag{5.7.10}$$

$$= \sqrt{\frac{24}{\pi}}\frac{1}{\sqrt{\hbar}}\frac{1}{4}\left(\int_{e^{i\delta}\mathbb{R}_+} - \int_{e^{-i\delta}\mathbb{R}_+}\right)dy \frac{\sinh(y)}{\sinh(3y/2)}e^{-\frac{3}{2}y^2/\hbar} \tag{5.7.11}$$

in the second line $y = \sqrt{\frac{2\pi i\xi}{m}}$ where $m = 6$. This corresponds to the asymptotic expansion of the non-holomorphic Eichler integral of the unary theta series associated to $\Psi_1^{6+2}(q)$. We will return to this point in section §5.7.2.

5.7.2 On the mock side

The mock theta function of interest here is Ramanujan's mock theta function $f(q)$

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - 5q^6 + 7q^7 + \dots \tag{5.7.12}$$

As mentioned in chapter 2, Ramanujan observed that any singularity of this function at even k-roots of unity disappears by subtracting a term of the form $(-1)^{k/2}b(q)$. Or in other words

$$f(q) - (-1)^{k/2}b(q) = O(1) \tag{5.7.13}$$

where $q^{-1/24}b(q)$ is a modular form and it is defined by

$$b(q) := (1-q)(1-q^3)(1-q^5)\dots(1-2q+2q^4-\dots)\,. \tag{5.7.14}$$

The mock theta function $f(q)$ defines the quantum modular form [55, 221]

$$Q(x) = \begin{cases} q^{-1/24}f(q) & q \text{ is an odd order root of unity} \\ q^{-1/24}(f(q)-b(q)) & q \text{ is a } (4k)\text{th-root of unity} \\ q^{-1/24}(f(q)+b(q)) & q \text{ is a } (4k+2)\text{th-root of unity} \end{cases} \tag{5.7.15}$$

where $q = e^{2\pi i\tau}$, $\tau = x + it$, and the right-hand side corresponds to the radial limit as q approaches $x \in \mathbb{Q}$. The perturbative expansion for $q \to 1$ ($x = 0$) is

$$Q(-t) = 4\left(\frac{1}{3} + \frac{5}{3^2}\left(-\frac{t}{24}\right) + \frac{153}{3^3 2!}\left(-\frac{t}{24}\right)^2 + \dots\right) = 4\sqrt{3}\sum_{n=0}^{\infty}\frac{\alpha_n}{n!}\left(\frac{3}{2}\right)^n(-t)^n\,. \tag{5.7.16}$$

Following [200], the perturbative expansion in (5.7.16) is derived by first considering a modular transformation of $f(q)$

$$q^{-1/24}f(q) = \frac{2}{\sqrt{-i\tau}}e^{-2\pi i/3\tau}\omega(e^{-\pi i/\tau}) + 4\sqrt{3}\sqrt{-i\tau}j_1(\tau) \tag{5.7.17}$$

where

$$\omega(q) := \sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}\,, \tag{5.7.18}$$

$$j_1(\tau) := \int_0^{\infty} e^{3\pi i\tau x^2}\frac{\sin(2\pi\tau x)}{\sin(3\pi\tau x)}dx \tag{5.7.19}$$

$$j_1(\tau) = (\sqrt{-i\tau})^{-1}j_2(-1/\tau)\,, \qquad j_2(\tau) = \int_0^{\infty} e^{3\pi i\tau x^2}\frac{\cos(\pi\tau x)}{\cos(3\pi\tau x)}dx \tag{5.7.20}$$

Define $t = -2\pi i\tau$, $t \in \mathbb{R}_{>0}$, then $q = e^{-t}$ and the modular transformation of $f(q)$ given in (5.7.17) takes the form

$$e^{t/24} f(e^{-t}) = \sqrt{\frac{8\pi}{t}} e^{-4\pi^2/3t} \omega(e^{-2\pi^2/t}) + \sqrt{\frac{24t}{\pi}} \int_0^\infty e^{-\frac{3}{2}tx^2} \frac{\sinh(tx)}{\sinh(\frac{3}{2}tx)} dx\,. \tag{5.7.21}$$

As t approaches 0 from above the first term in the r.h.s. decays exponentially and $f(q)$ can be approximated by the integral expression

$$q^{-1/24} f(q) \sim \sqrt{\frac{24t}{\pi}} \int_0^\infty e^{-\frac{3}{2}tx^2} \frac{\sinh(tx)}{\sinh(\frac{3}{2}tx)} dx = \sqrt{\frac{24}{\pi}} \frac{1}{\sqrt{t}} \int_0^\infty e^{-\frac{3}{2}y^2/t} \frac{\sinh(y)}{\sinh(\frac{3}{2}y)} dy, \tag{5.7.22}$$

where in the last equation $y = tx$. Using the modular transformation of $j_1(\tau)$ (5.7.20) we obtain

$$q^{-1/24} f(q) \sim 4\sqrt{3}\, \frac{2\pi}{t} \int_0^\infty e^{-3\pi i x^2/\tau} \frac{\cos(-\pi x/\tau)}{\cos(-3\pi x\tau)} dx \tag{5.7.23}$$

$$= 4\sqrt{3}\, \frac{2\pi}{t} \int_0^\infty e^{-6\pi^2 x^2/t} \frac{\cosh(2\pi^2 x/t)}{\cosh(6\pi^2 x/t)} dx \tag{5.7.24}$$

$$= 4\sqrt{3} \int_0^\infty e^{-\frac{3}{2}ty^2} \frac{\cosh(\pi y)}{\cosh(3\pi y)} dy \tag{5.7.25}$$

$$= 4\sqrt{3} \sum_{n\geq 0} \frac{1}{n!} \left(\frac{3}{2}\right)^n (-t)^n \int_0^\infty y^{2n} \frac{\cosh(\pi y)}{\cosh(3\pi y)} dy \tag{5.7.26}$$

in the second line we substituted $t = -2\pi i\tau$, while in the third line $y = 2\pi x/t$. This is exactly the asymptotic expansion given in equation (5.7.16) and it gives an explicit formula for the coefficients appearing in the asymptotic expansion in terms of $j_2(\tau)$. The latter can be related to the non-holomorphic Eichler integral of the vector-valued shadow $(g_0(z), g_1(z), g_2(z))$ as follows [200]

$$4\sqrt{3}(\sqrt{-i\tau})(j_2(\tau), -j_1(\tau), j_3(\tau)) = -2i\sqrt{3} \int_0^\infty \frac{(g_0(z), g_1(z), g_2(z))}{\sqrt{-i(z+\tau)}} dz \tag{5.7.27}$$

where

$$g_0(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n (n+1/3) e^{3\pi i(n+1/3)^2\tau} \tag{5.7.28}$$

$$g_1(\tau) = -\sum_{n\in\mathbb{Z}} (n+1/6) e^{3\pi i(n+1/6)^2\tau} \tag{5.7.29}$$

$$g_2(\tau) = \sum_{n\in\mathbb{Z}} (n+1/3) e^{3\pi i(n+1/3)^2\tau} \tag{5.7.30}$$

The veto $(g_0(z), g_1(z), g_2(z))$ defines the shadow of the mock modular form

$$\{q^{-1/24} f(q),\, 2q^{1/3}\omega(q^{1/2}),\, 2q^{1/3}\omega(-q^{1/2})\}\,. \tag{5.7.31}$$

The relation between the coefficients in the expansion of the quantum modular form in (5.7.16) and the non-holomorphic Eichler integral of $g(\tau)$ (the unary theta series) was first brought to light in [200].

Comparing (5.7.6) with (5.7.16), we observe that the false theta function given by $2\Psi_1^{6+2}(q)$ defines the same quantum modular form as $q^{-1/24}f(q)$. The same conclusion can be reached by comparing the integral expressions in (5.7.11) with (5.7.26).

5.8 The Rademacher approach

In section §5.8.1, we extend the construction of Rademacher sums, first introduced in chapter 2, to functions defined on the other side of the plane. As anticipated in the previous section, these are false theta functions. Through this method we obtain a neat map between false theta functions and mock theta functions, when the latter are examples of optimal mock theta functions, i.e. they have minimal possible growth in their coefficients. In section §5.8.2, we display this technique with an explicit example related to the Poincaré homology sphere.

5.8.1 A false Rademacher sum

In this section the main question is how to extend the Rademacher series to the lower half of the plane. Similarly to the case investigated first by Rademacher, and described in the introduction, we are interested in the extension of a function, naturally defined inside the unit disc, to the outside the unit disc. However, this time we do not have a modular form but a mock modular form. Surprisingly, the expansion of zero principle produces a non-vanishing function on the other side of the plane that in the case of a mock theta function of weight 1/2 is a false theta function.

The expansion of zero principle was later reviewed and extended to the context of mock modular forms (harmonic Maass forms to be precise) by Rhoades [222]. For further references on this topic see [55,58,61,223]. Here we describe how to connect the Rademacher sum of a weight 1/2 mock theta function to an explicit formula for the Fourier coefficients of a false theta function. The discussion is restricted to the examples of mock theta functions that comprise the space of optimal mock theta functions analyzed in [64]. The optimality condition[34] translates into the

[34] This is the same condition that makes the non-linear sigma model of K3 surfaces an extremal (non-chiral) conformal field theory.

requirement that each component of the vector-valued mock theta function as $\mathrm{Im}(\tau) \to \infty$ satisfies

$$h_r(\tau) = O(q^{-1/4m}), \qquad \text{for each } r \bmod 2m, \tag{5.8.1}$$

where m corresponds to the label of the Weil representation described in §5.2.3. In other words, only $h_1(\tau)$ has a pole of the form $q^{-1/4m}$, while all the others components have only positive q-powers in their Fourier expansion. The base of the space of optimal mock Jacobi form was proven to be labeled by genus zero subgroups of $SL_2(\mathbb{R})$ [64].

Following [222], we show how to define the Rademacher sum of a false theta function starting from the expression of the Rademacher sum for a mock theta function. To ease the notation in the following derivation we restrict to scalr-valued Rademacher sums and then generalize the result to the vector-valued case. The Rademacher series given in (2.5.20) can be specialized to the case of a scalar-valued weight 1/2 mock modular form as

$$\begin{aligned}\mathcal{R}^{(n)}_{\Gamma,1/2,\rho}(\tau) &= q^n + \sum_{\substack{k>0\\ k\in\mathbb{Z}+\mu}} \alpha_{\Gamma,1/2,\rho}(n,k) q^k \\ &= q^n + \sum_{c>0}\sum_{0<d\leq c} \frac{2\pi}{c} e\Big(n\frac{a}{c} + \mu\frac{d}{c}\Big)\, n^{1/2} q^{\mu}\, \rho(\gamma)^{-1} \times \\ &\qquad \times \sum_{\substack{k>0\\ k\in\mathbb{Z}}} e\Big(k\frac{d}{c} + k\tau\Big)\big(-(k+\mu)n\big)^{-1/4} I_{1/2}\Big(\frac{4\pi}{c}\sqrt{-(k+\mu)n}\Big)\end{aligned} \tag{5.8.2}$$

where for simplicity we assume $\mathfrak{r} = 0$. Define by $G_{c,d}(\tau)$ the first factor in the summation over c and d

$$G_{c,d}(\tau) := \begin{cases} \frac{2\pi}{c}\, e\big(n\frac{a}{c} + \mu\frac{d}{c}\big)\, n^{1/2} q^{\mu}\, \rho(\gamma)^{-1} & c>0\,, \\ q^n & c=0\,. \end{cases} \tag{5.8.3}$$

Moreover, we can express the Bessel function as

$$t^{\frac{-1}{4}} I_{1/2}\Big(\frac{4\pi}{c}\sqrt{t}\Big) = \frac{i^{-1/2}}{t^{1/4}} \sum_{m\geq 0} \frac{(-1)^m}{m!\Gamma(m+\frac{3}{2})}\left(\frac{i}{2}\Big(\frac{4\pi}{c}\sqrt{t}\Big)\right)^{2m+1/2} \tag{5.8.4}$$

$$= \frac{1}{2\pi i}\int_{|s|=r} ds\, e^{s\,t} \sum_{m\geq 0} \frac{1}{s^{m+1}} \frac{1}{\Gamma(m+\frac{3}{2})}\left(\frac{2\pi}{c}\right)^{2m+1/2}, \tag{5.8.5}$$

where the integral converges for any s as long as r is sufficiently small. Thanks to this integral representation and the properties of convergence of (5.8.2), the last line of equation (5.8.2) can be written as

$$\sum_{k>0} \widetilde{q}^{k} \big(-(k+\mu)n\big)^{-\frac{1}{4}} I_{1/2}\Big(\frac{4\pi}{c}\sqrt{-(k+\mu)n}\Big) = \tag{5.8.6}$$
$$\frac{1}{2\pi i}\int_{|s|=r} ds\, \frac{\widetilde{q}\, e^{-n(1+\mu)s}}{1-\widetilde{q}\, e^{-ns}} \sum_{m\geq 0} \frac{1}{s^{m+1}} \frac{1}{\Gamma(m+\frac{3}{2})} \Big(\frac{2\pi}{c}\Big)^{2m+1/2}$$

where $\widetilde{q} := e\big(\frac{d}{c}+\tau\big)$. Denote by $f_{c,d}(\tau;s)$ the above integrand expression, that is

$$f_{c,d}(\tau;s) := \frac{\widetilde{q}\, e^{-n(1+\mu)s}}{1-\widetilde{q}\, e^{-ns}} \sum_{m\geq 0} \frac{1}{s^{m+1}} \frac{1}{\Gamma(m+\frac{3}{2})} \Big(\frac{2\pi}{c}\Big)^{2m+1/2} \tag{5.8.7}$$

when c is strictly positive, and it is otherwise given by $f_{0,d}(\tau;s) = 1$.

The whole Rademacher expression can be succinctly written as

$$F(\tau) := \frac{1}{2\pi i}\int_{|s|=r} ds \sum_{c>0}\sum_{0<d\leq c} G_{c,d}(\tau)\, f_{c,d}(\tau;s)\,. \tag{5.8.8}$$

The convergence of this expression both in the upper and the lower half-plane is guaranteed by the definition of $f_{c,d}(\tau;s)$. Note that $G_{c,d}(\tau)$ has no issue of convergence. Indeed, the only term that has to be carefully expanded is the first term in the definition of $f_{c,d}(\tau;s)$, which is nothing but the geometric series

$$\frac{\widetilde{q}\, e^{-n(1+\mu)s}}{1-\widetilde{q}\, e^{-ns}} = \begin{cases} \sum_{k>0} \widetilde{q}^{k}\, e^{-n(k+\mu)s}\,, & |\widetilde{q}\, e^{-ns}| < 1 \\ \sum_{k\geq 0} \widetilde{q}^{-k}\, e^{ns(k-\mu)}\,, & |\widetilde{q}\, e^{-ns}| > 1 \end{cases} \tag{5.8.9}$$

By expanding $F(\tau)$ in the upper half-plane, that is using the first condition, we recover (5.8.2). In particular, inside the unit circle $f_{c,d}(\tau;s)$ takes the form

$$f^{in}_{c,d}(\tau;s) := \sum_{k>0} \widetilde{q}^{k}\, e^{-n(k+\mu)s} \sum_{m\geq 0} \frac{1}{s^{m+1}} \frac{1}{\Gamma(m+\frac{3}{2})} \Big(\frac{2\pi}{c}\Big)^{2m+1/2} \tag{5.8.10}$$

and therefore

$$\mathcal{R}^{(n)}_{\Gamma,\,1/2,\,\rho}(\tau) = \frac{1}{2\pi i}\int_{|s|=r} ds \sum_{c>0}\sum_{0<d\leq c} G_{c,d}(\tau)\, f^{in}_{c,d}(\tau;s)\,. \tag{5.8.11}$$

Alternatively, to define a function outside the unit circle we use the second equation in (5.8.9) and obtain

$$\mathcal{F}^{(n)}_{\Gamma,\,1/2,\,\rho}(\tau) = \frac{1}{2\pi i}\int_{|s|=r} ds \sum_{c>0}\sum_{0<d\leq c} G_{c,d}(\tau)\, f^{out}_{c,d}(\tau;s)\,, \tag{5.8.12}$$

where

$$f_c^{out}(\tau; s) := -\sum_{k>0} \widetilde{q}^{-k}\, e^{ns(k-\mu)} \sum_{m\geq 0} \frac{1}{s^{m+1}} \frac{1}{\Gamma(m+\frac{3}{2})} \left(\frac{2\pi}{c}\right)^{2m+1/2}. \tag{5.8.13}$$

In our class of examples the expression $\mathcal{F}^{(n)}_{\Gamma,\,1/2,\,\rho}(\tau)$ turns out to be a false theta function.

In terms of the Rademacher series (5.8.2), the coefficients of the mock modular form $\mathcal{R}^{(n)}_{\Gamma,\,1/2,\,\rho}(\tau) = q^n + \sum_k \alpha_{\Gamma,1/2,\rho}(n,k) q^k$ are

$$\alpha_{\Gamma,\rho,w}(n,k) = \sum_{c>0} \sum_{0<d<c} \frac{2\pi}{c}\, e\Big(n\frac{a}{c} + k\frac{d}{c}\Big) \rho(\gamma)^{-1} \Big(-\frac{k}{n}\Big)^{-\frac{1}{4}} I_{1/2}\Big(\frac{4\pi}{c}\sqrt{-kn}\Big), \tag{5.8.14}$$

where $k \in \mathbb{Z} + \mu$. Whereas in the case of $\mathcal{F}^{(n)}_{\Gamma,\,1/2,\,\rho}(\tau) = q^n + \sum_k \widetilde{\alpha}_{\Gamma,1/2,\rho}(n,k) q^k$ the coefficients are

$$\widetilde{\alpha}_{\Gamma,1/2,\rho}(n,k) = -\sum_{c>0} \sum_{0<d<c} \frac{2\pi}{c}\, e\Big(n\frac{a}{c} - k\frac{d}{c}\Big) \rho(\gamma)^{-1} \Big(\frac{k}{n}\Big)^{-\frac{1}{4}} I_{1/2}\Big(\frac{4\pi}{c}\sqrt{kn}\Big) \tag{5.8.15}$$

where $k \in \mathbb{Z} - \mu$. The difference between these two expressions is accounted for by a change in sign in k and μ and an overall sign.

The above derivation is easily generalizable to vector-valued Rademacher sums. Indeed, one can prove that the formula for the k_j q-power of the j-entry of the vector-valued false theta, whose Rademacher has polar term n_i is

$$\begin{aligned} \widetilde{\alpha}_{\Gamma,\rho,w}(n_i,k_j) = \sum_{c>0} \sum_{0<d<c} \frac{-2\pi}{c}\, e\Big(n_i\frac{a}{c} - k_j\frac{d}{c}\Big) \rho(\gamma)^{-1}_{ji} \times \\ \times \Big(-\frac{k_j}{n_i}\Big)^{\frac{w-1}{2}} J_{1-w}\Big(\frac{4\pi}{c}\sqrt{-k_j n_i}\Big) \end{aligned} \tag{5.8.16}$$

with $k_j \in \mathbb{Z} - \mu_j$.

The relation between weight 1/2 mock modular forms and their lower-half plane companion, $\mathcal{F}^{(n)}_{\Gamma,\,1/2,\,\rho}(\tau)$, can be analyzed from a different perspective following [89]. By definition 5.2.2 a false theta function is the Eichler integral of weight 3/2 cusp form, which is the shadow of the mock modular form. The Rademacher expression for this modular form is derived from the Rademacher expression of the mock modular form via Zagier and Eichler dualities [89]. The Eichler integral of the Rademacher sum of this cusp form directly reproduces equation (5.8.16). Therefore, one can think of the relation between a mock theta function and false theta function as based on the unary theta series of weight 3/2.

Notice that the components of vector-valued mock modular forms and false theta functions can be expressed as scalar-valued Rademacher sums. The scalar-valued Rademacher sum corresponding to the ℓ-th component is derived from the vector-valued expression by restricting the multiplier system to a single entry of the matrix defining the above multiplier and choosing the correct modular group. Due to the presence of a single pole in the first component of the mock modular form, one has to restrict to the $(\ell, 1)$-entry of the multiplier system of the vector-valued Rademacher sum. Therefore, denoting as $\psi_\ell(\gamma)$ the $(\ell, 1)$-entry of the matrix $\rho(\gamma)^{-1}$, we obtain

$$\begin{aligned} \mathcal{R}^{(n)}{}_{\Gamma,\psi_\ell,1/2}(\tau) = \delta_{\ell,1} q^n + \sum_{c>0} \sum_{0<d<c} \frac{2\pi}{c} e\Big(n\frac{a}{c} + \mu_\ell \frac{d}{c}\Big) \psi_\ell(\gamma) \times \\ \times \sum_{\substack{k_\ell > 0 \\ k_\ell \in \mathbb{Z}}} e\Big(k_\ell \frac{d}{c}\Big) \left(-\frac{k_\ell + \mu_\ell}{n}\right)^{-1/4} q^{k_\ell + \mu_\ell} I_{1/2}\Big(\frac{4\pi}{c}\sqrt{-(k_\ell + \mu_\ell)n}\Big). \end{aligned} \tag{5.8.17}$$

This allows recovering the false theta expression corresponding to the ℓ-component via (5.8.12). A concrete example is displayed below.

5.8.2 An example

The relation between the false theta function and the mock theta function of $M(-1; 1/2, 1/3, 1/2)$ can be found in the original work by [222]. Here we explain what happens in the case of the Poincaré homology sphere.

Similarly to section 5.5.3, if we consider $\Sigma(2, 3, 5)$ the set of irreducible representation is labeled by $\sigma^{30+6,10,15} = \{1, 7\}$. Therefore, we have a two component vector labeled by the elements of $\sigma^{30+6,10,15}$. Through equation (5.8.17), we can restrict to the first component of the Rademacher sum which can be expressed as independent scalar-valued ones with $\ell = 1$ and the multiplier system specified by the $(\ell, 1)$-entry of the matrices conjugated to (5.5.17) and (5.5.18). The first terms in the q-series of the two mock modular form are given by

$$\mathcal{R}^{(-1/120)}_{SL_2(\mathbb{Z}),\psi_1,1/2}(\tau) = q^{-1/120} + q^{119/120} + q^{239/120} + 2q^{359/120} + \ldots = q^{-1/120}\chi_0(q)$$

This corresponds to the first component of the vector-valued mock modular form that appear in umbral moonshine in relation with the Niemeier lattice E_8^3. The corresponding false theta function is

$$\mathcal{F}^{(-1/120)}_{SL_2(\mathbb{Z}),\psi_1,1/2}(\tau) = -\frac{3}{2}\big(q^{1/120} + q^{121/120} + q^{361/120} + \ldots\big) = -\frac{3}{2} q^{1/120} \widetilde{\chi}_0(q)$$

where $\widetilde{\chi}_0(q)$ is defined in (5.6.7) and the factor of $3/2$ is due to the choice of constant term in front of the shadow.

5.9 Conclusion

In this chapter, we investigate the role played by number theory in the context of 3-manifold invariants and the physics of the 3d-3d correspondence. The work restricts to a class of 3-manifold that are Seifert 3-manifold with three singular fibers. For this class of 3-manifold M_3 we have a prescription for computing the homological blocks of negative definite and indefinite linking forms, under the assumption that the value of the inverse linking matrix, in the diagonal entry corresponding to the high valency vertex, is positive. We show that this element governs the growth of the series, which is unbounded from below when the diagonal entry is negative. We observe that the homological blocks are given in this class of examples by false theta functions.

A new interpretation of the resurgence analysis of the Chern-Simons theory on M_3 performed in [68, 214] is given in this section. Our viewpoint sheds some new light on the invariants of Seifert manifolds by emphasizing the relation with number-theoretic objects such as false theta functions and projective representations of $SL_2(\mathbb{Z})$. The different ingredients of the resurgence analysis are described in terms of the T- and S-matrix spanning the Weil representation of $Mp_2(\mathbb{Z})$ and the projection operators folding the representation into a smaller dimensional representation.

This new perspective on the resurgence analysis of analytically continued Chern-Simons theory on M_3 leads to interesting predictions of certain topological data of M_3. In this chapter we restricted to the case of Brieskorn homology spheres, with a particular example the Poincaré homology sphere, more general results are reported in [3]. For general M_3 multiple topological data are governed by the combination of the Weil representation attached to M_3 with the $SL_2(\mathbb{Z})$ representation acting on abelian flat connections. This allows to efficiently extract topological data of non-abelian flat connections of 3-manifold invariants.

By restricting to a specific class of Seifert 3-manifolds, we can produce candidates for the homological blocks of $\overline{M}_3$, the companions of the homological blocks defined for M_3. The latter correspond to a suitable extension of the homological blocks of M_3 to the lower half of the plane. The map between one half of the plane and the other is implemented via the Rademacher expansion of an optimal mock theta function and a false theta function. Specifically, we generalize the treatment in [222] to the examples of optimal mock theta functions that appear in this context. This technique provides a neat map between the two sides of the plane which primarily depends on the shadow of the mock modular form. This leads to a realization of certain examples of the famous Ramanujan's mock theta functions as well as certain umbral moonshine mock theta functions in the context of 3-manifold invariants. Moreover, by identifying a natural candidate for $\widehat{Z}_a(q^{-1})$, we are able to infer an integral q-series expression for the superconformal index of the 3d $\mathcal{N} = 2$ theory $T[M_3]$.

We conclude with some comments and open questions that arose from our work:

- Further investigation is needed to systematically associate a Weil representation to any M_3. A topological or a geometrical perspective on these projective representations of $SL_2(\mathbb{Z})$ might provide new clues on the procedure hinging back to the form of a generic plumbing graph $\mathcal{G}$.
  More striking, the presence of various $SL_2(\mathbb{Z})$ representations in the study of homological blocks of $T[M_3]$ appears to play a fundamental role that hints at a deeper structure not yet discovered.

- Challenges remain in the definition of the homological blocks of $\overline{M}_3$ for mock theta functions that do not belong to the space of optimal mock theta functions. The map between the homological blocks of M_3 and the ones of $\overline{M}_3$ for more general Seifert manifolds with three singular fibers would be a non-trivial extension that needs further investigation.

- The integral expression appearing in the Borel resummation procedure corresponds to the asymptotic expansion of the fast theta function close to the real line, which thanks to the analytic property of Ψ^{m+K} can be extended to the upper half-plane. On the other hand, this integral corresponds to the asymptotic limit of the mock theta function once the divergences at rational points have been removed. Questions remain on the physical interpretation of the "modular correction" introduced to cut out the singularities of mock theta functions at the different cusps.

- It would be desirable to have an independent computation of the homological blocks, for instance through a localization computation for the 3d-2d coupled system, to prove or disprove our conjecture. An interesting aspect of the localization integral is that it can be directly related to the complex integral defined in the analysis of the Hardy-Littlewood-Ramanujan circle method described in chapter 2.

- On a different note, a direct extension of this work would be to consider Seifert examples with more than three singular fibers. A similar story, even if more involved from a number theory perspective, should hold for 4-singular fiber examples. We plan to add more on this in the near future.

- Another natural extension of our results is the treatment of the Chern-Simons theory for higher rank gauge groups.

- One more case merits mentioning here. A number-theoretical structure has been already discovered in the context of the Volume conjecture, which relates the specific asymptotic behaviors of the colored Jones polynomial of a knot $\mathcal{K}$ with the hyperbolic volume of the knot complement $S^3\backslash\mathcal{K}$, or, in its refined version, with complex Chern-Simons theory on the complement of the knot, see [224] and references therein. This physical system was related to Nahm's conjecture on the modularity of hypergeometric series [225], which identifies when a q-hypergeometric series is modular through the Bloch group and rational conformal field theories, see [226] for a proof of one arrow of the conjecture. It would be fascinating to find a similar conjecture for the q-hypergeometric series corresponding to mock theta functions that appear in this context.
- Concluding, as previously pointed out, there is a connection with umbral moonshine functions that would be interesting to unravel in the future.

Appendix A

A.1 A geometric view on elliptic functions and modular forms

The aim of this Appendix is twofold: firstly, to provide an intuitive and geometric description of the moduli space of elliptic curves, together with an account of functions and forms defined on this space; secondly, to introduce several modular objects used in the main text. The description of modular groups follows the book [81]. The brief introduction to modular forms follows the classical references [41, 227].

A.1.1 Modular groups

Denote by $\mathcal{H}$ the upper half-plane

$$\mathcal{H} := \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}\,,$$

The action of a generic element γ of the special linear group $SL_2(\mathbb{R})$ on the upper half-plane (or more generally on the complex plane) is given by fractional linear transformations

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}\,, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,. \tag{A.1.1}$$

This action stabilizes the upper, the lower half-plane and the real line $\mathbb{R} \cup \{\infty\}$and maps $\mathbb{R} \cup \{i\infty\}$ into itself. Notice that only $PSL_2(\mathbb{R})$, that is the quotient group $SL_2(\mathbb{R})/\pm\mathbb{I}$, acts faithfully on the Riemann sphere $(\mathbb{C} \cup \{i\infty\})$. Since the transformation in (A.1.1) preserves the angles it is also known as conformal mapping.

Elements of $PSL_2(\mathbb{R})$ are denominated differently depending on the number of fixed point in $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$: parabolic ($\gamma$ has one fixed point in $\mathbb{R} \cup \{\infty\}$), hyperbolic

(γ has two distinct fixed points on $\mathbb{R} \cup \{\infty\}$) or elliptic ($\gamma$ has one fixed point in $\mathcal{H}$ and the conjugate one in the lower half-plane). Clearly, this definition is invariant under conjugation in the group and therefore applies to the conjugacy class of γ. A point $x \in \mathbb{R} \cup \{\infty\}$ is called a cusp if there exists a parabolic element γ that satisfies $\gamma x = x$. Throughout the thesis when we refer to a cusp at x we mean the orbit of x under the action of Γ. The quotient $\mathcal{F} := \Gamma\backslash\mathcal{H}$ is the so-called fundamental domain. We obtain the compactified fundamental domain $\overline{\mathcal{F}}$ if we include the cusps of Γ: the compactified fundamental domain is defined as $\overline{\mathcal{F}} := \Gamma\backslash\overline{\mathcal{H}}$, where $\overline{\mathcal{H}} = \mathcal{H} \cup \{\text{cusps}\}$.

A subgroup $\Gamma < SL_2(\mathbb{R})$ is discrete if the induced topology is discrete, where the metric topology of the fundamental group is induced by the norm $|\gamma| = a^2 + b^2 + c^2 + d^2$. A group Γ acts discontinuously if the stabilizer group $\Gamma_\tau = \{\gamma \in \Gamma \,|\, \gamma\tau = \tau\}$ is finite for any $\tau \in \mathbb{H}$. A subgroup of $SL_2(\mathbb{R})$ acting discontinuously on the upper half-plane is a so-called Fuchsian group. A subgroup of $SL_2(\mathbb{R})$ is discrete if and only if it acts discontinuously on $\mathcal{H}$, for more details we refer to [81]. We restrict to Fuchsian groups of the first kind. In this case, every point on the boundary of the upper half-plane, i.e. $\mathbb{R} \cup \{\infty\}$, is a limit (in the $\mathbb{C}$-topology) of an orbit $\Gamma\tau$ for some $\tau \in \mathcal{H}$ (see e.g. [81]). In other words, a Fuchsian group of the first kind is a discrete subgroup of $SL_2(\mathbb{R})$ whose compactified fundamental domain $\overline{\mathcal{F}}$ is a compact Riemann surface.

In the following, we introduce different modular groups that we encounter in the main text. The modular group $SL_2(\mathbb{Z})$ is the discrete subgroup of $SL_2(\mathbb{R})$ generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{A.1.2}$$

To prove this statement one can follow the steps of the continued fraction expansion of $\frac{a}{c}$: Starting from a general matrix γ, we can apply T^n from the left to obtain

$$T^n\gamma = \begin{pmatrix} a + cn & b + dn \\ c & d \end{pmatrix} \tag{A.1.3}$$

if $c \neq 0$ this reduces the first entry of the first row to $0 \leq a < |c|$ by suitable choice of $n \in \mathbb{Z}$. Applying then S

$$S\gamma = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \tag{A.1.4}$$

we interchange the rows and flip the signs of the last row. Therefore by repeatedly applying these two operations we end up with an upper triangular matrix with 1's on the diagonal. This can be reduced to $\mathbb{I}$, the identity matrix, by first acting with a certain power of T and then, if necessary, by changing the overall sign via S^2.

An example of a normal subgroup of the modular group $SL_2(\mathbb{Z})$ is the principal congruence group of level N

$$\Gamma(N) := \left\{ \gamma \in SL_2(\mathbb{Z}) \middle| \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,(\mathrm{mod}\, N) \right\}. \tag{A.1.5}$$

Any subgroup of the modular group which contains $\Gamma(N)$ is called a congruence subgroup of level N. We denote by $\Gamma_0(N)$ the Hecke congruence subgroup of level N,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \det(\gamma) = 1, \quad c \in \mathbb{Z} \right\}. \tag{A.1.6}$$

The Atkin-Lehner involution for $\Gamma_0(N)$ is

$$W_e = \left\{ \begin{pmatrix} ae & b \\ cN & de \end{pmatrix} \in GL_2(\mathbb{Z}) \middle| \det(\gamma) = e,\, e||N \right\}, \tag{A.1.7}$$

where $||$ denotes that e is an exact divisor of N, i.e. e divides N, $e|N$, and $\left(e, \frac{N}{e}\right) = 1$. Moreover, the set of matrices W_{e_i} satisfies

$$W_e^2 = 1 \,\mathrm{mod}(\Gamma_0(N)), \tag{A.1.8}$$

$$W_{e_1} W_{e_2} = W_{e_2} W_{e_1} = W_{e_3} \,\mathrm{mod}(\Gamma_0(N)), \qquad e_3 = \frac{e_1 e_2}{(e_1 e_2)^2}. \tag{A.1.9}$$

An important example of Atkin-Lehner involution is the so-called Fricke involution W_N, which generates the transformation $\tau \to -1/N\tau$.

Next, we introduce the modular group $\Gamma_0(n|h)$, defined by

$$\Gamma_0(n|h) = \left\{ \begin{pmatrix} a & \frac{b}{h} \\ cn & d \end{pmatrix} \middle| \det(\gamma) = 1 \right\} \tag{A.1.10}$$

where $a, b, c, d \in \mathbb{Z}$, $h \in \mathbb{Z}$, $h^2|N$ and $N = nh$. For h the largest divisor of 24, $\Gamma_0(n|h)$ is a subgroup of the normalizer group $\mathcal{N}(N)$ (defined below). The corresponding Atkin-Lehner involution is

$$w_e = \left\{ \begin{pmatrix} ae & \frac{b}{h} \\ cN & de \end{pmatrix} \middle| \det(\gamma) = e,\, e||\frac{n}{h} \right\}; \tag{A.1.11}$$

this satisfies a closure condition similar to equation (A.1.9) for W_e with respect to $\Gamma_0(n|h)$ instead of $\Gamma_0(N)$. The normalizer group of $\Gamma_0(N)$ in $SL_2(\mathbb{R})$ is

$$\mathcal{N}(N) = \{\rho \in SL_2(\mathbb{R}) | \rho \Gamma_0(N) \rho^{-1} = \Gamma_0(N)\}. \tag{A.1.12}$$

$\mathcal{N}(N)$ is generated by $\Gamma_0(n|h)$ and its Atkin-Lehner involutions. For an explicit description of the normalizer group, the reader is referred to [67].

The groups Γ_g with $g \in \mathbb{M}$ are subgroups of $\mathcal{N}(N)$ of the form $\Gamma_0(n|h) + e_1, e_2, ..$ where $n = o(g)$ is the order of g, $h|24, h|n$ and $N = nh$. Here $\Gamma_0(n|h) + e_1, e_2, ..$ stands for the union of a particular set of Atkin-Lehner involutions $(w_{e_1}, w_{e_2}, ..)$ and $\Gamma_0(n|h)$. From this description, it is apparent that Γ_g is a subgroup of $\mathcal{N}(N)$ and contains $\Gamma_0(N)$.

Lastly, we define the group Γ_θ,

$$\Gamma_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c - d \equiv a - b \equiv 1 \pmod 2 \right\}, \tag{A.1.13}$$

whose Hauptmodul is

$$K(\tau) = \left(\frac{\eta^2(\tau)}{\eta\left(\frac{\tau}{2}\right)\eta(2\tau)} \right)^{24} = q^{-1/2} + 24 + 276q^{1/2} + \dots \tag{A.1.14}$$

A.1.2 Elliptic functions and modular forms

The following section gives a geometric interpretation of elliptic functions and modular forms restricting to elliptic curves over $\mathbb{C}$ and the modular group $SL_2(\mathbb{Z})$. The discussion is primarily based on [41, 227, 228]. At the end of this section we give the definition of a vector-valued modular form for more general modular groups and multiplier systems.

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be the lattice (free abelian group over $\mathbb{C}$) spanned by the oriented basis (ω_1, ω_2). To each lattice Λ is associated an elliptic curve $E = \mathbb{C}/\Lambda$, that is to say a Riemann surface of genus one endowed with an abelian group action. An elliptic curve is defined up to homotheties $\Lambda \to \lambda\Lambda\,,\ \lambda \in \mathbb{C}^*$; therefore, rescaling the lattice by a non-zero scalar produces an elliptic curve isomorphic to the original one. The lattice rescaled by $\lambda = \omega_2^{-1}$, for instance, gives rise to the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$, where $\tau = \omega_1/\omega_2$, which is isomorphic to E. In terms of elliptic curves, the transformation (A.1.1) amounts to a change of basis and rescale of the lattice Λ and, hence, from the elliptic curve E_τ it produces an isomorphic curve $E_{\gamma\tau}$. According to the above, the quotient space $\Gamma\backslash\mathcal{H}$ (the fundamental domain of Γ) represents the moduli space of elliptic curves over $\mathbb{C}$; each point of $\Gamma\backslash\mathcal{H}$, in fact, corresponds to an isomorphism class of elliptic curves.

A function $f : \mathbb{C} \to \mathbb{C}$ is elliptic with respect to Λ if it is a meromorphic function on $\mathbb{C}$ and it is periodic with periods in Λ

$$f(z + \omega) = f(z) \quad \forall \omega \in \Lambda\,. \tag{A.1.15}$$

Alternatively, $f(z)$ can be viewed as a meromorphic function on the torus $\mathbb{C}/\Lambda$.

An example of a non-constant elliptic function is the Weierstrass function

$$\wp(z) = u^{-2} + \sum_{\omega\in\Lambda}{}' ((z-\omega)^{-2} - \omega^{-2})\,. \tag{A.1.16}$$

This is an even periodic function with one pole of order two at the origin and residue zero. The periodicity is built in the function by the average over lattices, where the sum excludes the point $w = 0$. The importance of this function is partially given by the fact that it generates the subfield of all even elliptic functions in $\mathcal{E}(\Lambda)$. Together with its derivative $\wp'(z)$, it generates the field of all elliptic functions for a lattice Λ, i.e. $\mathbb{C}(\wp, \wp') = \mathcal{E}(\Lambda)$. From the definition of Weierstrass function

$$\wp(z) = u^{-2} + \sum_{m\geq 1} (m+1)\Big(\sum_{\omega\in\Lambda}{}' \omega^{-(m+2)}\Big) u^m\,. \tag{A.1.17}$$

A modular function is a complex-valued function defined on the quotient space $\Gamma\backslash\mathcal{H}$, alternatively it can be viewed as a function of lattices which is invariant under rescaling of the lattice and thus it satisfies $F(\lambda\Lambda) = F(\Lambda)\,,\ \forall\lambda\in\mathbb{C}$. A generalization of this concept is provided by modular forms. Let $F(\lambda\Lambda)$ be a homogenous function of degree $-k$ of a lattice Λ

$$F(\lambda\Lambda) = \lambda^{-k} F(\Lambda)\,,\ \forall\lambda\in\mathbb{C}\,. \tag{A.1.18}$$

Rescaling the lattice λ, we obtain the definition of a modular form[1] $f : \mathcal{H} \to \mathbb{C}$

$$f(\tau) := \omega_2^k\, F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)\,, \qquad \tau\in\mathcal{H}\,. \tag{A.1.19}$$

The functional equation satisfied by $f(\tau)$ is derived from the invariance of $F(\Lambda)$ under a change of basis and it is given by

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)\,. \tag{A.1.20}$$

The function $f(\tau)$ is a modular form of weight k with respect to $SL_2(\mathbb{Z})$.

A vector-valued modular form of weight $k\in\mathbb{Z}$, multiplier system ρ with respect to the group Γ is a function $\varphi : \mathcal{H} \to \mathbb{C}$ obeying the functional equation

$$\varphi(\gamma\tau) = j^{2k}(\gamma,\tau)\rho(\gamma).\varphi(\tau), \qquad \forall\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma, \quad \tau\in\mathcal{H}, \tag{A.1.21}$$

[1]The function $f(\tau)$ is called a modular form because $f(\tau)d^{-k/2}\tau$ is a holomorphic $(-k/2)$-form on the space $SL_2(\mathbb{Z})\backslash\mathcal{H}$; by contrast, a modular function $f(\tau)$ is a meromorphic function on the space $SL_2(\mathbb{Z})\backslash\mathcal{H}$.

where the dot corresponds to matrix multiplication. Here $j^2(\gamma,\tau)$ denotes the automorphic factor $(c\tau + d)$ and the multiplier system is a map $\rho : \Gamma \to SU(d)$ in the case of a vector-valued modular form of dimension d. If we restrict to a scalar-valued modular form $\rho : \Gamma \to S^1$. The slash-operator for a modular form is defined as

$$\varphi(\tau)|_k = j^{-2k}(\gamma,\tau)\varphi(\gamma\tau)\,. \tag{A.1.22}$$

Alternatively, one can use the following definition

$$\varphi(\tau)|_{k,\rho} = j^{-2k}(\gamma,\tau)\rho(\gamma)^{-1}.\varphi(\gamma\tau)\,. \tag{A.1.23}$$

A.2 Modular miscellaneous

We report here the definitions of several modular objects that are used throughout the thesis. First, we recall the definition of the Dedekind eta function,

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n)\,, \tag{A.2.1}$$

which is a modular form of weight 1/2 under $SL(2,\mathbb{Z})$ and has a multiplier system

$$\rho_\eta(\gamma) := \exp\Big[i\pi \sum_{\mu\,(\mathrm{mod}\,k)} \Big(\Big(\frac{\mu}{k}\Big)\Big)\Big(\Big(\frac{h\mu}{k}\Big)\Big)\Big]\,, \tag{A.2.2}$$

with $\gamma = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix} \in SL(2,\mathbb{Z})$, $hh' \equiv -1 \pmod k$, and

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R}\backslash\mathbb{Z}\,, \\ 0 & \text{if } x \in \mathbb{Z}\,. \end{cases} \tag{A.2.3}$$

We also introduce the Eisenstein series, which are defined as

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \frac{n\, q^n}{1-q^n}\,, \tag{A.2.4}$$

$$E_4(\tau) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3\, q^n}{1-q^n}\,, \tag{A.2.5}$$

$$E_6(\tau) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5\, q^n}{1-q^n}\,. \tag{A.2.6}$$

We define the Jacobi theta functions $\theta_i(\tau,z)$ as follows,

$$\theta_1(\tau,z) := \sum_{n\in\mathbb{Z}} (-1)^n\, q^{\frac{1}{2}(n-\frac{1}{2})^2}\, y^{n-\frac{1}{2}}\,, \tag{A.2.7}$$

$$\theta_2(\tau,z) := \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-\frac{1}{2})^2}\, y^{n-\frac{1}{2}}\,, \tag{A.2.8}$$

$$\theta_3(\tau,z) := \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}\, y^{n}\,, \tag{A.2.9}$$

$$\theta_4(\tau,z) := \sum_{n\in\mathbb{Z}} (-1)^n\, q^{\frac{n^2}{2}}\, y^{n}\,. \tag{A.2.10}$$

A fundamental object in our discussion is the weight 1/2 index m theta series, whose components are defined by

$$\theta_{m,r}(\tau,z) = \sum_{\substack{k\in\mathbb{Z}\\ k\equiv r \ (\mathrm{mod}\ 2m)}} q^{k^2/4m}\, y^k\,, \tag{A.2.11}$$

when $m\in\mathbb{Z}_{>0}$ and are otherwise given by

$$\theta_{m,r}(\tau,z) = \sum_{k=r \ (\mathrm{mod}\ 2m)} \mathrm{e}(\tfrac{k}{2})\, q^{k^2/4m} y^k, \tag{A.2.12}$$

for half-integer index m, and with $2m\in\mathbb{Z}_{>0}$ and $r-m\in\mathbb{Z}$. The modular properties of the theta series are dictated by its transformation under the generators of the modular group $SL(2,\mathbb{Z})$ (see equation (A.1.2)) and are thus represented by

$$\theta_{m,r}(\tau+1,z) = \rho(T)_{r,r'}\,\theta_{m,r'}(\tau,z)\,, \tag{A.2.13}$$

$$\theta_{m,r}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e\left(\frac{mz^2}{\tau}\right)\sqrt{-i\tau}\,\rho(S)_{r,r'}\,\theta_{m,r'}(\tau,z)\,, \tag{A.2.14}$$

where the $2m$-dimensional matrices $\rho(S)$ and $\rho(T)$ define its multiplier system. For $m\in\mathbb{Z}$, these take the form

$$\rho(T)_{r,r'} = e\left(\frac{r^2}{4m}\right)\delta_{r,r'}\,, \qquad \rho(S)_{r,r'} = \frac{1}{\sqrt{2m}}e\left(-\frac{rr'}{2m}\right), \tag{A.2.15}$$

whereas for $m\in\frac{1}{2}\mathbb{Z}$,

$$\rho(T)_{r,r'} = e\left(\frac{r^2}{4m}\right)\delta_{r,r'}\,, \qquad \rho(S)_{r,r'} = \frac{1}{\sqrt{2m}}e\left(-\frac{rr'}{2m}\right)e\left(\frac{r-r'}{2}\right). \tag{A.2.16}$$

A more precise description of theta functions with integer index m is given in section §2.2.3.

A.2.1 Hecke-like operators

We define three Hecke-like operators [8] which are needed to obtain the polar coefficients of the mixed mock modular forms appearing in section §4.3.

The first is an operator which sends a (mock) Jacobi form $\varphi(\tau, z)$ to $\varphi(\tau, sz)$,

$$U_s : \sum_{n,\ell} c(n,\ell)\, q^n\, y^\ell \quad \mapsto \quad \sum_{n,\ell} c(n,\ell)\, q^n\, y^{s\ell}\,, \tag{A.2.17}$$

or, in terms of its action on the Fourier coefficients of $\varphi(\tau, z)$,

$$c(\varphi|U_s\,;\, n, \ell) = c(\varphi\,;\, n, \ell/s)\,, \tag{A.2.18}$$

with the convention that $c(n, \ell/s) = 0$ if $s \not| \ \ell$.
The second operator sends a (mock) Jacobi form of weight w and index m to one of weight w and index tm and is denoted $V_{w,t}$, and its action on the Fourier coefficients is

$$c(\varphi|V_{w,t}\,;\, n, \ell) = \sum_{d|(n,\ell,t)} d^{w-1}\, c\Big(\varphi\,;\, \frac{nt}{d^2}, \frac{\ell}{d}\Big)\,. \tag{A.2.19}$$

Finally, we define a combination of these two operators, which also sends a (mock) Jacobi form of weight w and index m to one of weight w and index tm and is given by

$$\mathcal{V}^{(m)}_{w,t} = \sum_{\substack{s^2|t \\ (s,m)=1}} \mu(s)\, V_{w,t/s^2}\, U_s\,, \tag{A.2.20}$$

where $\mu(s)$ is related to the Möbius function, $\mu(s) = s\,\mu_M(s)$. We have in particular the values $\mu(1) = 1$ and $\mu(2) = -2$.

Appendix B

B.1 Holomorphic orbifolds

In this section we briefly review aspects of holomorphic orbifold CFTs which are relevant to chiral CFTs with a discrete symmetry group. Denote by $\phi(\tau)$ the partition function of a chiral CFT with Hilbert space $\mathcal{H}$ and central charge c,

$$\phi(\tau) = \mathrm{Tr}_{\mathcal{H}}(q^{L_0-c/24}), \tag{B.1.1}$$

where L_0 represents the Virasoro generator. The above partition function corresponds to a path integral on a torus with complex structure parameter τ and periodic boundary conditions along the two cycles. Given an automorphism group G of the theory, it is possible to define twining functions

$$\phi_g(\tau) = \mathrm{Tr}_{\mathcal{H}}(g\, q^{L_0-c/24}), \qquad \forall g \in G \tag{B.1.2}$$

where the g-insertion stands for the representation of the element g acting on the Hilbert space of the theory. Moreover, one can build the invariant subspace with respect to the action of g by defining a projection operator, $\mathcal{P}$, whose action for an element of order n is

$$\mathrm{Tr}_{\mathcal{H}}(\mathcal{P}\, q^{L_0-c/24}) = \frac{1}{n}\sum_{i=0}^{n-1} \mathrm{Tr}_{\mathcal{H}}(g^i\, q^{L_0-c/24}), \tag{B.1.3}$$

This is the first step in the construction of an orbifold partition function.

Additionally, one must include states arising from the g-twisted sectors, i.e.

$$\phi_{e,g}(\tau) = \mathrm{Tr}_{\mathcal{H}_g}(q^{L_0-c/24}), \qquad \forall\, g \in G\,. \tag{B.1.4}$$

The latter are defined as traces over twisted Hilbert spaces, $\mathcal{H}_g$, which consist of states defined modulo a g-transformation. Throughout we denote by e the identity element of the group under consideration. Analogously, on the torus twisting and twining correspond to changing the boundary conditions along one of the cycles of the torus. Thus, we are led to define a twisted-twined function, whose boundary conditions along the two cycles are dictated by elements of the group G. From a Hamiltonian approach, the twisted-twined function is defined as

$$\phi_{g,h}(\tau,z) = \mathrm{Tr}_{\mathcal{H}_h}(g\, q^{L_0-c/24}), \qquad g \in C_G(h),\ h \in G\,. \tag{B.1.5}$$

Since the action on the spectrum is well defined so long as g and h commute, the twining element g belongs to the centralizer of h in G, $C_G(h) = \{g \in G | gh = hg\}$. In the case of chiral CFTs these functions are class functions up to a phase. In order to obtain a consistent orbifold one has to impose certain constraints which prevent anomalous phases from appearing under modular transformations which fix the boundary conditions.

Different twisted-twined functions can be related to each other by modular transformations. In fact, $\phi_{g,h}$ satisfies the following functional equation

$$\phi_{g,h}(\gamma\tau) = \rho_{g,h}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \phi_{h^b g^d, h^a g^c}(\tau), \qquad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{g,h}\,, \tag{B.1.6}$$

defining a modular function with multiplier system ρ with respect to the modular group $\Gamma_{g,h}$ which fixes the pair (g,h).

The complete $\langle h\rangle$-orbifold partition function therefore takes the form

$$\phi_{orb}(\tau) = \frac{1}{|C_G(h)|} \sum_{[h]} \sum_{g \in C_G(h)} \phi_{g,h}(\tau), \tag{B.1.7}$$

where the first sum is over representatives of the conjugacy classes of h, and the second sum is over elements commuting with h.

Examples of holomorphic orbifolds are the ones obtained from the monster CFT, coined by Norton as *Generalized moonshine*. Before considering their properties in the next section, we generalize the above concepts to superconformal field theories.

A similar reasoning can be applied to SCFTs with a non-trivial current algebra. Instead of focusing on its partition function, we consider the elliptic genus (EG). The latter is defined for an $\mathcal{N} = 2$ SCFT by

$$\psi(\tau,z) = \mathrm{Tr}_{\mathcal{H}}((-1)^F q^{L_0-c/24} \overline{q}^{L_0-c/24} y^{J_0}) \tag{B.1.8}$$

where z is the $U(1)$-chemical potential and $y = e(z)$. Once again the modular properties of $\psi(\tau,z)$ and its twisted-twined companion $\psi_{g,h}(\tau,z)$ can be used to define the EG of the orbifolded theory, which this time depends on the two variables

τ and z. Under a modular transformation $\gamma \in \Gamma_{g,h}$, $\psi_{g,h}(\tau, z)$ transforms as a weight 0 index m Jacobi form

$$\psi_{g,h}(\gamma\tau, \gamma z) = e\Big(\frac{mcz^2}{c\tau + d}\Big)\, \psi_{h^b g^d, h^a g^c}(\tau, z)\,. \tag{B.1.9}$$

B.2 Mock and meromorphic Jacobi forms

The first instance of mock Jacobi form we consider is the so-called Appell–Lerch sum, defined as

$$f_u^{(m)}(\tau, z) = \sum_{k\in\mathbb{Z}} \frac{q^{mk^2} y^{2mk}}{1 - yq^k e^{-2\pi i u}}. \tag{B.2.1}$$

Its completion, following [6], is

$$\widehat{f}_u^{(m)}(\tau, \overline{\tau}, z) = f_u^{(m)}(\tau, z) - \frac{1}{2} \sum_{r\in\mathbb{Z}/\mathbb{Z}} R_{m,r}(\tau, u)\theta_{m,r}(\tau, z) \tag{B.2.2}$$

with

$$R_{m,r}(\tau, u) = \sum_{k\equiv r \pmod{2m}} \left(\mathrm{sgn}(k + \tfrac{1}{2}) - E\left(\sqrt{\frac{\mathrm{Im}\tau}{m}}\left(k + 2m\frac{\mathrm{Im}u}{\mathrm{Im}\tau}\right)\right)\right) q^{-\frac{k^2}{4m}} e^{-2\pi i k u},$$

$$E(z) = \mathrm{sgn}(z)\left(1 - \int_{z^2}^{\infty} dt\, t^{-1/2}\, e^{-\pi t}\right).$$

Moreover, $\widehat{f}_u^{(m)}(\tau, \overline{\tau}, z)$ transforms as a (non-holomorphic) Jacobi form of weight 1 and index m.

We denote by $\mu_{m;0}(\tau, z) = f_0^{(m)}(\tau, -z) - f_0^{(m)}(\tau, z)$. This specialization of the Appell–Lerch sum has the following relation to the modular group $SL_2(\mathbb{Z})$: let the (non-holomorphic) completion of $\mu_{m;0}(\tau, z)$ be

$$\widehat{\mu}_{m;0}(\tau, \overline{\tau}, z) = \mu_{m;0}(\tau, z) - \frac{1}{\sqrt{2m}} \sum_{r\in\mathbb{Z}/2m\mathbb{Z}} \theta_{m,r}(\tau, z) \int_{-\overline{\tau}}^{i\infty} \big(i(\tau' + \tau)\big)^{-1/2}\, \overline{S_{m,r}(-\overline{\tau}')}\, d\tau'. \tag{B.2.3}$$

Then $\widehat{\mu}_{m;0}$ transforms like a Jacobi form of weight 1 and index m for $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and it has a simple pole at $z = 0$. Here $S_m = (S_{m,r})$ is the vector-valued cusp form for $SL_2(\mathbb{Z})$ whose components are given by the unary theta series

$$S_{m,r}(\tau) = \frac{1}{2\pi i} \frac{\partial}{\partial z} \theta_{m,r}(\tau, z)\Big|_{z=0}. \tag{B.2.4}$$

Note that the explicit form of the theta series $S_{m,r}(\tau)$ changes depending on whether m is integer or half-integer because of equations (A.2.11), (A.2.12).

For later use, we define two weight one meromorphic Jacobi forms, $\Psi_{1,1}$ of index one, defined as

$$\Psi_{1,1}(\tau,z) = -i\,\frac{\theta_1(\tau,2z)\,\eta(\tau)^3}{(\theta_1(\tau,z))^2} = \frac{y+1}{y-1} - (y^2 - y^{-2})q + \cdots\,, \tag{B.2.5}$$

and $\Psi_{1,-\frac{1}{2}}$ of index $-\frac{1}{2}$, defined as

$$\Psi_{1,-\frac{1}{2}}(\tau,z) = -i\,\frac{\eta(\tau)^3}{\theta_1(\tau,z)} = \frac{1}{y^{1/2}-y^{-1/2}} + q\,(y^{1/2} - y^{-1/2}) + O(q^2).$$

B.3 Superconformal characters and modules

In this section we review the representation theory and character formulas of the $\mathcal{N}=4$, $\mathcal{N}=2$, and Spin(7) SCAs.

B.3.1 Characters of the Spin(7) algebra

Here we briefly review the representation theory of the $\mathcal{SW}(3/2,2)$ superconformal algebra with central charge 12—this is the algebra which arises on the worldsheet of type II string theory compactified on a manifold of Spin(7) holonomy [131].[1] This algebra is an extension of the $c=12$ $\mathcal{N}=1$ SCA by two additional generators: the stress-energy tensor of a $c=1/2$ Ising model (of dimension 2) and its superpartner (of dimension 5/2).

In [229] the unitary representations of the $\mathcal{SW}(3/2,2)$ SCA were classified. There are two algebras—NS and R—which correspond to whether the fermions are 1/2-integer (NS) or integer (R) graded. For our purposes it suffices to work in the NS sector; here the representations are uniquely specified by two quantum numbers and will be labeled $|a,h\rangle$, where a is the dimension of the internal Ising factor, $a\in\{0,1/16,1/2\}$, and the total dimension is h. The result is that there are three massless (BPS) representations with quantum numbers $|0,0\rangle$, $|1/16,1/2\rangle$, and $|1/2,1\rangle$, and two continuous families of massive (non-BPS) representations with quantum numbers $|0,n\rangle$, and $|1/16,1/2+n\rangle$, where $n\geq 1/2$.

Conjectural characters for each of these representations were computed in [130], to which we refer for more details and derivations, including a discussion of the characters in the Ramond sector. We define the following combination of functions

$$\widetilde{\theta}_{m,r}(\tau) = \theta_{m,r}(\tau) + \theta_{m,r-m}(\tau) \tag{B.3.1}$$

[1] Throughout this section, we follow the notation used in [121].

which satisfies $\widetilde{\theta}_{m,r} = \widetilde{\theta}_{m,-r} = \widetilde{\theta}_{m,r+m}$, $\widetilde{\theta}_{m,r}(\tau) = \theta_{m/2,r}(\tau/2)$, and

$$\widetilde{f}_u^{(m)}(\tau, z) = f_u^{(m)}(\tau, z) - f_u^{(m)}(\tau, -z). \tag{B.3.2}$$

We denote the character of the a non-BPS representation $|a, h\rangle$ by $\chi_{a,h}^{NS}(\tau)$, and the characters of the BPS representations by $\widetilde{\chi}_a^{NS}(\tau)$, as they are uniquely specified by their a eigenvalue. The result is that the non-BPS characters are given by

$$\chi_{0,h}^{NS}(\tau) = q^{h-\frac{49}{120}}\,\mathcal{P}(\tau)\,\Theta_0^{NS}(\tau) = q^h\,(q^{-1/2} + 1 + q^{1/2} + 3\,q + \dots) \tag{B.3.3}$$

and

$$\chi_{\frac{1}{16},h}^{NS}(\tau) = q^{h-\frac{61}{120}}\,\mathcal{P}(\tau)\,\Theta_{\frac{1}{16}}^{NS}(\tau) = q^h\,(q^{-1/2} + 2 + 3\,q^{1/2} + 5\,q + \dots) \tag{B.3.4}$$

where

$$\mathcal{P}(\tau) = \frac{\eta^2(\tau)}{\eta^2(\frac{\tau}{2})\eta^2(2\tau)}\,,$$

and we have defined

$$\Theta_0^{NS}(\tau) = \left(\widetilde{\theta}_{30,2}(\tau) - \widetilde{\theta}_{30,8}(\tau)\right), \tag{B.3.5}$$

$$\Theta_{\frac{1}{16}}^{NS}(\tau) = \left(\widetilde{\theta}_{30,4}(\tau) - \widetilde{\theta}_{30,14}(\tau)\right). \tag{B.3.6}$$

Furthermore, the BPS character of total dimension $h = \frac{1}{2}$ is given by

$$\widetilde{\chi}_{\frac{1}{2}}^{NS}(\tau) = \mathcal{P}(\tau)\,\mu^{NS}(\tau)\,, \tag{B.3.7}$$

where,

$$\mu^{NS}(\tau) = \left(q^{\frac{5}{8}}\,\widetilde{f}^{(5)}_{\frac{\tau}{2}+\frac{1}{2}}(6\tau, \tau) + q^{\frac{25}{8}}\,\widetilde{f}^{(5)}_{\frac{\tau}{2}+\frac{1}{2}}(6\tau, -2\tau)\right), \tag{B.3.8}$$

and the other two BPS characters can be found using the BPS relations which relate massless and massive characters:

$$\widetilde{\chi}_0^{NS} + \widetilde{\chi}_{\frac{1}{16}}^{NS} = q^{-n}\chi_{0,n}^{NS}\,, \qquad \widetilde{\chi}_{\frac{1}{16}}^{NS} + \widetilde{\chi}_{\frac{1}{2}}^{NS} = q^{-n}\chi_{\frac{1}{16},\frac{1}{2}+n}^{NS}\,. \tag{B.3.9}$$

Spin(7) modules

The partition function for a module of the Spin(7) superconformal algebra, i.e.

$$\mathcal{Z}_{\mathrm{NS}}^{\mathrm{Spin}(7)}(\tau) = \mathrm{Tr}_{NS}\,q^{L_0 - c/24}, \tag{B.3.10}$$

transforms as a weight zero modular function for the congruence subgroup Γ_θ. Furthermore, it follows from the explicit description of the Spin(7) characters above that such a function admits an expansion of the form

$$\mathcal{Z}_{\mathrm{NS}}^{\mathrm{Spin}(7)}(\tau) = \mathcal{P}(\tau)\left(A_0\mu^{NS}(\tau) + F_{\frac{1}{16}}(\tau)\Theta_{\frac{1}{16}}^{NS}(\tau) + F_0(\tau)\Theta_0^{NS}(\tau)\right) \tag{B.3.11}$$

where we can expand the function (F_j) as

$$F_{\frac{1}{16}}(\tau) \quad = \sum_{n\geq 0} b_j(n) q^{n-1/120} \tag{B.3.12}$$

$$F_0(\tau) \quad = \sum_{n\geq 0} c_j(n) q^{n-49/120}. \tag{B.3.13}$$

From the properties of the Appell–Lerch sums detailed in section B.2, it follows that $\underline{F} := (F_j)$ is a weight 1/2 vector-valued mock modular form for $SL_2(\mathbb{Z})$ with shadow given by $A_0\widetilde{\underline{S}}(\tau)$, where we have defined

$$\widetilde{\underline{S}} = \begin{pmatrix} S_1 \\ S_7 \end{pmatrix} \tag{B.3.14}$$

and $\widetilde{S}_\alpha(\tau) = \sum_{k\in\mathbb{Z}} k\epsilon^R_\alpha(k) q^{k^2/120}$ for $\alpha = 1, 7$ and

$$\epsilon^R_1(k) = \begin{cases} 1 & k = 1, 29 \pmod{60} \\ -1 & k = -11, -19 \pmod{60} \\ 0 & \text{otherwise} \end{cases} \tag{B.3.15}$$

$$\epsilon^R_7(k) = \begin{cases} 1 & k = -7, -23 \pmod{60} \\ -1 & k = 17, 13 \pmod{60} \\ 0 & \text{otherwise} \end{cases}. \tag{B.3.16}$$

See [121] for more details.

B.3.2 $\mathcal{N} = 2$ superconformal characters

The $\mathcal{N} = 2$ SCA with central charge $c = 3(2\ell + 1) = 3\hat{c}$, $\ell \in \frac{1}{2}\mathbb{Z}$, contains an affine $\widehat{u(1)}$ current algebra of level $\ell + \frac{1}{2}$. In this notation $m = \ell + \frac{1}{2}$. The unitary irreducible highest weight representations are labeled by the eigenvalues of L_0 and J_0, which we call h and Q, respectively [230, 231], and which we denote by $\mathcal{V}^{\mathcal{N}=2}_{\ell;h,Q}$. There are $2\ell + 1$ massless (BPS) representations with eigenvalues $h = \frac{c}{24} = \frac{\hat{c}}{8}$ and $Q \in \{-\frac{\hat{c}}{2} + 1, -\frac{\hat{c}}{2} + 2, \ldots, \frac{\hat{c}}{2} - 1, \frac{\hat{c}}{2}\}$, whereas there are $2\ell + 1$ continuous families of massive (non-BPS) representations with eigenvalues $h > \frac{\hat{c}}{8}$ and $Q \in \{-\frac{\hat{c}}{2} + 1, -\frac{\hat{c}}{2} + 2, \ldots, \frac{\hat{c}}{2} - 2, \frac{\hat{c}}{2} - 1, \frac{\hat{c}}{2}\}$, $Q \neq 0$.

We focus on the graded characters in the Ramond sector, which are defined as

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;h,Q}(\tau, z) = \mathrm{tr}_{\mathcal{V}^{\mathcal{N}=2}_{\ell;h,Q}} \left((-1)^{J_0} y^{J_0} q^{L_0 - c/24} \right). \tag{B.3.17}$$

In terms of functions in section B.2, the massive characters are

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;h,Q}(\tau, z) = \mathrm{e}(\tfrac{\ell}{2}) (\Psi_{1,-\frac{1}{2}}(\tau, z))^{-1} q^{h - \frac{c}{24} - \frac{j^2}{4\ell}} \theta_{\ell,j}(\tau, z), \quad j = \mathrm{sgn}(Q)\,(|Q| - 1/2), \tag{B.3.18}$$

and the massless ones (with $Q \neq \frac{\hat{c}}{2}$) are

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,Q}(\tau,z) = \mathrm{e}(\tfrac{\ell+Q+1/2}{2})(\Psi_{1,-\frac{1}{2}}(\tau,z))^{-1}\, y^{Q+\frac{1}{2}}\, f^{(\ell)}_u(\tau,z+u)\,, \quad u = \tfrac{1}{2} + \tfrac{(1+2Q)\tau}{4\ell}\,. \tag{B.3.19}$$

Furthermore, the character $\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,Q}(\tau,z)$ for $Q = \frac{\hat{c}}{2}$ can be determined by the relation

$$\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,\frac{\hat{c}}{2}} = q^{-n}\Big(\mathrm{ch}^{\mathcal{N}=2}_{\ell;n+c/24,\frac{\hat{c}}{2}} + \sum_{k=1}^{\frac{\hat{c}}{2}-1}(-1)^k\big(\mathrm{ch}^{\mathcal{N}=2}_{\ell,n+c/24,\frac{\hat{c}}{2}-k} + \mathrm{ch}^{\mathcal{N}=2}_{\ell,n+c/24,k-\frac{\hat{c}}{2}}\big)\Big) + (-1)^{\frac{\hat{c}}{2}}\,\mathrm{ch}^{\mathcal{N}=2}_{\ell;c/24,0}\,, \tag{B.3.20}$$

due to the fact that at the unitary bound, several BPS multiplets can combine into a non-BPS multiplet.

$\mathcal{N}=2$ modules

The graded partition function of a module for the $c = 6m$ $\mathcal{N}=2$ superconformal algebra in the Ramond sector, i.e.

$$\mathcal{Z}^{\mathcal{N}=2}_m(\tau,z) = \mathrm{Tr}_{\mathrm{R}}\left((-1)^{J_0} y^{J_0} q^{L_0-c/24}\right), \tag{B.3.21}$$

transforms as a weak Jacobi form of weight zero and index m for $SL_2(\mathbb{Z})$ as in the $\mathcal{N}=4$ case. Furthermore, from the representation theory discussed above we expect such a partition function to have an expansion

$$\mathcal{Z}^{\mathcal{N}=2}_m(\tau,z) = \mathrm{e}(\tfrac{\ell}{2})(\Psi_{1,-\frac{1}{2}})^{-1}\Big(C_0\,\widetilde{\mu}_{\ell;0}(\tau,z) + \sum_{j-\ell\in\mathbb{Z}/2\ell\mathbb{Z}} \widetilde{F}^{(\ell)}_j(\tau)\theta_{\ell,j}(\tau,z)\Big) \tag{B.3.22}$$

when the $\mathcal{N}=2$ SCA has even central charge, $c = 3(2\ell+1)$. (See [63] for more details.) In the last equation, we have defined

$$\widetilde{\mu}_{\ell;0} = \mathrm{e}(\tfrac{1}{4})\, y^{1/2} f^{(\ell)}_u(\tau,u+z)\,, \quad u = \frac{1}{2} + \frac{\tau}{4\ell}\,,$$

and the function $\widetilde{F}^{(\ell)}_j(\tau)$ satisfies

$$\widetilde{F}^{(\ell)}_j(\tau) = \widetilde{F}^{(\ell)}_{-j}(\tau) = \widetilde{F}^{(\ell)}_{j+2\ell}(\tau). \tag{B.3.23}$$

Through its relation to the Appell–Lerch sum, $\widetilde{\mu}_{\ell;0}$ admits a completion which transforms as a weight one, half-integral index Jacobi form under the Jacobi group. Defining $\widehat{\widetilde{\mu}}_{\ell;0}$ by replacing $\mu_{m;0}$ with $\widehat{\widetilde{\mu}}_{\ell;0}$ and the integer m with the half-integral ℓ in (B.2.3), we see that $\widehat{\widetilde{\mu}}_{\ell;0}$ transforms like a Jacobi form of weight 1 and index ℓ under the group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. Following the same computation as in the previous section, we hence conclude that $\widetilde{F}^{(\ell)} = (\widetilde{F}^{(\ell)}_j)$, where $j - 1/2 \in \mathbb{Z}/2\ell\mathbb{Z}$, is a vector-valued mock modular form with a vector-valued shadow $C_0\, S_\ell = C_0(S_{\ell,j}(\tau))$.

B.3.3 $\mathcal{N}=4$ superconformal characters

Let $m=\widetilde{m}-1$.The $\mathcal{N}=4$ SCA with central charge $c=6(\widetilde{m}-1)$, $\widetilde{m}>1$, contains a level $\widetilde{m}-1$ $\widehat{su(2)}$ current algebra (cf. [232]). We will label the unitary irreducible highest weight representations by the eigenvalues of L_0 and $\frac{1}{2}J_0^3$, which we denote by h and j, respectively. We discuss representations in the Ramond sector, where a representation with quantum numbers (h,j) will be denoted $\mathcal{V}^{\mathcal{N}=4}_{m;h,j}$. There are two types of representations: a discrete set of $\widetilde{m}$ massless (BPS) representations, and $\widetilde{m}-1$ continuous families of massive (non-BPS) representations (c.f. [233].)

The BPS representations have $h=\frac{c}{24}=\frac{\widetilde{m}-1}{4}$ and $j\in\{0,\frac{1}{2},\dots,\frac{\widetilde{m}-1}{2}\}$, and the non-BPS representations have $h>\frac{\widetilde{m}-1}{4}$ and $j\in\{\frac{1}{2},1,\dots,\frac{\widetilde{m}-1}{2}\}$. Their graded characters, defined as

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z)=\mathrm{Tr}_{\mathcal{V}^{\mathcal{N}=4}_{m;h,j}}\left((-1)^{J_0^3}y^{J_0^3}q^{L_0-c/24}\right), \tag{B.3.24}$$

were computed in [234] and can be written in terms of functions defined in section B.2 as

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z)=(\Psi_{1,1}(\tau,z))^{-1}\mu_{\widetilde{m};j}(\tau,z) \tag{B.3.25}$$

and

$$\mathrm{ch}^{\mathcal{N}=4}_{m;h,j}(\tau,z)=(\Psi_{1,1}(\tau,z))^{-1}\,q^{h-\frac{c}{24}-\frac{j^2}{\widetilde{m}}}\left(\theta_{\widetilde{m},2j}(\tau,z)-\theta_{\widetilde{m},-2j}(\tau,z)\right) \tag{B.3.26}$$

in the massless and massive cases, respectively.

$\mathcal{N}=4$ modules

The graded partition function of a module for the $c=6(\widetilde{m}-1)$ $\mathcal{N}=4$ SCA in the Ramond sector, i.e.

$$\mathcal{Z}^{\mathcal{N}=4}_m(\tau,z)=\mathrm{Tr}_{\mathrm{R}}\left((-1)^{J_0^3}y^{J_0^3}q^{L_0-c/24}\right), \tag{B.3.27}$$

transforms as a weak Jacobi form of weight zero and index m for $SL_2(\mathbb{Z})$. Moreover, the representation theory of the $\mathcal{N}=4$ SCA discussed above and the explicit description of the μ and θ functions in section B.2 allows one to rewrite the graded partition function as

$$\mathcal{Z}^{\mathcal{N}=4}_m(\tau,z)=(\Psi_{1,1}(\tau,z))^{-1}\Big(c_0\,\mu_{\widetilde{m};0}(\tau,z)+\sum_{r\in\mathbb{Z}/2\widetilde{m}\mathbb{Z}}F_r^{(\widetilde{m})}(\tau)\,\theta_{\widetilde{m},r}(\tau,z)\Big), \tag{B.3.28}$$

where the $F^{(\widetilde{m})}=(F_r^{(\widetilde{m})})$, $r\in\mathbb{Z}/2\widetilde{m}\mathbb{Z}$ obey

$$F_r^{(\widetilde{m})}(\tau)=-F_{-r}^{(\widetilde{m})}(\tau)=F_{r+2\widetilde{m}}^{(\widetilde{m})}(\tau). \tag{B.3.29}$$

See, for example, [63].

The way in which the functions $\mathcal{Z}_m^{\mathcal{N}=4}(\tau,z)$ and $\widehat{\mu}_{\widetilde{m};0}$ transform under the Jacobi group shows that the non-holomorphic function $\sum_{r\in\mathbb{Z}/2\widetilde{m}\mathbb{Z}} \widehat{F}_r^{(\widetilde{m})}(\tau)\,\theta_{\widetilde{m},r}(\tau,z)$ transforms as a Jacobi form of weight 1 and index $\widetilde{m}$ under $SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$, where

$$\widehat{F}_r^{(\widetilde{m})}(\tau) = F_r^{(\widetilde{m})}(\tau) + c_0\,\mathrm{e}(-\tfrac{1}{8})\,\frac{1}{\sqrt{2\widetilde{m}}}\int_{-\overline{\tau}}^{i\infty}(\tau'+\tau)^{-1/2}\,\overline{S_{\widetilde{m},r}(-\overline{\tau}')}\,\mathrm{d}\tau'.$$

In other words, $F^{(\widetilde{m})} = (F_r^{(\widetilde{m})})$, $r\in\mathbb{Z}/2\widetilde{m}\mathbb{Z}$ is a vector-valued mock modular form with a vector-valued shadow $c_0\,S_{\widetilde{m}}$, whose r-th component is given by $S_{\widetilde{m},r}(\tau)$, with the multiplier for $SL_2(\mathbb{Z})$ given by the inverse of the multiplier system of $S_{\widetilde{m}}$.

B.4 Cusp behavior of $h_g^{\mathcal{N}=2}$

In this section we discuss an intriguing property of the vector-valued mock modular forms $h_g^{\mathcal{N}=2}$ of $\mathcal{E}_{m=2}^{\mathcal{N}=2}(M_{23})$ for $\pi_g \in \{1^6\,3^6,\ 1^2\,2^2\,3^2\,6^2,\ 1.3.5.15\}$, i.e. $g\in\{3A,\ 6A,\ 15AB\}$ using the standard ATLAS notation [235] for these conjugacy classes. These are precisely the functions which are not Rademacher sums at the infinite cusp. They have poles at the cusp at zero, $\frac{1}{2}$, and $\frac{1}{5}$, respectively. However, the coefficients in the expansion of these functions around these cusps can be related to the coefficients in the expansion at the infinite cusp, via a relation with functions appearing in the M_{24} ($\ell=2$) case of umbral moonshine. First, let

$$H_{1A}(\tau) := \frac{1}{2}\widehat{H}_{1A}^{(2)}(\tau) = q^{-1/8}(-1+45q+231q^2+770q^3\ldots), \tag{B.4.1}$$

be the function such that $\widehat{H}_{1A}^{(2)}(\tau)$ is the single independent component of a weight $\frac{1}{2}$ vector-valued mock modular form for $SL_2(\mathbb{Z})$ whose coefficients encode the graded dimensions of an M_{24} module [36, 94, 99]. Furthermore, let

$$H_{g'}(\tau) := \frac{1}{2}\widehat{H}_{g'}^{(2)}(\tau) \tag{B.4.2}$$

be the corresponding (weight $\frac{1}{2}$, vector-valued) mock modular forms for $\Gamma_{g'}$ encoding the graded traces of g' in this module for all conjugacy classes $g'\in M_{24}$, where $\Gamma_{g'}$ is just equal to $\Gamma_0(o(g'))$. We also use below the fact that

$$H_{3A}(\tau) := \frac{1}{2}\widehat{H}_{3A}^{(2)}(\tau) = q^{-1/8}(-1+0q+-3q^2+5q^3\ldots) \tag{B.4.3}$$

for the conjugacy class $g'=3A$ in M_{24}. Note the following interesting relation between the functions $h_g^{\mathcal{N}=2}$ for all $g\in M_{23}$ and the functions $H_{g'}(\tau)$.

We introduce the notation $h^{\infty}_{g,r}$ to denote the rth component of $h^{\mathcal{N}=2}_g$ expanded about the cusp of Γ_g at $\tau = i\infty$. Similarly, we will use the notation $h^{\zeta}_{g,r}$ to denote the rth component of $h^{\mathcal{N}=2}_g$ expanded about the cusp of Γ_g at $\tau = \zeta$. Our first observation is that

$$\begin{aligned} h^{\infty}_{1A,\frac{1}{2}}(\tau) &= \frac{1}{3}\sum_{\alpha=0}^{2} e\left(\frac{\alpha}{24}\right) H_{1A}\left(\frac{\tau+\alpha}{3}\right) \\ &= q^{-1/24}(-1+770q+13915q^2+132825q^3\ldots) \\ h^{\infty}_{1A,\frac{3}{2}}(\tau) &= \frac{1}{3}\sum_{\alpha=0}^{2} e\left(-\frac{15\alpha}{24}\right) H_{1A}\left(\frac{\tau+\alpha}{3}\right) - H_{1A}(3\tau) \\ &= q^{-9/24}(1+231q+5796q^2+65505q^3\ldots) \end{aligned} \tag{B.4.4}$$

and

$$\begin{aligned} h^{\infty}_{3A,\frac{1}{2}}(\tau) &= \frac{1}{3}\sum_{\alpha=0}^{2} e\left(\frac{\alpha}{24}\right) H_{3A}\left(\frac{\tau+\alpha}{3}\right) \\ &= q^{-1/24}(-1+5q+10q^2+21q^3\ldots) \\ h^{\infty}_{3A,\frac{3}{2}}(\tau) &= -\frac{2}{3}\sum_{\alpha=0}^{2} e\left(-\frac{15\alpha}{24}\right) H_{3A}\left(\frac{\tau+\alpha}{3}\right) - H_{1A}(3\tau) \\ &= q^{-9/24}(1+6q+18q^2-15q^3\ldots). \end{aligned} \tag{B.4.5}$$

This encodes the relation of the 1A and 3A twining functions of $\mathcal{E}^{\mathcal{N}=2}_{m=2}(M_{23})$ to those of M_{24} umbral moonshine.[2]

Now let's look at the expansion of $h^{\mathcal{N}=2}_{3A}$ at $\zeta = 0$. We find that the components $h^{0}_{3A,\frac{1}{2}}(\tau), h^{0}_{3A,\frac{3}{2}}(\tau)$ can be expressed as linear combinations of the functions $h^{\infty}_{3A,\frac{1}{2}}(\tau), h^{\infty}_{3A,\frac{3}{2}}(\tau)$ and $H_{1A}(\tau)$. Explicitly, the relation is

$$\begin{aligned} h^{0}_{3A,\frac{1}{2}}(\tau) &= H_{1A}\left(\frac{\tau}{3}\right) - 3h^{\infty}_{3A,\frac{1}{2}}(\tau) = 2q^{-1/24}+45q^{7/24}+231q^{15/24}+\ldots \\ h^{0}_{3A,\frac{3}{2}}(\tau) &= -h^{\infty}_{3A,\frac{1}{2}}(\tau) - h^{\infty}_{3A,\frac{3}{2}}(\tau) - H_{1A}(3\tau) = q^{-1/24}-6q^{15/24}+\ldots \end{aligned}$$

Similarly, consider the following pairs of conjugacy classes: $(g',g) = (2A,6A)$ and $(g',g) = (5A,15AB)$ for $g' \in M_{24}$ and $g \in M_{23}$. Then we have a similar relation for the two other functions with additional poles given by

$$\begin{aligned} h^{\zeta_g}_{g,\frac{1}{2}}(\tau) &= H_{g'}\left(\frac{\tau}{3}\right) - 3h^{\infty}_{g,\frac{1}{2}}(\tau) \\ h^{\zeta_g}_{g,\frac{3}{2}}(\tau) &= -h^{\infty}_{g,\frac{1}{2}}(\tau) - h^{\infty}_{g,\frac{3}{2}}(\tau) - H_{g'}(3\tau), \end{aligned}$$

[2] The above equations in (B.4.4) and (B.4.5) look very much like the action of a Hecke operator on H_g. It would be interesting to explore this connection further.

where $\zeta_g = \frac{1}{2}$ for $g = 6A$ and $\zeta_g = \frac{1}{5}$ for $g = 15AB$.

It would be very interesting to understand the origin of these properties, and in particular why they behave similarly to the Hauptmoduln of monstrous moonshine for groups with Atkin-Lehner involutions. For example, consider the McKay-Thompson series for conjugacy class $g = 3A$ in the monster group, expanded at the infinite cusp

$$T_{3A}(\tau) = \frac{1}{q} + 783q + 8672q^2 + 65367q^3 + \ldots. \tag{B.4.6}$$

This is a Hauptmodul for the group $\Gamma_0(3) + 3$, which is defined in Appendix A and, in particular, contains the Fricke involution which takes $\tau \mapsto -\frac{1}{3\tau}$. Such an involution relates the cusp at infinity to the cusp at $\tau = 0$, and thus these cusps are equivalent with respect to $\Gamma_0(3)+3$. As a result, the expansion of the Hauptmodul at $\tau = 0$, which we will denote $T^0_{3A}(\tau)$, is given by

$$T^0_{3A}(\tau) = T_{3A}\left(-\frac{1}{3\tau}\right) = q^{-\frac{1}{3}} + 783q^{\frac{1}{3}} + 8672q^{\frac{2}{3}} + 65367q + \ldots = T_{3A}\left(\frac{\tau}{3}\right). \tag{B.4.7}$$

The properties we observe for certain $h^{\mathcal{N}=2}_g$ in equation (B.4.6) in this section are strikingly similar to this behavior.

B.5 Twining functions

In this section, we derive the twined partition functions of $\mathcal{E}^{\mathcal{N}=2}_{m=4}$ for all conjugacy classes $[g] \in M_{23}$, and we discuss a few such cases for $\mathcal{E}^{\mathcal{N}=4}_{m=4}$ and $[g] \in M_{11}$. To ease the notation introduced in section §3.5.3, let

$$\begin{aligned} F^{un}_g(\Lambda;\tau,z) &:= \mathrm{Tr}_{\mathscr{H}}\left(\frac{1-h}{2}\right) g(-1)^F q^{L_0-\frac{c}{24}} y^{J_0(J_3)} \\ &= \mathrm{Tr}_{\mathscr{H}}\left(\frac{1-h}{2}\right) g q^{L_0-\frac{c}{24}} y^{J_0(J_3)} \end{aligned} \tag{B.5.1}$$

be the g-twined trace which is the contribution of the untwisted sector[3] to the partition function $\mathcal{Z}^{\mathcal{N}=2(4)}_{m=4}(\tau,z)$, and let

$$F^{tw}_g(\Lambda;\tau,z) := \mathrm{Tr}_{\mathscr{H}^{tw}}\left(\frac{1+h}{2}\right) g(-1)^F q^{L_0-\frac{c}{24}} y^{J_0(J_3)} \tag{B.5.2}$$

be the corresponding g-twined contribution of the twisted sector.

[3] Note that all states in the untwisted Hilbert space are bosonic so we can drop the $(-1)^F$ in (B.5.1).

The untwisted sector

We start with $\Lambda = A_1^{24}$. To implement the trace in the untwisted sector, we need to know the action of h on the $\widehat{su(2)}_1$ modules, as well as their characters with the $U(1)$ charge included, which we will denote by $\mathrm{ch}_0(\tau, z)$ and $\mathrm{ch}_1(\tau, z)$. It is straightforward to see that these are given by

$$\mathrm{ch}_0(\tau, z) = \mathrm{Tr}_{[0]} q^{L_0 - c/24} y^{J_0} = \frac{\theta_3(2\tau, 4z)}{\eta(\tau)} := (\mathbf{z})_+, \quad \text{and} \tag{B.5.3}$$

$$\mathrm{ch}_1(\tau, z) = \mathrm{Tr}_{[1]} q^{L_0 - c/24} y^{J_0} = \frac{\theta_2(2\tau, 4z)}{\eta(\tau)} := (\widetilde{\mathbf{z}}), \tag{B.5.4}$$

where J_0 is the zero mode of the $U(1)$ current in equation (3.5.22). Furthermore, using the explicit description of h, it is easy to check that the characters with an h-insertion are given by

$$\mathrm{ch}_0^-(\tau, z) = \mathrm{Tr}_{[0]} h q^{L_0 - c/24} y^{J_0} = \frac{\theta_4(2\tau, 4z)}{\eta(\tau)} := (\mathbf{z})_-, \quad \text{and} \tag{B.5.5}$$

$$\mathrm{ch}_1^-(\tau, z) = \mathrm{Tr}_{[1]} h q^{L_0 - c/24} y^{J_0} = 0. \tag{B.5.6}$$

In order to write the (twined) partition function in terms of these characters, we introduce the following notation,

$$\begin{aligned} (\mathbf{n})_+^m &:= \mathrm{ch}_0(n\tau, 0)^m \\ (\mathbf{n})_-^m &:= \mathrm{ch}_0^-(n\tau, 0)^m \\ (\widetilde{\mathbf{n}})^m &:= \mathrm{ch}_1(n\tau, 0)^m. \end{aligned} \tag{B.5.7}$$

Given this we can evaluate the trace in equation (B.5.1) with $g = 1$ to compute the contribution of the untwisted states to the Ramond sector partition, which is

$$\begin{aligned} F^{un}(\Lambda; \tau, z) &= \frac{1}{2}\left((\mathbf{z})_+(\mathbf{1})_+^{23} - (\mathbf{z})_-(\mathbf{1})_-^{23}\right) + \frac{253}{2}\left((\mathbf{z})_+(\mathbf{1})_+^{7}(\widetilde{\mathbf{1}})^{16} + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^{7}(\mathbf{1})_+^{16}\right) \\ &+ 253\left((\mathbf{z})_+(\mathbf{1})_+^{15}(\widetilde{\mathbf{1}})^{8} + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^{15}(\mathbf{1})_+^{8}\right) \\ &+ 644\left((\mathbf{z})_+(\mathbf{1})_+^{11}(\widetilde{\mathbf{1}})^{12} + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^{11}(\mathbf{1})_+^{12}\right) \\ &+ \frac{1}{2}(\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^{23}, \end{aligned}$$

where we note that all of the untwisted states are bosonic and thus invariant under $(-1)^F$.

Furthermore, we can compute the g-twined trace of equation (B.5.1) using an explicit description of the action of M_{23} on the binary Golay code, which we obtain from GAP.[4] From this we compute the invariant vectors of the Golay code

[4] This open source program lives at https://www.gap-system.org/.

under the 24-dimensional permutation representation of g. The results for all conjugacy classes g in M_{23} are given in Table B.1 and B.2.

M_{23} $[g]$	Frame shape	$F_g^{un}(\Lambda;\tau,z)$
2A	$1^8 2^8$	$\frac{1}{2}\left((\mathbf{z})_+(\mathbf{1})_+^7 - (\mathbf{z})_-(\mathbf{1})_-^7 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^7\right)\left((\mathbf{2})_+^8 + (\widetilde{\mathbf{2}})^8\right)$ $+7\left((\mathbf{z})_+(\mathbf{1})_+^7 - (\mathbf{z})_-(\mathbf{1})_-^7 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^7\right)(\mathbf{2})_+^4(\widetilde{\mathbf{2}})^4$ $+14\left((\mathbf{z})_+(\mathbf{1})_+^3(\widetilde{\mathbf{1}})^4 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^3(\mathbf{1})_+^4\right)(\mathbf{2})_+^2(\widetilde{\mathbf{2}})^2\left((\mathbf{2})_+^2 + (\widetilde{\mathbf{2}})^2\right)^2$
3A	$1^6 3^6$	$\frac{1}{2}\left((\mathbf{z})_+(\mathbf{1})_+^5(\mathbf{3})_+^6 - (\mathbf{z})_-(\mathbf{1})_-^5(\mathbf{3})_-^6\right) + \frac{1}{2}(\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^5(\widetilde{\mathbf{3}})^6$ $+\frac{1}{2}\left((\mathbf{1})_+^5(\mathbf{3})_+(\widetilde{\mathbf{z}})(\widetilde{\mathbf{3}})^5 + (\widetilde{\mathbf{1}})^5(\widetilde{\mathbf{3}})(\mathbf{z})_+(\mathbf{3})_+^5\right)$ $+5\left((\mathbf{z})_+(\mathbf{1})_+^3(\mathbf{3})_+^4(\widetilde{\mathbf{1}})^2(\widetilde{\mathbf{3}})^2 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^3(\widetilde{\mathbf{3}})^4(\mathbf{1})_+^2(\mathbf{3})_+^2\right)$ $+\frac{5}{2}\left((\mathbf{z})_+(\mathbf{1})_+(\mathbf{3})_+^2(\widetilde{\mathbf{1}})^4(\widetilde{\mathbf{3}})^4 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})(\widetilde{\mathbf{3}})^2(\mathbf{1})_+^4(\mathbf{3})_+^4\right)$ $+\frac{5}{2}\left((\mathbf{z})_+(\mathbf{1})_+^4(\mathbf{3})_+(\widetilde{\mathbf{1}})(\widetilde{\mathbf{3}})^5 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^4(\widetilde{\mathbf{3}})(\mathbf{1})_+(\mathbf{3})_+^5\right)$ $+5\left((\mathbf{z})_+(\mathbf{1})_+^2(\widetilde{\mathbf{1}})^3 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^2(\mathbf{1})_+^3\right)(\mathbf{3})_+^3(\widetilde{\mathbf{3}})^3$
4A	$1^4 2^2 4^4$	$\frac{1}{2}\left(\left((\mathbf{z})_+(\mathbf{1})_+^3 - (\mathbf{z})_-(\mathbf{1})_-^3\right)(\mathbf{2})_+^2 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^3(\widetilde{\mathbf{2}})^2\right)\times$ $\times\left((\mathbf{4})_+^2 + (\widetilde{\mathbf{4}})^2\right)^2$ $+2\left(\left((\mathbf{z})_+(\mathbf{1})_+^3 - (\mathbf{z})_-(\mathbf{1})_-^3\right)(\widetilde{\mathbf{2}})^2 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^3(\mathbf{2})_+^2\right)(\mathbf{4})_+^2(\widetilde{\mathbf{4}})^2$ $+2\left((\mathbf{z})_+(\mathbf{1})_+(\widetilde{\mathbf{1}})^2 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})(\mathbf{1})_+^2\right)\times$ $\times(\mathbf{2})_+(\widetilde{\mathbf{2}})\left((\mathbf{4})_+(\widetilde{\mathbf{4}})^3 + (\mathbf{4})_+^3(\widetilde{\mathbf{4}})\right)$
5A	$1^4 5^4$	$\frac{1}{2}\left((\mathbf{z})_+(\mathbf{1})_+^3(\mathbf{5})_+^4 - (\mathbf{z})_-(\mathbf{1})_-^3(\mathbf{5})_-^4\right) + \frac{1}{2}(\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^3(\widetilde{\mathbf{5}})^4$ $+\frac{1}{2}\left((\mathbf{1})_+^3(\mathbf{5})_+(\widetilde{\mathbf{z}})(\widetilde{\mathbf{5}})^3 + (\widetilde{\mathbf{1}})^3(\widetilde{\mathbf{5}})(\mathbf{z})_+(\mathbf{5})_+^3\right)$ $+\frac{3}{2}\left((\mathbf{z})_+(\mathbf{1})_+^2(\mathbf{5})_+(\widetilde{\mathbf{1}})(\widetilde{\mathbf{5}})^3 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^2(\widetilde{\mathbf{5}})(\mathbf{1})_+(\mathbf{5})_+^3\right)$ $+\frac{3}{2}\left((\mathbf{z})_+(\mathbf{1})_+(\widetilde{\mathbf{1}})^2 + (\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})(\mathbf{1})_+^2\right)(\mathbf{5})_+^2(\widetilde{\mathbf{5}})^2$

Table B.1: The twining functions $F_g^{un}(A_1^{24};\tau,z)$ of the untwisted sector of $\mathcal{E}_{m=4}^{\mathcal{N}=2}$ under for the conjugacy classes $[g] \in M_{23}$ of order smaller than six. We label them by their Frame shapes corresponding to their embedding into the **24** of Co_0.

M_{23} $[g]$	Frame shape	$F_g^{un}(\Lambda;\tau,z)$
6A	$1^2 2^2 3^2 6^2$	$\frac{1}{2}\left((\mathbf{z})_+(\mathbf{1})_+(\mathbf{3})_+^2-(\mathbf{z})_-(\mathbf{1})_-(\mathbf{3})_-^2+(\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})(\widetilde{\mathbf{3}})^2\right)\times$ $\times\left((\mathbf{2})_+(\mathbf{6})_++(\widetilde{\mathbf{2}})(\widetilde{\mathbf{6}})\right)^2$ $+\frac{1}{2}\left((\mathbf{z})_+(\widetilde{\mathbf{1}})+(\widetilde{\mathbf{z}})(\mathbf{1})_+\right)(\mathbf{3})_+(\widetilde{\mathbf{3}})\left((\mathbf{2})_+(\widetilde{\mathbf{6}})+(\widetilde{\mathbf{2}})(\mathbf{6})_+\right)^2$
7A	$1^3 7^3$	$\frac{1}{2}\left((\mathbf{z})_+(\mathbf{1})_+^2(\mathbf{7})_+^3-(\mathbf{z})_-(\mathbf{1})_-^2(\mathbf{7})_-^3\right)+\frac{1}{2}(\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})^2(\widetilde{\mathbf{7}})^3$ $+\frac{1}{2}\left((\mathbf{z})_+(\widetilde{\mathbf{1}})^2(\widetilde{\mathbf{7}})^+(\widetilde{\mathbf{z}})(\mathbf{1})_+^2(\mathbf{7})_+\right)(\mathbf{7})_+(\widetilde{\mathbf{7}})$ $+\left((\mathbf{z})_+(\mathbf{7})_++(\widetilde{\mathbf{z}})(\widetilde{\mathbf{7}})\right)(\mathbf{1})_+(\mathbf{7})_+(\widetilde{\mathbf{1}})(\widetilde{\mathbf{7}})$
8A	$1^2 2.4.8^2$	$\frac{1}{2}\left((\mathbf{z})_+(\mathbf{1})_+-(\mathbf{z})_-(\mathbf{1})_-\right)(\mathbf{2})_+(\mathbf{4})_+(\mathbf{8})_+^2$ $+\frac{1}{2}(\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})(\widetilde{\mathbf{2}})(\widetilde{\mathbf{4}})(\widetilde{\mathbf{8}})^2+\frac{1}{2}(\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})(\widetilde{\mathbf{2}})(\widetilde{\mathbf{4}})(\mathbf{8})_+^2$ $+\frac{1}{2}\left((\mathbf{z})_+(\mathbf{1})_+-(\mathbf{z})_-(\mathbf{1})_-\right)(\mathbf{2})_+(\mathbf{4})_+(\widetilde{\mathbf{8}})^2$ $+\frac{1}{2}(\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})(\widetilde{\mathbf{2}})(\mathbf{4})_+(\mathbf{8})_+(\widetilde{\mathbf{8}})$ $+\frac{1}{2}\left((\mathbf{z})_+(\mathbf{1})_+-(\mathbf{z})_-(\mathbf{1})_-\right)(\mathbf{2})_+(\widetilde{\mathbf{4}})(\mathbf{8})_+(\widetilde{\mathbf{8}})$
11AB	$1^2 11^2$	$\frac{1}{2}\left((\mathbf{z})_+(\mathbf{1})_+(\mathbf{11})_+^2-(\mathbf{z})_-(\mathbf{1})_-(\mathbf{11})_-^2\right)+\frac{1}{2}(\widetilde{\mathbf{z}})(\widetilde{\mathbf{1}})(\widetilde{\mathbf{11}})^2$ $+\frac{1}{2}\Big((\mathbf{z})_+(\widetilde{\mathbf{1}})(\mathbf{11})_+(\widetilde{\mathbf{11}})+(\widetilde{\mathbf{z}})_+(\mathbf{1})_+(\mathbf{11})_+(\widetilde{\mathbf{11}})\Big)$
14AB	1.2.7.14	$\frac{1}{2}\left((\mathbf{z})_+(\mathbf{2})_+(\mathbf{7})_+(\mathbf{14})_+-(\mathbf{z})_-(\mathbf{2})_+(\mathbf{7})_-(\mathbf{14})_+\right)$ $+\frac{1}{2}(\widetilde{\mathbf{z}})(\widetilde{\mathbf{2}})(\widetilde{\mathbf{7}})(\widetilde{\mathbf{14}})$ $+\frac{1}{2}\left((\mathbf{z})_+(\widetilde{\mathbf{2}})(\mathbf{7})_+(\widetilde{\mathbf{14}})-(\mathbf{z})_-(\widetilde{\mathbf{2}})(\mathbf{7})_-(\widetilde{\mathbf{14}})+(\widetilde{\mathbf{z}})(\mathbf{2})_+(\widetilde{\mathbf{7}})(\mathbf{14})_+\right)$
15AB	1.3.5.15	$\frac{1}{2}\left((\mathbf{z})_+(\mathbf{3})_+(\mathbf{5})_+(\mathbf{15})_+-(\mathbf{z})_-(\mathbf{3})_-(\mathbf{5})_-(\mathbf{15})_-\right)$ $+\frac{1}{2}(\widetilde{\mathbf{z}})(\widetilde{\mathbf{3}})(\widetilde{\mathbf{5}})(\widetilde{\mathbf{15}})+\frac{1}{2}\left((\mathbf{z})_+(\widetilde{\mathbf{3}})(\widetilde{\mathbf{5}})(\mathbf{15})_++(\widetilde{\mathbf{z}})(\mathbf{3})_+(\mathbf{5})_+(\widetilde{\mathbf{15}})\right)$
23AB	1.23	$\frac{1}{2}\left((\mathbf{z})_+(\mathbf{23})_+-(\mathbf{z})_-(\mathbf{23})_-\right)$

Table B.2: The twining functions $F_g^{un}(A_1^{24};\tau,z)$ of the untwisted sector of $\mathcal{E}_{m=4}^{\mathcal{N}=2}$ under for the conjugacy classes $[g]\in M_{23}$ of order bigger than five. We label them by their Frame shapes corresponding to their embedding into the **24** of Co_0.

Similarly, we now consider the functions $F_g^{un}(\Lambda;\tau,z)$ for $\Lambda = A_2^{12}$. First we need the $\widehat{su(3)}$ characters including a chemical potential for the Cartan J_3 of the invariant $\widehat{su(2)}_4$. These are given by

$$\chi_0(\tau,z) = \mathrm{Tr}_{[0]} q^{L_0-c/24} y^{J_3} = \frac{\theta_3(2\tau,4z)\theta_3(6\tau)+\theta_2(2\tau,4z)\theta_2(6\tau)}{2\eta^2(\tau)} := [\mathbf{z}]_+ \quad \text{(B.5.8)}$$

for the vacuum character and

$$\chi_i(\tau,z) = \mathrm{Tr}_{[i]} q^{L_0-c/24} y^{J_3} = \frac{\theta_3(2\tau,4z)\theta_3\left(\frac{2\tau}{3}\right)+\theta_2(2\tau,4z)\theta_2\left(\frac{2\tau}{3}\right)}{2\eta^2(\tau)} - \frac{\chi_0(\tau,z)}{2} := [\widetilde{\mathbf{z}}] \quad \text{(B.5.9)}$$

for the nontrivial primaries with $i = 1, 2$. Finally, we also need the trace of h in these modules, which we compute to be

$$\chi_0^-(\tau,z) = \mathrm{Tr}_{[0]} h q^{L_0-c/24} y^{J_3} = \frac{\theta_4(2\tau,4z)\theta_4(2\tau)}{\eta^2(\tau)} := [\mathbf{z}]_- \quad \text{(B.5.10)}$$

and

$$\chi_i^-(\tau,z) = \mathrm{Tr}_{[i]} h q^{L_0-c/24} y^{J_3} = 0, \quad i = 1, 2. \quad \text{(B.5.11)}$$

Putting all of these components together, we compute the partition function of the orbifold theory in the untwisted sector with a projection onto anti-invariant states under h to be

$$\begin{aligned} F^{un}(\Lambda;\tau,z) &= \frac{1}{2}\left([\mathbf{z}]_+[\mathbf{1}]_+^{11} - [\mathbf{z}]_-[\mathbf{1}]_-^{11}\right) + 66\left([\mathbf{z}]_+[\mathbf{1}]_+^5[\widetilde{\mathbf{1}}]^6 + [\widetilde{\mathbf{z}}][\widetilde{\mathbf{1}}]^5[\mathbf{1}]_+^6\right) \\ &+ 55[\mathbf{z}]_+[\mathbf{1}]_+^2[\widetilde{\mathbf{1}}]^9 + 165[\widetilde{\mathbf{z}}][\mathbf{1}]_+^3[\widetilde{\mathbf{1}}]^8 + 12[\widetilde{\mathbf{z}}][\widetilde{\mathbf{1}}]^{11}. \end{aligned}$$

As an example, we consider elements in conjugacy classes $[g] \in \{3A, 5A, 11AB\}$ of M_{11}. Again we use GAP to obtain an action of M_{11} in its 11-dimensional permutation representation on the ternary Golay code, which we then use to compute the invariant vectors of the theory under this action. The results are reported in Table B.3.

$L_2(11)$ $[g]$	Frame shape	$F_g^{un}(\Lambda;\tau,z)$
3A	$1^6 3^6$	$\frac{1}{2}\left([\mathbf{z}]_+[\mathbf{1}]_+^2[\mathbf{3}]_+^3 - [\mathbf{z}]_-[\mathbf{1}]_-^2[\mathbf{3}]_-^3\right) + 3[\widetilde{\mathbf{z}}][\widetilde{\mathbf{1}}]^2[\widetilde{\mathbf{3}}]^3$ $+3[\mathbf{z}]_+[\mathbf{1}]_+^2[\mathbf{3}]_+[\widetilde{\mathbf{3}}]^2 + 3[\widetilde{\mathbf{z}}][\widetilde{\mathbf{1}}]^2[\widetilde{\mathbf{3}}][\mathbf{3}]_+^2$ $+3[\widetilde{\mathbf{z}}][\widetilde{\mathbf{1}}]^2[\widetilde{\mathbf{3}}]^2[\mathbf{3}]_+ + [\mathbf{z}]_+[\mathbf{1}]_+^2[\widetilde{\mathbf{3}}]^3$
5A	$1^4 5^4$	$\frac{1}{2}\left([\mathbf{z}]_+[\mathbf{1}]_+[\mathbf{5}]_+^2 - [\mathbf{z}]_-[\mathbf{1}]_-[\mathbf{5}]_-^2\right) + 2[\widetilde{\mathbf{z}}][\widetilde{\mathbf{1}}][\widetilde{\mathbf{5}}]^2$ $+[\mathbf{z}]_+[\mathbf{5}]_+[\widetilde{\mathbf{1}}][\widetilde{\mathbf{5}}] + [\mathbf{1}]_+[\mathbf{5}]_+[\widetilde{\mathbf{z}}][\widetilde{\mathbf{5}}]$
11AB	$1^2 11^2$	$\frac{1}{2}\left([\mathbf{z}]_+[\mathbf{11}]_+ - [\mathbf{z}]_-[\mathbf{11}]_-\right) + [\widetilde{\mathbf{z}}][\widetilde{\mathbf{11}}]$

Table B.3: The twining functions $F_g^{un}(A_2^{12};\tau,z)$ of the untwisted sector of $\mathcal{E}_{m=4}^{\mathcal{N}=4}$ for certain conjugacy classes $[g] \in M_{11}$. We label them by their Frame shapes corresponding to their embedding into the **24** of Co_0.

The twisted sector

Finally, we need a description of the twisted sector Hilbert space, and the action of h on the twisted states. After we include the $U(1)$ grading, the contribution of the twisted sector in (B.5.2) to the full partition function is

$$F^{tw}(\Lambda;\tau,z) = 2^{11}\frac{\theta_2(\tau,2z)}{\theta_2(\tau,0)}\left(\frac{\theta_3(\tau,2z)}{\theta_3(\tau,0)}\frac{\eta^{24}(\tau)}{\eta^{24}(\tau/2)} - \frac{\theta_4(\tau,2z)}{\theta_4(\tau,0)}\frac{\eta^{24}(2\tau)\eta^{24}(\tau/2)}{\eta^{48}(\tau)}\right) \tag{B.5.12}$$

for both $\Lambda = A_1^{24}$ and $\Lambda = A_2^{12}$.

The twisted sector Hilbert spaces of all $\mathbb{Z}_2$ orbifolds of a Niemeier CFT are isomorphic and have an action of the group Co_0. Once we grade by the additional $U(1)$ charge as in equation (B.5.12), the Co_0 symmetry is broken to subgroups which preserve a two-plane in the 24-dimensional representation. In particular, since both M_{23} and $L_2(11)$ satisfy this constraint, we can define a consistent action of elements of these groups on the twisted sector Hilbert spaces. The action for a given conjugacy class g of these groups follows from the 24-dimensional permutation representation of g as follows. Define

$$\eta_g(\tau) := q\prod_{n>0}\prod_{i=1}^{12}(1-\lambda_i^{-1}q^n)(1-\lambda_i q^n) \tag{B.5.13}$$

and

$$\eta_{-g}(\tau) := q\prod_{n>0}\prod_{i=1}^{12}(1+\lambda_i^{-1}q^n)(1+\lambda_i q^n) \tag{B.5.14}$$

where $\{\lambda_i\}$ are the 24 eigenvalues of g in its 24-dimensional permutation representation, specified by the Frame shape π_g as in equation (3.4.10). Then the trace of g in the twisted sector is given by

$$F_g^{tw}(\Lambda;\tau,z) = c_g \frac{\theta_2(\tau,2z)}{\theta_2(\tau,0)}\left(\frac{\theta_3(\tau,2z)}{\theta_3(\tau,0)}\frac{\eta_g(\tau)}{\eta_g(\tau/2)} - \frac{\theta_4(\tau,2z)}{\theta_4(\tau,0)}\frac{\eta_{-g}(\tau)}{\eta_{-g}(\tau/2)}\right) \quad \text{(B.5.15)}$$

where the constant c_g is defined by

$$c_g := 2^{\frac{1}{2}(\#\text{ of cycles of }\pi_g)-1}. \quad \text{(B.5.16)}$$

From this and the results in the previous section we can reconstruct all the twining functions of $\mathcal{E}^{\mathcal{N}=2}_{m=4}$ under elements of M_{23}. We present the first several coefficients of these functions and their decompositions into irreducible M_{23} representations in the tables in the next section.

B.6 Tables

In this section we present certain useful tables. In §B.6.1 we present character tables of certain groups mentioned in the text. In §B.6.2 we present the first several coefficients and decompositions of the vector-valued mock modular forms arising from $\mathcal{E}^{\mathcal{N}=2}_{m=4}$ for conjugacy classes $[g] \in M_{23}$.

B.6.1 Irreducible characters

Below, we make use of the following standard notation: $b_n = (-1+i\sqrt{n})/2$, $\overline{b_n} = (-1-i\sqrt{n})/2$, $\beta_n = (-1+\sqrt{n})/2$, $\overline{\beta_n} = (-1-\sqrt{n})/2$ and $a_n = i\sqrt{n}$, $\overline{a_n} = -i\sqrt{n}$.

B.6.2 Coefficients and decompositions

Table B.4: Character Table of M_{23}.

$[g]$	1A	2A	3A	4A	5A	6A	6B	7AB	8A	11A	11B	14A	14B	15A	15B	23A	23B
$[g^2]$	1A	1A	3A	2A	5A	3A	7A	7B	4A	11B	11A	7A	7B	15A	15B	23A	23B
$[g^3]$	1A	2A	1A	4A	5A	2A	7B	7A	8A	11A	11B	14B	14A	5A	5A	23A	23B
$[g^5]$	1A	2A	3A	4A	A	6A	7B	7A	8A	11A	11B	14B	14A	3A	3A	23B	23A
$[g^7]$	1A	2A	3A	4A	5A	6A	1A	1A	8A	11B	11A	2A	2A	15B	15A	23B	23A
$[g^{11}]$	1A	2A	3A	4A	5A	6A	7A	7B	8A	1A	1A	14A	14B	15B	15A	23B	23A
$[g^{23}]$	1A	2A	3A	4A	5A	6A	7A	7B	8A	11A	11B	14A	14B	15A	15B	1A	1A
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	6	4	2	2	0	1	1	0	0	0	-1	-1	-1	-1	-1	-1
χ_3	45	-3	0	1	0	0	b_7	$\overline{b_7}$	-1	1	1	-b_7	-$\overline{b_7}$	0	0	-1	-1
χ_4	45	-3	0	1	0	0	$\overline{b_7}$	b_7	-1	1	1	-$\overline{b_7}$	-b_7	0	0	-1	-1
χ_5	230	22	5	2	0	1	-1	-1	0	-1	-1	1	1	0	0	0	0
χ_6	231	7	6	-1	1	-2	0	0	-1	0	0	0	0	1	1	1	1
χ_7	231	7	-3	-1	1	1	0	0	-1	0	0	0	0	b_{15}	$\overline{b_{15}}$	1	1
χ_8	231	7	-3	-1	1	1	0	0	-1	0	0	0	0	$\overline{b_{15}}$	b_{15}	1	1
χ_9	253	13	1	1	-2	1	1	1	-1	0	0	-1	-1	1	1	0	0
χ_{10}	770	-14	5	-2	0	1	0	0	0	0	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
χ_{11}	770	-14	5	-2	0	1	0	0	0	0	0	0	0	0	0	$\overline{b_{23}}$	b_{23}
χ_{12}	896	0	-4	0	1	0	0	0	0	b_{11}	$\overline{b_{11}}$	0	0	1	1	-1	-1
χ_{13}	896	0	-4	0	1	0	0	0	0	$\overline{b_{11}}$	b_{11}	0	0	1	1	-1	-1
χ_{14}	990	-18	0	2	0	0	b_7	$\overline{b_7}$	0	0	0	b_7	$\overline{b_7}$	0	0	1	1
χ_{15}	990	-18	0	2	0	0	$\overline{b_7}$	b_7	0	0	0	$\overline{b_7}$	b_7	0	0	1	1
χ_{16}	1035	27	0	-1	0	0	-1	-1	1	1	1	-1	-1	0	0	0	0
χ_{17}	2024	8	-1	0	-1	-1	1	1	0	0	0	1	1	-1	-1	0	0

Table B.5: Character table of M_{11}.

$[g]$	1A	2A	3A	4A	5A	6A	8A	8B	11A	11B
$[g^2]$	1A	1A	3A	2A	5A	3A	4A	4A	11B	11A
$[g^3]$	1A	2A	1A	4A	5A	2A	8A	8B	11A	11B
$[g^5]$	1A	2A	3A	4A	1A	6A	8B	8A	11A	11B
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	10	2	1	2	0	-1	0	0	-1	-1
χ_3	10	-2	1	0	0	1	a_2	$\overline{a_2}$	-1	-1
χ_4	10	-2	1	0	0	1	$\overline{a_2}$	a_2	-1	-1
χ_5	11	3	2	-1	1	0	-1	-1	0	0
χ_6	16	0	-2	0	1	0	0	0	β_{11}	$\overline{\beta_{11}}$
χ_7	16	0	-2	0	1	0	0	0	$\overline{\beta_{11}}$	β_{11}
χ_8	44	4	-1	0	-1	1	0	0	0	0
χ_9	45	-3	0	1	0	0	-1	-1	1	1
χ_{10}	55	-1	1	-1	0	-1	1	1	0	0

Table B.6: The twined series for M_{23}. The table displays the Fourier coefficients multiplying $q^{-D/56}$ in the q-expansion of the function $\widetilde{h}^{\mathcal{N}=2}_{g,1}(\tau)$.

[g]	1A	2A	3A	4A	5A	6AB	7AB	8A	11AB	14AB	15AB	23AB
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
79	32890	490	76	22	10	4	4	0	0	0	1	0
159	2969208	10136	585	80	18	17	4	0	0	0	0	0
239	101822334	88670	3192	374	54	-16	5	2	-2	1	-3	0
319	2065775107	636803	12550	947	132	74	11	-1	0	-1	0	0
399	29747513059	3408531	42757	2399	269	-15	12	5	0	0	2	0
479	334821538370	16448690	136784	5582	530	80	11	-4	0	-1	-1	0
559	3122115821404	68126268	386305	12996	824	33	20	-4	-3	0	5	0
639	25061866943436	262901388	1026324	26780	1586	360	29	4	4	1	-1	1
719	177895424302751	922681999	2615528	53771	2666	-320	29	-7	1	1	-7	0
799	1138785187015234	3070987058	6274135	104846	4574	1079	31	4	0	-1	5	-1
879	6672991048411185	9574047505	14472639	201593	7415	-401	57	9	3	1	-1	0
959	36211921311763437	28624358621	32442711	369065	11122	1079	58	-5	-2	-2	1	0
1039	183681040795024267	81543759179	70065910	662651	17967	-70	61	19	0	1	0	-1
1119	877475502920966100	224506987348	147298461	1169604	27740	3409	88	-16	-3	0	-4	0

Table B.7: The twined series for M_{23}. The table displays the Fourier coefficients multiplying $q^{-D/56}$ in the q-expansion of the function $\widetilde{h}^{\mathcal{N}=2}_{g,2}(\tau)$.

$[g]$	1A	2A	3A	4A	5A	6AB	7AB	8A	11AB	14AB	15AB	23AB
-9	1	1	1	1	1	1	1	1	1	1	1	1
71	14168	392	74	20	8	2	0	2	0	0	-1	0
151	1659174	6278	465	94	24	5	-1	2	0	-1	0	0
231	63544239	70287	2367	279	59	27	-4	-1	0	0	2	0
311	1373777350	471990	10699	786	125	-21	1	0	0	1	-1	0
391	20649050170	2768410	36727	2114	200	115	-7	2	1	1	2	0
471	239838441957	13053893	113958	5229	457	-34	-5	-3	4	-1	-2	-1
551	2291638384937	56517657	337376	11397	842	120	-8	-1	0	0	-4	0
631	18760451739204	216334868	899886	23784	1479	50	-7	-6	-1	1	6	0
711	135352127137850	778525770	2278664	48830	2600	528	-7	12	-4	1	-1	0
791	878471971333176	2585630360	5566971	97056	3901	-469	-18	8	0	-2	1	1
871	5209082274923427	8188169219	12900135	183083	6807	1475	-9	-5	-1	-1	0	0
951	28562269988425239	24491271063	28872441	336679	10799	-567	-32	3	0	0	-4	1
1031	146211017617763307	70510224443	62961633	610623	17107	1481	-15	-3	0	1	-2	0
1111	704198296122633807	194427334975	132796005	1086555	26522	-191	-37	-15	0	-1	5	0

Table B.8: The twined series for M_{23}. The table displays the Fourier coefficients multiplying $q^{-D/56}$ in the q-expansion of the function $\widetilde{h}_{g,3}^{\mathcal{N}=2}(\tau)$.

[g]	1A	2A	3A	4A	5A	6AB	7AB	8A	11AB	14AB	15AB	23AB
-25	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
55	2024	120	26	12	4	6	1	2	0	1	1	0
135	485001	2953	234	41	11	-2	-1	1	0	-1	-1	0
215	23912778	37850	1704	206	38	8	1	-4	-1	1	-1	0
295	594404250	276954	7008	634	105	12	-1	-2	0	-1	3	1
375	9795220335	1719215	25389	1679	215	77	5	3	-1	1	-1	0
455	121610515928	8440360	85280	3852	333	-56	6	2	0	-2	0	0
535	1223045193953	37766625	248780	9089	693	264	9	1	0	1	0	-1
615	10431487439956	148238340	677512	19744	1221	-96	2	2	0	-2	-3	0
695	77848480769761	545254705	1771723	40485	2236	259	2	7	-2	2	-2	0
775	519869748402405	1843176725	4327128	78673	3720	56	2	3	5	-2	3	0
855	3159048430391220	5930043604	10148229	152428	5665	1033	10	-20	4	2	4	0
935	17694698437501954	17975169890	23094682	285770	9394	-910	15	-14	0	-1	-8	0
1015	92296742373818321	52381498417	50515790	519049	14871	2710	31	13	-2	3	0	0
1095	452022567897804867	145967611235	107402373	917355	23372	-1123	26	-1	-4	-2	8	-1

Table B.9: The twined series for M_{23}. The table displays the Fourier coefficients multiplying $q^{-D/56}$ in the q-expansion of the function $\widetilde{h}^{\mathcal{N}=2}_{g,4}(\tau)$.

$[g]$	1A	2A	3A	4A	5A	6AB	7AB	8A	11AB	14AB	15AB	23AB
-49	1	1	1	1	1	1	1	1	1	1	1	1
31	23	7	5	3	3	1	2	1	1	0	0	0
111	61984	1008	109	36	14	9	6	-2	-1	0	-1	-1
191	4994473	12841	814	105	23	-2	8	1	0	-4	-1	0
271	159121844	126116	3851	384	79	47	14	6	2	4	1	1
351	3066459912	791976	14742	1104	107	-18	12	4	-1	-4	2	0
431	42526230351	4396655	52188	2871	301	44	24	-5	0	4	-2	0
511	465019661864	19995832	157790	6236	549	34	-19	-2	-1	3	0	0
591	4237704983457	83898753	443403	14105	1002	267	36	-3	0	8	3	0
671	33383739990645	313694485	1187433	29821	1780	-215	44	1	0	-8	-2	0
751	233270628632745	1105509481	2962041	60217	2700	769	60	-7	0	-8	6	-1
831	1473401102910159	3610317407	7067235	114659	4804	-289	66	-3	-6	4	-5	0
911	8534324476198088	11260342856	16341254	218344	7803	758	92	24	7	0	-6	0
991	45845203718962384	33250870352	36222793	402864	12389	-31	100	16	0	0	8	0
1071	230465514424059585	94612982465	77950314	723905	19565	2570	-99	-15	0	1	-1	1

Table B.10: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}_{g,1}^{\mathcal{N}=2}(\tau)$ into irreducible representations χ_n of M_{23} for $1 \leq n \leq 6$.

$[g]$	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6
-1	-1	0	0	0	0	0
79	3	6	0	0	8	4
159	13	50	4	4	172	109
239	75	520	361	361	3132	2637
319	555	6234	8431	8431	52201	48823
399	4516	72901	127496	127496	699946	683877
479	39919	762273	1458831	1458831	7687965	7629252
559	334018	6894918	13697115	13697115	70963150	70890020
639	2561300	54661450	110264073	110264073	567249277	568242219
719	17798711	385781306	783730795	783730795	4018617602	4030917545
799	112816113	2462973668	5020155321	5020155321	25701388426	25795911388
879	657802189	14413077161	29426216833	29426216833	150534019095	151134807838
959	3560695812	78161432884	159711506399	159711506399	816701680872	820091947106
1039	18036997856	396321128501	810190283194	810190283194	4142106992628	4159658528152
1119	86103293155	1892920594014	3870600609373	3870600609373	19786191912051	19870958706758

Table B.11: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}_{g,1}^{\mathcal{N}=2}(\tau)$ into irreducible representations χ_n of M_{23} for $7 \leq n \leq 12$.

$[g]$	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
-1	0	0	0	0	0	0
79	1	1	4	1	1	2
159	84	84	128	184	184	249
239	2474	2474	2975	7288	7288	8876
319	48214	48214	54404	152910	152910	181177
399	681735	681735	754547	2228718	2228718	2611937
479	7622433	7622433	8384529	25191152	25191152	29406002
559	70870712	70870712	77765357	235322052	235322052	274222110
639	568190993	568190993	622851914	1890404790	1890404790	2201283134
719	4030786690	4030786690	4416567105	13423360272	13423360272	15625363600
799	25795597950	25795597950	28258570095	85943199892	85943199892	100024910245
879	151134084106	151134084106	165547158795	503648520358	503648520358	586120983088
959	820090325240	820090325240	898251754417	2733239573184	2733239573184	3180668696724
1039	4159655024839	4159655024839	4555976147351	13864390059718	13864390059718	16133598865348
1119	19870951342688	19870951342688	21763871927454	66233398639795	66233398639795	77072943845016

Table B.12: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}_{g,1}^{\mathcal{N}=2}(\tau)$ into irreducible representations χ_n of M_{23} for $13 \leq n \leq 17$.

$[g]$	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}
-1	0	0	0	0	0
79	2	1	1	7	7
159	249	225	225	400	614
239	8876	9311	9311	11209	20448
319	181177	196277	196277	215961	411688
399	2611937	2864311	2864311	3052375	5912176
479	29406002	32384527	32384527	34136386	66480999
559	274222110	302544979	302544979	317457009	619667980
639	2201283134	2430487575	2430487575	2545442969	4973369259
719	15625363600	17258520366	17258520366	18058720902	35299442572
799	100024910245	110498190645	110498190645	115573171916	225958546463
879	586120983088	647547609561	647547609561	677144766331	1324034573732
959	3180668696724	3514164059458	3514164059458	3674386668984	7184990032658
1039	16133598865348	17825641954808	17825641954808	18637288293732	36444893250715
1119	77072943845016	85157221728649	85157221728649	89031831245163	174102949680701

Table B.13: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}_{g,2}^{\mathcal{N}=2}(\tau)$ into irreducible representations χ_n of M_{23} for $1 \leq n \leq 6$.

$[g]$	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6
-9	1	0	0	0	0	0
71	2	5	0	0	7	3
151	10	37	3	3	108	67
231	60	371	211	211	2094	1692
311	401	4320	5558	5558	35182	32682
391	3347	51686	88067	88067	489393	475958
471	29191	549308	1043607	1043607	5517942	5468790
551	247970	5076761	10046497	10046497	52142001	52052092
631	1925413	40964481	82518209	82518209	424787070	425421787
711	13572600	293700302	596218175	596218175	3058208800	3067141035
791	87112694	1900466946	3872364194	3872364194	19828140356	19899850416
871	513770437	11252794339	22969949734	22969949734	117516030536	117981038495
951	2809241950	61654430595	125970832520	125970832520	644191847539	646855266217
1031	14359666110	315486247432	644909227030	644909227030	3297183714290	3311123599862

Table B.14: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}^{\mathcal{N}=2}_{g,2}(\tau)$ into irreducible representations χ_n of M_{23} for $7 \leq n \leq 12$.

$[g]$	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
-9	0	0	0	0	0	0
71	0	0	2	0	0	0
151	45	45	74	100	100	137
231	1580	1580	1932	4481	4481	5533
311	32142	32142	36420	101485	101485	120436
391	474150	474150	525770	1545133	1545133	1812904
471	5463084	5463084	6012239	18038594	18038594	21063676
551	52035254	52035254	57111733	172694257	172694257	201278319
631	425376804	425376804	466340794	1414993671	1414993671	1647801950
711	3067027234	3067027234	3360726669	10212802503	10212802503	11888586088
791	19899571950	19899571950	21800037596	66296462153	66296462153	77160348340
871	117980393857	117980393857	129233185927	393155335789	393155335789	457538780532
951	646853822454	646853822454	708508249448	2155841669413	2155841669413	2508762643117
1031	3311120452151	3311120452151	3626606693880	11036094133318	11036094133318	12842424391871
1111	15947023462552	15947023462552	17466176597842	53154055601266	53154055601266	61853163157258

Table B.15: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}_{g,2}^{\mathcal{N}=2}(\tau)$ into irreducible representations χ_n of M_{23} for $13 \leq n \leq 17$.

$[g]$	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}
-9	0	0	0	0	0
71	0	0	0	5	3
151	137	125	125	229	343
231	5533	5714	5714	7145	12797
311	120436	130213	130213	144101	273914
391	1812904	1985578	1985578	2122817	4105045
471	21063676	23189159	23189159	24465217	47625182
551	201278319	222025040	222025040	233079356	454856422
631	1647801950	1819248831	1819248831	1905627188	3722950603
711	11888586088	13130671614	13130671614	13740785934	26857884945
791	77160348340	85238123579	85238123579	89156647614	174307658489
871	457538780532	505484997007	505484997007	528601141162	1033572367727
951	2508762643117	2771795449104	2771795449104	2898203583666	5667189790804
1031	12842424391871	14189261724139	14189261724139	14835429985991	29010332044013
1111	61853163157258	68340924042730	68340924042730	71450643678386	139722462064585

Table B.16: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}_{g,3}^{\mathcal{N}=2}(\tau)$ into irreducible representations χ_n of M_{23} for $1 \leq n \leq 6$.

$[g]$	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6
-25	-1	0	0	0	0	0
55	2	2	0	0	3	0
135	4	17	0	0	44	26
215	35	192	70	70	910	692
295	228	2109	2333	2333	15904	14401
375	1815	25661	41343	41343	235739	227084
455	15674	283294	527166	527166	2813624	2778595
535	135681	2728183	5353413	5353413	27892476	27802131
615	1082201	22844512	45852109	45852109	236430984	236628378
695	7845613	169151973	342809579	342809579	1759756137	1764351037
775	51675284	1125397975	2291272507	2291272507	11736660597	11777354738
855	311949208	6826465150	13929053191	13929053191	71275567395	71552199331
935	1741436074	38202099659	78037446817	78037446817	399108330301	400742735902
1015	9067633922	199170750739	407094744895	407094744895	2081435347715	2090191121215
1095	44366697752	975187152952	1993866656330	1993866656330	10192904243304	10236401971933

Table B.17: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}_{g,3}^{\mathcal{N}=2}(\tau)$ into irreducible representations χ_n of M_{23} for $7 \leq n \leq 12$.

$[g]$	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
-25	0	0	0	0	0	0
55	0	0	1	0	0	0
135	14	14	27	25	25	38
215	609	609	788	1643	1643	2065
295	14053	14053	16128	43581	43581	52061
375	225834	225834	251423	731026	731026	859812
455	2774317	2774317	3057500	9137701	9137701	10679771
535	27789758	27789758	30517710	92128896	92128896	107420534
615	236594479	236594479	259438583	786646427	786646427	916233373
695	1764262516	1764262516	1933413743	5873450857	5873450857	6837772000
775	11777138395	11777138395	12902535131	39231891343	39231891343	45662596088
855	71551692177	71551692177	78378155440	238424129443	238424129443	277474384470
935	400741580942	400741580942	438943677467	1335557549215	1335557549215	1554211032206
1015	2090188596103	2090188596103	2289359341885	6966572026982	6966572026982	8106871287186
1095	10236396601532	10236396601532	11211583746696	34119303660620	34119303660620	39703341303712

Table B.18: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}_{g,3}^{\mathcal{N}=2}(\tau)$ into irreducible representations χ_n of M_{23} for $13 \leq n \leq 17$.

$[g]$	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}
-25	0	0	0	0	0
55	0	0	0	1	0
135	38	30	30	78	103
215	2065	2080	2080	2799	4845
295	52061	55872	55872	63071	118714
375	859812	939215	939215	1011049	1948452
455	10679771	11745983	11745983	12423389	24153700
535	107420534	118443822	118443822	124470503	242778649
615	916233373	1011381183	1011381183	1059877938	2070176920
695	6837772000	7551522153	7551522153	7904063364	15447739906
775	45662596088	50440859848	50440859848	52765036392	103154220956
855	277474384470	306544968511	306544968511	320579896279	626812915588
935	1554211032206	1717144636530	1717144636530	1795503032799	3510906233262
1015	8106871287186	8957019447270	8957019447270	9365049511187	18313001324552
1095	39703341303712	43867672430453	43867672430453	45864145536600	89687451269313

Table B.19: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}_{g,4}^{\mathcal{N}=2}(\tau)$ into irreducible representations χ_n of M_{23} for $1 \leq n \leq 6$.

$[g]$	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6
-49	1	0	0	0	0	0
31	1	1	0	0	0	0
111	4	10	0	0	15	6
191	15	69	10	10	245	172
271	109	746	572	572	4753	4045
351	719	8794	12676	12676	76096	71969
431	6211	102911	182780	182780	996449	976116
511	54133	1051489	2029244	2029244	10653186	10587505
591	449634	9337434	18600800	18600800	96246988	96195371
671	3396944	72726076	146918168	146918168	755302028	756828089
751	23297382	505622990	1027806326	1027806326	5268663130	5285372319
831	145823768	3185848487	6495661454	6495661454	33250375888	33374695435
911	840907061	18431146756	37635327650	37635327650	192515167760	193289021419
991	4506789615	98947558783	202202119656	202202119656	1033940285159	1038249102575
1071	22628187385	497248672021	1016558364226	1016558364226	5197058914779	5219123838308

Table B.20: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}^{\mathcal{N}=2}_{g,4}(\tau)$ into irreducible representations χ_n of M_{23} for $7 \leq n \leq 12$.

$[g]$	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
-49	0	0	0	0	0	0
31	0	0	0	0	0	0
111	3	3	8	1	1	4
191	131	131	192	326	326	422
271	3864	3864	4584	11441	11441	13896
351	71227	71227	79986	227680	227680	269022
431	973518	973518	1076328	3188386	3188386	3734146
511	10579624	10579624	11630930	35000973	35000973	40841472
591	96173267	96173267	105510368	319449559	319449559	372208482
671	756768664	756768664	829494146	2518305187	2518305187	2932230208
751	5285224408	5285224408	5790846500	17602308990	17602308990	20489229802
831	33374342002	33374342002	36560188886	111198256076	111198256076	129415838256
911	193288204547	193288204547	211719348700	644138990502	644138990502	749610922070
991	1038247291426	1038247291426	1137194846082	3460365519723	3460365519723	4026806724990
1071	5219119941435	5219119941435	5716368606934	17395758923417	17395758923417	20242906870222

Table B.21: The table shows the decomposition of the Fourier coefficients multiplying $q^{-D/56}$ in the function $\widetilde{h}_{g,4}^{\mathcal{N}=2}(\tau)$ into irreducible representations χ_n of M_{23} for $13 \leq n \leq 17$.

$[g]$	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}
-49	0	0	0	0	0
31	0	0	0	0	0
111	4	1	1	14	14
191	422	405	405	632	1024
271	13896	14621	14621	17398	31919
351	269022	292364	292364	319046	610695
431	3734146	4097893	4097893	4358824	8450520
511	40841472	44996504	44996504	47382036	92324399
591	372208482	410707009	410707009	430804236	841061633
671	2932230208	3237782827	3237782827	3390298956	6624684857
751	20489229802	22631442859	22631442859	23678982898	46287128401
831	129415838256	142968950388	142968950388	149529067462	292352194117
911	749610922070	828178149562	828178149562	866014527476	1693351770023
991	4026806724990	4449040146378	4449040146378	4651835981222	9096369338035
1071	20242906870222	22365973077886	22365973077886	23384220869164	45727565759616

Appendix C

C.1 A few mock Jacobi forms

We introduce two special Jacobi forms which are needed in the discussion of the optimal choice of a mock Jacobi form for the first non-prime power index $m = 6$ in section 4.3.
The mock Jacobi form of weight 2 and index 6 $\mathcal{F}_6(\tau, z)$ is defined via

$$\mathcal{F}_6(\tau,z) := \eta(\tau)\, h^{(6)}(\tau)\, \frac{\vartheta_1(\tau,4z)}{\vartheta_1(\tau,2z)}\,, \tag{C.1.1}$$

$$h^{(6)}(\tau) = \frac{12\, F_2^{(6)}(\tau) - E_2(\tau)}{\eta(\tau)}\,, \tag{C.1.2}$$

$$F_2^{(6)}(\tau) = -\sum_{r>s>0} \chi_{12}(r^2 - s^2)\, s\, q^{rs/6}\,. \tag{C.1.3}$$

where $\chi_{12}(n)$ denotes the Kronecker symbol

$$\chi_{12}(n) = \left(\frac{12}{n}\right) = \begin{cases} +1 & \text{if} \quad n \equiv \pm 1 \pmod{12} \\ -1 & \text{if} \quad n \equiv \pm 5 \pmod{12} \\ 0 & \text{if} \quad (n,12) = 1\,. \end{cases} \tag{C.1.4}$$

The mock Jacobi form of weight 2 and index 6 $\mathcal{K}_6(\tau, z)$ is

$$\mathcal{K}_6(\tau,z) := \frac{1}{12^5}\Big(E_4 A B^5 - 5\, E_6 A^2 B^4 + 10\, E_4^2 A^3 B^3 - 10\, E_4 E_6 A^4 B^2 + (5\, E_4^3 - \frac{1}{4} D) A^5 B - E_4^2 E_6 A^6\Big)\,, \tag{C.1.5}$$

with $D := 2^{11}\, 3^3\, \eta^{24}(\tau)$. We report here the explicit expressions in terms of Jacobi theta functions and the Dedekind function

$$\varphi_{-2,1}(\tau,z) := A(\tau,z) = \frac{\vartheta_1^2(\tau,z)}{\eta^6(\tau)}, \tag{C.1.6}$$

$$\varphi_{0,1}(\tau,z) := B(\tau,z) = 4\Big(\frac{\vartheta_2^2(\tau,z)}{\vartheta_2^2(\tau)} + \frac{\vartheta_3^2(\tau,z)}{\vartheta_3^2(\tau)} + \frac{\vartheta_3^2(\tau,z)}{\vartheta_3^2(\tau)}\Big), \tag{C.1.7}$$

$$\varphi_{10,1}(\tau,z) = \eta^{18}(\tau)\,\vartheta_1^2(\tau,z)\,. \tag{C.1.8}$$

The Jacobi forms $\varphi_{-2,1}(\tau,z)$ and $\varphi_{0,1}(\tau,z)$ generate the ring of all weak Jacobi forms of even weight over the ring of modular forms [31].

C.2 Asymptotic and convergence

In this section we provide the details of the proof of Theorem 4.3.3. Our starting point is (4.3.17) and we examine the behavior of the Fourier coefficients when the order of the Farey sequence $N \to \infty$.

Proof. To evaluate the contribution to the first term in (4.3.17), denoted Σ_1, we split the negative and positive powers of q in the expansion of $f_{m,j}$. We denote the contribution of the former by Σ_1^* and write

$$\begin{aligned}\Sigma_1 =& \Sigma_1^* + \sum_{\substack{0\le h<k\le N\\(h,k)=1}} e^{-2\pi i \frac{h}{k}\frac{\Delta}{4m}+\frac{h'}{k}}\,\psi(\gamma)_{\ell j}\times\\ &\times\sum_{n_+>0}\alpha_m(n_+,j)\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}}\mathrm{d}\phi\, z^{21/2}\, e^{\frac{2\pi}{k}\left(z\frac{\Delta}{4m}-\frac{1}{z}\frac{\Delta^+}{4m}\right)}\,,\end{aligned} \tag{C.2.1}$$

where a sum over $j\in\mathbb{Z}/2m\mathbb{Z}$ is implied, $\Delta = 4mn-\ell^2$ and $\Delta^+ = 4mn_+ - j^2$. From the theory of Farey fractions, it is known that

$$\frac{1}{k+k_j} \le \frac{1}{N+1}\,, \quad j\in\{1,2\}\,, \tag{C.2.2}$$

where h_1/k_1 is the Farey fraction antecedent h/k and h_2/k_2 is the consecutive one. Therefore,

$$\vartheta'_{h,k},\,\vartheta''_{h,k} \le \frac{1}{kN}\,. \tag{C.2.3}$$

Also, recalling that $z = \frac{k}{N^2} - \mathrm{i}\,k\phi$ and that $-\vartheta'_{h,k}\le\phi\le\vartheta''_{h,k}$, we have the bound

$$|z|^2 \le \frac{k^2}{N^4} + \frac{1}{N^2}\,. \tag{C.2.4}$$

Using these results, we obtain a bound on the integral

$$\left| \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{21/2}\, e^{\frac{2\pi}{k}\left(z\frac{\Delta}{4m}-\frac{1}{z}\frac{\Delta^+}{4m}\right)} \mathrm{d}\phi \right| \le \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} |z|^{21/2}\, e^{\frac{\pi}{2mN^2}\left(\Delta-\Delta^+|z|^{-2}\right)} \mathrm{d}\phi\,. \tag{C.2.5}$$

Since $\Delta^+ > 0$ for $n_+ > 0$ and $j \in \mathbb{Z}/2m\mathbb{Z}$, we have that, when $N \to \infty$, the positive powers of q in the expansion of $f_{m,j}$ contribute to Σ_1 a term of order

$$\Sigma_1 = \Sigma_1^* + \mathcal{O}\Big(\sum_{\substack{0\le h<k\le N\\(h,k)=1}} \frac{1}{kN}\, N^{-21/2}\Big) = \Sigma_1^* + \mathcal{O}\Big(N^{-21/2}\Big)\,, \tag{C.2.6}$$

where the last equality follows from $\sum_{\substack{0\le h<k\le N\\(h,k)=1}} \left(\frac{1}{k}\right) = \sum_{0<k\le N} 1 = N$. In conclusion, the dominant contribution to Σ_1 when $N \to \infty$ comes from the polar terms in the Fourier expansion of $f_{m,j}$. This polar contribution is given by

$$\begin{aligned}\Sigma_1^* = \sum_{\substack{\widetilde{n}\ge\widetilde{n}_0\\ \widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}\\ \widetilde{\Delta}<0}} \alpha_m(\widetilde{n},\widetilde{\ell}) \sum_{\substack{0\le h<k\le N\\(h,k)=1}} e^{2\pi i(-\frac{h}{k}\frac{\Delta}{4m}+\frac{h'}{k}\frac{\widetilde{\Delta}}{4m})} \times \\ \times\, \psi(\gamma)_{\ell\widetilde{\ell}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \mathrm{d}\phi\, z^{21/2} e^{\frac{2\pi}{k}\left(z\frac{\Delta}{4m}-\frac{1}{z}\frac{\widetilde{\Delta}}{4m}\right)}\,,\end{aligned} \tag{C.2.7}$$

with $\widetilde{\Delta} = 4m\widetilde{n} - \widetilde{\ell}^2$ and $\widetilde{n}_0$ given in (4.3.19). We can now write the integral in terms of a Bessel function when $N \to \infty$. To do so, one needs to first write the integral in a symmetric way:

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{kN}}^{\frac{1}{kN}} - \int_{-\frac{1}{kN}}^{-\vartheta'_{h,k}} - \int_{\vartheta''_{h,k}}^{\frac{1}{kN}}\,. \tag{C.2.8}$$

The second and third term contribute an error term which vanishes in the $N \to \infty$ limit [76], and we are left with only the first integral. Integrals of these shapes can be evaluated using the method presented in [76]. For $a > 0$ and $b \in \mathbb{R}^*$, they give

$$\begin{aligned}\int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^r\, e^{\frac{2\pi}{k}(a\,z+\frac{b}{z})}\, \mathrm{d}\phi = \\ = \begin{cases} \frac{2\pi}{k}\left(\frac{b}{\sqrt{ab}}\right)^{r+1} I_{r+1}\left(\frac{4\pi}{k}\sqrt{ab}\right) + \mathcal{O}\left(\frac{1}{kN^{r+1}}\right), & \text{for } b>0 \\ \mathcal{O}\left(\frac{1}{kN^{r+1}}\right), & \text{for } b<0 \end{cases}\end{aligned} \tag{C.2.9}$$

where the I-Bessel function $I_\rho(z)$ has the following integral representation for $z \in \mathbb{R}^*$,

$$I_\rho(z) = \frac{1}{2\pi i}\left(\frac{x}{2}\right)^\rho \int_{\epsilon-i\infty}^{\epsilon+i\infty} t^{-\rho-1}\, e^{t+\frac{z^2}{4t}}\, \mathrm{d}t\,, \tag{C.2.10}$$

and asymptotics

$$I_\rho(z) \underset{x\to\infty}{\sim} \frac{e^z}{\sqrt{2\pi z}}\Big(1-\frac{\mu-1}{8z}+\frac{(\mu-1)(\mu-3^2)}{2!(8z)^3}+\dots\Big)\,, \tag{C.2.11}$$

Using this result and in the limit $N\to\infty$, (C.2.6) shows that for $\Delta>0$,

$$\Sigma_1 = \sum_{\substack{\widetilde{n}\geq\widetilde{n}_0\\ \widetilde{\ell}\in\mathbb{Z}/2m\mathbb{Z}\\ \widetilde{\Delta}<0}} \alpha_m(\widetilde{n},\widetilde{\ell}) \sum_{k=1}^{\infty}\sum_{\substack{0\leq h<k\\(h,k)=1}} e^{2\pi i(-\frac{h}{k}\frac{\Delta}{4m}+\frac{h'}{k}\frac{\widetilde{\Delta}}{4m})}\,\psi(\gamma)_{\ell\widetilde{\ell}}\times$$
$$\times\,\frac{2\pi}{k}\left(\frac{|\widetilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\Big(\frac{\pi}{mk}\sqrt{|\widetilde{\Delta}|\Delta}\Big)\,. \tag{C.2.12}$$

We now turn to the second term in (4.3.17), the shadow contribution Σ_2. Before extracting its asymptotic, we write the Eichler integral $\mathcal{I}_{m,\ell}(x)$, (4.3.16) , in a new form in terms of hyperbolic functions. Despite the differences with [77, 171], we can adopt the same procedure to prove the following identity

$$\int_0^\infty \frac{\vartheta^0_{m,j}(iw-\frac{h'}{k})}{(w+x)^{3/2}}\mathrm{d}w \;= \tag{C.2.13}$$
$$= \sum_{\substack{g\,(2mk)\\ g\equiv j\,(2m)}} e^{-\pi i\frac{g^2h'}{2mk}}\left(\frac{2\delta_{0,g}}{\sqrt{x}}-\frac{1}{\sqrt{2m}\,\pi k^2 x}\int_{-\infty}^{+\infty} e^{-2\pi x m u^2} f_{k,g,m}(u)\,\mathrm{d}u\right).$$

Using Mittag-Leffler theory (see §C.2.1), $f_{k,g,m}(u)$ takes the form

$$f_{k,g,m}(u) := \begin{cases} \frac{\pi^2}{\sinh^2(\frac{\pi u}{k}-\frac{\pi i g}{2mk})} & \text{if } g\not\equiv 0\,(\mathrm{mod}\,2mk)\,,\\ \frac{\pi^2}{\sinh^2(\frac{\pi u}{k})}-\frac{k^2}{u^2} & \text{if } g\equiv 0\,(\mathrm{mod}\,2mk)\,, \end{cases} \tag{C.2.14}$$

This different representation of the $\mathcal{I}_{m,\ell}(x)$ integral gives rise to two contributions: one for $g\equiv 0\,(2mk)$ and one for $g\not\equiv 0\,(2mk)$. The first one has itself two contributions, coming from the polar and non-polar terms in $\eta(\tau)^{-24}$. The non-polar terms contribute an error of the type (C.2.6), while the polar term q^{-1} yields

$$\Sigma^*_{2,\,g\equiv 0(2mk)} = \sum_{\substack{0\leq h<k\leq N\\(h,k)=1}} \sqrt{\frac{m}{8\pi^2}}\,\frac{2\sqrt{k}}{\kappa}\, e^{2\pi i(-\frac{h}{k}\frac{\Delta}{4m}-\frac{h'}{k})}\times$$
$$\times\,\psi(\gamma)_{\ell 0}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{11}\, e^{\frac{2\pi z}{k}\frac{\Delta}{4m}+\frac{2\pi}{kz}}\,\mathrm{d}\phi\,. \tag{C.2.15}$$

Once again, the ϕ integrals are evaluated using (C.2.9), which in the limit $N\to\infty$ gives

$$\Sigma_{2,\,g\equiv 0(2mk)} = \frac{\sqrt{2m}}{\kappa}\sum_{k=1}^{\infty}\sum_{\substack{0\le h<k\\(h,k)=1}} e^{2\pi i(-\frac{h}{k}\frac{\Delta}{4m}-\frac{h'}{k})}\times$$
$$\times\,\psi(\gamma)_{\ell 0}\frac{1}{\sqrt{k}}\left(\frac{4m}{\Delta}\right)^{6} I_{12}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta}\right). \qquad \text{(C.2.16)}$$

The second piece for $g\not\equiv 0\ (2mk)$ requires an analysis similar to the one conducted in [77] above Lemma 3.2. Introducing, for $b>0$ and $g\in\mathbb{Z}$,

$$\mathcal{J}_{k,g,m,b}(z) = e^{\frac{2\pi b}{kz}}\, z^{23/2}\int_{-\sqrt{b}}^{\sqrt{b}} e^{-\frac{2\pi m u^2}{kz}}\, f_{k,g,m}(u)\,\mathrm{d}u\,, \qquad \text{(C.2.17)}$$

one can show that, in the limit $N\to\infty$,

$$\Sigma^{*}_{2,\,g\not\equiv 0(2mk)} = -\frac{1}{4\pi^2\kappa}\sum_{\substack{0\le h<k\le N\\(h,k)=1}}\frac{1}{k}\sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z}\\ g(2mk)\\ g\equiv j(2m)}} e^{2\pi i\left(-\frac{h}{k}\frac{\Delta}{4m}-\frac{h'}{k}\left(1+\frac{g^2}{4m}\right)\right)}\times$$
$$\times\,\psi(\gamma)_{\ell g}\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi}{k}z\frac{\Delta}{4m}}\mathcal{J}_{k,g,m,1}(z)\,\mathrm{d}\phi\,. \qquad \text{(C.2.18)}$$

We now evaluate the ϕ integral as usual. Using (C.2.9) and in the limit $N\to\infty$, this truncates the integration range over u to the region where $1-mu^2$ is positive, i.e.

$$\Sigma_{2,\,g\not\equiv 0(2mk)} =$$
$$= \frac{-1}{2\pi\kappa}\sum_{k=1}^{\infty}\sum_{\substack{j\in\mathbb{Z}/2m\mathbb{Z}\\ g(2mk)\\ g\equiv j(2m)}}\sum_{\substack{0\le h<k\\(h,k)=1}} e^{2\pi i\left(-\frac{h}{k}\frac{\Delta}{4m}-\frac{h'}{k}\left(1+\frac{g^2}{4m}\right)\right)}\psi(\gamma)_{\ell g}\frac{1}{k^2}\left(\frac{4m}{\Delta}\right)^{25/4}\times$$
$$\times\int_{-1/\sqrt{m}}^{+1/\sqrt{m}} f_{k,g,m}(u)\, I_{25/2}\left(\frac{2\pi}{k\sqrt{m}}\sqrt{\Delta(1-mu^2)}\right)(1-mu^2)^{25/4}\,\mathrm{d}u\,. \qquad \text{(C.2.19)}$$

Adding the three contributions Σ_1, $\Sigma_{2,\,g\equiv 0(2mk)}$ and $\Sigma_{2,\,g\not\equiv 0(2mk)}$ gives the claim of Theorem 4.3.3. ∎

C.2.1 Mittag-Leffler theory

To extract the contribution of the shadow to the asymptotic of the function, we write the non-holomorphic Eichler integral of the shadow of h_j in a new form in terms of hyperbolic functions. In particular, we prove the following[1]

$$\int_0^\infty \frac{\Theta_j(iw - \frac{h'}{k})}{(w+x)^{3/2}}dw = \\ = \sum_{g\ (2k), g\equiv j(2)} e\left(-\frac{g^2h'}{4k}\right)\left(\frac{2\delta_{0,g}}{\sqrt{x}} - \frac{1}{\sqrt{2\pi k^2 x}}\int_{-\infty}^{+\infty} e^{-2\pi x u^2} f_{k,g}(u)du\right) \tag{C.2.20}$$

where

$$f_{k,g}(u) := \begin{cases} \frac{\pi^2}{\sinh^2(\frac{\pi u}{k} - \frac{\pi i g}{2k})} & \text{if } g \not\equiv 0 \,(\text{mod}\, 2k) \\ \frac{\pi^2}{\sinh^2(\frac{\pi u}{k})} - \frac{k^2}{u^2} & \text{if } g \equiv 0 \,(\text{mod}\, 2k) \end{cases} \tag{C.2.21}$$

and the theta function is defined by

$$\Theta_j\left(iw - \frac{h'}{k}\right) = \sum_{g\ (2k), g\equiv j(2)} e^{-\frac{g^2h'}{4k}} \sum_{n\in\mathbb{Z}} e^{-2\pi w(\frac{g}{2}+nk)^2}. \tag{C.2.22}$$

Notice that

$$\frac{1}{2\sinh^2(\frac{\pi u}{k} - \frac{\pi i g}{2k})} = \mathcal{L}\left(\frac{1}{1 - e^{2\pi i r - 2\pi(\frac{u}{k} - \frac{ig}{2k})}} - \frac{1}{1 - e^{-2\pi i r + 2\pi(\frac{u}{k} - \frac{ig}{2k})}}\right)$$

where $\mathcal{L}$ is an operator defined by $\mathcal{L}(h) = \frac{1}{2\pi i}(\partial_r h)|_{r=0}$. Thus, we define

$$J^{\pm}(u;r) := \frac{1}{1 - e^{\pm 2\pi i r \mp 2\pi(\frac{u}{k} - \frac{ig}{2k})}}, \\ J(u) := \mathcal{L}(J^+(u;r) - J^-(u;r)) = \frac{1}{2\sinh^2(\frac{\pi u}{k} - \frac{\pi i g}{2k})}.$$

The poles of $J^{\pm}(u;r)$ are located at $\frac{u}{k} = i(\frac{g}{2k} + r + n)$ where $n \in \mathbb{Z}$ and from the residues of $J^{\pm}$ we obtain the following equality

$$\frac{1}{1 - e^{2\pi i r - 2\pi(\frac{u}{k} - \frac{ig}{2k})}} - \frac{1}{1 - e^{-2\pi i r + 2\pi(\frac{u}{k} - \frac{ig}{2k})}} = \\ \frac{-1}{2\pi}\left(\frac{1}{\frac{u}{k} - i(\frac{g}{2k} + r + n)} + \frac{1}{\frac{u}{k} - i(\frac{g}{2k} + r + n)}\right) \tag{C.2.23}$$

[1]For more details the reader is referred to [76].

We now focus on the $g \not\equiv 0(2k)$ term on the right-hand side of equation (C.2.20) and elide the first exponential factor (being superfluous to the proof),

$$
\begin{aligned}
&-\sum_{g\,(2k),g\equiv j(2)}\left(\frac{1}{\sqrt{2}\pi k^2 x}\int_{-\infty}^{+\infty} e^{-2\pi x u^2}\frac{\pi^2}{\sinh^2(\frac{\pi u}{k}-\frac{\pi i g}{2k})}du\right) = \\
&= -\sum_{g\,(2k),g\equiv j(2)}\left(\frac{\sqrt{2}}{\pi k^2 x}\int_{-\infty}^{+\infty} e^{-2\pi x u^2}\frac{\pi^2}{2\sinh^2(\frac{\pi u}{k}-\frac{\pi i g}{2k})}du\right) \\
&= -\sum_{g\,(2k),g\equiv j(2)}\sum_{n\in\mathbb{Z}}\left(\frac{\sqrt{2}}{\pi k^2 x}\int_{-\infty}^{+\infty} e^{-2\pi x u^2}\frac{-\pi^2}{2\pi^2(\frac{u}{k}-i(\frac{g}{2k}+n))^2}du\right) \\
&= \sum_{g\,(2k),g\equiv j(2)}\sum_{n\in\mathbb{Z}}\left(\frac{\sqrt{2}}{\pi k^2 x}\int_{-\infty}^{+\infty} e^{-2\pi x u^2}\frac{1}{2(\frac{u}{k}-i(\frac{g}{2k}+n))^2}du\right) \\
&= \sum_{g\,(2k),g\equiv j(2)}\sum_{n\in\mathbb{Z}}\left(\frac{1}{\pi\sqrt{2x}}\int_{-\infty}^{+\infty} \frac{e^{-2\pi x u^2}}{(u-i(\frac{g}{2}+nk))^2}du\right) \qquad \text{(C.2.24)}\\
&= \sum_{g\,(2k),g\equiv j(2)}\sum_{n\in\mathbb{Z}}\int_{0}^{+\infty} \frac{e^{-2\pi w(\frac{g}{2}+nk)^2}}{(w+x)^{3/2}}dw\,. \qquad \text{(C.2.25)}
\end{aligned}
$$

In equation (C.2.24) we used the identity

$$
\int_{\mathbb{R}}\frac{e^{-2\pi x u^2}}{(u-is)^2}du = \sqrt{2}\pi x\int_0^{\infty}\frac{e^{-2\pi w s^2}}{(w+x)^{3/2}}dw\,. \qquad \text{(C.2.26)}
$$

Therefore, we obtain

$$
\begin{aligned}
&-\sum_{g\,(2k),g\equiv j(2)}\left(\frac{1}{\sqrt{2}\pi k^2 x}\int_{-\infty}^{+\infty} e^{-2\pi x u^2}\frac{\pi^2}{\sinh^2(\frac{\pi u}{k}-\frac{\pi i g}{2k})}du\right) = \\
&\qquad\qquad = \sum_{g\,(2k),g\equiv j(2)}\sum_{n\in\mathbb{Z}}\int_{0}^{+\infty} \frac{e^{-2\pi w(\frac{g}{2}+nk)^2}}{(w+x)^{3/2}}du
\end{aligned}
$$

The rest of the identity originates from the term with $g \equiv 0(2k)$. Following the same steps as above we obtain a contribution of the form $\frac{\pi^2}{\sinh^2(\frac{\pi u}{k})}$, except for the term with g identically 0 which needs special attention. First, we eliminate the $g = 0$ contribution to the sinh which is $-\frac{k^2}{u^2}$ and then we compute the integral with $g = 0$ which is a simple integral of the form

$$
\int_0^{\infty}\frac{e^{-2\pi w(nk)^2}}{(w+x)^{3/2}}dw = \frac{2}{\sqrt{x}}. \qquad \text{(C.2.27)}
$$

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Contributions to publications

The author participated in all the conceptual discussions for all the publications. The order of the authors does not correspond to the contribution to the paper, they are listed in alphabetical order.

[1] F. Ferrari, S. M. Harrison,
"Properties of Extremal CFTs with small central charge",
`arXiv:1710.10563 [hep-th]`.

The author contributed to Section 2, and part of Section 3 performed the calculations in Section 6 except Section 6.4 and performed the computations for Appendix C. The author contributed to the writing of the manuscript.

[2] F. Ferrari, V. Reys,
"Mixed Rademacher and BPS black holes",
JHEP 1707 (2017) 094, `arXiv:1702.02755 [hep-th]`.

The author contributed to parts of Section 2, contributed to the results of Section 3 and Appendix B. The author contributed to the writing of the manuscript.

[3] M. C. N. Cheng, S. Chun, F. Ferrari, S. Gukov, S. M. Harrison,
to appear.

The author contributed to the extension of the Rademacher sums, the analysis of q-hypergeometric series and part of the calculations of the examples.

Summary

> *"What is mathematics?*
> *It is notoriously dangerous to attempt to formulate definitions but,*
> *on the other hand, it must not be forgotten that a bad definition may provoke a good one.*
> *I would therefore venture provisionally to suggest that:*
> *Mathematics is the science by which we investigate those characteristics of any subject-matter of thought which are due to the conception that it consists of a number of differing and non-differing individuals and pluralities.*
> *If the result of this attempt be only to elicit conclusive proof that mathematics is something else, and an indication of what it really is, my main object in this brief address will have been attained."*
>
> L. J. Rogers, *Third Memoir on the Expansion of certain Infinite Products*

In a world (full) of diversity, that most of the time we are unable to comprehend, one of the main tools to address questions and problems is to abstract. This is, roughly, what mathematics does: it looks for unifying themes by abstracting concepts. In doing so, mathematics conveys harmony to the plurality of entities it strives to describe. Historically, physics and mathematics have deeply influenced and inspired each other. The force of this inspiration comes from the fact that they are guided by different principles, questions and, perhaps most importantly, intuitions.

Interestingly, string theory, born as a physical theory, turned out to be deeply connected to different branches of mathematics. The original playground of string theory was quantum chromodynamics, but soon after it has been implemented to unify the theory describing particle physics (the Standard Model) with the theory of general relativity. However, no experiment has demonstrated yet that string theory is the right framework to describe quantum gravity. Not yet constrained by experimental proves, string theory lives "unchained". The beauty (and the

danger) of it is that we can sail on an ocean of possibilities in the search for new inspirations.

The work presented here lies at the intersection between the branch of mathematics called number theory and the physics of quantum field theories and string theories. Number theory guides us through the various chapters, from the beginning to the end of this thesis: "A glimpse of 2d and 3d quantum field theories through number theory".

Even though string theory is often the framework of this work, we primarily focus on two-dimensional conformal field theories. These can be thought to describe physical systems where distances do not matter and only angles do; from a string theory perspective, a conformal field theory is the theory living on the (worldsheet of) the string. Consider, for instance, a closed string. This is given by a loop that in a finite amount of time delineates a cylinder (the worldsheet of the string). Imagine now to take the initial time equal to the final time. In this way, we create out of the cylinder a two-dimensional surface: a torus. A torus is a genus one surface with a hole and no boundary.

Thanks to the fact that the physical system living on the torus is described by a conformal field theory, we can change distances, rescale the torus and still measure the same physical quantities. In other words, the physical observables correspond to functions invariant under these transformations and defined on the torus. These functions are fundamental objects in number theory and are known as modular functions. The main character of this thesis – a mock modular form – generalizes the concept of modular functions to objects that are almost invariant under conformal transformations.

Till now, we have focused on a 2d theory living on the torus. However, we could view this surface as the boundary of another space. The latter is 3-dimensional (one dimension more than its boundary) and it corresponds to the interior of the torus displayed above. By analogy, we can view a two-dimensional conformal field theory as boundary theory of a three-dimensional bulk theory. These 2d and 3d theories appear in different disguises in the various chapters of this thesis.

***

Results and future directions

In the following, we summarize the contribution of this thesis and some possible future directions.

Chapter 3
In this chapter, we examine properties of extremal (chiral) conformal field theories with small central charge $c \leq 24$. We primarily focus on the analysis of the Rademacher sums of their twining functions. Most of the theories considered have a few twining functions that are not expressible as Rademacher sums at the infinity cusp. The only extremal conformal field theories (ECFTs) whose twining functions satisfy the aforementioned property are certain $c = 12$, $\mathcal{N} = 4$ ECFTs with symmetry groups M_{22} and M_{11}. These results suggest that the connection between sporadic symmetry groups, mock modular forms, and 2d CFTs does not hinge on the strict Rademacher summability properties at the infinite cusp present in most cases of moonshine. An interesting question is whether there exists a 3d quantum gravity interpretation to the expression of the Rademacher sums where multiple cusps are included. An intriguing feature of specific ECFTs, such as the monster CFT, is the presence of genus zero subgroups of $SL_2(\mathbb{R})$, which has not yet a clear physical interpretation.

Chapter 4
Chapter 4 is dedicated to the study of the exact entropy of certain supersymmetric black holes. We focus on the microscopic degeneracy of 1/4 BPS black holes in Type IIB string theory compactified on $\text{K3} \times T^2$. This black hole system provides a rare opportunity to compare the sub-leading terms in the entropy of the brane system with the macroscopic entropy provided by the supergravity theory. We derive an explicit formula for the Fourier coefficients of the microscopic counting function. This corresponds to a Rademacher sum for a mixed mock modular form. This work paves the way for a complete matching between the microscopic degeneracy and quantum entropy. A topic that deserves future investigation. Another interesting question that deserves further analysis is what is the role of automorphic forms in black hole physics and their connection to generalized Kac-Moody algebras and Siegel modular forms. Further research on these topics might provide important hints for what is behind the appearance of sporadic groups in physics.

Chapter 5
We investigate the resurgence analysis of analytically continued Chern-Simons theory on a 3-manifold. Through the analysis of different $SL_2(\mathbb{Z})$ representations, we derive explicit formulae to efficiently extract information about irreducible flat connections of Brieskorn homology spheres, Seifert fibered 3-manifolds with three singular fibers and trivial first homology group. For general Seifert fibered 3-

manifolds, we define the homological blocks associated to indefinite 3-manifolds, thus extending the works on negative definite 3-manifolds. We show under which condition the homological block defines a convergent q-series inside the unit disc. The latter corresponds to a false theta function analyzed in the resurgence analysis. In addition, we provide explicit q-series candidates for the homological blocks of 3-manifolds with opposite orientation when the resulting homological block is an optimal mock theta function. The map between a homological block for a 3-manifold and the one associated to a 3-manifold with opposite orientation is not canonical. However, we show that the Rademacher sum provides a neat map between false theta functions and mock theta functions for the example considered. Mysterious is the appearance of projective representations of $SL_2(\mathbb{Z})$ that are attached to each 3-manifold, this topic deserves further investigation. Quite surprisingly mock modular forms, including the examples that appear in Ramanujan's work and umbral moonshine, also appear in this context; it would be interesting to study further this connection.

***

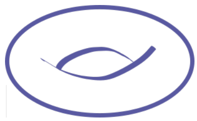
Samenvatting

In een wereld vol verscheidenheid, die meestal niet te bevatten valt, is abstractie één van de belangrijkste hulpmiddelen om vragen en problemen aan te pakken. Dit is grofweg wat wiskunde doet: het zoekt naar overkoepelende thema's door concepten te abstraheren. Hierdoor schept wiskunde harmonie in deze veelvoud van entiteiten. Natuurkunde en wiskunde hebben elkaar vanouds sterk beïnvloed en geïnspireerd. De kracht van deze inspiratie schuilt erin dat beide gebieden gebaseerd zijn op verschillende principes, verschillende vragen en in het bijzonder verschillende intuïties.

Interessant genoeg is gebleken dat snaartheorie, een van oorsprong natuurkundige theorie, op een diepe manier gerelateerd is aan verschillende gebieden in de wiskunde. Snaartheorie vindt zijn oorsprong in de sterke kernkracht, maar werd al snel geïmplementeerd om de theorie van de elementaire deeltjes (het standaard model) en de algemene relativiteitstheorie te unificeren. Er is echter nog geen experiment waaruit is gebleken dat snaatheorie het juiste raamwerk biedt om kwantum zwaartekracht te beschrijven. Nog niet geketend door experimentele bewijzen leeft snaarhteorie een vrij bestaan. De schoonheid (en ook het gevaar) van dit is dat we kunnen zeilen op een oceaan van mogelijkheden, op zoek naar nieuwe inspiraties.

Het werk bevat in deze scriptie ligt op het raakvlak tussen het gebied in de wiskunde genaamd getaltheorie en de natuurkunde van quantumveldentheorieën en snaartheorieën. Getaltheorie is als een onzichtbare draad van begin tot eind door de hoofdstukken gewoven van het proefschrift: "Een glimp van 2d en 3d quantumveldentheorieën via getaltheorie".

Ook al is snaatheorie vaak het raamwerk van dit proefschrift focussen we voornamelijk op twee-dimensionale conforme velden theorieën. Dit soort theorieën beschrijft fysische systemen waar afstanden er niet toedoen, slechts hoeken. Vanuit het perspectief van snaartheorie is een conforme velden theorie de theorie die leeft op het wereldvlak van een snaar. Bekijk bijvoorbeeld een gesloten snaar. Dit is een lus die in een eindige hoeveelheid tijd een cylinder omsluit (het wereldvlak van de snaar). Stel je nu voor dat je de begintijd van dit tijdsinterval gelijk neemt aan de eindtijd. Op deze manier creëren we vanuit de cylinder een twee-dimensionaal oppervlak: een torus. Een torus is een genus één oppervlak met een gat en geen rand.



Dankzij het feit dat het fysische systeem op de torus beschreven wordt door een conforme velden theorie kunnen we de afstanden veranderen, de torus herschalen, en nog steeds dezelfde fysische grootheden meten. Met andere woorden, de fysische grootheden corresponderen met functies die invariant zijn onder dit soort transformaties en gedefiniëerd zijn op de torus. Deze functies zijn fundamentele objecten in getaltheorie en staan bekend als modulaire functies. De hoofdrolspeler van dit proefschrift, een "bedriegelijke modulaire vorm", generaliseert het concept van modulaire functies tot objecten die bijna invariant zijn onder conforme transformaties.

Tot zover hebben we gefocust op een 2d theorie op een torus. We kunnen dit oppervlak echter ook zien als de rand van een andere ruimte. Laatstgenoemde is dan een drie-dimensionale ruimte (één dimensie meer dan de rand) en correspondeert bijvoorbeeld met het inwendige van de torus. Analoog hieraan kunnen we een twee-dimensionale conforme velden theorie beschouwen als een randtheorie van een drie-dimensionale volume theorie. De 2d en 3d theorieën komen tevoorschijn in verschillende gedaantes in de verscheidene hoofdstukken van dit proefschrift.

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