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Noether Symmetries and Conservation Laws in Non-Static Plane Symmetric Spacetime

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Abstract: In this paper, we find all nonstatic plane symmetric spacetime metrics whose corresponding Lagrangians possess Noether symmetries. The set of determining equations is analyzed through a Maple algorithm that restricts the metric coefficients to satisfy certain conditions. These restrictions on metric coefficients, while using them to solve the determining equations, give rise to a number of plane symmetric metrics admitting 4-, 5-, 6-, 7-, 8-, 9-, 11-, and 17-dimensional Noether algebras. The Noether theorem is used to find a conserved quantity corresponding to each Noether symmetry. Some physical implications are discussed by finding bounds for different energy conditions for the obtained metrics.

Keywords: non-static plane symmetric spacetime; Noether symmetries; energy conditions; conservation laws

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1. Introduction

Among all theories of gravitation in modern physics, the theory of general relativity plays a dominant role and it is considered to be the most fundamental and accurate theory. This theory was proposed by Albert Einstein in 1915 as a generalization of his previous scientific theory, the special theory of relativity. As a result of the development of this theory, Newton's law of universal gravitation was refined, and gravity was treated as a distortion of spacetime rather than a force of attraction between massive objects. The distortion in spacetime is known as the curvature and occurs because of the presence of matter in some region of spacetime. This curvature is measured with a tensor quantity G_{ab} known as the Einstein tensor, which is defined as $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$. Here, $R_{ab} =$ is the Ricci tensor, $g_{ab} =$ is the metric tensor, and R is the Ricci scalar. On the other hand, the physics of spacetime is given by another tensor quantity, T_{ab}, called energy-momentum tensor. The tensors representing the geometry and physics of spacetime are related as follows: $G_{ab} = kT_{ab}$, where $k = \frac{8\pi G}{c^4}$ is a coupling constant with $G =$ Newtonian constant of gravitation, and c denotes the speed of light. Indices a and b vary from 0 to 3, and because of the symmetric nature of G_{ab} and T_{ab}, equation $G_{ab} = kT_{ab}$ gives rise to a system of 10 tensor equations, Einstein's field equations (EFEs) [1].

Though EFEs are highly nonlinear, and it is very challenging to find their exact solutions, some physically important solutions of these equations, including Schwarzschild, Kerr, Friedmann, and plane wave solutions are found in the literature. These solutions of EFEs play a key role in the study of black holes, the field of cosmology, and in gravitational radiation. In order to find the exact solutions of EFEs, one always needs some assumptions to be imposed on the metric or Riemann tensors. Out of these assumptions, the most common is the symmetry restriction on g_{ab}, which is defined in terms of a smooth vector field V (called the Killing vector field, KVF) satisfying the following condition [2]:

$$\mathcal{L}_V g_{\mu\nu} = g_{\mu\nu,\lambda} V^{\lambda} + g_{\mu\lambda} V^{\lambda}_{,\nu} + g_{\nu\lambda} V^{\lambda}_{,\mu} = 0, \tag{1}$$

where $\mathcal{L}$ denotes a Lie derivative operator. Killing vector fields are also used for finding the conservation laws in a physical system and particularly in spacetimes. In particular, if a spacetime possesses a timelike KVF, it predicts the conservation of energy. Similarly, KVFs representing spacial translations and rotations give rise to the conservation of linear and angular momenta, respectively.

Another important spacetime symmetry is interpreted in terms of homothetic vector fields (HVFs), which are similarly defined as KVFs are, with the difference that the right-hand side of Equation (1) is replaced with $2\alpha g_{\mu\nu}$, where α is a constant. Moreover, if α is a function dependent on four coordinates of a spacetime, then V defines a conformal vector field. In the literature, these three symmetries are thoroughly discussed [3–16].

For every spacetime metric given by $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$, the corresponding Lagrangian $L = L(s, x^a, \dot{x^a})$ is defined as $L = g_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds}$, where s denotes the arc length parameter, and the derivative of x^a with respect to s is denoted by a dot over it. If V is a vector field of the form $V = \xi \partial_s + V^a \partial_{x^a}$, and $V^{[1]} = V + V^a_{,s} \partial_{\dot{x}^a}$ signifies its first prolongation, then V defines a Noether symmetry of L if there exists some function $F(s, x^a)$, known as the gauge function, such that:

$$V^{[1]} L + (D_s \xi) L = D_s F, \tag{2}$$

where $D_s = \partial_s + \dot{x^a} \partial_{x^a}$; and ξ and V^a depend on s and x^a. If V is a Noether symmetry vector field, then by the Noether theorem, its associated conservation law is expressed as follows [17]:

$$I = \xi L + (V^a - \dot{x^a} \xi) \frac{\partial L}{\partial \dot{x^a}} - F. \tag{3}$$

Noether symmetries play a pivotal role in the classification of spacetimes and in solving differential equations. In the case of complicated higher-order ordinary differential equations, these symmetries help in reducing their orders; for partial differential equations, they reduce the complexity of equations by reducing the number of independent variables. Apart from this, these symmetries are also helpful in the linearization of nonlinear differential equations [18–20].

There exist some well-known relations between the symmetries of a spacetime metric and the Noether symmetries of its corresponding Lagrangian. Every KVF admitted by a spacetime metric is also a Noether symmetry of its corresponding Lagrangian, but the converse is not true. Moreover, if a spacetime metric admits a homothetic symmetry V, then the associated Lagrangian possesses a Noether symmetry of the form $V + 2\alpha s \partial_s$, where α represents the homothety constant. Conversely, if the Lagrangian admits a Noether symmetry of the form $V + 2\alpha s \partial_s$, then V is a homothetic symmetry of the corresponding metric [21]. By 'proper Noether symmetry', we mean a Noether symmetry that neither is it associated with a HVF nor is it a KVF. Moreover, there exist no general relation between conformal and Noether symmetries except for the flat Minkowski metric that admits 15 conformal vector fields, and the set of these 15 conformal vector fields is a proper subset of the set of 17 Noether symmetries for the Lagrangian of the Minkowski metric. For the details of Noether symmetries admitted by some well-known Lagrangians, we refer to [22–29].

In this paper, we explore Noether symmetries of the Lagrangian associated to the most general nonstatic plane symmetric spacetime metric. For this purpose, we follow a recently developed approach that uses a Maple algorithm for analyzing the determining equations and obtaining all possible metrics possessing these symmetries. After that, the set of determining equations are solved for all these metrics to obtain the final form of Noether symmetries.

2. Determining Equations

The general nonstatic plane symmetric metric can be written in the following form [1]:

$$ds^2 = -A^2(t,x)\,dt^2 + B^2(t,x)\,dx^2 + C^2(t,x)\,(dy^2 + \,dz^2) \tag{4}$$

with three minimal KVFs given by $V_{(1)} = \partial_y$, $V_{(2)} = \partial_z$ and $V_{(3)} = z\partial_y - y\partial_z$. The main reason behind considering the nonstatic plane symmetric metric for the current study is that it contains some well-known important classes of solutions to the field equations. In the case when A, B, and C depend only on x, this metric represents static plane symmetric spacetime that admits an additional KVF given by ∂_t. Static plane symmetric metrics are crucial from many physical perspectives; for example, these metrics are helpful in finding Kasner's spatially homogeneous solutions of the field equations [30] and Taub's well-known solution [31]. These solutions are further used for the study of kinematics and dynamics of nonrotating rigid bodies. Similarly, for $A = A(t)$, $B = B(t)$ and $C = C(t)$, the metric (4) gives the locally rotationally symmetric Bianchi type I metric for which an additional KVF is given by ∂_x. Bianchi-type metrics are well-known homogeneous but not necessarily isotropic cosmological models that are the solutions of field equations, and their types depend upon different choices of scale factors. In particular, if $A = B = C = A(t)$, Metric (4) becomes the well-known Friedmann metric that is widely used in the field of cosmology. Thus, the present study also classifies the static plane symmetric Bianchi type I and Friedmann metrics according to their Noether symmetries as a subcase. Moreover, as stated in the introduction, Killing and homothetic algebras form subsets of Noether algebra, so the present work also provides a complete classification of these three metrics and the general nonstatic plane symmetric metric via Killing and homothetic symmetries. Below is the Lagrangian associated with Metric (4):

$$L = -A^2\,\dot{t}^2 + B^2\,\dot{x}^2 + C^2\,[\dot{y}^2 + \dot{z}^2], \tag{5}$$

and the corresponding geodesic equations are:

$$\begin{aligned} A^2\,\ddot{t} + AA_{,t}\,\dot{t}^2 + 2AA_{,x}\,\dot{t}\,\dot{x} + BB_{,t}\,\dot{x}^2 + CC_{,t}\,(\dot{y}^2 + \dot{z}^2) &= 0, \\ B^2\,\ddot{x} + AA_{,x}\,\dot{t}^2 + 2BB_{,t}\,\dot{t}\,\dot{x} + BB_{,x}\,\dot{x}^2 - CC_{,x}\,(\dot{y}^2 + \dot{z}^2) &= 0, \\ C\,\ddot{y} + 2C_{,t}\,\dot{t}\,\dot{y} + 2C_{,x}\,\dot{x}\,\dot{y} &= 0, \\ C\,\ddot{z} + 2C_{,t}\,\dot{t}\,\dot{z} + 2C_{,x}\,\dot{x}\,\dot{z} &= 0. \end{aligned} \tag{6}$$

We used Lagrangian (5) in Equation (2) to obtain the following determining equations:

$$\xi_{,t} = \xi_{,x} = \xi_{,y} = \xi_{,z} = F_{,s} = 0, \tag{7}$$

$$2A_{,t}V^0 + 2A_{,x}\,V^1 + 2AV^0_{,t} = A\xi_s, \tag{8}$$

$$2B_{,t}V^0 + 2B_{,x}\,V^1 + 2BV^1_{,x} = B\xi_s, \tag{9}$$

$$2C_{,t}V^0 + 2C_{,x}\,V^1 + 2CV^2_{,y} = C\xi_s, \tag{10}$$

$$2C_{,t}V^0 + 2C_{,x}\,V^1 + 2CV^3_{,z} = C\xi_s, \tag{11}$$

$$A^2V^0_{,x} - B^2V^1_{,t} = 0, \tag{12}$$

$$A^2V^0_{,y} - C^2V^2_{,t} = 0, \tag{13}$$

$$A^2V^0_{,z} - C^2V^3_{,t} = 0, \tag{14}$$

$$B^2V^1_{,y} + C^2V^2_{,x} = 0, \tag{15}$$

$$B^2V^1_{,z} + C^2V^3_{,x} = 0, \tag{16}$$

$$V^2_{,z} + V^3_{,y} = 0, \tag{17}$$

$$2A^2V^0_{,s} = -F_t, \tag{18}$$

$$2B^2V^1_{,s} = F_x, \tag{19}$$

$$2C^2V^2_{,s} = F_y, \tag{20}$$

$$2C^2V^3_{,s} = F_z. \tag{21}$$

In order to find the explicit form of vector field V, we need to solve these determining equations. However, these equations cannot be solved generally unless some conditions are imposed on the metric coefficients. In the literature, such systems are usually solved by imposing some conditions on A, B, and C. However, there are no proper criteria under which these conditions should be imposed on A, B, and C, such that the resulting equations could be completely solved. Thus, this approach of solving determining equations does not provide a complete list of metrics possessing Noether symmetries. There is another approach, the Rif tree approach [21], which uses a Maple (Rif) algorithm for finding all possible metrics that admit some Noether symmetries. These metrics are then used to solve the determining equations, and the final forms of Noether symmetries are obtained. The list of metrics, which is obtained as a result of analyzing determining equations through the Rif algorithm, is given in terms of the branches of a tree called the Rif tree. Each branch of the Rif tree restricts functions A, B, and C to satisfy some constraints that are then used to solve the Noether symmetry equations, giving a complete classification of the Lagrangian associated with a nonstatic plane symmetric metric via Noether symmetries. The Rif tree given in Figure 1 was obtained after analyzing determining Equations (7)–(21) through the Rif algorithm.

The nodes (pivots) of the Rif tree are obtained as follows:

$$
\begin{aligned}
p1 &= C_{,t} \\
p2 &= C_{,t}B_{,x} - C_{,x}B_{,t} \\
p3 &= C_{,t}A_{,x} - C_{,x}A_{,t} \\
p4 &= C_{,t}C_{,tx} - C_{,tt}C_{,x} \\
p5 &= (C_{,t})^2C_{,xx} - 2C_{,tx}C_{,x} + C_{,tt}(C_{,x})^2 \\
p6 &= (C_{,t})^2C_{,ttx} - 2C_{,t}C_{,tt}C_{,tx} - C_{,t}C_{,x}C_{,ttt} + 2(C_{,tt})^2C_{,x} \\
p7 &= (C_{,t})^2C_{,xx} - C_{,tt}(C_{,x})^2 \\
p8 &= C_{,x} \\
p9 &= B_{,t} \\
p10 &= A_{,t} \\
p11 &= B_{,t}A_{,x} - B_{,x}A_{,t} \\
p12 &= B_{,t}B_{,tx} - B_{,x}B_{,tt} \\
p13 &= -(B_{,t})^2B_{,xx} + 2B_{,t}B_{,x}B_{,tx} - (B_{,x})^2B_{,tt} \\
p14 &= B_{,ttt}B_{,t}B_{,x} - (B_{,t})^2B_{,ttx} + 2B_{,t}B_{,tt}B_{,tx} - 2B_{,x}(B_{,tt})^2 \\
p15 &= (B_{,t})^2B_{,xx} - (B_{,x})^2B_{,tt} \\
p16 &= B_{x} \\
p17 &= A_{,t}A_{,tx} - A_{,x}A_{,tt} \\
p18 &= (A_{,t})^2A_{,xx} - 2A_{,t}A_{,x}A_{,tx} + (A_{,x})^2A_{,tt} \\
p19 &= (A_{,t})^2A_{,ttx} - A_{,t}A_{,x}A_{,ttt} - 2A_{,t}A_{,tx}A_{,tt} + 2A_{,x}(A_{,tt})^2 \\
p20 &= (A_{,t})^2A_{,xx} - (A_{,x})^2A_{,tt} \\
p21 &= A_{,x}
\end{aligned}
$$

Symbols $=$ and $<>$ in the Rif tree show whether the corresponding p_i is zero or nonzero. For example, in Branch 1, we have $p1 = C_{,t} \neq 0$ and $p2 = C_{,t}B_{,x} - C_{,x}B_{,t} \neq 0$. Similarly, each branch of the Rif tree imposes some conditions on A, B and C that are then used to solve Equations (7)–(21). As a result, we obtained 4-, 5-, 6-, 7-, 8-, 9-, 11- and 17-dimensional Noether algebras admitted by different metrics. These results are summarized in Sections 3–10.

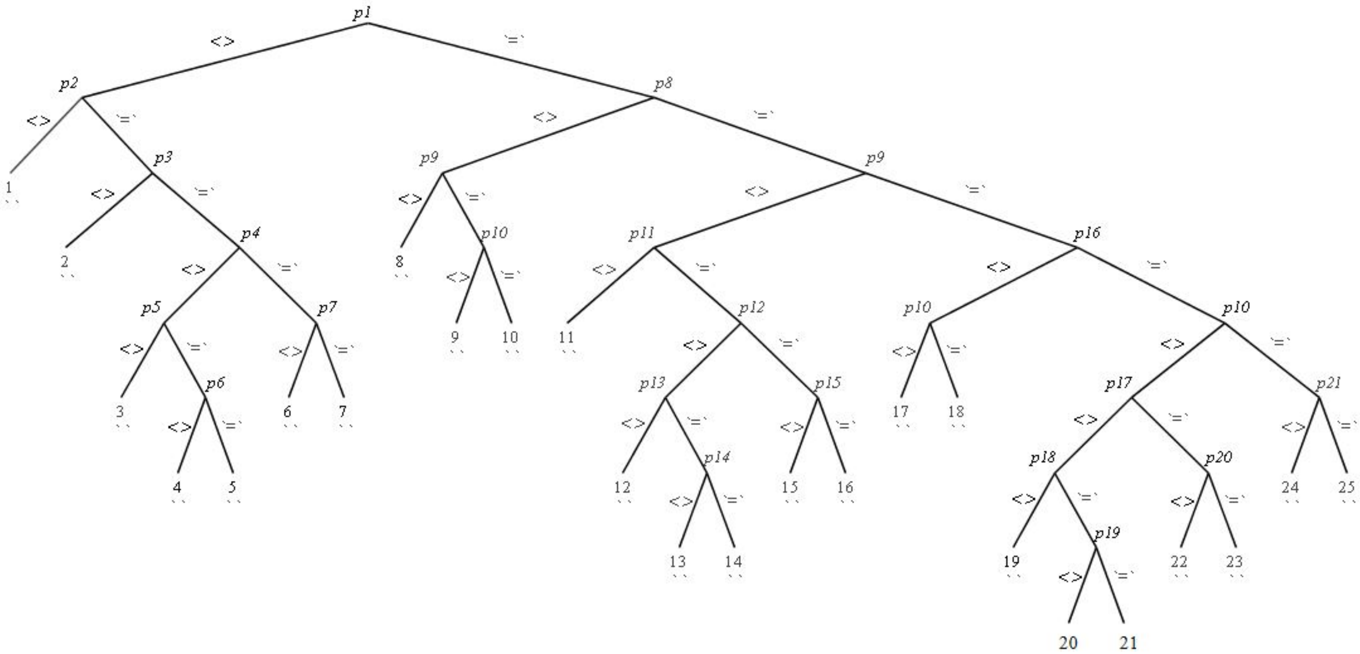

Figure 1. Rif tree.

3. Four Noether Symmetries

The minimal number of Noether symmetries for the spacetime under consideration was 4. These four symmetries contained the minimal three KVFs and an additional trivial Noether symmetry, given by $V_{(0)} = \partial_s$. There were only two branches of the Rif tree, namely Branches 4 and 6, whose associated metrics possessed these four Noether symmetries. The conserved quantities corresponding to these four symmetries $V_{(0)}, \ldots, V_{(3)}$ were obtained as follows:

$$\begin{aligned} I_0 &= A^2\dot{t}^2 - B^2\dot{x}^2 - C^2(\dot{y}^2 + \dot{z}^2), \\ I_1 &= 2C^2(z\dot{y} - y\dot{z}), \\ I_2 &= 2C^2\dot{y}, \\ I_3 &= 2C^2\dot{z}. \end{aligned}$$

4. Five Noether Symmetries

There are many branches that give rise to metrics possessing one additional symmetry along with four minimal ones. In Table 1, we present all such metrics along with their extra symmetry, denoted by $V_{(4)}$, and the corresponding conserved quantity. In cases where $V_{(4)}$ is independent of variable s, it represents a KVF, while in the remaining cases, $V_{(4)}$ is of the form $s\partial_s + X$, where X represents the corresponding HVF. All the metrics of this section were nonstatic except 5j, 5k, and 5r. The first two of these three metrics are Bianchi Type I models, while 5r represents a static plane symmetric metric.

Table 1. Metrics with five Noether symmetries.

Metric No./ Branch No.	Metric Coefficients	Additional Symmetry	Conserved Quantity
5a 2	$A = A(t)$, $B =$ const., $C = -\frac{(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}}{2a_3(a_1-2a_3)} + (a_1\int Adt + 2a_4)^{1-\frac{2a_3}{a_1}}$ where $A_{,t} \neq 0$ $a_1 \neq 2a_3$ and $a_1 \neq 0,\ a_3 \neq 0$	$V_{(4)} = s\partial_s + \frac{\int Adt}{2A}\partial_t + \frac{x}{2}\partial_x$,	$I_4 = -sL - A\int Adt\dot{t} + x\dot{x}$

Table 1. *Cont.*

Metric No./ Branch No.	Metric Coefficients	Additional Symmetry	Conserved Quantity
5b 2	$A=(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}$; $B=$const.; $C=(2a_3t+2a_4)^{\frac{a_1-2a_5}{2a_3}}$ where $a_1,a_3\neq 0,\ a_1\neq 2a_3\neq 2a_5$	$V_{(4)}=s\partial_s+\frac{x}{2}\partial_x$	$I_4=-sL+x\dot{x}$
5c 2	$A=a_1x+a_2,\ B=$const. $C=e^{\frac{(a_3-2a_4)t}{2a_5}}$ where $a_3\neq 2a_4$, and $a_1\neq 0$	$V_{(4)}=s\partial_s+\frac{A}{2a_1}\partial_x$,	$I_4=-sL+\frac{A}{a_1}\dot{x}$
5d 2	$A=a_1x+a_2,\ B=$const., $C=C(t)\neq e^{\frac{(a_3-2a_4)t}{2a_5}}$, $C_{,t}\neq 0$ and $a_1\neq 0$	$V_{(4)}=s\partial_s+\frac{A}{2a_1}\partial_x+\frac{y}{2}\partial_y+\frac{z}{2}\partial_z$,	$I_4=-sL+\frac{A}{a_1}\dot{x}$ $+C^2(y\dot{y}+z\dot{z})$.
5e 2	$A=\sqrt{2a_1x+2a_2},\ B=$ const. $C=e^{a_3t}$, where $a_1,a_3\neq 0$,	$V_{(4)}=-\frac{C}{\dot{C}}\partial_t+y\partial_y+z\partial_z$,	$I_4=\frac{2A^2C}{\dot{C}}\dot{t}+C^2(y\dot{y}+z\dot{z})$
5f 2	$A=\sqrt{2a_1x+2a_2}$, $B=$ const., $C=(a_3t+4a_4)^{2-\dfrac{4a_5}{a_3}}$, where $a_3\neq 2a_5$ and $a_1,a_3\neq 0$	$V_{(4)}=s\partial_s+\frac{t}{4}\partial_t+\frac{A}{4A_{,x}}\partial_x.$,	$I_4=-sL-\frac{A^2t}{2}\dot{t}+\frac{A}{2A_{,x}}\dot{x}$.
5g 3	$A=$ const., $B=$ const., $C=\frac{(a_1t+2a_2)^{1-\frac{2a_3}{a_1}}}{a_1-2a_3}+(a_1x+2a_4)^{1-\frac{2a_3}{a_1}}$, where $a_1\neq 2a_3$ and $a_1,a_3\neq 0$	$V_{(4)}=s\partial_s+\frac{t}{2}\partial_t+\frac{x}{2}\partial_x$,	$I_4=-sL-t\dot{t}+x\dot{x}$
5h 3	$A=$ const., $B=$ const., $C=a_1(\frac{t^2-x^2}{2})+a_2x+a_3t+a_4$, where $a_1,a_2,a_3\neq 0$.	$V_{(4)}=-\frac{C_{,x}}{C_{,xx}}\partial_t+\frac{C_{,t}}{C_{,tt}}\partial_x$,	$I_4=\frac{2C_{,x}}{C_{,xx}}\dot{t}+\frac{2C_{,t}}{C_{,tt}}\dot{x}$.
5i 5	$A=$const., $B=$ const., $C=\frac{1}{a_1}\ln(\frac{-a_1x+a_2}{a_1t+a_3})$, where $a_1,a_2,a_3\neq 0$.	$V_{(4)}=s\partial_s+\frac{(a_1t+a_3)}{2a_1}\partial_t$ $-\frac{(-a_1x+a_2)}{2a_1}\partial_x+\frac{y}{2}\partial_y+\frac{z}{2}\partial_z$	$I_4=-sL-\frac{(a_1t+a_3)}{a_1}\dot{t}$ $-\frac{(-a_1x+a_2)}{a_1}\dot{x}+C^2(y\dot{y}+z\dot{z})$
5j 7	$A=$ const.; $B=a_2t$; where $a_2\neq 0$, $C=C(t)\neq(a_1t)^{1-\frac{2a_3}{a_1}}$, $C_{,t}\neq 0$	$V_{(4)}=\partial_x$	$I_4=2B^2\dot{x}$
5k 7	$A=$ const.; $B=B(t)\neq(a_1t+2a_2)^{1-\frac{2a_3}{a_1}}$; $C=C(t)\neq(a_1t+2a_2)^{1-\frac{2a_4}{a_1}}$, $C_{,t}\neq 0$	$V_{(4)}=\partial_x$	$I_4=2B^2\dot{x}$
5l 8	$A=$ const., $B=(a_1t+2a_2)^{1-a_3+\frac{2a_3a_5}{a_1}}$, $C=(a_3x+a_4)^{\dfrac{1}{a_3}}$, where $a_1,a_3\neq 0$.	$V_{(4)}=s\partial_s+\frac{t}{2}\partial_t$ $+\frac{(a_3x+a_5)}{2}\partial_x$	$I_4=-sL-t\dot{t}$ $+B^2(a_3x+a_5)\dot{x}$.
5m 8	$A=$const., $B=a_1t+2a_2$, $C=C(x)\neq(a_3x+a_4)^{\dfrac{1}{a_3}}$, where $a_1\neq 0$ and $C_{,x}\neq 0$	$V_{(4)}=s\partial_s+\frac{t}{2}\partial_t+\frac{y}{2}\partial_y+\frac{z}{2}\partial_z$,	$I_4=-sL-t\dot{t}+C^2(y\dot{y}+z\dot{z})$
5n 8	$A=$ const., $B=\sqrt{2a_1t+2a_2}$, $C=(a_3x+a_4)^{\dfrac{1}{a_3}}$, where $a_1,a_3\neq 0$	$V_{(4)}=s\partial_s+\frac{B^2}{4a_1}\partial_t$, $+\frac{C}{4a_3C_{,x}}\partial_x+\frac{2a_3-1}{4a_3}$ $(y\partial_y+z\partial_z)$.	$I_4=-sL-\frac{B^2}{2a_1}\dot{t}+\frac{CB^2}{2a_3C_{,x}}\dot{x}$ $+\frac{C^2(2a_3-1)}{2a_3}(y\dot{y}+z\dot{z})$.
5o 8	$A=$ const., $B=\sqrt{2a_1t+2a_2}$, $C=e^{a_3x}$, where $a_1,a_3\neq 0$	$V_{(4)}=-\frac{C}{C_{,x}}\partial_x+y\partial_y+z\partial_z$,	$I_4=-\frac{2B^2C}{C_{,x}}\dot{x}+2C^2(y\dot{y}+z\dot{z})$
5p 8	$A=$ const., $B=t^{\frac{2a_2}{a_1}}$, $C=[(a_1-2a_2)x]^{\frac{a_1-2a_3}{a_1-2a_2}}$ where $a_1\neq 0$, $a_1\neq 2a_2$ and $a_1\neq 2a_3$	$V_{(4)}=s\partial_s+\frac{t}{2}\partial_t+\frac{x}{2}\partial_x$,	$I_4=-sL-t\dot{t}+B^2x\dot{x}$

Table 1. *Cont.*

Metric No./ Branch No.	Metric Coefficients	Additional Symmetry	Conserved Quantity
5q 9	$A=(a_1t+a_2)^{-1+\frac{a_3}{2a_1}}+\frac{(a_3x+2a_4)^{1-\frac{2a_1}{a_3}}}{a_3-2a_1}$, $B=\text{const}, C=(a_3x+2a_4)^{1-\frac{2a_5}{a_3}}$, where $a_3\neq 2a_1\neq 2a_5$,	$V_{(4)}=s\partial_s+\frac{x}{2}\partial_x$,	$I_4=-sL+B^2x\dot{x}$
5r 10	$A=A(x)\neq(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}$, $B=\text{const.}$, $C=C(x)\neq(a_1x+2a_2)^{1-\frac{2a_4}{a_1}}, C_{,x}\neq 0$	$V_{(4)}=\partial_t$,	$I_4=-2A^2\dot{t}$

5. Six Noether Symmetries

Like the previous section, there are some branches of the Rif tree whose corresponding metrics admit six Noether symmetries; four being the minimum ones and two extra symmetries for all these metrics (denoted by $V_{(4)}$ and $V_{(5)}$), their associated conserved quantities and the explicit form of the metrics are listed in Table 2. Out of the two extra symmetries, some are independent of s which represent KVFs, some are of the form $s\partial_s+X$, giving a corresponding HVF X, while the remaining symmetries are proper Noether symmetries. The metrics labeled 6c, 6d, and 6e were Bianchi Type I metrics, while metrics 6g and 6h were static plane symmetric metrics. All the remaining metrics were nonstatic.

Table 2. Metrics with six Noether symmetries.

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
6a 1	$A=\text{const.}, B=a_1t+a_2$, $C=\frac{L(x)B}{a_2}$, where $a_1,a_2\neq 0$	$V_{(4)}=s\partial_s+\frac{B}{2a_1}\partial_t$ $V_{(5)}=\frac{s^2}{2}\partial_s+\frac{sB}{2a_1}\partial_t, F=-\frac{t^2}{2}-\frac{a_2B}{a_1^2}$	$I_5=-sL-\frac{B}{a_1}\dot{t}$ $I_4=-\frac{s^2}{2}L-\frac{sB}{a_1}\dot{t}+\frac{t^2}{2}+\frac{a_2B}{a_1^2}$
6b 1	$A=\text{const.}, B=B(x), B_{,x}\neq 0$, $C=C(t)\neq\frac{(a_1t+2a_2)^{1+\frac{a_3}{a_1}}}{a_1+a_3}, C_{,t}\neq 0$	$V_{(4)}=\frac{1}{B}\partial_x$ $V_{(5)}=\frac{s}{2B}\partial_x, F=\int Bdx$	$I_4=2B\dot{x}$ $I_5=sB\dot{x}-\int Bdx$
6c 7	$A=\text{const.}, B=a_2t$, $C=(a_1t)^{1-\frac{2a_3}{a_1}}$, where $a_1,a_2\neq 0$ and $a_1\neq 2a_3$	$V_{(4)}=s\partial_s+\frac{t}{2}\partial_t$, $V_{(5)}=\partial_x$	$I_4=-sL-t\dot{t}$ $I_5=2B^2\dot{x}$
6d 7	$A=\text{const.}, B=(a_1t+2a_2)^{1-\frac{2a_3}{a_1}}$, $C=(a_1t+2a_2)^{1-\frac{2a_4}{a_1}}$, where $a_1\neq 0, a_1\neq 2a_3, 2a_4$	$V_{(4)}=s\partial_s+\frac{t}{2}\partial_t$, $V_{(5)}=\partial_x$	$I_4=-sL-t\dot{t}$ $I_5=2B^2\dot{x}$
6e 7	$A=\text{const.}, B=\text{const.}$, $C=C(t)\neq(a_1t+2a_2)^{1-\frac{2a_3}{a_1}}$, where $C_{,t}\neq 0$	$V_{(4)}=\partial_x$ $V_{(5)}=\frac{s}{2}\partial_x, F=x$	$I_4=2\dot{x}$ $I_5=s\dot{x}-x$
6f 9	$A=A(t), B=\text{const.}$, $C=C(x)\neq(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}$, where $A_{,t\neq 0}, C_{,x}\neq 0$,	$V_{(4)}=\frac{1}{A}\partial_t$ $V_{(5)}=-\frac{s}{2A}\partial_t, F=\int Adt$	$I_4=-2A\dot{t}$ $I_5=sA\dot{t}-\int Adt$
6g 10	$A=(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}, B=\text{const.}$, $C=(a_1x+2a_2)^{1-\frac{2a_4}{a_1}}$, where $a_1\neq 0, a_1\neq 2a_3, 2a_4$	$V_{(4)}=s\partial_s+\frac{x}{2}\partial_x$, $V_{(5)}=\partial_t$	$I_4=-sL+x\dot{x}$ $I_5=-2A^2\dot{t}$
6h 10	$A=\text{const.}, B=\text{const.}$, $C=C(x)\neq(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}, C_{,x}\neq 0$	$V_{(4)}=\partial_t$ $V_{(5)}=-\frac{s}{2}\partial_t, F=t$	$I_4=-2\dot{t}$ $I_5=s\dot{t}-t$
6i 11	$A=c_1t+c_2x,\ B=c_3t+c_4x,\ C=\text{const.}$, where $c_1,c_2,c_3,c_4\neq 0$ and $c_1c_4-c_2c_3\neq 0$	$V_{(4)}=s\partial_y, F=2y$ $V_{(5)}=s\partial_z, F=2z$	$I_4=2s\dot{y}-2y$ $I_5=2s\dot{z}-2z$

Table 2. *Cont.*

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
6j 12	$A = t^2 + x,\ B = e^{t^2+x},\ C = \text{const.}$	$V_{(4)} = s\partial_y, F = 2y$ $V_{(5)} = s\partial_z, F = 2z$	$I_4 = 2s\dot{y} - 2y$ $I_5 = 2s\dot{z} - 2z$
6k 13	$A = B = e^{c_1 t + c_2 x}\ C = \text{const.}$, where $c_1 \neq 0$ and $c_2 \neq 0$	$V_{(4)} = s\partial_y, F = 2y$ $V_{(5)} = s\partial_z, F = 2z$	$I_4 = 2s\dot{y} - 2y$ $I_5 = 2s\dot{z} - 2z$
6l 14	$A = B = \ln(\frac{x}{t}),\ C = \text{const.}$,	$V_{(4)} = s\partial_y, F = 2y$ $V_{(5)} = s\partial_z, F = 2z$	$I_4 = 2s\dot{y} - 2y$ $I_5 = 2s\dot{z} - 2z$
6m 15	$A = B = c_1 t + c_2 x^2,\ C = \text{const.}$, where $c_1 \neq 0$ and $c_2 \neq 0$	$V_{(4)} = s\partial_y, F = 2y$ $V_{(5)} = s\partial_z, F = 2z$	$I_4 = 2s\dot{y} - 2y$ $I_5 = 2s\dot{z} - 2z$
6n 19	$A = c_1 t^2 + c_2 x,,\ B = \text{const.},\ C = \text{const.}$, where $c_1 \neq 0$ and $c_2 \neq 0$	$V_{(4)} = s\partial_y, F = 2y$ $V_{(5)} = s\partial_z, F = 2z$	$I_4 = 2s\dot{y} - 2y$ $I_5 = 2s\dot{z} - 2z$
6o 20	$A = e^{c_1 t + c_2 x},\ B = \text{const.},\ C = \text{const.}$, where $c_1 \neq 0$ and $c_2 \neq 0$	$V_{(4)} = s\partial_y, F = 2y$ $V_{(5)} = s\partial_z, F = 2z$	$I_4 = 2s\dot{y} - 2y$ $I_5 = 2s\dot{z} - 2z$
6p 21	$A = \ln(\frac{x}{t}),\ B = \text{const.},\ C = \text{const.}$	$V_{(4)} = s\partial_y, F = 2y$ $V_{(5)} = s\partial_z, F = 2z$	$I_4 = 2s\dot{y} - 2y$ $I_5 = 2s\dot{z} - 2z$
6q 22	$A = c_1 t + c_2 x^2,\ B = \text{const.},\ C = \text{const.}$, where $c_1 \neq 0$ and $c_2 \neq 0$	$V_{(4)} = s\partial_y, F = 2y$ $V_{(5)} = s\partial_z, F = 2z$	$I_4 = 2s\dot{y} - 2y$ $I_5 = 2s\dot{z} - 2z$

6. Seven Noether Symmetries

The solution of determining equations for some branches give rise to metrics possessing seven Noether symmetries. Out of these seven, four are the minimal ones, while three additional symmetries, denoted by $V_{(4)}$, $V_{(5)}$ and $V_{6)}$, and the corresponding conserved quantities are listed in Table 3. Like the previous sections, some of the three extra symmetries are KVFs or correspond to HVFs, while the remaining are proper Noether symmetries. Out of the obtained metrics, those labeled 7b, 7c and 7d were Bianchi type I, metrics 7f, 7g, 7h, and 7i were static, while the remaining were nonstatic plane symmetric metrics.

Table 3. Metrics with seven Noether symmetries.

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
7a 1	$A = \text{const.}, B = B(x), C = \frac{(a_1 t + 2a_2)^{1+\frac{a_3}{a_1}}}{a_1 + a_3}$, where $B_{,x} \neq 0$ and $a_1 \neq 0,\ a_1 \neq -a_3$	$V_{(4)} = s\partial_s + \frac{t}{2}\partial_t + \frac{\int Bdx}{2B}\partial_x$, $V_{(5)} = \frac{1}{B}\partial_x$ $V_{(6)} = \frac{s}{2B}\partial_x, F = \int Bdx$	$I_4 = -sL - t\dot{t} + B\int Bdx\dot{x}$ $I_5 = 2B\dot{x}$ $I_6 = sB\dot{x} - \int Bdx$
7b 7	$A = \text{const.}, B = C \neq (a_1 t + 2a_2)^{1-\frac{2a_3}{a_1}}$	$V_{(4)} = -y\partial_x + x\partial_y$, $V_{(5)} = -z\partial_x + x\partial_z$, $V_{(6)} = \partial_x$	$I_4 = 2C^2(-\dot{x}y + \dot{y}x)$ $I_5 = 2C^2(-\dot{x}z + \dot{z}x)$ $I_6 = 2C^2\dot{x}$
7c 7	$A = \text{const.}, B = \text{const.}, C = (a_1 t + 2a_2)^{1-\frac{2a_3}{a_1}}$, where $a_1 \neq 0,\ a_1 \neq 2a_3$	$V_{(4)} = s\partial_s + \frac{t}{2}\partial_t + \frac{x}{2}\partial_x$, $V_{(5)} = \partial_x$ $V_{(6)} = \frac{s}{2}\partial_x, F = x$	$I_4 = -sL - t\dot{t} + x\dot{x}$ $I_5 = 2\dot{x}$ $I_6 = s\dot{x} - x$
7d 7	$A = \text{const.}, B = \text{const.}, C = a_1 t + a_2$, where $a_1 \neq 0$	$V_{(4)} = s\partial_s + \frac{C}{2a_1}\partial_t + \frac{x}{2}\partial_x$, $V_{(5)} = \partial_x$ $V_{(6)} = \frac{s}{2}\partial_x, F = x$	$I_4 = -sL - \frac{C}{a_1}\dot{t} + x\dot{x}$ $I_5 = 2\dot{x}$ $I_6 = s\dot{x} - x$
7e 9	$A = A(t), B = \text{const.}, C = (a_1 x + 2a_2)^{1-\frac{2a_3}{a_1}}$, where $a_1 \neq 2a_3,\ A_{,t} \neq 0$,	$V_{(4)} = s\partial_s + \frac{\int Adt}{2A}\partial_t + \frac{x}{2}\partial_x$ $V_{(5)} = \frac{1}{A}\partial_t$ $V_{(6)} = -\frac{s}{2A}\partial_t, F = \int Adt$	$I_4 = -sL - A\int Adt\dot{t}$ $I_5 = -2A\dot{t}$ $I_6 = sA\dot{t} - \int Adt$

Table 3. *Cont.*

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
7f 10	$A = C \neq (a_1 x + 2a_2)^{1-\frac{2a_3}{a_1}}$, $B =$ const., where $A_{,x} \neq 0$	$V_{(4)} = y\partial_t + t\partial_y$, $V_{(5)} = z\partial_t + t\partial_z$, $V_{(6)} = \partial_t$	$I_4 = 2A^2(-\dot{t}y + \dot{y}t)$ $I_5 = 2A^2(-\dot{t}z + \dot{z}t)$ $I_6 = -2A^2\dot{t}$
7g 10	$A =$ const., $B =$ const., $C = (a_1 x + 2a_2)^{1-\frac{2a_3}{a_1}}$, where $a_1 \neq 0$, $a_1 \neq 2a_3$	$V_{(4)} = s\partial_s + \frac{t}{2}\partial_t + \frac{x}{2}\partial_x$, $V_{(5)} = \partial_t$ $V_{(6)} = \frac{-s}{2}\partial_t$, $F = t$	$I_4 = -sL - t\dot{t} + x\dot{x}$ $I_5 = -2\dot{t}$ $I_6 = s\dot{t} - t$
7h 18	$A = A(x) \neq (a_1 \int Bdx + 2a_2)^{1-\frac{2a_3}{a_1}}$, $B = B(x)$, $C =$ const., where $B_x \neq 0$	$V_{(4)} = \partial_t$ $V_{(5)} = s\partial_y$, $F = 2y$ $V_{(6)} = s\partial_z$, $F = 2z$	$I_4 = -2A^2\dot{t}$ $I_5 = 2s\dot{y} - 2y$ $I_6 = 2s\dot{z} - 2z$
7i 24	$A = A(x) \neq (a_1 x + 2a_2)^{1-\frac{2a_3}{a_1}}$, $B =$ const., $C =$ const., where $A_{,x} \neq 0$	$V_{(4)} = \partial_t$ $V_{(5)} = s\partial_y$, $F = 2y$ $V_{(6)} = s\partial_z$, $F = 2z$	$I_4 = -2A^2\dot{t}$ $I_5 = 2s\dot{y} - 2y$ $I_6 = 2s\dot{z} - 2z$

7. Eight Noether Symmetries

This section contains those metrics that admit four minimal Noether symmetries along with four additional symmetries, denoted with $V_{(4)}$, $V_{(5)}$, $V_{(6)}$ and $V_{(7)}$ in Table 4. In each case, $V_{(4)}$ corresponds to a HVF, and $V_{(5)}$ is a KVF. For metrics 8e and 8h, $V_{(6)}$ and $V_{(7)}$ are KVFs, while for the remaining metrics, they represent proper Noether symmetries. Metric 8e is the well-known Friedmann metric, metric 8f is a Bianchi Type I model, metrics 8h–8k are static plane symmetric, and the remaining are nonstatic plane symmetric metrics.

Table 4. Metrics with eight Noether symmetries.

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
8a 1	$A =$ const., $B = a_1 t + a_2$, $C = (a_3 e^{a_1 x} + a_4 e^{-a_1 x})\frac{B}{a_2}$, where $a_i \neq 0; i = 1, \ldots, 4$	$V_{(4)} = s\partial_s + \frac{B}{2a_1}\partial_t$, $V_{(5)} = -\frac{a_2 C_{,x}}{a_1 a_4 B}\partial_t + \frac{a_2 C}{a_4 B^2}\partial_x$, $V_{(6)} = \frac{a_2 s C_{,x}}{2a_1 a_4 B}\partial_t - \frac{a_2 s C}{2a_4 B^2}\partial_x$, $F = \frac{-a_2 C_{,x}}{a_1^2 a_4}$ $V_{(7)} = \frac{s^2}{2}\partial_s + s\frac{B}{2a_1}\partial_t$, $F = \frac{-t^2}{2} - \frac{a_2 B}{a_1^2}$	$I_4 = -sL - \frac{B}{a_1}\dot{t}$ $I_5 = \frac{2a_2 C_{,x}}{a_1 a_4 B}\dot{t} + \frac{2a_2 C}{a_4}\dot{x}$ $I_6 = -\frac{a_2 s C_{,x}}{a_1 a_4 B}\dot{t} - \frac{a_2 s C}{a_4}\dot{x} + \frac{a_2 C_{,x}}{a_1^2 a_4}$ $I_7 = -\frac{s^2}{2}L - \frac{sB}{a_1}\dot{t} + \frac{t^2}{2} + \frac{a_2 B}{a_1^2}$
8b 1	$A =$ const., $B = B(x)$, $C = a_1 t - a_2$, where $B_{,x} \neq 0$ and $a_1 \neq 0$,	$V_{(4)} = s\partial_s + \frac{C}{2C_{,t}}\partial_t + \frac{\int Bdx}{2B}\partial_x$ $V_{(5)} = \frac{1}{B}\partial_x$ $V_{(6)} = \frac{s}{2B}\partial_x$, $F = \int Bdx$ $V_{(7)} = \frac{s^2}{2}\partial_s + \frac{st}{2}\partial_t + \frac{s\int Bdx}{2B}\partial_x$, $F = \frac{(\int Bdx)^2 - t^2}{2}$,	$I_4 = -sL - \frac{C}{C_{,t}}\dot{t} + B\int Bdx\dot{x}$ $I_5 = 2B\dot{x}$ $I_6 = sB\dot{x} - \int Bdx$ $I_7 = -\frac{s^2}{2}L - st\dot{t}$ $+sB\int Bdx\dot{x} - (\frac{\int Bdx)^2 - t^2}{2})$
8c 2	$A = A(t)$, $B =$ const., $C = a_1 x + a_2 \int Adt$, where $A_{,t} \neq 0$ $a_1 \neq a_2$ and $a_1, a_2 \neq 0$.	$V_{(4)} = s\partial_s + \frac{\int Adt}{2A}\partial_t + \frac{x}{2}\partial_x$, $V_{(5)} = \frac{1}{A}\partial_t - \frac{a_2}{a_1}\partial_x$ $V_{(6)} = \frac{-s}{2A}\partial_t + \frac{a_2 s}{2a_1}\partial_x$, $F = \frac{a_2 x}{a_1} + \int Adt$ $V_{(7)} = \frac{s^2}{2}\partial_s + s\frac{\int Adt}{2A}\partial_t + s\frac{x}{2}\partial_x$, $F = \frac{x^2 - (\int Adt)^2}{2}$	$I_4 = -sL - A\int Adt\dot{t} + x\dot{x}$ $I_5 = -2A\dot{t} - \frac{2a_2}{a_1}\dot{x}$ $I_6 = sA\dot{t} + \frac{a_2 s}{a_1}\dot{x} - (\frac{a_2}{a_1}x + \int Adt)$ $I_7 = -\frac{s^2}{2}L - sA\int Adt\dot{t}$ $+sx\dot{x} - (\frac{x^2 - (\int Adt)^2}{2})$
8d 7	$A =$ const., $B =$ const., $C = a_1 x + a_2 t$, where $a_1 \neq a_2$ and $a_1, a_2 \neq 0$.	$V_{(4)} = s\partial_s + \frac{t}{2}\partial_t + \frac{x}{2}\partial_x$ $V_{(5)} = -\frac{a_2}{a_1}\partial_t + \partial_x$ $V_{(6)} = \frac{-sa_2}{2a_1}\partial_t + \frac{s}{2}\partial_x$, $F = x + \frac{a_2}{a_1}t$ $V_{(7)} = \frac{s^2}{2}\partial_s + s\frac{t}{2}\partial_t + s\frac{x}{2}\partial_x$, $F = \frac{x^2 - t^2}{2}$	$I_4 = -sL - t\dot{t} + x\dot{x}$ $I_5 = \frac{2a_2}{a_1}\dot{t} + 2\dot{x}$ $I_6 = s\frac{a_2}{a_1}\dot{t} + s\dot{x} - (x + \frac{a_2}{a_1}t)$ $I_7 = -\frac{s^2}{2}L - st\dot{t}$ $+sx\dot{x} - (\frac{x^2 - t^2}{2})$

Table 4. *Cont.*

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
8e 7	$A=$ const., $B=C=(a_1t+2a_2)^{1-\frac{2a_3}{a_1}}$, where $a_1\neq 2a_3$, $a_1\neq 0$,	$V_{(4)}=s\partial_s+\frac{t}{2}\partial_t$, $V_{(5)}=-y\partial_x+x\partial_y$ $V_{(6)}=-z\partial_x+x\partial_z$, $V_{(7)}=\partial_x$	$I_4=-sL-A\int Adt\dot{t}$ $I_5=2B^2(-y\dot{x}+x\dot{y})$ $I_6=2B^2(-z\dot{x}+x\dot{z})$ $I_7=2B^2\dot{x}$
8f 7	$A=$ const., $B=$ const., $C=a_1t-a_2$, where $a_1\neq 0$,	$V_{(4)}=s\partial_s+\frac{C}{2C}\partial_t+\frac{x}{2}\partial_x$, $V_{(5)}=\partial_x$ $V_{(6)}=\frac{s}{2}\partial_x$, $F=x$ $V_{(7)}=\frac{s^2}{2}\partial_s+\frac{st}{2}\partial_t+\frac{sx}{2}\partial_x$, $F=\frac{x^2-t^2}{2}$	$I_4=-sL-\frac{C}{C}\dot{t}+x\dot{x}$ $I_5=2\dot{x}$ $I_6=s\dot{x}-x$ $I_7=-\frac{s^2}{2}L-st\dot{t}$ $+sx\dot{x}-(\frac{x^2-t^2}{2})$
8g 9	$A=A(t)$, $B=$const., $C=a_1x+a_2$, where $A_{,t}\neq 0$ and $a_1\neq 0$,	$V_{(4)}=s\partial_s+\frac{\int Adt}{2A}\partial_t+\frac{C}{2a_1}\partial_x$ $V_{(5)}=\frac{1}{A}\partial_t$ $V_{(6)}=-\frac{s}{2A}\partial_t$, $F=\int Adt$ $V_{(7)}=\frac{s^2}{2}\partial_s+s\frac{\int Adt}{2A}\partial_t+s\frac{x}{2}\partial_x$, $F=\frac{x^2-(\int Adt)^2}{2}$	$I_4=-sL-A\int Adt\dot{t}+\frac{C}{a_1}\dot{x}$ $I_5=-2A\dot{t}$ $I_6=sA\dot{t}-\int Adt$ $I_7=-\frac{s^2}{2}L-sA\int Adt\dot{t}$ $+sx\dot{x}-(\frac{x^2-(\int Adt)^2}{2})$
8h 10	$A=C=(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}$, $B=$ const., where $a_1\neq 2a_3$ and $a_1\neq 0$,	$V_{(4)}=s\partial_s+\frac{x}{2}\partial_x$, $V_{(5)}=y\partial_t+t\partial_y$ $V_{(6)}=z\partial_t+t\partial_z$, $V_{(7)}=\partial_t$	$I_4=-sL+x\dot{x}$ $I_5=2A^2(-y\dot{t}+t\dot{y})$ $I_6=2A^2(-z\dot{t}+t\dot{z})$ $I_7=-2A^2\dot{t}$
8i 10	$A=$ const., $B=$ const., $C=a_1x+a_2$, where $a_1\neq 0$,	$V_{(4)}=s\partial_s+\frac{t}{2}\partial_t+\frac{C}{2a_1}\partial_x$ $V_{(5)}=\partial_t$ $V_{(6)}=-\frac{s}{2}\partial_t$, $F=t$ $V_{(7)}=\frac{s^2}{2}\partial_s+s\frac{t}{2}\partial_t+s\frac{x}{2}\partial_x$, $F=\frac{x^2-t^2}{2}$	$I_4=-sL-t\dot{t}+\frac{C}{a_1}\dot{x}$ $I_5=-2\dot{t}$ $I_6=s\dot{t}-t$ $I_7=-\frac{s^2}{2}L-st\dot{t}+sx\dot{x}-(\frac{x^2-t^2}{2})$
8j 18	$A=(a_1\int Bdx+2a_2)^{1-\frac{2a_3}{a_1}}$, $B=B(x)$, $C=$ const., where $B_{,x}\neq 0$ $a_1\neq 2a_3$ and $a_1\neq 0$.	$V_{(4)}=s\partial_s+\frac{\int Bdx}{2B}\partial_x+\frac{y}{2}\partial_y+\frac{z}{2}\partial_z$, $V_{(5)}=\partial_t$ $V_{(6)}=s\partial_y$, $F=2y$ $V_{(7)}=s\partial_z$, $F=2z$	$I_4=-sL+B\int Bdx\dot{x}+y\dot{y}+z\dot{z}$ $I_5=-2A^2\dot{t}$ $I_6=2s\dot{y}-2y$ $I_7=2s\dot{z}-2z$
8k 24	$A=(a_1x+2a_2)^{1-\frac{2a_3}{a_1}}$, $B=$const., $C=$ const., where $a_1\neq 2a_3$ and $a_1\neq 0$.	$V_{(4)}=s\partial_s+\frac{x}{2}\partial_x+\frac{y}{2}\partial_y+\frac{z}{2}\partial_z$, $V_{(5)}=\partial_t$ $V_{(6)}=s\partial_y$, $F=2y$ $V_{(7)}=s\partial_z$, $F=2z$	$I_4=-sL+x\dot{x}+y\dot{y}+z\dot{z}$ $I_5=-2A^2\dot{t}$ $I_6=2s\dot{y}-2y$ $I_7=2s\dot{z}-2z$

8. Nine Noether Symmetries

Branches 1, 7, 9, 10, and 24 yield different metrics that admit nine Noether symmetries, four being the minimal ones, and five are additional symmetries. In Table 5, we present all these metrics, their additional symmetries, and associated conservation laws. Metrics 9c and 9d were Bianchi Type I, 9f, 9g, and 9h were static plane symmetric, and metrics 9a, 9b, and 9e were nonstatic plane symmetric metrics. For metrics 9a, 9c, and 9f, $V_{(4)}$ is associated with a HVF, while for all other metrics, it gives a KVF. Moreover, $V_{(5)}$, $V_{(6)}$ and $V_{(7)}$ represent KVFs for all metrics except the metric 9h for which $V_{(5)}$ and $V_{(6)}$ are KVFs and $V_{(7)}$ is a proper Noether symmetry. Finally, for all metrics $V_{(8)}$ gives a proper Noether symmetry.

Table 5. Metrics with nine Noether symmetries.

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
9a 1	$A=$ const., $B=a_1t+a_2$, $C=e^{a_3x}\frac{B}{a_2}$, where $a_1\neq 0$, $a_2\neq 0, a_3\neq 0$	$V_{(4)}=s\partial_s+\frac{B}{2a_1}\partial_t$ $V_{(5)}=\partial_x-a_4y\partial_y-a_4z\partial_z$ $V_{(6)}=\frac{y}{a_4}\partial_t+(\frac{z^2-y^2}{2}+\frac{a_2^2}{2a_3^2a_4^2e^{2a_4x}})\partial_y-yz\partial_z$ $V_{(7)}=-\frac{z}{a_4}\partial_t+yz\partial_y+(\frac{z^2-y^2}{2}-\frac{a_2^2}{2a_3^2a_4^2e^{2a_4x}})\partial_z$ $V_{(8)}=\frac{s^2}{2}\partial_s+s\frac{B}{2a_1}\partial_t, F=-\frac{t^2}{2}-\frac{Ba_2}{a_1^2}$	$I_4=-sL-\frac{B}{a_1}\dot{t}$ $I_5=2B^2\dot{x}-2a_4C^2(y\dot{y}+z\dot{z})$ $I_6=-\frac{2y}{a_4}\dot{t}+(z^2-y^2+\frac{a_2^2}{a_3^2a_4^2e^{2a_4x}})C^2\dot{y}$ $-2yzC^2\dot{z}$ $I_7=-\frac{2z}{a_4}\dot{t}+2yzC^2\dot{y}+(z^2-y^2$ $-\frac{a_2^2}{a_3^2a_4^2e^{2a_4x}})C^2\dot{z}$ $I_8=-\frac{s^2}{2}L-s\frac{B}{a_1}\dot{t}+(\frac{t^2}{2}+\frac{Ba_2}{a_1^2})$
9b 1	$A=$const., $B=B(x)$, $C=e^{a_1t}$, where $B_{,x}\neq 0$, and $a_1\neq 0$.	$V_{(4)}=\frac{1}{B}\partial_x$ $V_{(5)}=\frac{yC}{C_{,t}}\partial_t+(\frac{z^2-y^2}{2}-\frac{1}{2a_1^2C^2})\partial_y$ $-yz\partial_z$ $V_{(6)}=-\frac{zC}{C_{,t}}\partial_t$ $+yz\partial_y+(\frac{z^2-y^2}{2}+\frac{1}{2a_1^2C^2})\partial_z$ $V_{(7)}=\partial_t-a_1y\partial_y-a_1z\partial_z$ $V_{(8)}=\frac{s}{2B}\partial_x, F=\int Bdx$	$I_4=2B\dot{x}$ $I_5=-\frac{2y}{a_1}\dot{t}+(z^2-y^2-\frac{1}{a_1^2C^2})C^2\dot{y}$ $-2yzC^2\dot{z}$ $I_6=\frac{2z}{a_2}\dot{t}$ $+2yzC^2\dot{y}+(z^2-y^2+\frac{1}{a_1^2C^2})C^2\dot{z}$ $I_7=-2\dot{t}-2C^2a_1(y\dot{y}+z\dot{z})$ $I_8=sB\dot{x}-\int Bdx$
9c 7	$A=$ const., $B=a_1t$, $C=\frac{1}{a_2}t$, where $a_1\neq 0$, and $a_2\neq 0$.	$V_{(4)}=s\partial_s+\frac{t}{2}\partial_t$ $V_{(5)}=\partial_x$ $V_{(6)}=-\frac{y}{a_1^2a_2^2}\partial_x+x\partial_y$ $V_{(7)}=-\frac{z}{a_1^2a_2^2}\partial_x+x\partial_z$ $V_{(8)}=\frac{s^2}{2}\partial_s+s\frac{t}{2}\partial_t, F=-\frac{t^2}{2}$	$I_4=-sL-t\dot{t}$ $I_5=2B^2\dot{x}$ $I_6=-\frac{2B^2y}{a_1^2a_2^2}\dot{x}+2C^2x\dot{y}$ $I_7=-\frac{2B^2z}{a_1^2a_2^2}\dot{x}+2C^2x\dot{z}$ $I_8=-\frac{s^2}{2}L-st\dot{t}+\frac{t^2}{2}$
9d 7	$A=$ const., $B=$ const., $C=e^{a_1t}$, where $a_1\neq 0$.	$V_{(4)}=\partial_x$ $V_{(5)}=\frac{yC}{C_{,x}}\partial_t$ $+(\frac{z^2-y^2}{2}+\int\frac{1}{C^2}dt)\partial_y-yz\partial_z$ $V_{(6)}=-\frac{zC}{C_{,x}}\partial_t$ $+yz\partial_y+(\frac{z^2-y^2}{2}-\int\frac{1}{C^2}dt)\partial_z$ $V_{(7)}=\partial_t-a_2y\partial_y-a_2z\partial_z$ $V_{(8)}=\frac{s}{2}\partial_x, F=x$	$I_4=2\dot{x}$ $I_5=-\frac{2y}{a_2}\dot{t}+(\frac{z^2-y^2}{2}$ $+\int\frac{1}{C^2}dx)2C^2\dot{y}-2yzC^2\dot{z}$ $I_6=\frac{2z}{a_2}\dot{t}+2yzC^2\dot{y}$ $+(\frac{z^2-y^2}{2}-\int\frac{1}{C^2}dt)2C^2\dot{z}$ $I_7=-\frac{2}{a_2}\dot{t}-2C^2(y\dot{y}+z\dot{z})$ $I_8=s\dot{x}-x$
9e 9	$A=A(t)$, $B=$ const., $C=e^{a_1x}$, where $A_{,t}\neq 0$ and $a_1\neq 0$	$V_{(4)}=\frac{1}{A}\partial_t$ $V_{(5)}=\frac{yC}{C_{,x}}\partial_x+(\frac{z^2-y^2}{2}-\int\frac{1}{a_2C^2}dx)\partial_y-yz\partial_z$ $V_{(6)}=-\frac{zC}{C_{,x}}\partial_x+yz\partial_y+(\frac{z^2-y^2}{2}+\int\frac{1}{a_2C^2}dx)\partial_z$ $V_{(7)}=-\frac{C}{C_{,x}}\partial_x+y\partial_y+z\partial_z$ $V_{(8)}=-\frac{s}{2A}\partial_t, F=t$	$I_4=-2A\dot{t}$ $I_5=\frac{2y}{a_2}\dot{x}+(\frac{z^2-y^2}{2}$ $-\frac{1}{a_2}\int\frac{1}{C^2}dx)2C^2\dot{y}-2yzC^2\dot{z}$ $I_6=-\frac{2z}{a_2}\dot{x}+2yzC^2\dot{y}+(\frac{z^2-y^2}{2}$ $+\frac{1}{a_2}\int\frac{1}{C^2}dx)2C^2\dot{z}$ $I_7=-\frac{2}{a_2}\dot{x}+2C^2(y\dot{y}+z\dot{z})$ $I_8=sA\dot{t}-t$
9f 10	$A=a_1x+a_2$, $C=a_1x+a_2$, $B=$ const., where $a_1\neq 0$.	$V_{(4)}=s\partial_s+\frac{A}{2a_1}\partial_x$ $V_{(5)}=\partial_t$ $V_{(6)}=y\partial_t+t\partial_y$ $V_{(7)}=z\partial_t+t\partial_z$ $V_{(8)}=\frac{s^2}{2}\partial_s+s\frac{x}{2}\partial_x, F=\frac{x^2}{2}$	$I_4=-sL+\frac{A}{a_1}\dot{x}$ $I_5=-2A^2\dot{t}$ $I_6=2A^2(-y\dot{t}+t\dot{y})$ $I_7=2A^2(-z\dot{t}+t\dot{z})$ $I_8=-\frac{s^2}{2}L+sx\dot{x}-\frac{x^2}{2}$
9g 10	$A=$ const., $B=$ const., $C=e^{a_1x}$, where $a_1\neq 0$,	$V_{(4)}=\partial_t$ $V_{(5)}=\frac{yC}{C_{,x}}\partial_x+(\frac{z^2-y^2}{2}-\int\frac{1}{a_2C^2}dx)\partial_y-yz\partial_z$ $V_{(6)}=-\frac{zC}{C_{,x}}\partial_x+yz\partial_y+(\frac{z^2-y^2}{2}+\int\frac{1}{a_2C^2}dx)\partial_z$ $V_{(7)}=\partial_x-a_2y\partial_y-a_2z\partial_z$ $V_{(8)}=-\frac{s}{2}\partial_t, F=t$	$I_4=-2\dot{t}$ $I_5=\frac{2y}{a_2}\dot{x}+(\frac{z^2-y^2}{2}$ $-\frac{1}{a_2}\int\frac{1}{C^2}dx)2C^2\dot{y}-2yzC^2\dot{z}$ $I_6=-\frac{2z}{a_2}\dot{x}+2yzC^2\dot{y}+(\frac{z^2-y^2}{2}$ $+\frac{1}{a_2}\int\frac{1}{C^2}dx)2C^2\dot{z}$ $I_7=-\frac{2}{a_2}\dot{x}+2C^2(y\dot{y}+z\dot{z})$ $I_8=s\dot{t}-t$

Table 5. *Cont.*

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
9h 24	$A = a_1 e^{a_2 x} + a_3 e^{-a_2 x}$, $B =$const., $C =$ const., where $a_i \neq 0, i = 1,2,3.$	$V_{(4)} = \frac{A_{,x}}{mA} cos(mt)\partial_t + sin(mt)\partial_x$, $V_{(5)} = -\frac{A_{,x}}{mA} sin(mt)\partial_t + cos(mt)\partial_x$ $V_{(6)} = \partial_t$ $V_{(7)} = s\partial_z$, $F = 2z$ $V_{(8)} = s\partial_y$, $F = 2y$ where $m = 2a_2\sqrt{a_1 a_3}$,	$I_4 = -\frac{2AA_{,x}}{m} cos(mt)\dot{t} + 2sin(mt)\dot{x}$ $I_5 = \frac{2AA_{,x}}{m} sin(mt)\dot{t} + 2cos(mt)\dot{x}$ $I_6 = -2A^2\dot{t}$ $I_7 = 2s\dot{z} - 2z$ $I_8 = 2s\dot{y} - 2y$

9. Eleven Noether Symmetries

There exist only two metrics, obtained from the constraints of Branches 7 and 10, possessing 11-dimensional Noether algebra. For each of these two metrics, we obtained seven additional symmetries (KVFs) denoted with $V_{(4)}, \ldots, V_{(10)}$ in Table 6. Metric 11a is Bianchi type I, while metric 11b is a static plane symmetric metric.

Table 6. Metrics with 11 Noether symmetries.

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
11a 7	$A =$ const., $B = C = e^t$.	$V_{(4)} = y\partial_t - yx\partial_x$ $+(\frac{z^2-y^2}{2} + \frac{e^{-2t}}{2} + \frac{x^2}{2})\partial_y - yz\partial_z$ $V_{(5)} = -z\partial_t + xz\partial_x +$, $+yz\partial_y + (\frac{z^2-y^2}{2} - \frac{e^{-2t}}{2} - \frac{x^2}{2})\partial_z$ $V_{(6)} = -x\partial_t - (\frac{y^2+z^2}{2} - \frac{e^{-2t}}{2}$ $-\frac{x^2}{2})\partial_x + xy\partial_y + xz\partial_z$ $V_{(7)} = -\partial_t + x\partial_x + y\partial_y + z\partial_z$ $V_{(8)} = -y\partial_x + x\partial_y$, $V_{(9)} = -z\partial_x + x\partial_z$, $V_{(10)} = \partial_x$	$I_4 = -2y\dot{t} - 2B^2 yx\dot{x}$ $+(z^2 - y^2 + e^{-2t} + x^2)C^2\dot{y} - 2C^2 yz\dot{z}$ $I_5 = 2z\dot{t} + 2B^2 zx\dot{x} + 2C^2 zy\dot{y}$ $+(z^2 - y^2 - e^{-2t} - x^2)C^2\dot{z}$ $I_6 = 2x\dot{t} - (y^2 + z^2 - e^{-2t}$ $-x^2)B^2\dot{x} + 2C^2 x(y\dot{y} + z\dot{z})$ $I_7 = 2\dot{t} + 2B^2(x\dot{x} + y\dot{y} + z\dot{z})$ $I_8 = 2B^2(-y\dot{x} + x\dot{y})$ $I_9 = 2B^2(-z\dot{x} + x\dot{z})$ $I_{10} = 2B^2\dot{x}$
11b 10	$A = C = e^{a_1 x}$, $B =$const., where $a_1 \neq 0$.	$V_{(4)} = [\frac{y^2+z^2}{2} + \frac{1}{2(C_{,x})^2} + \frac{t^2}{2}]\partial_t - \frac{tC}{C_{,x}}\partial_x + yt\partial_y$, $+zt\partial_z$ $V_{(5)} = -yt\partial_t + \frac{yC}{C_{,x}}\partial_x +$, $[\frac{z^2-y^2}{2} + \frac{1}{2(C_{,x})^2} - \frac{t^2}{2}]\partial_y - yz\partial_z$ $V_{(6)} = zt\partial_t - \frac{zC}{C_{,x}}\partial_x +$, $+yz\partial_y + [\frac{z^2-y^2}{2} - \frac{1}{2(C_{,x})^2} + \frac{t^2}{2}]\partial_z$ $V_{(7)} = -t\partial_t + \frac{C}{C_{,x}}\partial_x - y\partial_y - z\partial_z$ $V_{(8)} = y\partial_t + t\partial_y$, $V_{(9)} = z\partial_t + t\partial_z$, $V_{(10)} = \partial_t$	$I_4 = -A^2[y^2 + z^2 + \frac{1}{(C_{,x})^2} + t^2]\dot{t} - \frac{2tC}{C_{,x}}\dot{x}$ $+2C^2 t(y\dot{y} + z\dot{z})$ $I_5 = C^2[z^2 - y^2 + \frac{1}{(C_{,x})^2} - t^2]\dot{y}$ $+\frac{2yC}{C_{,x}}\dot{x} + 2C^2 y(t\dot{t} - z\dot{z})$ $I_6 = 2C^2 z(-t\dot{t} + y\dot{y}) - \frac{2zC}{C_{,x}}\dot{x}$ $+C^2[z^2 - y^2 - \frac{1}{(C_{,x})^2} - t^2]\dot{z}$ $I_7 = 2C^2(t\dot{t} - y\dot{y} - z\dot{z}) + 2\frac{C}{C_{,x}}\dot{x}$ $I_8 = 2C^2(-y\dot{t} + t\dot{y})$ $I_9 = 2C^2(-z\dot{t} + t\dot{z})$ $I_{10} = -2A^2\dot{t}$

10. Seventeen Noether Symmetries

During the solution of determining equations for Branches 1, 2, 16, 17, 18, 23, and 24, we obtained many metrics with 17 Noether symmetries. The set of these 17 symmetries contained four minimal symmetries along with 13 additional symmetries denoted with $V_{(4)}, \ldots, V_{(16)}$ in Table 7. The exact form of these metrics and the conserved quantities associated with $V_{(4)}, \ldots, V_{(16)}$ are also given in the same tables. Metrics 17c and 17g awerere Bianchi type I, metrics 17e, 17f, and 17h were static plane symmetric, and the remaining were nonstatic plane symmetric metrics. For all the metrics of this section, $V_{(4)}$ corresponds to a HVF, $V_{(5)}, \ldots, V_{(11)}$ are KVFs and $V_{12}, \ldots, V_{16}$ are proper Noether symmetries.

Table 7. Metrics with 17 Noether symmetries.

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
17a 1	$A=\text{const.}$, $B=a_1t+a_2$, $C=e^{a_1x}\frac{B}{a_2}$, where $a_1\neq 0$, and $a_2\neq 0$,	$V_{(4)}=s\partial_s+\frac{B}{2a_1}\partial_t$, $V_{(5)}=\partial_x-a_1y\partial_y-a_1z\partial_z$ $V_{(6)}=e^{a_1x}\partial_t-\frac{e^{a_1x}}{B}\partial_x$ $V_{(7)}=[(y^2+z^2)\frac{a_1^2a_3^2e^{a_1x}}{a_2^2}+e^{-a_1x}]\partial_t$ $+[-(y^2+z^2)\frac{a_1^2a_3^2e^{a_1x}}{a_2^2B}+\frac{e^{-a_1x}}{B}]\partial_x$ $-\frac{2ya_1e^{-a_1x}}{B}\partial_y-\frac{2za_1e^{-a_1x}}{B}\partial_z$, $V_{(8)}=ye^{a_1x}\partial_t-\frac{ye^{a_1x}}{B}\partial_x-\frac{a_2^2e^{-a_1x}}{a_3^2a_1B}\partial_y$, $V_{(9)}=ze^{a_1x}\partial_t-\frac{ye^{a_1x}}{B}\partial_x-\frac{a_2^2e^{-a_1x}}{a_3^2a_1B}\partial_y$, $V_{(10)}=\frac{y}{a_1}\partial_x+(\frac{z^2-y^2}{2}+\frac{a_2^2e^{-2a_1x}}{2a_1^2a_3^2})\partial_y$ $-yz\partial_z$ $V_{(11)}=-\frac{z}{a_1}\partial_x+yz\partial_y$ $+(\frac{z^2-y^2}{2}-\frac{a_2^2e^{-2a_1x}}{2a_1^2a_3^2})\partial_z$ $V_{(12)}=[-(\frac{y^2+z^2}{2})\frac{sa_1^2a_3^2e^{a_1x}}{a_2^2}-\frac{se^{-a_1x}}{2}]\partial_t$ $+[(\frac{y^2+z^2}{2})\frac{sa_1^2a_3^2e^{a_1x}}{a_2^2B}-\frac{se^{-a_1x}}{2B}]\partial_x$ $+\frac{sya_1a_3}{a_2C}\partial_y+\frac{sza_1a_3}{a_2C}\partial_z$, $F=(\frac{y^2+z^2}{2})\frac{2Ca_1a_3}{a_2}+\frac{Be^{-a_1x}}{a_1}$ $V_{(13)}=-\frac{sya_1a_3e^{a_1x}}{a_2}\partial_t+\frac{sya_1a_3e^{a_1x}}{a_2B}\partial_x$ $+\frac{s}{C}\partial_y$, $F=2Cy$ $V_{(14)}=-\frac{sza_1a_3e^{a_1x}}{a_2}\partial_t+\frac{sza_1a_3e^{a_1x}}{a_2B}\partial_x$ $+\frac{s}{C}\partial_z$, $F=2Cz$ $V_{(15)}=-\frac{se^{a_1x}}{2}\partial_t+\frac{se^{a_1x}}{2B}\partial_x$, $F=\frac{Be^{a_1x}}{a_1}$ $V_{(16)}=\frac{s^2}{2}\partial_s+\frac{sB}{2a_1}\partial_t$, $F=-\frac{t^2}{2}-\frac{a_2B}{a_1^2}$	$I_4=-sL+\frac{BA^2}{a_1}\dot{t}$ $I_5=2B^2\dot{x}-2a_1C^2y\dot{y}-2a_1C^2z\dot{z}$ $I_6=-2e^{a_1x}A^2\dot{t}-2e^{-a_1x}B\dot{x}$ $I_7=-2A^2[(y^2+z^2)\frac{a_1^2a_3^2e^{a_1x}}{a_2^2}+e^{-a_1x}]\dot{t}$ $+2[-(y^2+z^2)\frac{a_1^2a_3^2e^{a_1x}}{a_2^2}+e^{-a_1x}]B\dot{x}$ $-\frac{4ya_1e^{-a_1x}}{B}C^2\dot{y}-\frac{4za_1e^{-a_1x}}{B}C^2\dot{z}$ $I_8=-2A^2e^{a_1x}y\dot{t}-2Be^{a_1x}y\dot{x}-\frac{2C^2a_2^2e^{-a_1x}}{a_1a_3^2B}\dot{y}$ $I_9=-2A^2e^{a_1x}z\dot{t}-2Be^{a_1x}z\dot{x}-\frac{2C^2a_2^2e^{-a_1x}}{a_1a_3^2B}\dot{z}$ $I_{10}=\frac{2B^2y}{a_1}\dot{x}+(z^2-y^2$ $+\frac{a_2^2e^{-2a_1x}}{a_1^2a_3^2})C^2\dot{y}-2C^2yz\dot{z}$ $I_{11}=-\frac{2B^2z}{a_1}\dot{x}+2C^2yz\dot{y}$ $+(z^2-y^2-\frac{a_2^2e^{-2a_1x}}{a_1^2a_3^2})C^2\dot{z}$ $I_{12}=[(y^2+z^2)\frac{sa_1^2a_3^2e^{a_1x}}{a_2^2}$ $+se^{-a_1x}]A^2\dot{t}+[\frac{(y^2+z^2)sa_1^2a_3^2e^{a_1x}}{a_2^2}$ $-se^{-a_1x}]B\dot{x}+\frac{2sya_1a_3}{a_2}C\dot{y}$ $+\frac{2sza_1a_3}{a_2}C\dot{z}-(\frac{y^2+z^2}{2})\frac{2Ca_1a_3}{a_2}$ $-\frac{Be^{-a_1x}}{a_1}$ $I_{13}=\frac{2A^2sya_1a_3e^{a_1x}}{a_2}\dot{t}$ $+\frac{2Bsya_1a_3e^{a_1x}}{a_2}\dot{x}+2sC\dot{y}-2Cy$ $I_{14}=\frac{2A^2sza_1a_3e^{a_1x}}{a_2}\dot{t}$ $+\frac{2Bsza_1a_3e^{a_1x}}{a_2}\dot{x}+2sC\dot{z}-2Cz$ $I_{15}=sA^2e^{a_1x}\dot{t}+sBe^{a_1x}\dot{x}-\frac{Be^{a_1x}}{a_1}$ $I_{16}=-\frac{s^2}{2}L-\frac{sBA^2}{a_1}\dot{t}+\frac{t^2}{2}+\frac{a_2B}{a_1^2}$
17b 2	$A=A(t)$, $B=\text{const.}$, $C=a_1(x+\int Adt)$, where $A_{,t}\neq 0$ and $a_1\neq 0$	$V_{(4)}=s\partial_s+\frac{\int Adt}{2A}\partial_t+\frac{x}{2}\partial_x$, $V_{(5)}=-\frac{x}{A}\partial_t-\int Adt\partial_x+y\partial_y+z\partial_z$ $V_{(6)}=(\frac{a_1^2(y^2+z^2)}{2A}+\frac{1}{A})\partial_t-a_1^2\partial_x$ $-\frac{a_1y}{C}\partial_y-\frac{a_1z}{C}\partial_z$ $V_{(7)}=\frac{a_1^2}{A}\partial_t+(\frac{-a_1^2(y^2+z^2)}{2}+1)\partial_x-a_1^2\partial_x$ $-\frac{a_1y}{C}\partial_y-\frac{a_1z}{C}\partial_z$ $V_{(8)}=\frac{y}{A}(-\int Adt+\frac{1}{a_1})\partial_t+y\int Adt\partial_x$ $+[\frac{z^2-y^2}{2}+\frac{\int Adt}{a_1C}]\partial_y-yz\partial_z$ $V_{(9)}=\frac{z}{A}(\int Adt-\frac{1}{a_1})\partial_t+z\int Adt\partial_x$ $+yz\partial_y+[\frac{z^2-y^2}{2}-\frac{\int Adt}{a_1C}]\partial_z$ $V_{(10)}=\frac{y}{A}\partial_t-y\partial_x-\frac{1}{a_1C}\partial_y$ $V_{(11)}=\frac{z}{A}\partial_t-z\partial_x-\frac{1}{a_1C}\partial_z$ $V_{(12)}=-[(\frac{y^2+z^2}{2}\frac{sa_1^2}{2A})+\frac{s}{2A}]\partial_t+(\frac{y^2+z^2}{2}\frac{sa_1^2}{2})\partial_x$ $+\frac{sya_1}{2C}\partial_y+\frac{sza_1}{2C}\partial_z$, $F=a_1C(\frac{y^2+z^2}{2})+\int Adt$ $V_{(13)}=(\frac{y^2+z^2}{2}\frac{sa_1^2}{2A})\partial_t-[(\frac{y^2+z^2}{2}\frac{sa_1^2}{2})-\frac{s}{2}]\partial_t$ $-\frac{sya_1}{2C}\partial_y-\frac{sza_1}{2C}\partial_z$, $F=-a_1C(\frac{y^2+z^2}{2})+x$ $V_{(14)}=-\frac{a_1sy}{A}\partial_t+a_1sy\partial_x+\frac{s}{C}\partial_y$, $F=2Cy$ $V_{(15)}=-\frac{a_1sz}{A}\partial_t+a_1sz\partial_x+\frac{s}{C}\partial_z$, $F=2Cz$ $V_{(16)}=\frac{s^2}{2}\partial_s+\frac{s\int Adt}{2A}\partial_t+\frac{sx}{2}\partial_x$, $F=\frac{x^2-(\int Adt)^2}{2}$	$I_4=-sL-A\int Adt\dot{t}+B^2x\dot{x}$ $I_5=2Ax\dot{t}-2B^2\int Adt\dot{x}$ $+2C^2(y\dot{y}+z\dot{z})$ $I_6=-2A(\frac{a_1^2(y^2+z^2)}{2}+1)\dot{t}$ $-2a_1^2B^2\dot{x}-2a_1C(y\dot{y}+z\dot{z})$ $I_7=-2Aa_1^2\dot{t}-2B^2(\frac{a_1^2(y^2+z^2)}{2}$ $-1)\dot{x}-2a_1C(y\dot{y}+z\dot{z})$ $I_8=2Ay(\int Adt-\frac{1}{a_1})\dot{t}$ $+2B^2y\int Adt\dot{x}+2[\frac{z^2-y^2}{2}$ $+\frac{\int Adt}{a_1C}]C^2\dot{y}-2C^2zy\dot{z}$ $I_9=2Az(-\int Adt+\frac{1}{a_1})\dot{t}$ $+2B^2z\int Adt\dot{x}+2C^2zy\dot{y}$ $+2[\frac{z^2-y^2}{2}-\frac{\int Adt}{a_1C}]C^2\dot{z}$ $I_{10}=-2Ay\dot{t}-2B^2y\dot{x}-\frac{2C}{a_1}\dot{y}$ $I_{11}=-2Az\dot{t}-2B^2z\dot{x}-\frac{2C}{a_1}\dot{z}$ $I_{12}=[(\frac{y^2+z^2}{2}sa_1^2)+s]A\dot{t}$ $+(\frac{y^2+z^2}{2}sa_1^2)B^2\dot{x}+sCa_1(y\dot{y}+z\dot{z})$ $-(a_1C(\frac{y^2+z^2}{2})+\int Adt)$ $I_{13}=-(\frac{y^2+z^2}{2}sa_1^2)A\dot{t}$ $-[(\frac{y^2+z^2}{2}sa_1^2)-s]B^2\dot{x}-sCa_1(y\dot{y}$ $+z\dot{z})+a_1C(\frac{y^2+z^2}{2})-x$ $I_{14}=2a_1Asy\dot{t}+2a_1B^2sy\dot{x}$ $+2Cs\dot{y}-2Cy$ $I_{15}=2a_1Asz\dot{t}+2a_1B^2sz\dot{x}$ $+2Cs\dot{z}-2Cz$ $I_{16}=-\frac{s^2}{2}L-sA\int Adt\dot{t}+sB^2x\dot{x}$ $-(\frac{x^2-(\int Adt)^2}{2})$

Table 7. *Cont.*

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
17c 16	A =const., $B = a_1 t + a_2$, C = const., where $a_1 \neq 0$	$V_{(4)} = s\partial_s + \frac{B}{2a_1}\partial_t + \frac{y}{2}\partial_y + \frac{z}{2}\partial_z$, $V_5 = ye^{a_1 x}\partial_t + \frac{Be^{a_1 x}}{a_1}\partial_y$ $V_6 = ye^{-a_1 x}\partial_t + \frac{Be^{-a_1 x}}{a_1}\partial_y$ $V_7 = ze^{a_1 x}\partial_t + \frac{Be^{a_1 x}}{a_1}\partial_z$ $V_8 = ze^{-a_1 x}\partial_t + \frac{Be^{-a_1 x}}{a_1}\partial_z$ $V_9 = e^{a_1 x}\partial_t - \frac{e^{a_1 x}}{B}\partial_x$ $V_{10} = e^{-a_1 x}\partial_t + \frac{-e^{a_1 x}}{B}\partial_x$ $V_{11} = \partial_x$ $V_{(12)} = s\partial_y, F = 2y$ $V_{(13)} = s\partial_z, F = 2y$ $V_{(14)} = -\frac{se^{a_1 x}}{2}\partial_t + \frac{se^{a_1 x}}{2B}\partial_x, F = \frac{Be^{a_1 x}}{a_1}$ $V_{(15)} = -\frac{se^{-a_1 x}}{2}\partial_t - \frac{se^{-a_1 x}}{2B}\partial_x, F = \frac{Be^{-a_1 x}}{a_1}$ $V_{(16)} = \frac{s^2}{2}\partial_s + \frac{sB}{2a_1}\partial_t + \frac{sy}{2}\partial_y + \frac{sz}{2}\partial_z$, $F = \frac{-t^2+y^2+z^2}{2} - \frac{a_2 B}{a_1^2}$	$I_4 = -sL - \frac{B}{a_1}\dot{t} + (y\dot{y} + z\dot{z})$ $I_5 = -2ye^{a_1 x}\dot{t} + \frac{2Be^{a_1 x}}{a_1}\dot{y}$ $I_6 = -2ye^{-a_1 x}\dot{t} + \frac{2Be^{-a_1 x}}{a_1}\dot{y}$ $I_7 = -2ze^{a_1 x}\dot{t} + \frac{2Be^{a_1 x}}{a_1}\dot{z}$ $I_8 = -2ze^{-a_1 x}\dot{t} + \frac{2Be^{-a_1 x}}{a_1}\dot{z}$ $I_9 = -2e^{a_1 x}\dot{t} - 2Be^{a_1 x}\dot{x}$ $I_{10} = -2e^{-a_1 x}\dot{t} + 2Be^{-a_1 x}\dot{x}$ $I_{11} = 2B^2\dot{x}$ $I_{12} = 2s\dot{y} - 2y$ $I_{13} = 2s\dot{z} - 2z$ $I_{14} = se^{a_1 x}\dot{t} + sBe^{a_1 x}\dot{x} - \frac{Be^{a_1 x}}{a_1}$ $I_{15} = se^{-a_1 x}(\dot{t} - B\dot{x}) - \frac{Be^{-a_1 x}}{a_1}$ $I_{16} = -\frac{s^2}{2}L - \frac{sB}{a_1}\dot{t} + sy\dot{y}$ $+sc^2 z\dot{z} - [\frac{-t^2+y^2+z^2}{2} - \frac{a_2 B}{a_1^2}]$
17d 17	$A = A(t)$, $B = B(x)$, C = const., where $A_{,t} \neq 0$ and $B_{,x} \neq 0$	$V_{(4)} = s\partial_s + \frac{\int Adt}{2A}\partial_t + \frac{\int Bdx}{2}\partial_x + \frac{y}{2}\partial_y + \frac{z}{2}\partial_z$, $V_{(5)} = \frac{y}{A}\partial_t + \int Adt\partial_y$, $V_{(6)} = \frac{z}{A}\partial_t + \int Adt\partial_z$, $V_{(7)} = \frac{\int Bdx}{A}\partial_t + \frac{\int Adt}{B}\partial_x$, $V_{(8)} = -\frac{y}{B}\partial_x + \int Bdx\partial_y$, $V_{(9)} = -\frac{z}{B}\partial_x + \int Bdx\partial_z$, $V_{(10)} = \frac{1}{A}\partial_t$ $V_{(11)} = \frac{1}{B}\partial_x$ $V_{(12)} = s\partial_y, F = 2y$ $V_{(13)} = s\partial_z, F = 2y$ $V_{(14)} = \frac{s}{2B}\partial_x, F = \int Bdx$ $V_{(15)} = -\frac{s}{2A}\partial_t, F = \int Adt$ $V_{(16)} = \frac{s^2}{2}\partial_s + \frac{s\int Adt}{2A}\partial_t + \frac{s\int Bdx}{2B}\partial_x + \frac{sy}{2}\partial_y$, $+\frac{sz}{2}\partial_z$, $F = \frac{-(\int Adt)^2 + (\int ABdx)^2 + y^2 + z^2}{2}$	$I_4 = -sL - A\int Adt\dot{t} + B\int Bdx\dot{x}$ $+y\dot{y} + z\dot{z}$ $I_5 = -2Ay\dot{t} + 2\int Adt\dot{y}$ $I_6 = -2Az\dot{t} + 2\int Adt\dot{z}$ $I_7 = -2A\int Bdx\dot{t} + 2B\int Adt\dot{x}$ $I_8 = -2By\dot{x} + 2\int Bdx\dot{y}$ $I_9 = -2Bz\dot{x} + 2\int Bdx\dot{z}$ $I_{10} = -2A\dot{t}$ $I_{11} = 2B\dot{x}$ $I_{12} = 2s\dot{y} - 2y$ $I_{13} = 2s\dot{z} - 2z$ $I_{14} = sB\dot{x} - \int Bdx$ $I_{15} = sA\dot{t} - \int Adt$ $I_{16} = -\frac{s^2}{2}L - sA\int Adt\dot{t} + sB\int Bdx\dot{x}$ $+s(y\dot{y} + z\dot{z})$ $-[\frac{-(\int Adt)^2 + (\int ABdx)^2 + y^2 + z^2}{2}]$
17e 18	$A = a_1 \int Bdx$, $B = B(x)$, C = const., where $a_1 \neq 0$ and $B_{,x} \neq 0$	$V_{(4)} = s\partial_s + \frac{\int Bdx}{2B}\partial_x + \frac{y}{2}\partial_y + \frac{z}{2}\partial_z$, $V_{(5)} = \frac{ya_1 e^{a_1 t}}{A}\partial_t - \frac{ya_1 e^{a_1 t}}{B}\partial_x + Ae^{a_1 t}\partial_y$, $V_{(6)} = -\frac{ya_1 e^{-a_1 t}}{A}\partial_t - \frac{ya_1 e^{-a_1 t}}{B}\partial_x + Ae^{-a_1 t}\partial_y$, $V_{(7)} = \frac{za_1 e^{a_1 t}}{A}\partial_t - \frac{za_1 e^{a_1 t}}{B}\partial_x + Ae^{a_1 t}\partial_z$, $V_{(8)} = -\frac{za_1 e^{-a_1 t}}{A}\partial_t - \frac{za_1 e^{-a_1 t}}{B}\partial_x + Ae^{-a_1 t}\partial_z$, $V_{(9)} = -\frac{e^{a_1 t}}{\int Bdx}\partial_t + \frac{a_1 e^{a_1 t}}{B}\partial_x$, $V_{(10)} = -\frac{e^{-a_1 t}}{\int Bdx}\partial_t - \frac{a_1 e^{-a_1 t}}{B}\partial_x$, $V_{(11)} = \partial_t$ $V_{(12)} = s\partial_y, F = 2y$ $V_{(13)} = s\partial_z, F = 2y$ $V_{(14)} = -\frac{sa_1 e^{a_1 t}}{2A}\partial_t + \frac{sa_1 e^{a_1 t}}{2B}\partial_x, F = Ae^{a_1 t}$ $V_{(15)} = \frac{sa_1 e^{-a_1 t}}{2A}\partial_t + \frac{sa_1 e^{-a_1 t}}{2B}\partial_x, F = Ae^{-a_1 t}$ $V_{(16)} = \frac{s^2}{2}\partial_s + \frac{s\int Bdx}{2B}\partial_x + \frac{sy}{2}\partial_y + \frac{sz}{2}\partial_z$, $F = \frac{(\int Bdx)^2 + y^2 + z^2}{2}$	$I_4 = -sL + B\int Bdx\dot{x}$ $I_5 = -2a_1 yAe^{a_1 t}\dot{t} - 2a_1 yBe^{a_1 t}\dot{x}$ $+2Ae^{a_1 t}\dot{y}$ $I_6 = 2a_1 yAe^{-a_1 t}\dot{t} - 2a_1 yBe^{-a_1 t}\dot{x}$ $+2Ae^{-a_1 t}\dot{y}$ $I_7 = -2a_1 zAe^{a_1 t}\dot{t} - 2a_1 zBe^{a_1 t}\dot{x}$ $+2Ae^{a_1 t}\dot{z}$ $I_8 = 2a_1 zAe^{-a_1 t}\dot{t} - 2a_1 zBe^{-a_1 t}\dot{x}$ $+2Ae^{-a_1 t}\dot{z}$ $I_9 = \frac{2A^2 e^{a_1 t}}{\int Bdx}\dot{t} + 2a_1 Be^{a_1 t}\dot{x}$ $I_{10} = \frac{2A^2 e^{-a_1 t}}{\int Bdx}\dot{t} - 2a_1 Be^{-a_1 t}\dot{x}$ $I_{11} = -2A^2\dot{t}$ $+y\dot{y} + z\dot{z}$ $I_{12} = 2s\dot{y} - 2y$ $I_{13} = 2s\dot{z} - 2z$ $I_{14} = a_1 sAe^{a_1 t}\dot{t} + a_1 sBe^{a_1 t}\dot{x}$ $-Ae^{a_1 t}$ $I_{15} = -a_1 sAe^{-a_1 t}\dot{t} + a_1 sBe^{-a_1 t}\dot{x}$ $-Ae^{-a_1 t}$ $I_{16} = -\frac{s^2}{2}L + sB\int Bdx\dot{x} + sy\dot{y}$ $+sz\dot{z} - [\frac{(\int Bdx)^2 + y^2 + z^2}{2}]$

Table 7. *Cont.*

Metric No./ Branch No.	Metric Coefficients	Additional Symmetries and Gauge Function	Conserved Quantities
17f 18	$A=$const., $B=B(x)$, $C=$ const., where $B_{,x}\neq 0$	$V_{(4)}=s\partial_s+\frac{t}{2}\partial_t+\frac{\int Bdx}{2B}\partial_x+\frac{y}{2}\partial_y+\frac{z}{2}\partial_z$, $V_{(5)}=y\partial_t+t\partial_y$, $V_{(6)}=z\partial_t+t\partial_z$, $V_{(7)}=\int Bdx\partial_t+t\partial_x$, $V_{(8)}=-\frac{y}{B}\partial_x+\int Bdx\partial_y$, $V_{(9)}=-\frac{z}{B}\partial_x+\int Bdx\partial_z$, $V_{(10)}=\partial_t$ $V_{(11)}=\frac{1}{B}\partial_x$ $V_{(12)}=s\partial_y$, $F=2y$ $V_{(13)}=s\partial_z$, $F=2y$ $V_{(14)}=\frac{s}{2B}\partial_x$, $F=\int Bdx$ $V_{(15)}=-\frac{s}{2}\partial_t$, $F=t$ $V_{(16)}=\frac{s^2}{2}\partial_s+\frac{st}{2}\partial_t+\frac{s\int Bdx}{2B}\partial_x+\frac{sy}{2}\partial_y$, $+\frac{sz}{2}\partial_z$, $F=\frac{-t^2+(\int ABdx)^2+y^2+z^2}{2}$	$I_4=-sL-t\dot{t}+B\int Bdx\dot{x}$ $+y\dot{y}+z\dot{z}$ $I_5=-2y\dot{t}+2t\dot{y}$ $I_6=-2z\dot{t}+2t\dot{z}$ $I_7=-2A^2\int Bdx\dot{t}+2Bt\dot{x}$ $I_8=-2By\dot{x}+2\int Bdx\dot{y}$ $I_9=-2Bz\dot{x}+2\int Bdx\dot{z}$ $I_{10}=-2A^2\dot{t}$ $I_{11}=2B\dot{x}$ $I_{12}=2s\dot{y}-2y$ $I_{13}=2s\dot{z}-2z$ $I_{14}=sB\dot{x}-\int Bdx$ $I_{15}=s\dot{t}-t$ $I_{16}=-\frac{s^2}{2}L-st\dot{t}+sB\int Bdx\dot{x}$ $+s(y\dot{y}+z\dot{z})$ $-[\frac{-t^2+(\int ABdx)^2+y^2+z^2}{2}]$
17g 23	$A=A(t)$, $B=$const. $C=$const., where $A_{,t}\neq 0$	$V_{(4)}=s\partial_s+\frac{\int Adt}{2A}\partial_t+\frac{x}{2}\partial_x+\frac{y}{2}\partial_y+\frac{z}{2}\partial_z$, $V_{(5)}=\frac{y}{A}\partial_t+\int Adt\partial_y$, $V_{(6)}=\frac{z}{A}\partial_t+\int Adt\partial_z$, $V_{(7)}=\frac{x}{A}\partial_t+\int Adt\partial_x$, $V_{(8)}=-y\partial_x+x\partial_y$, $V_{(9)}=-zB\partial_x+x\partial_z$, $V_{(10)}=\frac{1}{A}\partial_t$ $V_{(11)}=\partial_x$ $V_{(12)}=s\partial_y$, $F=2y$ $V_{(13)}=s\partial_z$, $F=2y$ $V_{(14)}=\frac{s}{2}\partial_x$, $F=x$ $V_{(15)}=-\frac{s}{2A}\partial_t$, $F=\int Adt$ $V_{(16)}=\frac{s^2}{2}\partial_s+\frac{s\int Adt}{2A}\partial_t+\frac{sx}{2}\partial_x+\frac{sy}{2}\partial_y$, $+\frac{sz}{2}\partial_z$, $F=\frac{-(\int Adt)^2+x^2+y^2+z^2}{2}$	$I_4=-sL-A\int Adt\dot{t}+x\dot{x}+y\dot{y}+z\dot{z}$ $I_5=-2Ay\dot{t}+2\int Adt\dot{y}$ $I_6=-2Az\dot{t}+2\int Adt\dot{z}$ $I_7=-2Ax\dot{t}+2\int Adt\dot{x}$ $I_8=-2y\dot{x}+2x\dot{y}$ $I_9=-2z\dot{x}+2x\dot{z}$ $I_{10}=-2A\dot{t}$ $I_{11}=2\dot{x}$ $I_{12}=2s\dot{y}-2y$ $I_{13}=2s\dot{z}-2z$ $I_{14}=s\dot{x}-x$ $I_{15}=sA\dot{t}-\int Adt$ $I_{16}=-\frac{s^2}{2}L-sA\int Adt\dot{t}+sx\dot{x}$ $+sc^2(y\dot{y}+z\dot{z})-[\frac{-(\int Adt)^2+x^2+y^2+z^2}{2}]$
17h 24	$A=a_1x+a_2$, $B=$ const., $C=$ const., where $a_1\neq 0$	$V_{(4)}=s\partial_s+\frac{A}{2a_1}\partial_x+\frac{y}{2}\partial_y+\frac{z}{2}\partial_z$, $V_{(5)}=ye^{a_1t}\partial_t+\frac{e^{a_1x}}{a_1}\partial_y$ $V_{(6)}=ye^{-a_1x}\partial_t+\frac{e^{-a_1x}}{a_1}\partial_y$ $V_{(7)}=ze^{a_1x}\partial_t+\frac{e^{a_1x}}{a_1}\partial_z$ $V_{(8)}=ze^{-a_1x}\partial_t+\frac{e^{-a_1x}}{a_1}\partial_z$ $V_{(9)}=e^{a_1x}\partial_t-e^{a_1x}\partial_x$ $V_{(10)}=e^{-a_1x}\partial_t+e^{a_1x}\partial_x$ $V_{(11)}=\partial_x$ $V_{(12)}=s\partial_y$, $F=2y$ $V_{(13)}=s\partial_z$, $F=2y$ $V_{(14)}=-\frac{sa_1e^{a_1t}}{2A}\partial_t+\frac{sa_1e^{a_1t}}{2}\partial_x$, $F=Ae^{a_1}t$ $V_{(15)}=\frac{sa_1e^{-a_1t}}{2}\partial_t-\frac{se^{-a_1t}}{2}\partial_x$, $F=Ae^{-a_1t}$ $V_{(16)}=\frac{s^2}{2}\partial_s+\frac{sB}{2a_1}\partial_x+\frac{sy}{2}\partial_y+\frac{sz}{2}\partial_z$, $F=\frac{x^2+y^2+z^2}{2}+\frac{a_2x}{a_1}$	$I_4=-sL-\frac{A}{a_1}\dot{x}+y\dot{y}+z\dot{z}$ $I_5=-2yA^2e^{a_1x}\dot{t}+\frac{2e^{a_1x}}{a_1}\dot{y}$ $I_6=-2yA^2e^{-a_1x}\dot{t}+\frac{2e^{-a_1x}}{a_1}\dot{y}$ $I_7=-2zA^2e^{a_1x}\dot{t}+\frac{2e^{a_1x}}{a_1}\dot{z}$ $I_8=-2zA^2e^{-a_1x}\dot{t}+\frac{2e^{-a_1x}}{a_1}\dot{z}$ $I_9=-2A^2e^{a_1x}\dot{t}-2e^{a_1x}\dot{x}$ $I_{10}=-2A^2e^{-a_1x}\dot{t}+2e^{-a_1x}\dot{x}$ $I_{11}=2\dot{x}$ $I_{12}=2s\dot{y}-2y$ $I_{13}=2s\dot{z}-2z$ $I_{14}=Ase^{a_1t}\dot{t}+se^{a_1t}\dot{x}-Ae^{a_1t}$ $I_{15}=-se^{-a_1t}(A\dot{t}-\dot{x})-Ae^{-a_1x}$ $I_{16}=-\frac{s^2}{2}L-\frac{s}{a_1}\dot{x}+sy\dot{y}$ $+sz\dot{z}-[\frac{x^2+y^2+z^2}{2}+\frac{a_2x}{a_1}]$

11. Physical Implications

Solving the Noether symmetry equations for plane symmetric spacetime, we obtained a number of Lorentzian metrics possessing different dimensional Noether algebras. A Lorentzian metric represents an exact solution of the EFEs if it satisfies the field equations, and its corresponding energy-momentum tensor is associated with some known matter. For each of these metrics, one can easily find Ricci tensor R_{ab}, Ricci scalar $R=g^{ab}R_{ab}$, and Einstein tensor $G_{ab}=R_{ab}-\frac{1}{2}Rg_{ab}$, which then can be used in EFEs to find the corresponding T_{ab}. If the resulting T_{ab} is associated with some known matter, metric g_{ab} may be regarded as an exact solution of the field equations. Moreover, to see which of the obtained metrics are physically realistic, we found their energy-momentum tensor and used it to check different energy conditions. The nonzero components of T_{ab} for Metric (4) are as follows:

$$T_{00} = -\frac{1}{B^3C^2}\Big[2BA^2CC_{,xx} - B^3(C_{,t})^2 - 2B^2CB_{,t}C_{,t} + A^2B(C_{,x})^2 - 2A^2B_{,x}CC_{,x}\Big],$$

$$T_{01} = -\frac{1}{ABC}\Big[2ABC_{,tx} - A_{,x}BC_{,t} - AB_{,t}C_{,x}\Big],$$

$$T_{11} = -\frac{1}{A^3C^2}\Big[2AB^2CC_{,tt} + AB^2(C_{,t})^2 - 2A_{,t}B^2CC_{,t} - A^3(C_{,x})^2 - 2A^2A_{,x}CC_{,x}\Big],$$

$$T_{22} = T_{33} = \frac{C}{A^3B^3}\Big[A^3BC_{,xx} - AB^3C_{,tx} - AB^2CB_{,tt} + A^2A_{,xx}BC + A_{,t}B^3C_{,t}$$
$$-AB^2B_{,t}C_{,t} + A_{,t}B^2B_{,t}C + A^2A_{,x}BC_{,x} - A^3B_{,x}C_{,x} - A^2A_{,x}B_{,x}C\Big].$$

For different known sources of matter, T_{ab} obtains a particular form. For example, if $\rho =$ is energy density, $u_a =$ is four-velocity, $n_b =$ is a spacelike unit vector, such that $u_bu^b = -1$, $n_bn^b = 1$ and $u_bn^b = 0$,, and the pressures parallel and perpendicular to n_b are denoted by $p_{\parallel}$ and $p_{\perp}$, then T_{ab} represents an anisotropic fluid if $T_{ab} = (\rho + p_{\perp})u_au_b + (p_{\parallel} - p_{\perp})n_an_b + p_{\perp}g_{ab}$ [32]. Similarly, T_{ab} for a perfect fluid with density ρ, pressure p, and four-velocity u_a is of the form $T_{ab} = (p + \rho)u_au_b + pg_{ab}$. For Metric (4), if the source of the matter is assumed to be anisotropic fluid, then T_{ab} is obtained as $T_{00} = \rho A^2$, $T_{11} = p_{\parallel}B^2$, $T_{22} = T_{33} = p_{\perp}C^2$, $T_{01} = 0$, which gives a perfect fluid if $p_{\parallel} = p_{\perp} = p$. Thus, out of the obtained metrics of our classification, all those metrics for which the off-diagonal component of T_{ab} vanishes represent anisotropic or perfect fluids. For such metrics, one can easily find ρ, $p_{\parallel}$ and $p_{\perp}$ as:

$$\rho = \frac{T_{00}}{A^2},\ p_{\parallel} = \frac{T_{11}}{B^2},\ p_{\perp} = \frac{T_{22}}{C^2}. \tag{22}$$

Out of the metrics of Section 4, the off-diagonal component of T_{ab} did not vanish for metrics 5b–5f and 5l–5p, showing that these models were not anisotropic or perfect fluids. However, for metrics 5b–5f, we have $T_{00} \geq 0$, giving physically realistic metrics. Metrics 5l and 5n are physically realistic if condition $2a_3 \geq 3$ holds. Similarly, for metrics 5m and 5p, we must have $2CC_{,xx} + (C_{,x})^2 \geq 0$ and $2CC_{,xx} + (C_{,x})^2 \leq 0$, respectively. Metric 5o is an unrealistic metric with $T_{00} < 0$. Besides this, metric 8a in Section 7, metric 9a in Section 8, and metric 17a in Section 10 were also not anisotropic or perfect fluids. For metrics 8a and 9a, the positive energy condition held if $a_3a_4 \geq 0$ and $a_1^2 \geq a_3^2$ respectively. Lastly, for model 17a, we have $T_{00} = 0$, which gives a physically meaningful metric.

For all other metrics obtained during our classification, we have $T_{01} = 0$, giving anisotropic or perfect fluids. For these metrics, one can find ρ, $p_{\parallel}$ and $p_{\perp}$ by using Equation (22) and then using these quantities in the expressions for some well-known energy conditions, including weak (WEC, $\rho \geq 0, \rho + p_{\parallel} \geq 0, \rho + p_{\perp} \geq 0$), null (NEC, $\rho + p_{\parallel} \geq 0,\ \rho + p_{\perp} \geq 0$,) strong (SEC, $\rho + p_{\parallel} \geq 0, \rho + p_{\perp} \geq 0, \rho + p_{\parallel} + 2p_{\perp} \geq 0$) and dominant energy conditions (DEC, $\rho \geq 0, \rho \geq |p_{\parallel}|, \rho \geq |p_{\perp}|$).

Metrics 7d, 8b, 8f, 8g, and 17b–17h identically satisfied all these energy conditions. In particular, $T_{ab} = 0$ for metrics 17b–17h, giving vacuum solutions. Moreover, there were some metrics for which none of the energy conditions was satisfied. Such metrics are represented by 8i, 9b, 9d, 9e and 9g in Sections 8 and 9. However, the energy density was positive for metrics 9b and 9d. The remaining three were unrealistic metrics with negative energy density.

The energy conditions for the remaining metrics were conditionally satisfied, and restricted the metric coefficients and the parameters involved therein to satisfy certain constraints. In Table 8, we list such constraints for some of the metrics where the coefficients are explicitly known, and the expressions for energy conditions are simple. One may similarly find bounds for energy conditions for the remaining metrics

Table 8. Energy conditions.

Metric No.	Physical Terms	Energy Conditions
6j, 6l, 6m	$\rho = p_{\parallel} = 0,\ p_{\perp} = -\frac{B_{,tt}}{B}$	(1) NEC, WEC, SEC: satisfied if $\frac{B_{,tt}}{B} \leq 0$ (2) DEC: satisfied if $B_{,tt} = 0$
6k	$\rho = p_{\parallel} = 0,\ p_{\perp} = -\frac{B_{,tt}}{B}$	(1) NEC, WEC, SEC: satisfied if $\frac{B_{,tt}}{B} < 0$ (2) DEC: failed
6n–6q, 7i	$\rho = p_{\parallel} = 0,\ p_{\perp} = \frac{A_{,xx}}{A}$	(1) NEC, WEC, SEC: satisfied if $\frac{A_{,xx}}{A} \geq 0$ (2) DEC: satisfied if $A_{,xx} = 0$
8c, 8d	$\rho = \frac{a_2^2 - a_1^2}{C^2},\ p_{\parallel} = \frac{a_1^2 - a_2^2}{C^2},\ p_{\perp} = 0$	All energy conditions are satisfied if $a_2^2 \geq a_1^2$
8k	$\rho = p_{\parallel} = 0,\ p_{\perp} = \frac{-2a_3(a_1 - 2a_3)}{(a_1 x + 2a_(2)^2}$	(1) NEC, WEC, SEC: satisfied if $a_3(a_1 - 2a_3) \leq 0$ (2) DEC: satisfied if $a_3 = 0$
9f	$\rho = -\frac{a_1^2}{A^2},\ p_{\parallel} = \frac{3a_1^2}{A^2},\ p_{\perp} = \frac{a_1^2}{A^2}$	(1) NEC, SEC: satisfied (2) WEC, DEC: failed
9h	$\rho = p_{\parallel} = 0$ $p_{\perp} = a_2^2$	(1) NEC, WEC, SEC: satisfied (2) DEC: failed
11a	$\rho = 3,\ p_{\parallel} = p_{\perp} = -3$	(1) NEC, WEC, DEC: satisfied (2) SEC: failed
11b	$\rho = -3a_1^2,\ p_{\parallel} = p_{\perp} = 3a_1^2$	(1) NEC, SEC: satisfied (2) WEC, DEC: failed

12. Conclusions

Considering the most general nonstatic plane symmetric metric, we investigated its Noether symmetries by the Rif tree approach; thereby, we found several metrics with different Noether algebras of dimensions 4, 5, 6, 7, 8, 9, 11, and 17. The expressions for the conserved quantities associated with all the obtained Noether symmetries are found by using Noether's theorem. The current classification also provides a complete list of nonstatic plane symmetric metrics, and their Killing and homothetic symmetries. Moreover, as static plane symmetric, locally rotationally symmetric Bianchi type I, and Friedmann metrics are special forms of the nonstatic plane symmetric metric, the present study also covered the classification of these spacetime metrics via their Killing, homothetic, and Noether symmetries.

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