Arithmetic Structures in Rational Conformal Field Theories

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Abstract

This thesis investigates the arithmetic structures underlying those (supersymmetric) two-dimensional conformal field theories that are exactly solvable by algebraic methods — so-called rational conformal field theories. A particular emphasis is placed on their connection to Hodge theory and number theory. Building on the established correspondence between $\mathcal{N} = (2,2)$ supersymmetric toroidal rational conformal field theories and Hodge structures of complex multiplication (CM) type, the thesis extends this correspondence to a broader class of exactly solvable and strongly interacting $\mathcal{N} = (2,2)$ rational superconformal field theories, exemplified by so-called Gepner models.

It is shown in detail how the presence of a Galois symmetry in $\mathcal{N} = (2,2)$ rational superconformal field theories equips the associated Hodge structures, under certain assumptions, with sufficient symmetry — precisely in the sense required for them to be of CM-type. As many $\mathcal{N} = (2,2)$ rational superconformal field theories describe infrared fixed points of supersymmetric non-linear sigma models, which play a central role in worldsheet descriptions of superstring theory, these results provide a controlled setting to study the emergence of arithmetic structures in strongly coupled regimes in superstring theory moduli spaces.

A substantial portion of the thesis is devoted to a systematic analysis of arithmetic structures in $\mathcal{N} = (2,2)$ toroidal rational conformal field theories. In particular, this thesis derives an explicit expansion of their partition functions in terms of ray class theta functions, revealing a novel number-theoretic interpretation of rational toroidal partition functions, and elucidating their relation to class field theory. This construction suggests a deep interplay between modular invariance in conformal field theory and class field theory in algebraic number theory.

The structure of the thesis reflects these two results. After a detailed review of foundational aspects of two-dimensional conformal field theories — including the role of extended chiral symmetry algebras as defining properties of exactly solvable rational conformal field theories, as well as discussions of boundary conditions and Galois theory — a particular emphasis is placed on $\mathcal{N} = (2,2)$ supersymmetric conformal field theories and their relation to Hodge structures. Subsequently, the thesis focuses on arithmetic structures in $\mathcal{N} = (2,2)$ toroidal non-linear sigma models, analysing rational points in their moduli spaces and deriving their partition functions explicitly as products of generalised theta functions, which are then related to ray class theta functions. Later chapters explore the fundamental number-theoretic concept of complex multiplication in the context of elliptic curves and higher-dimensional abelian varieties, showing how $\mathcal{N} = (2,2)$ toroidal rational conformal field theories naturally endow the associated Hodge structures with the CM-type property. Finally, the correspondence between $\mathcal{N} = (2,2)$ toroidal rational conformal field theories and CM-type Hodge structures is extended to more general $\mathcal{N} = (2,2)$ rational superconformal field theories, exemplified by Gepner models and $\mathcal{N} = (2,2)$ minimal models, establishing that both geometric and non-geometric models can realise arithmetic structures of CM-type.

Statement of Originality

Except where explicitly stated otherwise, the work presented in this thesis is the result of original research conducted by the author. It includes substantial contributions to the following collaborative works:

[1] **"Minimally extended current algebras of toroidal conformal field theories"** by H. Jockers, M. Sarve, I. G. Zadeh, *JHEP* 07 (2024) 187

[2] **"Hodge Structures of Complex Multiplication Type from Rational Conformal Field Theories"** by H. Jockers, P. Kuusela, M. Sarve, (*to appear*)

All ideas, results, figures, or text originating from the work of others are clearly identified and properly attributed in the main body or bibliography. This thesis has not been submitted for the award of any other academic degree or professional qualification.

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Contents

Chapter 1

Introduction

Exact solutions in quantum field theory are remarkably rare. Most physically relevant field theories are strongly coupled and lack general analytic control. Strikingly, two-dimensional conformal field theories stand out as a class of models where exact results can often be achieved. These are quantum field theories that are invariant under conformal transformations — that is, local scale transformations and angle-preserving deformations of the spacetime manifold. While in higher dimensions the conformal symmetry algebra is finite-dimensional, in two dimensions it enhances to an infinite-dimensional symmetry algebra, known as the *Virasoro algebra*. This rich symmetry structure renders many two-dimensional conformal field theories exactly solvable in a precise sense: one can explicitly determine the full spectrum of the theory, i.e., its Hilbert space, as well as all correlation functions, without relying on a conventional Lagrangian or path integral formulation.[1]

At first glance, two-dimensional conformal field theories might appear as mathematical curiosities or toy models, constrained by low-dimensional kinematics. However, over the past few decades, they have emerged as a central pillar in modern theoretical and mathematical physics. Their relevance spans a wide range of disciplines: from the description of critical phenomena in two-dimensional statistical and condensed matter systems, to their foundational role in perturbative string theory — governing the worldsheet dynamics of strings — and further to their surprising connections with four-dimensional supersymmetric gauge theories via the so-called *AGT duality* and their generalisations. Moreover, as we will explore in detail, two-dimensional conformal field theories have deep links to number theory and arithmetic geometry, especially in so-called *rational conformal field theories*. This remarkable interplay between physics and pure mathematics continues to inspire profound developments on both sides.

In the following introduction, we will summarise some of the key physical contexts where

[1]Here, "exactly solvable" refers to the complete determination of the Hilbert space and correlation functions using algebraic and representation-theoretic methods. The necessary data to achieve this will be described in detail in chapter 2.

two-dimensional conformal field theories arise naturally, including their role in statistical mechanics, in the worldsheet formulation of string theory, and in dualities with four-dimensional gauge theories. This will set the stage for the main focus of this thesis: the detailed investigation of arithmetic and number-theoretic structures that emerge from the rich algebraic data underlying exactly solvable rational conformal field theories.

1.1 From Ising-Model to Conformal Field Theory

To illustrate the relevance of two-dimensional conformal field theory in describing statistical mechanical models near second-order phase transitions, we briefly review the planar Ising model and qualitatively highlight its connection to Euclidean quantum field theory and conformal field theory. Throughout, we follow the notation of the standard textbooks [3, 4].

The Ising model consists of a configuration of spins $\sigma \in \{\pm\frac{1}{2}\}$ arranged on a two-dimensional $N \times N$ lattice with lattice spacing a, where each spin interacts only with its nearest neighbors. For simplicity, we assume that the coupling between all neighboring spins is uniform, i.e., $J_{ij} = J$ for all pairs $\{i, j\}$. The Hamiltonian of the system in an external magnetic field h is given by

$$H = -\sum_{ij} J_{ij}\sigma_i\,\sigma_j - h\sum_i \sigma_i\,. \tag{1.1.1}$$

A central object in the statistical mechanics description of any classical system is the partition function,

$$Z = \sum_i e^{-\beta E_i}\,, \tag{1.1.2}$$

where $\beta = \frac{1}{T}$ denotes the inverse temperature. The partition function acts as a normalisation factor in the Boltzmann distribution, which gives the probability that a particular microstate (e.g., a spin configuration) occurs,

$$P_i = \frac{1}{Z}e^{-\beta E_i}, \quad \text{with} \quad \sum_i P_i = 1\,. \tag{1.1.3}$$

Thermodynamic quantities such as the internal energy $U = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}$ and the free energy $F = -T\ln Z$ follow directly from the partition function. From these, one can derive observable quantities such as the magnetisation,

$$M = -\frac{1}{N}\frac{\partial F}{\partial h} = \langle \sigma_i \rangle\,, \tag{1.1.4}$$

and the magnetic susceptibility, which measures the response of the system to an external magnetic field,

$$\chi = \frac{\partial M}{\partial h}\Big|_{h=0}\,. \tag{1.1.5}$$

The magnetic susceptibility is directly related to the pair correlation function,

$$\begin{aligned} \Gamma(i-j) &= \langle \sigma_i\sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle = \frac{1}{\beta^2}\frac{\partial^2}{\partial h(i)\,\partial h(j)}\ln Z\Big|_{h=0} \\ \chi &= \beta\sum_j \Gamma(i-j)\,, \end{aligned} \tag{1.1.6}$$

which already hints at a close relation to quantum field theory, where correlation functions are the fundamental observables.

Many statistical systems exhibit phase transitions, where macroscopic quantities change dramatically as a control parameter (such as the temperature) crosses a critical value. A phase transition is called first-order if the macroscopic quantity itself changes discontinuously, and second-order if its derivative exhibits a discontinuity or divergence. A standard example of a first-order phase transition is the condensation of water vapor to liquid water. The two-dimensional Ising model, however, features a second-order phase transition characterised by a diverging magnetic susceptibility at the critical temperature.

Within the framework of block spin renormalisation group theory [5], one can derive the scaling behavior of physical quantities close to the critical point $t = \frac{T-T_c}{T_c} = 0$,

$$\begin{array}{llcl} \text{Heat capacity:} & C & \sim & |t|^{-\alpha} \\ \text{Magnetisation:} & M & \sim & |t|^{\beta} \\ \text{Magnetic susceptibility:} & \chi & \sim & |t|^{-\gamma} \\ \text{Equation of state:} & M & \sim & h^{1/\delta}\,, \end{array}$$

where the notation $\sim$ indicates the dominant singular behavior as $t \to 0$, with $h = 0$ assumed for the first three relations. The exponents $\alpha, \beta, \gamma, \delta$ are known as critical exponents and are universal quantities that depend only on broad features such as the spatial dimension and symmetries of the system, but not on microscopic details.

The pair correlation function near criticality also exhibits universal scaling,

$$\Gamma(i-j) \to |i-j|^{-\eta} e^{-\frac{|i-j|}{\zeta}}\,, \tag{1.1.7}$$

with the correlation length

$$\zeta \sim |t|^{-\nu}\,. \tag{1.1.8}$$

The critical exponents characterise the universal behavior of the physical system near the phase transition. Within the framework of block spin renormalisation group theory [5], one can derive relations between the critical exponents. In fact, all exponents can be expressed in terms of two independent ones, commonly chosen to be η and ν. These are directly linked to the scaling behavior of the spin-spin correlation function and the correlation function of the local energy density $E(\vec{r}) = s(r_1)s(r_1')$,

$$\begin{aligned} \Gamma(i-j) &\sim |i-j|^{-\eta} \\ \langle E(\vec{r}_1)E(\vec{r}_2)\rangle &\sim |\vec{r}_1 - \vec{r}_2|^{-4+2/\nu}\,. \end{aligned} \tag{1.1.9}$$

To fully determine the behavior of the system at the critical point, we are left to compute these correlation functions. Deriving them from first principles requires lifting the discrete lattice model to a continuum quantum field theory. Although the details are technical and beyond the scope of this qualitative introduction, the key idea is that the two-dimensional classical Ising model can be mapped to a one-dimensional quantum spin chain [6, 7]. Through the Jordan–Wigner and Bogoliubov transformations [8, 9], this spin chain can be rewritten in terms of free fermionic creation and annihilation operators. Taking the continuum limit $a \to 0$ then yields a free fermion quantum field theory that exactly describes the Ising model's universal behavior near the critical point.

At the critical point, the diverging correlation length ζ indicates the absence of any intrinsic length scale. Physically, this means that the system becomes scale invariant. In the Ising model, this can be seen qualitatively via block spin renormalisation: at the critical point, successive coarse-graining transformations leave the long-distance physics invariant. In two dimensions, scale invariance in quantum field theories is automatically enhanced to conformal invariance under mild physical assumptions [10, 11]. Hence, the continuum limit of the Ising model at criticality is exactly described by a two-dimensional conformal field theory of free fermions. The scaling relations (1.1.9) thus follow directly from the symmetries and operator content of the underlying conformal field theory. In the two-dimensional Ising model, the operator content consists of the identity operator, the energy operator $\epsilon(z, \bar{z})$, which has *conformal weight* $\frac{1}{2}$ and arises as the composite of the two free fermionic fields Ψ and $\overline{\Psi}$, and the spin operator $\sigma(z, \bar{z})$, which has conformal weight $\frac{1}{16}$. From these conformal weights, one directly obtains the critical exponents $\eta = \frac{1}{4}$ and $\nu = 1$. The methods for computing these exact correlation functions, as well as the general structure underlying two-dimensional conformal field theories, will be discussed in detail in chapter 2.

1.2 Two-Dimensional Conformal Field Theories in String Theory

The main results presented in this thesis arise from the investigation of arithmetic structures of two-dimensional (super)conformal field theories, particularly those that emerge in the formulation of (super)string theory. In perturbative string theory, the fundamental dynamical object is not a point particle, as in conventional quantum field theory, but a one-dimensional extended object — a string. As the string propagates through a so-called *target space* $\mathcal{M}$, it sweeps out a two-dimensional surface Σ known as the *worldsheet*, in analogy to the worldline traced out by a point particle. The dynamics of this worldsheet is governed by a two-dimensional quantum field theory, which, under appropriate gauge choices, becomes a two-dimensional conformal field theory. In the following review, we adopt the notation of the standard textbooks [12–15], which we also refer to for a comprehensive introduction to (super)string theory.

Worldsheet Conformal Field Theory

The classical action for the bosonic string is described by the *Polyakov action* [16–18],

$$S_P = -\frac{T}{2}\int_\Sigma d^2\xi\sqrt{-\det h}\,h^{ab}(\xi)\partial_a X^\mu(\xi)\partial_b X^\nu(\xi)\eta_{\mu\nu}\,, \tag{1.2.1}$$

where (ξ^1, ξ^2) are local coordinates on the two-dimensional worldsheet Σ, and we assume for simplicity a flat, D-dimensional target space $\mathcal{M}$ with local coordinates X^μ. The parameter $T = \frac{1}{2\pi\alpha'}$ denotes the string tension, which is related to the string length $\ell_s = 2\pi\sqrt{\alpha'}$, and h_{ab} is the worldsheet metric. This action describes D two-dimensional scalar fields $X^\mu(\xi)$ coupled to a dynamical worldsheet metric h_{ab}; in other words, it is a scalar field theory coupled to two-dimensional gravity. A key feature is that the Polyakov action is invariant under both local diffeomorphisms and local Weyl transformations on the worldsheet. In particular, invariance under Weyl transformations,

$$h_{ab} \to h_{ab} + 2\Lambda(\xi)h_{ab} + \mathcal{O}(\Lambda^2)h_{ab}\,, \tag{1.2.2}$$

is special to two dimensions and plays a pivotal role in the consistent quantisation of the theory [18]. Weyl invariance implies that the worldsheet energy-momentum tensor is traceless, $T^a_a = 0$, which already signals the presence of classical conformal symmetry. By exploiting both local diffeomorphism and Weyl invariance, one can locally choose the so-called *conformal gauge*, in which the metric takes the flat form $h_{ab} = \eta_{ab}$. In this gauge, the Polyakov action reduces to a theory of free scalar fields

$$S_P = \frac{T}{2}\int_\Sigma d\tau d\sigma\left((\partial_\tau X)^2 - (\partial_\sigma X)^2\right)\,. \tag{1.2.3}$$

Even after fixing the gauge $h_{ab} = \eta_{ab}$, a large residual symmetry remains: these are the *conformal Killing transformations* of the worldsheet. A convenient choice of coordinates is the lightcone coordinates

$$\xi^+ = \tau + \sigma\,, \qquad \xi^- = \tau - \sigma\,. \tag{1.2.4}$$

In these coordinates, the conformal Killing vector equations decouple: the transformations are generated by arbitrary holomorphic functions $\epsilon^+(\xi^+)$ and $\epsilon^-(\xi^-)$,

$$\xi^+ \mapsto \xi^+ - \epsilon^+(\xi^+), \quad \xi^- \mapsto \xi^- - \epsilon^-(\xi^-)\,. \tag{1.2.5}$$

These residual coordinate transformations can be compensated for by local Weyl transformations. The infinitely many conserved charges associated with the conformal Killing vectors take the form,

$$L_{\epsilon^+} = \int T_{++}(\xi^+)\epsilon^+(\xi^+)d\sigma\,, \tag{1.2.6}$$

where T_{++} is the holomorphic component of the energy-momentum tensor. By choosing an appropriate Fourier basis for $\epsilon^\pm$, one defines the *Virasoro generators*,

$$\begin{aligned} L_m &\equiv -\frac{l}{4\pi^2}\int_0^l d\sigma T_{--}\exp(-\mathrm{i}m\frac{2\pi}{l}\sigma) \\ \overline{L}_m &\equiv -\frac{l}{4\pi^2}\int_0^l d\sigma T_{++}\exp(\mathrm{i}m\frac{2\pi}{l}\sigma)\,. \end{aligned} \tag{1.2.7}$$

These generate the residual conformal transformations via the Poisson bracket,

$$\{L_m, X^\mu\} = -\frac{l}{2\pi}\exp(\mathrm{i}m\frac{2\pi}{l}\xi^-)\partial_- X^\mu\,,\;\; \{\overline{L}_m, X^\mu\} = -\frac{l}{2\pi}\exp(\mathrm{i}m\frac{2\pi}{l}\xi^+)\partial_+ X^\mu\,. \tag{1.2.8}$$

A straightforward calculation shows that the classical Virasoro generators close into the *Witt algebra*,

$$\begin{aligned} \{L_m, L_n\} &= -i(m-n)L_{m+n} \\ \{\overline{L}_m, \overline{L}_n\} &= -i(m-n)\overline{L}_{m+n}\,. \end{aligned} \tag{1.2.9}$$

From the explicit form of the energy momentum tensor $T_{--} = -\frac{1}{\alpha}\eta_{\mu\nu}\partial_- X^\mu \partial_- X^\nu$ and by using the oscillator expansion of the closed string,

$$\partial_- X^\mu = \frac{2\pi}{l}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-\mathrm{i}2\pi/l\, n\xi^-}\,, \tag{1.2.10}$$

we get

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}}\alpha_{m-n}\cdot\alpha_n\,. \tag{1.2.11}$$

Upon quantisation, the Fourier modes α_n^μ become operators, and commutators replace Poisson brackets. The ordering ambiguity for operators is resolved by introducing normal ordering,

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\, \alpha_{m-n}\cdot\alpha_n\, : \quad \text{with} \quad :\alpha_m^\mu\alpha_n^\nu := \begin{cases}\alpha_m^\mu\alpha_n^\nu\,, & m\le n\,,\\ \alpha_n^\nu\alpha_m^\mu\,, & n<m\,.\end{cases} \tag{1.2.12}$$

The quantum Virasoro generators then satisfy the Virasoro algebra,

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \delta_{m,-n}\frac{c}{12}(m^3 - m) \\ \left[L_m, \overline{L}_n\right] &= 0\,, \qquad l, m \in \mathbb{Z}\,, \end{aligned} \tag{1.2.13}$$

with central charge $c = \eta^\mu_\mu = D$, the number of target space dimensions. The appearance of a nonzero central charge reflects a quantum anomaly of the classical Weyl symmetry. Since Weyl invariance acts as a gauge redundancy, consistency as a quantum theory demands the cancellation of this anomaly[2].

In the modern covariant quantisation scheme [19–22], one evaluates the string path integral by dividing out the overcounting from gauge-equivalent configurations of h_{ab}. Writing the path integral measure $\mathcal{D}h$ in terms of a reference metric $\tilde{h}$ and gauge parameters ζ leads to a factor corresponding to the volume of the diffeomorphism and Weyl gauge group. The associated Faddeev–Popov determinant is computed by introducing anti-commuting ghost fields b, c, yielding the total path integral,

$$Z \sim \int \mathcal{D}X\mathcal{D}b\mathcal{D}c\, e^{i(S_m + S_g)}\,. \tag{1.2.14}$$

The total Virasoro generator combines matter and ghost contributions,

$$L^{\text{tot}}_m = L^{(m)}_m + L^{(g)}_m\,, \tag{1.2.15}$$

with total central charge $c^{\text{tot}} = c^{(m)} + c^{(g)} = c^{(m)} - 26$. The condition for a consistent quantum theory is that the total Weyl anomaly, which is proportional to c^{tot}, vanishes. In a flat target space $\mathcal{M} = \mathbb{R}^{1,D-1}$, this implies the *critical dimension* $D = 26$ for the bosonic string. To incorporate fermionic degrees of freedom into the theory, an analogous analysis shows that one obtains an $\mathcal{N} = 1$ superconformal symmetry on the worldsheet, which replaces the purely bosonic conformal symmetry of the bosonic string. The requirement that this superconformal symmetry remains anomaly-free imposes the condition $c^{(m)} = 15$ for the matter–sector. For a flat target space $\mathcal{M}$, this in turn implies the critical spacetime dimension $D = 10$. While it is often stated that string theory requires for consistency a specific number of spacetime dimensions (26 for bosonic string string theory and 10 for superstring theory), more precisely, it is the total central charge $c^{(m)}$ of the matter–sector of the worldsheet (super)conformal field theory that must take a critical value, which is $c^{(m)} = 26$ for superstring theory and $c^{(m)} = 15$ for superstring theory. Only when the string propagates in flat spacetime $\mathbb{R}^{1,D-1}$, this condition translates to the usual critical spacetime dimensions. But this is in no means the most general situation. Typically, one considers a *compactification*, i.e., target spaces of the form $\mathcal{M} = \mathbb{R}^{1,3} \times \mathcal{C}$ with $\mathcal{C}$, in superstring theory, being a 6-dimensional internal manifold. The "internal" conformal field theory on $\mathcal{C}$ for generic manifolds cannot be solved exactly because it is not described by a free Lagrangian.

[2]Gauge anomalies signal the breakdown of a gauge symmetry upon quantisation. Non-vanishing Gauge anomalies typically render a quantum field theory non-unitary or ill-defined.

However, as we will see in later chapters, if $\mathcal{C}$ is a so-called *Calabi–Yau threefold*, which is a compact complex three-dimensional manifold with vanishing first Chern class, the internal conformal field theory can sometimes be solved explicitly. Important examples falling into this category are so-called *Gepner models*, which we will discuss in detail in section 4.4.

From an abstract point of view, it is the two-dimensional conformal field theory defined on the worldsheet that constitutes the fundamental object in perturbative string theory. Spacetime itself, as well as the spectrum of particle excitations and interactions, are *emergent* phenomena encoded in the structure of the underlying two-dimensional conformal field theory.[3] The different vibrational modes of the string correspond to various fields in spacetime — such as the graviton, Kalb-Ramond field, dilaton, and gauge bosons — each of which arises from specific vertex operators in the two-dimensional conformal field theory with well-defined conformal weights and operator product expansions (OPEs).

Closed string states correspond to conformal field theories defined on compact two-dimensional surfaces without boundary (e.g., spheres, tori, and higher genus surfaces), and their vertex operators must be mutually local and BRST invariant. These states mediate gravitational interactions, and the graviton itself arises from the massless spin-2 excitation in the closed string spectrum. On the other hand, gauge degrees of freedom are described by open strings, whose endpoints are constrained to lie on dynamical objects known as *D-branes*. The worldsheet theory in this case is a conformal field theory defined on surfaces with boundaries, and its formulation requires a careful treatment of boundary conditions consistent with conformal invariance. The framework of *boundary conformal field theory* provides the tools to analyze such setups by studying the spectrum and dynamics of boundary states and open string vertex operators.

Double Expansion in Perturbative String Theory

A distinctive feature of perturbative string theory is its *double expansion*, which reflects the presence of two fundamental expansion parameters: the worldsheet expansion in α', controlling the *non-linear sigma model* (NLSM) interactions on the string worldsheet, and the spacetime loop expansion in the string coupling g_s, which arises dynamically as the vacuum expectation value of the dilaton field. This structure distinguishes string theory from conventional quantum field theory and encodes both the local geometry of the target space and the sum over worldsheet topologies.

In a curved target space, the Polyakov action (1.2.1) generalises to the non-linear sigma model,

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{h}\, h^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \dots \,, \tag{1.2.16}$$

[3] This perspective is complemented in non-perturbative string theory by the inclusion of objects such as D-branes, NS5-branes, and more exotic configurations. These are not described purely by two-dimensional worldsheet conformal field theories, as they do not couple directly to the string worldsheet, but their low-energy effective dynamics can often be captured by studying worldsheets with boundaries and the corresponding so-called boundary conformal field theories.

where $G_{\mu\nu}(X)$ is the target space metric and the dots indicate additional background fields, such as the Kalb-Ramond B-field, and the dilaton.

Expanding around a classical background solution X_0^μ with small fluctuations Y^μ,

$$X^\mu(\xi) = X_0^\mu + \sqrt{\alpha'}\, Y^\mu(\xi)\,, \tag{1.2.17}$$

one finds that the expansion parameter controlling interactions on the worldsheet is of order $\sqrt{\alpha'}/R$, where R denotes the typical curvature radius of the target space. If $\sqrt{\alpha'}/R \ll 1$, the non-linear sigma model is *weakly coupled* and a perturbative approach is justified. For curvature radii of the order of the string length, the non-linear sigma model is *strongly coupled*, and perturbation theory breaks down. In the latter case, one needs to solve the worldsheet conformal field theory exactly.

Unlike point particle field theory, where interactions arise from local vertices, string interactions are governed by the topology of the worldsheet. The worldsheet path integral includes a sum over all possible genera and boundaries. The string coupling g_s arises dynamically as

$$g_s = e^{\Phi}\,, \tag{1.2.18}$$

where Φ denotes the vacuum expectation value of the dilaton field. In the path integral, the sum over all possible worldsheet topologies gives a perturbative expansion in powers of g_s. Specifically, for closed strings, the Euler characteristic χ of a compact, orientable worldsheet is

$$\chi = 2 - 2g - b\,, \tag{1.2.19}$$

with genus g and number of boundaries b. Using the *Riemann-Roch-theorem*, one can express this topological invariant as

$$\chi = \frac{1}{4\pi}\int_\Sigma d^2\xi\,\sqrt{h}\,R^{(2)} + \frac{1}{2\pi}\int_{\partial\Sigma} ds\,k\,, \tag{1.2.20}$$

where $R^{(2)}$ is the Ricci scalar curvature of the worldsheet metric h_{ab}, ds denotes the line element along the boundary, and k is the geodesic curvature of the boundary. The Euler characteristic appears naturally in the string path integral because one can add to the Polyakov action the topological term

$$S_\lambda = \lambda\,\chi\,, \qquad \lambda \in \mathbb{R}\,. \tag{1.2.21}$$

This term does not affect the local dynamics on the worldsheet since the Euler characteristic depends only on the global topology of Σ. Instead, it acts as a bookkeeping device that weights each worldsheet by its topological class in the full path integral. Concretely, its contribution exponentiates to a factor $e^{\lambda\chi}$, which organises the expansion over genera and boundaries. In the context of string theory, this topological coupling is provided by the *dilaton* field $\Phi(X)$. The contribution of the dilaton to the Polyakov action reads

$$S_\Phi = \frac{1}{4\pi}\int_\Sigma d^2\xi\,\sqrt{h}\,R^{(2)}\,\Phi(X)\,. \tag{1.2.22}$$

When the dilaton takes a constant background value $\Phi(X) = \Phi_0$, this reduces to

$$S_\Phi = \Phi_0 \, \chi \,. \tag{1.2.23}$$

Comparing with the general topological term S_λ shows that the dilaton expectation value plays the role of the coupling λ. Therefore, the string coupling is identified as

$$g_s = e^{\Phi_0} \,. \tag{1.2.24}$$

This demonstrates that, in string theory, the spacetime coupling constant is not a fixed external parameter but a dynamical modulus determined by the vacuum expectation value of the dilaton. Fluctuations around this background give rise to a scalar degree of freedom in the target space, reinforcing the fact that the string coupling is part of the dynamical content of the theory. Before gauge fixing, the scattering amplitude that describes the scattering of n string states reads schematically

$$\mathcal{S}_{j_1 \dots j_n}(k_1, \dots, k_n) = \sum_{\substack{\text{compact} \\ \text{topologies}}} \frac{1}{\text{Vol(Diff.} \times \text{Weyl)}} \int \mathcal{D}X \, \mathcal{D}h \; e^{-S_X - \lambda\chi} \prod_{i=1}^{n} V_{j_i}(k_i) \,, \tag{1.2.25}$$

where we integrate over the insertion of the vertex operator $V_{j_i}(k_i, \xi_i)$ to ensure worldsheet diffeomorphism invariance of the amplitude,

$$V_{j_i}(k_i) \sim \int_\Sigma d^2\xi_i \sqrt{h(\xi_i)} V_{j_i}(k_i, \xi_i) \,. \tag{1.2.26}$$

The contribution of each topology to the scattering amplitude is weighted by

$$g_s^{-\chi} = g_s^{2g+b-2} \,. \tag{1.2.27}$$

Thus, in the closed string, each additional handle ($\chi \to \chi - 2$) corresponds to a higher-loop correction, weighted by extra powers of g_s.

Combining both expansions, perturbative string theory is schematically organised by

$$\text{String Amplitude} \sim \sum_{g,b} g_s^{2g+b-2} \big(\text{Worldsheet CFT correlator expansion in } \alpha'\big) \,, \tag{1.2.28}$$

For this to define a perturbative series in the large volume regime $\sqrt{\alpha'}/R \ll 1$, we need $g_s \ll 1$.

To probe the theory at fixed genus in the stringy regime, where the extended nature of the string plays an important role and where a perturbative expansion in α' breaks down, having access to exactly solvable, strongly interacting two-dimensional conformal or superconformal field theories is thus of central importance in perturbative string theory. However, the worldsheet approach is inherently perturbative in the string coupling g_s: it does not by itself capture non-perturbative spacetime effects such as D-brane instantons or spacetime topology change.

In certain regimes, however, strong–weak coupling dualities — such as S-duality in type IIB string theory [23–25], or the duality between heterotic string theory and type IIA string theory [25–27] — relate strongly coupled sectors of the theory to weakly coupled ones in a dual frame. Remarkably, these dualities can map non-perturbative α'-effects in one description to non-perturbative g_s-effects in its dual [28–35]. In this way, exact worldsheet methods in one frame may provide non-perturbative information about the string coupling expansion in a dual frame. This is a striking example of how the structure of string dualities extends the predictive power of perturbative worldsheet techniques beyond their naive range of validity.

1.3 Conformal Field Theory and Quantum Gauge Theories

Four-dimensional quantum gauge theories remain the most accurate framework for describing the fundamental interactions of elementary particles, with the Standard Model as their prime example. To make contact with experimental observables, such as scattering amplitudes, one needs to compute correlation functions and partition functions, which form the basic building blocks of any observable quantity in a quantum field theory.

In general quantum field theories, and in particular for the Standard Model, these quantities can typically only be computed perturbatively through Feynman diagram expansions, using the gauge coupling as the expansion parameter. This perturbative approach is valid only when the coupling constant is sufficiently small compared to the relevant energy scale of the process. For instance, in Quantum Chromodynamics (QCD), *asymptotic freedom* ensures that the effective coupling becomes weak at high energies, justifying perturbative expansions. However, at low energies QCD becomes strongly coupled, leading to *confinement*, where the fundamental degrees of freedom — quarks and gluons — are no longer observed as free particles but appear only in bound states such as hadrons. Describing this low-energy regime requires a full understanding of non-perturbative effects, since the perturbative expansion breaks down in the strong coupling limit.

Exact, fully non-perturbative solutions for general four-dimensional gauge theories, such as QCD, remain out of reach. Nonetheless, certain special classes of four-dimensional quantum field theories, namely those with extended supersymmetry, admit more analytic control. In these theories, the symmetry algebra of Minkowski spacetime — the Poincaré algebra — is extended by additional fermionic generators, introducing a symmetry between bosonic and fermionic degrees of freedom. These supersymmetric quantum field theories often allow exact computations of quantities that are inaccessible in generic gauge theories.

A prominent example of such exact results are *instantons*, which correspond to non-trivial, finite-action solutions to the Euclidean equations of motion. More precisely, they are field configurations connecting two topologically distinct vacua. Instantons are non-perturbative contributions to physical observables and cannot be captured by ordinary perturbation theory. In $\mathcal{N} = 2$ supersymmetric gauge theories, intricate connections between four-dimensional gauge theories and two-dimensional conformal field theories have been discovered, most notably through the Alday–Gaiotto–Tachikawa (AGT) correspondence [36] (for a comprehensive review, we refer to [37]).

The AGT correspondence asserts that the instanton partition function of $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group SU(2) coupled to four massive hypermultiplets can be expressed in terms of quantities from the *Liouville conformal field theory*, namely *conformal blocks* and *structure constants*. Explicitly, up to known prefactors, one finds the schematic relation

$$Z_{\text{inst}} \sim N_{21}N_{43}\,\mathcal{F}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_2\end{bmatrix}, \tag{1.3.1}$$

where the left-hand side denotes the four-dimensional instanton partition function, and the right-hand side involves the conformal block $\mathcal{F}$ appearing in the four-point correlation function of the Liouville theory:

$$\left\langle e^{2\alpha_4\phi(\infty)}e^{2\alpha_3\phi(1)}e^{2\alpha_2\phi(q)}e^{2\alpha_1\phi(0)}\right\rangle \sim \int_{\mathbb{R}^+} dp\; C_{21}(p)\,C_{43}(-p)\left|\mathcal{F}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_2\end{bmatrix}\right|^2, \tag{1.3.2}$$

with structure constants $|N_{ij}|^2 = C_{ij}$.

The original AGT correspondence has since been generalised to a broader class of dualities [38–44], e.g., relating $\mathcal{N}=2$ supersymmetric gauge theories with gauge group $\mathrm{SU}(N)$ defined on orbifolds $\mathbb{R}^4/\mathbb{Z}_p$ to coset conformal field theories associated with the affine coset algebra

$$\frac{\widehat{\mathfrak{su}}(N)_k \otimes \widehat{\mathfrak{su}}(N)_p}{\widehat{\mathfrak{su}}(N)_{k+p}}\,. \tag{1.3.3}$$

At the present stage, such correspondences are not yet practical tools for computing new physical observables in four-dimensional supersymmetric gauge theories purely from the conformal field theory side. Nonetheless, these remarkable connections have deepened our understanding of dualities and non-perturbative phenomena in supersymmetric gauge theories by exposing a hidden two-dimensional (super)conformal symmetry structure.

1.4 Rational Conformal Field Theories

In the study of two-dimensional conformal field theories — especially those describing internal worldsheet dynamics in string theory — one is often interested in the structure and classification of entire families of such theories. These families, called *conformal manifolds*, are parameter spaces in which each point corresponds to a distinct conformal field theory, and points are connected via deformations by *exactly marginal operators* that preserve conformal symmetry [10]. A particularly illustrative example of a conformal manifold arises from the theory of a free bosonic field compactified on a circle of radius R, described by the action

$$S = \frac{R^2}{\pi}\int d^2z\partial\phi\bar{\partial}\phi\,. \tag{1.4.1}$$

In this example, the exactly marginal operator generates a continuous family of theories. Perturbing the action by a small deformation parameter

$$S(\epsilon) = S + \epsilon\int d^2z\partial\phi\bar{\partial}\phi\,, \tag{1.4.2}$$

yields a theory with a shifted compactification radius R', where $R' = R+\delta R$. The spectrum and correlation functions of the deformed theory can be computed perturbatively in $\delta \ll R$, and the conformal manifold is one-dimensional, parametrised by the radius R.

Another central feature of two-dimensional conformal field theories is the existence of extended symmetries beyond the Virasoro algebra. In the free boson theory (1.4.1), the modes of the chiral current

$$j(z) = i\partial\phi(z)\,, \tag{1.4.3}$$

generate the so-called *Heisenberg algebra*, which is an infinite-dimensional affine Lie algebra. In the present case, it is the affine extension of $\mathfrak{u}(1)$. Such affine extensions of current algebras are called Kac–Moody algebras and are common in two-dimensional conformal field theories. They often serve as building blocks for *rational models*.

The compatibility of these extended chiral symmetries with the Virasoro symmetry is encoded in the *Sugawara construction* [45–51], which embeds the Virasoro algebra into the vertex algebra of the Kac–Moody algebra such that any module of the affine algebra carries a natural Virasoro module structure.

At generic values of the radius R, the chiral algebra of the free boson is the Heisenberg algebra, but at special points where $R^2 \in \mathbb{Q}$, the chiral algebra is enlarged by additional chiral vertex operators,

$$\Gamma^{\pm}(z) = e^{\pm i\sqrt{C(R)}\phi(z)}\,, \tag{1.4.4}$$

for some radius dependent constant $C(R)$. These chiral vertex operators enlarge the chiral symmetry algebra. A famous example is the theory at $R = 1$, which gives rise to a Kac-Moody algebra $\widehat{\mathfrak{su}}(2)_1$, which is the affine extension of $\mathfrak{su}(2)$ at level 1. In such cases, the infinite number of Virasoro modules reorganize into a finite number of modules of the extended chiral algebra. Such theories are called *rational conformal field theories* and are of central importance. This is because rational conformal field theories provide a large class of non-trivial two-dimensional conformal field theories that are exactly solvable by algebraic methods. A feature that generic two-dimensional conformal field theories do not possess. In string theory, two-dimensional conformal field theories often arise as the RG fixed point in a strong-coupling regime of some non-linear sigma model, where perturbative methods break down. In this context, exact solvability is highly desirable, and rational conformal field theories play an important role.

Two natural questions arise:

(i) *Can we classify all rational conformal field theories?* Tremendous progress has been made over the past three decades in classifying rational conformal field theories with affine Lie algebra symmetries, W-algebras and their associated modular invariants [52–79] and using modular tensor categories [80–84], but a complete constructive classification remains elusive.

(ii) *How are rational conformal field theories distributed within a given conformal manifold?* For the compactified free boson on S^1, the rational points at $R^2 \in \mathbb{Q}$ form a dense subset of the moduli space, due to the density of $\mathbb{Q}$ in $\mathbb{R}$. Thus, in principle, any non-rational theory on S^1 can be approximated arbitrarily closely by a nearby rational theory via conformal perturbation theory [85–91].

The next step in complexity is the compactification of two bosons on a complex one-dimensional torus, yielding a higher-dimensional conformal manifold. Even in this case, the rational points form a dense subset. In [92, 93] the authors revealed a deep correspondence between rational conformal field theory on complex tori and the theory of *complex multiplication* (CM), an arithmetic property where the endomorphism algebra of the torus is larger than at generic points in the moduli space of complex tori. This leads to a striking interplay between rationality in conformal field theory and arithmetic geometry, which was later conjectured to hold more generally [94].

In this setting, the classification and distribution of rational conformal field theory can be studied through the lens of *Hodge theory*: the Hodge structures of complex tori with complex multiplication reflect the enhanced symmetry content of the underlying rational conformal field theory. Conversely, the presence of extra symmetries in the Hodge structure of a target space (e.g., a Calabi–Yau manifold) can inform us about the rationality of the corresponding worldsheet conformal field theory.

1.5 Outline

Thesis Goals. This thesis investigates the arithmetic structures in two-dimensional rational conformal field theories, with a particular focus on their relationship to Hodge theory and number theory. A central objective is to extend the well-established correspondence between $\mathcal{N} = (2, 2)$ toroidal rational conformal field theories and Hodge structures admitting complex multiplication beyond the class of toroidal theories. We demonstrate, under certain technical assumptions, that this correspondence holds for a broader class of exactly solvable $\mathcal{N} = (2, 2)$ rational superconformal field theories. We will exemplify our general model-independent findings with so-called Gepner models. These models are believed to describe infrared (IR) fixed points of supersymmetric non-linear sigma models with Calabi–Yau target spaces, thereby providing a window into the emergence of arithmetic structures in strongly coupled regimes in superstring theory.

An important component of this investigation of arithmetic structures in two-dimensional rational conformal field theories is a detailed analysis of toroidal rational conformal field theories. In particular, we derive an explicit expansion of the partition function for these theories in terms of *ray class theta functions* — special theta series associated with ray class groups that encode the arithmetic of ideals in the ring of integers of a number field. This construction provides a novel number-theoretic interpretation of the rational toroidal partition function, while also highlighting a deep interplay between modular invariance in rational conformal field theory and class field theory in algebraic number theory.

Structure of the Thesis. The thesis is organised as follows. We begin by reviewing the foundational aspects of two-dimensional conformal field theory, with emphasis on the structure of chiral symmetry algebras and the defining characteristics of rational conformal field theories. Next, we discuss the embedding of two-dimensional conformal field theories into string theory, focusing on the role of $\mathcal{N} = (2, 2)$ supersymmetric conformal field theories and their connection to Hodge theory.

We then turn to toroidal non-linear sigma models and provide a detailed analysis of rational points in their moduli spaces. This includes the derivation of explicit formulas for their partition functions in terms of products of generalised theta functions. These, in turn, will be related to ray class theta functions, which yield an interpretation of the toroidal rational partition function in terms of number-theoretic objects.

Following this, we review the theory of complex multiplication in the context of elliptic curves and higher-dimensional abelian varieties and work out their relation to rational conformal field theory. We will then extend this correspondence to generic $\mathcal{N} = (2, 2)$ rational superconformal field theories and exemplify our findings with Gepner models with a geometrically realised CM-type Hodge structure and non-geometric $\mathcal{N} = (2, 2)$ minimal models.

Chapter 2

Aspects of Two-Dimensional Conformal Field Theories

In this chapter, we will review the basic structures and concepts of two-dimensional (super) conformal field theories, including their symmetry algebra, operator content, correlation functions, and Galois symmetry. We explain in precise terms what it means to *solve* a two-dimensional conformal field theory and why rational conformal field theories are often regarded as exactly solvable, due to their enhanced chiral symmetry algebra. Finally, we conclude this section with an introduction to the essential elements of boundary conformal field theory, which will play a significant role in subsequent sections, along with $\mathcal{N} = (2, 2)$ superconformal field theories. For a more comprehensive introduction to the subject, we refer the reader to the standard physics textbooks [95–97], and [98] for a more mathematical approach.

2.1 Basics of Two-Dimensional Conformal Field Theories

Conformal Group and Conformal Algebra

Conformal transformations are local diffeomorphisms that preserve angles between tangent vectors at each point. Intuitively, they form the group of transformations that locally rescale the metric tensor by a smooth, positive function. More concretely, let $\mathcal{M}$ be a flat d-dimensional manifold with arbitrary signature (k, l) satisfying $k + l = d$, and let $\eta_{\mu\nu}$ denote its flat metric. A transformation $x \mapsto x'(x)$ is said to be conformal if it rescales the metric according to

$$\eta_{\mu\nu} \mapsto \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} = \Lambda(x) \eta_{\mu\nu} \, , \tag{2.1.1}$$

where $\Lambda(x) > 0$ is a smooth and positive function. To derive the local constraints imposed by (2.1.1), we consider infinitesimal transformations of the form

$$\begin{aligned} x'^{\mu} &= x^{\mu} - \epsilon^{\mu}(x) + \mathcal{O}(\epsilon^2) \\ \Lambda(x) &= e^{\omega(x)} = 1 + \omega(x) + \dots\,, \end{aligned} \tag{2.1.2}$$

with $\epsilon^{\mu} = \epsilon v^{\mu}$. Now, plugging into (2.1.1), keeping only terms of first order in $\epsilon \ll 1$, we find the equations,

$$\begin{aligned} \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} &= \omega(x)\eta_{\mu\nu} \\ \omega(x) &= \frac{2}{d}\partial_{\mu}\epsilon^{\mu} \\ \left(\eta_{\mu\nu}\partial^2 + (d-2)\partial_{\mu}\partial_{\nu}\right)\omega(x) &= 0 \\ (d-1)\,\partial^2\omega(x) &= 0\,. \end{aligned} \tag{2.1.3}$$

Note that when $d > 2$, we have that $\partial_{\mu}\partial_{\nu}\left(\partial_{\beta}\epsilon^{\beta}\right) = 0$, which implies that all third and higher derivatives of $\epsilon^{\mu}(x)$ must vanish. Consequently, the most general infinitesimal conformal transformation is a second-degree polynomial in the coordinates x^{μ}. This significantly restricts the conformal group in dimensions higher than two.

Solving the constraints in $d > 2$ reveals that the conformal group is generated by

- Translations: $x^{\mu} \to x^{\mu} + a^{\mu}$
- Lorentz rotations: $x^{\mu} \to x^{\mu} + \hat{\omega}^{\mu}{}_{\nu}x^{\nu}, \quad \hat{\omega}_{\mu\nu} = -\hat{\omega}_{\nu\mu}$
- Dilations: $x^{\mu} \to (1+\sigma)x^{\mu}$
- Special conformal transformation: $x^{\mu} \to x^{\mu} + b^{\mu}x_{\beta}x^{\beta} - 2x^{\mu}b_{\alpha}x^{\alpha}$.

The global special conformal transformation can be shown to be

$$x^{\mu} \to \frac{x^{\mu} + b^{\mu}x_{\alpha}x^{\alpha}}{1 + 2b_{\beta}x^{\beta} + b_{\gamma}b^{\gamma}x_{\sigma}x^{\sigma}}\,. \tag{2.1.4}$$

Since there exist transformations for which $x^{\mu} \to \infty$, the special conformal transformations cannot be defined globally on $\mathbb{R}^{k,l}$. Instead, one performs a so-called *conformal compactification*, whereby $\mathbb{R}^{k,l}$ is embedded into a larger manifold $N^{k,l}$ by an embedding map $\iota : \mathbb{R}^{k,l} \hookrightarrow N^{k,l}$ such that all conformal transformations on $\mathbb{R}^{k,l}$ are lifted to globally defined smooth diffeomorphisms on $N^{k,l}$. For $d > 2$ the so-called conformal compactification manifold $N^{k,l}$ of $\mathbb{R}^{k,l}$ can be constructed as the closure of the embedding of $\mathbb{R}^{k,l}$ into real projective space $\mathbb{RP}^{d+1}$. The group of all such globally defined conformal diffeomorphisms on the compactification manifold $N^{k,l}$ is referred to as the *conformal group* of the space. The structure and properties of this group depend not only on the dimension d but also on the signature (k,l) of the metric and the topology of the compactification manifold. For $d > 2$ one can construct an explicit one-to-one map between the matrix representation of $\mathrm{SO}(k+1, l+1)$ acting diffeomorphic on $\mathbb{RP}^{d+1}$, and the conformal transformations on $\mathbb{R}^{k,l}$.

Let us now study the conformal group in $d = 2$. For $d = 2$, the constraint (2.1.1) can be solved most conveniently in Euclidean signature $\eta_{\mu\nu} = \delta_{\mu\nu}$, and in the complex coordinates,

$$\begin{aligned} z &= x^0 + \mathrm{i}x^1, & \bar{z} &= x^0 - \mathrm{i}x^1 \\ \epsilon &= \epsilon^0 + \mathrm{i}\epsilon^1, & \bar{\epsilon} &= \epsilon^0 - \mathrm{i}\epsilon^1 \\ \partial_0 &= \partial_z + \partial_{\bar{z}}, & \partial_1 &= \mathrm{i}(\partial_z - \partial_{\bar{z}}) \,, \end{aligned} \tag{2.1.5}$$

from which we get:

$$\partial_z \bar{\epsilon}(z, \bar{z}) = 0 \,, \qquad \partial_{\bar{z}} \epsilon(z, \bar{z}) = 0 \,. \tag{2.1.6}$$

We deduce that $\epsilon = \epsilon(z)$ and $\bar{\epsilon} = \bar{\epsilon}(\bar{z})$. Hence, in $d = 2$, the local conformal transformations are generated by all meromorphic functions $\epsilon(z)$ and anti-meromorphic functions $\bar{\epsilon}(\bar{z})$. Therefore, the conformal algebra is infinite-dimensional, which is the root of many simplifying features in two-dimensional conformal field theories that do not exist in higher dimensions. Expanding $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ in Laurent series,

$$\begin{aligned} \epsilon(z) &= \sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1} \,, \\ \bar{\epsilon}(\bar{z}) &= \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n \bar{z}^{n+1} \,, \end{aligned} \tag{2.1.7}$$

the corresponding infinitesimal generators of conformal transformations are given by

$$l_n = -z^{n+1} \partial_z \,, \qquad \bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}} \,, \qquad n \in \mathbb{Z} \,. \tag{2.1.8}$$

These generators obey the commutation relations

$$\begin{aligned} [l_m, l_n] &= (m - n) l_{m+n} \\ [\bar{l}_m, \bar{l}_n] &= (m - n) \bar{l}_{m+n} \\ [\bar{l}_m, l_n] &= 0 \,. \end{aligned} \tag{2.1.9}$$

This algebra is known as the *Witt algebra* and is the classical analog of the quantum *Virasoro algebra* in two-dimensional conformal field theories.

While the local conformal transformations form an infinite-dimensional algebra, only a finite-dimensional subgroup corresponds to globally well-defined transformations on the conformally compactified complex plane. The conformal compactification of the complex plane is given by the Riemann sphere $\mathbb{CP}^1$, which adds the point at infinity. On this compact space, only the transformations generated by the modes l_{-1}, l_0, and l_1 extend to globally defined holomorphic automorphisms. These three generators correspond to

- l_{-1}: Translations, $z \mapsto z + a$, $a \in \mathbb{C}$
- l_0: Dilations (scaling), $z \mapsto \lambda z$, $\lambda \in \mathbb{C}$
- l_1: Special conformal transformations, $z \mapsto z/(1 + cz)$.

Together, these generate the Möbius transformations of the form

$$z \mapsto \frac{az+b}{cz+d}\,, \qquad ad-bc \neq 0\,. \tag{2.1.10}$$

The Möbius group is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$, the group of complex projective linear transformations. These are the globally defined conformal transformations on the Riemann sphere and form the *global conformal group* in two dimensions. All other transformations generated by higher modes l_n with $|n| > 1$ are only locally defined and exhibit singular behavior at points $z = 0$ or $z = \infty$.

Virasoro Algebra

In the context of two-dimensional quantum conformal field theories, states are elements of a complex projective Hilbert space $\mathbb{P} = \mathbb{P}(\mathcal{H})$. To quantise the classical conformal symmetry, we are looking for representations of the classical symmetry group in $U(\mathbb{P})$ – the group of unitary projective transformations on $\mathbb{P}$. In general, such *projective representations* cannot be induced from unitary representations of the classical Witt algebra acting on the Hilbert space $\mathcal{H}$, but rather one needs to consider the central extension of the Witt algebra [98].

A central extension of a Lie algebra is the following. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$, and let $\mathfrak{a}$ be an abelian Lie algebra over the same field, meaning that the Lie bracket on $\mathfrak{a}$ is trivial:

$$[X,Y] = 0 \quad \text{for all } X, Y \in \mathfrak{a}\,. \tag{2.1.11}$$

A *central extension* of $\mathfrak{g}$ by $\mathfrak{a}$ is a short exact sequence of Lie algebra homomorphisms,

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0\,, \tag{2.1.12}$$

such that

- $\mathfrak{a}$ is embedded as a Lie subalgebra of $\mathfrak{h}$,
- $\mathfrak{a}$ is in the center of $\mathfrak{h}$, i.e.,

$$[X,Y] = 0 \quad \text{for all } X \in \mathfrak{a},\ Y \in \mathfrak{h}\,, \tag{2.1.13}$$

- the projection $\pi : \mathfrak{h} \to \mathfrak{g}$ is a surjective Lie algebra homomorphism with kernel $\mathfrak{a}$, so that

$$\mathfrak{g} \cong \mathfrak{h}/\mathfrak{a}\,. \tag{2.1.14}$$

Up to equivalence, the central extension of the Witt algebra $\mathcal{W}$ by $\mathbb{C}$ is unique and can be constructed via the linear map

$$\omega : \mathcal{W} \times \mathcal{W} \to \mathbb{C}\,, \tag{2.1.15}$$

given by [98]

$$\omega(L_n, L_m) := \delta_{n+m,0} \cdot \frac{1}{12}(n^3 - n)\,. \tag{2.1.16}$$

The resulting centrally extended algebra is called *Virasoro algebra*,

$$\mathrm{Vir} := \mathcal{W} \oplus \mathbb{C} \cdot c\,, \qquad c \in \mathbb{C}\,^{1}\,. \tag{2.1.17}$$

We denote the generators of the Virasoro algebra by L_m and $\overline{L}_m$ respectively, which satisfy the commutation relations

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \delta_{m,-n}\frac{c}{12}(m^3 - m) \\ \left[L_m, \overline{L}_n\right] &= 0\,, \qquad l, m \in \mathbb{Z}\,. \end{aligned} \tag{2.1.18}$$

Physically, a non-zero central charge signals the presence of a quantum anomaly; in the present case, it is a measure for the breakdown of classical conformal symmetry by passing to the quantum theory.

Spectrum of a Conformal Field Theory and Operator-State Correspondence

We proceed by introducing the essential ingredients required to specify a Euclidean, unitary two-dimensional conformal field theory. Unless explicitly stated otherwise, the theory is defined on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ (i.e., the conformal compactification of the complex plane). We begin with the description of the spectrum of the theory. The spectrum consists of local fields, among which the most important are the so-called *primary fields*, which we denote by $\phi(z, \bar{z})$. These fields correspond to irreducible, unitary highest-weight representations of the Virasoro algebra. Under global scaling transformations $z \mapsto \lambda z$, a conformal field $\phi(z, \bar{z})$ with conformal weights $(h, \overline{h})$ transforms as

$$\phi(z, \bar{z}) \mapsto \phi'(\lambda z, \overline{\lambda}\bar{z}) = \lambda^{-h}\overline{\lambda}^{-\overline{h}}\phi(z, \bar{z})\,. \tag{2.1.19}$$

A conformal field is called a *primary field* if it transforms under any conformal transformation $z \mapsto f(z)$, $\bar{z} \mapsto \overline{f}(\bar{z})$, as

$$\phi(z, \bar{z}) \mapsto \phi'(f(z), \overline{f}(\bar{z})) = \left(\frac{\partial f}{\partial z}\right)^{-h} \left(\frac{\partial \overline{f}}{\partial \bar{z}}\right)^{-\overline{h}} \phi(z, \bar{z})\,. \tag{2.1.20}$$

If a conformal field transforms in this way only under global conformal transformations (i.e., Möbius transformations), it is called a *quasi-primary field*. Every primary field is quasi-primary, but the converse does not necessarily hold. The analysis of quasi-primary fields generalises to higher dimensions. What is unique to two-dimensional conformal field theories is the concept of a primary field.

Throughout this thesis, we will use the terms *primary fields* and *primary states* interchangeably, which is justified by the existence of the *operator-state correspondence* in two-dimensional conformal field theories. To explain this, let us consider a two-dimensional

[1] Since in two-dimensional conformal field theories we are mainly interested in studying representations of the Virasoro algebra on a complex vector space, the central element can be viewed as a complex multiple of the identity map on the complex vector space.

Euclidean conformal field theory defined on flat space with coordinates (x^0, x^1). We compactify the spatial coordinate x^1 on a unit circle to place the theory on an infinitely long cylinder with complex coordinates

$$\omega = x^0 + \mathrm{i}x^1, \quad \text{and periodicity} \quad \omega \sim \omega + 2\pi\mathrm{i}\,. \tag{2.1.21}$$

Any field $\phi(w, \overline{w})$ admits the mode expansion

$$\phi(\omega, \overline{\omega}) = \sum_{m,n\in\mathbb{Z}} \phi_{m,n} e^{-m\omega} e^{-n\overline{\omega}}\,. \tag{2.1.22}$$

We can then perform a conformal map to the complex plane (or rather its conformal compactification $\mathbb{C} \cup \{\infty\}$) given by

$$z = e^{\omega}\,. \tag{2.1.23}$$

Under this conformal map, the infinite past $x^0 \to -\infty$ is mapped to the origin $z = 0$ and the infinite future $x^0 \to +\infty$ is mapped to $z = \infty$. Applying the conformal map (2.1.23) to the mode expansion (2.1.22) we get, according to (2.1.20),

$$\phi(z, \bar{z}) = \sum_{m,n\in\mathbb{Z}} \phi_{m,n} z^{-m-h} \bar{z}^{-n-\overline{h}}\,. \tag{2.1.24}$$

The *operator-state correspondence* arises from the fact that the insertion of a local operator at the origin $z = 0$ creates a state defined on the asymptotic spatial slice at $x^0 = -\infty$. Concretely, the asymptotic in-state created by the field ϕ acting on the vacuum $|0\rangle$ is given by

$$|\phi_{\text{in}}\rangle := \lim_{x^0\to-\infty} \phi(\omega, \overline{\omega})\,|0\rangle = \phi_{0,0}\,|0\rangle\,. \tag{2.1.25}$$

For this expression to be regular, we need to demand

$$\phi_{m,n}\,|0\rangle = 0\,, \qquad m > -h,\; n > -\overline{h}\,. \tag{2.1.26}$$

The Hermitian conjugate of the in-state defines the out-state,

$$\langle\phi_{\text{out}}| := |\phi_{\text{in}}\rangle^{\dagger}\,. \tag{2.1.27}$$

Note that Hermitian conjugation reverses the Euclidean time direction and keeps the coordinate x^1 fixed, which implies $z \mapsto 1/\bar{z}$. On fields, Hermitian conjugation is defined as

$$\phi^{\dagger}(z, \bar{z}) = \bar{z}^{-2h} z^{-2\overline{h}} \phi(\frac{1}{\bar{z}}, \frac{1}{z})\,, \tag{2.1.28}$$

from which one finds for the modes

$$(\phi_{n,m})^{\dagger} = \phi_{-n,-m}\,. \tag{2.1.29}$$

In general quantum field theories, the correspondence between local operators and states is not bijective, because local fields are defined pointwise, whereas quantum states are projectively defined on entire spatial slices. What makes two-dimensional conformal field theories special is the presence of an infinite-dimensional conformal symmetry that allows one to map the infinite spatial slice at $x^0 = -\infty$ to the single point $z = 0$ on the complex plane.

The regularity of the vacuum under the action of the stress energy tensor in the limit $z \to 0$ imposes conditions on the action of the Virasoro generators L_n,

$$\begin{aligned} L_n \left|0\right\rangle &= 0, \quad n \geq -1, \\ \left\langle 0\right| L_n &= 0, \quad n \leq 1\,, \end{aligned} \tag{2.1.30}$$

i.e., the vacuum must be annihilated by the generators of the group of global conformal transformations L_{-1}, L_0, and L_1.

A *primary state* $\left|\phi\right\rangle$ created by a primary field $\phi(z)$ satisfies

$$\begin{aligned} L_0 \left|\phi\right\rangle &= h \left|\phi\right\rangle, \\ L_n \left|\phi\right\rangle &= 0, \quad n > 0\,. \end{aligned} \tag{2.1.31}$$

The descendants of the primary are generated by the action of the negative modes L_{-n} with $n > 0$:

$$L_0(L_{-n}\left|\phi\right\rangle) = (n+h)(L_{-n}\left|\phi\right\rangle)\,. \tag{2.1.32}$$

Thus, the Hilbert space of a two-dimensional conformal field theory naturally decomposes into a direct sum of highest weight representations (HWR) of the Virasoro algebra,

$$\mathcal{H} = \bigoplus_i V_{h_i}\,, \tag{2.1.33}$$

where each V_{h_i} is the Verma module built on the primary state of conformal weight h_i. The Verma module is explicitly spanned by descendant states of the form

$$L_{-k_1} L_{-k_2} \cdots L_{-k_j} \left|\phi\right\rangle, \quad k_i > 0\,. \tag{2.1.34}$$

For conformal field theories with central charge $c \geq 1$, it is a classic result (see, e.g., [99–101]) that there always exists an infinite tower of inequivalent irreducible unitary highest weight representations. This means that the sum in (2.1.33) is always over an infinite number of Verma modules for $c \geq 1$. Something interesting happens for $0 \leq c < 1$. In [101] it has been shown, using previous technical results from [102] and [103], that for a given central charge,

$$c = 1 - \frac{6}{m(m+1)}\,, \qquad m \geq 2\,, m \in \mathbb{N}\,, \tag{2.1.35}$$

there exists only a finite number of unitary irreducible highest weight representations of the Virasoro algebra with conformal weights given by

$$h_{r,s}(m) = \frac{\left((m+1)r - ms\right)^2 - 1}{4m(m+1)}\,, \qquad (1 \leq r < m\,,\, 1 \leq s < r)\,. \tag{2.1.36}$$

This is the discrete series of unitary *Virasoro minimal models*. In fact, all unitary irreducible highest weight representations of the Virasoro algebra with $c < 1$ are classified by the central charge (2.1.35) and the conformal weight (2.1.36). Some Virasoro minimal models provide well-known and exact conformal field theory descriptions of certain statistical mechanical systems at their second-order phase transition points. For example, the minimal model with parameter $m = 3$ corresponds precisely to the two-dimensional Ising model at criticality, which was described in the introduction, while the case $m = 5$ describes the critical point of the three-state Potts model. Another interesting example is the tricritical Ising model at $m = 4$. This model possesses, in fact, an $\mathcal{N} = 1$ supersymmetric extension of the Virasoro algebra, the so-called *superconformal algebra*. Supersymmetric extensions of the Virasoro algebra and their minimal models will play a central role in later sections.

Correlation Functions

The central observables to calculate in two-dimensional conformal field theory are the correlation functions of local quantum fields. Indeed, to "solve" a conformal field theory means to determine its spectrum of local fields and to compute all of their correlation functions consistent with the underlying symmetry algebra. We denote an n-point correlation function by

$$\left\langle\, \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \,\right\rangle . \tag{2.1.37}$$

As in any quantum field theory, the product of fields inside a correlation function is time-ordered (in the present case, on the cylinder). Under the conformal map from the cylinder to the complex plane, time ordering along the cylinder maps to *radial ordering* on the plane:

$$\mathcal{R}\big(\,\phi_1(z_1)\phi_2(z_2)\big) := \begin{cases} \phi_1(z_1)\,\phi_2(z_2) & |z_1| > |z_2|\,, \\ \phi_2(z_2)\,\phi_1(z_1) & |z_2| > |z_1|\,. \end{cases} \tag{2.1.38}$$

The $\mathrm{PSL}(2,\mathbb{C})$ invariance of the vacuum implies that all correlation functions of quasi-primary fields must be invariant as well. Let us now restrict for simplicity to chiral quasi-primary fields and show how the structure of correlation functions is constrained by invariance under $\mathrm{PSL}(2,\mathbb{C})$. For a one-point function,

$$\big\langle \phi(z) \big\rangle = G(z)\,, \tag{2.1.39}$$

we can use a translation $f(z) = z + a$ to find

$$\big\langle \phi'(z) \big\rangle = \big\langle \phi(z+a) \big\rangle = G(z+a)\,. \tag{2.1.40}$$

Invariance under translations then implies that $G(z) = C_\phi$ for some constant C_ϕ. Using invariance under dilations $f(z) = \lambda z$ further restricts to $C_\phi = 0$ for $h \neq 0$. Hence, the only non-vanishing one-point function involves the vacuum primary field.

Next, consider the two-point function and use invariance under translations, which fixes the functional form to be

$$\big\langle \phi_1(z_1)\phi_2(z_2) \big\rangle = G(z_1 - z_2)\,. \tag{2.1.41}$$

Under dilations $f(z) = \lambda z$, the two-point function transforms as

$$\big\langle \phi_1(z_1)\phi_2(z_2) \big\rangle \to \lambda^{h_1}\lambda^{h_2} G(\lambda(z_1 - z_2))\,, \tag{2.1.42}$$

which further restricts to

$$G(z_1 - z_2) = \frac{c_{12}}{(z_1 - z_2)^{h_1+h_2}}\,, \tag{2.1.43}$$

for some constant c_{12}. Under a special conformal transformation $f(z) = -\frac{1}{z}$ we find

$$\big\langle \phi_1(z_1)\phi_2(z_2) \big\rangle \to z_1^{-h_1+h_2} z_2^{-h_2+h_1} \frac{c_{12}}{(z_1 - z_2)^{h_1+h_2}}\,, \tag{2.1.44}$$

from which we conclude that for non-trivial two-point functions we must have $h_1 \overset{!}{=} h_2$.

For the three-point function, we again use invariance under the three global conformal transformations above and find:

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) \rangle = \frac{C_{123}}{(z_1 - z_2)^{h_1+h_2-h_3}(z_2 - z_3)^{h_2+h_3-h_1}(z_1 - z_3)^{h_1+h_3-h_2}}\,. \tag{2.1.45}$$

The constants C_{123} are a priori not fixed by conformal symmetry and have to be determined by other means.

Operator Product Expansion (OPE) and Conformal Ward Identities

The *operator product expansion* (OPE) is a very powerful concept in conformal field theories. It allows one to systematically reduce any n-point correlation function to an $(n-1)$-point function by successively expanding the product of two local fields. Through repeated application, all higher-point correlators can, in principle, be expressed in terms of two-point and three-point functions. Consequently, solving a two-dimensional conformal field theory amounts to determining its full spectrum of primary fields, together with the set of three-point structure constants C_{ijk} that completely specify the OPE algebra.

The OPE between two (quasi-)primary fields is an asymptotic series with radius of convergence determined by the distance to the nearest other field insertion inside a correlation function. It is defined as

$$\phi_1(z,\bar{z})\phi_2(0,0) = \sum_p \sum_{\{k,\bar{k}\}} C_{12}^{p\,\{k,\bar{k}\}} z^{h_p-h_1-h_2+K}\,\bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2+\bar{K}} \phi_p^{\{k,\bar{k}\}}(0,0)\,, \tag{2.1.46}$$

where the sum runs over all primary fields ϕ_p and their descendants labelled by the multi-indices $\{k\}$ and $\{\bar{k}\}$, with $K = \sum_i k_i$ and $\bar{K} = \sum_i \bar{k}_i$. The coefficients $C_{12}^{p\,\{k,\bar{k}\}}$ factorise according to

$$C_{12}^{p\,\{k,\bar{k}\}} = C_{12}^{p}\beta_{12}^{p\{k\}}\bar{\beta}_{12}^{p\{\bar{k}\}}\,. \tag{2.1.47}$$

Here, C_{12}^p is the OPE coefficient for the primary fields, and the β-coefficients encode the contributions of the descendant fields within the Verma module of ϕ_p. The relation between the three-point coupling constant C_{ijk} appearing in correlation functions and the OPE coefficients is

$$C_{ijk} = C^i_{jk}\, c_{ii}, \tag{2.1.48}$$

where c_{ii} is a normalisation factor associated with the two-point function of the primary ϕ_i. The β-coefficients that describe the coupling of descendant fields can be calculated by acting with both sides of equation (2.1.46) on the vacuum and then applying on both sides the operator L_n. Using the commutation relations of the Virasoro algebra, the resulting relation determines all β-coefficients recursively in terms of the conformal weights of the primary field.

To determine the OPE of the energy-momentum tensor $T(z)$, we recall its mode expansion

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \tag{2.1.49}$$

$$L_n = \frac{1}{2\pi \mathrm{i}} \oint_{C_0} dz z^{n+1} T(z)\,, \tag{2.1.50}$$

with C_0 denoting a contour around $z = 0$. Using the known Virasoro commutation relations (2.1.18), one finds the OPE of $T(z)T(z')$ by plugging the above expansion into the commutator $[L_m, L_n]$,

$$T(z)T(z') = \frac{c/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial_{z'} T(z')}{z-z'} + \text{regular terms}\ . \tag{2.1.51}$$

Thus, the singular part of the OPE of the energy momentum tensor $T(z)$ encodes the commutation relations of the modes L_m. This can be seen by explicitly deriving the Virasoro commutation relations from the given OPE (2.1.51)

$$\begin{aligned}[L_m, L_n] &= \oint_{C_0} \frac{dz'}{2\pi \mathrm{i}} (z')^{n+1} \oint_{C_{z'}} \frac{dz}{2\pi \mathrm{i}} z^{m+1} \mathcal{R}\left(T(z)T(z')\right) \\ &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}\,.\end{aligned} \tag{2.1.52}$$

Another important consequence of the Virasoro symmetry is that correlation functions involving descendant fields can always be expressed in terms of correlation functions involving only the associated primaries. Consider a correlation function of the form

$$\langle \phi^{-n}(z) X \rangle\ , \tag{2.1.53}$$

where $\phi^{-n}(z)$ is a descendant field corresponding to the state $L_{-n}\,|h\rangle$, for a primary state $|h\rangle$, and X is a chain of primary fields. One can write this descendant as a contour integral involving the stress-energy tensor $T(z)$,

$$\phi^{-n}(z) = \frac{1}{2\pi i} \oint_z d\tilde{z} \frac{1}{(\tilde{z}-z)^{n-1}} T(\tilde{z})\phi(z)\,, \tag{2.1.54}$$

so that we can replace in the above correlation function

$$\langle \phi^{-n}(z)X\rangle = \frac{1}{2\pi i}\oint_z d\tilde{z}\frac{1}{(\tilde{z}-z)^{n-1}}\langle T(\tilde{z})\phi(z)X\rangle \ . \tag{2.1.55}$$

Evaluating this contour integral amounts to picking up the singular terms from the OPE of $T(\tilde{z})$ with $\phi(z)$,

$$T(\tilde{z})\,\phi(z) \sim \frac{h}{(\tilde{z}-z)^2}\,\phi(z) + \frac{1}{\tilde{z}-z}\,\partial_z\phi(z) + \dots . \tag{2.1.56}$$

As a result, the descendant correlator can be written as a differential operator acting on the primary correlator,

$$\mathcal{L}_{-n}\,\langle\phi(z)X\rangle\ , \qquad \mathcal{L}_{-n} = \sum_i \left(\frac{(n-1)h_i}{(z_i-z)^n} - \frac{1}{(z_i-z)^{n-1}}\partial_{z_i}\right)\ , \tag{2.1.57}$$

with positions z_i of the fields in the chain X. Using a combination of the OPE and contour manipulations as described above, one can systematically relate any n-point correlation function involving descendant fields to expressions involving only two-point and three-point functions of primary fields. As a result, solving a two-dimensional conformal field theory essentially means specifying its full spectrum of irreducible representations of the chiral symmetry algebra, together with the OPE-algebra of its primary fields, which is fully determined by the three-point structure constants C_{ijk}. Once these data are known, all higher-point functions and correlation functions involving descendant fields can be calculated explicitly. Note that in the derivation above, we heavily made use of the so-called *conformal Ward-Takahashi identities*. In complex coordinates, the holomorphic and anti-holomorphic components of the stress-energy tensor are denoted by

$$T_{zz}(z) = T(z), \quad T_{\bar{z}\bar{z}}(\bar{z}) = \overline{T}(\bar{z}). \tag{2.1.58}$$

The Noether current associated with a conformal transformation

$$z \mapsto z+\epsilon(z), \quad \bar{z} \mapsto \bar{z}+\bar{\epsilon}(\bar{z})\,, \tag{2.1.59}$$

is given by

$$J_z(z) = \epsilon(z)T(z), \quad J_{\bar{z}}(\bar{z}) = \bar{\epsilon}(\bar{z})\,\overline{T}(\bar{z}). \tag{2.1.60}$$

An application of the general conservation law for the Noether current yields the conformal Ward–Takahashi identity in its contour-integral form,

$$\delta_{\epsilon,\bar{\epsilon}}\,\mathcal{O}(w,\overline{w}) = -\frac{1}{2\pi i}\oint_{C_w} dz\,\big(\epsilon(z)T(z) + \bar{\epsilon}(\bar{z})\,\overline{T}(\bar{z})\big)\,\mathcal{O}(w,\overline{w})\,, \tag{2.1.61}$$

where C_w is a counter-clockwise contour in z and $\bar{z}$ encircling w. This implies that the effect of a conformal transformation generated by $T(z)$ on a local field $\mathcal{O}(w,\overline{w})$ is fully encoded in the singular terms of their OPE. Specifically, if $\mathcal{O}(w,\overline{w})$ is a primary field of conformal weights $(h,\overline{h})$, the holomorphic OPE directly follows by evaluating (2.1.61), which yields

$$T(z)\,\mathcal{O}(w,\overline{w}) \ = \ \frac{h\,\mathcal{O}(w,\overline{w})}{(z-w)^2} \ + \ \frac{\partial_w\mathcal{O}(w,\overline{w})}{z-w} \ + \ \text{regular terms}\,. \tag{2.1.62}$$

Conformal Blocks and Conformal Bootstrap

Let us assume that the field content of a given two-dimensional conformal field theory is known. We have seen that "solving" a two-dimensional conformal field theory then boils down to determining the three-point structure constants C_{ijk}. Arguably, the most important objects in a two-dimensional conformal field theory that constrain the possible three-point structure constants C_{ijk} are so-called *conformal blocks*. To stress their importance, let us consider a generic four-point function between primary fields ϕ_i,

$$\langle\phi_1(z,\bar{z})\phi_2(z,\bar{z})\phi_3(z,\bar{z})\phi_4(z,\bar{z})\rangle \ . \tag{2.1.63}$$

As we have seen in the previous section 2.1, conformal invariance strongly restricts the coordinate dependence of such a correlator. In particular, global conformal symmetry implies that the four-point function depends only on the cross-ratio,

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \tag{2.1.64}$$

and its anti-holomorphic counterpart $\overline{x}$. Using conformal invariance, one can map three of the insertion points to fixed positions by applying suitable Möbius transformations. A convenient standard choice is

$$z_1 \to \infty, \quad z_2 \to 1, \quad z_4 \to 0, \tag{2.1.65}$$

leaving the coordinate $z_3 = x$ as the only nontrivial position. This yields the normalised four-point function,

$$\lim_{z_1,\bar{z}_1\to\infty} z_1^{2h_1}\bar{z}_1^{2\overline{h}_1} \langle\phi_1(z_1,\bar{z}_1)\phi_2(1,1)\phi_3(x,\overline{x})\phi_4(0,0)\rangle = G_{34}^{21}(x,\overline{x})\,, \tag{2.1.66}$$

with

$$G_{34}^{21}(x,\overline{x}) = \langle h_1,\overline{h}_1|\,\phi_2(1,1)\phi_3(x,\overline{x})\,|h_4,\overline{h}_4\rangle \ . \tag{2.1.67}$$

To evaluate this explicitly, one inserts the OPE between ϕ_3 and ϕ_4, which generates an expansion over intermediate conformal families labeled by p. One finds [95],

$$\begin{aligned} G_{34}^{21}(x,\overline{x}) &= \sum_p C_{34}^p C_{12}^p A_{34}^{21}(p\,|\,x,\overline{x}) \\ A_{34}^{21}(p\,|\,x,\overline{x}) &= (C_{12}^p)^{-1} x^{h_p-h_3-h_4}\overline{x}^{\overline{h}_p-\overline{h}_3-\overline{h}_4} \langle h_1,\overline{h}_1|\,\phi_2(1,1)\psi_p(x,\overline{x}\,|\,0,0)\,|0\rangle \\ \psi_p(x,\overline{x}\,|\,0,0) &= \sum_{\{k,\overline{k}\}} \beta_{34}^{p\{k\}}\overline{\beta}_{34}^{p\{\overline{k}\}} x^K\overline{x}^{\overline{K}}\phi_p^{\{k,\overline{k}\}}(0,0)\,, \qquad K = \sum_i k_i\,. \end{aligned} \tag{2.1.68}$$

with the holomorphic factorisation of $A_{34}^{21}(p\,|\,x,\overline{x})$ given by

$$\begin{aligned} A_{34}^{21}(p\,|\,x,\overline{x}) &= \mathcal{F}_{34}^{21}(p\,|\,x)\overline{\mathcal{F}}_{34}^{21}(p\,|\,\overline{x}) \\ \mathcal{F}_{34}^{21}(p\,|\,x) &= x^{h_p-h_3-h_4}\sum_{\{k\}}\beta_{34}^{p\{k\}}x^K\frac{\langle h_1|\,\phi_2(1)L_{-k_1}\dots L_{-k_N}\,|h_p\rangle}{\langle h_1|\,\phi_2(1)\,|h_p\rangle}\,. \end{aligned} \tag{2.1.69}$$

The holomorphic function $\mathcal{F}^{21}_{34}(p\,|\,x)$ is called *conformal block* and is clearly independent of the undetermined structure constants. This is because the three-point correlation function in the numerator and denominator differ only by the application of a differential operator of the form (2.1.57), so that the structure constants cancel out. Thus, the conformal block depends only on the central charge and the conformal weights of the fields involved, and is the piece of the four-point function determined solely by conformal symmetry. In practice, it is notoriously difficult to calculate them from definition (2.1.69). Nevertheless, once these conformal blocks are determined, one must ensure consistency under different possible OPE orderings of the fields within the four-point function. For a four-point function, one can channel the OPE expansion in different ways, e.g., $(1,2)(3,4)$ or $(1,3)(2,4)$ or $(1,4)(2,3)$, each corresponding to a different choice of intermediate states. Since the physical four-point function must be single-valued and independent of how we choose to fuse operators, these expansions must agree. This self-consistency is called *crossing symmetry*, which implies powerful constraints on the structure constants, e.g.,

$$\sum_p C^p_{21} C^p_{34} \mathcal{F}^{21}_{34}(p\,|\,x) \overline{\mathcal{F}}^{21}_{34}(p\,|\,\overline{x}) = \sum_q C^q_{41} C^q_{32} \mathcal{F}^{41}_{32}(q\,|\,1-x) \overline{\mathcal{F}}^{41}_{32}(q\,|\,1-\overline{x})\,. \tag{2.1.70}$$

We remark that this system of crossing relations is the basis of the so-called *bootstrap hypothesis* for “solving” a two-dimensional conformal field theory: the consistency conditions arising from crossing symmetry, together with the constraints imposed by the conformal symmetry itself, are so strong that they fully determine the theory’s structure constants and spectrum in favorable cases. For a rough estimate, suppose that the theory contains N distinct conformal families. Then, the number of independent three-point structure constants is of order $\mathcal{O}(N^3)$, while crossing symmetry for four-point functions imposes $\mathcal{O}(N^4)$ constraints on them. This program has been successfully carried out for the Virasoro minimal models, which contain only a finite number of Virasoro conformal families. By contrast, for unitary theories with central charge $c \geq 1$, the number of Virasoro conformal families is infinite, and a complete solution purely by the conformal bootstrap becomes impractical in most situations. Exceptions are theories with a free Lagrangian as the free boson, free fermion, and orbifolds thereof. For most theories with an infinite number of conformal families, solvability relies on extra symmetries (e.g., affine symmetry in the free boson theory or Wess-Zumino-Witten models) or the existence of so-called degenerate representations that are associated with null-states (zero norm states), which decouple from the theory and thereby constrain non-trivial correlation functions through differential equations. The latter, for instance, is responsible for the exact solvability of Virasoro minimal models.

2.2 Rational Conformal Field Theories

Definition

We have argued that the bootstrap hypothesis provides a systematic framework to algebraically and exactly solve a two-dimensional conformal field theory; that is, to determine all structure constants, or equivalently, the OPE coefficients of the primary fields. Once these are known, the Ward identities can then be used to determine the correlation functions of any n-point function involving the remaining descendant states as well.

The bootstrap approach is particularly powerful when applied to theories whose Hilbert space decomposes into a finite sum of conformal families. A standard result in the representation theory of the Virasoro algebra, as reviewed above (2.1.35), shows that the theories of fixed central charge c whose spectrum consists of finitely many unitary irreducible Virasoro Verma modules are classified by the Virasoro minimal models with $0 \leq c < 1$.

By contrast, for central charge $c \geq 1$, there exists no conformal field theory whose spectrum remains finite when organised solely by the Virasoro algebra. However, for any central charge, there may exist an *extended chiral algebra* $\mathcal{A}$ that contains the Virasoro algebra as a subalgebra, such that the Hilbert space decomposes into a finite direct sum of irreducible modules of $\mathcal{A}$:

$$\mathcal{H} = \bigoplus_{i,j} M_{i,j}\, V_{h_i} \otimes \overline{V_{h_j}}, \tag{2.2.1}$$

where V_{h_i} and $\overline{V_{h_j}}$ denote the modules built from irreducible highest weight representations of the extended chiral algebras $\mathcal{A}$ and $\overline{\mathcal{A}}$. In particular, the infinite number of Virasoro primaries are packaged into a finite number of extended modules, such that these Virasoro primaries become descendant states with respect to $\mathcal{A}$ and $\overline{\mathcal{A}}$.

There exist various definitions in the literature for what is meant by a *rational conformal field theory.* Common to most definitions, however, are the following properties (see [104] and references therein):

- The theory possesses only a finite number of irreducible highest weight representations.
- The one-loop partition function is expressible as a finite sum of the form

$$Z = \sum_{i,j} \mathcal{M}_{i,j}\, \chi_i(q)\, \overline{\chi_j(q)}\,, \tag{2.2.2}$$

  where the $\mathcal{A}$-characters $\chi_i(q)$ converge for $|q| = |e^{2\pi \mathrm{i} \tau}| < 1$ and the multiplicity matrix $\mathcal{M}_{i,j}$ has non-negative integer entries.
- The fusion coefficients $\mathcal{N}_{ij}^{k}$ describing the fusion of three irreducible highest weight representations are all finite.

Fusion Rules

One important feature of rational conformal field theories is that they have *finite* fusion coefficients. While the notion of fusion rules also makes sense in non-rational conformal field theories, it is a generic feature in those cases that the fusion coefficients become infinite.

Recall that the operator product expansion (OPE) of two primary fields is an expansion in terms of all primary fields and their descendants. In particular, the OPE coefficient C_{ij}^{k} encodes whether the three-point correlation function among the primaries ϕ_i, ϕ_j, and ϕ_k is non-vanishing.

To capture this information, if a conformal family appears in the OPE of two primaries (or more generally conformal or extended families) leads to the concept of *Fusion rules* denoted

$$[\phi_i] \times [\phi_j] = \sum_k \mathcal{N}_{ij}^{k} \, [\phi_k] \,, \tag{2.2.3}$$

which formalises the concept of non-vanishing three-point functions between three Verma modules $[\phi_i]$. We say that a fusion of two Verma modules $[\phi_i]$, $[\phi_j]$ onto a third $[\phi_k]$ is possible if there is a non-vanishing correlation function between fields of the three Verma modules. One can define an algebra structure whose identity element is the identity field, whose generators are the primaries ϕ_i, and whose multiplication law is given by the fusion product $\times$. By the commutativity and associativity of the OPE algebra, the same properties hold for the fusion algebra.

Denoting by

$$(\mathcal{N}_i)_{jk} \,, \tag{2.2.4}$$

the $N \times N$ matrix, with N being the number of (extended) conformal families, whose entries are given by $\mathcal{N}_{ij}^{k}$, commutativity and associativity imply the matrix relations

$$\mathcal{N}_i \, \mathcal{N}_j = \mathcal{N}_j \, \mathcal{N}_i \,, \tag{2.2.5}$$

and

$$\mathcal{N}_i \, \mathcal{N}_j = \sum_k \mathcal{N}_{ij}^{k} \, \mathcal{N}_k \,. \tag{2.2.6}$$

This shows that the fusion matrices furnish a representation of the fusion algebra.

It is worth noting that the fusion coefficients $\mathcal{N}_{ij}^{k}$ can be any natural number greater than or equal to one, as they count the number of independent couplings between three Verma modules, modulo singular vector relations and Ward identities. Such multiplicities naturally arise in rational conformal field theories with affine Lie algebras as extended chiral symmetry. In this context, they can be understood as coming from multiplicities in the direct sum decomposition of tensor products of finite Lie algebra highest weight representations. In fact, the fusion coefficients can be seen as a truncation of this tensor product expansion. The truncation arises from additional constraints from singular vectors in degenerate highest weight representations [53]. The coefficients $\mathcal{N}_{ij}^k$ in theories with affine symmetry can be computed explicitly by the Kac-Walton formula [105–108].

Whenever a rational conformal field theory has fusion coefficients $\mathcal{N}_{ij}^k > 1$ for the extended chiral algebra $\mathcal{A}$, it means that there are multiple independent couplings between three (extended) conformal families. This is in sharp contrast to non-rational conformal field theories, where Ward identities are always sufficient to reduce correlation functions involving descendant fields to correlation functions only involving Virasoro primary fields (see eq. (2.1.57)). If $\mathcal{A}$ is an affine Lie algebra, affine Ward identities can be used to solve the theory exactly, i.e., determine all independent couplings between three (extended) conformal families [53, 109, 110]. In this case the structure constants C_{ij}^k appearing in the OPE are replaced by $C_{ij}^{k,(1)}\eta_{ij}^{k,(1)} + C_{ij}^{k,(2)}\eta_{ij}^{k,(2)} + \dots$, where η is an invariant tensor of the zero-mode subalgebra of $\mathcal{A}$ [110, 111].

In cases where the conformal field theory possesses a more general W-algebra, reducing exact solvability to the determination of primary correlation functions, becomes more intricate. This is because, in general, Ward identities alone are insufficient to express all descendant correlation functions in terms of those of the corresponding primaries. Nonetheless, in certain cases — such as the three-state Potts minimal model that possesses an extended W_3-*symmetry algebra* — a finite subset of correlation functions suffices, from which all others can be systematically derived. For further details, we refer to [104, 112–117].

Partition Function

So far, we have discussed two-dimensional conformal field theories defined on the complex plane, or equivalently, the Riemann sphere. This setting naturally allows for a factorisation of many quantities of interest into holomorphic and anti-holomorphic parts. When a theory possesses an extended chiral algebra $\mathcal{A}$ (and its anti-holomorphic counterpart $\overline{\mathcal{A}}$), it is necessary to understand how the irreducible highest weight representations of $\mathcal{A}$ and $\overline{\mathcal{A}}$, denoted by $\mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$, combine to form the full spectrum of (extended) primary fields $\phi_i(z, \bar{z})$.

A standard method to analyse the spectrum in any quantum field theory is to study loop diagrams. In the framework of conformal field theory, this corresponds to considering the theory on compact Riemann surfaces of genus g. The simplest non-trivial example is the torus with $g = 1$, whose zero-point function — the *partition function* — encodes the spectrum of the theory. Since the torus enjoys modular invariance, the partition function must be invariant under modular transformations. This imposes a fundamental constraint on the admissible Hilbert spaces of two-dimensional conformal field theories.

Recall that the theory defined on the complex plane can be obtained from an infinite cylinder with complex coordinate $w = x^0 + \mathrm{i}x^1$, via the conformal map $z = e^w$. To obtain the torus, one cuts out a finite segment and identifies the boundaries (with possible twist). Thereby, we obtain both periodic space and time coordinates. As depicted in figure 2.1, the torus can be represented as a two-dimensional lattice in the complex plane spanned by the periods 1 and τ, where $\tau = \tau_1 + \mathrm{i}\tau_2 \in \mathbb{H}$ is the complex structure parameter.

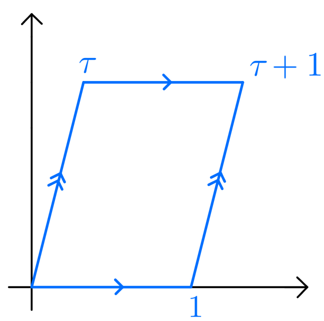

Figure 2.1: **Fundamental lattice of the torus.** The complex torus can be constructed as the quotient of the complex plane by the lattice Λ generated by the basis vectors 1 and $\tau = \tau_1 + \mathrm{i}\tau_2$, which is the complex structure parameter.

The lattice is invariant under the action of the modular group

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})/\mathbb{Z}_2\,. \tag{2.2.7}$$

The generators of the modular group are the transformations

$$\mathrm{T}: \quad \tau \mapsto \tau + 1\,, \quad \text{and} \quad \mathrm{S}: \quad \tau \mapsto -\frac{1}{\tau}\,, \tag{2.2.8}$$

which together generate all modular transformations. In matrix notation, the modular group is generated by the two standard 2×2 matrices S and T given by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{2.2.9}$$

These generators satisfy the well-known relations $S^4 = \mathbb{I}_{2\times 2}$ and $(ST)^3 = S^2$. A rational conformal field theory provides a unitary representation

$$\rho : \mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{U}(N) , \tag{2.2.10}$$

with $N := |\mathrm{HWR}_{\mathcal{H}}(\mathcal{A})|$, and the images $\rho(S) \equiv \mathrm{S}$ and $\rho(T) \equiv \mathrm{T}$ are referred to as the modular S- and T-matrices of the theory.

Following the axiomatic framework of ref. [57], the S-matrix in the image of ρ is symmetric and satisfies

$$\mathrm{S}^2 = \mathrm{C} , \tag{2.2.11}$$

where C denotes the charge conjugation matrix that interchanges each primary field with its conjugate representation[2]. We will denote the corresponding operator that acts on representations by

$$\mathcal{C} : \mathrm{HWR}_{\mathcal{H}}(\mathcal{A}) \to \mathrm{HWR}_{\mathcal{H}}(\mathcal{A}),\ i \mapsto \mathcal{C}(i) . \tag{2.2.12}$$

Looking at figure 2.1, we see that a closed time loop in τ_2 direction also involves a change in space direction τ_1. Accordingly, the partition function is defined as

$$Z(\tau,\overline{\tau}) = \mathrm{Tr}_{\mathcal{H}} \left(e^{-2\pi\tau_2 H} e^{2\pi\tau_1 P} \right) , \tag{2.2.13}$$

where H is the Hamiltonian generating time translations and P is the momentum operator generating spatial translations. Using the relations

$$\begin{aligned} (H)_{\mathrm{cyl}} &= (L_0)_{\mathrm{cyl}} + (\overline{L}_0)_{\mathrm{cyl}} \\ (P)_{\mathrm{cyl}} &= \mathrm{i}\big((L_0)_{\mathrm{cyl}} - (\overline{L}_0)_{\mathrm{cyl}}\big) , \end{aligned} \tag{2.2.14}$$

and substituting $(L_0)_{\mathrm{cyl}} = L_0 - \frac{c}{24}$, we arrive at

$$Z(q,\overline{q}) = \mathrm{Tr}_{\mathcal{H}} \left(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} \right) , \qquad q = e^{2\pi\mathrm{i}\tau} . \tag{2.2.15}$$

In rational conformal field theories, the partition function can be decomposed into characters $\chi_i(q)$ associated with the irreducible highest weight representations of the extended chiral algebra $\mathcal{A}$:

$$\chi_i(q) := \mathrm{Tr}_{\mathcal{H}_i}\big(q^{L_0 - \frac{c}{24}}\big). \tag{2.2.16}$$

[2] The geometric action of $S^2 = -\mathbb{I}_{2\times 2}$ reverses the orientation of all homology one-cycles of the torus T^2. The resulting unitary action $\mathrm{S}^2 = \rho(-\mathbb{I}_{2\times 2})$ corresponds to the charge conjugation operator C in the conformal field theory, which permutes each representation with its conjugate under the chiral symmetry algebra. In simple cases such as affine Lie algebra extensions, this may amount to a sign change for the conserved spin-1 currents; more generally, it implements the appropriate automorphism for the extended symmetry algebra.

Consequently, the partition function takes the form

$$Z(q,\overline{q}) = \sum_{i,j} \mathcal{M}_{i,j}\, \chi_i(q)\, \overline{\chi_j(q)}\,, \tag{2.2.17}$$

which is a finite sum. The characters transform under the modular group as

$$\chi_i\,(\tau+1) = \sum_{j\in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \mathrm{T}_{ij}\, \chi_j(\tau), \quad \chi_i\left(-\frac{1}{\tau}\right) = \sum_{j\in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \mathrm{S}_{ij}\, \chi_j(\tau)\ , \tag{2.2.18}$$

with the diagonal matrix $\mathrm{T}_{ij} = \delta_{ij} e^{2\pi \mathrm{i}(h_i - c/24)}$ and a unitary symmetric matrix S, which together must satisfy the relation

$$\mathrm{S}^2 = (\mathrm{ST})^3 = \mathrm{C}\,. \tag{2.2.19}$$

Using the symmetry and unitarity of the S-matrix and the above relation, it also follows that

$$\mathrm{S}^*\mathrm{S} = \mathrm{Id}_{N\times N}\,, \qquad \mathrm{S}_{i\mathcal{C}(j)} = \mathrm{S}_{ik}\mathrm{C}_{kj} = \mathrm{S}^*_{ij}\,. \tag{2.2.20}$$

The requirement of modular invariance of the partition function translates to

$$\mathrm{S}^\dagger M \mathrm{S} = M\,, \qquad \mathrm{T}^\dagger M \mathrm{T} = M\,, \tag{2.2.21}$$

hence M must commute with both S and T. Clearly, the choice $M = \mathrm{Id}$ (i.e., the *diagonal modular invariant*) always yields a modular invariant partition function for any extended chiral algebra $\mathcal{A}$. Similarly, using (2.2.19), one finds that $M = \mathrm{C}$ is also a solution to (2.2.21) (the *charge conjugate modular invariant*).

One of the most profound results in the study of two-dimensional conformal field theories is the Verlinde formula [118], proven in [119, 120], which connects the modular S-matrix — a property of the chiral algebra, such as an affine Lie algebra or a W-algebra — to the fusion coefficients,

$$\mathcal{N}_{ij}^{k} = \sum_{m\in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \mathrm{S}_{im}\, \frac{\mathrm{S}_{jm}}{\mathrm{S}_{0m}}\, \mathrm{S}^*_{km}\,. \tag{2.2.22}$$

This remarkable relation demonstrates how modular properties encode the algebraic structure of fusion in rational conformal field theories.

Using the matrix notation (2.2.4), and recalling that the modular S-matrix S is unitary and symmetric,

$$\mathrm{S}^\dagger = \mathrm{S}^* = \mathrm{S}^{-1}\,, \tag{2.2.23}$$

the Verlinde formula admits a matrix representation, which will be useful for later reference:

$$\mathcal{N}_a = \mathrm{S}\,\Lambda_a\,\mathrm{S}^{-1} \quad \text{with} \quad \Lambda_a := \mathrm{diag}\big(\lambda_{a,0},\, \lambda_{a,i_1},\, \ldots,\, \lambda_{a,i_{N-1}}\big), \quad \lambda_{a,i} := \frac{\mathrm{S}_{ai}}{\mathrm{S}_{0i}}\,. \tag{2.2.24}$$

Here, the labels $0, i_1, \ldots, i_{N-1}$ run over the N inequivalent irreducible highest weight representations $\mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$ of the rational conformal field theory.

Let us remark that, although the genus-one partition function determines the spectrum of the theory, it contains no information about the dynamics encoded in the OPE algebra. Moreover, distinct conformal field theories can share the same genus-one partition function while differing in their underlying structure. A prominent example is provided by the $c = 16$ theories associated with the symmetry groups $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$. These theories have identical torus partition functions, yet they are physically distinct due to their differing Lie symmetries, and hence, distinct correlation functions [121].

In string theory in particular, one is interested not only in genus-one partition functions but also in correlation functions and partition functions on higher-genus worldsheets. In [72, 120, 122], it was argued that modular invariance at genus one, together with the consistency of four-point functions under crossing symmetry at genus zero, ensures modular invariance at all genera. In particular, all correlation functions on higher-genus surfaces can be reduced to expressions involving only correlation functions on the plane; see, for example, [123] and references therein.

For more details on the foundational framework of conformal field theory on higher-genus surfaces, we refer to [72, 120, 122, 124–127]. For applications in string theory, see [128–131], and for a modern perspective, consult [132, 133].

Simple Current Extension

A convenient and powerful method to construct new consistent rational conformal field theories from a known one is the *simple current extension* [56, 134]. The idea is that if a rational conformal field theory admits a special type of primary field — a so-called *simple current* — one can systematically build new modular invariant partition functions by extending the original chiral symmetry algebra by this simple current. This process effectively reorganises the original spectrum into orbits under the action of the simple current, projects onto sectors with integer monodromy charge, and yields consistent modular invariants. Simple current techniques thus provide a systematic way to generate new theories, with enlarged symmetry algebra, starting from a known rational conformal field theory. An important example of such a construction is a so-called *Gepner model*, which we will introduce in section 4.4.

In the following, we will use the notation of [96]. Suppose we have a rational conformal field theory containing a primary field with conformal family $[J_a]$ that acts on other primary fields via the fusion product as

$$[J_a] \times [\phi_i] = \left[\phi_{J_a(i)}\right] . \tag{2.2.25}$$

Such a field J_a is called a *simple current* if its conformal weight is integral and its fusion with any other primary produces exactly one primary field. Since the conformal field theory has a finite number of primaries, it follows that

$$\exists\, N_a \quad \text{such that} \quad J_a^{N_a} = 1\,, \tag{2.2.26}$$

where N_a is called the *length* of the simple current. The set $\{J_a, J_a^2, \ldots, J_a^{N_a}\}$ forms a finite abelian group isomorphic to $\mathbb{Z}_{N_a}$ within the fusion algebra.

A simple current J_a organises the set of primaries into orbits under its action. The length of each orbit is $N_a^i = N_a/p$ for some divisor p of N_a. The orbit of a primary ϕ_i then reads

$$\{\phi_i, J_a \times \phi_i, J_a^2 \times \phi_i, \ldots, J_a^{N_a^i - 1} \times \phi_i\}\,. \tag{2.2.27}$$

The operator product expansion (OPE) between the simple current J_a and any primary field ϕ_i takes the form

$$J_a(z)\,\phi_i(w) \sim \frac{\phi_{J_a(i)}(w)}{(z-w)^{Q_i^{(a)}}} + \text{descendants}\,, \tag{2.2.28}$$

where $Q_i^{(a)}$ is the *monodromy charge*. Upon analytic continuation $z - w \mapsto e^{2\pi \mathrm{i}}(z-w)$, the OPE picks up a phase factor $e^{-2\pi \mathrm{i} Q_i^{(a)}}$. The monodromy charge has the general form

$$Q_i^{(a)} = \frac{t_a^i}{N_a} = h(\phi_i) + h(J_a) - h(J_a\phi_i) \mod 1\,, \quad t_a^i \in \mathbb{Z}\,, \tag{2.2.29}$$

and one can show that all states within a given orbit of J_a share the same monodromy charge.

An important relation is how the S-matrix of the original chiral algebra transforms under the action of a simple current J_a [57, 134],

$$\mathrm{S}_{iJ_a(j)} = \mathrm{e}^{2\pi\mathrm{i}Q_i^{(a)}}\mathrm{S}_{ij} \quad \text{for any } i,j \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A}) \; . \tag{2.2.30}$$

A modular invariant partition function can be constructed by projecting onto states with integer monodromy charge with respect to the simple current J_a. The partition function of the extended theory reads

$$Z_J(\tau,\overline{\tau}) = \frac{1}{C} \sum_{\substack{(\alpha,i) \\ Q(\phi_i)\in\mathbb{Z}}} \sum_{\beta=0}^{N_a-1} \chi_{(\alpha+\beta,\,i)}(\tau)\,\overline{\chi_{(\alpha,\,i)}(\tau)}\,, \tag{2.2.31}$$

where (α, i) denotes the representation corresponding to the primary $J_a^\alpha\phi_i$, and the sum is taken over all primaries ϕ_i with integer monodromy charge as well as all orbit elements $J_a^\alpha\phi_i$ with $\alpha \in \{0, 1, \ldots, N_a^i - 1\}$. The normalisation constant C is chosen in such a way that the vacuum appears with multiplicity one in the partition function. A direct calculation shows that this partition function is indeed modular invariant [56, 96]. Note that in (2.2.31) the sum over β always runs over the maximal length N_a. This means that shorter orbits appear with multiplicity N_a/N_a^i in the partition function. In the absence of short orbits, or *fixed points*, the S-matrix of the extended theory is given by

$$(\mathrm{S}_{\mathrm{ext}})_{[\lambda],[\mu]} = N_a\,\mathrm{S}_{\lambda\mu} \; . \tag{2.2.32}$$

If we extend with respect to a collection of simple currents, N_a is replaced by the order of the simple current group.

Example: Characters, Modular S-Matrix, Fusion Rules and Simple Current Extension for $\widehat{\mathfrak{su}}(2)_k$.

Consider an extended chiral algebra $\mathcal{A}$ given by $\widehat{\mathfrak{su}}(2)_k$. The characters of the irreducible, integral highest weight representations of $\widehat{\mathfrak{su}}(2)_k$ can be derived using the Weyl–Kac character formula [95, 108],

$$\chi_l^{(k)}(\tau,z) = \frac{\Theta_{l+1,k+2}(\tau,z) - \Theta_{-l-1,k+2}(\tau,z)}{\Theta_{1,2}(\tau,z) - \Theta_{-1,2}(\tau,z)}, \quad 0 \le l \le k\,. \tag{2.2.33}$$

The generalised theta functions $\Theta_{l,k}(\tau, z)$ are defined as

$$\Theta_{l,k}(\tau,z) = \sum_{n\in\mathbb{Z}+\frac{l}{2k}} q^{kn^2} e^{-2\pi iknz}, \quad q = e^{2\pi i\tau}. \tag{2.2.34}$$

From the explicit form of these characters, one obtains the modular S-matrix, which describes how the characters transform under the modular S-transformation,

$$S^{(k)}_{ll'} = \sqrt{\frac{2}{k+2}}\sin\Big(\frac{(l+1)(l'+1)\pi}{k+2}\Big), \quad l,\, l' = 0, \ldots, k\,. \tag{2.2.35}$$

Having determined the S-matrix, one can then compute the fusion coefficients of $\widehat{\mathfrak{su}}(2)_k$ using the Verlinde formula (2.2.22),

$$\phi_{l_1} \times \phi_{l_2} = \sum_{l_3=0}^{k} \mathcal{N}_{l_1 l_2}^{l_3}\, \phi_{l_3}\,. \tag{2.2.36}$$

The explicit form of the fusion rules reads

$$\mathcal{N}_{l_1 l_2}^{l_3} = \begin{cases} 1, & \text{if } |l_1 - l_2| \le l_3 \le \min(l_1 + l_2,\, 2k - l_1 - l_2) \text{ and } l_1 + l_2 + l_3 \equiv 0 \pmod 2\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{2.2.37}$$

The primary field ϕ_l has conformal weight

$$h_l = \frac{l(l+2)}{4(k+2)}\,. \tag{2.2.38}$$

Let us now study a simple current extension of the diagonal $\widehat{\mathfrak{su}}(2)_4$ theory. From the explicit form of the fusion coefficients (2.2.37), we find that the field ϕ_4 is a simple current, since

$$\mathcal{N}_{4,l}^{\,m} = \delta_{l+m,\,4}\,. \tag{2.2.39}$$

This implies that fusing with $J = \phi_4$ permutes the primaries with

$$[\phi_4] \times [\phi_l] = [\phi_{4-l}]\,, \tag{2.2.40}$$

and moreover, as $[J]^2 = 1$, it follows that the simple current has order two.

As we argued previously, the modular invariant partition function must include only those orbits that have integral monodromy charge. For the fields ϕ_l, the monodromy charge can be calculated using eq. (2.2.29),

$$Q_{\phi_l} = \frac{l}{2}\,. \tag{2.2.41}$$

Hence, the combinations that have integral monodromy charge are

$$\tilde{\chi}_0 = \chi_0 + \chi_4\,, \quad \tilde{\chi}_2 = \chi_2\,. \tag{2.2.42}$$

The shorter length orbit for χ_2 is a fixed point of the simple current action. From the S-matrix (2.2.35) we find the transformation behavior of the two orbits,

$$\tilde{S} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}. \tag{2.2.43}$$

This matrix is not symmetric due to the unresolved fixed point, a subtlety that can be resolved with a more careful treatment of fixed-point resolutions, see e.g., [134].

Combining all ingredients, the simple current extended partition function takes the form

$$Z_J(\tau,\overline{\tau}) = \frac{1}{C}\sum_{\alpha,\beta=0}^{1} \chi_{(\alpha+\beta,0)}\overline{\chi}_{(\alpha,0)} + \chi_{(\beta,2)}\overline{\chi}_{(0,2)}\,. \tag{2.2.44}$$

Explicitly,

$$Z_J(\tau,\overline{\tau}) = |\chi_0+\chi_4|^2 + 2\,|\tilde{\chi}_2|^2 = |\tilde{\chi}_0|^2 + 2\,|\tilde{\chi}_2|^2\,. \tag{2.2.45}$$

The normalisation $C = 1$ is chosen to ensure the vacuum appears exactly once.

Finally, this can be written in matrix form:

$$Z_J(\tau,\overline{\tau}) = \begin{pmatrix}\tilde{\chi}_0\\ \tilde{\chi}_2\end{pmatrix}^T \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} \begin{pmatrix}\overline{\tilde{\chi}}_0\\ \overline{\tilde{\chi}}_2\end{pmatrix}. \tag{2.2.46}$$

A direct check shows that the modular invariance under S-transformation holds,

$$\tilde{S}^T M \tilde{S}^* = M\,. \tag{2.2.47}$$

Note that the simple current extended partition function (2.2.46) is non-diagonal with respect to the characters of $\widehat{\mathfrak{su}}(2)_4$, but diagonal with respect to the characters of the extended chiral algebra. Through affine branching rules, the characters $\tilde{\chi}_0$ and $\tilde{\chi}_2$ can be shown to correspond to the irreducible, integral highest weight representations of $\widehat{\mathfrak{su}}(3)_1$, see e.g., [95]. For a further decomposition into characters appearing in the free boson theory compactified on a circle of rational square-radius, see eq. (3.4.7).

2.3 Galois Groups in Rational Conformal Field Theories

In preparation for section 4.2, we will show now, using the Verlinde formula (2.2.24), how one can associate a Galois group to any rational conformal field theory. This Galois group, and its normal subgroups, will play a central role in section 4.2, where we derive a correspondence between rationality in $\mathcal{N} = (2, 2)$ superconformal field theories and the arithmetic property of complex multiplication in Hodge structures. In this section, we will follow the arguments of refs. [135, 136] and [2] to demonstrate explicitly the important fact that the Galois group associated with any rational conformal field theory is abelian.

Let us denote the N characteristic polynomials of the $N \times N$ fusion matrices $\mathcal{N}_a$ by

$$P_a(X) := \det(\mathcal{N}_a - X\, \mathrm{I}_{N\times N}) \ . \tag{2.3.1}$$

They are degree-N polynomials $P_a \in \mathbb{Z}[X] \subset \mathbb{Q}[X]$ with integral coefficients.

Using the Verlinde formula (2.2.24), we can define a finite field extension by adjoining the roots of the characteristic polynomials $P_a(X)$, i.e., the eigenvalues of $\mathcal{N}_a$ which are $\lambda_{a,i}$ as defined in eq. (2.2.24)

$$\mathbb{Q}(\lambda) := \mathbb{Q}\left(\{\lambda_{a,i}\}_{a,i\in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})}\right) \ . \tag{2.3.2}$$

By construction, the field $\mathbb{Q}(\lambda)$, generated over $\mathbb{Q}$ by all the eigenvalues $\lambda_{a,i}$ of the fusion matrices, is the *splitting field* of the product polynomial

$$\prod_{a\in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} P_a(X) \in \mathbb{Q}[X] \, , \tag{2.3.3}$$

where each $P_a(X)$ is the characteristic polynomial of a fusion matrix $\mathcal{N}_a$.

Since $\mathbb{Q}(\lambda)$ is the smallest field extension of $\mathbb{Q}$ over which this polynomial splits completely into linear factors (i.e., all its roots lie in the field), it is by definition a *normal extension* of $\mathbb{Q}$. A field extension is called *normal* if every irreducible polynomial over the base field that has at least one root in the extension field actually splits completely over that field.

Furthermore, because we are working over $\mathbb{Q}$, which is a field of characteristic zero, all algebraic extensions are automatically separable. That is, every algebraic element over $\mathbb{Q}$ is a root of a separable polynomial (a polynomial with distinct roots). Therefore, the extension $\mathbb{Q}(\lambda)/\mathbb{Q}$ is also *separable*.

An extension that is both normal and separable is called a *Galois extension*. Hence, $\mathbb{Q}(\lambda)$ is a Galois extension of $\mathbb{Q}$.

Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$ be an element of the Galois group of the field extension $\mathbb{Q}(\lambda)$. Since the coefficients of each polynomial P_a are rational, they are fixed under the Galois action. That is, for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$, we have

$$\sigma(P_a(\lambda_{a,i})) = P_a(\sigma(\lambda_{a,i})) \, . \tag{2.3.4}$$

Moreover, since $\lambda_{a,i}$ is a root of P_a, we know $P_a(\lambda_{a,i}) = 0$, and therefore

$$\sigma(P_a(\lambda_{a,i})) = \sigma(0) = 0\,. \tag{2.3.5}$$

Thus,

$$P_a(\sigma(\lambda_{a,i})) = 0\,. \tag{2.3.6}$$

This shows that $\sigma(\lambda_{a,i})$ is also a root of the same polynomial P_a. Note that this is a general feature of Galois theory: Galois automorphisms permute the roots of a polynomial with coefficients in the base field (in this case $\mathbb{Q}$).

Therefore, the action of σ maps the root $\lambda_{a,i}$ to another root $\lambda_{a,j}$ of P_a, which we denote as:

$$\sigma(\lambda_{a,i}) = \lambda_{a,\varsigma_a(i)}\,, \tag{2.3.7}$$

where $\varsigma_a \in \mathrm{Sym}(\mathrm{HWR}_{\mathcal{H}}(\mathcal{A}))$ is a permutation of the index set $\mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$.

At this point, the permutation ς_a could a priori depend on a, i.e., on the specific fusion matrix $\mathcal{N}_a$. However, a crucial observation due to [135] is that this permutation is independent of a. To understand this, we make use of the *Verlinde formula*, which expresses the eigenvalues $\lambda_{a,i}$ of the fusion matrices in terms of the modular S-matrix as:

$$\lambda_{a,i} = \frac{\mathrm{S}_{a,i}}{\mathrm{S}_{0,i}}\,. \tag{2.3.8}$$

Using this expression, one finds that the eigenvalues satisfy the identity:

$$\lambda_{a,i}\lambda_{b,i} = \sum_{c\in\mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \mathcal{N}_{ab}^{c}\lambda_{c,i}\,, \tag{2.3.9}$$

which reflects the fusion algebra structure in diagonalised form.

Now, we apply the Galois automorphism σ to both sides of equation (2.3.9). Using the action in equation (2.3.7), we obtain:

$$\begin{aligned} \sigma(\lambda_{a,i}\lambda_{b,i}) &= \sigma\left(\sum_{c\in\mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \mathcal{N}_{ab}^{c}\lambda_{c,i}\right) \\ \lambda_{a,\varsigma_a(i)}\lambda_{b,\varsigma_b(i)} &= \sum_{c\in\mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \mathcal{N}_{ab}^{c}\lambda_{c,\varsigma_c(i)} \end{aligned} \tag{2.3.10}$$

For this identity to hold for arbitrary a and b, the indices on both sides must match. In general, this is only possible if

$$\varsigma_a(i) = \varsigma_b(i) = \varsigma_c(i)\,, \tag{2.3.11}$$

for all $a, b, c \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$. That is, the permutation of the index i must be the same across all fusion matrices. We thus conclude that:

$$\varsigma_a = \varsigma\,, \quad \text{independent of } a\,. \tag{2.3.12}$$

Therefore, the Galois group acts on the eigenvalues $\lambda_{a,i}$ by a single, common permutation $\varsigma \in \mathrm{Sym}(\mathrm{HWR}_{\mathcal{H}}(\mathcal{A}))$ that depends only on the Galois automorphism σ, and not on the representation a. We summarise the action as:

$$\sigma(\lambda_{a,i}) = \lambda_{a,\varsigma(i)} \quad \text{for all } a \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})\,. \tag{2.3.13}$$

This means that each Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$ determines a permutation $\varsigma \in \mathrm{Sym}(\mathrm{HWR}_{\mathcal{H}}(\mathcal{A}))$ that governs how it permutes the eigenvalues of all fusion matrices simultaneously.

Next, we note that the complex conjugates of the eigenvalues $\lambda_{a,i}$ are themselves contained in the splitting field $\mathbb{Q}(\lambda)$. This follows because each $\lambda_{a,i}$ is a root of the characteristic polynomial $P_a \in \mathbb{Q}[X]$, and since P_a has real coefficients (being in $\mathbb{Q}[X]$), its complex conjugate roots are also roots of P_a. Therefore,

$$\lambda^*_{a,i} \in \mathbb{Q}(\lambda)\,. \tag{2.3.14}$$

Using the unitarity and symmetry of the modular S-matrix,

$$\mathrm{S}\mathrm{S}^* = \mathrm{I}_{N\times N}, \quad \mathrm{S}^\top = \mathrm{S}\,, \tag{2.3.15}$$

together with the non-vanishing of the entries $\mathrm{S}_{0,i}$, we derive the following identity for the inverse square of $\mathrm{S}_{0,i}$:

$$\frac{1}{\mathrm{S}^2_{0,i}} = \sum_{a\in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \frac{\mathrm{S}_{a,i}}{\mathrm{S}_{0,i}} \left(\frac{\mathrm{S}_{a,i}}{\mathrm{S}_{0,i}}\right)^* = \sum_{a\in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \lambda_{a,i}\lambda^*_{a,i}\,. \tag{2.3.16}$$

This shows that the values $\mathrm{S}^2_{0,i}$ (as well as their inverses) lie in the field extension $\mathbb{Q}(\lambda)$. Additionally, since

$$\lambda_{a,i} = \frac{\mathrm{S}_{a,i}}{\mathrm{S}_{0,i}}\,, \tag{2.3.17}$$

we can express the square of each entry $\mathrm{S}_{a,i}$ as

$$\mathrm{S}^2_{a,i} = \lambda^2_{a,i}\mathrm{S}^2_{0,i}\,, \tag{2.3.18}$$

and since both $\lambda_{a,i}$ and $\mathrm{S}_{0,i}$ belong to $\mathbb{Q}(\lambda)$, it follows that:

$$\mathrm{S}^2_{a,i} \in \mathbb{Q}(\lambda) \quad \text{for all } a, i \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})\,. \tag{2.3.19}$$

Let us now determine the Galois action on the squares $\mathrm{S}^2_{a,i}$. First we note that complex conjugation is an automorphism of the field $\mathbb{Q}(\lambda)$, and since it fixes $\mathbb{Q}$, it defines an element $\iota \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$.

From the general Galois action on eigenvalues (see eq. (2.3.13)), we know that

$$\lambda^*_{a,i} = \iota(\lambda_{a,i}) = \lambda_{a,\iota(i)}\,, \tag{2.3.20}$$

where $\iota(i)$ denotes the index obtained from applying the permutation associated with ι.

Now, for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$, we apply σ to both sides of equation (2.3.16):

$$\begin{aligned} \sigma\left(\frac{1}{\mathrm{S}_{0,i}^2}\right) &= \sum_{a\in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \lambda_{a,\varsigma(i)}\lambda^*_{a,\iota\circ\varsigma\circ\iota(i)} \\ &= \sum_{a\in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \frac{\mathrm{S}_{a,\varsigma(i)}\mathrm{S}^*_{a,\iota\circ\varsigma\circ\iota(i)}}{\mathrm{S}_{0,\varsigma(i)}\mathrm{S}^*_{0,\iota\circ\varsigma\circ\iota(i)}}\,. \end{aligned} \tag{2.3.21}$$

Using the fact that $\mathrm{S}_{0,j} \in \mathbb{R}\setminus\{0\}$ for all $j \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$, together with the unitarity of the S-matrix,

$$\sum_{a\in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \mathrm{S}_{a,j}\cdot \mathrm{S}^*_{a,k} = \delta_{j,k}\,, \tag{2.3.22}$$

we obtain

$$\sigma\left(\frac{1}{\mathrm{S}_{0,i}^2}\right) = \frac{\delta_{\varsigma(i),\iota\circ\varsigma\circ\iota(i)}}{\mathrm{S}_{0,\varsigma(i)}\mathrm{S}_{0,\iota\circ\varsigma\circ\iota(i)}}\,. \tag{2.3.23}$$

Since the left-hand side is non-zero for all i, we must have $\delta_{\varsigma(i),\iota\circ\varsigma\circ\iota(i)} = 1$, meaning:

$$\varsigma(i) = \iota\circ\varsigma\circ\iota(i) \quad \text{for all } i \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})\,. \tag{2.3.24}$$

This implies that the permutations ς and $\iota\circ\varsigma\circ\iota$ are equal:

$$\varsigma = \iota\circ\varsigma\circ\iota\,, \tag{2.3.25}$$

and $\sigma(S_{0,i}^2) = S_{0,\varsigma(i)}^2$. From eq. (2.3.13) and the relation $\mathrm{S}_{a,i}^2 = \lambda_{a,i}^2\mathrm{S}_{0,i}^2$, it follows that the Galois action on $\mathrm{S}_{a,i}^2$ is given by

$$\sigma(\mathrm{S}_{a,i}^2) = \mathrm{S}_{a,\varsigma(i)}^2 \quad \text{for all } a \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})\,. \tag{2.3.26}$$

The fact that complex conjugation ι is a Galois automorphism implies that it generates a normal subgroup of the Galois group $\mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$. If ι is trivial, then the field $\mathbb{Q}(\lambda)$ is totally real. If ι is non-trivial, the fixed field of the corresponding normal subgroup is a totally real subfield of $\mathbb{Q}(\lambda)$. In this case, the extension $\mathbb{Q}(\lambda)/\mathbb{Q}$ is a quadratic totally imaginary extension of a totally real subfield — that is, a *complex multiplication* (CM) field.

An example of the first case is the family of non-supersymmetric Virasoro minimal models, and an example of the second case is the series of $\mathcal{N} = (2,2)$ supersymmetric minimal models, which will be introduced in section 4.3.

Let us now show that the Galois group $\mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$ is in fact abelian [135]. Using the definition of $\lambda_{a,i}$, i.e.,

$$\lambda_{a,i} = \frac{S_{a,i}}{S_{0,i}}\,, \tag{2.3.27}$$

together with the symmetry of the S-matrix, we find the following identity

$$\lambda_{a,i} = \lambda_{i,a} \cdot \frac{\lambda_{a,0}}{\lambda_{i,0}}, \quad \text{for all } a, i \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})\,. \tag{2.3.28}$$

Taking the square root of eq. (2.3.26)) and using the symmetry of the S-matrix gives the identity

$$\lambda_{\varsigma(a),i} \cdot \lambda_{i,0} = \pm\lambda_{\varsigma(i),a} \cdot \lambda_{a,0}, \quad \text{for all } \sigma \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})\,. \tag{2.3.29}$$

Let now $\sigma_1, \sigma_2 \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$ be two Galois automorphisms with associated permutations ς_1 and ς_2, respectively. We want to compute $\sigma_1\sigma_2(\lambda_{a,i})$. First, from the identity (2.3.28), it follows that

$$\sigma_2(\lambda_{a,i}) = \sigma_2\left(\lambda_{i,a} \cdot \frac{\lambda_{a,0}}{\lambda_{i,0}}\right) = \lambda_{i,\varsigma_2(a)} \cdot \frac{\lambda_{a,\varsigma_2(0)}}{\lambda_{i,\varsigma_2(0)}}\,. \tag{2.3.30}$$

Now applying σ_1 to this result and using $\lambda_{i,\varsigma_2(a)} = \lambda_{\varsigma_2(a),i}\frac{\lambda_{i,0}}{\lambda_{\varsigma_2(a),0}}$, $\lambda_{a,\varsigma_2(0)} = \lambda_{\varsigma_2(0),a}\frac{\lambda_{a,0}}{\lambda_{\varsigma_2(0),0}}$ and $\frac{\lambda_{i,0}}{\lambda_{i,\varsigma_2(0)}\lambda_{\varsigma_2(0),0}} = \frac{1}{\lambda_{\varsigma_2(0),i}}$, we get

$$\begin{aligned}
\sigma_1\sigma_2(\lambda_{a,i}) &= \sigma_1\left(\lambda_{i,\varsigma_2(a)}\frac{\lambda_{a,\varsigma_2(0)}}{\lambda_{i,\varsigma_2(0)}}\right) \\
&= \sigma_1\left(\frac{\lambda_{\varsigma_2(a),i}}{\lambda_{\varsigma_2(0),i}}\frac{\lambda_{a,0}}{\lambda_{\varsigma_2(a),0}}\lambda_{\varsigma_2(0),a}\right) = \frac{\lambda_{\varsigma_2(a),\varsigma_1(i)}}{\lambda_{\varsigma_2(0),\varsigma_1(i)}}\sigma_1\left(\frac{\lambda_{\varsigma_2(0),a}}{\lambda_{\varsigma_2(a),0}}\lambda_{a,0}\right) \\
&= \frac{\lambda_{\varsigma_2(a),\varsigma_1(i)}}{\lambda_{\varsigma_2(0),\varsigma_1(i)}}\frac{\lambda_{\varsigma_2(0),a}}{\lambda_{\varsigma_2(a),0}}\lambda_{a,0} = \sigma_2\left(\frac{\lambda_{\varsigma_1(i),a}}{\lambda_{\varsigma_1(i),0}}\lambda_{a,0}\right) = \sigma_2\left(\lambda_{a,\varsigma_1(i)}\right) = \sigma_2\sigma_1\left(\lambda_{a,i}\right)\,.
\end{aligned} \tag{2.3.31}$$

In the last line of the computation the identity (2.3.29) is used for $i = 0$, which shows that the term within σ_1 is ± 1, hence does not transform under $\mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$.

We now define the field extension generated by the entries of the modular S-matrix. We denote this field extension by

$$\mathbb{Q}(S) := \mathbb{Q}\left(\{\mathrm{S}_{a,i}\}_{a,i\in\mathrm{HWR}_{\mathcal{H}}(\mathcal{A})}\right)\,, \tag{2.3.32}$$

which is Galois over the intermediate field $\mathbb{Q}(\lambda)$. This follows from the fact that $\mathbb{Q}(S)$ is the splitting field of a family of separable polynomials,

$$P_{a,i}(x) := x^2 - \mathrm{S}_{a,i}^2 \in \mathbb{Q}(\lambda)[x]\,. \tag{2.3.33}$$

Indeed, each $\mathrm{S}_{a,i}$ is a root of the corresponding polynomial $P_{a,i}$, and since the polynomials are quadratic with distinct roots $\pm\mathrm{S}_{a,i}$ whenever $\mathrm{S}_{a,i} \neq 0$, they are separable. Thus, $\mathbb{Q}(S)$ is the splitting field of a separable set of polynomials over $\mathbb{Q}(\lambda)$, implying that the extension is normal and separable, i.e., Galois.

Moreover, the extension $\mathbb{Q}(S)$ is also Galois over the base field $\mathbb{Q}$. To show this, we consider any $\mathbb{Q}$-automorphism σ of any field containing $\mathbb{Q}(S)$ and investigate its action on $\mathbb{Q}(S)$.

Since the $\lambda_{a,i}$ are algebraic numbers and roots of polynomials with coefficients in $\mathbb{Q}$, it follows that any such $\mathbb{Q}$-automorphism σ restricts to an automorphism of the subfield $\mathbb{Q}(\lambda)$ with

$$\sigma(\mathbb{Q}(\lambda)) \subseteq \mathbb{Q}(\lambda)\,. \tag{2.3.34}$$

This fact ensures that σ acts compatibly on the generators $\lambda_{a,i}$ and hence on $\mathrm{S}_{a,i}$, which are related by

$$\mathrm{S}_{a,i} = \mathrm{S}_{0,i}\lambda_{a,i}\,. \tag{2.3.35}$$

Now observe that for fixed $i \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$, the entry $\mathrm{S}_{0,i}^2$ lies in $\mathbb{Q}(\lambda)$, and thus so does $\sigma(\mathrm{S}_{0,i})^2$. Since $\mathrm{S}_{0,i}$ is real and nonzero (for unitary two-dimensional conformal field theory), it follows that $\sigma(\mathrm{S}_{0,i})$ must be of the form

$$\sigma(\mathrm{S}_{0,i}) = \epsilon_\sigma(i)\mathrm{S}_{0,\varsigma(i)}\,, \tag{2.3.36}$$

where $\epsilon_\sigma(i) \in \{+1,-1\}$ and $\varsigma \in \mathrm{Sym}(\mathrm{HWR}_{\mathcal{H}}(\mathcal{A}))$ is a permutation of the representation labels, determined by the action of σ restricted to $\mathbb{Q}(\lambda)$. Consequently, the full S-matrix elements transform under σ as

$$\sigma(\mathrm{S}_{a,i}) = \sigma(\mathrm{S}_{0,i}\lambda_{a,i}) = \epsilon_\sigma(i)\mathrm{S}_{0,\varsigma(i)}\lambda_{a,\varsigma(i)} = \epsilon_\sigma(i)\mathrm{S}_{a,\varsigma(i)}\,, \tag{2.3.37}$$

with $\epsilon_\sigma(i)$ independent of $a \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$. This transformation law confirms that the action of any $\mathbb{Q}$-automorphism σ, as defined above, on the set $\{\mathrm{S}_{a,i}\}$ remains within $\mathbb{Q}(S)$, i.e.,

$$\sigma(\mathbb{Q}(S)) \subseteq \mathbb{Q}(S)\,. \tag{2.3.38}$$

Therefore, the field extension $\mathbb{Q}(S)/\mathbb{Q}$ is also normal and separable, and hence Galois [136].

Finally, due to the symmetry of the S-matrix, $\mathrm{S}_{a,i} = \mathrm{S}_{i,a}$, the action of the Galois group preserves this symmetry. Using the transformation law (2.3.37), one can verify that the composition of two automorphisms corresponds to the composition of their associated permutations and sign functions in a commutative way. Thus, the Galois group $\mathrm{Gal}(\mathbb{Q}(S)/\mathbb{Q})$ is abelian as well [136].

2.4 Boundary Conformal Field Theory

In most physical systems of finite extent, boundaries play a crucial role in determining their dynamics. In string theory, for example, open strings have endpoints constrained to lie on higher-dimensional hypersurfaces known as D-branes. This motivates the study of two-dimensional conformal field theories defined on surfaces with boundaries. The study of consistent *boundary conformal field theories* remains an active and rich area of research, the full depth of which extends well beyond the focus of this section. Here, we aim to introduce only those essential concepts and constructions that will be required in later chapters. For a comprehensive treatment of boundary conformal field theories, we refer to [97, 137].

When restricting a conformal field theory defined on the full complex plane to a half-plane, one generically breaks part of the conformal symmetry. However, for a consistent conformal field theory, one copy of the Virasoro algebra (and potentially an extended chiral algebra) must remain preserved, which leads to relations between holomorphic and anti-holomorphic quantities.

A notable feature of boundary conformal field theories is that correlation functions can develop additional singularities when fields approach the boundary. From general quantum field theory arguments, this signals the presence of additional operators, the so-called *boundary fields*. In string theory, these naturally arise as the open string excitations attached to D-branes.

Gluing Condition and Ishibashi States

Requiring the boundary condition to be invariant under conformal transformations leads to the constraint on the energy-momentum tensor at the boundary,

$$T(z) = \overline{T}(\bar{z})\,, \tag{2.4.1}$$

which physically expresses the absence of momentum flow across the boundary. If the conformal field theory has an extended chiral algebra $\mathcal{A}$ with additional generators W, then one may more generally require that the boundary condition preserves the whole algebra or a suitable subalgebra. This requirement is encoded in the generalised gluing condition,

$$W(z) = \omega(\overline{W})(\bar{z})\,, \tag{2.4.2}$$

where ω is a pointwise acting automorphism of the extended chiral algebra satisfying $\omega(T) = T$. Defining an extended chiral field on the full complex plane by

$$\widetilde{W}(z) = \begin{cases} W(z) & \text{for } \operatorname{Im}(z) \geq 0\,, \\ \omega\overline{W}(\bar{z}) & \text{for } \operatorname{Im}(z) < 0\,, \end{cases} \tag{2.4.3}$$

one obtains Ward identities for correlation functions with boundaries, analogous to those on the full plane. At the level of states, the gluing conditions read

$$(L_n - \overline{L}_{-n})\,|B\rangle = 0\,, \tag{2.4.4}$$

$$(W_n - (-1)^{h_W}\omega\,\overline{W}_{-n})\,|B\rangle = 0\,, \quad \forall\, n \in \mathbb{Z}\,. \tag{2.4.5}$$

In the case of a trivial automorphism ω, a solution to these conditions was given in [138] for each highest-weight representation α of $\mathcal{A}$,

$$|\alpha\rangle\rangle = \sum_{N=0}^{\infty}\sum_{k=1}^{d_\alpha(N)} |\alpha, N, k\rangle \otimes U\,\overline{|\alpha, N, k\rangle} \tag{2.4.6}$$

$$|\alpha\rangle\rangle_\omega := (\mathrm{Id}\otimes V_\omega)\,|\alpha\rangle\rangle\,, \tag{2.4.7}$$

where $\{|\alpha, N, k\rangle\}$ denotes a $d_\alpha(N)$-dimensional orthonormal basis of the Hilbert space $\mathcal{H}_\alpha$ built on the highest-weight representation $\alpha \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$ at level N. The anti-unitary operator U satisfies

$$U\,\overline{W}_n = (-1)^{h_W}\omega\left(\overline{W}_n\right)\,U\,. \tag{2.4.8}$$

In particular, U maps the anti-chiral state to the conjugate state $\overline{|\alpha^+\rangle}$, so that the (un-twisted) Ishibashi state lies in $\mathcal{H}_\alpha \otimes \overline{\mathcal{H}}_{\alpha^+}$. The Ishibashi states (2.4.6) are normalised as

$$\langle\langle\beta|\,q^{L_0-\frac{c}{24}}\,|\alpha\rangle\rangle = \delta_{\alpha,\beta}\,\chi_\alpha(q)\,, \qquad \chi_\alpha(q) = \mathrm{Tr}_{\mathcal{H}_\alpha}(q^{L_0-\frac{c}{24}})\,. \tag{2.4.9}$$

For twisted Ishibashi states $|\alpha\rangle\rangle_\omega$ corresponding to non-trivial automorphism ω, let us denote by π_α the action of the generators W on the Hilbert space H_α for some representation $\alpha \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$. We can define a new action of W on the same Hilbert space by composing with ω. Concretely, this means that for any generator W and state $|h\rangle \in H_\alpha$,

$$\pi_\alpha^\omega(W)\,|h\rangle := \pi_\alpha(\omega W)\,|h\rangle\,. \tag{2.4.10}$$

Equipped with this new action, the space H_α is again as an $\mathcal{A}$-module isomorphic to some unique representation space $H_{\Omega(\alpha)}$,

$$\pi_\alpha \circ \omega \cong \pi_{\Omega(\alpha)}\,. \tag{2.4.11}$$

If the automorphism ω is an *outer* automorphism (that is, not equivalent to the identity by an inner symmetry), then the resulting representation $\Omega(\alpha)$ is inequivalent to α. We define a unitary operator $V_\omega\,:\,\mathcal{H}_{\Omega(\alpha)} \to \mathcal{H}_\alpha$ such that

$$\pi_\alpha^\omega(W) = V_\omega \pi_{\Omega(\alpha)}(W) V_\omega^{-1}\,, \tag{2.4.12}$$

and we assume that V_ω commutes with U, i.e.,

$$V_\omega U = U V_\omega\,. \tag{2.4.13}$$

A straight forward calculation, using the vanishing of

$$\langle\beta, M, \ell| \otimes \langle V_\omega U(\gamma; M', \ell')| \left(W_m - (-1)^{h(W)}\, \omega(\overline{W}_{-m})\right) |\alpha\rangle\!\rangle_\omega = 0\,, \tag{2.4.14}$$

for all orthonormal basis vectors $|\beta, M, \ell\rangle \otimes |\gamma, M', \ell'\rangle$ in the total Hilbert space $\mathcal{H}$ verifies that (2.4.6) and (2.4.7) indeed satisfy the gluing conditions (2.4.4). If we want to construct boundary states that preserve only a subalgebra $\mathcal{A}_0 \subset \mathcal{A}$, the relevant Ishibashi states must be constructed from the highest-weight representations of $\mathcal{A}_0$ appearing in each $\mathcal{A}$-module.

Cardy Constraint

Let us now consider a general boundary state as a linear combination of Ishibashi states,

$$|B_a\rangle = \sum_{\alpha \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} B^a_\alpha\, |\alpha\rangle\!\rangle\,. \tag{2.4.15}$$

Analogous to how the bulk Hilbert space spectrum is encoded in the one-loop torus partition function, the spectrum in a boundary conformal field theory is encoded in the cylinder partition function

$$Z_{ab} = \mathrm{Tr}_{\mathcal{H}_{ab}} \left(q^{L_0 - \frac{c}{24}}\right)\,, \tag{2.4.16}$$

where $\mathcal{H}_{ab}$ denotes the Hilbert space of open string states in a theory defined on a finite cylinder with boundary conditions B_a and B_b on the opposite edges, and $q = e^{-2\pi t}$. Analogous to the torus whose shape is parameterised by a complex parameter τ, we denote by $0 \le t < \infty$ the real modular parameter of the cylinder. The cylinder is parameterised as

$$\{(\tau, \sigma)\,:\, 0 \le \sigma \le \pi\,,\, 0 \le \tau \le 2\pi t\}\,. \tag{2.4.17}$$

The Hilbert space $\mathcal{H}_{ab}$ decomposes into irreducible highest weight representations of the extended chiral algebra $\mathcal{A}$

$$\mathcal{H}_{ab} = \bigoplus_{\alpha \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \mathcal{H}_\alpha^{\oplus n^\alpha_{ab}}\,, \tag{2.4.18}$$

with multiplicities n^α_{ab}. The partition function can be expanded in terms of the characters of the highest weight representations of $\mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$ as

$$Z_{ab}(t) = \sum_{\alpha \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} n^\alpha_{ab}\, \chi_\alpha(\mathrm{i}t)\,. \tag{2.4.19}$$

In [137], it was argued that the same physical amplitude can alternatively be computed in the closed string channel. In this picture, the process is reinterpreted as the emission of a closed string of length 2π from the boundary B_a, which then propagates for a Euclidean time l under the closed-string Hamiltonian

$$H_{\mathrm{cl}} = L_0 + \overline{L}_0 - \frac{c + \overline{c}}{24}\,, \tag{2.4.20}$$

before being absorbed at the boundary B_b. See figure 2.2 for an illustration.

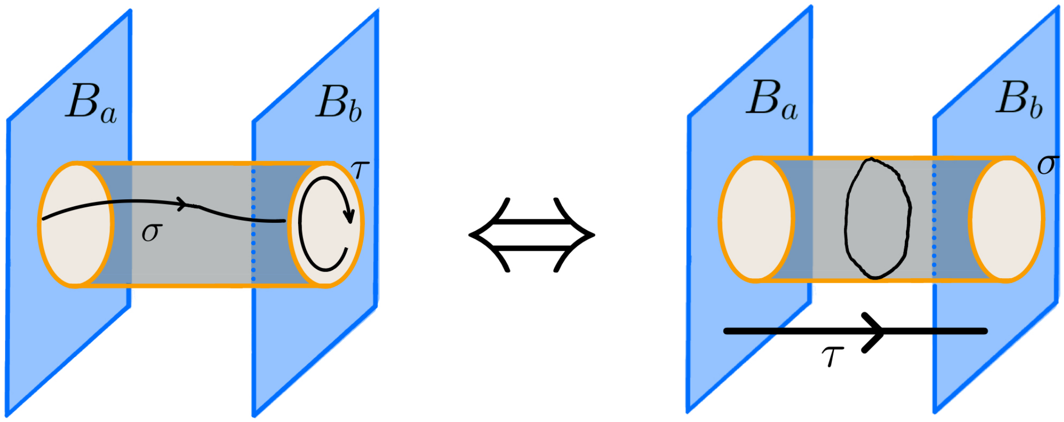

Figure 2.2: **Open-closed worldsheet duality.** The open string cylinder partition function on the left is identified as the tree-level closed string exchange process on the right. This is known as worldsheet duality between open and closed strings.

The closed string amplitude reads

$$\widetilde{Z}_{ab}(l) = \langle \Theta B_a \,|\, e^{-2\pi l (L_0 + \overline{L}_0 - \frac{c+\bar{c}}{24})} \,|\, B_b \rangle = \sum_{\alpha \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} (B^a_\alpha)^* B^b_\alpha \chi_\alpha(2\mathrm{i}l)\,, \tag{2.4.21}$$

where Θ denotes the combined action of charge conjugation, parity, and time reversal, needed to account for the opposite orientation of the two boundaries. We fix the action of Θ by

$$\Theta\,|a\rangle = \sum_{\alpha \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} (B^a_\alpha)^* \,|\alpha^+\rangle\!\rangle\,, \tag{2.4.22}$$

where α^+ denotes the representation conjugate to α. The two descriptions are related by a change of spatial and (Euclidean) time coordinate without changing the modulus of the cylinder (ratio of circumference and height), which is $\frac{2\pi}{2\pi l}$ for the closed string tree diagram and $\frac{2\pi t}{\pi}$ for the open string loop diagram, hence we find the relation

$$t = \frac{1}{2l}\,. \tag{2.4.23}$$

To map the closed string tree level amplitude to the open string loop amplitude, we perform a modular S-transformation,

$$\widetilde{Z}_{ab}(l) \to \widetilde{Z}_{ab}(\frac{1}{2t}) = \sum_{\alpha,\beta \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} (B^a_\alpha)^* B^b_\alpha \,\mathrm{S}_{\alpha\beta}\, \chi_\beta(it) = \sum_{\beta \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} n^\beta_{ab}\, \chi_\beta(it) = Z_{ab}(t)\,. \tag{2.4.24}$$

Equating the two channels yields the celebrated *Cardy condition*

$$n^\beta_{ab} = \sum_{\alpha \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} (B^a_\alpha)^* B^b_\alpha \,\mathrm{S}_{\alpha\beta} \in \mathbb{N}\,. \tag{2.4.25}$$

One sees that if one were to use individual Ishibashi states alone, the Cardy condition would generally fail, since the S-matrix elements are not guaranteed to be non-negative integers. Thus, the non-trivial linear combinations encoded by the coefficients B_i^a are crucial for consistency.

For a theory with a charge conjugate modular invariant partition function, a solution to the Cardy condition is given by

$$B_\alpha^a = \frac{\mathrm{S}_{a\alpha}}{\sqrt{\mathrm{S}_{0\alpha}}}\,. \tag{2.4.26}$$

This follows directly from the Verlinde formula (2.2.22),

$$n_{ab}^{\beta} = \sum_{\alpha\in\mathrm{HWR}_{\mathcal{H}}(\mathcal{A})} \frac{(\mathrm{S}_{a\alpha})^*\mathrm{S}_{b\alpha}\mathrm{S}_{\beta\alpha}}{\mathrm{S}_{0\alpha}} = \mathcal{N}_{\beta b}^{a} \in \mathbb{N}\,. \tag{2.4.27}$$

2.5 $\mathcal{N} = (2,2)$ Supersymmetric Conformal Field Theories

In this section, we review main aspects of two-dimensional $\mathcal{N} = (2,2)$ superconformal field theories, intending to establish notation and conventions that will be used heavily in section 4.2. The content presented here follows standard notations found in the literature, and we refer to [96, 139–141] for more details.

The chiral (left-moving) sector of a two-dimensional $\mathcal{N} = (2,2)$ superconformal field theory is described by the $\mathcal{N} = 2$ super Virasoro algebra, also known as the *super Virasoro algebra* SVir. This algebra extends the usual Virasoro algebra Vir present in non-supersymmetric conformal field theories, whose generators L_m correspond to the Laurent modes of the holomorphic stress-energy tensor $T(z)$,

$$T(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}, \tag{2.5.1}$$

by incorporating additional supersymmetry and R-symmetry currents.

Specifically, the $\mathcal{N} = 2$ super Virasoro algebra includes:

- A U(1) R-current $J(z)$ of spin-one, whose modes are denoted by J_m, defined via

$$J(z) = \sum_{m\in\mathbb{Z}} J_m z^{-m-1}. \tag{2.5.2}$$

- Two fermionic (Grassmann-valued) supercurrents $G^{\pm}(z)$ of spin 3/2, with U(1)-charges ± 1, whose modes are denoted by $G_r^{\pm}$:

$$G^{\pm}(z) = \sum_{r} G_r^{\pm}\, z^{-r-3/2}. \tag{2.5.3}$$

These operators satisfy the following (anti-)commutation relations, which define the $\mathcal{N} = 2$ super Virasoro algebra:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0}\,, \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\,, \\
\{G_r^{\pm}, G_s^{\pm}\} &= 0\,, \\
[L_m, J_n] &= -nJ_{m+n}\,, \qquad [L_m, G_r^{\pm}] = \left(\frac{m}{2} - r\right)G_{m+r}^{\pm}\,, \\
[J_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm}\,.
\end{aligned} \tag{2.5.4}$$

Note that one can explicitly verify that the energy momentum tensor $T(z)$ together with the combination $G = \left(e^{\mathrm{i}\gamma}G^{+} + e^{-\mathrm{i}\gamma}G^{-}\right)$ generate a one-parameter family of $\mathcal{N}=1$ subalgebras for $\gamma \in \mathbb{R}$. Furthermore, they can be combined into an $\mathcal{N}=1$ super energy-momentum tensor, which can be described in a so-called superspace formalism, see e.g., [12].

In a two-dimensional superconformal field theory, the mode expansion of fermionic fields — including the supercurrents $G^{\pm}(z)$ — depends on the *spin structure* on the worldsheet. A spin structure specifies the boundary conditions imposed on fermionic fields when transported around non-contractible cycles of the worldsheet. The two types of spin structures are:

- **Ramond (R) sector**: fermions are *antiperiodic* on the complex plane,

$$\psi(e^{2\pi i}z) = -\psi(z)\,, \tag{2.5.5}$$

  leading to a mode expansion with integer modes:

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n\, z^{-n-\frac{1}{2}}\,. \tag{2.5.6}$$

- **Neveu–Schwarz (NS) sector**: fermions are *periodic* on the complex plane,

$$\psi(e^{2\pi i}z) = \psi(z)\,, \tag{2.5.7}$$

  giving rise to half-integer modes:

$$\psi(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r\, z^{-r-\frac{1}{2}}\,. \tag{2.5.8}$$

The supercurrents $G^{\pm}(z)$ are fermionic operators of conformal weight $3/2$, and hence their mode expansions are similarly determined by the spin structure.

Note that upon mapping the complex plane to the cylinder, and subsequently to the torus, the periodicity of the boundary conditions changes: the Ramond (R) sector corresponds to periodic fields, while the Neveu–Schwarz (NS) sector corresponds to antiperiodic fields on both the cylinder and the torus.

An $\mathcal{N} = (2,2)$ supersymmetric rational conformal field theory with extended chiral algebra $\mathcal{A}$ possesses both left- and right-moving $\mathcal{N}=2$ super Virasoro algebras. These combine to form the symmetry algebra

$$\mathrm{SVir} \times \overline{\mathrm{SVir}}\,, \tag{2.5.9}$$

where SVir and $\overline{\mathrm{SVir}}$ denote the holomorphic and antiholomorphic copies of the $\mathcal{N}=2$ super Virasoro algebra, respectively.

It is important to note, however, that the full super Virasoro algebra SVir is *not* part of the chiral algebra $\mathcal{A}$ of a rational conformal field theory. This is because the supercurrents

$G^{\pm}(z)$ are fermionic operators with non-integral conformal weight $h = \frac{3}{2}$, which renders SVir a *super*-Lie algebra rather than an ordinary Lie algebra. As a result, the chiral algebra $\mathcal{A}$ of an $\mathcal{N} = (2,2)$ rational superconformal field theory is generated solely by Grassmann-even operators, i.e., the bosonic subalgebra [122, 140, 142, 143].

The $\mathcal{N} = 2$ super Virasoro algebras in the Ramond and Neveu–Schwarz sectors, denoted by $\mathrm{SVir}_{\mathrm{R}}$ and $\mathrm{SVir}_{\mathrm{NS}}$ respectively, are in fact *isomorphic* as Lie superalgebras [144]. This isomorphism can be explicitly constructed and is known as the *spectral flow* isomorphism [139–141, 144].

Let χ be an element of $\mathrm{SVir}_{\mathrm{NS}}$ and χ' its image in $\mathrm{SVir}_{\mathrm{R}}$. Then the isomorphism is given by the unitary map $\mathcal{U}$:

$$\mathrm{SVir}_{\mathrm{NS}} \xrightarrow{\simeq} \mathrm{SVir}_{\mathrm{R}}\,, \quad \chi \mapsto \chi' = \mathcal{U}\chi\mathcal{U}^{-1}\,, \tag{2.5.10}$$

with the generators mapping as follows

$$\begin{aligned} L'_m &= L_m + \frac{1}{2}J_m + \frac{c}{24}\delta_{m,0}\,, \\ J'_m &= J_m + \frac{c}{6}\delta_{m,0}\,, \\ G'^{\pm}_r &= G^{\pm}_{r\pm\frac{1}{2}}\,. \end{aligned} \tag{2.5.11}$$

This algebraic isomorphism also induces an isomorphism between representations of the Ramond and Neveu–Schwarz sectors. That is, the spectral flow operator $\mathcal{U}$ defines a map,

$$\mathcal{U} : \mathcal{H}_{\mathrm{NS}} \longrightarrow \mathcal{H}_{\mathrm{R}}\,, \tag{2.5.12}$$

which takes a state in a representation of the NS–sector to a corresponding state in a representation of the R–sector.

Analogously to the non-supersymmetric case, we define a *superconformal primary state* as a highest-weight state with respect to the $\mathcal{N} = 2$ super Virasoro algebra. Let $|\phi\rangle$ be a state in the Hilbert space of the theory. Then $|\phi\rangle$ is called a *superconformal primary state* if it satisfies the following annihilation conditions:

$$\boxed{\begin{aligned} L_n\,|\phi\rangle &= 0, \qquad \text{for all } n > 0\,, \\ J_n\,|\phi\rangle &= 0, \qquad \text{for all } n > 0\,, \\ G^{\pm}_r\,|\phi\rangle &= 0, \qquad \text{for all } r > 0\,, \end{aligned}} \tag{2.5.13}$$

where L_n are the Virasoro generators, J_n the modes of the $\mathrm{U}(1)_R$ current, and $G^{\pm}_r$ the fermionic supercurrent modes.

A superconformal primary state generates an entire highest-weight module (a *super-Verma module*) under the action of the negative modes of the algebra. Its descendants are obtained by acting with the operators L_{-n}, J_{-n}, and $G^{\pm}_{-r}$ for $n, r > 0$.

In $\mathcal{N} = (2,2)$ superconformal field theories, the operator-state correspondence associates each superconformal primary state $|\phi\rangle$ with a local field $\phi(z,\bar{z})$ on the worldsheet. The field $\phi(z,\bar{z})$ is then referred to as a *superconformal primary field*.

An important object in $\mathcal{N} = (2,2)$ superconformal field theories, which will play a key role in the construction of Hodge structures in section 4.2, is the so-called chiral ring. The following discussion is based on [145]. A superconformal primary field $\phi(z,\bar{z})$ is called a *chiral primary field* if its operator product expansion (OPE) with the spin-$\frac{3}{2}$ current $G^+(z)$ is regular. Analogously, it is called an *anti-chiral primary field* if the OPE with $G^-(z)$ is regular.

Via the operator-state correspondence, a field $\phi(z,\bar{z})$ corresponds to a state $|\phi\rangle$ in the Hilbert space. In this language, a superconformal primary state is chiral if it is annihilated by the fermionic mode $G^+_{-1/2}$, and anti-chiral if it is annihilated by $G^-_{-1/2}$:

$$\text{Chiral primary: } G^+_{-1/2}\,|\phi\rangle = 0\,, \qquad \text{Anti-chiral primary: } G^-_{-1/2}\,|\phi\rangle = 0\,. \tag{2.5.14}$$

A similar characterisation applies in the anti-holomorphic sector, with the modes $\overline{G}^{\pm}_{-1/2}$. A superconformal primary field that satisfies chirality constraints in both sectors is then classified into one of four types, according to the conditions:

$$\begin{aligned}
&\text{(c,c) state:} \quad G^+_{-1/2}\,|\phi\rangle = 0\,, \quad \overline{G}^+_{-1/2}\,|\phi\rangle = 0\,,\\
&\text{(a,a) state:} \quad G^-_{-1/2}\,|\phi\rangle = 0\,, \quad \overline{G}^-_{-1/2}\,|\phi\rangle = 0\,,\\
&\text{(c,a) state:} \quad G^+_{-1/2}\,|\phi\rangle = 0\,, \quad \overline{G}^-_{-1/2}\,|\phi\rangle = 0\,,\\
&\text{(a,c) state:} \quad G^-_{-1/2}\,|\phi\rangle = 0\,, \quad \overline{G}^+_{-1/2}\,|\phi\rangle = 0\,.
\end{aligned} \tag{2.5.15}$$

Here, the labels (c/a, c/a) indicate whether the chirality condition is imposed in the holomorphic and/or anti-holomorphic sectors, with "c" denoting chiral and "a" anti-chiral.

A fundamental property of such (c,c), (a,a), (c,a), and (a,c) fields is that their OPEs with one another are regular and preserve the chirality type. Consequently, the product of two such fields remains in the same class and defines a well-behaved operator product in the limit where their insertion points coincide. This structure equips the space of these fields with a ring structure, referred to respectively as the (c,c)-ring, the (a,a)-ring, the (c,a)-ring, and the (a,c)-ring of the theory [145].

Let us determine the conformal weights and U(1)-charges of chiral and anti-chiral primary states. Denoting the left- and right-moving conformal weights and charges by

$$L_0\,|\phi\rangle = h_L(|\phi\rangle)\,|\phi\rangle\,, \quad \overline{L}_0\,|\phi\rangle = h_R(|\phi\rangle)\,|\phi\rangle\,, \quad J_0\,|\phi\rangle = q_L(|\phi\rangle)\,|\phi\rangle\,, \quad \overline{J}_0\,|\phi\rangle = q_R(|\phi\rangle)\,|\phi\rangle\,, \tag{2.5.16}$$

one finds, from the anti-commutation relations of the superconformal algebra, namely

$$\{G^{\pm}_{-1/2}, G^{\mp}_{1/2}\} = 2L_0 \mp J_0 \,, \tag{2.5.17}$$

that chiral and anti-chiral primaries satisfy:

$$h_{L/R}(|\phi\rangle) = \pm\frac{1}{2} q_{L/R}(|\phi\rangle) \,, \tag{2.5.18}$$

with the positive (negative) sign holding for chiral (anti-chiral) states.

Moreover, unitarity of the superconformal field theory implies bounds on the charges and conformal weights. From the anticommutator

$$\{G^{\pm}_{-3/2}, G^{\mp}_{3/2}\} = 2L_0 \pm J_0 + \frac{2}{3}c \,, \tag{2.5.19}$$

and the requirement of positive-definiteness of the norm, one obtains:

$$0 \leq h_{L/R}(|\phi\rangle) \leq \frac{c}{6} \quad \Longleftrightarrow \quad 0 \leq |q_{L/R}(|\phi\rangle)| \leq \frac{c}{3} \,, \tag{2.5.20}$$

with $q_{L/R} > 0$ for chiral primaries and $q_{L/R} < 0$ for anti-chiral primaries. Note that for $\mathcal{N} = (2,2)$ superconformal field theories associated with compact Calabi–Yau manifolds, the chiral rings are finite-dimensional, as they are related to the finite-dimensional Dolbeault cohomology rings of the manifold [145]. More generally, due to the bound (2.5.20), the chiral rings are finite for compact $\mathcal{N} = (2,2)$ superconformal field theories[3].

Spectral Flow and Ramond Ground States

Acting with the spectral flow operator $\mathcal{U}$ on a chiral primary state $|\phi\rangle$, we use the properties of the isomorphism defined in equation (2.5.12) to see that the resulting state $|\alpha\rangle := \mathcal{U}\,|\phi\rangle$ satisfies

$$L_0\,(|\alpha\rangle) = \frac{c}{24}\;|\alpha\rangle \;. \tag{2.5.21}$$

Similarly, an anti-chiral primary state in the NS–sector flows under $\mathcal{U}^{-1}$ to a state in the R–sector of conformal weight $c/24$. Using the anti-commutator

$$\{G^{+}_{0}, G^{-}_{0}\} = 2L_0 - \frac{c}{12} \,, \tag{2.5.22}$$

together with $\left(G^{+}_{0}\right)^{\dagger} = G^{-}_{0}$, we find

$$\langle\alpha|\,\{G^{+}_{0}, G^{-}_{0}\}\,|\alpha\rangle = 0 = \|G^{+}_{0}|\alpha\rangle\|^2 + \|G^{-}_{0}|\alpha\rangle\|^2 \,. \tag{2.5.23}$$

By the unitarity of the $\mathcal{N} = (2,2)$ superconformal field theory, it then follows that

$$G^{\pm}_{0}\,|\alpha\rangle = 0 \,. \tag{2.5.24}$$

[3]A conformal field theory is called compact if its spectrum of conformal weights is discrete.

Thus, the image of the spectral flow operator $\mathcal{U}$ acting on chiral states, and the image of $\mathcal{U}^{-1}$ acting on anti-chiral states, is annihilated by the Ramond zero-modes and is thus a *Ramond ground state*.

More generally, a state $|\varphi\rangle$ in the NS–sector with U(1)-charges $q_{L/R}(|\varphi\rangle)$ and conformal weights $h_{L/R}(|\varphi\rangle)$ flows under $\mathcal{U}$ to a state in the R–sector, with

$$q_{L/R}(\mathcal{U}\,|\varphi\rangle) = q_{L/R}(|\varphi\rangle) - \frac{c}{6} \tag{2.5.25}$$

$$h_{L/R}(\mathcal{U}\,|\varphi\rangle) = h_{L/R}(|\varphi\rangle) - \frac{1}{2}q_{L/R}(|\varphi\rangle) + \frac{c}{24}\,, \tag{2.5.26}$$

where the subscript refers to whether the spectral flow is applied in the left- or right-moving sector.

Finally, the charge conjugation operator $\mathcal{C}$, induced by the charge conjugation matrix (2.2.11), induces the transformation on states

$$q_{L/R}(\mathcal{C}\,|\alpha\rangle) = -q_{L/R}(|\alpha\rangle)\,, \tag{2.5.27}$$

with the conformal weights staying fixed.

Chiral Rings and Hodge Decomposition

In the context of $\mathcal{N} = (2,2)$ superconformal field theories arising as conformal fixed points in non-linear sigma models on complex Kähler manifolds that admit a Ricci-flat metric (e.g., Calabi–Yau manifolds), the chiral ring, or equivalently the R–ground states, have a geometric realisation in terms of (p,q)-forms and Dolbeault cohomology [146].

In a unitary theory, any state in the NS–sector admits a unique orthogonal decomposition of the form

$$|\phi\rangle = |\phi_0\rangle + G^+_{-1/2}|\phi_1\rangle + G^-_{1/2}|\phi_2\rangle\,, \tag{2.5.28}$$

where $|\phi_0\rangle$ is a chiral primary state. If $|\phi\rangle$ has conformal weight h and U(1) charge q, then $|\phi_1\rangle$ and $|\phi_2\rangle$ possess weights and charges $(h-\frac{1}{2}, q-1)$ and $(h+\frac{1}{2}, q+1)$, respectively.

Assuming that decomposition (2.5.28) holds, orthogonality of the terms is straightforward to verify. Uniqueness also follows similarly: if two distinct decompositions existed, their difference would yield a nontrivial combination of the form $0 = |\phi_0'\rangle + G^+_{-1/2}|\phi_1'\rangle + G^-_{1/2}|\phi_2'\rangle$. Then, acting with $\langle\phi_2'|\,G^+_{-1/2}$, and using that $|\phi_0'\rangle$ is a chiral primary together with the identity $G^+_{-1/2}G^+_{-1/2} = 0$, we find that $G^-_{1/2}|\phi_2'\rangle = 0$. Analogously, we find $G^+_{-1/2}|\phi_1'\rangle = 0$, and hence also $|\phi_0'\rangle = 0$, so that all components must vanish individually.

To establish the existence of the decomposition (2.5.28), we need to show that any state $|\phi\rangle$, that is not a chiral primary state, can be decomposed as

$$|\phi\rangle = G^+_{-1/2}\,|\phi_1\rangle + G^-_{1/2}\,|\phi_2\rangle\;. \tag{2.5.29}$$

Suppose now that $|\phi\rangle$ is not a chiral primary state. Then, $|\phi\rangle$ satisfies

$$G^+_{-1/2}|\phi\rangle \neq 0\,, \tag{2.5.30}$$

i.e., it is not annihilated by $G^+_{-1/2}$.

From the $\mathcal{N}=2$ super Virasoro algebra, the anticommutator relation reads:

$$\{G^+_{-1/2}, G^-_{1/2}\} = 2L_0 - J_0\,. \tag{2.5.31}$$

Acting on the state $|\phi\rangle$, we obtain:

$$\{G^+_{-1/2}, G^-_{1/2}\}|\phi\rangle = (2L_0 - J_0)|\phi\rangle = (2h-q)|\phi\rangle\,. \tag{2.5.32}$$

Since $|\phi\rangle$ is not a chiral primary, we know that

$$2h - q \neq 0\,. \tag{2.5.33}$$

Therefore, we can express $|\phi\rangle$ as

$$|\phi\rangle = \frac{1}{2h-q}\{G^+_{-1/2}, G^-_{1/2}\}|\phi\rangle\,. \tag{2.5.34}$$

Defining

$$|\eta\rangle = \frac{1}{2h-q}|\phi\rangle\,, \tag{2.5.35}$$

we can write

$$|\phi\rangle = \{G^+_{-1/2}, G^-_{1/2}\}|\eta\rangle = G^+_{-1/2}G^-_{1/2}|\eta\rangle + G^-_{1/2}G^+_{-1/2}|\eta\rangle\,. \tag{2.5.36}$$

Finally, setting

$$|\phi_1\rangle = G^-_{1/2}|\eta\rangle, \quad |\phi_2\rangle = G^+_{-1/2}|\eta\rangle\,, \tag{2.5.37}$$

we conclude that

$$|\phi\rangle = G^+_{-1/2}|\phi_1\rangle + G^-_{1/2}|\phi_2\rangle\,. \tag{2.5.38}$$

This establishes the desired decomposition.

In the special case where $|\phi\rangle$ is chiral, but not a primary, the decomposition simplifies to

$$|\phi\rangle = |\phi_0\rangle + G^+_{-1/2}|\phi_1\rangle\,, \tag{2.5.39}$$

as $|\phi_2\rangle$ must vanish in this case, which can be seen upon acting with $G^+_{-1/2}$ on (2.5.28).

Assuming for simplicity a left-right diagonal theory, this decomposition strongly resembles the Hodge decomposition of (p,q)-forms on compact complex Kähler manifolds. This relation can be made precise in geometric $\mathcal{N}=(2,2)$ non-linear sigma models on Calabi–Yau manifolds, as reviewed in detail in [141, 145]. Schematically, one identifies the nilpotent operators

$$(G^+_{-1/2}, G^-_{-1/2}) \quad \leftrightarrow \quad (\partial, \partial^*)\,, \tag{2.5.40}$$

where (∂, ∂^*) denotes the dual pair (with respect to the standard inner product on $A^{p,q}$) of *Dolbeault operators*

$$\begin{aligned} \partial &: A^{p,q}(M) \to A^{p+1,q}(M) \\ \partial^* &: A^{p,q}(M) \to A^{p-1,q}(M)\,, \end{aligned} \tag{2.5.41}$$

and $A^{p,q}(M)$ denotes the space of (p,q)-forms on some complex Kähler manifold M, with coefficients in $\mathbb{C}$. The operator

$$\{G^+_{-1/2}, G^-_{1/2}\} = 2L_0 - \frac{1}{2}J_0\,, \tag{2.5.42}$$

mirrors the Laplacian relation in cohomology

$$\Delta_\partial = \partial\partial^* + \partial^*\partial\,. \tag{2.5.43}$$

A similar correspondence holds for the right-moving sector

$$(\overline{G}^+_{-1/2}, \overline{G}^-_{-1/2}) \quad \leftrightarrow \quad (\overline{\partial}, \overline{\partial}^*), \tag{2.5.44}$$

with:

$$\{\overline{G}^+_{-1/2}, \overline{G}^-_{1/2}\} = 2\overline{L}_0 + \frac{1}{2}\overline{J}_0\,. \tag{2.5.45}$$

In this identification, we have

- Chiral fields correspond to forms annihilated by ∂ — *closed forms*.
- Chiral primary fields correspond to forms annihilated by Δ_∂ — *harmonic forms*.

A standard result in Hodge theory is that each class in the Dolbeault cohomology group,

$$H^{p,q}(M,\mathbb{C}) = \frac{\overline{\partial}\text{-closed}}{\overline{\partial}\text{-exact}}\,, \tag{2.5.46}$$

has a unique harmonic representative. This follows from *Hodge's theorem*. Furthermore, for Kähler manifolds we have $\Delta_\partial = \Delta_{\overline{\partial}}$. Thus, there is a one-to-one correspondence between those elements in the rings that have left-right symmetric, or left-right antisymmetric U(1)-charges, and classes in $H^{p,q}(M,\mathbb{C})$. For superstring theory compactifications to four dimensions, $\mathcal{N} = (2,2)$ superconformal field theories with central charges $c = 9$ are particularly relevant. In these theories, chiral primary fields satisfy

$$h_L \le \frac{3}{2}\,, \qquad h_R \le \frac{3}{2}\,, \tag{2.5.47}$$

with $|q_L| = 2h_L$ and $|q_R| = 2h_R$. Fields with integer U(1)-charges form a subring of the chiral ring. These charges satisfy

$$q_L, q_R \in \{-3,-2,-1,0,1,2,3\}\,, \tag{2.5.48}$$

where $q_L, q_R \geq 0$ in the (c,c) ring and $q_L \leq 0$, $q_R \geq 0$ in the (a,c) ring.

Consequently, the range of integer charges in the (c,c) ring matches precisely the range of (p,q) indices in the Dolbeault cohomology groups $H^{p,q}(M,\mathbb{C})$ of a complex three-dimensional manifold M. The dimensions of these cohomology groups are denoted as $h^{p,q}(M)$.

Furthermore, since the field with $q = 2h = 3$ is unique, as it maps to the unique NS–sector ground state under spectral flow with parameter $\eta = 1$, this correspondence selects manifolds M with

$$h^{0,0}(M) = 1\,, \qquad h^{3,0}(M) = h^{0,3}(M) = h^{3,3}(M) = 1\,. \tag{2.5.49}$$

These properties characterise Calabi–Yau threefolds, which will appear later in section 4.2 as geometric examples of $\mathcal{N} = (2,2)$ superconformal field theories.

Fermionic Spin Structures and BPS Boundary States

For superconformal field theories, the open string cylinder amplitude (2.4.16) and the closed string tree-level amplitude (2.4.21) need to be adjusted, due to the choice of spin structures on the spatial circle S^1 [147, 148] for fermionic fields. To compute the open string cylinder partition function (2.4.16), we take the trace over fermions with anti-periodic boundary conditions, which under the open/closed worldsheet duality maps to closed string fermions in the Neveu–Schwarz sector, also with anti-periodic boundary conditions. In contrast, inserting $(-1)^F$ — where F denotes the fermion number operator — modifies the open string trace to enforce periodic boundary conditions for the fermions. This then corresponds to closed string fermions in the Ramond sector. As a result, the open and closed string channels relate as follows [147–150],

$$\widetilde{Z}^{(\mathrm{R})}_{ab}(l) = \mathrm{Tr}_{\mathcal{H}_{ab}}(-1)^F q^{L_0-\frac{c}{24}}\ , \quad \widetilde{Z}^{(\mathrm{NS})}_{ab}(l) = \mathrm{Tr}_{\mathcal{H}_{ab}}\, q^{L_0-\frac{c}{24}}\ , \tag{2.5.50}$$

where the superscripts distinguish the Ramond and Neveu–Schwarz sectors of the resulting closed string matrix elements.

Let us now specialise to Cardy boundary states in left-right symmetric $\mathcal{N} = (2,2)$ supersymmetric rational conformal field theories, which contain both left- and right-moving copies of the $\mathcal{N} = 2$ super Virasoro algebra, $\mathrm{SVir} \times \overline{\mathrm{SVir}}$. BPS boundary states preserve a diagonal $\mathcal{N} = 2$ subalgebra of $\mathrm{SVir} \times \overline{\mathrm{SVir}}$, which can be determined by an automorphism $\omega : \mathrm{SVir} \to \mathrm{SVir}$.

There are two standard choices for supersymmetric boundary conditions, *A-type* and *B-type*. A-type boundary conditions correspond to an automorphism that sends the U(1) R-current J to $-J$ and exchanges the supercurrents $G^\pm$ by a phase rotation, $G^\pm \mapsto e^{\mp i\alpha} G^\mp$. B-type conditions, on the other hand, preserve J and rotate the supercurrents by the phase rotation $G^\pm \mapsto e^{\pm i\beta} G^\pm$. Together with the usual Virasoro condition $(L_m - \overline{L}_{-m})\,|B\rangle = 0$, the full gluing condition for the boundary state reads [147, 148, 150]:

$$\begin{aligned} &\text{A-type:} \quad (J_m - \overline{J}_{-m})\,|B\rangle = 0\ , \quad (G^\pm_r + i\, e^{\mp i\alpha}\overline{G}^\mp_{-r})\,|B\rangle = 0\ , \\ &\text{B-type:} \quad (J_m + \overline{J}_{-m})\,|B\rangle = 0\ , \quad (G^\pm_r + i\, e^{\pm i\beta}\overline{G}^\pm_{-r})\,|B\rangle = 0\ , \end{aligned} \tag{2.5.51}$$

for all allowed modes m and r.

For these BPS boundary conditions, the open string Hilbert space $\mathcal{H}_{ab}$ organises into supersymmetric multiplets with respect to the open string Hamiltonian $H_{\text{open}} = L_0 - \frac{c}{24}$. For each positive energy level, there are equal numbers of bosonic and fermionic states. Because the trace in the open string index includes $(-1)^F$, the contributions of such paired states cancel, leaving only the index $\text{Tr}_{\mathcal{H}_{ab}}(-1)^F$, which counts unpaired open string zero-energy states.

By the open/closed consistency condition (2.5.50), the closed string channel matrix element must likewise be independent of $\tilde{q} = e^{-2\pi l}$.

The bra state in the closed string tree-level amplitude must be chosen carefully to reflect the same supersymmetry condition as the ket state and the fermionic modes in the open string index, to be consistent with the worldsheet open/closed duality. For an A-type boundary state with phase α, the Hermitian conjugate of (2.5.51) shifts the phase by π,

$$\langle B_a| \, G_r^{\pm} + i\, e^{\mp i(\alpha-\pi)} \overline{G}_{-r}^{\mp} = 0\,, \tag{2.5.52}$$

where we used $(G_r^{\pm})^{\dagger} = G_{-r}^{\mp}$ hence $\langle B_a|$ preserves a different linear-combination of the supercharges than $|B_a\rangle$. To restore the original condition, one acts with the operator $e^{i\pi J_0}$, where J_0 is the left-moving zero-mode of the R-symmetry current[4]. This ensures that bra and ket boundary states preserve compatible combinations of the supercharges [148, 150],

$$\langle B_a| \, e^{i\pi J_0} (G_r^{\pm} + i\, e^{\mp i\alpha} \overline{G}_{-r}^{\mp}) = 0\,, \tag{2.5.53}$$

where we used the relation $e^{\mathrm{i}\beta J_0} G_r^{\pm} = e^{\pm \mathrm{i}\beta} G_r^{\pm} e^{\mathrm{i}\beta J_0}$ for $\beta = \pi$. Similar considerations hold for B-type boundary states. Consequently, the relevant closed string channel matrix element reads[5]

$$\widetilde{Z}_{ab}^{(\mathrm{R})}(l) = \langle B_a| \, e^{i\pi J_0} \tilde{q}^{L_0 + \overline{L}_0 - \frac{c}{12}} \, |B_b\rangle\,. \tag{2.5.54}$$

Since this is $\tilde{q}$-independent, it may be computed in the limit $l \to \infty$, where only Ramond–Ramond ground states contribute. The result relates the open string index to an overlap of boundary states projected onto the RR–sector:

$$\text{Tr}_{\mathcal{H}_{ab}}(-1)^F = {}_{\mathcal{P}}\langle B_a| \, e^{i\pi J_0} \, |B_b\rangle_{\mathcal{P}}\,. \tag{2.5.55}$$

The distinction between A-type and B-type boundary conditions reflects the action of the *mirror automorphism* of the $\mathcal{N} = 2$ super Virasoro algebra [153]. This automorphism

[4]Viewing the ket boundary state $|B_a\rangle$ as a D-brane p, a shift of the phase parameter α by π implies that the corresponding bra boundary state $\langle B_a|$ describes the boundary condition of the anti-D-brane $\overline{p}$ [151]. Multiplying the bra boundary state $\langle B_a|$ by $(-1)^{F_L}$ reverses this operation, converting the anti-D-brane $\overline{p}$ back into the original D-brane p.

[5]Since the combined operator $(-1)^{F_L} \mathrm{e}^{\mathrm{i}\pi J_0}$, with F_L denoting the left-moving fermion number, is central in the theory [152], one may equivalently define an index using the insertion ${}_{\mathcal{P}}\langle B_a| \, (-1)^{F_L} \, |B_b\rangle_{\mathcal{P}}$ [147, 150]. Compared to the insertion of $\mathrm{e}^{\mathrm{i}\pi J_0}$, the contributions to the index may differ by an overall phase. In this work, the insertion $e^{\mathrm{i}\pi J_0}$ is used.

exchanges the right-moving R-current $\overline{J} \mapsto -\overline{J}$ and swaps the right-moving supercurrents $\overline{G}^{\pm} \mapsto \overline{G}^{\mp}$. Thus, mirror symmetry interchanges A-type and B-type boundary conditions [148]. In the latter part of the thesis, we focus primarily on A-type boundary states, which are built from combinations of states with left-right symmetric U(1) charges, typically containing (c,c)–primaries. Under the mirror map, these are mapped to B-type states, which are linear combinations of (c,a)–primaries.

Finally, let us remark that for the unitary rational $\mathcal{N} = (2,2)$ superconformal field theories and their boundary states studied in section 4.2, we assume

(i) The maximal chiral algebra $\mathcal{A}$ contains a subalgebra $\mathcal{A}'$, which includes the purely bosonic part $\mathcal{A}_b$ of the super Virasoro algebra generated by the Grassmann even Generators of SVir, such that the representations $\mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$ decompose into finitely many irreducible representations of $\mathcal{A}'$. We therefore have the hierarchy:

$$\mathcal{A}_b \subset \mathcal{A}' \subset \mathcal{A}. \tag{2.5.56}$$

This ensures that the $\mathcal{N} = (2,2)$ superconformal field theory is rational with respect to $\mathcal{A}'$ as well.

(ii) Each irreducible representation in $\mathrm{HWR}_{\mathcal{H}}(\mathcal{A}')$ contains at most one (c,c) primary of the $\mathcal{N} = 2$ super Virasoro algebra.

Chapter 3

Toroidal Rational Conformal Field Theories

This chapter pursues two principal objectives. The first is to derive explicit expansions of the partition functions of toroidal rational conformal field theories in terms of products of the same fundamental building blocks that appear in the partition functions of rational conformal field theories on a circle S^1, as well as in non-supersymmetric Virasoro minimal models. These building blocks are the characters of minimal extensions of $\widehat{\mathfrak{u}}(1)$ current algebras.

While the decomposition of the partition function of toroidal rational conformal field theories into characters of their maximally extended chiral algebra is well understood — particularly for even-dimensional target space tori — the systematic decomposition into characters of minimal extensions of $\widehat{\mathfrak{u}}(1)$ current algebras is less explored in the literature. For rational toroidal conformal field theories with target space tori T^D, where $D > 1$, such decompositions are finer than those with respect to the full extended chiral algebra; however, they are not unique. As discussed in the main text, this non-uniqueness stems from the fact that the decomposition depends on a choice of a sublattice of the even self-dual lattice $\Gamma^{D,D}$ of signature (D, D) that characterises the toroidal conformal field theory.

Our motivation for considering such minimal decompositions is twofold. First, such minimal decompositions are universally applicable for target space tori of any dimension, as long as the target space torus is related to a rational toroidal conformal field theory. Second, as these decompositions do not rely on the symmetry of the entire extended chiral algebra, such finer decompositions can be useful to explicitly describe specific orbifolds or orientifolds of such rational toroidal conformal field theories. As will be shown in section 4.2, in the context of boundary conformal field theory, boundary states often preserve only a proper subalgebra of the maximally extended chiral algebra. Consequently, a detailed understanding of the family of possible subalgebras is highly desirable. The framework developed here makes it possible to describe such families explicitly for toroidal rational conformal field theories.

The second objective of this chapter is to demonstrate how, given an expansion of toroidal partition functions into characters of minimal extensions of $\widehat{\mathfrak{u}}(1)$ current algebras, one obtains an equivalent expansion in terms of so-called *Ray class theta functions*. This representation provides the partition function of a toroidal rational conformal field theory with an explicit interpretation in the language of algebraic number theory and class field theory, thereby illustrating the deep connection between rational conformal field theory and the arithmetic structure of number fields.

3.1 Rational Conformal Field Theory on a Circle

The characters of the extensions of the $\widehat{\mathfrak{u}}(1)$ current algebras relevant for this work already appear in the context of two-dimensional conformal field theories describing a real free boson ϕ compactified on a circle. This topic is discussed in detail, for instance, in Refs. [95, 96, 154]. The partition function for the free boson on a circle reads[1]

$$Z_{S^1}(\tau;R) = \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{1}{2}(\frac{m}{R}+\frac{Rn}{2})^2} \overline{q}^{\frac{1}{2}(\frac{m}{R}-\frac{Rn}{2})^2} , \qquad q = e^{2\pi i\tau} , \tag{3.1.1}$$

where τ denotes the modular parameter of the worldsheet torus, which is valued in the Siegel upper half plane, $\eta(\tau)$ is the Dedekind eta function, and R is the radius of the circle S^1. It is well known that for radii of the form $R = \sqrt{\frac{2p'}{p}}$, with p and p' positive coprime integers, the conformal field theory becomes rational. This rationality arises because the $\widehat{\mathfrak{u}}(1)$ current algebra of the free boson, generated by the chiral primary field $\partial\phi$, is extended by the chiral primary fields $e^{\pm i\sqrt{2pp'}\,\phi}$. The precise form of the exponent is fixed by imposing the condition of mutual locality. Following ref. [95], we denote this extension of the $\widehat{\mathfrak{u}}(1)$ current algebra by $\widehat{\mathfrak{u}}(1)_{pp'}$, and we often refer to it as a minimal extension of the $\widehat{\mathfrak{u}}(1)$ current algebra. Note, however, that for *abelian* Kac-Moody algebras, such as the current algebra of a free boson $\widehat{\mathfrak{u}}(1)$, the notion of a level has no intrinsic meaning, as it can be arbitrarily changed by a rescaling of the generators. Therefore, the notation $\widehat{\mathfrak{u}}(1)_{pp'}$ should not be confused with the level of an (non-abelian) affine Lie algebra. Thus, the notation $\widehat{\mathfrak{u}}(1)_{pp'}$ does not indicate a level, but rather labels the specific minimal extension occurring for the rational compactification radius $R = \sqrt{\frac{2p'}{p}}$.

For these special radii, the partition function (3.1.1) simplifies to

$$Z^{\text{rat}}_{S^1}(\tau;p,p') = \sum_{\lambda\in\{0,1,\ldots,2pp'-1\}} \mathcal{K}_{\omega\lambda,pp'}(\tau)\, \overline{\mathcal{K}_{\lambda,pp'}(\tau)} , \tag{3.1.2}$$

where $\omega = pr + p's$ for integers r and s satisfying the *Bézout identity* $pr - p's = 1$. The functions $\mathcal{K}_{\lambda,\alpha}(\tau)$ are the characters of the irreducible highest weight representations of $\widehat{\mathfrak{u}}(1)_\alpha$ defined for any positive integer α by

$$\mathcal{K}_{\lambda,\alpha}(\tau) = \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{\frac{(2\alpha n+\lambda)^2}{4\alpha}} , \qquad \lambda \in \mathbb{Z} , \tag{3.1.3}$$

with the symmetry properties

$$\mathcal{K}_{\lambda,\alpha}(\tau) = \mathcal{K}_{\lambda+2\alpha,\alpha}(\tau) , \qquad \mathcal{K}_{\lambda,\alpha}(\tau) = \mathcal{K}_{-\lambda,\alpha}(\tau) . \tag{3.1.4}$$

[1]Throughout this chapter we adopt the convention $\alpha' = 2$.

The characters transform under modular transformations as

$$K_{\lambda,\alpha}(\tau+1) = e^{2\pi \mathrm{i}\left(\frac{\lambda^2}{4\alpha}-\frac{1}{24}\right)} K_{\lambda,\alpha}(\tau) \tag{3.1.5}$$

$$K_{\lambda,\alpha}(-1/\tau) = \sum_{\mu=0}^{2\alpha-1} \frac{1}{\sqrt{2\alpha}} e^{\frac{\pi \mathrm{i}\lambda\mu}{\alpha}} K_{\mu,\alpha}(\tau)\,. \tag{3.1.6}$$

The structure of these extended characters can be understood as follows: The factor $\frac{q^{h_\lambda}}{\eta(\tau)}$ with conformal weight $h_\lambda = \frac{\lambda^2}{4pp'}$ together with the Dedekind η function, encodes the Virasoro conformal family generated by the free boson oscillators acting on the corresponding primary state. For rational squared radius, the chiral algebra is extended by additional generators $\Gamma^{\pm}$. The requirement of mutual locality in the operator product expansion (OPE) with the generators of the extended chiral algebra, forces the primary fields to be of the form $e^{\mathrm{i}\gamma\phi}$, with $\gamma = \frac{\lambda}{\sqrt{2pp'}}$, $\lambda \in \mathbb{Z}$. A shift $\lambda \mapsto \lambda + 2pp'$ corresponds to multiplying the primary by $e^{\mathrm{i}\sqrt{2pp'}\,\phi}$, which inserts the extended generator Γ^+ and thus produces an extended descendant state. Therefore, the set of extended primary fields for the extended chiral algebra is naturally labeled by $0 \leq \lambda < 2pp'$. All values of λ outside this range correspond to extended descendants built from primaries within this range. Each character $\mathcal{K}_{\lambda,pp'}$, with $\lambda \in \{0,\ldots,2pp'-1\}$, thus describes one extended primary and its entire tower of extended descendants. This illustrates explicitly how the infinite set of Virasoro primaries in the free boson theory compactified on S^1 organises itself into a finite number of extended primaries — one of the defining features of a rational conformal field theory.

Let us now show explicitly how to derive the expansion (3.1.2), starting from the general form (3.1.1). In eq. (3.1.1), the sum runs over a two-dimensional lattice spanned by the linear combinations $pm + p'n$ and $pm - p'n$, where m and n parametrize the standard $\mathbb{Z}^2$ lattice. We wish to find an integral basis transformation matrix M, with $\det M = 1$, such that the transformed lattice basis has a triangular form,

$$\begin{pmatrix} p & p' \\ p & -p' \end{pmatrix} \cdot M = \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix}\,. \tag{3.1.7}$$

A suitable choice is

$$M = \begin{pmatrix} -p' & r \\ -p & s \end{pmatrix}\,, \tag{3.1.8}$$

where (r,s) is a Bézout pair satisfying

$$pr - p's = \gcd(p,p') = 1\,, \tag{3.1.9}$$

since by assumption p and p' are coprime. With this transformation, the basis takes the form

$$\begin{pmatrix} p & p' \\ p & -p' \end{pmatrix} \cdot M = \begin{pmatrix} -2pp' & \omega \\ 0 & 1 \end{pmatrix}\,, \tag{3.1.10}$$

where we defined $pr + p's = \omega$.

The transformed lattice sum in the partition function then takes the form

$$Z^{\text{rat}}_{S^1}(\tau;p,p') = \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{1}{4pp'}(-2pp'(m-\frac{\omega n}{2pp'}))^2} \overline{q}^{\frac{1}{4pp'}(n)^2} \,. \tag{3.1.11}$$

Finally, performing a change of summation variables,

$$n \mapsto 2pp'n + \delta\,, \qquad \delta \in \{\,0,1,\ldots,2pp'-1\,\}\,, \tag{3.1.12}$$

yields the desired factorised form of the partition function

$$Z^{\text{rat}}_{S^1}(\tau;p,p') = \frac{1}{|\eta(\tau)|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{1}{4pp'}(-2pp'm+\omega\delta)^2} \overline{q}^{\frac{1}{4pp'}(2pp'n+\delta)^2} \tag{3.1.13}$$

$$Z^{\text{rat}}_{S^1}(\tau;p,p') = \sum_{\lambda\in\{\,0,1,\ldots,2pp'-1\,\}} K_{\omega\lambda,pp'}(\tau)\overline{K_{\lambda,pp'}(\tau)}\,. \tag{3.1.14}$$

From this expansion, we can immediately identify the spectrum of the finite set of extended primary fields encoded in the extended characters $\mathcal{K}_{\lambda,pp'}$. Each such character has the lowest conformal weight $h_\lambda = \frac{\lambda^2}{4pp'}$ and includes all corresponding descendant contributions with respect to the extended chiral algebra $\widehat{\mathfrak{u}}(1)_{pp'}$. In particular, we see that the partition function is diagonal whenever $\omega = 1 \bmod 2pp'$. This occurs for radii of the form $R_{S^1} = \sqrt{2k}$, with $k \in \mathbb{N}$. Note that for this class of rational conformal field theories, the charge conjugation modular invariant coincides with the diagonal modular invariant, since the charge conjugation matrix satisfies $C_{ij} = \delta_{j,\,2pp'-i}$, and the symmetry $\mathcal{K}_{\lambda,pp'} = \mathcal{K}_{2pp'-\lambda,pp'}$ holds.

3.2 Toroidal Partition Function

Let us now turn to the class of toroidal compactifications of the 26-dimensional bosonic string, that is, to the case of D free bosonic fields compactified on a D-dimensional flat torus T^D. Such compactifications result in an effective $(26-D)$-dimensional theory. We follow the standard conventions and notation as, for instance, presented in [15]. The relevant worldsheet theory is described by the bosonic non-linear sigma model in conformal gauge,

$$S = -\frac{1}{8\pi}\int d^2\sigma \left(\eta^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\right)\,, \tag{3.2.1}$$

where $G_{\mu\nu}$ denotes the target space metric, $B_{\mu\nu}$ is a constant antisymmetric two-form field (the *B-field*), and $\epsilon^{\alpha\beta}$ is the Levi-Civita symbol in two dimensions (with $\epsilon^{01} = -1$). Let us focus on the internal compact directions, labeled by capital indices X^I. Due to the periodic identification on the torus T^D, the internal coordinates obey

$$\begin{aligned} X^I &\sim X^I + 2\pi\sum_{i=1}^{D} n^i e^I_i = X^I + 2\pi L^I \\ L^I &= \sum_{i=1}^{D} n^i e^I_i\,,\ n_i \in \mathbb{Z}\,, \end{aligned} \tag{3.2.2}$$

where $\mathbf{e}_i = \{e_i^I\}$ denotes a basis vector of the D-dimensional lattice Λ defining the torus as the quotient

$$T^D = \mathbb{R}^D/2\pi\Lambda\,. \tag{3.2.3}$$

The vector L^I encodes the *winding numbers* of the string around the compact directions.

A mode expansion of the bosonic string consistent with this periodicity is given by

$$\begin{aligned} X^I &= X_L^I + X_R^I \\ X_R^I &= x_R^I + p_R^I(\tau+\rho) + \mathrm{i}\sum_{n\neq 0}\frac{1}{n}\overline{\alpha}_n^I e^{-\mathrm{i}n(\tau+\rho)} \\ X_L^I &= x_L^I + p_L^I(\tau-\rho) + \mathrm{i}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-\mathrm{i}n(\tau-\rho)}\,, \end{aligned} \tag{3.2.4}$$

where p_R^I and p_L^I denote the right- and left-moving momenta along the compact directions, respectively. They are determined by the compactification data as

$$\begin{aligned} (p_I)_{R,L} &= G_{IJ}p^J + \frac{1}{2}B_{IJ}L^J \pm \frac{1}{2}\left(G_{IJ} \mp B_{IJ}\right)L^J \\ (p_I)_{R,L} &= \pi_I \pm \frac{1}{2}\left(G_{IJ} \mp B_{IJ}\right)L^J\,, \end{aligned} \tag{3.2.5}$$

where π_I is the center of mass momentum along the internal compact directions, which generates translations. The condition that the wavefunction $e^{\mathrm{i}x^I\pi_I}$ be single-valued under the lattice identification requires that $L^I\pi_I \in \mathbb{Z}$, implying that the allowed momenta π_I lie in the dual lattice Λ^*,

$$\pi_I = m_i e_I^{*i}\,,\ m_i \in \mathbb{Z}\,, \tag{3.2.6}$$

where the dual basis $\mathbf{e}^{*i} = \{e_I^{*i}\}$ satisfies $\mathbf{e}_i \cdot \mathbf{e}^{*j} = \delta_i^{\ j}$. We can thus rewrite (3.2.5) as

$$(p_I)_{R,L} = e_I^{*i}\left(m_i \pm \frac{1}{2}g_{ij}n^j - \frac{1}{2}b_{ij}n^j\right)\,, \tag{3.2.7}$$

where $g_{ij} = e_i^I G_{IJ} e_j^J$ and $b_{ij} = e_i^I B_{IJ} e_j^J$ are the metric and B-field, expressed in the lattice basis. The squared norms of the momenta then read, in matrix notation,

$$p_{R,L}^2 = \mathbf{m}^T\mathbf{g}^{-1}\mathbf{m} + \frac{1}{4}\mathbf{n}^T(\mathbf{g}-\mathbf{b}\mathbf{g}^{-1}\mathbf{b})\mathbf{n} + \mathbf{n}^T\mathbf{b}\mathbf{g}^{-1}\mathbf{m} \pm \mathbf{n}^T\mathbf{m}\,. \tag{3.2.8}$$

They are related to the L_0 zero-mode Virasoro generator by

$$\begin{aligned} L_0 &= N_L + \frac{\mathbf{p_L}^2}{2} \\ \overline{L}_0 &= N_R + \frac{\mathbf{p_R}^2}{2}\,, \end{aligned} \tag{3.2.9}$$

where $N_{L,R}$ denote the left- and right-moving oscillator excitation numbers.

One can now combine the left- and right-moving momenta into the $2D$-dimensional lattice vector $\mathbf{P} = (\mathbf{p_L}, \mathbf{p_R})$, which spans the lattice $\Gamma^{D,D}$. Equipped with the inner product in Lorentzian signature,

$$\mathbf{P} \cdot \mathbf{P}' = \sum_I (p_L^I p_L'^J - p_R^I p_R'^J)\delta_{IJ}\,, \tag{3.2.10}$$

the lattice is even and integral:

$$\mathbf{P} \cdot \mathbf{P}' = \sum_{i=1}^{D} (m_i n'^i + m_i' n^i)\,. \tag{3.2.11}$$

This bilinear form does not depend explicitly on the choice of the metric g_{ij} or the B-field b_{ij} and thus defines a unique lattice structure for each compactification. For instance, taking $e_i^I = \sqrt{2}\delta_i{}^I$ and $b_{ij} = 0$ yields the factorised form

$$\Gamma^{D,D} = \bigotimes_{i=1}^{D} \Gamma_i^{1,1}, \quad \text{where} \quad \Gamma_i^{1,1} = \langle \tfrac{1}{\sqrt{2}}(1,1),\, \tfrac{1}{\sqrt{2}}(1,-1)\rangle\,. \tag{3.2.12}$$

This lattice is manifestly self-dual since its basis also generates its dual lattice, showing that $\Gamma^{D,D}$ is an even self-dual Lorentzian lattice for all choices of g_{ij} and b_{ij}. All self-dual even Lorentzian lattices $\Gamma^{D,D}$ can be obtained by an $O(D,D)$ transformations of a reference lattice (e.g., the lattice obtained for the choice $g_{ij} = 2\delta_{ij}$, $b_{ij} = 0$). The bosonic string theory is also invariant under separate $O(D)$ transformations of $\mathbf{p_L}$, $\mathbf{p_R}$. Additionally, the theory admits discrete duality transformations acting on g_{ij} and b_{ij}, which can be shown to generate the group $O(D,D,\mathbb{Z})$. The moduli space of bosonic toroidal compactifications is hence

$$\mathcal{M}_{T^D} = \frac{O(D,D)}{O(D)\times O(D)}/O(D,D,\mathbb{Z})\,, \tag{3.2.13}$$

and is called *Narain moduli space* [155, 156].

Using eq. (3.2.9) and recalling the definition of the partition function (2.2.15), the partition function for a toroidal compactification is given by

$$Z_{T^D}(\tau,\overline{\tau}) = \frac{1}{|\eta(\tau)|^{2D}} \sum_{\boldsymbol{P}=(\boldsymbol{p_L},\boldsymbol{p_R})\in\Gamma^{D,D}} q^{\frac{1}{2}\boldsymbol{p_L}^2}\overline{q}^{\frac{1}{2}\boldsymbol{p_R}^2}\,, \qquad q = e^{2\pi i\tau}\,. \tag{3.2.14}$$

For the compactified theory to describe a rational conformal field theory, it must be possible to decompose the partition function into a finite sum of products of holomorphic and anti-holomorphic characters. Concretely, this requires that the lattice $\Gamma^{D,D}$ admits a decomposition into a direct sum of mutually orthogonal even D-dimensional lattices $\Gamma^{L,0}$ and $\Gamma^{0,R}$, such that their sum is a sublattice

$$\Gamma^{L,0} \oplus \Gamma^{0,R} \subseteq \Gamma^{D,D}\,. \tag{3.2.15}$$

By the well-known gluing lattice construction [157, 158], one can decompose the full lattice $\Gamma^{D,D}$ into cosets with respect to $\Gamma^{L,0} \oplus \Gamma^{0,R}$ using so-called glue vectors $\boldsymbol{v} = (\boldsymbol{v_L}, \boldsymbol{v_R})$,

$$\Gamma^{D,D} = \bigoplus_{\boldsymbol{v}_i \in \Gamma^{D,D}/(\Gamma^{L,0} \oplus \Gamma^{0,R})} \left(\Gamma^{L,0} \oplus \Gamma^{0,R} + \boldsymbol{v}_i\right) , \tag{3.2.16}$$

where the number of cosets is equal to the index of $\Gamma^{L,0} \oplus \Gamma^{0,R}$ inside $\Gamma^{D,D}$, i.e., the number of equivalence classes $|\Gamma^{D,D}/\left(\Gamma^{L,0} \oplus \Gamma^{0,R}\right)|$.

The partition function for rational toroidal theories can then be decomposed as

$$Z^{\mathrm{rat}}_{T^D}(\tau, \overline{\tau}) = \frac{1}{|\eta(\tau)|^{2D}} \sum_{(\boldsymbol{v_L}, \boldsymbol{v_R}) \in \Gamma^G} \sum_{\substack{\boldsymbol{w_L} \in \Gamma^{L,0} \\ \boldsymbol{w_R} \in \Gamma^{0,R}}} q^{\frac{1}{2}(\boldsymbol{w_L}+\boldsymbol{v_L})^2} \overline{q}^{\frac{1}{2}(\boldsymbol{w_R}+\boldsymbol{v_R})^2} , \tag{3.2.17}$$

where $\Gamma^G = \Gamma^{D,D}/\left(\Gamma^{L,0} \oplus \Gamma^{0,R}\right)$ and $\boldsymbol{v_L} \in (\Gamma^{L,0})^*/\Gamma^{L,0}$, $\boldsymbol{v_R} \in (\Gamma^{0,R})^*/\Gamma^{0,R}$.

Defining lattice theta functions for the holomorphic and anti-holomorphic sectors, respectively,

$$\Theta^{\Gamma^{L,0}}_{\boldsymbol{v_L}}(\tau) = \frac{1}{\eta(\tau)^D} \sum_{\boldsymbol{w_L} \in \Gamma^{L,0}} q^{\frac{1}{2}(\boldsymbol{w_L}+\boldsymbol{v_L})^2} , \quad \overline{\Theta}^{\Gamma^{0,R}}_{\boldsymbol{v_R}}(\overline{\tau}) = \frac{1}{\eta(\overline{\tau})^D} \sum_{\boldsymbol{w_R} \in \Gamma^{0,R}} \overline{q}^{\frac{1}{2}(\boldsymbol{w_R}+\boldsymbol{v_R})^2} , \tag{3.2.18}$$

the partition function (3.2.17) takes the form

$$Z^{\mathrm{rat}}_{T^D}(\tau) = \sum_{\boldsymbol{v}=(\boldsymbol{v_L}, \boldsymbol{v_R}) \in \Gamma^G} \Theta^{\Gamma^{L,0}}_{\boldsymbol{v_L}}(\tau) \, \overline{\Theta}^{\Gamma^{0,R}}_{\boldsymbol{v_R}}(\overline{\tau}) . \tag{3.2.19}$$

In this way, the partition function is written explicitly as a finite sum of products of holomorphic and anti-holomorphic theta functions. These theta functions are naturally interpreted as the characters of the maximally extended chiral and anti-chiral algebras of the rational toroidal conformal field theory. The lattices $\Gamma^{L,0}$ and $\Gamma^{0,R}$ serve as the root lattices for these extended chiral algebras. The associated chiral algebra is a W-algebra of the form $\mathcal{W}(2, |\boldsymbol{\kappa}_1|^2, \ldots, |\boldsymbol{\kappa}_D|^2)$ labelled in terms of the squares of the generators $\sqrt{2}\boldsymbol{\kappa}_i$, $i = 1, \ldots, D$, of the lattice $\Gamma^{L,0}$, see for instance ref. [93, 96]. Analogously, the anti-chiral W-algebra $\mathcal{W}(2, |\tilde{\boldsymbol{\kappa}}_1|^2, \ldots, |\tilde{\boldsymbol{\kappa}}_D|^2)$ is associated with the root vectors $\sqrt{2}\tilde{\boldsymbol{\kappa}}_i$, $i = 1, \ldots, D$, of the lattice $\Gamma^{0,R}$.

3.3 Decomposition of Rational Toroidal Partition Functions

In this section, our goal is to derive an explicit decomposition of the partition function (3.2.19) for rational toroidal conformal field theories with target space T^D into a finite sum of products of the characters (3.1.3). This result appeared in the work [1]. Concretely, we wish to express the partition function in the form

$$Z^{\text{rat}}_{T^D}(\tau) = \sum_{\lambda \in I} \mathcal{K}_{\lambda_1,\alpha_1}(\tau)\dots\mathcal{K}_{\lambda_D,\alpha_D}(\tau)\,\overline{\mathcal{K}_{\tilde{\lambda}_1,\tilde{\alpha}_1}(\tau)}\dots\overline{\mathcal{K}_{\tilde{\lambda}_D,\tilde{\alpha}_D}(\tau)}\ , \tag{3.3.1}$$

where $\alpha_1,\dots,\alpha_D,\tilde{\alpha}_1,\dots,\tilde{\alpha}_D$ are positive integers and I is a finite set of $2D$-tuples $(\lambda_1,\dots,\lambda_D,\tilde{\lambda}_1,\dots,\tilde{\lambda}_D)$.

The central idea behind the decomposition (3.3.1) is the existence of an orthogonal (not necessarily primitive) sublattice $O_L \oplus O_R$, contained in the root lattice $\Gamma^{L,0}\oplus\Gamma^{0,R}$ of the even self-dual charge lattice $\Gamma^{D,D}$ of signature (D,D), which characterises the rational conformal field theory with target space T^D. We demonstrate that the individual summands in (3.3.1) are in one-to-one correspondence with the cosets of $\Gamma^{D,D}$ modulo $O_L \oplus O_R$.

It follows immediately that there is always a distinguished, universal term in this expansion that corresponds to the sublattice $O_L \oplus O_R$ itself, representing the contribution from the generators of the associated subalgebra. We present an explicit constructive method for determining such decompositions in any dimension D. As outlined in the main text, this construction amounts to identifying a suitable $2D$-tuple of positive integers $(\alpha_1,\dots,\alpha_D,\tilde{\alpha}_1,\dots,\tilde{\alpha}_D)$ and a $2D\times 2D$ integer matrix $\boldsymbol{H}$ derived from the charge lattice of the toroidal rational conformal field theory. By computing the Hermite normal form of $\boldsymbol{H}$ — a standard procedure implemented in many computer algebra systems — one can then unambiguously read off the expansion (3.3.1). We illustrate this approach with explicit examples in low dimensions, focusing on the cases $D = 2$ and $D = 3$ for clarity.

For a generic point in the Narain moduli space of toroidal conformal field theories, a non-trivial sublattice $\Gamma^{L,0} \oplus \Gamma^{0,R}$ does not exist, and the theory is thus non-rational. In general, a toroidal conformal field theory has an infinite tower of Virasoro primary fields: each lattice vector in $\Gamma^{D,D}$ corresponds to a Virasoro primary field with conformal weights given by $(h_L, h_R) = (\frac{\boldsymbol{p_L}^2}{2}, \frac{\boldsymbol{p_R}^2}{2})$.

A point in the Narain moduli space is fully specified by the Riemannian flat torus metric $\boldsymbol{g} = \boldsymbol{\Lambda}^T\boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}$ is the basis matrix whose columns are the generators of the torus lattice $\Lambda \cong H_1(T^D,\mathbb{Z})$, and by the antisymmetric B-field, written as $\boldsymbol{b} = \boldsymbol{\Lambda}^T\boldsymbol{B}\boldsymbol{\Lambda}$. Comparing to (3.2.7) and following the notation of [93], the left- and right-moving charge vectors take the explicit form

$$\boldsymbol{p_L} = \frac{1}{\sqrt{2}}\left(\boldsymbol{\mu} - \boldsymbol{B}\boldsymbol{\lambda} - \boldsymbol{\lambda}\right)\ , \qquad \boldsymbol{p_R} = \frac{1}{\sqrt{2}}\left(\boldsymbol{\mu} - \boldsymbol{B}\boldsymbol{\lambda} + \boldsymbol{\lambda}\right)\ , \tag{3.3.2}$$

with $(\boldsymbol{\mu},\boldsymbol{\lambda}) \in \Lambda^* \oplus \Lambda$.

Building on the results in [93, 94, 159, 160], it follows that a toroidal conformal field theory is rational if and only if all conformal weights (h_L, h_R) of the Virasoro primaries are rational. Equation (3.2.8) then implies that the torus metric and B-field must be rational-valued, when written in the lattice basis, i.e.,

$$\boldsymbol{g} \in \mathrm{Sym}_D(\mathbb{Q}) \ , \qquad \boldsymbol{b} \in \mathrm{Skew}_D(\mathbb{Q}) \ , \tag{3.3.3}$$

where $\mathrm{Sym}_D(\mathbb{Q})$ and $\mathrm{Skew}_D(\mathbb{Q})$ denote the spaces of $D \times D$ rational symmetric and anti-symmetric matrices, respectively.

Using the rationality of all conformal weights and the explicit form of the charge vectors (3.3.2), any partition function (3.2.14) of a rational toroidal conformal field theory has the general form

$$Z^{\mathrm{rat}}_{T^D}(\tau) = \frac{1}{|\eta(\tau)|^{2D}} \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^D} \prod_{i=1}^{D} q^{\frac{1}{4a_i}(\boldsymbol{a}_i^T\boldsymbol{m}+\boldsymbol{b}_i^T\boldsymbol{n})^2} \prod_{j=1}^{D} \overline{q}^{\frac{1}{4\tilde{a}_j}(\tilde{\boldsymbol{a}}_i^T\boldsymbol{m}+\tilde{\boldsymbol{b}}_i^T\boldsymbol{n})^2} \ , \tag{3.3.4}$$

with integers $a_i, \tilde{a}_j \in \mathbb{Z}_{>0}$ and integral vectors $\boldsymbol{a}_i, \tilde{\boldsymbol{a}}_j, \boldsymbol{b}_i, \tilde{\boldsymbol{b}}_j \in \mathbb{Z}^D$ for $i, j = 1, \ldots, D$.

The lattices $\Gamma^{L,0}$ and $\Gamma^{0,R}$, which are the root lattices of the maximally extended chiral algebra of the toroidal rational conformal field theory, contain finite index sublattices O_L and O_R whose generators are pairwise orthogonal. In the following we denote the generators of O_L and O_R by $\boldsymbol{o}_1, \ldots, \boldsymbol{o}_D$ and $\tilde{\boldsymbol{o}}_1, \ldots, \tilde{\boldsymbol{o}}_D$, respectively. Due to the evenness of $\Gamma^{D,D}$, these generators have even norm. Note that the orthogonal sublattice O_L and O_R are the root lattices of products of $\hat{\mathfrak{u}}(1)_{\alpha_i}$ algebras for some integers α_i. The following summarises the inclusion of sublattices,

$$O_L \oplus O_R \subset \Gamma^{L,0} \oplus \Gamma^{0,R} \subset \Gamma^{D,D} \ . \tag{3.3.5}$$

The contribution from the direct product of the $\widehat{\mathfrak{u}}(1)_\alpha$ subalgebras associated with $O_L \oplus O_R$ to the partition function is

$$Z^{O_L\oplus O_R}_{T^D}(\tau) = \frac{1}{|\eta(\tau)|^{2D}} \prod_{i=1}^{D} \left(\sum_{k\in\mathbb{Z}} q^{\alpha_i k^2}\right) \left(\sum_{k\in\mathbb{Z}} \overline{q}^{\tilde{\alpha}_i k^2}\right) = \prod_{i=1}^{D} \mathcal{K}_{0,\alpha_i}(\tau)\overline{\mathcal{K}_{0,\tilde{\alpha}_i}(\tau)} \ . \tag{3.3.6}$$

where $\mathcal{K}_{\lambda,\alpha}$ are the $\widehat{\mathfrak{u}}(1)_\alpha$ characters (3.1.3) and $\alpha_i := \frac{1}{2}\boldsymbol{o}_i^2$ and $\tilde{\alpha}_i := -\frac{1}{2}\tilde{\boldsymbol{o}}_i^2$, $i = 1, \ldots, D$.

To complete the lattice sum in the partition function (3.3.4), we must include those lattice points $\Gamma^{D,D}$ that do not reside in the sublattice $O_L \oplus O_R$. Denote by $n+1$ the index of the sublattice $O_L \oplus O_R$ embedded in $\Gamma^{D,D}$, and let $\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_n \in \Gamma^{D,D}$ and $\boldsymbol{\rho}_0 \in O_L \oplus O_R$ represent the cosets of the quotient $\Gamma^{D,D}/O_L \oplus O_R$. Then for all lattice points in $\boldsymbol{p} \in \Gamma^{D,D}$, $\boldsymbol{p} \notin O_L \oplus O_R$, we clearly have $\boldsymbol{p} - \boldsymbol{\rho}_a \in O_L \oplus O_R$ for some $a \in \{1, \ldots n\}$. We can expand the coset representatives $\boldsymbol{\rho}_a$ in terms of the generators $\boldsymbol{o}_i$ and $\tilde{\boldsymbol{o}}_i$ as

$$\boldsymbol{\rho}_a = \rho_{a,1}\boldsymbol{o}_1 + \ldots + \rho_{a,n}\boldsymbol{o}_n + \tilde{\rho}_{a,1}\tilde{\boldsymbol{o}}_1 + \ldots + \tilde{\rho}_{a,n}\tilde{\boldsymbol{o}}_n \ , \qquad a = 0, \ldots, n \ , \tag{3.3.7}$$

with rational coefficients $\rho_{a,i}$ and $\tilde{\rho}_{a,i}$. Clearly without loss of generality, we can assume $\rho_{a,i}, \tilde{\rho}_{a,i} \in [0, 1)$, as an integer shift to any of these coefficients does not change the equivalence class of $\boldsymbol{\rho}_a$ in the quotient $\Gamma^{D,D}/O_L \oplus O_R$. We also have that the product of $\boldsymbol{\rho}_a$ with the generators $\boldsymbol{o}_i$ and $\tilde{\boldsymbol{o}}_j$ must be an integer, since $\boldsymbol{\rho}_a \in \Gamma^{D,D}$. Then, using $\boldsymbol{o}_i^2 = 2\alpha_i$ and $\tilde{\boldsymbol{o}}_i^2 = -2\tilde{\alpha}_i$, we obtain

$$\boldsymbol{\rho}_a \boldsymbol{o}_i = 2\rho_{a,i}\alpha_i \in \mathbb{Z} \ , \qquad \tilde{\boldsymbol{\rho}}_a \boldsymbol{o}_i = -2\tilde{\rho}_{a,i}\tilde{\alpha}_i \in \mathbb{Z} \ , \tag{3.3.8}$$

and hence

$$\begin{aligned} \rho_{a,i} &= \frac{\lambda_{a,i}}{2\alpha_i} \quad \text{with} \quad \lambda_{a,i} \in \{0,\ldots,2\alpha_i - 1\} \ , \\ \tilde{\rho}_{a,i} &= \frac{\tilde{\lambda}_{a,i}}{2\tilde{\alpha}_i} \quad \text{with} \quad \tilde{\lambda}_{a,i} \in \{0,\ldots,2\tilde{\alpha}_i - 1\} \ . \end{aligned} \tag{3.3.9}$$

We conclude that the D-dimensional vector $(\rho_{a,1},\cdots,\rho_{a,D})$ in the basis of $\boldsymbol{o}_i$ is an element of O_L^*, respectively $(\tilde{\rho}_{a,1},\cdots,\tilde{\rho}_{a,D}) \in O_R^*$, where O_L^* and O_R^* are the dual lattices.

We can now rewrite the lattice sum in the partition function (3.3.4) as a sum over the sublattice $O_L \oplus O_R$ (3.3.6), together with the contribution from remaining lattice points in $\Gamma^{D,D}$, which is a sum over $\boldsymbol{\rho}_a + (O_L \oplus O_R)$ for all $a = 1,\ldots,n$. The partition function then reads

$$\begin{aligned} Z_{T^D}^{\text{rat}}(\tau) &= Z_{T^D}^{O_L\oplus O_R}(\tau) + \frac{1}{|\eta(\tau)|^{2D}} \sum_{a=1}^{n} \prod_{i=1}^{D} \left(\sum_{k\in\mathbb{Z}} q^{\alpha_i \left(k+\frac{\lambda_{a,i}}{2\alpha_i}\right)^2} \right) \left(\sum_{k\in\mathbb{Z}} \overline{q}^{\tilde{\alpha}_i \left(k+\frac{\tilde{\lambda}_{a,i}}{2\tilde{\alpha}_i}\right)^2} \right) \\ &= \sum_{a=0}^{n} \prod_{i=1}^{D} \mathcal{K}_{\lambda_{a,i},\alpha_i}(\tau)\, \overline{\mathcal{K}_{\tilde{\lambda}_{a,i},\tilde{\alpha}_i}(\tau)} \ , \end{aligned} \tag{3.3.10}$$

which is the general form (3.3.1) we were after.

To recapitulate, the derivation of the expansion (3.3.10) proceeds in two steps. First, one determines the generators $\boldsymbol{o}_i$ and $\tilde{\boldsymbol{o}}_i$, $i = 1,\ldots,D$, spanning the sublattice $O_L \oplus O_R$. Subsequently, one constructs the set of coset representatives $\boldsymbol{\rho}_a$, $a = 1,\ldots,n$. This data uniquely determines the expansion (3.3.10) for the given choice of orthogonal sublattice. However, the expansion (3.3.10) of the rational toroidal partition function is not canonical, as it depends explicitly on the choice of $O_L \oplus O_R$, following eq. (3.3.5). Distinct choices of the orthogonal sublattice yield inequivalent expansions (3.3.1), expressed as different products of the characters (3.1.3) corresponding to extended $\widehat{\mathfrak{u}}(1)$ current algebras.

If the sublattice O_L is isometric to $\Gamma^{L,0}$, then the lattice generators $\sqrt{2}\boldsymbol{\kappa}_i$ of $\Gamma^{L,0}$ can be identified with the generators $\boldsymbol{o}_i$ — see above eq. (3.3.5). In this case, the holomorphic characters of the maximal extended chiral algebra defined in eq. (3.2.18) factorise into the characters (3.1.3) of extensions of the $\widehat{\mathfrak{u}}(1)$ current algebra, namely

$$\Theta_{\boldsymbol{w_L}}^{O_L}(\tau) = \prod_{i=1}^{D} \mathcal{K}_{\lambda_i,|\boldsymbol{\kappa}_i^2|}(\tau) \ , \tag{3.3.11}$$

where the labels λ_i are obtained from the expansion $\boldsymbol{w_L} = \frac{1}{\boldsymbol{\kappa}_i^2}(\lambda_1\boldsymbol{o}_i + \ldots + \lambda_D\boldsymbol{o}_D)$. If in addition, the sublattice O_R is isometric to $\Gamma^{0,R}$, then our construction coincides with the gluing construction [157, 158].

If however the lattice O_L is not isometric to the lattice $\Gamma^{L,0}$, then the holomorphic characters $\Theta^{\Gamma^{L,0}}_{\boldsymbol{w_L}}(\tau)$ are not equal to products of characters $\mathcal{K}_{\lambda,\alpha}(\tau)$ of extended current algebras $\widehat{\mathfrak{u}}(1)_\alpha$, but instead can be expanded as a sum of such products. In this case, the more general construction described above must be applied to arrive at the decomposition (3.3.1).

Geometrically, factorisations of the form (3.3.11) of the extended chiral characters always occur if the metric $\boldsymbol{G}$ of the target space torus T^D is both rational and diagonal, and if the B-field $\boldsymbol{B}$ vanishes. The torus T^D is then a product $T^D \simeq S^1 \times \ldots \times S^1$ of D circles as a Riemannian manifold, and $\mathcal{K}_{\lambda_i,|\boldsymbol{\kappa}_i^2|}(\tau)$ are the characters of the extended current algebra $\widehat{\mathfrak{u}}(1)_{|\boldsymbol{\kappa}_i^2|}$ associated with each rational circle S^1. However, factorisations (3.3.11) can also occur for more general target space tori and choices of the B-field.

Minimally Extended $\widehat{\mathfrak{u}}(1)$ Current Algebra Decomposition for T^2

To illustrate the general decomposition (3.3.10) concretely, let us now consider the explicit example of a toroidal conformal field theory with target space T^2. As reviewed in (3.2.13), such a theory is uniquely specified by a constant Riemannian metric whose matrix representation in the torus lattice basis we denote by $\boldsymbol{g}$ and an anti-symmetric B-field $B \in H^2(T^2,\mathbb{R})/H^2(T^2,\mathbb{Z})$ that appeares in the calculation of the momentum charges $\boldsymbol{p_{L,R}}$ in the form $b_{ij} = e_i^I B_{IJ} e_j^J$, for lattice basis vectors $\boldsymbol{e}_i$. Equivalently, the real two-torus can be described as a complex torus $T^2 = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z} \oplus u\mathbb{Z}$, for some complex structure parameter in the upper-half plane $u \in \mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\}$ [161].

In this complex setting, the Kähler metric expressed in the lattice basis takes the explicit form

$$\boldsymbol{g} = \frac{2t_2}{u_2} \begin{pmatrix} 1 & u_1 \\ u_1 & |u|^2 \end{pmatrix} , \tag{3.3.12}$$

where $t \in \mathcal{H}$ is the complexified Kähler modulus, whose imaginary part t_2 specifies the volume of T^2 and hence the Kähler class of the Kähler form $\boldsymbol{\omega} = \boldsymbol{gI}$. One verifies directly that the metric g is compatible with the complex structure defined by the period matrix $\Pi = (1 \;\; u)$, since the associated complex structure I in the lattice basis is given by

$$\boldsymbol{I} = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = -\frac{1}{\mathrm{Im}(u)} \begin{pmatrix} \mathrm{Re}(u) & u\overline{u} \\ -1 & -\mathrm{Re}(u) \end{pmatrix} . \tag{3.3.13}$$

A short computation then shows the compatibility condition

$$\boldsymbol{I}^T \boldsymbol{g} \boldsymbol{I} = \boldsymbol{g} \,. \tag{3.3.14}$$

In terms of the complexified Kähler modulus t, the skew-symmetric B-field $\boldsymbol{b}$ of the target space torus T^2 is

$$\boldsymbol{b} = 2 \begin{pmatrix} 0 & t_1 \\ -t_1 & 0 \end{pmatrix} , \tag{3.3.15}$$

where $u_1 = \mathrm{Re}\, u$, $u_2 = \mathrm{Im}\, u$, $t_1 = \mathrm{Re}\, t$, and $t_2 = \mathrm{Im}\, t$.

Substituting this into (3.2.8), the explicit squares of the left- and right-moving momenta become

$$\boldsymbol{p_L}^2 = \frac{1}{2t_2u_2}|m_2 - um_1 + \bar{t}(n_1 + un_2)|^2 \ , \quad \boldsymbol{p_R}^2 = \frac{1}{2t_2u_2}|m_2 - um_1 + t(n_1 + un_2)|^2 \ . \tag{3.3.16}$$

Thus, the toroidal partition function (3.2.14) sums over all allowed conformal weights given by $\boldsymbol{p_L}^2$ and $\boldsymbol{p_R}^2$ as explicit functions of u and t.

The rationality condition eq. (3.3.3) implies for the two-torus, by using eq. (3.3.12) and eq. (3.3.15), that both complex structure modulus u and the complexified Kähler modulus t take values in an imaginary quadratic number field $\mathbb{Q}(\sqrt{-D'})$, for some positive square-free integer D' [93, 160, 162],

$$\begin{aligned} &u = a + b\sqrt{-D'} \ , && t = c + d\sqrt{-D'} \ , \\ &a, c \in \mathbb{Q}, \quad b, d \in \mathbb{Q}_{>0}, && D' \in \mathbb{Z}_{>0} \text{ and } D' \text{ square-free} \ . \end{aligned} \tag{3.3.17}$$

Complex one-dimensional tori for which the complex structure parameter satisfies the above condition are said to admit *complex multiplication* — a key number-theoretic property which implies extra automorphisms of the lattice Λ. Attached to each rational toroidal conformal field theory on T^2 is therefore a *mirror-pair* of complex multiplication tori T^2_u, T^2_t. This is the first instance of the appearance of the intricate arithmetic structure of complex multiplication in rational conformal field theories [92–94]. The geometric and arithmetic aspects of complex multiplication, along with its connection to $\mathcal{N} = (2,2)$ rational superconformal field theories and mirror symmetry, will be discussed in detail in chapter 4.

Inserting the explicit parameterisation (3.3.17) into eq. (3.3.16), we find

$$\begin{aligned} \boldsymbol{p_L}^2 &= \frac{(-am_1 + m_2 + cn_1 + (ac + D'bd)n_2)^2}{2bdD'} + \frac{(bm_1 + dn_1 + (ad - bc)n_2)^2}{2bd} \ , \\ \boldsymbol{p_R}^2 &= \frac{(-am_1 + m_2 + cn_1 + (ac - D'bd)n_2)^2}{2bdD'} + \frac{(-bm_1 + dn_1 + (bc + ad)n_2)^2}{2bd} \ . \end{aligned} \tag{3.3.18}$$

By clearing denominators, we see that this takes the generic form

$$\begin{aligned} \boldsymbol{p_L}^2 &= \frac{\left(\boldsymbol{a}_1^T\boldsymbol{m} + \boldsymbol{b}_1^T\boldsymbol{n}\right)^2}{2a_1} + \frac{\left(\boldsymbol{a}_2^T\boldsymbol{m} + \boldsymbol{b}_2^T\boldsymbol{n}\right)^2}{2a_2} \ , && a_i \in \mathbb{Z}_{>0} \ , \ \boldsymbol{a}_i, \boldsymbol{b}_i \in \mathbb{Z}^2 \ , \\ \boldsymbol{p_R}^2 &= \frac{\left(\tilde{\boldsymbol{a}}_1^T\boldsymbol{m} + \tilde{\boldsymbol{b}}_1^T\boldsymbol{n}\right)^2}{2\tilde{a}_1} + \frac{\left(\tilde{\boldsymbol{a}}_2^T\boldsymbol{m} + \tilde{\boldsymbol{b}}_2^T\boldsymbol{n}\right)^2}{2\tilde{a}_2} \ , && \tilde{a}_i \in \mathbb{Z}_{>0} \ , \ \tilde{\boldsymbol{a}}_i, \tilde{\boldsymbol{b}}_i \in \mathbb{Z}^2 \ , \end{aligned} \tag{3.3.19}$$

where the constants $a_i, \tilde{a}_i$, $i = 1, 2$, and vectors $\boldsymbol{a}_i, \boldsymbol{b}_i, \tilde{\boldsymbol{a}}_i, \tilde{\boldsymbol{b}}_i$, $i = 1, 2$, are linear functions of the rational numbers a, b, c, d, which define the moduli u and t in eq. (3.3.17). We thus explicitly see that the partition function $Z^{\text{rat}}_{T^2}(\tau; u, t)$ is of the form (3.3.4). It should be emphasised that eq. (3.3.19) holds for target spaces T^D in any dimension.

Decomposition Construction

Starting with eq. (3.3.19), let us now derive the decomposition (3.3.10) explicitly for T^2. We define 4×4-matrices $\boldsymbol{H}$ and $\boldsymbol{D}$

$$\boldsymbol{H} = \begin{pmatrix} \boldsymbol{a}_1^T & \boldsymbol{b}_1^T \\ \boldsymbol{a}_2^T & \boldsymbol{b}_2^T \\ \tilde{\boldsymbol{a}}_1^T & \tilde{\boldsymbol{b}}_1^T \\ \tilde{\boldsymbol{a}}_2^T & \tilde{\boldsymbol{b}}_2^T \end{pmatrix}, \qquad \boldsymbol{D} = \begin{pmatrix} 2a_1 & & & \\ & 2a_2 & & \\ & & -2\tilde{a}_1 & \\ & & & -2\tilde{a}_2 \end{pmatrix}, \tag{3.3.20}$$

which by construction have only integral entries and their determinants are non-vanishing.

The integral intersection pairing

$$\boldsymbol{\Sigma} : \mathbb{Z}^4 \times \mathbb{Z}^4 \to \mathbb{Z}\,, \tag{3.3.21}$$

of the even self-dual lattice $\Gamma^{2,2}$ is induced by the norm $\|(\boldsymbol{m},\boldsymbol{n})\|^2 = \boldsymbol{p}_{\boldsymbol{L}}^2(\boldsymbol{m},\boldsymbol{n}) - \boldsymbol{p}_{\boldsymbol{R}}^2(\boldsymbol{m},\boldsymbol{n})$ for $(\boldsymbol{m},\boldsymbol{n}) \in \mathbb{Z}^4$. Hence, we find that the pairing $\boldsymbol{\Sigma}$ can be explicitly written as the integral symmetric 4×4-matrix

$$\boldsymbol{\Sigma} = \boldsymbol{H}^T \boldsymbol{D}^{-1} \boldsymbol{H} \in \mathrm{Mat}_4(\mathbb{Z})\ . \tag{3.3.22}$$

Using the matrix $\boldsymbol{H}$, we can write the partition function (3.3.4) for rational toroidal conformal field theory with target space T^2 as

$$Z_{T^2}^{\mathrm{rat}}(\tau;u,t) = \frac{1}{|\eta(\tau)|^2} \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^2} e^{(\boldsymbol{m},\boldsymbol{n})\boldsymbol{H}^T\boldsymbol{T}(\tau)\boldsymbol{H}\begin{pmatrix}\boldsymbol{m}\\ \boldsymbol{n}\end{pmatrix}}, \tag{3.3.23}$$

with the diagonal matrix

$$\boldsymbol{T}(\tau) = \begin{pmatrix} \frac{i\pi\tau}{2a_1} & & & \\ & \frac{i\pi\tau}{2a_2} & & \\ & & -\frac{i\pi\bar{\tau}}{2\tilde{a}_1} & \\ & & & -\frac{i\pi\bar{\tau}}{2\tilde{a}_2} \end{pmatrix}. \tag{3.3.24}$$

As the lattice $\Gamma^{2,2}$ is unimodular, i.e., even, integral, and self-dual, the inverse matrix $\boldsymbol{\Sigma}^{-1}$ is integral as well. Then, from eq. (3.3.22) we get

$$\boldsymbol{\Sigma}^{-1} = \boldsymbol{H}^{-1}\boldsymbol{D}\left(\boldsymbol{H}^{-1}\right)^T \in \mathrm{Mat}_4(\mathbb{Z})\ , \tag{3.3.25}$$

which implies

$$\boldsymbol{D} = \boldsymbol{H}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{H}^T\right)\ . \tag{3.3.26}$$

We thus explicitly see that the columns of the diagonal matrix $\boldsymbol{D}$ are integral linear combinations of the columns of the matrix $\boldsymbol{H}$, with integral coefficients given by the column entries of the integral matrix $\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{H}^T\right)$.

We can now interpret the columns of the matrix $\boldsymbol{H}$ as generators of the lattice $\Gamma^{2,2}$ together with the intersection pairing given by the inverse matrix $\boldsymbol{D}^{-1}$. As a consequence of eq. (3.3.26), the columns of the matrix $\boldsymbol{D}$ span a sublattice $O_L \oplus O_R$ of $\Gamma^{2,2}$, because it has mutually orthogonal generators given by integral-linear combinations of the columns of $\boldsymbol{H}$. This sublattice $O_L \oplus O_R$ clearly respects the chain of inclusions (3.3.5) that are needed for the decomposition of the partition function into the characters $\mathcal{K}_{\lambda,\alpha}$ of the form (3.3.10). The contribution of the sublattice $O_L \oplus O_R$ to the partition function (3.3.23), i.e., the contribution (3.3.6) reads

$$Z_{T^2}^{O_L\oplus O_R}(\tau;u,t) = \frac{1}{|\eta(\tau)|^2}\sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^2} e^{\begin{pmatrix}\boldsymbol{m},\boldsymbol{n}\end{pmatrix}\boldsymbol{D}\boldsymbol{T}(\tau)\boldsymbol{D}\begin{pmatrix}\boldsymbol{m}\\ \boldsymbol{n}\end{pmatrix}} = \prod_{i=1}^{2}\mathcal{K}_{0,a_i}(\tau)\overline{\mathcal{K}_{0,\tilde{a}_i}(\tau)} \ . \tag{3.3.27}$$

The remaining terms in the partition function stem from the cosets of the sublattice $O_L \oplus O_R$ inside $\Gamma^{2,2}$. Let us denote the index of the sublattice generated by the columns of $\boldsymbol{D}$ within the lattice generated by the columns of $\boldsymbol{H}$ by $n+1$. As discussed in section 3.3, we need to construct $n+1$ vectors $\boldsymbol{\rho}_0,\ldots,\boldsymbol{\rho}_n$ representing the $n+1$ distinct cosets of the quotient of the lattice of $\boldsymbol{H}$ by the sublattice of $\boldsymbol{D}$. Explicitly, we find for the index of this sublattice

$$n+1 = \frac{\det\boldsymbol{D}}{\det\boldsymbol{H}} = 2^4\frac{a_1a_2\tilde{a}_1\tilde{a}_2}{\det\boldsymbol{H}} = \det\boldsymbol{H} \ , \tag{3.3.28}$$

where we use $\det\boldsymbol{\Sigma} = 1$, which follows from the unimodular property of $\Gamma^{2,2}$, together with eq. (3.3.26).

Without loss of generality we can assume that the integral matrix $\boldsymbol{H}$ is in (column) Hermite normal form[2]

$$\boldsymbol{H} = \begin{pmatrix} h_{11} & 0 & 0 & 0 \\ h_{21} & h_{22} & 0 & 0 \\ h_{31} & h_{32} & h_{33} & 0 \\ h_{41} & h_{42} & h_{43} & h_{44}\end{pmatrix} \quad \in \ \mathrm{Mat}_4(\mathbb{Z}) \ . \tag{3.3.29}$$

The diagonal entries of the inverse matrix $\boldsymbol{H}^{-1}$ are

$$\boldsymbol{H}^{-1} = \begin{pmatrix} \frac{1}{h_{11}} & \cdots & \cdots & \cdots \\ 0 & \frac{1}{h_{22}} & \cdots & \cdots \\ 0 & 0 & \frac{1}{h_{33}} & \cdots \\ 0 & 0 & 0 & \frac{1}{h_{44}}\end{pmatrix} \ . \tag{3.3.30}$$

[2]For any integral matrix $\boldsymbol{M}$ there exists a unimodular matrix $\boldsymbol{U}$ such that the matrix product $\boldsymbol{M}\boldsymbol{U} = \boldsymbol{H}$ is in Hermite normal form. As the multiplication with a unimodular matrix realises a lattice automorphism, any lattice generated by the column vectors of $\boldsymbol{M}$ can be represented by a matrix $\boldsymbol{H}$ in Hermite normal form.

From eq. (3.3.26) it follows that the matrix product $\boldsymbol{H}^{-1}\boldsymbol{D}$ is integral with diagonal entries

$$\boldsymbol{H}^{-1}\boldsymbol{D} = \begin{pmatrix} \frac{2a_1}{h_{11}} & \cdots & \cdots & \cdots \\ 0 & \frac{2a_2}{h_{22}} & \cdots & \cdots \\ 0 & 0 & -\frac{2\tilde{a}_1}{h_{33}} & \cdots \\ 0 & 0 & 0 & -\frac{2\tilde{a}_2}{h_{44}} \end{pmatrix} \in \ \mathrm{Mat}_4(\mathbb{Z}) \ , \tag{3.3.31}$$

from which we obtain the important result

$$s_1 = \frac{2a_1}{h_{11}},\ s_2 = \frac{2a_2}{h_{22}},\ \tilde{s}_1 = \frac{2\tilde{a}_1}{h_{33}},\ \tilde{s}_2 = \frac{2\tilde{a}_2}{h_{44}} \in \mathbb{Z} \ . \tag{3.3.32}$$

We now give an Ansatz for a set of vectors $\boldsymbol{\rho}_0, \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_n$ representing the cosets associated with the sublattice of the diagonal matrix $\boldsymbol{D}$,

$$\mathcal{C} = \left\{ \boldsymbol{H}\begin{pmatrix} r_1 \\ r_2 \\ \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix} \middle| \, r_i \in \{0,\ldots,s_i-1\},\ \tilde{r}_i \in \{0,\ldots,\tilde{s}_i-1\} \ \text{ for } \ i=1,2 \right\} \ . \tag{3.3.33}$$

Now we need to show that all elements of the set $\mathcal{C}$ mutually represent distinct cosets. To show that, let us choose two lattice vectors $\boldsymbol{\rho}, \boldsymbol{\sigma} \in \mathcal{C}$, which are obtained from the integers $r_1, r_1' \in \{0,\ldots,s_i-1\}, \ldots, \tilde{r}_1, \tilde{r}_1' \in \{0,\ldots,s_i-1\}$, respectively. These two lattice vectors live in the same coset if and only if their difference $\boldsymbol{\rho}-\boldsymbol{\sigma}$ is an element in $O_L \oplus O_R$, i.e., the sublattice generated by the columns of the diagonal matrix $\boldsymbol{D}$. For the difference, we explicitly find

$$\boldsymbol{\rho}-\boldsymbol{\sigma} = \boldsymbol{H}\begin{pmatrix} \Delta r_1 \\ \Delta r_2 \\ \Delta\tilde{r}_1 \\ \Delta\tilde{r}_2 \end{pmatrix} = \begin{pmatrix} h_{11}\Delta r_1 \\ h_{21}\Delta r_1 + h_{22}\Delta r_2 \\ h_{31}\Delta r_1 + h_{32}\Delta r_2 + h_{33}\Delta r_3 \\ h_{41}\Delta r_1 + h_{42}\Delta r_2 + h_{43}\Delta r_3 + h_{44}\Delta r_4 \end{pmatrix} \ , \tag{3.3.34}$$

where $\Delta r_1 = r_1 - r_1', \ldots, \Delta\tilde{r}_2 = \tilde{r}_2 - \tilde{r}_2'$. Using the relations (3.3.32) we clearly have that $|\Delta r_1| < s_1 = \frac{2a_1}{h_{11}}$. Hence, the absolute value of the first entry of the vector $\boldsymbol{\rho}-\boldsymbol{\sigma}$ is smaller than $2a_1$. It follows that the vector $\boldsymbol{\rho}-\boldsymbol{\sigma}$ can only be an element of the sublattice of $\boldsymbol{D}$, i.e., an integral multiple of the first column vector of $\boldsymbol{D}$, if $\Delta r_1 = 0$. Let us therefore assume that $\Delta r_1 = 0$ (otherwise we can already conclude that the vectors represent different cosets). Looking at the second entry of the vector $\boldsymbol{\rho}-\boldsymbol{\sigma}$ and using $\Delta r_1 = 0$, we see that the term proportional to h_{21} vanishes, and we can repeat the same argument as for the first entry. This means that the second entry is smaller in absolute value than a_2. By considering the first and the second entry, we can now conclude that the difference $\boldsymbol{\rho}-\boldsymbol{\sigma}$ can only be an element of the sublattice $O_L \oplus O_R$ if both $\Delta r_1 = 0$ and $\Delta r_2 = 0$. From the explicit structure of the difference vector, see eq. (3.3.34), we can repeat this argument inductively for the remaining entries and it follows that $\boldsymbol{\rho}-\boldsymbol{\sigma}$ is only an element of the sublattice $O_L \oplus O_R$, if and only if $\Delta r_1 = \Delta r_2 = \Delta \tilde{r}_1 = \Delta \tilde{r}_2 = 0$, which is equivalent to $\boldsymbol{\rho} = \boldsymbol{\sigma}$. We have thus shown that all the lattice vectors in the set $\mathcal{C}$ defined in eq. (3.3.33), describe distinct cosets for the sublattice of $O_L \oplus O_R \subseteq \Gamma^{2,2}$.

We can calculate the cardinality of the set $\mathcal{C}$ by using equations (3.3.28) and (3.3.32),

$$\left|\mathcal{C}\right| = s_1 s_2 \tilde{s}_1 \tilde{s}_2 = 2^4 \frac{a_1 a_2 \tilde{a}_1 \tilde{a}_2}{h_{11} h_{22} h_{33} h_{44}} = n + 1 \ . \tag{3.3.35}$$

Therefore, the set $\mathcal{C}$ incorporates all the cosets of the sublattice $\boldsymbol{D}$ within $\Gamma^{2,2}$. Let us denote the $n+1$ elements of $\mathcal{C}$ in the following by the lattice vectors $\boldsymbol{\rho}_0, \ldots, \boldsymbol{\rho}_n$, with $\boldsymbol{\rho}_0$ being the zero vector of the set $\mathcal{C}$.

We can now explicitly decompose the full partition function $Z^{\text{rat}}_{T^2}(\tau; u, t)$ in terms of the characters $\mathcal{K}_{\lambda,\alpha}$ as

$$Z^{\text{rat}}_{T^2}(\tau; u, t) = \sum_{b=0}^{n} \prod_{i=1}^{2} \mathcal{K}_{\rho_{b,i},a_i}(\tau) \overline{\mathcal{K}_{\rho_{b,i+2},\tilde{a}_i}(\tau)} \ , \tag{3.3.36}$$

where $\rho_{b,i}$ denotes the entries of the column vectors $\boldsymbol{\rho}_b$, $a = 0, \ldots, n$. Note again that from this decomposition we can immediately read off the contribution of the generators of the $\bigotimes_{i=1}^{2} \widehat{\mathfrak{u}}(1)_{a_i} \otimes \bigotimes_{j=1}^{2} \widehat{\mathfrak{u}}(1)_{\tilde{a}_j}$ subalgebra of the maximally extended chiral algebra to the partition function, i.e., the part (3.3.27) which corresponds to the trivial coset given by the null vector $\boldsymbol{\rho}_0$.

The presented explicit construction for the decomposition (3.3.36) of rational toroidal conformal field theories with target space T^2 can be applied without any further modification to any rational toroidal conformal field theories with target space T^D, for any dimension D. Starting with the partition function $Z^{\text{rat}}_{T^D}(\tau)$ in the form (3.3.4), we can always read off a $2D \times 2D$-dimensional matrix $\boldsymbol{H}$ and the $2D \times 2D$-dimensional diagonal matrix $\boldsymbol{D} = \text{Diag}\,(2a_1, \ldots, 2a_D, -2\tilde{a}_1, \ldots, -2\tilde{a}_D)$. As we have shown for the target space T^2, the columns of $\boldsymbol{D}$ generate a sublattice of the lattice $\Gamma^{D,D}$ generated by the columns of $\boldsymbol{H}$ for a general target space dimension D. Then by calculating the Hermite normal form of $\boldsymbol{H}$, we can read off (analogously as for $D = 2$) the character expansion of the form (3.3.10). We illustrate this construction explicitly with an example of a rational toroidal conformal field theory with target space T^3 in the following section.

3.4 Examples

In this section, we illustrate the general decomposition construction outlined in the previous section by presenting explicit representative examples. We focus on rational toroidal conformal field theories whose target spaces are two- and three-dimensional tori.

Target Space T^2: Extended $\widehat{\mathfrak{su}}(3)_1$ Chiral Algebra

Consider the diagonal partition function associated with the affine Lie algebra $\widehat{\mathfrak{su}}(3)_1$ at level one, taken as the extended chiral algebra. In this case, the lattices $\Gamma^{L,0}$ and $\Gamma^{0,R}$ are both given by the root lattice of $\mathfrak{su}(3)$. This partition function is realised by setting both the complex structure parameter and the complexified Kähler modulus of the target torus T^2 to $u = t = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. The resulting partition function then explicitly takes the form

$$Z_{T^2}^{\widehat{\mathfrak{su}}(3)_1}(\tau) = \frac{1}{|\eta(\tau)|^4} \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^2} q^{\frac{1}{12}(m_1+2m_2-n_1+2n_2)^2} q^{\frac{1}{4}(m_1+n_1)^2} \cdot \overline{q}^{\frac{1}{12}(m_1+2m_2-n_1-n_2)^2} \overline{q}^{\frac{1}{4}(m_1-n_1+n_2)^2} \ . \quad (3.4.1)$$

From eq. (3.3.4) we identify the constants

$$\begin{aligned} &a_1 = \tilde{a}_1 = 3 \ , \quad a_2 = \tilde{a}_2 = 1 \ , \quad \boldsymbol{a}_1 = \tilde{\boldsymbol{a}}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ , \quad \boldsymbol{a}_2 = \tilde{\boldsymbol{a}}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \\ &\boldsymbol{b}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \ , \quad \boldsymbol{b}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad \tilde{\boldsymbol{b}}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \ , \quad \tilde{\boldsymbol{b}}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \ , \end{aligned} \quad (3.4.2)$$

which determine the diagonal 4×4-matrix $\boldsymbol{D}$ and an integral 4×4-matrix according to eq. (3.3.20). The latter matrix reads

$$\boldsymbol{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix} \ . \quad (3.4.3)$$

Observe that the integers $a_1, a_2, \tilde{a}_1, \tilde{a}_2$ satisfy the relation (3.3.28), and together with eq. (3.3.32) yields the integers

$$s_1 = 6 \ , \quad s_2 = 1 \ , \quad \tilde{s}_1 = 2 \ , \quad \tilde{s}_2 = 1 \ , \quad (3.4.4)$$

such that the partition function decomposes as

$$Z_{T^2}^{\widehat{\mathfrak{su}}(3)_1}(\tau) = \sum_{r_1=0}^{5} \sum_{\tilde{r}_1=0}^{1} \mathcal{K}_{r_1,3}(\tau)\, \mathcal{K}_{r_1,1}(\tau) \overline{\mathcal{K}_{r_1+3\tilde{r}_1,3}(\tau)}\, \overline{\mathcal{K}_{r_1+\tilde{r}_1,1}(\tau)} \ . \quad (3.4.5)$$

Exploiting the symmetries (3.1.4) of the characters $\mathcal{K}_{\lambda,\alpha}$, which for example imply the identity $\mathcal{K}_{2,1} \equiv \mathcal{K}_{0,1}$, $\mathcal{K}_{5,3} \equiv \mathcal{K}_{1,3}$, $\mathcal{K}_{4,3} \equiv \mathcal{K}_{2,3}$, we obtain the explicit expansion

$$\begin{aligned} Z_{T^2}^{\widehat{\mathfrak{su}}(3)_1}(\tau) = \ & \mathcal{K}_{0,1}\mathcal{K}_{0,3}\overline{\mathcal{K}}_{0,1}\overline{\mathcal{K}}_{0,3} + \mathcal{K}_{1,1}\mathcal{K}_{3,3}\overline{\mathcal{K}}_{0,1}\overline{\mathcal{K}}_{0,3} + 2\mathcal{K}_{1,1}\mathcal{K}_{1,3}\overline{\mathcal{K}}_{1,1}\overline{\mathcal{K}}_{1,3} \\ & + 2\mathcal{K}_{0,1}\mathcal{K}_{2,3}\overline{\mathcal{K}}_{1,1}\overline{\mathcal{K}}_{1,3} + 2\mathcal{K}_{1,1}\mathcal{K}_{1,3}\overline{\mathcal{K}}_{0,1}\overline{\mathcal{K}}_{2,3} + 2\mathcal{K}_{0,1}\mathcal{K}_{2,3}\overline{\mathcal{K}}_{0,1}\overline{\mathcal{K}}_{2,3} \\ & + \mathcal{K}_{0,1}\mathcal{K}_{0,3}\overline{\mathcal{K}}_{1,1}\overline{\mathcal{K}}_{3,3} + \mathcal{K}_{1,1}\mathcal{K}_{3,3}\overline{\mathcal{K}}_{1,1}\overline{\mathcal{K}}_{3,3} \\ = \ & |\chi_{100}(\tau)|^2 + |\chi_{010}(\tau)|^2 + |\chi_{001}(\tau)|^2 \ , \end{aligned} \quad (3.4.6)$$

where

$$\begin{aligned} \chi_{100}(\tau) &= \mathcal{K}_{0,1}(\tau)\mathcal{K}_{0,3}(\tau) + \mathcal{K}_{1,1}(\tau)\mathcal{K}_{3,3}(\tau) \ , \\ \chi_{010}(\tau) &= \mathcal{K}_{1,1}(\tau)\mathcal{K}_{1,3}(\tau) + \mathcal{K}_{0,1}(\tau)\mathcal{K}_{2,3}(\tau) \ , \\ \chi_{001}(\tau) &= \mathcal{K}_{1,1}(\tau)\mathcal{K}_{1,3}(\tau) + \mathcal{K}_{0,1}(\tau)\mathcal{K}_{2,3}(\tau) \ . \end{aligned} \quad (3.4.7)$$

We denote by $\chi_{100}(\tau)$, $\chi_{010}(\tau)$, $\chi_{001}(\tau)$ the irreducible specialised characters of the affine Lie algebra $\widehat{\mathfrak{su}}(3)_1$ at level one. These characters can be derived from the Kac–Weyl character formula for $\widehat{\mathfrak{su}}(3)_1$ upon taking an appropriate limit, as for instance discussed in ref. [95].

The Gram matrices $\boldsymbol{G_L}$ and $\boldsymbol{G_R}$ for the chiral and anti-chiral lattices $\Gamma^{L,0}$ and $\Gamma^{0,R}$ coincide with the Cartan matrix of the simple Lie algebra $\mathfrak{su}(3)$, i.e.,

$$\boldsymbol{G_L} = \boldsymbol{G_R} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \ . \tag{3.4.8}$$

Consequently, the generators of the lattices $\Gamma^{L,0}$ and $\Gamma^{0,R}$ cannot be chosen to be mutually orthogonal. Therefore, the specialised characters of $\widehat{\mathfrak{su}}(3)_1$ do not factor into characters of the extended current algebras $\widehat{u}(1)_\alpha$ individually. Instead, they can be expressed as a sum of such characters, as given in eq. (3.4.7). These particular expansions arise from the maximally diagonal sublattices O_L and O_R of $\Gamma^{L,0}$ and $\Gamma^{0,R}$ with mutually orthogonal generators and Gram matrices

$$\boldsymbol{G}_{O_L} = \boldsymbol{G}_{O_R} = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \ . \tag{3.4.9}$$

The decompositions (3.4.7) can be determined as follows. For example, the vacuum character $\chi_{100}(\tau)$ has the expansion

$$\chi_{100}(\tau) = \frac{1}{\eta(\tau)^2} \sum_{m_1,m_2\in\mathbb{Z}} q^{\frac{1}{4}m_1^2} q^{\frac{3}{4}(m_1+2m_2)^2} \ , \tag{3.4.10}$$

which by substituting the summation index m_1 by $2m_1' + \alpha$ for $\alpha \in \{0,1\}$ yields

$$\chi_{100}(\tau) = \frac{1}{\eta(\tau)^2} \sum_{\substack{m_1',m_2'\in\mathbb{Z} \\ \alpha\in\{0,1\}}} q^{\left(m_1'+\frac{\alpha}{2}\right)^2} q^{3\left(m_2'+\frac{3\alpha}{6}\right)^3} = \mathcal{K}_{0,1}(\tau)\mathcal{K}_{0,3}(\tau) + \mathcal{K}_{1,1}(\tau)\mathcal{K}_{3,3}(\tau) \ . \tag{3.4.11}$$

This agrees with eq. (3.4.7).

Target Space $\boldsymbol{T^2}$: Moduli $\boldsymbol{u = t = \frac{1}{2} + \frac{\sqrt{-5}}{2}}$

For the next example, we choose the complex structure modulus and complexified Kähler modulus of the complex torus T^2 to be $u = t = \frac{1}{2} + \frac{\sqrt{-5}}{2}$. This choice leads to a maximally extended chiral algebra which is not given by an affine semi-simple Lie algebra, but instead by a more general W-algebra. The following data specifies the partition function

$$\begin{aligned} &a_1 = \tilde{a}_1 = 5 \ , \quad a_2 = \tilde{a}_2 = 1 \ , \quad \boldsymbol{a}_1 = \tilde{\boldsymbol{a}}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \ , \quad \boldsymbol{a}_2 = \tilde{\boldsymbol{a}}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \\ &\boldsymbol{b}_1 = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \ , \quad \boldsymbol{b}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad \tilde{\boldsymbol{b}}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \ , \quad \tilde{\boldsymbol{b}}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \ , \end{aligned} \tag{3.4.12}$$

from which we calculate the 4×4-matrix $\boldsymbol{H}$ in Hermite normal form

$$\boldsymbol{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 6 & 5 & 10 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} . \tag{3.4.13}$$

The decomposition construction then yields

$$\begin{aligned} Z^{\text{rat}}_{T^2}(\tau; u, t) = \; & \mathcal{K}_{0,1}\mathcal{K}_{0,5}\overline{\mathcal{K}}_{0,1}\overline{\mathcal{K}}_{0,5} + \mathcal{K}_{0,1}\mathcal{K}_{5,5}\overline{\mathcal{K}}_{0,5}\overline{\mathcal{K}}_{1,1} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{4,5}\overline{\mathcal{K}}_{0,1}\overline{\mathcal{K}}_{1,5} \\ & + 2\,\mathcal{K}_{1,1}\mathcal{K}_{1,5}\overline{\mathcal{K}}_{1,1}\overline{\mathcal{K}}_{1,5} + 2\,\mathcal{K}_{0,1}\mathcal{K}_{2,5}\overline{\mathcal{K}}_{0,1}\overline{\mathcal{K}}_{2,5} + 2\,\mathcal{K}_{0,1}\mathcal{K}_{3,5}\overline{\mathcal{K}}_{1,1}\overline{\mathcal{K}}_{2,5} \\ & + 2\,\mathcal{K}_{1,1}\mathcal{K}_{2,5}\overline{\mathcal{K}}_{0,1}\overline{\mathcal{K}}_{3,5} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{3,5}\overline{\mathcal{K}}_{1,1}\overline{\mathcal{K}}_{3,5} + 2\,\mathcal{K}_{0,1}\mathcal{K}_{4,5}\overline{\mathcal{K}}_{0,1}\overline{\mathcal{K}}_{4,5} \\ & + 2\,\mathcal{K}_{0,1}\mathcal{K}_{1,5}\overline{\mathcal{K}}_{1,1}\overline{\mathcal{K}}_{4,5} + \mathcal{K}_{1,1}\mathcal{K}_{0,5}\overline{\mathcal{K}}_{0,1}\overline{\mathcal{K}}_{5,5} + \mathcal{K}_{1,1}\mathcal{K}_{5,5}\overline{\mathcal{K}}_{1,1}\overline{\mathcal{K}}_{5,5} \; . \end{aligned} \tag{3.4.14}$$

Target Space T^2: Moduli $u = t = \frac{1}{4} + \frac{\sqrt{-3}}{4}$

The next example with $u = t = \frac{1}{4} + \frac{\sqrt{-3}}{4}$ illustrates that generically the mutually orthogonal generators of the orthogonal sublattice $O_L \oplus O_R$ have distinct norm, which leads to factors of $\mathcal{K}_{\lambda,\alpha}$ with distinct indices α in the decomposition of the partition function. The following data determines the partition function

$$\begin{gathered} a_1 = 3 \; , \qquad a_2 = 1 \; , \qquad \tilde{a}_1 = 12 \; , \qquad \tilde{a}_2 = 4 \; , \\ \boldsymbol{a}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} , \quad \tilde{\boldsymbol{a}}_1 = \begin{pmatrix} 2 \\ -8 \end{pmatrix} , \quad \boldsymbol{a}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \tilde{\boldsymbol{a}}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} , \\ \boldsymbol{b}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} , \quad \tilde{\boldsymbol{b}}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} , \quad \boldsymbol{b}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \tilde{\boldsymbol{b}}_2 = \begin{pmatrix} -2 \\ -1 \end{pmatrix} , \end{gathered} \tag{3.4.15}$$

from which we calculate the 4×4-matrix $\boldsymbol{H}$ in Hermite normal form

$$\boldsymbol{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 3 & 6 & 0 \\ 3 & 1 & 2 & 8 \end{pmatrix} . \tag{3.4.16}$$

The decomposition of the partition function then reads

$$\begin{aligned} Z^{\text{rat}}_{T^2}(\tau; u, t) = \; & \mathcal{K}_{0,1}\mathcal{K}_{0,3}\overline{\mathcal{K}}_{0,4}\overline{\mathcal{K}}_{0,12} + 2\,\mathcal{K}_{0,1}\mathcal{K}_{1,3}\overline{\mathcal{K}}_{1,4}\overline{\mathcal{K}}_{1,12} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{1,3}\overline{\mathcal{K}}_{2,4}\overline{\mathcal{K}}_{2,12} \\ & + 2\,\mathcal{K}_{0,1}\mathcal{K}_{2,3}\overline{\mathcal{K}}_{2,4}\overline{\mathcal{K}}_{2,12} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{2,3}\overline{\mathcal{K}}_{3,4}\overline{\mathcal{K}}_{1,12} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{0,3}\overline{\mathcal{K}}_{1,4}\overline{\mathcal{K}}_{3,12} \\ & + 2\,\mathcal{K}_{0,1}\mathcal{K}_{3,3}\overline{\mathcal{K}}_{3,4}\overline{\mathcal{K}}_{3,12} + \mathcal{K}_{1,1}\mathcal{K}_{3,3}\overline{\mathcal{K}}_{4,4}\overline{\mathcal{K}}_{0,12} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{1,3}\overline{\mathcal{K}}_{0,4}\overline{\mathcal{K}}_{4,12} \\ & + 2\,\mathcal{K}_{0,1}\mathcal{K}_{2,3}\overline{\mathcal{K}}_{4,4}\overline{\mathcal{K}}_{4,12} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{2,3}\overline{\mathcal{K}}_{1,4}\overline{\mathcal{K}}_{5,12} + 2\,\mathcal{K}_{0,1}\mathcal{K}_{1,3}\overline{\mathcal{K}}_{3,4}\overline{\mathcal{K}}_{5,12} \\ & + 2\,\mathcal{K}_{0,1}\mathcal{K}_{0,3}\overline{\mathcal{K}}_{2,4}\overline{\mathcal{K}}_{6,12} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{3,3}\overline{\mathcal{K}}_{2,4}\overline{\mathcal{K}}_{6,12} + 2\,\mathcal{K}_{0,1}\mathcal{K}_{1,3}\overline{\mathcal{K}}_{1,4}\overline{\mathcal{K}}_{7,12} \\ & + 2\,\mathcal{K}_{1,1}\mathcal{K}_{2,3}\overline{\mathcal{K}}_{3,4}\overline{\mathcal{K}}_{7,12} + 2\,\mathcal{K}_{0,1}\mathcal{K}_{2,3}\overline{\mathcal{K}}_{0,4}\overline{\mathcal{K}}_{8,12} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{1,3}\overline{\mathcal{K}}_{4,4}\overline{\mathcal{K}}_{8,12} \\ & + 2\,\mathcal{K}_{0,1}\mathcal{K}_{3,3}\overline{\mathcal{K}}_{1,4}\overline{\mathcal{K}}_{9,12} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{0,3}\overline{\mathcal{K}}_{3,4}\overline{\mathcal{K}}_{9,12} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{1,3}\overline{\mathcal{K}}_{2,4}\overline{\mathcal{K}}_{10,12} \\ & + 2\,\mathcal{K}_{0,1}\mathcal{K}_{2,3}\overline{\mathcal{K}}_{2,4}\overline{\mathcal{K}}_{10,12} + 2\,\mathcal{K}_{1,1}\mathcal{K}_{2,3}\overline{\mathcal{K}}_{1,4}\overline{\mathcal{K}}_{11,12} + 2\,\mathcal{K}_{0,1}\mathcal{K}_{1,3}\overline{\mathcal{K}}_{3,4}\overline{\mathcal{K}}_{11,12} \\ & + \mathcal{K}_{1,1}\mathcal{K}_{3,3}\overline{\mathcal{K}}_{0,4}\overline{\mathcal{K}}_{12,12} + \mathcal{K}_{0,1}\mathcal{K}_{0,3}\overline{\mathcal{K}}_{4,4}\overline{\mathcal{K}}_{12,12} \; . \end{aligned} \tag{3.4.17}$$

We can apply the modular $\mathrm{PSL}(2,\mathbb{Z})$-transformation $z \mapsto \frac{z-1}{z}$ to both moduli u and t, which yields $u' = t' = \sqrt{-3}$. This shows that the toroidal conformal field theory with T^2 target is realised as a product of circles $S^1 \times S^1$, with radii $\sqrt{2}$ and $\sqrt{6}$ and vanishing B-field. Through this duality, the partition function (3.4.17) can also be written as

$$\begin{aligned} Z^{\mathrm{rat}}_{T^2}(\tau; u, t) &= Z^{\mathrm{rat}}_{T^2}(\tau; \sqrt{-3}, \sqrt{-3}) \\ &= \left(\mathcal{K}_{0,1}\overline{\mathcal{K}}_{0,1} + \mathcal{K}_{1,1}\overline{\mathcal{K}}_{1,1}\right)\left(\mathcal{K}_{0,3}\overline{\mathcal{K}}_{0,3} + 2\mathcal{K}_{1,3}\overline{\mathcal{K}}_{1,3} + 2\mathcal{K}_{2,3}\overline{\mathcal{K}}_{2,3} + \mathcal{K}_{3,3}\overline{\mathcal{K}}_{3,3}\right) \ , \end{aligned} \tag{3.4.18}$$

from which we explicitly see the factorisation into two factors of the rational conformal field theory on S^1 (3.1.2) with radii $\sqrt{2}$ and $\sqrt{6}$, respectively.

Target Space T^3: Extended $\widehat{\mathfrak{su}}(4)_1$ Chiral Algebra

As a last example, we consider the rational toroidal conformal field theory on a three-dimensional target space torus T^3 with diagonal partition function for the maximally extended chiral algebra $\widehat{\mathfrak{su}}(4)_1$. For this theory, the sublattices $\Gamma^{L,0}$ and $\Gamma^{0,R}$ are the root lattices of the simple Lie algebra $\mathfrak{su}(4)$. A straightforward calculation shows that this theory is obtained for the following metric $\boldsymbol{g}$ and B-field $\boldsymbol{b}$ in the lattice basis

$$\boldsymbol{g} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \ , \qquad \boldsymbol{b} = \begin{pmatrix} 0 & \frac{3}{2} & 1 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} & 0 \end{pmatrix} \ . \tag{3.4.19}$$

Following ref. [93], we calculate the following integers that specify the partition function

$$\begin{gathered} a_1 = \tilde{a}_1 = 1 \ , \quad a_2 = \tilde{a}_2 = 3 \ , \quad a_3 = \tilde{a}_3 = 6 \ , \\ \boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \boldsymbol{a}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} , \quad \boldsymbol{a}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} , \\ \boldsymbol{b}_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} , \quad \boldsymbol{b}_2 = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} , \quad \boldsymbol{b}_3 = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} , \\ \tilde{\boldsymbol{a}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \tilde{\boldsymbol{a}}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} , \quad \tilde{\boldsymbol{a}}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} , \\ \tilde{\boldsymbol{b}}_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} , \quad \tilde{\boldsymbol{b}}_2 = \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} , \quad \tilde{\boldsymbol{b}}_3 = \begin{pmatrix} 6 \\ 0 \\ -4 \end{pmatrix} . \end{gathered} \tag{3.4.20}$$

The integral 6×6-matrix $\boldsymbol{H}$ in Hermite normal reads

$$\boldsymbol{H} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 9 & 6 & 3 & 4 & 8 & 12 \end{pmatrix} , \tag{3.4.21}$$

and we verify that the integers $a_1, a_2, a_3, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ satisfy the relation (3.3.28). Using eq. (3.3.32) yields

$$s_1 = 2\ , \quad s_2 = 3\ , \quad s_3 = 4\ , \quad \tilde{s}_1 = 2\ , \quad \tilde{s}_2 = 3\ , \quad \tilde{s}_3 = 1\,. \tag{3.4.22}$$

The partition function consequently decomposes as

$$\begin{aligned} Z_{T^3}^{\widehat{\mathfrak{su}}(4)_1}(\tau) = \sum_{r_1,\tilde{r}_1=0}^{1} \sum_{r_2,\tilde{r}_2=0}^{2} \sum_{r_3=0}^{3} \mathcal{K}_{r_1,1}(\tau)\, \mathcal{K}_{r_1+2r_2,3}(\tau)\, \mathcal{K}_{r_1+2r_2+3r_3,6}(\tau) \\ \cdot \overline{\mathcal{K}_{\tilde{r}_1,1}(\tau)}\, \overline{\mathcal{K}_{\tilde{r}_1+2\tilde{r}_2,3}(\tau)}\, \overline{\mathcal{K}_{9r_1+6r_2+3r_3+4\tilde{r}_1+8\tilde{r}_2,6}(\tau)}\ . \end{aligned} \tag{3.4.23}$$

The partition function can be written in diagonal form in terms of the specialised characters of the affine Lie algebra $\widehat{\mathfrak{su}}(4)_1$,

$$Z_{T^3,\,\widehat{\mathfrak{su}}(4)_1}(\tau) = |\chi_{1000}(\tau)|^2 + |\chi_{0100}(\tau)|^2 + |\chi_{0010}(\tau)|^2 + |\chi_{0001}(\tau)|^2\,, \tag{3.4.24}$$

and we find

$$\begin{aligned} \chi_{1000}(\tau) &= \mathcal{K}_{0,1}\mathcal{K}_{0,3}\mathcal{K}_{0,6} + \mathcal{K}_{1,1}\mathcal{K}_{3,3}\mathcal{K}_{0,6} + 2\mathcal{K}_{1,1}\mathcal{K}_{1,3}\mathcal{K}_{4,6} + 2\mathcal{K}_{0,1}\mathcal{K}_{2,3}\mathcal{K}_{4,6}\ , \\ \chi_{0100}(\tau) &= \mathcal{K}_{1,1}\mathcal{K}_{1,3}\mathcal{K}_{1,6} + \mathcal{K}_{0,1}\mathcal{K}_{2,3}\mathcal{K}_{1,6} + \mathcal{K}_{0,1}\mathcal{K}_{0,3}\mathcal{K}_{3,6} \\ &\quad + \mathcal{K}_{1,1}\mathcal{K}_{3,3}\mathcal{K}_{3,6} + \mathcal{K}_{1,1}\mathcal{K}_{1,3}\mathcal{K}_{5,6} + \mathcal{K}_{0,1}\mathcal{K}_{2,3}\mathcal{K}_{5,6}\ , \\ \chi_{0010}(\tau) &= \chi_{0100}(\tau)\ , \\ \chi_{0001}(\tau) &= 2\mathcal{K}_{1,1}\mathcal{K}_{1,3}\mathcal{K}_{2,6} + 2\mathcal{K}_{0,1}\mathcal{K}_{2,3}\mathcal{K}_{2,6} + \mathcal{K}_{0,1}\mathcal{K}_{0,3}\mathcal{K}_{6,6} + \mathcal{K}_{1,1}\mathcal{K}_{3,3}\mathcal{K}_{6,6}\ . \end{aligned} \tag{3.4.25}$$

Similarly to (3.4.7), the specialised characters of $\widehat{\mathfrak{su}}(4)_1$ can be calculated by the Kac–Weyl character for the affine Lie algebra $\widehat{\mathfrak{su}}(4)_1$ evaluated in a certain limit [95]. The intersection form of the sublattices $\Gamma^{L,0}$ and $\Gamma^{0,R}$ of the maximally extended chiral algebra $\mathfrak{su}(4)$ is the Cartan matrix

$$\boldsymbol{G_L} = \boldsymbol{G_R} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\ . \tag{3.4.26}$$

The densest sublattices O_L and O_R of $\Gamma^{L,0}$ and $\Gamma^{0,R}$ with pairwise orthogonal generators have the 3×3 Gram matrices

$$\boldsymbol{G}_{O_L} = \boldsymbol{G}_{O_R} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{pmatrix}\ . \tag{3.4.27}$$

3.5 Ray Class Theta Functions and Rational Toroidal Partition Functions

We now aim to provide the building blocks $\mathcal{K}_{m,k}\mathcal{K}_{m',k'}$, that appear in the partition function of any toroidal rational conformal field theories, a number-theoretic interpretation in terms of so-called *Ray class theta functions*. In particular, we explicitly write products of the form $\mathcal{K}_{m,k}\mathcal{K}_{m',k'}$ in terms of Ray class theta functions associated with quadratic imaginary field extensions $\mathbb{Q}(\sqrt{-D'})$, for some positive square-free integer D'. This yields a fully number-theoretic interpretation of the partition function (3.3.10), for even-dimensional rational toroidal conformal field theories. Our presentation follows the conventions of [163]. We begin by defining the generalised theta function $\Theta_{m,k}$ as $\mathcal{K}_{m,k} = \eta(q)^{-1}\,\Theta_{m,k}$. It is then convenient to interpret the generalised theta function $\Theta_{m,k}$ as a coset theta function of the form,

$$\theta(X,d) = \sum_{x\in X} q^{|x|^2/d}\,, \tag{3.5.1}$$

where X is a sparse subset of a (real or hermitian) positive definite, inner product space V and $d \in \mathbb{R}^+$. For the generalised theta function, we have

$$\Theta_{m,k} = \theta(\frac{m}{2k} + \mathbb{Z}, \frac{1}{k})\,, \tag{3.5.2}$$

and hence $V = \mathbb{R}$ and X are the cosets $\mathbb{Z} + \frac{m}{2k}$ of $\mathbb{Z}$. To express the product $\Theta_{m,k}\Theta_{m',k'}$ as a single coset theta function, we observe that if we have two distinct inner product spaces V and V', with respective lattices $\Lambda \subset V$, $\Lambda \subset V'$, then clearly $\Lambda\times\Lambda'$ is a lattice in $V \oplus V'$ and cosets obey $(v+\Lambda)\times(v'+\Lambda') = (v,v') + \Lambda\times\Lambda'$, which translates at the level of the coset theta function to the identity

$$\theta((v,v') + \Lambda\times\Lambda', d) = \theta(v+\Lambda, d)\theta(v'+\Lambda', d)\,. \tag{3.5.3}$$

For the product $\Theta_{m,k}\Theta_{m',k'}$ we take $V = \mathbb{R}$ and $V' = i\mathbb{R}$, noting that identifying in the second factor $\mathbb{R}$ with $i\mathbb{R}$ does not alter the lattice sum. Then we take the direct sum $V \oplus V' \cong \mathbb{C}$ as inner product space, and find

$$\begin{aligned}\Theta_{m,k}\Theta_{m',k'} &= \theta((\frac{m}{2\sqrt{k}} \pm i(\frac{m'}{2\sqrt{k'}})) + (\sqrt{k}\mathbb{Z} + i\sqrt{m'}\mathbb{Z}), 1)\\ &= \theta((\frac{m}{\mu\sqrt{k_0 h}} \pm i\frac{m'}{\lambda\sqrt{k_0' h}}) + 2(\mu\sqrt{k_0 h}\mathbb{Z} + i\lambda\sqrt{k_0' h}\mathbb{Z}), 4)\\ &= \theta((m\lambda k_0' \pm m'\mu\sqrt{-D'}) + 2\lambda\mu h k_0'(\mu k_0\mathbb{Z} + \lambda\sqrt{-D'}\mathbb{Z}), 4kk'k_0'/h)\,,\end{aligned} \tag{3.5.4}$$

where we used the rescaling identity $\theta(\alpha X, |\alpha|^2 d) = \theta(X,d)$ and introduced the variables $\gcd(k,k') = h$, $\frac{k}{h} = \mu^2 k_0$, $\frac{k'}{h} = \lambda^2 k_0'$, with k_0, k_0' positive and square-free, and $D' = k_0 k_0'$. The product $\Theta_{m,k}\Theta_{m',k'}$ is therefore a coset theta function with inner product space $\mathbb{C}$ and cosets $\alpha + J$, for the lattice $J = 2\lambda\mu h k_0'\langle \mu k_0\mathbb{Z}, \lambda\sqrt{-D'}\mathbb{Z}\rangle$ and $\alpha = (m\lambda k_0' \pm m'\mu\sqrt{-D'})$.

We now wish to interpret the coset theta function (3.5.4) as a Ray class theta function, which is a standard object in algebraic number theory and class field theory. To this end, we first recall several basic definitions from algebraic number theory, which we will need in what follows (for a detailed introduction, see e.g., [164]).

A *quadratic imaginary extension* of $\mathbb{Q}$ is a subfield of the complex numbers which, when regarded as a vector space over $\mathbb{Q}$, has dimension two. Such fields have the explicit form

$$K = \mathbb{Q}(\sqrt{-D'}) = \{a + b\sqrt{-D'}\,,\ a, b \in \mathbb{Q}\}\,, \tag{3.5.5}$$

where $D' > 0$ is a square-free integer. We denote by $\mathcal{O}_K$ the ring of integers in K, consisting of those elements of K which are roots of monic polynomials with coefficients in $\mathbb{Z}$. For imaginary quadratic fields, the ring of integers is explicitly given by

$$\mathcal{O}_K = \begin{cases} \langle 1, \frac{1}{2}(1+\sqrt{-D'})\rangle\ , & -D' \equiv 1 \mod 4 \\ \langle 1, \sqrt{-D'}\rangle\,, & \text{otherwise}\,. \end{cases} \tag{3.5.6}$$

The invertible elements in K and $\mathcal{O}_K$ are called *units* and are typically denoted by $K^\times$ and $\mathcal{O}_K^\times$, respectively. For quadratic imaginary fields, the units are precisely the roots of unity contained in K, i.e.,

$$\mathcal{O}_K^\times = \begin{cases} \{\pm 1,\, \pm i\} & \text{if } D' = 1, \\ \{\pm 1,\, \pm\omega,\, \pm\omega^2\} & \text{if } D' = 3, \quad \omega = e^{2\pi i/3}, \\ \{\pm 1\} & \text{otherwise}\,. \end{cases} \tag{3.5.7}$$

A *fractional ideal* I of $\mathcal{O}_K$ is an $\mathcal{O}_K$-submodule of K generated by finitely many elements $e_1, \ldots, e_n \in K$:

$$I = \{\sum_{i=1}^{n} \alpha_i e_i \,|\, \alpha_i \in \mathcal{O}_K\}\,. \tag{3.5.8}$$

If I can be generated by a single element, i.e., $I = \alpha\,\mathcal{O}_K$ for some $\alpha \in K$, then I is called a *principal ideal*. We denote by $\mathcal{I}(K)$ the group of all fractional ideals of K and by $\mathcal{P}(K)$ the subgroup of principal fractional ideals. The fractional ideals form an abelian group under *ideal multiplication*, which is defined by

$$I.J = \Big\{\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}\, e_i\, e'_j \mid \alpha_{ij} \in \mathcal{O}_K\Big\}, \tag{3.5.9}$$

where $I = \langle e_1, \ldots, e_n\rangle_{\mathcal{O}_K}$ and $J = \langle e'_1, \ldots, e'_m\rangle_{\mathcal{O}_K}$. The neutral element under ideal multiplication is $\mathcal{O}_K$. The quotient

$$\mathcal{C}(K) = \mathcal{I}(K)/\mathcal{P}(K)\,, \tag{3.5.10}$$

is called the *ideal class group* of K and measures the failure of unique factorisation of elements into irreducibles in $\mathcal{O}_K$. It plays a fundamental role in algebraic number theory and class field theory, governing the arithmetic structure of K. A classical example that illustrates the failure of unique factorisation is the ring of integers $\mathbb{Z}(\sqrt{-5})$. In this ring, the element 6 admits two distinct factorisations into irreducible elements

$$6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})\,. \tag{3.5.11}$$

One checks directly that none of the factors 2, 3, $1+\sqrt{-5}$, or $1-\sqrt{-5}$ can be written as a product of non-unit elements, and clearly none of them are associated. Such elements are called *irreducible*: an element $\alpha \neq 0$ in an integral domain is called irreducible if it is not a unit and whenever $\alpha = \beta\gamma$, then either β or γ must be a unit. This demonstrates that $\mathbb{Z}(\sqrt{-5})$ is not a *unique factorisation domain* (UFD). The ideal class group of $K = \mathbb{Q}(\sqrt{-5})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, as one can check that the non-principal ideal $I = \left(2, 1+\sqrt{-5}\right)$ satisfies $I^2 = (2)$ which is principal.

A fractional ideal $I \in \mathcal{I}(K)$ is called *integral* if $I \subseteq \mathcal{O}_K$ and *prime* if it is integral and not contained in any other ideal but $\mathcal{O}_K$, i.e., it is *maximal* with respect to set inclusion. Note that the ring of integers $\mathcal{O}_K$ of any number field is a so-called *Dedekind domain*, for which every non-zero prime ideal is a maximal ideal. More generally, an ideal $\mathfrak{p}$ of a commutative ring R is called *prime* if $\mathfrak{p}$ is not the whole ring R and if $a, b \in R$ with $ab \in \mathfrak{p}$ then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Similar to the prime factorisation of integers, every fractional ideal $I \in \mathcal{I}(K)$ factorises uniquely into prime ideals $\mathfrak{p}_i$[3]

$$I = \prod_{i=1}^{n} \mathfrak{p}_i^{v_{\mathfrak{p}_i}(I)}\,, \tag{3.5.12}$$

where each $\mathfrak{p}_i$ is a distinct prime ideal of $\mathcal{O}_K$ and $v_{\mathfrak{p}_i}(I) \in \mathbb{Z}$ is the *valuation* of I at $\mathfrak{p}_i$. For any given I, only finitely many valuations are non-zero. Revisiting the classical example $\mathbb{Z}(\sqrt{-5})$, one can show that unique factorisation (up to permutation) is restored at the level of ideals,

$$(6) = (2)\,.\,(3) = \mathfrak{p}^2.\mathfrak{q}.\overline{\mathfrak{q}} = (\mathfrak{p}.\mathfrak{q})\,.\,(\mathfrak{p}.\overline{\mathfrak{q}}) = \left(1+\sqrt{-5}\right).\left(1-\sqrt{-5}\right)\,, \tag{3.5.13}$$

where $\mathfrak{p} = (2, 1+\sqrt{-5})$, $\mathfrak{q} = (3, 1+\sqrt{-5})$, $\overline{\mathfrak{q}} = (2, 1-\sqrt{-5})$ are prime ideals. Clearly (2), (3), $(1 \pm \sqrt{-5})$ are not prime ideals by eq. (3.5.11), as in any integral domain a principal ideal is prime if and only if it is generated by a *prime element*, i.e., an element p in a commutative ring R which is not a unit and for which it follows that whenever p divides a product ab with $a, b \in R$, then p divides a or b.

Given two fractional ideals $I, J \in \mathcal{I}(K)$, we say that I divides J, written $I \mid J$, if $J.I^{-1}$ is integral. The least common multiple and greatest common divisor (highest common factor) of two ideals are defined respectively by

$$\mathrm{lcm}(I,J) = I \cap J\,, \qquad \mathrm{hcf}(I,J) = I + J = \{a + b \,|\, a \in I, b \in J\}\,. \tag{3.5.14}$$

[3] Unique factorisation into prime ideals holds in all Dedekind domains.

Two ideals I and J are said to be *coprime* if their highest common factor equals the unit ideal, i.e., $\mathrm{hcf}(I,J)=\mathcal{O}_K$. The *norm* of an integral ideal $\mathcal{N}(I)$ is defined to be the finite number of elements in the quotient group $\mathcal{O}_K/I$. For principal ideals $\mathcal{I}=\alpha\mathcal{O}_K$ we have $\mathcal{N}(I)=|\alpha|^2$. For a fractional ideal $J=I.(I')^{-1}$, we define $\mathcal{N}(J)=\mathcal{N}(I)/\mathcal{N}(I')$, where I and I' are integral ideals. One verifies that the norm is multiplicative.

Let us now fix an integral ideal $F\subseteq\mathcal{O}_K$, which we refer to as the *conductor*. We define the set of nonzero fractional ideals of quotients of ideals which are coprime to F by

$$\mathcal{I}_F(K)=\Big\{I\in\mathcal{I}(K)\;\Big|\;v_{\mathfrak{p}_i}(I)=0\text{ for all prime ideals }\mathfrak{p}_i\mid F\Big\}\,. \tag{3.5.15}$$

Next, we introduce the set of principal ideals generated by quotients of elements in $\mathcal{O}_K$ that are congruent modulo F and coprime to F, i.e.,

$$K_{1,F}=\Big\{\lambda/\mu\;\big|\;\lambda-\mu\in F\,,\;\mu\mathcal{O}_K+F=\mathcal{O}_K\,,\;\lambda\mu\neq0\,,\;\lambda,\mu\in\mathcal{O}_K\Big\}\,, \tag{3.5.16}$$

and set $\mathcal{P}_F(K)=\{\alpha\mathcal{O}_K\;\big|\;\alpha\in K_{1,F}\}$. We then define the *ray class group* of $\mathcal{O}_K$ with respect to the conductor F to be the quotient

$$\mathcal{C}_F(K)=\frac{\mathcal{I}_F(K)}{\mathcal{P}_F(K)}\,. \tag{3.5.17}$$

The elements of the finite group $\mathcal{C}_F(K)$ are the cosets $I\mathcal{P}_F(K)$ for some $I\in\mathcal{I}_F(K)$, which we denote by $[I]_F$. To each *ray class* $[I]_F$ we associate the *ray class theta function*

$$\theta([I]_F;d)=\sum_{v\in[I]_F,\,v\subset\mathcal{O}_K}q^{\mathcal{N}(v)/d}\,,\;d\in\mathbb{R}^+\,. \tag{3.5.18}$$

The connection to the coset theta function (3.5.4) is as follows. The quadratic imaginary field extension K is fixed by the discriminant D' in (3.5.4). Let us assume for simplicity that the lattice J in (3.5.4) is an ideal of $\mathcal{O}_K$. This is clearly not the most general situation. Recall that the lattice J is a rank-2 $\mathbb{Z}$-module in K. It is an ideal of $\mathcal{O}_K$ iff it is closed under multiplication by all elements of $\mathcal{O}_K$. The $\mathbb{Z}$-module $J=x\mathbb{Z}\oplus(y+z\omega)\,\mathbb{Z}$ with $x,z\in\mathbb{N}$, $y\in\mathbb{Z}$ and ω the generator of $\mathcal{O}_K$, is an ideal of $\mathcal{O}_K$ iff $z\mid y$, $z\mid x$ and $x\mid zN(a+\omega)$ with $y=az$, $a\in\mathbb{Z}$ (for a straightforward proof see for instance [164, 165]).

The norm N of an element $\alpha=a+b\sqrt{-D'}$, $a,b\in\mathbb{Q}$ in K is defined as $N(\alpha):=a^2+b^2D'$. For the lattice J in (3.5.4), this is clearly not always satisfied. Take, for instance, $k=1$, $k'=3$. However, looking at the definition of the lattice J in (3.5.4) we see that there always exists a sublattice in J given by $d\mathcal{O}_K$, for some integer $d\in\mathbb{N}$, which is an integral ideal of $\mathcal{O}_K$. For the example $k=1$, $k'=3$ the smallest integer d such that $d\mathcal{O}_K\subset J$ is $d=12$. Hence, even when J itself is not an ideal in $\mathcal{O}_K$, it can always be decomposed in terms of a coset sum with respect to an integral ideal $d\mathcal{O}_K$,

$$J=\bigsqcup_{w\in J/d\mathcal{O}_K}(w+b)\,,\qquad b\in d\mathcal{O}_K\,. \tag{3.5.19}$$

Let us now continue with the assumption that J is an ideal in $\mathcal{O}_K$ (otherwise one replaces J by $d\mathcal{O}_K$). In this case, $J \in \mathcal{I}(K)$ and $\alpha \in K \setminus J$, with α assumed to be nonzero. We set $H = \text{hcf}(\alpha\mathcal{O}_K, J)$ and the conductor $F = J.H^{-1}$. For this choice we find that $[\alpha H^{-1}] \in \mathcal{I}_F(K)$, as

$$\text{hcf}(\alpha H^{-1}, F) = \alpha H^{-1} + F = (\alpha\mathcal{O}_K + F.H).H^{-1} = \mathcal{O}_K\,, \tag{3.5.20}$$

where we use that $\mathcal{O}_K$ is the neutral element under ideal multiplication and

$$\alpha\mathcal{O}_K + F.H = \alpha\mathcal{O}_K + J.H^{-1}.H = H\,. \tag{3.5.21}$$

We denote by $(\mathcal{O}_K^\times)_F$ the group of units of $\mathcal{O}_K$ which are also contained in $K_{1,F}$ and by w_F its order, then we have the following correspondence between the coset theta function and ray class theta function [163]

$$\theta(\alpha + J, d) = w_F \theta([\alpha H^{-1}]_F, d/\mathcal{N}(H))\,. \tag{3.5.22}$$

This relation provides a concrete correspondence between the norm of the lattice points in the coset theta function on the left, and the norm of elements in the ray class $[\alpha H^{-1}]_F$ on the right. We have shown in (3.3.10) that a rational toroidal partition function for even-dimensional tori can be decomposed in terms of products of coset theta functions. With eq. (3.5.22) we can thus explicitly interpret the conformal weights of the states in the Hilbert space of a given rational toroidal conformal field theory as a sum of the (scaled) norm of integral ideals in specific ray classes. These ray classes are determined purely by conformal field theoretic datum, according to (3.5.22), which depends explicitly on the precise location of the rational toroidal conformal field theory in the moduli space.

Example

As an example, let us consider the rational toroidal conformal field theory on T^2 with fixed moduli $u = \frac{\sqrt{-2}}{3}$ and $t = 2\sqrt{-2}$, following the conventions used in section 3.3. This theory is equivalent to the product of two rational S^1 theories with radii $R_1 = \sqrt{12}$ and $R_2 = \sqrt{6}$ respectively. The partition function follows directly from (3.1.2) and takes the form

$$\begin{aligned} Z_{T^2} =&\big(\mathcal{K}_{0,6}\,\overline{\mathcal{K}}_{0,6} + 2\,\mathcal{K}_{1,6}\,\overline{\mathcal{K}}_{1,6} + 2\,\mathcal{K}_{2,6}\,\overline{\mathcal{K}}_{2,6} + 2\,\mathcal{K}_{3,6}\,\overline{\mathcal{K}}_{3,6} \\ &+ 2\,\mathcal{K}_{4,6}\,\overline{\mathcal{K}}_{4,6} + 2\,\mathcal{K}_{5,6}\,\overline{\mathcal{K}}_{5,6} + \mathcal{K}_{6,6}\,\overline{\mathcal{K}}_{6,6}\big)\times \\ &\times \big(\mathcal{K}_{0,12}\,\overline{\mathcal{K}}_{0,12} + 2\,\mathcal{K}_{7,12}\,\overline{\mathcal{K}}_{1,12} + 2\,\mathcal{K}_{10,12}\,\overline{\mathcal{K}}_{2,12} \\ &+ 2\,\mathcal{K}_{3,12}\,\overline{\mathcal{K}}_{3,12} + 2\,\mathcal{K}_{4,12}\,\overline{\mathcal{K}}_{4,12} + 2\,\mathcal{K}_{11,12}\,\overline{\mathcal{K}}_{5,12} \\ &+ 2\,\mathcal{K}_{6,12}\,\overline{\mathcal{K}}_{6,12} + 2\,\mathcal{K}_{1,12}\,\overline{\mathcal{K}}_{7,12} + 2\,\mathcal{K}_{8,12}\,\overline{\mathcal{K}}_{8,12} \\ &+ 2\,\mathcal{K}_{9,12}\,\overline{\mathcal{K}}_{9,12} + 2\,\mathcal{K}_{2,12}\,\overline{\mathcal{K}}_{10,12} + 2\,\mathcal{K}_{5,12}\,\overline{\mathcal{K}}_{11,12} + \mathcal{K}_{12,12}\,\overline{\mathcal{K}}_{12,12}\big)\,. \end{aligned} \tag{3.5.23}$$

Let us, for illustration, write the coset theta function in $\mathcal{K}_6\mathcal{K}_{1,12}$ as a ray class theta function. From eq. (3.5.4) we find

$$\Theta_{1,6}\Theta_{1,12} = \theta\left(J + \alpha, d\right)\,, \tag{3.5.24}$$

with $D' = 2$, $J = 24\mathcal{O}_K$, $d = 96$ and $\alpha = 2 + \sqrt{-D'}$. In imaginary quadratic number fields, all prime ideals occur as prime factors of the principal ideal $p\mathcal{O}_K$ for some prime number p, and they need to have norm p or p^2. Either there is a prime ideal P with norm p, then $p\mathcal{O}_K = P.\overline{P}$, or if there is none, then $p\mathcal{O}_K$ is itself a prime ideal with norm p^2. In the letter case, the prime p is called *innert*. We find the following factorisations in the ring of integers of $K = \sqrt{-2}$ as $(2)\mathcal{O}_K = P_2^2$ with $P_2 := (\sqrt{-2})\mathcal{O}_K$, i.e., the prime 2 *ramifies*, and $(3)\mathcal{O}_K = P_3.\overline{P_3}$, with $P_3 := (1 + \sqrt{-2})\mathcal{O}_K$. The factorisation of J then follows as

$$J = (24)\mathcal{O}_K = (2^3)(3)\mathcal{O}_K = P_2^6.P_3.\overline{P}_3 \,. \tag{3.5.25}$$

Next we need to determine the ideal $H = \mathrm{hcf}(J, \alpha\mathcal{O}_{\mathcal{K}})$. The prime ideal factorisation of $\alpha\mathcal{O}_{\mathcal{K}}$ can be found by rewriting $\alpha = 2 + \rho = \rho(1 - \rho)$ with $\rho = \sqrt{-2}$, and hence

$$\alpha\mathcal{O}_K = P_2.\overline{P}_3 \,. \tag{3.5.26}$$

Thus we find $H = P_2.\overline{P}_3$ with $\mathcal{N}(H) = 6$. Finally, the conductor is

$$F = J.H^{-1} = P_2^5.P_3 = 4P_2.P_3 \,, \tag{3.5.27}$$

hence we get

$$\Theta_{1,6}\Theta_{1,12} = \theta\left([(1)]_{4P_2.P_3}\,, 16\right) \,. \tag{3.5.28}$$

To evaluate the expression on the right-hand side, we need to determine the set $K_{1,4P_2.P_3}$, see eq. (3.5.16), as the above sums over the norm of the corresponding principal integral ideals in $\mathcal{P}_{4P_2.P_3}(K)$. Thus, we are looking for all non-zero elements $\lambda, \mu \in \mathcal{O}_K$ with the property that their difference belongs to $4P_2.P_3$ and $\mu\mathcal{O}_K + 4P_2.P_3 = \mathcal{O}_K$. Given these two conditions, the set $K_{1,4P_2.P_3}$ takes the general form

$$K_{1,4P_2.P_3} = \left\{ 1 + \frac{\delta}{\mu} \;\middle|\; \delta \in 4P_2.P_3,\ \mu \in \mathcal{O}_K,\ \mu \neq 0,\ \mu\mathcal{O}_K + 4P_2.P_3 = \mathcal{O}_K \right\} . \tag{3.5.29}$$

Now we use $\delta = 4(-2 + \sqrt{-2})(a + b\sqrt{-2})$ and $\mu = 1$ to restrict to the corresponding integral principal ideals in $\mathcal{P}_{4P_2.P_3}(K)$, to finally get

$$\theta\left([(1)]_{4P_2.P_3}\,, 16\right) = \sum_{a,b\in\mathbb{Z}} q^{\frac{1}{16}(1-16a+96a^2-16b+192b^2)} = \Theta_{1,6}\Theta_{1,12} \,. \tag{3.5.30}$$

Recall that the contribution from the generators of the $\widehat{\mathfrak{u}}(1)_6 \otimes \widehat{\mathfrak{u}}(1)_{12}$ current algebra to the partition function comes from

$$\Theta_{0,6}\Theta_{0,12} \,. \tag{3.5.31}$$

The coefficient α in (3.5.22) is therefore zero, but note that the zero ideal (0) by definition does not specify a ray class. Hence, formula (3.5.22) cannot be applied directly. However, (3.5.2) enjoys the identity $\Theta_{m,k} = \Theta_{m+2k,k}$, therefore

$$\Theta_{0,6}\Theta_{0,12} = \Theta_{12,6}\Theta_{0,12} \,, \tag{3.5.32}$$

hence $\alpha = 24$, $H = (\alpha) = J$ and consequently the conductor is $F = \mathcal{O}_K$. Comparing to (3.5.22) and (3.5.18), the corresponding ray class theta function is a summation over the norm of all principal integral ideals $(a + b\sqrt{-2})$ of $\mathcal{O}_{\mathcal{K}}$ with scale factor $d = 1/6$, i.e.,

$$\theta\left([(1)]_{\mathcal{O}_K}, 1/6\right) = \sum_{a,b\in\mathbb{Z}} q^{6(a^2+2b^2)} = \Theta_{0,6}\Theta_{0,12}\,. \tag{3.5.33}$$

In this way, one can explicitly write every term in the partition function of a rational toroidal conformal field theory associated with even dimensional tori, using eq. (3.5.22), in terms of a product of an overall inverse Dedekind eta function raised to some power times a sum of products of ray class theta functions, see eq. (3.3.6). This expansion, in particular the Ideal calculations presented above, can be easily implemented in standard computer algebra software, such as SageMath or PARI/GP. We have therefore achieved our goal in this section by providing a fully number-theoretic explanation of the partition function of rational toroidal conformal field theories associated with even-dimensional tori.

Chapter 4

Rational Conformal Field Theories and CM–Hodge Structures

In this chapter, we explore the classification of two-dimensional rational conformal field theories using number-theoretic methods, with a particular emphasis on the structure and distribution of these theories within the moduli space of general two-dimensional conformal field theories. The classification of rational conformal field theories is a longstanding problem of great significance in both mathematical and theoretical physics. From a physical perspective, rational conformal field theories are among the few quantum field theories that can be solved exactly; that is, the spectrum, correlation functions, and operator product expansions can be computed explicitly. The solvability makes them ideal candidates for exploring explicit string vacua in the framework of (perturbative) string theory in small volume regimes where classical supergravity approximations break down. As such, they automatically incorporate worldsheet instanton corrections and, through string dualities, sometimes even capture non-perturbative (in the string coupling constant g_s) features of the theory, thus offering rare windows into regimes where standard perturbative tools break down.

A central question in this context concerns the distribution of rational conformal field theories within moduli spaces, i.e., whether they are dense in these spaces or if they are located at isolated points. The question of denseness was investigated in various works through methods of conformal perturbation theory, see for instance [89–91]. In [92, 94] a rather different approach was suggested. It was observed that for $\mathcal{N} = (2,2)$ superconformal field theories with complex one-dimensional tori as target space, the rationality condition can be formulated as a number-theoretic property of the underlying rational Hodge structure of the complex torus. Namely, it was found that an $\mathcal{N} = (2,2)$ superconformal field theory compactified on a complex one-dimensional torus is rational iff the complex torus with fixed complex structure and its mirror complex torus, which specify the parameter of the theory, are of *complex multiplication type* (CM-type), with endomorphism algebras being isomorphic to the same quadratic imaginary number field. As the authors in [94] concluded, this suggests that the presence of a CM-type Hodge structure is connected to the rationality in the underlying $\mathcal{N} = (2,2)$ superconformal field theory. Note that an $\mathcal{N} = (2,2)$ superconformal field theory compactified on a complex torus is rational, iff the bosonic conformal field theory on the underlying real torus is rational [93], see also eq. (4.1.1).

This chapter aims to develop and generalise this connection to generic Hodge structures associated with $\mathcal{N} = (2,2)$ rational superconformal field theories. After reviewing the well-established results linking CM-type elliptic curves and CM-type abelian varieties to rational conformal field theories [93, 94, 166–168], we will explore how these ideas extend to more general geometric and non-geometric settings. We will exemplify our construction with $\mathcal{N} = (2,2)$ rational superconformal field theories associated with Calabi–Yau threefolds (so-called *Gepner models*), which play a central role in supersymmetric string compactifications, and also consider more exotic Hodge structures that do not necessarily arise from algebraic varieties but still exhibit CM-like properties. The latter case will be exemplified with $\mathcal{N} = (2,2)$ minimal models. Under mild assumptions, we will show how a given $\mathcal{N} = (2,2)$ rational superconformal field theory endows the associated Hodge structure with additional symmetries, specifically at the level of endomorphism algebras, which are shown to satisfy the conditions of CM-type Hodge structures. This approach suggests a possible organising principle for classifying certain $\mathcal{N} = (2,2)$ rational superconformal field theories and indicates a connection between number theory and quantum field theory. In particular, it offers a framework for further exploring the relationship between number theory and rational conformal field theory, as preliminarily discussed in the previous chapter.

4.1 Rational Conformal Field Theories and CM-type Abelian Varieties

Overview

We have seen in eq. (3.3.3) that the rationality condition of a toroidal conformal field theory with target space $T^D = \mathbb{R}^D/\Lambda$ can be formulated in terms of a constant Riemannian metric g taking rational values on $\Lambda \cong H_1(T^D, \mathbb{Z})$, together with a rational valued B-field, when restricted to the lattice Λ. We now want to understand this criterion Hodge theoretically. Let us therefore restrict to D even and add to the bosonic action (3.2.1) a number of D Majorana fermionic fields, which are the superpartners of the bosonic abelian currents $j_1, \ldots, j_D$. The resulting theory can be shown to possess (two copies of) an $\mathcal{N} = 2$ superconformal structure with U(1) R-current J and supercurrents $G^{\pm}$ inducing a complex structure on T^D, and thus an isomorphism[1] $T^D \cong \mathbb{C}^{D/2}/\Lambda$ [93, 141, 146]. The partition

[1] Starting with a bosonic toroidal conformal field theory with fixed Riemannian metric g and B-field B, the uplift to an $\mathcal{N} = (2,2)$ toroidal conformal field theory requires a choice of a complex structure I compatible with g such that g becomes a Kähler metric. This choice, for fixed g and B, is not unique

function of an $\mathcal{N}=(2,2)$ toroidal theory reads

$$Z_{\mathcal{N}=2}(\tau,z) = Z_\Gamma(\tau)\cdot\frac{1}{2}\sum_{i=1}^{4}\left|\frac{\vartheta_i(\tau,z)}{\eta(\tau)}\right|^D, \tag{4.1.1}$$

with classical theta functions,

$$\begin{aligned}
\vartheta_1(\tau,z) &:= \mathrm{i}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2}y^{n-\frac{1}{2}}\\
\vartheta_2(\tau,z) &:= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2}y^{n-\frac{1}{2}}\\
\vartheta_3(\tau,z) &:= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}}y^{n}\\
\vartheta_4(\tau,z) &:= \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n^2}{2}}y^{n}\,, \qquad y=e^{2\pi\mathrm{i}z}\,,
\end{aligned} \tag{4.1.2}$$

and $Z_\Gamma(\tau)$ denotes the Narain lattice sum of the underlying bosonic toroidal conformal field theory, which depends only on the chosen metric g and B-field B. The classical theta-functions ϑ_i encode the various combinations of spin-structure along the two non-trivial one-cycles of the worldsheet torus. We thus see from (4.1.1) that the $\mathcal{N}=(2,2)$ toroidal conformal field theory is rational iff the corresponding bosonic toroidal conformal field theory is rational. We observed in the previous section that for complex one-dimensional tori with complex structure parameter $u\in\mathcal{H}$ and complexified Kähler parameter $t\in\mathcal{H}$, the rationality criterion can be formulated as

$$u,t\in\mathbb{Q}(\sqrt{-D'})\,, \tag{4.1.3}$$

for some positive square-free integer D'. In [94], the $\mathcal{N}=2$ structure of the toroidal conformal field theory was used to formulate the criterion (4.1.3) in terms of *mirror symmetry*. Mirror symmetry [153, 170] is the observation in string theory that two different Calabi–Yau manifolds X and Y give rise to the same physical theory, when used as internal compactification manifolds. On the level of topological string theory, it states that the topological A-model on X is equivalent to the topological B-model on Y. Using the full $\mathcal{N}=2$ internal worldsheet conformal field theory, it states that the $\mathcal{N}=(2,2)$ superconformal field theory on X is equivalent to the $\mathcal{N}=(2,2)$ superconformal field theory on Y. We will be more precise with the definition of mirror symmetry in the case where X is an abelian variety momentarily. For the present discussion it suffices to note that, from an $\mathcal{N}=(2,2)$ superconformal field theory perspective, mirror symmetry is an algebra automorphism $\overline{J}\to-\overline{J}$ and $\overline{G}^{\pm}\to\overline{G}^{\mp}$. For an $\mathcal{N}=(2,2)$ toroidal conformal field

(unless D=2). However, one can show that all compatible choices of I lead to isomorphic $\mathcal{N}=(2,2)$ toroidal conformal field theories in the sense of [169] — the moduli space of toroidal superconformal field theories is isomorphic to the moduli space of bosonic toroidal conformal field theories [93].

theory on T^2_u with complexified Kähler modulus t this mirror automorphism exchanges the complex structure parameter u and t, which by inspecting eq. (3.3.16) indeed preserves the spectrum (under appropriate change of winding and momentum numbers). One can show that this is, in fact, a symmetry of the full conformal field theory. We conclude that for $\mathcal{N} = (2,2)$ toroidal conformal field theories on T^2_u, the rationality condition imposes a certain compatibility on the mirror pair (T^2_u, T^2_t), namely (4.1.3). Complex one-dimensional tori, for which the complex structure parameter is valued in a quadratic imaginary field extension, are known to be of *complex multiplication type*.

Complex multiplication is a property of the endomorphism algebra of rational Hodge structures and first occurred in the study of elliptic curves. In [94], the authors conjectured that the relation between rationality in $\mathcal{N} = (2,2)$ superconformal field theory, complex multiplication of rational Hodge structures and mirror symmetry should hold more generally for $\mathcal{N} = (2,2)$ superconformal field theories used in compactifications in string theory on Calabi–Yau manifolds X. They conjectured that an $\mathcal{N} = (2,2)$ superconformal field theory compactified on a Calabi–Yau manifold X is rational iff both X and the mirror manifold Y are of CM-type with isomorphic CM fields. In [166–168], the authors pointed out several shortcomings of the original conjecture by explicitly constructing counterexamples to the mirror condition.

In [166], the following theorem was proven:

An abelian variety is of CM-type iff it admits a constant rational Kähler metric. (4.1.4)

This theorem, together with the result in [93] that a ($\mathcal{N} = (2,2)$) toroidal conformal field theory is rational iff T^D is isogenous to a product of CM-type complex one-dimensional tori for even D, establishes a precise relation between CM-type abelian varieties and $\mathcal{N} = (2,2)$ rational superconformal field theories. Recall that an abelian variety over the complex numbers $\mathbb{C}$ is a complex torus $X = \mathbb{C}^d/\Lambda$, with $d = D/2$, that can be embedded into projective space $\mathbb{P}^n$ for some integer n. Equivalently, it is a complex torus that admits a positive definite Riemann form, i.e., it is polarisable. Complex tori, which are isogenous to products of elliptic curves, are abelian varieties with naturally induced polarisation [171]. In this way, all rational toroidal conformal field theories are associated with CM-type abelian varieties. Recall that an *isogeny* is a surjective holomorphic group homomorphism with finite kernel,

$$f \,:\, A \to B\,, \tag{4.1.5}$$

which preserves the algebraic structure and, in the context of abelian varieties, relates different abelian varieties, which are equivalent up to a finite group action. For two complex tori $X = \mathbb{C}^d/\Pi\mathbb{Z}^{2d}$ and $X' = \mathbb{C}^d/\Pi'\mathbb{Z}^{2d}$ with $(d \times 2d)$-period matrices Π, Π', every isogeny can be described by a complex matrix $C \in \mathrm{GL}(d,\mathbb{C})$ and a rational matrix $\gamma \in \mathrm{GL}(2d,\mathbb{Q})$, such that

$$\Pi' = C\Pi\gamma\,. \tag{4.1.6}$$

In a toroidal conformal field theory, an operation that relates theories compactified on isogenous complex tori is given by so-called shift orbifolds [93], i.e., an orbifold by the group generated by shift symmetries of the Narain lattice.

As pointed out in [166], while all toroidal rational conformal field theories are related to CM-type abelian varieties, the converse — particularly the connection with mirror symmetry — is more subtle. Specifically, it was demonstrated that one can construct a pair of mirror abelian varieties of CM-type over the same CM–field K, for which the underlying $\mathcal{N} = (2,2)$ toroidal conformal field theory is not rational. To clarify this important observation, we briefly summarise the relevant construction below, following the notation of [166, 169].

Generalised Kähler Structure and Mirror Symmetry

Let X be a complex abelian variety of complex dimension $D/2$, that is, a complex torus which admits an embedding into projective space and hence a polarisation. To formalise mirror symmetry for abelian varieties, it is convenient to introduce the framework of generalised complex geometry, which unifies symplectic and complex geometry under one structure [172].

A *generalised complex structure* on a smooth real manifold M is a linear map $\mathcal{I} : TM \oplus T^*M \to TM \oplus T^*M$ which is an almost complex structure and furthermore satisfies

(i) $\mathcal{I}^2 = -\mathrm{id}$,

(ii) $\mathcal{I}$ preserves the canonical symmetric bilinear form q on $TM \oplus T^*M$

$$q((v,\xi),(w,\eta)) := \eta(v) + \xi(w) = q(\mathcal{I}((v,\xi)), \mathcal{I}((w,\eta)))\,. \tag{4.1.7}$$

We then define a *generalised Kähler structure* as a pair $(\mathcal{I}_1, \mathcal{I}_2)$ of commuting generalised complex structures which satisfy that $\mathcal{G} := q\,(\cdot, -\mathcal{I}_1\mathcal{I}_2\cdot)$ defines a positive definite metric on $TM \oplus T^*M$.

We call a real torus $T^D = \mathbb{R}^D/\Lambda$ equipped with a generalised Kähler structure a *generalised complex torus*. For a given complex torus with constant Kähler metric g, complex structure I and constant B-field $B \in H^2(T, \mathbb{R})$ one obtains an *induced generalised complex torus* from

$$\begin{aligned} \mathcal{I} &:= \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I^T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \\ \mathcal{J} &:= \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}, \end{aligned} \tag{4.1.8}$$

with Kähler form $\omega = g(\cdot, I\cdot)$. Positivity of the metric $q(\cdot, -\mathcal{I}\mathcal{J})$ follows from

$$\begin{aligned} \mathcal{I}\mathcal{J} &= \begin{pmatrix} g^{-1}B & -g^{-1} \\ Bg^{-1}B - g & -Bg^{-1} \end{pmatrix} \\ -q\mathcal{I}\mathcal{J} &= \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}^T. \end{aligned} \tag{4.1.9}$$

We call two generalised complex tori $(T^D = \mathbb{R}^D/\Lambda, \mathcal{I}, \mathcal{J})$ and $(T'^D = \mathbb{R}^D/\Lambda', \mathcal{I}', \mathcal{J}')$ mirror to each other if there is a lattice isomorphism

$$\varphi \; : \; \Lambda \oplus \Lambda^* \to \Lambda' \oplus \Lambda^{*\prime} \,, \tag{4.1.10}$$

with $q(\cdot,\cdot) = q'(\varphi\cdot, \varphi\cdot)$, $\mathcal{I}' = \varphi\mathcal{J}\varphi^{-1}$, $\mathcal{J}' = \varphi\mathcal{I}\varphi^{-1}$. In physical terms, this reflects the duality between $\mathcal{N} = (2,2)$ toroidal conformal field theories, exchanging complex structure and Kähler structure of mirror pairs, and flipping the right-moving $\mathrm{U}(1)_R$-charge. In particular the $\mathcal{N} = (2,2)$ toroidal conformal field theory on the generalised complex torus $(T^D = \mathbb{R}^D/\Lambda, \mathcal{I}, \mathcal{J})$ is equivalent to the $\mathcal{N} = (2,2)$ toroidal conformal field theory on the generalised complex torus $(T'^D = \mathbb{R}^D/\Lambda', \mathcal{I}', \mathcal{J}')$.

Recall that a Narain lattice specifies the bosonic part of an $\mathcal{N} = (2,2)$ toroidal conformal field theory with elements being the momentum-winding vectors $p = (p_L, p_R)$, with pairing

$$p \cdot p' := p_L p'_L - p_R p'_R \,. \tag{4.1.11}$$

Choosing coordinates $x := \frac{1}{\sqrt{2}}(p_L + p_R)$ and $y := \frac{1}{\sqrt{2}}(p_L - p_R)$ we have $\Gamma \subset \mathbb{R}^D \oplus (\mathbb{R}^D)^*$ with $(x, y) \cdot (x', y') = xy' + x'y$, which illustrates the connection between the inner product on the Narain lattice Γ and the canonical inner product on $\mathbb{R}^D \oplus (\mathbb{R}^D)^*$, where $\mathbb{R}^D$ is the tangent space of the torus. Comparing to eq. (4.1.9), the rationality condition can be reformulated in terms of the generalised Kähler structure: an $\mathcal{N} = (2,2)$ toroidal conformal field theory with generalised complex torus $(T^D = \mathbb{R}^D/\Lambda, \mathcal{I}, \mathcal{J})$ is rational iff $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$ — that is, it preserves the rational lattice $(\Lambda \oplus \Lambda^*)_{\mathbb{Q}}$.

CM-type Abelian Varieties

Having briefly reviewed mirror symmetry for complex tori, we now turn to the concept of complex multiplication, which plays a central role in the conjecture proposed in [94].

Let $X = \mathbb{C}^{D/2}/\Lambda$ be an abelian variety of complex dimension $D/2$. An abelian variety is called *simple* if it contains no nontrivial abelian subvarieties. We define the endomorphism algebra of X over $\mathbb{Q}$ by

$$\begin{aligned} \mathrm{End}_{\mathbb{Q}}(X) &:= \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \mathrm{End}(X) &= \{ f \; : \; \mathbb{C}^{D/2} \to \mathbb{C}^{D/2} \mid f \text{ is } \mathbb{C}\text{-linear and } f(\Lambda) \subset \Lambda \} \,. \end{aligned} \tag{4.1.12}$$

Let us now define a complex multiplication (CM) type abelian variety.

(i) A *simple* abelian variety X is said to be of *CM-type*, if there exists an embedding

$$K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(X) \,, \tag{4.1.13}$$

of a number field K of degree $[K : \mathbb{Q}] = D$ into the endomorphism algebra of X. Such a field K is necessarily a CM field [173], i.e., a totally imaginary quadratic extension of a totally real number field (which means that the image of every embedding of the base field into $\mathbb{C}$ lies in $\mathbb{R}$, and there is no embedding of K into $\mathbb{R}$).

(ii) An arbitrary abelian variety X (not necessarily simple) is of *CM-type* if X is isogenous to a product of simple abelian varieties,

$$X_1 \times X_2 \times \cdots \times X_r\,, \tag{4.1.14}$$

each of which is of CM-type in the above sense [173].

This definition reflects the fact that CM-type abelian varieties are those whose endomorphism algebra is as large as possible, and they play a central role in arithmetic geometry and Hodge theory.

The following theorem, due to [166], highlights the central role that CM-type abelian varieties play in the context of toroidal rational conformal field theories:

Theorem 4.1.1. Let $X = V/\Lambda$ be a complex abelian variety of dimension $D/2$, with $V \cong \mathbb{C}^{D/2}$ and $\Lambda \subset V$ is a full lattice. Then X is of CM-type if and only if there exists a compatible constant Kähler metric g on X such that

$$g(\gamma,\delta) \in \mathbb{Q}\,, \quad \text{for all } \gamma,\delta \in \Lambda\,. \tag{4.1.15}$$

But from eq. (3.3.3), we know that this is precisely the condition for an ($\mathcal{N} = (2,2)$) toroidal conformal field theory to be rational. This establishes the significant role of CM-type abelian varieties as target spaces in ($\mathcal{N} = (2,2)$) toroidal conformal field theories. A proof of this important result can be found in [166]. The idea is that every CM-type simple abelian variety is isogenous to another CM-type simple abelian variety, which admits a Riemann form of a specific type:

$$E(z,w) = \sum_{j=1}^{D/2} \sigma_j(\beta)\left(\sigma_j(z)\overline{\sigma_j(w)} - \overline{\sigma_j(z)}\sigma_j(w)\right)\,, \tag{4.1.16}$$

where $\Phi = \{\sigma_i, \ldots, \sigma_{D/2}\}$ is a *CM-type* — that is, $\Phi \cup \overline{\Phi}$ constitutes the full set of embeddings of a number field K into $\mathbb{C}$. The element $\beta \in \mathcal{O}_K$ satisfies $K = K_0(\beta)$, where K_0 is a totally real field and $-\beta^2$ is totally positive (i.e., its image under every real embedding is positive), with $\mathrm{Im}(\sigma_j(\beta)) > 0$ for all j.

In particular, the Riemann form is rational valued on $\mathcal{O}_K$,

$$E(z,w) = \mathrm{Tr}_{K/\mathbb{Q}}(\beta z\overline{w}) \in \mathbb{Q}\,, \qquad \forall z,w \in \mathcal{O}_K\,. \tag{4.1.17}$$

Then define the bilinear form,

$$g(z,w) := E(z,\beta w) = \mathrm{Tr}_{K/\mathbb{Q}}(-\beta^2 z\overline{w})\,, \tag{4.1.18}$$

which is *positive definite* because $-\beta^2$ is totally positive. The form g is also *rational*, *symmetric* and, as seen from (4.1.18), *compatible* with the complex structure I of the abelian variety. Hence, for any CM-type abelian variety, there exists a compatible rational

Kähler metric[2]. The converse direction is more subtle, and we refer to [166] for a detailed discussion.

Theorem 4.1.1 establishes a rigorous connection between toroidal rational conformal field theory and CM-type abelian varieties, without referring to mirror symmetry.

[2] The existence of a compatible rational Kähler metric is preserved under isogenies.

CM-type Abelian Varieties, Mirror Symmetry and Rational Conformal Field Theory

We now present an explicit example illustrating that the mirror symmetry aspect of the original conjecture in [94] does not straightforwardly generalise beyond the case of complex one-dimensional $\mathcal{N} = (2,2)$ toroidal conformal field theories. Specifically, we will see that there exist mirror pairs of abelian varieties, both of CM-type over the same number field K, which nevertheless yield non-rational $\mathcal{N} = (2,2)$ toroidal conformal field theories. This demonstrates that not every mirror pair of algebraic varieties, which are of CM-type over the same number field, gives rise to a rational $\mathcal{N} = (2,2)$ superconformal field theory.

Let us consider the complex torus $X = \mathbb{C}^2/\Lambda$, where the lattice Λ is constructed as in [166]. Specifically, let $K = \mathbb{Q}(\zeta_5)$ be the cyclotomic field generated by a primitive fifth root of unity $\zeta_5 = e^{\frac{2\pi \mathrm{i}}{5}}$, so that $\zeta_5^5 = 1$. The ring of integers of K is given by

$$\mathcal{O}_K = \mathbb{Z}(\zeta_5)\,, \tag{4.1.19}$$

for which we choose the following generators, viewed as a $\mathbb{Z}$-module:

$$\mathcal{O}_K = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\left(\zeta_5 + \zeta_5^{-1}\right) \oplus \mathbb{Z}\left(\zeta_5 - \zeta_5^{-1}\right) \oplus \mathbb{Z}\left(\zeta_5^2 - \zeta_5^{-2}\right)\,. \tag{4.1.20}$$

The embedding maps, $\sigma_i \,:\, K \to \mathbb{C}$, are given by

$$\sigma_i \,:\, \zeta_5 \mapsto \zeta_5^i\,, \qquad i \in \{1,\ldots,4\}\,. \tag{4.1.21}$$

We choose a CM-type, $\Phi = \{\sigma_1, \sigma_2\}$. Then the lattice Λ is defined as

$$\Lambda := \Phi\left(\mathcal{O}_K\right) = \begin{pmatrix} 1 & \zeta_5 + \zeta_5^{-1} & \zeta_5 - \zeta_5^{-1} & \zeta_5^2 - \zeta_5^{-2} \\ 1 & \zeta_5^2 + \zeta_5^{-2} & \zeta_5^2 - \zeta_5^{-2} & \zeta_5^4 - \zeta_5^{-4} \end{pmatrix} \mathbb{Z}^4\,. \tag{4.1.22}$$

We separate the real part and imaginary parts of the lattice by writing

$$\Lambda = (Z \;\; A\mathrm{i})\,\mathbb{Z}^4\,, \tag{4.1.23}$$

where Z and A are real, invertible (2×2)-matrices. The rows of Z correspond to the real part of the embedding, and those of A to the imaginary part. Let us now bring the period matrix Λ into standard form by the change of coordinates $z \mapsto Z^{-1}z$ with $z \in \mathbb{C}^2$,

$$(Z \;\; A\mathrm{i}) \mapsto Z^{-1}\,(Z \;\; A\mathrm{i}) = \left(\mathrm{Id}_{2\times 2} \;\; Z^{-1}A\mathrm{i}\right)\,. \tag{4.1.24}$$

Recall that a complex d-dimensional torus with period matrix Π is an abelian variety iff the Riemann Bilinear relations are fulfilled, i.e.,

$$\begin{aligned} &\Pi E^{-1}\Pi^T = 0 \\ &i\Pi E^{-1}\overline{\Pi}^T > 0\,, \end{aligned} \tag{4.1.25}$$

for some non-degenerate, alternating, integer-valued $(2d\times 2d)$-matrix E.

For a $(d \times 2d)$-period matrix Π of the form

$$\Pi = (\mathrm{Id}_{d\times d} \;\; \Omega) \ , \tag{4.1.26}$$

we can always choose

$$E = \begin{pmatrix} 0_{d\times d} & \mathrm{Id}_{d\times d} \\ -\mathrm{Id}_{d\times d} & 0_{d\times d} \end{pmatrix} \ , \tag{4.1.27}$$

under which the Riemann Bilinear relations reduce to the requirement $\Omega \in \mathbb{H}_d$, the Siegel upper-half space:

$$\mathbb{H}_d = \left\{ \tau \in \mathbb{C}^{d\times d} \,\middle|\, \tau^\top = \tau,\ \mathrm{Im}(\tau) > 0 \right\} \ . \tag{4.1.28}$$

For the period matrix in (4.1.24), one verifies that $Z^{-1}A \in \mathcal{H}_2$, and hence the complex torus,

$$X = \mathbb{C}^2 / \left(\mathrm{Id}_{2\times 2} \;\; Z^{-1}A\mathrm{i} \right) \mathbb{Z}^4 \,, \tag{4.1.29}$$

is an abelian variety.

Furthermore, since $\Lambda = \Phi(\mathcal{O}_K)$, multiplication by $\{\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}$ acts as an endomorphism. Thus, the $\mathbb{Q}$-endomorphism algebra of this abelian variety is

$$\mathrm{End}_{\mathbb{Q}}(X) \cong \mathbb{Q}(\zeta_5) \,, \tag{4.1.30}$$

which is a CM-field of degree 4. Therefore, the abelian variety X is of CM-type over $\mathbb{Q}(\zeta_5)$.

We now consider the complex structure related to the period matrix Π via

$$I = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{i}\,\mathrm{Id}_{d\times d} & 0 \\ 0 & -\mathrm{i}\,\mathrm{Id}_{d\times d} \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} \ , \tag{4.1.31}$$

which satisfies $\mathrm{i}\,\Pi = \Pi I$ and $I^2 = -\mathrm{Id}_{2d\times 2d}$. For the present example, the matrix representation of I is explicitly given by

$$I = \begin{pmatrix} 0 & 0 & -\sqrt{2+\frac{2}{\sqrt{5}}} & -\sqrt{1-\frac{2}{\sqrt{5}}} \\ 0 & 0 & -\sqrt{1-\frac{2}{\sqrt{5}}} & -\sqrt{1+\frac{2}{\sqrt{5}}} \\ \frac{1}{5}\sqrt{5+2\sqrt{5}} & -\frac{1}{5}\sqrt{5-2\sqrt{5}} & 0 & 0 \\ -\frac{1}{5}\sqrt{5-2\sqrt{5}} & \frac{1}{5}\sqrt{5+2\sqrt{5}} & 0 & 0 \end{pmatrix} . \tag{4.1.32}$$

Let us choose the metric

$$g = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5+\frac{2}{\sqrt{5}} & 2-\frac{1}{\sqrt{5}} \\ 0 & 0 & 2-\frac{1}{\sqrt{5}} & 3-\frac{2}{\sqrt{5}} \end{pmatrix} \ , \tag{4.1.33}$$

which is compatible with I, i.e.,

$$I^T g I = g \,. \tag{4.1.34}$$

The associated anti-symmetric Kähler form is given by

$$\omega = gI = \begin{pmatrix} 0 & 0 & -\sqrt{8+\frac{8}{\sqrt{5}}} & -2\sqrt{1-\frac{2}{\sqrt{5}}} \\ 0 & 0 & -\sqrt{1-\frac{2}{\sqrt{5}}} & -\sqrt{1+\frac{2}{\sqrt{5}}} \\ \sqrt{8+\frac{8}{\sqrt{5}}} & \sqrt{1-\frac{2}{\sqrt{5}}} & 0 & 0 \\ 2\sqrt{1-\frac{2}{\sqrt{5}}} & \sqrt{1+\frac{2}{\sqrt{5}}} & 0 & 0 \end{pmatrix} . \tag{4.1.35}$$

From eq. (4.1.8) and setting $B = 0$, we obtain an induced generalised Kähler structure

$$\begin{aligned} \mathcal{I} &= \begin{pmatrix} 0 & -Z^{-1}A & 0 & 0 \\ A^{-1}Z & 0 & 0 & 0 \\ 0 & 0 & 0 & -Z^T(A^{-1})^T \\ 0 & 0 & A^T(Z^{-1})^T & 0 \end{pmatrix} \\ \mathcal{J} &= \begin{pmatrix} 0 & 0 & 0 & \rho^{-1}Z^T(A^{-1})^T \\ 0 & 0 & -A^{-1}Z\rho^{-1} & 0 \\ 0 & \rho Z^{-1}A & 0 & 0 \\ -A^T(Z^{-1})^T\rho & 0 & 0 & 0 \end{pmatrix} , \end{aligned} \tag{4.1.36}$$

with $\rho = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$. Now consider an isogenous abelian variety defined via

$$C = \rho \,, \qquad \gamma = \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \mathrm{Id}_{2\times 2} \end{pmatrix} , \tag{4.1.37}$$

leading to the period matrix

$$\Pi' = C\Pi\gamma = \begin{pmatrix} \mathrm{Id}_{2\times 2} & \rho Z^{-1}A\mathrm{i} \end{pmatrix} . \tag{4.1.38}$$

This gives rise to a new complex structure I' and a compatible Kähler metric g',

$$I' = \gamma^{-1}I\gamma = \begin{pmatrix} 0 & -\rho Z^{-1}A \\ A^{-1}Z\rho^{-1} & 0 \end{pmatrix} , \qquad g' = \gamma^T g\gamma = \begin{pmatrix} -\rho^{-1} & 0 \\ 0 & -A^T(Z^{-1})^T\rho Z^{-1}A \end{pmatrix} . \tag{4.1.39}$$

Again setting $B' = 0$, the induced generalised Kähler structure reads

$$\begin{aligned} \mathcal{I}' &= \begin{pmatrix} 0 & -\rho Z^{-1}A & 0 & 0 \\ A^{-1}Z\rho^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\rho^{-1}Z^T(A^{-1})^T \\ 0 & 0 & A^T(Z^{-1})^T\rho & 0 \end{pmatrix} \\ \mathcal{J}' &= \begin{pmatrix} 0 & 0 & 0 & Z^T(A^{-1})^T \\ 0 & 0 & -A^{-1}Z & 0 \\ 0 & Z^{-1}A & 0 & 0 \\ -A^T(Z^{-1})^T & 0 & 0 & 0 \end{pmatrix} . \end{aligned} \tag{4.1.40}$$

One can explicitly verify by direct calculation that the above constructed isogenous abelian varieties $X = \mathbb{C}^2/\Pi\mathbb{Z}^4$ and $X' = \mathbb{C}^2/\Pi'\mathbb{Z}^4$ are mirror to each other, with mirror map given by the unimodular matrix

$$\varphi = \begin{pmatrix} 0 & 0 & \mathrm{Id}_{2\times 2} & 0 \\ 0 & -\mathrm{Id}_{2\times 2} & 0 & 0 \\ \mathrm{Id}_{2\times 2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Id}_{2\times 2} \end{pmatrix} . \tag{4.1.41}$$

This induces an isomorphism between the Narain lattices $\Gamma^{4,4}$ and $\Gamma'^{4,4}$ associated with the respective generalised abelian varieties, as φ preserves the canonical symmetric bilinear form q and the decomposition into holomorphic and anti-holomorphic part of the charge vectors [166]. Furthermore, $\mathcal{I}' = \varphi\mathcal{J}\varphi^{-1}$ and $\mathcal{J}' = \varphi\mathcal{I}\varphi^{-1}$.

Hence, this provides an explicit counterexample to the conjecture proposed in [94] concerning the role of mirror symmetry. In this case, we have two abelian varieties X, X' that are mirror to one another and isogenous, implying that the mirror abelian variety X' is also of CM-type over the same number field $K = \mathbb{Q}(\zeta_5)$, since the endomorphism algebra $\mathrm{End}(X)_{\mathbb{Q}}$ is preserved under isogenies. However, examining (4.1.42) and (4.1.39), we find that the Kähler metrics are not defined over $\mathbb{Q}$, and therefore the associated $\mathcal{N} = (2,2)$ toroidal conformal field theory is not rational.

This stands in stark contrast to the situation in $\mathcal{N} = (2,2)$ toroidal conformal field theories on complex one-dimensional tori, where any mirror pair of complex tori, which are of CM-type over the same number field, automatically yields a rational theory. By theorem (4.1.1), we nonetheless know that for a CM-type abelian variety, a compatible $\mathbb{Q}$-valued Kähler metric always exists. For the example at hand, one such metric compatible with the complex structure I is

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} . \tag{4.1.42}$$

More generally, it has been shown in [166] that if two mirror generalised abelian varieties, induced by (X, g, B) and (X', g', B'), are equipped with rational valued g and B, then both X and X' are isogenous and of CM-type.

4.2 CM-type Hodge Structure from Rational Conformal Field Theory

In the previous section, we explored and reviewed how $\mathcal{N} = (2,2)$ toroidal rational conformal field theories are linked to abelian varieties of complex multiplication (CM) type. We highlighted that for any $\mathcal{N} = (2,2)$ toroidal rational conformal field theory, the associated mirror pair of abelian varieties is of CM-type over isomorphic number fields K. In strong contrast to the case of complex one-dimensional abelian varieties (i.e., elliptic curves), the converse direction of the conjecture in [94] is more subtle. We have seen a simple example in the case of complex two-dimensional abelian varieties where a mirror pair of isogenous CM-type abelian varieties does not give rise to an $\mathcal{N} = (2,2)$ rational superconformal field theory. It is therefore not clear in general what extra compatibility conditions we need to impose on a mirror pair such that the underlying $\mathcal{N} = (2,2)$ superconformal field theory is rational.

In this section, we want to systematically explore one direction of the conjecture [94], which passed all tests in the realm of $\mathcal{N} = (2,2)$ toroidal conformal field theories. In particular, given an abstract $\mathcal{N} = (2,2)$ rational superconformal field theory, we will construct explicitly the associated rational Hodge structure in terms of $\mathcal{N} = (2,2)$ superconformal field theory data. The motivation for this construction stems from the well-known case of non-linear sigma models on complex Kähler manifolds X, where the Dolbeault cohomology groups are naturally related to the chiral rings of the associated $\mathcal{N} = (2,2)$ superconformal field theory (see e.g., [141, 145] and references therein). However, our approach goes beyond the classical setting by emphasizing the role of boundary conformal field theory and the arithmetic structure encoded in boundary states. In particular, the Hodge structure that we construct need not have a non-linear sigma model origin. We then show how the rationality property of the $\mathcal{N} = (2,2)$ superconformal field theory equips the Hodge structure with an enlarged rational endomorphism algebra that satisfies the CM-type criteria in the case of polarisable Hodge structures. Examples illustrating this construction will be discussed in subsequent sections, including Gepner models, which furnish exact type IIA worldsheet descriptions of specific Calabi–Yau threefold compactifications, as well as non-geometric $N = (2,2)$ minimal models, which serve as simpler yet instructive examples of $\mathcal{N} = (2,2)$ rational superconformal field theories with rich arithmetic structure. This section largely follows the results presented in the work [2].

Definition of Rational Hodge Structures

A *rational Hodge structure of weight m* is defined by a finite-dimensional $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$, together with a decomposition of its complexification,

$$V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=m} V_{\mathbb{C}}^{p,q} \ , \tag{4.2.1}$$

such that the decomposition is preserved under complex conjugation in the following sense:

$$\overline{V_{\mathbb{C}}^{p,q}} = V_{\mathbb{C}}^{q,p} \ . \tag{4.2.2}$$

The dimensions $h^{p,q} = \dim_{\mathbb{C}} V_{\mathbb{C}}^{p,q}$ are referred to as the *Hodge numbers*.

A subspace $W_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ is called a *Hodge substructure* if it inherits a compatible Hodge decomposition, i.e.,

$$W_{\mathbb{C}} = W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=m} W_{\mathbb{C}} \cap V_{\mathbb{C}}^{p,q} \ . \tag{4.2.3}$$

A Hodge structure is said to be *simple* if it contains no proper non-trivial Hodge substructures.

In an analogous fashion, a *real Hodge structure of weight m* is defined by replacing the rational vector space $V_{\mathbb{Q}}$ with a real vector space $V_{\mathbb{R}}$ in the definitions above.

A rational Hodge structure of weight m is called *polarised* if there exists a non-degenerate bilinear form

$$Q : V_{\mathbb{Q}} \times V_{\mathbb{Q}} \to \mathbb{Q} \, , \tag{4.2.4}$$

called the *polarisation*, such that its complex-linear extension to $V_{\mathbb{C}}$ satisfies the following conditions:

$$Q(v,w) = (-1)^m Q(w,v) \ , \tag{4.2.5}$$

$$Q(v,w) = 0 \quad \text{if } v \in V_{\mathbb{C}}^{p,q}, \ w \in V_{\mathbb{C}}^{p',q'} \text{ and } p+p' \neq m \ , \tag{4.2.6}$$

$$i^{p-q} Q(v,\overline{v}) > 0 \quad \text{for all non-zero } v \in V_{\mathbb{C}}^{p,q} \ . \tag{4.2.7}$$

It follows immediately that any Hodge substructure of a polarised Hodge structure is itself polarised, where the polarisation is obtained by restricting Q to the corresponding subspace.

CM-type Rational Hodge Structure

We now introduce the notion of rational Hodge structures of *complex multiplication* (CM) *type*. Several equivalent definitions of CM-type Hodge structures appear in the literature; in this work, we adopt a formulation that is particularly well-suited to explicit constructions. A broader discussion of alternative definitions is provided in [2, 174].

Let $(V_{\mathbb{Q}}, V_{\mathbb{C}})$ be a pure rational Hodge structure of weight m. We begin by defining the concept of a *Hodge endomorphism*. These are vector space endomorphisms $\varphi : V_{\mathbb{Q}} \to V_{\mathbb{Q}}$, whose complexification $\varphi_{\mathbb{C}}$ preserves the Hodge decomposition, i.e.,

$$\varphi_{\mathbb{C}}(V_{\mathbb{C}}^{p,q}) \subseteq V_{\mathbb{C}}^{p,q}, \qquad \text{for all } \ p+q=m \ . \tag{4.2.8}$$

The set of all Hodge endomorphisms forms a $\mathbb{Q}$-algebra with multiplication being function composition,

$$\mathrm{End}_{\mathrm{Hdg}}(V_{\mathbb{Q}}) = \{\varphi \in \mathrm{Hom}(V_{\mathbb{Q}}, V_{\mathbb{Q}}) \mid \varphi_{\mathbb{C}}(V_{\mathbb{C}}^{p,q}) \subseteq V_{\mathbb{C}}^{p,q}\} \ . \tag{4.2.9}$$

Following standard terminology, we say that an irreducible Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}})$ admits *E-multiplication* if there exists a field embedding,

$$E \hookrightarrow \mathrm{End}_{\mathrm{Hdg}}(V_{\mathbb{Q}}) \,, \tag{4.2.10}$$

of a number field E. If, moreover, $[E : \mathbb{Q}] = \dim_{\mathbb{Q}} V_{\mathbb{Q}}$ and the Hodge structure is polarisable, then we say that $(V_{\mathbb{Q}}, V_{\mathbb{C}})$ is of *complex multiplication (CM) type*.

In this situation, one can show that the Hodge endomorphism algebra $\mathrm{End}_{\mathrm{Hdg}}(V_{\mathbb{Q}})$ is a commutative semisimple algebra, and in fact isomorphic to a CM field [175]. Recall that a *CM field* E is defined as a totally imaginary quadratic extension E/F, where F is a totally real number field. That is, all embeddings $\psi : F \to \mathbb{C}$ satisfy $\psi(F) \subset \mathbb{R}$, whereas no embedding of E into $\mathbb{R}$ exists.

When the Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}})$ is not irreducible, it admits a finite decomposition into irreducible *Hodge substructures*:

$$V_{\mathbb{Q}} = \bigoplus_{\alpha \in \mathcal{I}} W_{\mathbb{Q}}^{\alpha}, \qquad \text{with} \quad W_{\mathbb{C}}^{\alpha} = \bigoplus_{p+q=m} W_{\mathbb{C}}^{\alpha,(p,q)} \,. \tag{4.2.11}$$

Following terminology from the theory of abelian varieties (cf. [176]), we say that a Hodge structure has *sufficiently many complex multiplications* if each irreducible Hodge substructure $W_{\mathbb{Q}}^{\alpha}$ admits a multiplication by a number field E^{α}, satisfying

$$E^{\alpha} \hookrightarrow \mathrm{End}_{\mathrm{Hdg}}(W_{\mathbb{Q}}^{\alpha}), \qquad \text{with} \quad [E^{\alpha} : \mathbb{Q}] = \dim_{\mathbb{Q}} W_{\mathbb{Q}}^{\alpha} \,. \tag{4.2.12}$$

If, in addition, each Hodge substructure $W_{\mathbb{Q}}^{\alpha}$ is polarisable and of CM-type as above, then we say that the entire Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}})$ is *of CM-type*. The distinction between polarisable and non-polarisable Hodge structures is important for those Hodge structures arising in general $\mathcal{N} = (2,2)$ superconformal field theories, as a priori there is no reason to expect in non-geometric examples that a polarisation exists.

4.2.1 Rational Hodge Structure

In the following, we will demonstrate that any two-dimensional $\mathcal{N}=(2,2)$ superconformal field theory naturally gives rise to two real Hodge structures, which we refer to as the *A-type* and *B-type* Hodge structures. When the theory allows a geometric interpretation as a non-linear sigma model on a Calabi–Yau manifold, these structures correspond to the vertical and horizontal parts of the Hodge decomposition, respectively. The construction of these Hodge structures is based on particular subsectors of the theory: the A-type structure is built from the chiral–chiral (c,c) ring, while the B-type structure arises from the chiral–anti-chiral (c,a) ring. We assume that the chiral rings are finite-dimensional, which is satisfied, e.g., for non-linear sigma models on smooth and compact Calabi–Yau manifolds, and for compact $\mathcal{N}=(2,2)$ superconformal field theories with discrete spectrum [145].

A-Type Hodge Structure

The A-type Hodge structure is defined in terms of the vector space generated by states in the (c,c)-ring. For simplicity, we restrict to the subring $(\text{c,c})_{q_L=q_R}$, which contains only those states whose left- and right-moving U(1) charges coincide. As the left- and right-moving quantum numbers match in this subsector, we often suppress explicit mention of the right-moving part, with the understanding that operations are extended symmetrically.

As reviewed in section 2.5, spectral flow by the operator $\mathcal{U}$ maps highest-weight states in the $(\text{c,c})_{q_L=q_R}$ ring to Ramond–Ramond (RR) ground states $|\alpha\rangle$. Let $\mathcal{R}$ denote the finite set of representations associated with these states. We define the complex vector space as

$$V_{\mathbb{C}} := \langle\, |\alpha\rangle \,\rangle_{\mathbb{C}}^{\alpha\in\mathcal{R}} \ . \tag{4.2.13}$$

We choose an orthonormal basis $\{|\alpha\rangle_{\mathcal{N}}\}_{\alpha\in\mathcal{R}}$ such that

$$\langle\alpha|\beta\rangle_{\mathcal{N}} = \delta_{\alpha\beta} \ . \tag{4.2.14}$$

To define complex conjugation, we introduce an anti-linear involution $\iota : V_{\mathbb{C}} \to V_{\mathbb{C}}$, defined by

$$\iota\left(\sum_{\alpha\in\mathcal{R}} a_\alpha\, |\alpha\rangle_{\mathcal{N}}\right) := \sum_{\alpha\in\mathcal{R}} \operatorname{sgn}\left(\mathrm{S}_{0\mathcal{C}(\alpha)}\right) a_\alpha^* \left(|\mathcal{C}(\alpha)\rangle_{\mathcal{N}}\right) \ , \tag{4.2.15}$$

where $a_\alpha \in \mathbb{C}$ are arbitrary complex coefficients. This construction ensures agreement with complex conjugation in geometric settings.

From the charge conjugation relation (2.5.27), it follows that the U(1)-charge of a complex conjugated state transforms as

$$q_L(\iota(|\alpha\rangle_{\mathcal{N}})) = -q_L(|\alpha\rangle_{\mathcal{N}}) \ , \qquad q_R(\iota(|\alpha\rangle_{\mathcal{N}})) = -q_R(|\alpha\rangle_{\mathcal{N}}) \ . \tag{4.2.16}$$

By recalling how the conformal weight and U(1)-charge transforms under $\mathcal{U}$ and $\mathcal{C}$ (see eq. (2.5.25)), one can show that $\mathcal{U}^{-1}\iota(|\alpha\rangle)$ is a chiral state, hence $\iota(|\alpha\rangle) \in \mathcal{R}$. We can thus define the real vector space

$$V_{\mathbb{R}} = \langle v \in V_{\mathbb{C}} \,|\, \iota(v) = v \rangle_{\mathbb{R}} \ , \tag{4.2.17}$$

so that $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

We now proceed to construct the grading $V_{\mathbb{C}}^{p,q} \subset V_{\mathbb{C}}$ by assigning to each basis element $|\alpha\rangle_{\mathcal{N}}$ a pair of integers (p, q) that determine its Hodge type. This grading must be compatible with complex conjugation, as specified by equation (4.2.2), which leads to

$$p(-q_L(|\alpha\rangle)) \ = \ q(q_L(|\alpha\rangle)) \qquad \text{for all } \alpha \in \mathcal{R}\,. \tag{4.2.18}$$

To ensure the resulting Hodge structure is non-degenerate and minimal, we impose the following additional conditions on the pair (p, q). First, we require that the set of (p, q)-values for a given theory generically includes zero, i.e.,

$$p(q_L(|\alpha\rangle)) = q(q_L(|\beta\rangle)) = 0 \quad \text{for some } \alpha, \beta \in \mathcal{R}\ . \tag{4.2.19}$$

Furthermore, we require that the (p, q)-values are minimal, in the sense that the collection of integers arising from the charge assignment has no common divisor other than one,

$$\gcd\left(\{p(q_L(|\alpha\rangle)),\, q(q_L(|\alpha\rangle)) \mid \alpha \in \mathcal{R}\}\right) = 1\,. \tag{4.2.20}$$

These requirements, together with demanding the (p, q)-values to be linear in the U(1) charge, such that in geometric settings the Hodge structure agrees with the one obtained from geometry (e.g., Gepner models), give rise to the following (up to conjugation) unique assignment,

$$p(q_L) := l\left(\frac{c}{6} + q_L\right) \ , \qquad q(q_L) := l\left(\frac{c}{6} - q_L\right) \ , \tag{4.2.21}$$

where the normalisation factor l is chosen such that both p and q are guaranteed to be integers for all states $|\alpha\rangle_{\mathcal{N}} \in \mathcal{R}$. It is given by the least common multiple of the denominators appearing in the linear expressions

$$l := \operatorname{lcm}\left(\left\{\operatorname{denom}\left(\frac{c}{6} \pm q_L(|\alpha\rangle_{\mathcal{N}})\right) \,\middle|\, \alpha \in \mathcal{R}\right\}\right) \ . \tag{4.2.22}$$

This ensures that the Hodge structure is not trivially related to a simpler one. Moreover, the positivity of the indices p and q, as well as the existence of states with $p = 0$ or $q = 0$, follows from the unitarity bound (2.5.20) and the spectral flow transformation (2.5.25), which together imply

$$|q_L(|\alpha\rangle)| \leq \frac{c}{6} \qquad \text{for all } \alpha \in \mathcal{R}\ . \tag{4.2.23}$$

In particular, the state $\mathcal{U}\,|0\rangle$, where $|0\rangle$ denotes the vacuum state, saturates the lower bound with $q_L(\mathcal{U}\,|0\rangle) = -c/6$, while its conjugate $\iota(\mathcal{U}\,|0\rangle)$ attains the upper bound $q_L = c/6$.

Finally, we define the graded components of the complexified vector space as follows:

$$V_{\mathbb{C}}^{p,q} := \left\{ |\alpha\rangle \in V_{\mathbb{C}} \;\middle|\; \frac{c}{6} + q_L(|\alpha\rangle) = \frac{p}{l}, \quad \frac{c}{6} - q_L(|\alpha\rangle) = \frac{q}{l} \right\} . \tag{4.2.24}$$

Under this assignment, the total weight of the Hodge structure is $m = p + q = lc/3$, and the pair $(V_{\mathbb{R}}, V_{\mathbb{C}})$, along with the Hodge decomposition into $V_{\mathbb{C}}^{p,q}$, defines a Hodge structure of that weight.

B-Type Hodge Structure

We now turn to the construction of the B-type Hodge structures, which arise from applying the mirror symmetry transformation to the A-type case. This symmetry operation, as discussed in [153], effectively inverts at the level of the $\mathcal{N} = (2,2)$ superconformal field theory the sign of the right-moving U(1) current and interchanges the roles of the (c,c) and (c,a) rings. Accordingly, the B-type Hodge structure is naturally built from the (c,a)-ring. To simplify the construction, we focus on the subring $(\text{c,a})_{q_L=-q_R}$, which consists of states for which the left-moving U(1) charge is the negative of the right-moving one.

Given that the A-type Hodge structure depends only on the left-moving sector, and that the mirror automorphism merely alters the sign of the right-moving charge, it is to be expected that the construction of B-type structures will follow closely the same lines.

To remain consistent with the conventions established in the A-type case, we define the action of the spectral flow to be consistent with mirror symmetry. In particular, we have

$$\begin{aligned} q_L(\mathcal{U}\,|\alpha\rangle) &= q_L(|\alpha\rangle) - \frac{c}{6} \,, \qquad & q_R(\mathcal{U}\,|\alpha\rangle) &= q_R(|\alpha\rangle) + \frac{c}{6} \,, \\ q_L(\mathcal{C}\,|\alpha\rangle) &= -q_L(|\alpha\rangle) \,, & q_R(\mathcal{C}\,|\alpha\rangle) &= -q_R(|\alpha\rangle) \,. \end{aligned} \tag{4.2.25}$$

Under this definition, the spectral flow again maps states in the $(\text{c,a})_{q_L=-q_R}$ subring to Ramond–Ramond ground states. Let $\mathcal{S}$ denote again the corresponding set of representations. The associated complex vector space is then defined analogously to the A-type case as

$$V_{\mathbb{C}} := \langle |\alpha\rangle \rangle_{\mathbb{C}}^{\alpha\in\mathcal{S}} \;. \tag{4.2.26}$$

We assign a Hodge decomposition to this space by defining graded subspaces according to the same rule:

$$V_{\mathbb{C}}^{p,q} := \left\{ |\alpha\rangle \in V_{\mathbb{C}} \;\middle|\; \alpha \in \mathcal{S}, \quad \frac{c}{6} + q_L(|\alpha\rangle) = \frac{p}{l}, \quad \frac{c}{6} - q_L(|\alpha\rangle) = \frac{q}{l} \right\} , \tag{4.2.27}$$

where the scaling factor l ensures integrality as in the A-type case.

The complex conjugation operator ι, defined in (4.2.15), satisfies

$$V_{\mathbb{C}}^{p,q} = \overline{V_{\mathbb{C}}^{q,p}} \;. \tag{4.2.28}$$

Thus, the pair $(V_{\mathbb{R}}, V_{\mathbb{C}})$, together with its decomposition into the components $V_{\mathbb{C}}^{p,q}$, once again defines a real Hodge structure of total weight $m = lc/3$.

Rational Structure from Boundary States

In the context of rational conformal field theories, the real Hodge structure we have constructed can be endowed with an explicit rational structure in terms of boundary states. As briefly reviewed in section 2.4, for rational conformal field theories, explicit boundary states can be constructed via the Cardy construction [137]. Although Cardy states do not exhaust all consistent boundary conditions in a general rational conformal field theory, they are particularly tractable and often form a complete basis in known examples. In the following, we will assume for simplicity that sufficiently many boundary states are given by Cardy states and have the form (2.4.26). The more general problem of classifying all boundary states remains an open and challenging question. Nonetheless, our construction can often be extended beyond Cardy states, and we demonstrate such a generalisation in section 4.4 for a Gepner model that includes non-Cardy boundary states. A more systematic extension to generic boundary conditions is left for future investigation.

In what follows, we focus on the construction of rational Hodge structures associated with the A-type case. As explained earlier, the B-type Hodge structures can be obtained via mirror symmetry.

To define a rational structure on the complex vector space $V_{\mathbb{C}}$ of RR–ground states, we begin by projecting the Cardy boundary states (2.4.26) onto $V_{\mathbb{C}}$. This is achieved using the orthogonal projection operator,

$$\mathcal{P}_{\mathcal{R}} \;=\; \sum_{\alpha\in\mathcal{R}} |\alpha\rangle_{\mathcal{N}}\,{}_{\mathcal{N}}\langle\alpha| \;\; . \tag{4.2.29}$$

with $\{|\alpha\rangle_{\mathcal{N}}\}$ an orthonormal basis of RR–ground states.

The *projected boundary states* are then defined by applying this projection to the Cardy boundary states,

$$|B_A\rangle_{\mathcal{P}} := \mathcal{P}_{\mathcal{R}}\,|B_A\rangle \;=\; \sum_{\alpha\in\mathcal{R}} |\alpha\rangle_{\mathcal{N}}\,{}_{\mathcal{N}}\langle\alpha|B_A\rangle \;\; , \tag{4.2.30}$$

for each highest weight representation $A \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A}')$.

We introduce the *intersection matrix* as [150],

$$\widetilde{\Sigma}_{AB} := {}_{\mathcal{P}}\langle B_A|\; e^{\mathrm{i}\pi J_0}\,|B_B\rangle_{\mathcal{P}} = \mathrm{Tr}_{\mathcal{H}_{AB}}(-1)^F\, . \tag{4.2.31}$$

Due to the relation to the open string Witten index, which we discussed in eq. (2.5.55), the intersection matrix takes values in $\mathbb{Q}$ (for suitably normalised boundary states even in $\mathbb{Z}$). We assume that the matrix $\widetilde{\Sigma}_{AB}$ has maximal rank, i.e., it is equal to $d := |\mathcal{R}| = \dim_{\mathbb{C}} V_{\mathbb{C}}$.

We now define the rational vector space $V_{\mathbb{Q}}$ as the $\mathbb{Q}$-linear span of the projected boundary states,

$$V_{\mathbb{Q}} := \big\langle\; |B_A\rangle_{\mathcal{P}}\;\big\rangle_{\mathbb{Q}}^{A\in\mathrm{HWR}_{\mathcal{H}}(\mathcal{A}')_{\mathrm{NS}}}\, . \tag{4.2.32}$$

To relate this rational vector space to $V_{\mathbb{C}}$ with basis $\{|\alpha\rangle_{\mathcal{N}}\}$, select any subset of boundary states $\mathcal{B}$ such that $|\mathcal{B}| = |\mathcal{R}|$ and $\Sigma := \widetilde{\Sigma}|_{\mathcal{B}}$ has full rank. Then the set $\{|B_A\rangle_{\mathcal{P}} \mid A \in \mathcal{B}\}$ forms a basis for both $V_{\mathbb{C}}$ and $V_{\mathbb{Q}}$.[3]

Using this basis, the identity operator on $V_{\mathbb{C}}$ can be written as:

$$\mathrm{Id}_{V_{\mathbb{C}}} = \mathcal{P}_{\mathcal{B}} := |B_A\rangle_{\mathcal{P}}\, \Sigma^{AB}\, {}_{\mathcal{P}}\langle B_B|\, \mathrm{e}^{\pi \mathrm{i} J_0} := |B_A\rangle_{\mathcal{P}}\, {}_{\mathcal{P}}\langle B^A|\, \mathrm{e}^{\pi \mathrm{i} J_0} \ , \tag{4.2.33}$$

where Σ^{AB} is the inverse of the matrix Σ_{AB}, and we define ${}_{\mathcal{P}}\langle B^A| := \Sigma^{AB}\, {}_{\mathcal{P}}\langle B_B| \in V_{\mathbb{Q}}$.

With this identity, any RR–ground state in the complex vector space $V_{\mathbb{C}}$ can be expanded in terms of the projected boundary states

$$|\alpha\rangle_{\mathcal{N}} = \mathcal{P}_{\mathcal{B}}\, |\alpha\rangle_{\mathcal{N}} = e^{-\pi i q_L(\alpha)}\, \overline{{}_{\mathcal{N}}\langle \alpha | B_A\rangle_{\mathcal{P}}}\ |B^A\rangle_{\mathcal{P}} \ . \tag{4.2.34}$$

This expression makes explicit that the RR–ground states are complex linear combinations of boundary states, and hence the rational vector space $V_{\mathbb{Q}}$ defined in eq. (4.2.32) satisfies

$$V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \ . \tag{4.2.35}$$

Compatibility with Complex Conjugation

We will now show that the pair $(V_{\mathbb{Q}}, V_{\mathbb{C}})$ as constructed above indeed defines a rational Hodge structure. Specifically, we need to show that the rational vector space $V_{\mathbb{Q}}$, constructed from projected boundary states, is preserved under the complex conjugation map $\iota : V_{\mathbb{C}} \to V_{\mathbb{C}}$, defined in eq. (4.2.15). This ensures compatibility between the rational and real structures, and hence allows us to again use the definition of the Hodge decomposition in (4.2.27) for the pair $(V_{\mathbb{Q}}, V_{\mathbb{C}})$.

To establish this compatibility, it suffices to demonstrate that each projected boundary state $|B_A\rangle_{\mathcal{P}}$, for $A \in \mathcal{B}$, is invariant under the action of ι. Given the form of the Cardy boundary state $|B_A\rangle$, its projection to the space of RR–ground states is given by

$$|B_A\rangle_{\mathcal{P}} = \mathcal{P}_{\mathcal{R}}\, |B_A\rangle = \kappa \sum_{\alpha \in \mathcal{R}} \frac{\mathrm{S}_{A\alpha}}{\sqrt{\mathrm{S}_{0\alpha}}}\, |\alpha\rangle_{\mathcal{N}} \ , \tag{4.2.36}$$

where $\kappa \in \mathbb{R}$ is a normalisation constant, independent of A and α, chosen such that the intersection matrix Σ_{AB} is rational.[4]

[3] This follows by the maximal rank of $\widetilde{\Sigma}$. Using the identity operator of $V_{\mathbb{C}}$ (4.2.33), any projected boundary state can be expressed as a rational linear combination of projected boundary states of the set $\{|B_A\rangle_{\mathcal{P}} \mid A \in \mathcal{B}\}$. Specifically, $|B_M\rangle_{\mathcal{P}} = |B_A\rangle_{\mathcal{P}}\, \Sigma^{AB} \widetilde{\Sigma}_{BM}$. We therefore conclude that for any $M \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A}')_{\mathrm{NS}}$, the projected boundary state $|B_M\rangle_{\mathcal{P}}$ is a rational linear combination of the basis states.

[4] The normalisation κ can be taken real without loss of generality, since any overall complex phase cancels in the definition of Σ_{AB}.

We now compute the action of the complex conjugation map ι on this projected state. Applying eq. (4.2.15), we find

$$\iota(|B_A\rangle_{\mathcal{P}}) = \kappa \sum_{\alpha\in\mathcal{R}} \operatorname{sgn}\left(\mathrm{S}_{0\mathcal{C}(\alpha)}\right) \frac{\mathrm{S}^*_{A\alpha}}{\left(\sqrt{\mathrm{S}_{0\alpha}}\right)^*} |\mathcal{C}(\alpha)\rangle_{\mathcal{N}} \ . \tag{4.2.37}$$

Since $\mathcal{C}$ is an invertible map on the set of RR–ground states $\mathcal{R}$, we can relabel the sum by the images $\beta := \mathcal{C}(\alpha)$, yielding

$$\iota(|B_A\rangle_{\mathcal{P}}) = \kappa \sum_{\beta\in\mathcal{R}} \operatorname{sgn}(\mathrm{S}_{0\beta}) \frac{\mathrm{S}^*_{A\mathcal{C}(\beta)}}{\left(\sqrt{\mathrm{S}_{0\mathcal{C}(\beta)}}\right)^*} |\beta\rangle_{\mathcal{N}} \ . \tag{4.2.38}$$

To simplify this expression, we use eq. (2.2.20), and $\mathrm{S}_{0\beta} \in \mathbb{R}$,

$$\left(\sqrt{\mathrm{S}_{0\beta}}\right)^* = \operatorname{sgn}(\mathrm{S}_{0\beta})\sqrt{\mathrm{S}_{0\beta}} \ , \tag{4.2.39}$$

to obtain

$$\iota(|B_A\rangle_{\mathcal{P}}) = \kappa \sum_{\beta\in\mathcal{R}} \frac{\mathrm{S}_{A\beta}}{\sqrt{\mathrm{S}_{0\beta}}} |\beta\rangle_{\mathcal{N}} = |B_A\rangle_{\mathcal{P}} \ . \tag{4.2.40}$$

This confirms that each projected boundary state $|B_A\rangle_{\mathcal{P}}$ is invariant under complex conjugation, and hence the rational vector space $V_{\mathbb{Q}}$ is contained in the real subspace $V_{\mathbb{R}} \subset V_{\mathbb{C}}$ fixed by ι.

Consequently, the Hodge grading previously introduced in eq. (4.2.24) remains applicable. We thus obtain a rational Hodge structure with Hodge decomposition

$$V_{\mathbb{Q}} := \left\langle \ |B_A\rangle_{\mathcal{P}} \ \right\rangle_{\mathbb{Q}}^{A\in\mathrm{HWR}_{\mathcal{H}}(\mathcal{A}')_{\mathrm{NS}}} \ , \qquad V_{\mathbb{C}}^{p,q} := \left\{ |\alpha\rangle \in V_{\mathbb{C}} \ \middle| \ \frac{c}{6} + q_L(\alpha) = \frac{p}{l}, \ \frac{c}{6} - q_L(\alpha) = \frac{q}{l} \right\} \ , \tag{4.2.41}$$

and weight $m = p + q = lc/3$, as before.

Polarisation

We will now address the question of polarisability of the constructed rational Hodge structure. For theories with integral U(1)-charges (for instance any consistent GSO-projected $\mathcal{N} = (2,2)$ worldsheet theory in Type II string theory) subject to the assumptions made in the previous section, the intersection matrix defined in eq. (4.2.31) gives rise to a polarisation. For theories with $(\mathrm{c,c})_{q_L=q_R}$-states with fractional U(1)-charges, we will define a twisted version of the open string Witten index, which polarises the associated rational Hodge structure in particular cases. For theories where the twisted version of the open string Witten index does not lead to a polarisation, it will be shown in an example in section 4.3 that the corresponding rational Hodge structure can still be polarisable.

The intersection matrix (4.2.31), through the relation to the open string Witten index, yields a $\mathbb{Q}$-bilinear function $V_{\mathbb{Q}} \times V_{\mathbb{Q}} \to \mathbb{Q}$. Using the results in [152], we can twist the index by an unitary operator $\mathcal{V}$ central to the $\mathcal{N}=2$ super Virasoro algebra, with $[e^{\mathrm{i}\pi J_0}, \mathcal{V}] = 0$. Furthermore, we demand that $\mathcal{V}$ restricts to a linear map on $V_{\mathbb{Q}}$,

$$\mathcal{V}|_{\mathbb{Q}} : V_{\mathbb{Q}} \to V_{\mathbb{Q}} \ , \tag{4.2.42}$$

thereby implementing an orthogonal transformation on $V_{\mathbb{Q}}$. Equipped with such an operator $\mathcal{V}$, we introduce a twisted Witten index, defining a $\mathbb{Q}$-bilinear pairing $\Sigma : V_{\mathbb{Q}} \times V_{\mathbb{Q}} \to \mathbb{Q}$ by

$$\Sigma(\,|B_A\rangle_{\mathcal{P}}\,, |B_B\rangle_{\mathcal{P}}) := {}_{\mathcal{P}}\langle B_A|\, \mathrm{e}^{\mathrm{i}\pi J_0}\mathcal{V}\, |B_B\rangle_{\mathcal{P}} \ . \tag{4.2.43}$$

For this bilinear form Σ to serve as a polarisation of the Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}}^{p,q})$ with weight $m = \frac{lc}{3}$, several conditions must be satisfied. Firstly, Σ must satisfy condition (4.2.5), namely

$$\Sigma(\,|B_B\rangle_{\mathcal{P}}\,, |B_A\rangle_{\mathcal{P}}) = (-1)^{-\frac{lc}{3}}\Sigma(\,|B_A\rangle_{\mathcal{P}}\,, |B_B\rangle_{\mathcal{P}}) \ . \tag{4.2.44}$$

Taking into account that l has been chosen such that $\frac{lc}{6} + lq_L(|\alpha\rangle) \in \mathbb{Z}$ for all $\alpha \in \mathcal{R}$, as specified in eq. (4.2.24), we derive the condition

$$\mathrm{e}^{-2\mathrm{i}\pi(J_0+\frac{lc}{6})}\mathcal{V}^{-1} = \mathrm{e}^{2\mathrm{i}\pi(l-1)J_0}\mathcal{V}^{-1} = \mathcal{V} \quad \Longrightarrow \quad \mathcal{V}^2 = \mathrm{e}^{2\mathrm{i}\pi(l-1)J_0} \ . \tag{4.2.45}$$

However, satisfying (4.2.45) alone does not guarantee that Σ defines a polarisation. It remains to verify conditions (4.2.6) and (4.2.7). To this end, we extend Σ to the complexified space $V_{\mathbb{C}}$. Due to the complex anti-linearity in the first argument in the hermitian inner product on the Hilbert space of states, the definition (4.2.43) naturally extends to a pairing on $V_{\mathbb{C}}$ with complex anti-linearity in the first argument as well. To obtain a bilinear form suitable for defining a polarisation, we introduce the function

$$Q(|\alpha\rangle\,, |\beta\rangle) \ := \ \Sigma(\iota(|\alpha\rangle), |\beta\rangle) = \iota(\langle\alpha|)\mathrm{e}^{\mathrm{i}\pi J_0}\mathcal{V}\,|\beta\rangle \ , \qquad |\alpha\rangle\,, |\beta\rangle \in V_{\mathbb{C}} \ , \tag{4.2.46}$$

where $\iota : V_{\mathbb{C}} \to V_{\mathbb{C}}$ denotes the anti-linear complex conjugation map. This definition ensures that Q is complex-linear in both arguments and provides the $\mathbb{C}$-bilinear extension of Σ from $V_{\mathbb{Q}}$ to $V_{\mathbb{C}}$, as $V_{\mathbb{Q}}$ is fixed by ι.

Evaluating Q on the normalised basis states $|\alpha\rangle_{\mathcal{N}}$, spanning $V_{\mathbb{C}}$, and using (4.2.34), we find

$$Q(\,|\alpha\rangle_{\mathcal{N}}\,, |\beta\rangle_{\mathcal{N}}) = \mathrm{e}^{\mathrm{i}\pi q_L(|\beta\rangle)}\,{}_{\mathcal{N}}\langle\mathcal{C}(\alpha)|\,\mathcal{V}\,|\beta\rangle_{\mathcal{N}} \ . \tag{4.2.47}$$

From eq. (4.2.14) and using that $[J_0, \mathcal{V}] = 0$, we observe that $Q(\,|\alpha\rangle_{\mathcal{N}}\,, |\beta\rangle_{\mathcal{N}})$ can only be non-zero when the states $|\mathcal{C}(\alpha)\rangle_{\mathcal{N}}$ and $|\beta\rangle_{\mathcal{N}}$ carry identical U(1)-charges. Consequently, applying (2.5.25) and (2.5.27), we conclude:

$$q_L(|\alpha\rangle_{\mathcal{N}}) = -q_L(|\beta\rangle_{\mathcal{N}})\,, \tag{4.2.48}$$

which implies $|\alpha\rangle_{\mathcal{N}} \in V_{\mathbb{C}}^{p,q}$ and $|\beta\rangle_{\mathcal{N}} \in V_{\mathbb{C}}^{q,p}$. Thus, condition (4.2.6) is satisfied.

Considering now condition (4.2.7), we compute for any state in the graded vector space $|\alpha\rangle \in V_{\mathbb{C}}^{p,q}$

$$\mathrm{i}^{p-q} Q(|\alpha\rangle\,, \iota(|\alpha\rangle)) = \mathrm{i}^{2lq_L(|\alpha\rangle)}\overline{Q(\iota(|\alpha\rangle), |\alpha\rangle)} = \overline{\langle\alpha|\, \mathrm{e}^{-\mathrm{i}\pi(l-1)J_0}\mathcal{V}\,|\alpha\rangle}\ . \tag{4.2.49}$$

Therefore, in order for Q to define a polarisation on $(V_{\mathbb{Q}}, V_{\mathbb{C}})$, the operator $\mathrm{e}^{-\mathrm{i}\pi(l-1)J_0}\mathcal{V}$ must be positive definite, requiring

$$\langle\alpha|\, \mathrm{e}^{-\mathrm{i}\pi(l-1)J_0}\mathcal{V}\,|\alpha\rangle > 0 \qquad \text{for all non-zero } \ |\alpha\rangle \in V_{\mathbb{C}}^{p,q}\ . \tag{4.2.50}$$

A solution satisfying both (4.2.45) and the positivity condition (4.2.50) is provided by the choice

$$\mathcal{V} = \mathrm{e}^{\mathrm{i}\pi(l-1)J_0}\ . \tag{4.2.51}$$

However, it must be noted that $\mathcal{V}$ commutes with the supercurrents $G^{\pm}$ only when l is an odd integer, and only in this case $\mathcal{V}$ is central to the $\mathcal{N}=2$ super Virasoro algebra. In section 4.3 we will see that A-type $\mathcal{N}=(2,2)$ minimal models at odd levels k give rise to odd l. In such cases, the twisted Witten index defined via $\mathcal{V} = \mathrm{e}^{\mathrm{i}\pi(l-1)J_0}$ indeed induces a well-defined operator on $V_{\mathbb{Q}}$, fulfilling all polarisation criteria. Note that when all states in the $\text{(c,c)}_{q_L=q_R}$-ring possess integral U(1)-charges, implying $l=1$, the operator $\mathcal{V}$ reduces to the identity. Consequently, in this scenario, the untwisted open string Witten index itself realises a valid polarisation.

4.2.2 Rational Hodge Structure and CM

We will now explicitly demonstrate that the rational Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}})$, constructed in section 4.2.1, admits sufficiently many complex multiplications provided a certain natural Galois-theoretic compatibility condition is satisfied by the associated $\mathcal{N}=(2,2)$ rational superconformal field theory. This compatibility condition, formulated precisely in eq. (4.2.65), requires that the set $\mathcal{R}$ of Ramond–Ramond (RR) representations appearing in the A-type Hodge structure be stable under the Galois action,

$$\mathrm{Gal}\left(\mathbb{Q}(S_\alpha)/\mathbb{Q}\right) \curvearrowright \mathcal{R}\,, \tag{4.2.52}$$

where $\mathbb{Q}(S_\alpha)$ denotes the number field generated by the modular S-matrix entries $\mathrm{S}_{\alpha A}$ for a fixed $\alpha \in \mathcal{R}$ and $A \in \mathcal{B}$, with $\mathcal{B}$ denoting a basis of $V_{\mathbb{Q}}$. Under this assumption we will show that the resulting Hodge structure is of complex multiplication (CM) type in the sense defined around eq. (4.2.10), if a polarisation exists[5]. Otherwise, the Hodge structure will have sufficiently many complex multiplications.

The key idea of the argument is that the presence of a Galois group and suitable Galois subgroups in rational conformal field theories leads to a splitting of the Hodge structure

[5]Note that for $\mathcal{N}=(2,2)$ superconformal field theories that are used as worldsheet theroies in Type IIA string theory, i.e., GSO-projected theories, the U(1)-charges are integral and hence, as recalled in section 4.2.1, a polarisation always exists

into irreducible substructures, each admitting an action by a number field of maximal degree. Specifically, there exists a tower of Galois extensions,

$$\mathbb{Q}(S) \supseteq \mathbb{Q}(S_\alpha) \supseteq \mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)} \supseteq \mathbb{Q}\,, \tag{4.2.53}$$

where $\mathbb{Q}(S)$ is the field generated by all entries of the modular S-matrix, and $\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}$ is the fixed field under the stabiliser subgroup of the Galois group that leaves $\alpha \in \mathcal{R}$ invariant.

Using this setup, we then show that the Galois action corresponding to the intermediate field $\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}$ permutes a collection of suitably normalised basis vectors,

$$\{|\alpha\rangle_S \mid \alpha \in \mathcal{R}\} \subset V_\mathbb{C}\,. \tag{4.2.54}$$

Since each of these states has a well-defined U(1)-charge, they are contained in individual Hodge summands $V_\mathbb{C}^{p,q}$ as defined in eq. (4.2.24).

We then define the subspace spanned by the Galois conjugates

$$W_\mathbb{C}^\alpha = \big\langle\, \sigma(|\alpha\rangle_S) \mid \sigma \in \mathrm{Gal}\left(\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}/\mathbb{Q}\right) \big\rangle_\mathbb{C}\,, \tag{4.2.55}$$

and demonstrate that $W_\mathbb{C}^\alpha \subset V_\mathbb{C}$ is the complexification of a rational sub-Hodge structure. In this way, we obtain a decomposition of $V_\mathbb{Q}$ into irreducible Hodge components.

Finally, to establish the existence of sufficiently many Hodge endomorphisms, we show that scalar multiplication by a primitive generator ζ of the cyclotomic field $\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)} = \mathbb{Q}(\zeta)$ defines a Hodge endomorphism on the subspace $W_\mathbb{Q}^\alpha$, yielding an embedding

$$\mathbb{Q}(\zeta) \hookrightarrow \mathrm{End}_{\mathrm{Hdg}}(W_\mathbb{Q}^\alpha)\,. \tag{4.2.56}$$

Since the dimension of $W_\mathbb{Q}^\alpha$ matches the degree of the extension $[\mathbb{Q}(\zeta) : \mathbb{Q}]$, we conclude that each irreducible substructure admits an E^α-multiplication with E^α being a number field of maximal possible degree. If the Hodge structure admits a polarisation, then each $W_\mathbb{Q}^\alpha$ is of CM-type, and hence the full structure $(V_\mathbb{Q}, V_\mathbb{C})$ is of CM-type as well.

The Galois Groups and Their Actions

Let now $(V_\mathbb{Q}, V_\mathbb{C})$ be a Hodge structure constructed as in section 4.2.1, with $\{|B_A\rangle\}_{A\in\mathcal{B}}$ denoting a basis of the rational vector space $V_\mathbb{Q}$, and let $\{|\alpha\rangle_\mathcal{N}\}_{\alpha\in\mathcal{R}}$ denote the basis of normalised RR–ground states which correspond to elements in the ring $(\mathrm{c,c})_{q_L=q_R}$ by spectral flow.

Our first goal is to demonstrate that these normalised ground states can be rescaled in such a way that the resulting vectors lie in the vector space $V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{Q}(S)$, where $\mathbb{Q}(S)$ is the number field generated by all entries of the modular S-matrix,

$$\mathbb{Q}(S) := \mathbb{Q}\big(\{\mathrm{S}_{ij} \mid i,j \in \mathrm{HWR}_\mathcal{H}(\mathcal{A}')\}\big)\,. \tag{4.2.57}$$

To see this, recall the expansion of a normalised RR–ground state in terms of projected boundary states in eq. (4.2.34), and upon substituting the general form of a Cardy state (2.4.26) we get

$$|\alpha\rangle_{\mathcal{N}} = e^{-\pi i q_L(\alpha)}\kappa \sum_{A\in\mathcal{B}} \frac{\mathrm{S}^*_{\alpha A}}{\sqrt{\mathrm{S}_{0\alpha}}^*}\,|B^A\rangle_{\mathcal{P}}\ . \tag{4.2.58}$$

Note that we are using the normalisation ${}_{\mathcal{N}}\langle\alpha|i\rangle\rangle = \delta_{\alpha i}$ between the normalised RR–ground states and the Ishibashi states $|i\rangle\rangle$, $i \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A}')$, see (2.4.6) and the assumption in 2.5. Using this expansion, we define the rescaled states

$$|\alpha\rangle_{\mathcal{I}} := e^{\pi i q_L(|\alpha\rangle)}\kappa^{-1}\sqrt{\mathrm{S}_{0\alpha}}^*\,|\alpha\rangle_{\mathcal{N}} = \sum_{A\in\mathcal{B}} \mathrm{S}^*_{\alpha A}\,|B^A\rangle_{\mathcal{P}}\,. \tag{4.2.59}$$

By construction, these states lie in the vector space $V_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{Q}(S_\alpha)$, where

$$\mathbb{Q}(S_\alpha) := \mathbb{Q}\big(\{\mathrm{S}_{\alpha A}\mid A\in\mathcal{B}\}\big) \subseteq \mathbb{Q}(S)\,. \tag{4.2.60}$$

Since $\mathbb{Q}(S)$ is a Galois extension of $\mathbb{Q}$ with an abelian Galois group, its subextension $\mathbb{Q}(S_\alpha)$ is also Galois. In particular, standard Galois theory ensures the existence of a natural restriction morphism (see e.g., [177, Thm. VI.1.10]):

$$\cdot\,|_{\mathbb{Q}(\mathrm{S}_i)} : \mathrm{Gal}(\mathbb{Q}(S)/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})\,. \tag{4.2.61}$$

This means that every automorphism $\rho \in \mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})$ can be realised as the restriction $\rho = \sigma|_{\mathbb{Q}(S_\alpha)}$ of some $\sigma\in\mathrm{Gal}(\mathbb{Q}(S)/\mathbb{Q})$.

The action of a Galois automorphism ρ on the rescaled states $|\alpha\rangle_{\mathcal{I}} \in V_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{Q}(S_\alpha)$ extends naturally by acting trivially on $V_{\mathbb{Q}}$ and linearly on the coefficients. Applying the known Galois action on the S-matrix entries (see eq. (2.3.37)), we have

$$\rho(\mathrm{S}_{\alpha A}) = \epsilon_\sigma(\alpha)\mathrm{S}_{\varsigma(\alpha)A} := \epsilon_\rho(\alpha)\mathrm{S}_{\varrho(\alpha)A}, \qquad \epsilon_\rho(\alpha)\in\{\pm1\}\,. \tag{4.2.62}$$

Substituting this into the expression for $|\alpha\rangle_{\mathcal{I}}$ and using that the Galois group $\mathrm{Gal}(\mathbb{Q}(S)/\mathbb{Q})$ is abelian, hence commutes with complex conjugation, we obtain the Galois action on the normalised RR–ground states,

$$\rho(|\alpha\rangle_{\mathcal{I}}) = \sum_{A\in\mathcal{B}}\rho(\mathrm{S}^*_{\alpha A})\,|B^A\rangle_{\mathcal{P}} = \epsilon_\rho(\alpha)\sum_{A\in\mathcal{B}}\mathrm{S}^*_{\varrho(\alpha)A}\,|B^A\rangle_{\mathcal{P}} = \epsilon_\rho(\alpha)\,|\varrho(\alpha)\rangle_{\mathcal{I}}\,. \tag{4.2.63}$$

Note that the last equality makes only sense if the representation $\varrho(\alpha)$ lies in the set $\mathcal{R}$ of RR–ground states such that there exists a normalised highest-weight state $|\varrho(\alpha)\rangle_{\mathcal{I}}$. Since we do not have a general proof that this is always the case, we introduce this as a working assumption. Specifically, we assume that the set of representations

$$\mathcal{R}_\alpha \ :=\ \{\ \varrho(\alpha)\mid\rho\in\mathrm{Gal}(\mathbb{Q}(\mathrm{S}_\alpha)/\mathbb{Q})\ \}\ =\ \mathrm{Im}(\varrho)\ \subset\ \mathrm{HWR}_{\mathcal{H}}(\mathcal{A}')\,, \tag{4.2.64}$$

is a subset of $\mathcal{R}$,

$$\mathcal{R}_\alpha \subset \mathcal{R} \quad \text{for any} \quad \alpha\in\mathcal{R}\ . \tag{4.2.65}$$

We refer to this as the *Galois compatibility condition* on R–ground states, which is satisfied in all cases known to us.

Splitting of the Hodge Structure

Let us now investigate how the Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}})$, constructed in section 4.2.1, admits a natural decomposition into Hodge substructures. Specifically, for each state $|\alpha\rangle_{\mathcal{I}} \in V_{\mathbb{C}}$, where $\alpha \in \mathcal{R}$, we define $W^{\alpha}_{\mathbb{Q}}$ to be the smallest subset of $V_{\mathbb{Q}}$ such that its complexification contains the state $|\alpha\rangle_{\mathcal{I}}$

$$|\alpha\rangle_{\mathcal{I}} \in W^{\alpha}_{\mathbb{C}} := W^{\alpha}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \ . \tag{4.2.66}$$

We now give an explicit construction of the subspace $W^{\alpha}_{\mathbb{Q}}$ and demonstrate that it defines a Hodge substructure.

To proceed, recall that $|\alpha\rangle_{\mathcal{I}}$ is constructed as a linear combination of boundary states with coefficients in the field extension $\mathbb{Q}(S_\alpha)$, which is generated by the S-matrix elements $\mathrm{S}_{\alpha A}$ for $A \in \mathcal{B}$. By the primitive element theorem, this field extension is generated by a single element $\zeta \in \mathbb{Q}(S_\alpha)$, which satisfies an irreducible polynomial of degree

$$n := [\mathbb{Q}(S_\alpha) : \mathbb{Q}] = |\mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})| \ . \tag{4.2.67}$$

This implies that each matrix element $\mathrm{S}^*_{\alpha A}$ can be expressed as a rational linear combination of the powers of ζ. Consequently, the state $|\alpha\rangle_{\mathcal{I}}$ can be written in the form:

$$|\alpha\rangle_{\mathcal{I}} = \sum_{k=0}^{n-1} \zeta^k \, |Q^{\alpha}_k\rangle \ , \qquad \text{with} \quad |Q^{\alpha}_k\rangle \in V_{\mathbb{Q}} \ , \tag{4.2.68}$$

where the states $|Q^{\alpha}_k\rangle$ are uniquely determined rational linear combinations of the boundary states $|B_A\rangle$, $A \in \mathcal{B}$. We define the subspace

$$W^{\alpha}_{\mathbb{Q}} := \mathrm{span}_{\mathbb{Q}} \left\{ |Q^{\alpha}_0\rangle , \ldots , |Q^{\alpha}_{n-1}\rangle \right\} \ , \tag{4.2.69}$$

so that $W^{\alpha}_{\mathbb{C}}$ contains the state $|\alpha\rangle_{\mathcal{I}}$ as well as all of its Galois conjugates, $\rho\,|\alpha\rangle_{\mathcal{I}}$.

Our next goal is to prove that these Galois conjugates of $|\alpha\rangle_{\mathcal{I}}$ span the complexified subspace $W^{\alpha}_{\mathbb{C}}$. Recall that the dimension of $W^{\alpha}_{\mathbb{Q}}$ is equal to the complex dimension of $W^{\alpha}_{\mathbb{C}}$, as the boundary states $|B_A\rangle$, $A \in \mathcal{B}$ are linearly independent over $\mathbb{C}$ (see footnote 3). Consider then the set $\{\rho(|\alpha\rangle_{\mathcal{I}})\}_{\rho \in \mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})}$. If this set contains n linearly independent vectors over $\mathbb{C}$, it must form a basis of $W^{\alpha}_{\mathbb{C}}$ since the dimension of $W^{\alpha}_{\mathbb{Q}}$ is at most $n = |\mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})|$, as observed from equation (4.2.69). Therefore, to complete the argument, it suffices to show that the Galois conjugates $\rho(|\alpha\rangle_{\mathcal{I}})$ are linearly independent for all $\rho \in \mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})$.

Let us assume that the set of Galois conjugates $\{\rho(|\alpha\rangle_{\mathcal{I}})\}_{\rho \in \mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})}$ is linearly dependent over $\mathbb{C}$. That is, suppose there exists a finite subset of non-trivial automorphisms $\{\rho_0, \ldots, \rho_{n'}\} \subset \mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})$, along with constants $c_s \in \mathbb{C}$, such that

$$|\alpha\rangle_{\mathcal{I}} = \sum_{s=0}^{n'} c_s \rho_s(|\alpha\rangle_{\mathcal{I}}) = \sum_{s=0}^{n'} c_s \epsilon_{\rho_s}(\alpha) \, |\varrho_s(\alpha)\rangle_{\mathcal{I}} \ . \tag{4.2.70}$$

Recall that by construction, each state $|\alpha\rangle_{\mathcal{I}}$ is associated with a unique representation $\alpha \in \mathcal{R}$, and that two such states $|\alpha\rangle_{\mathcal{I}}$, $|\beta\rangle_{\mathcal{I}}$ are linearly independent over $\mathbb{C}$ unless $\alpha = \beta$. Therefore, for the expression above to hold, it must be that at least one of the automorphisms ρ_* in the set satisfies $\varrho_*(\alpha) = \alpha$. This motivates us to define the subgroup of $\mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})$ that stabilises the label α,

$$\mathrm{Stab}(\alpha) := \{\rho_* \in \mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q}) \mid \varrho_*(\alpha) = \alpha\} \ . \tag{4.2.71}$$

From the transformation rule (4.2.62) governing the Galois action on the S-matrix entries, we know that any $\rho_* \in \mathrm{Stab}(\alpha)$ must satisfy

$$\rho_*(\mathrm{S}_{\alpha A}) = \mathrm{S}_{\alpha A} \quad \text{or} \quad \rho_*(\mathrm{S}_{\alpha A}) = -\mathrm{S}_{\alpha A} \qquad \text{for all } A \in \mathcal{B} \ . \tag{4.2.72}$$

If the action is trivial on all $\mathrm{S}_{\alpha A}$, then ρ_* must be the identity automorphism. On the other hand, if the action introduces a global sign flip, then ρ is a non-trivial automorphism of order two. We now argue that this is the only possibility for a non-trivial stabiliser.

Suppose there exist two such automorphisms ρ_* and ρ'_* satisfying $\rho_*(\mathrm{S}_{\alpha A}) = -\mathrm{S}_{\alpha A}$ and similarly for ρ'_*. Then their composition must satisfy:

$$\rho_* \circ \rho'_*(\mathrm{S}_{\alpha A}) = \mathrm{S}_{\alpha A} \quad \Rightarrow \quad \rho_* \circ \rho'_* = \mathrm{id} \quad \Rightarrow \quad \rho'_* = \rho_*^{-1} = \rho_* \ , \tag{4.2.73}$$

implying that $\mathrm{Stab}(\alpha)$ is either trivial or cyclic of order two,

$$\mathrm{Stab}(\alpha) = \langle \rho_* \rangle \quad \text{with} \quad \mathrm{ord}(\rho_*) \leq 2 \ . \tag{4.2.74}$$

Since $\mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})$ is Abelian (being a subgroup of the Abelian Galois group $\mathrm{Gal}(\mathbb{Q}(S)/\mathbb{Q})$), the subgroup $\mathrm{Stab}(\alpha)$ is normal. Thus, we can define the quotient group

$$\mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})/\mathrm{Stab}(\alpha) = \mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})/\langle \rho_* \rangle \ . \tag{4.2.75}$$

Now if $\mathrm{Stab}(\alpha)$ is trivial, then all Galois conjugates $\rho(|\alpha\rangle_{\mathcal{I}})$ are by definition linearly independent, which leads to a contradiction. It remains to treat the case where $\mathrm{ord}(\rho_*) = 2$. In this case, we have

$$\rho_*(\mathrm{S}_{\alpha A}) = -\mathrm{S}_{\alpha A} \quad \text{for all } A \in \mathcal{B} \ , \tag{4.2.76}$$

which implies that

$$\rho_*(|\alpha\rangle_{\mathcal{I}}) = -\,|\alpha\rangle_{\mathcal{I}} \ . \tag{4.2.77}$$

The fundamental theorem of Galois theory then implies the existence of a tower of field extensions

$$\mathbb{Q}(S_\alpha) \supset \mathbb{Q}(S_\alpha)^{\langle \rho_* \rangle} \supset \mathbb{Q} \ , \quad \text{with} \quad \left[\mathbb{Q}(S_\alpha) : \mathbb{Q}(S_\alpha)^{\langle \rho_* \rangle}\right] = 2 \ , \quad \left[\mathbb{Q}(S_\alpha)^{\langle \rho_* \rangle} : \mathbb{Q}\right] = \frac{n}{2} \ . \tag{4.2.78}$$

Applying the primitive element theorem successively, we obtain

$$\mathbb{Q}(S_\alpha) = \mathbb{Q}(\eta, \xi) \ , \tag{4.2.79}$$

where η generates the fixed field $\mathbb{Q}(S_\alpha)^{\langle\rho_*\rangle}$ and is the root of a minimal polynomial of order $n/2$, and ξ satisfies a quadratic equation over $\mathbb{Q}(\eta)$ of the form

$$\xi^2 + \kappa = 0 \ , \qquad \kappa \in \mathbb{Q}(\eta) \ . \tag{4.2.80}$$

Given this, the state $|\alpha\rangle_\mathcal{I}$ admits the expansion

$$|\alpha\rangle_\mathcal{I} = \sum_{a=0}^{1} \sum_{b=0}^{n/2-1} \xi^a \eta^b \, |Q^\alpha_{a,b}\rangle \ , \qquad \text{with} \quad |Q^\alpha_{a,b}\rangle \in V_\mathbb{Q} \ . \tag{4.2.81}$$

The automorphism ρ_*, by definition, fixes all elements in $\mathbb{Q}(\eta)$ and acts as $\rho_*(\xi) = -\xi$. Applying this to the expansion above and recalling that $\rho_*(|\alpha\rangle_\mathcal{I}) = -\,|\alpha\rangle_\mathcal{I}$, we find

$$\sum_{b=0}^{n/2-1} \eta^b \, |Q^\alpha_{0,b}\rangle - \sum_{b=0}^{n/2-1} \xi\eta^b \, |Q^\alpha_{1,b}\rangle = -\sum_{b=0}^{n/2-1} \eta^b \, |Q^\alpha_{0,b}\rangle - \sum_{b=0}^{n/2-1} \xi\eta^b \, |Q^\alpha_{1,b}\rangle \ . \tag{4.2.82}$$

Comparing coefficients of η^b and $\xi\eta^b$ independently, we conclude that

$$|Q^\alpha_{0,b}\rangle = 0 \quad \text{for all } b = 0, \ldots, \frac{n}{2} - 1 \ . \tag{4.2.83}$$

Thus, the subspace $W^\alpha_\mathbb{Q}$ reduces to

$$W^\alpha_\mathbb{Q} = \Big\langle \, |Q^\alpha_{1,b}\rangle \, \Big| \, b = 0, \ldots, \frac{n}{2} - 1 \Big\rangle_\mathbb{Q} \ , \tag{4.2.84}$$

with dimension at most $n/2 = |\mathrm{Gal}(\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}/\mathbb{Q})|$.

Now consider the state

$$\xi^{-1} \, |\alpha\rangle_\mathcal{I} \in V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{Q}(\eta) = V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)} \ . \tag{4.2.85}$$

We define an action of the Galois group $\mathrm{Gal}(\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}/\mathbb{Q})$ on this space by requiring that the vectors in $V_\mathbb{Q}$ remain fixed. By the restriction property of Galois automorphisms (cf. (4.2.61)), this coincides with the restriction of the Galois action from the full field $\mathbb{Q}(S_\alpha)$.

We now examine the set of Galois conjugates

$$\left\{ \theta \left(\xi^{-1} \, |\alpha\rangle_\mathcal{I} \right) \, \middle| \, \theta \in \mathrm{Gal}(\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}/\mathbb{Q}) \right\} \ . \tag{4.2.86}$$

Let $\theta \in \mathrm{Gal}(\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}/\mathbb{Q})$ and choose a lift $\rho \in \mathrm{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})$ such that

$$\rho|_{\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}} = \theta \, . \tag{4.2.87}$$

Then, using the expansion (4.2.81) and $|Q^\alpha_{0,b}\rangle = 0$, we have

$$\begin{aligned}\theta\left(\xi^{-1}\,|\alpha\rangle_{\mathcal{I}}\right) &= \sum_{b=0}^{n/2-1} \theta(\eta^b)\,|Q^\alpha_{1,b}\rangle = \sum_{b=0}^{n/2-1} \rho(\eta^b)\,|Q^\alpha_{1,b}\rangle \\ &= \rho(\xi^{-1})\rho\left(\sum_{b=0}^{n/2-1} \xi\eta^b\,|Q^\alpha_{1,b}\rangle\right) = \rho(\xi^{-1})\rho(|\alpha\rangle_{\mathcal{I}}) \\ &= \rho(\xi^{-1})\epsilon_\rho(\alpha)\,|\varrho(\alpha)\rangle_{\mathcal{I}}\ .\end{aligned} \tag{4.2.88}$$

To determine the linear independence of the set (4.2.86), suppose to the contrary that a non-trivial linear dependence exists among them. Then some θ would correspond to a ρ for which $\varrho(\alpha) = \alpha$, i.e., $\rho \in \mathrm{Stab}(\alpha)$. But this would contradict the assumption that θ acts non-trivially on $\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}$, since ρ restricts to the identity, i.e., $\theta = \rho|_{\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}} = \mathrm{Id}$. Hence, all elements in (4.2.86) must be linearly independent over $\mathbb{C}$.

We conclude that the number of independent Galois conjugates equals

$$\left|\mathrm{Gal}(\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}/\mathbb{Q})\right| = \frac{n}{2}\ , \tag{4.2.89}$$

saturating the bound in (4.2.84). Thus, the Galois conjugates of $\xi^{-1}\,|\alpha\rangle_{\mathcal{I}}$ span the complexification $W^\alpha_{\mathbb{C}}$, and we obtain

$$\dim_{\mathbb{Q}} W^\alpha_{\mathbb{Q}} = \dim_{\mathbb{C}} W^\alpha_{\mathbb{C}} = \frac{n}{2}\,. \tag{4.2.90}$$

To simplify the notation and unify the treatment of different cases, we introduce a common definition for the basis elements associated with each representation $\alpha \in \mathcal{R}$. Specifically, we define the state $|\alpha\rangle_{\mathcal{S}}$ by

$$|\alpha\rangle_{\mathcal{S}} = \begin{cases} |\alpha\rangle_{\mathcal{I}} & \text{if } |\mathrm{Stab}(\alpha)| = 1\ , \\ \xi^{-1}\,|\alpha\rangle_{\mathcal{I}} & \text{if } |\mathrm{Stab}(\alpha)| = 2\ , \end{cases} \tag{4.2.91}$$

so that in both cases, $|\alpha\rangle_{\mathcal{S}}$ lies in the extended space $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}$.

Let $n = |\mathrm{Gal}(\mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}/\mathbb{Q})|$ and let $\zeta \in \mathbb{Q}(S_\alpha)^{\mathrm{Stab}(\alpha)}$ be a primitive generator of the field extension. Then the scaled state $|\alpha\rangle_{\mathcal{S}}$ admits the expansion

$$|\alpha\rangle_{\mathcal{S}} = \sum_{k=0}^{n-1} \zeta^k\,|Q^\alpha_k\rangle\ , \qquad \text{with } |Q^\alpha_k\rangle \in V_{\mathbb{Q}}\ . \tag{4.2.92}$$

The reason to introduce the scaled basis vectors is that, regardless of the structure of the stabiliser subgroup $\mathrm{Stab}(\alpha)$, the Galois conjugates of the state $|\alpha\rangle_{\mathcal{S}}$ span the complex vector space $W_{\mathbb{C}}^{\alpha}$, which is the complexification of the smallest rational subspace $W_{\mathbb{Q}}^{\alpha} \subset V_{\mathbb{Q}}$ containing $|\alpha\rangle_{\mathcal{S}}$. We can therefore write

$$W_{\mathbb{C}}^{\alpha} = \left\langle\, \theta(|\alpha\rangle_{\mathcal{S}}) \,\middle|\, \theta \in \mathrm{Gal}\left(\mathbb{Q}(S_{\alpha})^{\mathrm{Stab}(\alpha)}/\mathbb{Q}\right) \right\rangle_{\mathbb{C}} \ . \tag{4.2.93}$$

To verify that $W_{\mathbb{Q}}^{\alpha}$ defines a valid Hodge substructure of $(V_{\mathbb{Q}}, V_{\mathbb{C}})$, we examine the action of the Galois group $\mathrm{Gal}(\mathbb{Q}(S_{\alpha})^{\mathrm{Stab}(\alpha)}/\mathbb{Q})$ on the scaled states. From equations (4.2.63) and (4.2.88), we find that

$$\theta\left(|\alpha\rangle_{\mathcal{S}}\right) = \begin{cases} \epsilon_{\rho}(\alpha)\ |\varrho(\alpha)\rangle_{\mathcal{I}} & \text{if } |\mathrm{Stab}(\alpha)| = 1 \ , \\ \rho(\xi^{-1})\, \epsilon_{\rho}(\alpha)\ |\varrho(\alpha)\rangle_{\mathcal{I}} & \text{if } |\mathrm{Stab}(\alpha)| = 2 \ , \end{cases} \tag{4.2.94}$$

where $\rho \in \mathrm{Gal}(\mathbb{Q}(S_{\alpha})/\mathbb{Q})$ restricts to $\theta \in \mathrm{Gal}(\mathbb{Q}(S_{\alpha})^{\mathrm{Stab}(\alpha)}/\mathbb{Q})$, and $\varrho(\alpha)$ denotes the representation to which α is mapped under the Galois action.

In both cases, $\theta(|\alpha\rangle_{\mathcal{S}})$ is proportional to a state $|\varrho(\alpha)\rangle_{\mathcal{N}}$ which, by construction, lies in a definite graded component $V_{\mathbb{C}}^{p,q}$ determined by its U(1)-charges. This shows that $W_{\mathbb{Q}}^{\alpha}$ satisfies the grading condition (4.2.3) and hence defines a genuine Hodge substructure of the full Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}})$.

Existence of Hodge Endomorphisms

To complete the construction, we now show that the Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}})$ possesses sufficiently many complex multiplications. In particular, if the structure is polarised, it is of CM-type. To demonstrate this, it suffices to show that each irreducible Hodge substructure $W_{\mathbb{Q}}^{\alpha} \subset V_{\mathbb{Q}}$ admits an E^{α}-multiplication, in the sense of definition (4.2.10), by the number field $E^{\alpha} \cong \mathbb{Q}(S_{\alpha})^{\mathrm{Stab}(\alpha)}$ of degree $[E^{\alpha} : \mathbb{Q}] = \dim_{\mathbb{Q}} W_{\mathbb{Q}}^{\alpha}$.

We construct these endomorphisms acting on the subspaces $W_{\mathbb{Q}}^{\alpha}$ by the induced maps corresponding to multiplication of the scaled state $|\alpha\rangle_{\mathcal{S}}$ by powers of a primitive generator ζ of the field $\mathbb{Q}(S_{\alpha})^{\mathrm{Stab}(\alpha)} = \mathbb{Q}(\zeta)$, as introduced previously. Denoting $n = [\mathbb{Q}(\zeta) : \mathbb{Q}]$, we define a set of endomorphisms associated with the elements ζ^{i}, $i = 0, \ldots, n-1$, which give an embedding $E^{\alpha} \simeq \mathbb{Q}(\zeta) \hookrightarrow \mathrm{End}_{\mathrm{Hdg}}(W_{\mathbb{Q}}^{\alpha})$.

To construct the Hodge endomorphisms, recall that the field $\mathbb{Q}(\zeta)$ can be regarded as a $\mathbb{Q}$-vector space with basis $\{1, \zeta, \ldots, \zeta^{n-1}\}$. Multiplication by ζ acts $\mathbb{Q}$-linearly on this space and can thus be represented by a matrix $\mathrm{T}(\zeta) \in \mathrm{Mat}(n \times n, \mathbb{Q})$ such that

$$\zeta^{k} \mapsto \zeta^{k+1} = \sum_{s=0}^{n-1} \mathrm{T}(\zeta)^{k}{}_{s}\, \zeta^{s} \ . \tag{4.2.95}$$

Applying this to the expansion of $|\alpha\rangle_{\mathcal{S}}$,

$$|\alpha\rangle_{\mathcal{S}} = \sum_{k=0}^{n-1} \zeta^{k}\, |Q_{k}^{\alpha}\rangle \ , \tag{4.2.96}$$

we obtain

$$\zeta\,|\alpha\rangle_{\mathcal{S}} = \sum_{k,s=0}^{n-1} \zeta^s\,\mathrm{T}(\zeta)^k{}_s\,|Q_k^\alpha\rangle \ . \tag{4.2.97}$$

This motivates the definition of a rational-linear map $\varphi_\zeta^\alpha : W_\mathbb{Q}^\alpha \to W_\mathbb{Q}^\alpha$ by

$$\varphi_\zeta^\alpha(|Q_k^\alpha\rangle) := \sum_{s=0}^{n-1} \mathrm{T}(\zeta)^k{}_s\,|Q_s^\alpha\rangle \ , \tag{4.2.98}$$

whose complexification acts, due to eq. (4.2.97), as

$$\varphi_{\zeta,\mathbb{C}}^\alpha(|\alpha\rangle_{\mathcal{S}}) = \zeta\,|\alpha\rangle_{\mathcal{S}} \ . \tag{4.2.99}$$

Now, consider the basis of $W_\mathbb{C}^\alpha$ given by the Galois conjugates $\{\rho(|\alpha\rangle_{\mathcal{S}})\}_{\rho\in\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})}$. Using that the Galois action leaves the coefficients $|Q_k^\alpha\rangle \in V_\mathbb{Q}$ invariant, we find for the action of the Hodge endomorphisms

$$\begin{aligned}\varphi_{\zeta,\mathbb{C}}^\alpha\,(\rho\,|\alpha\rangle_\mathbb{C}) \;&=\; \sum_{k=0}^{n-1}\rho(\zeta^k)\varphi_\zeta^\alpha(|Q_k^\alpha\rangle) \;=\; \sum_{k,s=0}^{n-1}\rho(\zeta^k)\mathrm{T}(\zeta)^s{}_k\,|Q_s^\alpha\rangle \;=\; \sum_{k,s=0}^{n-1}\rho\left(\mathrm{T}(\zeta)^s{}_k\zeta^k\right)|Q_s^\alpha\rangle \\ &=\; \sum_{k=0}^{n-1}\rho(\zeta^{k+1})\,|Q_k^\alpha\rangle \;=\; \rho(\zeta)\sum_{k=0}^{n-1}\rho(\zeta^k)\,|Q_k^\alpha\rangle \;=\; \rho(\zeta)\rho\,|\alpha\rangle_{\mathcal{S}} \ .\end{aligned} \tag{4.2.100}$$

Hence, $\varphi_{\zeta,\mathbb{C}}^\alpha$ is diagonalised in this basis, with eigenvalues given by the Galois conjugates of ζ. Since each $\rho(|\alpha\rangle_{\mathcal{S}})$ lies within a well-defined (p,q)-component and collectively they span these spaces, this implies that $\varphi_{\zeta,\mathbb{C}}^\alpha$ preserves the Hodge decomposition,

$$\varphi_{\zeta,\mathbb{C}}^\alpha(W_\mathbb{C}^{\alpha,(p,q)}) \subseteq W_\mathbb{C}^{\alpha,(p,q)} \ . \tag{4.2.101}$$

Moreover, repeated application of φ_ζ^α corresponds to multiplication by powers of ζ,

$$\left(\varphi_\zeta^\alpha\right)^k = \varphi_{\zeta^k}^\alpha \ , \qquad \text{for all } k\in\mathbb{Z} \ . \tag{4.2.102}$$

Thus, the collection $\{\varphi_{\zeta^k}^\alpha\}_{k=0}^{n-1}$ forms a set of commuting endomorphisms. To verify that they are $\mathbb{Q}$-linearly independent, suppose for contradiction that

$$\sum_{k=0}^{n-1} a_k\varphi_{\zeta^k}^\alpha = 0 \qquad \text{for some } a_k\in\mathbb{Q} \ . \tag{4.2.103}$$

Applying the complexification on both sides to $|\alpha\rangle_{\mathcal{S}}$, we obtain

$$\sum_{k=0}^{n-1} a_k\zeta^k\,|\alpha\rangle_{\mathcal{S}} = 0 \qquad \Rightarrow \qquad \sum_{k=0}^{n-1} a_k\zeta^k = 0 \ , \tag{4.2.104}$$

which contradicts the assumption that ζ has minimal polynomial of degree n over $\mathbb{Q}$. Therefore, the set $\{\varphi^\alpha_{\zeta^k}\}$ is $\mathbb{Q}$-linearly independent and generates a commutative subalgebra E^α of $\mathrm{End}_{\mathrm{Hdg}}(W^\alpha_{\mathbb{Q}})$ with

$$\dim_{\mathbb{Q}} E^\alpha = \dim_{\mathbb{Q}} W^\alpha_{\mathbb{Q}} = n \ , \tag{4.2.105}$$

which is by the map $\varphi^\alpha_\zeta \mapsto \zeta$ isomorphic to $\mathbb{Q}(\zeta)$. Hence $W^\alpha_{\mathbb{Q}}$ admits an E^α-multiplication, with $E^\alpha \cong \mathbb{Q}(\zeta)$.

We conclude that if the compatibility condition (4.2.65) holds for an $\mathcal{N} = (2,2)$ rational superconformal field theory, then the associated Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}})$ as constructed in this section has sufficiently many complex multiplications. In particular, if the Hodge structure is polarisable, it is of complex multiplication type.

4.3 Example: $\mathcal{N}$=(2,2) Minimal Models

As the first example, we show that the Hodge structure constructed in section 4.2.1 has sufficiently many complex multiplications for all $\mathcal{N} = (2,2)$ minimal models with diagonal one-loop partition function with respect to characters of the finitely many irreducible highest weight representations of the $\mathcal{N} = 2$ Virasoro algebra. Apart from providing an important class of examples, this section also serves to illustrate that our construction is independent of the conformal field theory having a geometric interpretation.

General aspects of $\mathcal{N} = (2,2)$ Minimal Models

Analogous to the discrete series of unitary Virasoro minimal models with central charge $c < 1$, which are rational conformal field theories for the Virasoro algebra, the unitary $\mathcal{N} = (2,2)$ minimal models form a discrete series with $c < 3$, and constitute rational conformal field theories for the $\mathcal{N} = 2$ super Virasoro algebra [178–180]. The finitely many Virasoro modules of the $\mathcal{N} = (2,2)$ minimal model at level k and central charge $c = \frac{3k}{k+2}$ are most conveniently realised through a GKO coset construction [181, 182] with coset algebra

$$\mathcal{A} = \frac{\widehat{\mathfrak{su}}(2)_k \otimes \widehat{\mathfrak{u}}(1)_2}{\widehat{\mathfrak{u}}(1)_{k+2}} \,. \tag{4.3.1}$$

The characters of the coset algebra (4.3.1) can be derived using the branching rules

$$\left(\lambda_{\widehat{\mathfrak{su}}(2)_k}\right) \otimes \left(\lambda_{\widehat{\mathfrak{u}}(1)_2}\right) = \bigoplus_{\lambda_{\widehat{\mathfrak{u}}(1)_{k+2}}} \left(\lambda_{\widehat{\mathfrak{u}}(1)_{k+2}}\right) \otimes \left(\lambda_{\widehat{\mathfrak{su}}(2)_k \otimes \widehat{\mathfrak{u}}(1)_2 / \widehat{\mathfrak{u}}(1)_{k+2}}\right) \,, \tag{4.3.2}$$

from which we find the corresponding character relation

$$\chi_l^{\widehat{\mathfrak{su}}(2)_k}(\tau)\chi_s^{\widehat{\mathfrak{u}}(1)_2}(\tau) = \sum_{m=0}^{2(k+2)-1} \chi_m^{\widehat{\mathfrak{u}}(1)_{k+2}}(\tau)\chi_{m,s}^l(\tau)\,. \tag{4.3.3}$$

The characters of the coset algebra can subsequently be expressed in the form [142, 180, 183]

$$\chi_{m,s}^l(\tau,z) = \sum_{j=1}^{k} C_{l,-m+4j+s}^{(k)}(\tau)\Theta_{2m-(4j+s)(k+2),2k(k+2)}(\tau,kz) \tag{4.3.4}$$

with the string functions $C_{l,-m+4j+s}^{(k)}(\tau)$ defined as the embedding of the $\widehat{\mathfrak{u}}(1)_k$ characters into the $\widehat{\mathfrak{su}}(2)_k$ characters,

$$\begin{aligned} \chi_l^{\widehat{\mathfrak{su}}(2)_k} &= \sum_{\substack{m=0,\\ l+m\equiv 0 \bmod 2}}^{2k-1} C_{l,m}^{(k)}(\tau)\Theta_{m,k}(\tau) \\ \Theta_{m,k}(\tau,z) &= \sum_{n\in\mathbb{Z}} q^{k(n+\frac{m}{2k})^2} y^{(n+\frac{m}{2k})}\,, \qquad q := e^{2\pi i\tau}\,,\ \ y := e^{2\pi i z}\,. \end{aligned} \tag{4.3.5}$$

The irreducible representations of the coset chiral algebra (4.3.1) are labeled by

$$\Big\{(l,m,s)\;\Big|\;0\le l\le k\;,\;-k-1\le m\le k+2\;,\;s\in\{0,2,\pm1\}\Big\}\;, \tag{4.3.6}$$

where s is defined modulo 4 and m is defined modulo $2(k+2)$, subject to the constraint $l+m+s\equiv 0 \mod 2$. For $s\in\{0,2\}$ the corresponding state is in the NS–sector, while $s\in\{\pm1\}$ corresponds to the R–sector. From the coset construction and the properties of the string functions, one obtains the following identification rules:

$$(l,m,s)\sim(k-l,m+k+2,s+2)\,. \tag{4.3.7}$$

Together with the constraint $l+m+s\equiv 0 \mod 2$, this determines the total number of independent representations of the coset algebra[6],

$$N=(k+1)2(k+2)(4)\frac{1}{2}\frac{1}{2}=2(k+1)(k+2)\,. \tag{4.3.8}$$

Under modular transformations, the characters (4.3.4) transform as

$$\begin{aligned}\chi^l_{m,s}(\tau+1,z)&=e^{2\pi\mathrm{i}\left(\frac{l(l+2)-m^2}{4(k+2)}+\frac{s^2}{8}-\frac{c}{24}\right)}\chi^l_{m,s}(\tau,z)\\ \chi^l_{m,s}(-1/\tau,z/\tau)&=\sum_{l',m',s'}\mathrm{S}_{(l,m,s),(l',m',s')}\,\chi^{l'}_{m',s'}(\tau,z)\,.\end{aligned} \tag{4.3.9}$$

The modular S-matrix elements $\mathrm{S}_{(l,m,s),(l',m',s')}$ are given by

$$\begin{aligned}\mathrm{S}_{(l,m,s),(l',m',s')}=&\;\frac{1}{(k+2)}\sin\left(\frac{\pi}{k+2}(l+1)(l'+1)\right)\exp\left(\mathrm{i}\frac{\pi}{k+2}mm'\right)\exp\left(-\mathrm{i}\frac{\pi}{2}ss'\right)\\ =&-\frac{\mathrm{i}}{2(k+2)}\left(\zeta_{2(k+2)}^{(l+1)(l'+1)}-\zeta_{2(k+2)}^{-(l+1)(l'+1)}\right)\zeta_{2(k+2)}^{mm'}\mathrm{e}^{-\mathrm{i}\pi ss'/2}\;,\end{aligned} \tag{4.3.10}$$

where $\zeta_n=\mathrm{e}^{2\pi\mathrm{i}/n}$. Using the coset algebra form (4.3.1), we can combine the results from each individual theory to determine the fusion rules with the help of the Verlinde formula (2.2.22),

$$[(l_1,m_1,s_1)]\times[(l_1,m_1,s_1)]=\sum_{l_3,m_3,s_3}\mathcal{N}^{l_3}_{l_1l_2}\,\delta^{(2k+4)}_{m_1+m_2,m_3}\,\delta^{(4)}_{s_1+s_2,s_3}\,[(l_3,m_3,s_3)]\,, \tag{4.3.11}$$

where $\mathcal{N}^{l_3}_{l_1l_2}$ denotes the $\widehat{\mathfrak{su}}(2)_k$ fusion coefficients (2.2.37).

[6] For fixed (l,m), define $a:=l+m \mod 2\in\{0,1\}$. Then the selection rule $l+m+s\equiv 0 \mod 2$ becomes $a+s\equiv 0 \mod 2$, which implies $s\equiv -a \mod 2$. Since s takes values in $\mathbb{Z}_4=\{0,1,2,3\}$, exactly half of these satisfy the congruence for each value of a. Therefore, the constraint removes precisely half of the initially allowed (l,m,s) combinations.

The one-loop modular invariant partition functions of the $\mathcal{N}=(2,2)$ minimal models admit an ADE-classification scheme, inherited from the ADE-classification of $\widehat{\mathfrak{su}}(2)_k$ theories [59, 69]. For a detailed discussion of the classification of $\mathcal{N}=(2,2)$ minimal model modular invariant partition functions, we refer to [184], and references therein. In this work, we always restrict to the A-type modular invariant partition function, for which the NS-part takes the form

$$Z_{\mathrm{NS}}(\tau,\overline{\tau},z,\bar{z}) = \frac{1}{2}\sum_{\substack{0\le l\le k\\ m=-k-1,\dots,k+2\\ l+m\equiv 0 \mod 2}} \left(\chi^l_{m,0}(\tau,z)+\chi^l_{m,2}(\tau,z)\right)\left(\overline{\chi^l_{m,0}(\tau,z)}+\overline{\chi^l_{m,2}(\tau,z)}\right). \tag{4.3.12}$$

The total supersymmetric partition function is

$$Z(\tau,\overline{\tau},z,\bar{z}) = \frac{1}{2}\left(Z_{\mathrm{NS}}(\tau,\overline{\tau},z,\bar{z})+Z_{\widetilde{\mathrm{NS}}}(\tau,\overline{\tau},z,\bar{z})+Z_{\mathrm{R}}(\tau,\overline{\tau},z,\bar{z})+Z_{\widetilde{\mathrm{R}}}(\tau,\overline{\tau},z,\bar{z})\right), \tag{4.3.13}$$

where each sector is defined as

$$\begin{aligned}
Z_{\mathrm{NS}}(\tau,\overline{\tau},z,\bar{z}) &:= \mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\left(q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}y^{J_0}\overline{y}^{\overline{J}_0}\right)\\
Z_{\widetilde{\mathrm{NS}}}(\tau,\overline{\tau},z,\bar{z}) &:= \mathrm{Tr}_{\mathcal{H}_{\mathrm{NS}}}\left((-1)^F q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}y^{J_0}\overline{y}^{\overline{J}_0}\right)\\
Z_{\mathrm{R}}(\tau,\overline{\tau},z,\bar{z}) &:= \mathrm{Tr}_{\mathcal{H}_{\mathrm{R}}}\left(q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}y^{J_0}\overline{y}^{\overline{J}_0}\right)\\
Z_{\widetilde{\mathrm{R}}}(\tau,\overline{\tau},z,\bar{z}) &:= \mathrm{Tr}_{\mathcal{H}_{\mathrm{R}}}\left((-1)^F q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}y^{J_0}\overline{y}^{\overline{J}_0}\right).
\end{aligned} \tag{4.3.14}$$

Here, $(-1)^F$ denotes the fermion number operator and enforces periodic boundary conditions for fermions in the time-direction of the worldsheet torus. This operator is included in the partition functions $Z_{\widetilde{\mathrm{NS}}}$, $Z_{\widetilde{\mathrm{R}}}$ to ensure that the total partition function includes a sum over all possible spin structures along both cycles of the worldsheet torus. Under modular transformations, the individual parts transform as

$$\begin{aligned}
\tau\to\tau+1: &\quad Z_{\mathrm{NS}}\to Z_{\widetilde{\mathrm{NS}}}\to Z_{\mathrm{NS}}\,,\quad Z_{\mathrm{R}}\to Z_{\mathrm{R}}\,,\quad Z_{\widetilde{\mathrm{R}}}\to Z_{\widetilde{\mathrm{R}}}\\
\tau\to-\frac{1}{\tau}: &\quad Z_{\mathrm{NS}}\to Z_{\mathrm{NS}}\,,\quad Z_{\widetilde{\mathrm{NS}}}\to Z_{\mathrm{R}}\,,\quad Z_{\mathrm{R}}\to Z_{\widetilde{\mathrm{NS}}}\,,\quad Z_{\widetilde{\mathrm{R}}}\to Z_{\widetilde{\mathrm{R}}}\,,
\end{aligned} \tag{4.3.15}$$

so that the total supersymmetric partition function is modular invariant. In fact, one can show that for all $\mathcal{N}=(2,2)$ superconformal field theories that are invariant under left-right symmetric spectral flow (i.e., for theories such that $q_L-q_R\in\mathbb{Z}$ for all states, and $q_L-q_R\in 2\mathbb{Z}$ for bosonic states [93]), it always suffices to calculate Z_{NS}, as the other parts are determined by spectral flow and modular transformations [93],

$$\begin{aligned}
Z_{\mathrm{R}}(\tau,\overline{\tau},z,\bar{z}) &= (q\overline{q})^{\frac{c}{24}}(y\overline{y})^{\frac{c}{6}}Z_{\mathrm{NS}}(\tau,\overline{\tau},z+\frac{\tau}{2},\bar{z}+\frac{\overline{\tau}}{2})\\
Z_{\widetilde{\mathrm{NS}}}(\tau,\overline{\tau},z,\bar{z}) &= Z_{\mathrm{NS}}(\tau,\overline{\tau},z+\frac{1}{2},\bar{z}+\frac{1}{2})\\
Z_{\widetilde{\mathrm{R}}}(\tau,\overline{\tau},z,\bar{z}) &= Z_{\mathrm{R}}(\tau,\overline{\tau},z+\frac{1}{2},\bar{z}+\frac{1}{2})\,.
\end{aligned} \tag{4.3.16}$$

The conformal weight h and the U(1)-charge q of a state is given by

$$h(|l,m,s\rangle) = \frac{l(l+2)-m^2}{4(k+2)} + \frac{s^2}{8} \mod 1 \tag{4.3.17}$$

$$q(|l,m,s\rangle) = -\frac{m}{k+2} + \frac{s}{2} \mod 2 \,. \tag{4.3.18}$$

To find the exact conformal weights, one needs to bring the labels (l,m,s) through the field identification rule into the range $|m-s| \leq l$. If neither representative lies in the range $|m-s| \leq l$, one applies the above formula for a representative that satisfies $m-s = l-2$ and adds one to the result, or if this does not exist, the one with $m-s = l+2$ and adds one [93]. For the U(1)-charge we reduce mod 2, so that the absolute value is smaller than or equal to 1.

In the NS–sector, the chiral states are labeled by $|l,-l,0\rangle_{\mathcal{N}}$. Under the spectral flow, which as a simple current corresponds to the state $|0,-1,-1\rangle_{\mathcal{N}}$, the chiral states are related to the R–ground states $|l,-(l+1),-1\rangle_{\mathcal{N}}$. The R–ground states have U(1)-charge

$$q_L(|l,-(l+1),-1\rangle) = -\frac{k-2l}{4+2k} \,. \tag{4.3.19}$$

Note that the combination $(l,m,s)+(l,m,s+2)$ realises a full irreducible $\mathcal{N}=2$ super Virasoro module [142]. Each individual (l,m,s) representation corresponds to a primary field with respect to the bosonic subalgebra of the $\mathcal{N}=2$ super Virasoro algebra. In particular, the state associated with the field $G_\pm$ is also a primary with respect to this bosonic subalgebra. It is precisely this bosonic subalgebra that qualifies as a chiral algebra in the standard sense of rational conformal field theory, as defined in [122].

The complex vector space $V_{\mathbb{C}}$ and rational vector space $V_{\mathbb{Q}}$

For illustrative purposes, we now carry out the steps of the general construction described in the previous section, specialising to the case of the (diagonal) $\mathcal{N}=(2,2)$ minimal models.

The complex vector space $V_{\mathbb{C}}$, for the $\mathcal{N}=(2,2)$ minimal model at level k, is given by

$$\begin{aligned} V_{\mathbb{C}} &:= \langle\, |l,-(l+1),-1\rangle_{\mathcal{N}}\, \rangle_{\mathbb{C}} \\ V_{\mathbb{C}}^{p,q} &:= \left\{ |l,-(l+1),-1\rangle_{\mathcal{N}} \;\middle|\; \ell\left(\frac{l}{k+2}\right) = p \;,\quad \ell\left(\frac{k-l}{k+2}\right) = q \right\} , \end{aligned} \tag{4.3.20}$$

with $\ell = k+2$ chosen in such a way that p and q are integral. The complex conjugation map $\iota : V_{\mathbb{C}} \to V_{\mathbb{C}}$ is defined as

$$\iota\left(\sum_{l=0}^{k} a_{(l,-(l+1),-1)}\, |l,-(l+1),-1\rangle_{\mathcal{N}}\right) := \sum_{l=0}^{k} \mathrm{sgn}(\mathrm{S}_{(0,0,0),(l,l+1,1)})\, a^*_{(l,-(l+1),-1)}\, |l,l+1,1\rangle_{\mathcal{N}} \;, \tag{4.3.21}$$

where, comparing to the general definition (4.2.15), we used

$$\mathcal{C}\left(|l,-(l+1),-1\rangle_{\mathcal{N}}\right) = |l,l+1,1\rangle_{\mathcal{N}} \ . \tag{4.3.22}$$

By using the identifications (4.3.6) and $(l,m,s) \sim (k-l, m+k+2, s+2)$, we find

$$\mathcal{U}^{-1}\left(|l,l+1,1\rangle_{\mathcal{N}}\right) = |k-l,-(k-l),0\rangle_{\mathcal{N}} \ . \tag{4.3.23}$$

Hence, the state $|l,l+1,1\rangle_{\mathcal{N}}$ is an R–ground state. Furthermore,

$$q_L(|l,l+1,1\rangle) = \frac{k-2l}{4+2k} = -q_L(|l,-(l+1),-1\rangle)\,, \tag{4.3.24}$$

and therefore, in comparison with the definition of $V_{\mathbb{C}}^{p,q}$ we see that the complex conjugation map ι indeed exchanges p and q. The boundary states are labelled by the highest-weight representations of the coset chiral algebra (4.3.1), and the projected boundary states (4.2.30) are given by [150]

$$|B_{(L,M,S)}\rangle_{\mathcal{P}} := \sqrt{2}\sum_{l=0}^{k} \frac{\mathrm{S}_{(L,M,S),(l,-(l+1),-1)}}{\sqrt{\mathrm{S}_{(0,0,0),(l,-(l+1),-1)}}}\,|l,-(l+1),-1\rangle_{\mathcal{N}} \ . \tag{4.3.25}$$

The intersection matrix Σ, given by

$$\Sigma_{(L,M,S),(L',M',S')} := {}_{\mathcal{P}}\langle B_{(L,M,S)}|\,(-1)^{J_0}\,|B_{(L',M',S')}\rangle_{\mathcal{P}} \ , \tag{4.3.26}$$

is integral when the boundary states are restricted to those with even values of S and S' [150]. For mixed combinations where S is even and S' is odd (or vice versa), the matrix Σ is generically not rational. It is always possible to choose a collection of boundary states

$$\{|B_{(L,M,S)}\rangle_{\mathcal{P}} \mid (L,M,S) \in \mathcal{B}\}\,, \tag{4.3.27}$$

with $|\mathcal{B}| = k+1$ and $\mathcal{B} \subset \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})_{\mathrm{NS}}$, such that the matrix Σ, when evaluated on this basis, has full rank. In particular, Σ is then invertible, and its inverse is also rational. A basis of the rational vector space $V_{\mathbb{Q}}$ is hence given by (4.3.27).

Using the property

$$\mathrm{S}^*_{(L,M,S),\mathcal{C}(l,m,s)} = \mathrm{S}_{(L,M,S),(l,m,s)}\,, \tag{4.3.28}$$

for all $(L,M,S),(l,m,s) \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A})$, one easily verifies that the rational vector space $V_{\mathbb{Q}}$ is compatible with the complex conjugation map ι, i.e.,

$$\iota\left(|B_{(L,M,S)}\rangle_{\mathcal{P}}\right) = |B_{(L,M,S)}\rangle_{\mathcal{P}} \ . \tag{4.3.29}$$

Recalling the discussion around (4.2.46), for $\mathcal{N} = (2,2)$ minimal models with odd k we can always construct a polarisation by

$$Q(|l,-(l+1),-1\rangle\,,|l',-(l'+1),-1\rangle) := \iota(\langle l,-(l+1),-1|)(-1)^{(k+2)J_0}\,|l',-(l'+1),-1\rangle \ . \tag{4.3.30}$$

For even values of k, we were unable to identify a general form of a polarisation that applies universally. Nevertheless, in all examples we examined, a polarisation was always found to exist. At the end of this section, we will illustrate the construction of a polarisation in the case $k = 4$.

Next we define the intermediate rescaled R–ground states $|l, -(l+1), -1\rangle_{\mathcal{I}}$ as

$$|l, -(l+1), -1\rangle_{\mathcal{I}} := \frac{1}{\sqrt{2}} \sqrt{\mathrm{S}_{(0,0,0),(l,-(l+1),-1)}}^{\,*} \mathrm{e}^{\mathrm{i}\pi q_L(l)} \, |l, -(l+1), -1\rangle_{\mathcal{N}} \ , \tag{4.3.31}$$

so that

$$|l, -(l+1), -1\rangle_{\mathcal{I}} \ = \ \mathrm{S}^*_{(l,-(l+1),-1),(L,M,S)} \Sigma^{(L',M',S'),(L,M,S)} \, |B_{(L',M',S')}\rangle_{\mathcal{P}} \ , \tag{4.3.32}$$

with $\Sigma^{(L',M',S'),(L,M,S)}$ denoting the components of the inverse intersection matrix Σ^{-1}.

Hodge Substructure and Hodge Endomorphisms

We proceed to explicitly construct the Hodge substructures of the Hodge structure associated with the $\mathcal{N} = (2, 2)$ minimal models at level k. To do so, we need to determine the following tower of Galois extensions[7]

$$\mathbb{Q}(S) \supseteq \mathbb{Q}(S_{(l,-(l+1),-1)}) \supseteq \mathbb{Q}(S_{(l,-(l+1),-1)})^{\mathrm{Stab}((l,-(l+1),-1))} \, . \tag{4.3.33}$$

To determine the number field $\mathbb{Q}(S)$ generated by the entries of the S-matrix (4.3.10), we consider the following combination of S-matrix elements[8]

$$\begin{aligned} \zeta_{2(k+2)} &= \left(1 + \frac{\mathrm{S}_{(0,0,0),(1,-1,0)}}{\mathrm{S}_{(0,-2,0),(1,1,0)}}\right) \left(\frac{\mathrm{S}_{(0,-2,0),(0,0,0)}}{\mathrm{S}_{(0,0,0),(1,-1,0)}}\right) \\ \mathrm{i} &= \frac{\mathrm{S}_{(1,1,0),(0,2,0)}}{1 - \frac{\mathrm{S}_{(0,-2,0),(0,-2,0)}}{\mathrm{S}_{(0,-2,0),(0,0,0)}}} \, . \end{aligned} \tag{4.3.34}$$

Then from the expression (4.3.10), it follows that

$$\mathbb{Q}(S) \ = \ \mathbb{Q}(\zeta_{2(k+2)}, \mathrm{i}) \ . \tag{4.3.35}$$

For representations $(l, m, s) \in \mathcal{R}$, i.e., $(l, m, s) = (l, -(l+1), -1)$ with $0 \leq l \leq k$, the corresponding S-matrix entries are given by

$$\mathrm{S}_{(l,-(l+1),-1),(L,M,S)} \ = \ \frac{(-\mathrm{i})(-1)^{-S/2}}{2(k+2)} \left(\zeta_{2(k+2)}^{(l+1)(L+1-M)} - \zeta_{2(k+2)}^{-(l+1)(L+1+M)}\right) \ . \tag{4.3.36}$$

[7]Note that since $\mathbb{Q}(S)$ is Galois and $\mathrm{Gal}(\mathbb{Q}(S)/\mathbb{Q})$ is abelian, all subfields are Galois over $\mathbb{Q}$

[8]Note that (0,0,0), (1,-1,0), (0,-2,0), (1,1,0), (0,2,0) are representations contained in any minimal model for arbitrary level $k \geq 1$

Let us introduce the integers

$$\begin{aligned} g &:= \gcd(l+1, 2(k+2)) \\ \kappa &:= \frac{l+1}{g} \\ k' &:= \frac{2(k+2)}{g}\,, \qquad \gcd(\kappa, k') = 1\,, \end{aligned} \tag{4.3.37}$$

so that we can rewrite (4.3.36) in the form

$$\mathrm{S}_{(l,-(l+1),-1),(L,M,S)} \;=\; \frac{(-\mathrm{i})(-1)^{-S/2}}{2(k+2)}\left(\zeta_{k'}^{\kappa(L+1-M)} - \zeta_{k'}^{-\kappa(L+1+M)}\right)\;. \tag{4.3.38}$$

Consider now the following quotient of S-matrix elements:

$$\frac{\mathrm{S}_{(l,-(l+1),-1),(0,0,0)}}{\mathrm{S}_{(l,-(l+1),-1),(0,2,0)}} = \zeta_{k'}^{2\kappa}\,. \tag{4.3.39}$$

For k' odd, a comparison with eq. (4.3.38) reveals that

$$\mathbb{Q}(S_{l,-(l+1),-1}) = \mathbb{Q}(\zeta_{k'}, \mathrm{i})\,. \tag{4.3.40}$$

Similarly for $k'/2$ odd we get the number field $\mathbb{Q}(S_{l,-(l+1),-1}) = \mathbb{Q}(\zeta_{k'}, \mathrm{i})$ by using the identity $\mathbb{Q}(\zeta_{k'/2}) = \mathbb{Q}(\zeta_{k'})$, which follows from $\zeta_{k'} = -\zeta_{\frac{k'}{2}}^{(\frac{k'}{2}+1)/2}$. For the remaining cases $k'/4$ even and $k'/4$ odd we rewrite (4.3.38) as

$$\mathrm{S}_{(l,-(l+1),-1),(L,M,S)} = \frac{(-1)^{-S/2}}{2(k+2)}(-\zeta_{k'}^{\frac{k'}{4}})\,\zeta_{k'}^{-\kappa}\,\zeta_{\frac{k'}{2}}^{-\kappa\frac{(L+M)}{2}}\left(-1+\zeta_{\frac{k'}{2}}^{\kappa(1+L)}\right). \tag{4.3.41}$$

Using (4.3.39) together with

$$\frac{\left(\mathrm{S}_{(l,-(l+1),-1),(0,0,0))}\right)^2}{\mathrm{S}_{(l,-(l+1),-1),(0,-2,0)} - \mathrm{S}_{(l,-(l+1),-1),(0,0,0)}} = \frac{-\mathrm{i}}{2(k+2)}\zeta_{k'}^{-\kappa}\,, \tag{4.3.42}$$

and (4.3.41) it follows that for $k'/4$ even we have $\mathbb{Q}(S_{l,-(l+1),-1}) = \mathbb{Q}(\zeta_{k'})$ and for $k'/4$ odd that $\mathbb{Q}(S_{l,-(l+1),-1}) = \mathbb{Q}(\zeta_{k'/2})$. We can summarise all possible cases by

$$\mathbb{Q}(S_{(l,-(l+1),-1)}) = \begin{cases} \mathbb{Q}(\zeta_{\frac{k'}{2}})\,, & \frac{k'}{4} \text{ odd} \\ \mathbb{Q}(\zeta_{\frac{4}{f}k'})\,, & \text{else} \end{cases}\,, \tag{4.3.43}$$

where we introduced the integer $f := \gcd(k', 4)$ and used

$$\begin{aligned} \mathbb{Q}(\zeta_x, \zeta_y) &= \mathbb{Q}(\zeta_{\mathrm{lcm}(x,y)}) \\ \mathrm{lcm} &= (\frac{|xy|}{\gcd(x,y)})\,. \end{aligned} \tag{4.3.44}$$

Let us now determine the stabiliser groups $\mathrm{Stab}((l,-(l+1),-1))$, i.e., we are looking for Galois automorphisms $\rho_* \in \mathrm{Gal}(\mathbb{Q}(\mathrm{S}_{(l,-(l+1),-1)})/\mathbb{Q})$ whose induced permutation ϱ_* on the set of representations $(l,-(l+1),-1)$ is the identity,

$$\begin{aligned}\rho_* \left|l,-(l+1),-1\right\rangle_{\mathcal{I}} &= \epsilon_{\rho_*}((l,-(l+1),-1)) \left|\varrho_*(l,-(l+1),-1)\right\rangle_{\mathcal{I}} \\ &= \epsilon_{\rho_*}((l,-(l+1),-1)) \left|l,-(l+1),-1\right\rangle_{\mathcal{I}} \ .\end{aligned} \tag{4.3.45}$$

Writing the S-matrix in terms of the primitive element $\zeta_{\frac{4}{f}k'}$,

$$\mathrm{S}_{(l,-(l+1),-1),(L,M,S)} \ = \ \frac{-\left(\zeta_{\frac{4}{f}k'}\right)^{\frac{k'}{f}}(-1)^{-S/2}}{2(k+2)}\left(\left(\zeta_{\frac{4}{f}k'}\right)^{\frac{4}{f}\kappa(L+1-M)} - \left(\zeta_{\frac{4}{f}k'}\right)^{-\frac{4}{f}\kappa(L+1+M)}\right) , \tag{4.3.46}$$

and recalling eq. (4.3.32) we find that $\rho_* \in (\mathbb{Z}/\frac{4}{f}k'\mathbb{Z})^\times$ with $\rho_* = \frac{2}{f}k'+1 \mod \frac{4}{f}k'$ [9].

The fixed field of $\mathbb{Q}(\zeta_{\frac{4}{f}k'})$ with respect to the order-two automorphism ρ_* is $\mathbb{Q}(\zeta_{\frac{2}{f}k'})$ which is $\mathbb{Q}(\zeta_{\frac{k'}{2}})$ for $f=2$ and $f=4$, and $\mathbb{Q}(\zeta_{k'})$ for $f=1$.

Let us now introduce a rescaled basis of RR–ground states as

$$\left|l,-(l+1),-1\right\rangle_{\mathcal{S}} := \begin{cases} \mathrm{i}\left|l,-(l+1),-1\right\rangle_{\mathcal{I}} \in V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{k'}) \,, & f=1\,,2 \\ \zeta_{k'}^{\kappa}\left|l,-(l+1),-1\right\rangle_{\mathcal{I}} \in V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{\frac{k'}{2}}) \,, & \frac{k'}{4} \text{ is even} \\ \left|l,-(l+1),-1\right\rangle_{\mathcal{I}} \in V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{\frac{k'}{2}}) \,, & \frac{k'}{4} \text{ is odd}\end{cases} , \tag{4.3.47}$$

and show how the elements in $\mathrm{Gal}(\mathbb{Q}(S_{(l,-(l+1),-1)})^{\mathrm{Stab}((l,-(l+1),-1))}/\mathbb{Q})$ act on the states (4.3.47) to construct the Hodge substructures.

For $f=2$ and $f=4$, the relevant Galois group is given by

$$\mathrm{Gal}\left(\mathbb{Q}\left(\zeta_{\frac{k'}{2}}\right)/\,\mathbb{Q}\right) \cong \left(\mathbb{Z}\,/\,\frac{k'}{2}\,\mathbb{Z}\right)^{\times} \ , \tag{4.3.48}$$

the multiplicative Abelian group of integers modulo $\frac{k'}{2}$ that are coprime to $\frac{k'}{2}$.

Consider an element

$$a \in \left(\mathbb{Z}\,/\,\frac{k'}{2}\,\mathbb{Z}\right)^{\times} \ . \tag{4.3.49}$$

The corresponding Galois action is then given by

$$\rho_a \ : \ \zeta_{\frac{k'}{2}} \longmapsto \zeta_{\frac{k'}{2}}^{a} \ . \tag{4.3.50}$$

[9] This Galois automorphism acts as $\zeta_{\frac{4}{f}k'} \mapsto -\zeta_{\frac{4}{f}k'}$. Such an automorphism exists for all $\mathbb{Q}(\zeta_n)$ with $\gcd(n,4)=4$. For even n we have $\zeta_n^{n/2}=-1$ and since we want to find an a such that $\zeta_n^a = -\zeta_n \to \zeta_n^{a-1} = -1$ it follows $a-1 \equiv n/2 \mod n$. From the obvious identity $\gcd(a+mb,b)=\gcd(a,b)$ it follows that $\gcd(\frac{n}{2}+1,n)=1$ for $\frac{n}{2}$ even.

The Galois element ρ_a acts on the state $|l,-(l+1),-1\rangle_{\mathcal{S}}$ in a natural way: it leaves the rational vector space $V_{\mathbb{Q}}$ invariant and acts nontrivially on the factor $\mathbb{Q}(\zeta_{\frac{k'}{2}})$, which appears raised to the power κ, as seen from eq. (4.3.41),

$$\left(\zeta_{\frac{k'}{2}}\right)^{\kappa} \mapsto \left(\zeta^{a}_{\frac{k'}{2}}\right)^{a\kappa} . \tag{4.3.51}$$

For $f=1$ the same holds with $\zeta_{\frac{k'}{2}}$ replaced by $\zeta_{k'}$.

From eq. (4.3.32), (4.3.47) and (4.3.51) it then follows that the resulting state, upon acting with the automorphism ρ_a, is again an RR–ground state $|l',-(l'+1),-1\rangle_{\mathcal{S}}$, with l' label given by

$$l' \equiv a(l+1)-1 \mod k+2 . \tag{4.3.52}$$

Note that a priori l' is valued in the range $0 \le l' \le k+1$. However, as we will show now, the value $l'=k+1$ leads to a contradiction. Suppose $l'=k+1$. Then, using eq. (4.3.52), it follows that

$$a(l+1) \equiv 0 \mod k+2 . \tag{4.3.53}$$

Recall that $g=\gcd(2(k+2),l+1)$, thus $l+1=gm$ for some integer m. Then from eq. (4.3.53) it follows that $k+2 \mid agm$. For $f=2$ and $f=4$ we have that $g \mid k+2$, and so $\frac{k+2}{g} \mid am$. Using that $\gcd(a,\frac{k+2}{g})=1$, we conclude $\frac{k+2}{g} \mid m$, thus $m=\frac{k+2}{g}M$ for some integer M. But then $l+1=(k+2)M$, and hence $l+1\equiv 0 \mod k+2$. This is a contradiction, as we know that $0\le l\le k$, hence $l'=k+1$ is not possible.

For $f=1$, one can argue analogously by using $g=2x$ for some integer x. We therefore find in all cases

$$\rho_a(|l,-(l+1),-1\rangle_{\mathcal{S}}) = \epsilon_{\rho_a}((l,-(l+1),-1))\ |l',-(l'+1),-1\rangle_{\mathcal{S}} , \tag{4.3.54}$$

with $0\le l'\le k$.

For the remainder of the discussion concerning the construction of Hodge substructures, we will use the notation $\mathbb{Q}(\zeta_{k_i})$ with $i=1,2$, where $k_1=k'$ and $k_2=\frac{k'}{2}$, in order to uniformly cover the possible subfields appearing in eq. (4.3.47). We now expand eq. (4.3.47) in terms of the generator ζ_{k_i} of the cyclotomic field $\mathbb{Q}(\zeta_{k_i})$ as

$$|l,-(l+1),-1\rangle_{\mathcal{S}} = \sum_{m=0}^{d-1} (\zeta_{k_i})^m\ |Q_m^{(l,-(l+1),-1)}\rangle , \tag{4.3.55}$$

where $|Q_m^{(l,-(l+1),-1)}\rangle \in V_{\mathbb{Q}}$ and $d=|\mathrm{Gal}(\mathbb{Q}(\zeta_{k_i})/\mathbb{Q})|=\phi(k_i)$. Here, $\phi(k_i)$ denotes Euler's totient function , i.e., the number of integers between 1 and k_i that are coprime to k_i. We then define the following subspace:

$$W_{\mathbb{Q}}^{(l,-(l+1),-1)} = \langle |Q_0^{(l,-(l+1),-1)}\rangle , \ldots , |Q_{d-1}^{(l,-(l+1),-1)}\rangle\rangle_{\mathbb{Q}} . \tag{4.3.56}$$

Clearly, the complexification of this space contains all Galois conjugates of the state $|l,-(l+1),-1\rangle_{\mathcal{S}}$, that is, the states $\rho_a\,|l,-(l+1),-1\rangle_{\mathcal{S}}$ with $\rho_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{k_i})/\mathbb{Q})$. Since the cyclotomic field $\mathbb{Q}(\zeta_{k_i})$ is the fixed field under the stabiliser group $\mathrm{Stab}((l,-(l{+}1),-1))$, the Galois conjugates $\rho_a\,|l,-(l+1),-1\rangle_{\mathcal{S}}$ are linearly independent over $\mathbb{C}$. Hence, they form a basis of the complexification of the subspace $W_{\mathbb{Q}}^{(l,-(l+1),-1)}$,

$$W_{\mathbb{C}}^{(l,-(l+1),-1)} = \langle \rho_a(|l,-(l+1),-1\rangle_{\mathcal{S}}) \mid \rho_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{k_i})/\mathbb{Q})\rangle_{\mathbb{C}} \ . \tag{4.3.57}$$

By eq. (4.3.54), it follows that each Galois conjugate $\rho_a\,|l,-(l+1),-1\rangle_{\mathcal{S}}$ lies in one of the subspaces $V^{p,q}$ defined in eq. (4.3.20). Thus, the subspace defined in eq. (4.3.56) satisfies the condition (4.2.3), and therefore constitutes a Hodge substructure.

Finally, we establish the existence of sufficiently many Hodge endomorphisms on these Hodge substructures. On the rational vector space $W_{\mathbb{Q}}^{(l,-(l+1),-1)}$, we define morphisms

$$\varphi_{\zeta_{k_i}}^{(l,-(l+1),-1)} \ : \ W_{\mathbb{Q}}^{(l,-(l+1),-1)} \to W_{\mathbb{Q}}^{(l,-(l+1),-1)} \ , \tag{4.3.58}$$

corresponding to multiplication of $|l,-(l+1),-1\rangle_{\mathcal{S}}$ by ζ_{k_i}. The field extension $\mathbb{Q}(\zeta_{k_i})$ can be regarded as a $\mathbb{Q}$-vector space with basis

$$\mathbb{Q}(\zeta_{k_i}) = \langle 1, \zeta_{k_i}, \ldots, (\zeta_{k_i})^{d-1}\rangle_{\mathbb{Q}} \ . \tag{4.3.59}$$

Multiplication by ζ_{k_i} is then described by

$$(\zeta_{k_i})^m \mapsto (\zeta_{k_i})^{m+1} = \sum_{n=0}^{d-1} \mathrm{T}(\zeta_{k_i})^m{}_n \ (\zeta_{k_i})^n \ , \tag{4.3.60}$$

where $\mathrm{T}(\zeta_{k_i}) \in \mathrm{Mat}(d{\times}d, \mathbb{Q})$ is the companion matrix of the minimal polynomial of ζ_{k_i}.

We now define an endomorphism on the rational subspace (4.3.56) by

$$\varphi_{\zeta_{k_i}}^{(l,-(l+1),-1)} \left(|Q_m^{(l,-(l+1),-1)}\rangle\right) := \mathrm{T}(\zeta_{k_i})^s{}_m \, |Q_s^{(l,-(l+1),-1)}\rangle \ , \tag{4.3.61}$$

and obtain the corresponding action of its complexification $\varphi_{\zeta_{k_i},\mathbb{C}}^{(l,-(l+1),-1)}$ on the complex subspace $W_{\mathbb{C}}^{(l,-(l+1),-1)}$,

$$\begin{aligned}
\varphi_{\zeta_{k_i},\mathbb{C}}^{(l,-(l+1),-1)} \left(\rho_a\,|l,-(l+1),-1\rangle_{\mathcal{S}}\right) &= \sum_{m,n=0}^{d-1} \rho_a\left(\zeta_{k_i}^n\right) \mathrm{T}(\zeta_{k_i})^m{}_n \ |Q_m^{(l,-(l+1),-1)}\rangle \\
&= \sum_{m,n=0}^{d-1} \rho_a\left(\mathrm{T}(\zeta_{k_i})^m{}_n \, \zeta_{k_i}^n\right) |Q_m^{(l,-(l+1),-1)}\rangle \\
&= \rho_a\left(\zeta_{k_i}\right) \sum_{m=0}^{d-1} \rho_a\left(\zeta_{k_i}^m\right) |Q_m^{(l,-(l+1),-1)}\rangle \\
&= \rho_a\left(\zeta_{k_i}\right) \rho_a\,|l,-(l+1),-1\rangle_{\mathcal{S}} \ .
\end{aligned} \tag{4.3.62}$$

We conclude that, in the basis of $W_{\mathbb{C}}^{(l,-(l+1),-1)}$ spanned by the Galois conjugates

$$\rho_a \, |l, -(l+1), -1\rangle_{\mathcal{S}} \ , \qquad \rho_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{k_i})/\mathbb{Q}) \, , \tag{4.3.63}$$

the endomorphism $\varphi_{\zeta_{k_i},\mathbb{C}}^{(l,-(l+1),-1)}$ is diagonalised and acts on the basis states by multiplication with $\rho(\zeta_{k_i})$. In particular, it preserves the Hodge decomposition,

$$\varphi_{\zeta_{k_i},\mathbb{C}}^{(l,-(l+1),-1)} \left(W_{\mathbb{C}}^{(l,-(l+1),-1),(p,q)} \right) \subseteq W_{\mathbb{C}}^{(l,-(l+1),-1),(p,q)} \, , \tag{4.3.64}$$

and is therefore a Hodge endomorphism. Moreover, the Hodge endomorphisms $\varphi_{\zeta_{k_i}}^{(l,-(l+1),-1)}$ satisfy the composition property

$$\left(\varphi_{\zeta_{k_i}}^{(l,-(l+1),-1)} \right)^n = \varphi_{\zeta_{k_i}^n}^{(l,-(l+1),-1)} \, , \qquad \text{for all } n \in \mathbb{Z} \, . \tag{4.3.65}$$

We have thus constructed $d = \dim_{\mathbb{Q}} W_{\mathbb{Q}}^{(l,-(l+1),-1)}$ commuting Hodge endomorphisms given by the set

$$\left\{ \varphi_{\zeta_{k_i}^0}^{(l,-(l+1),-1)}, \varphi_{\zeta_{k_i}^1}^{(l,-(l+1),-1)}, \ldots, \varphi_{\zeta_{k_i}^{d-1}}^{(l,-(l+1),-1)} \right\} . \tag{4.3.66}$$

The Hodge endomorphisms in (4.3.66) are clearly linearly independent over $\mathbb{Q}$; otherwise, any rational relation among them would induce a rational relation among the corresponding powers of the root of unity $\{\zeta_{k_i}^0, \zeta_{k_i}^1, \ldots, \zeta_{k_i}^{d-1}\}$. This would contradict the fact that the degree of the cyclotomic field $\mathbb{Q}(\zeta_{k_i})$ is d.

Hence, the set (4.3.66) generates an endomorphism algebra isomorphic to $\mathbb{Q}(\zeta_{k_i})$, under the natural isomorphism $\varphi_{\zeta_{k_i}}^{(l,-(l+1),-1)} \mapsto \zeta_{k_i}$. We therefore conclude that the Hodge structures associated with the diagonal $\mathcal{N} = (2,2)$ minimal models admit sufficiently many complex multiplications.

For odd values of k, a polarisation can always be constructed (see (4.3.30)), implying that the corresponding Hodge structure is of CM-type. As we will demonstrate in example 4.3, a polarisation can also exist in the case of even k, although it is not of the form given in (4.3.30).

Example: $k = 3$

Let us investigate the Hodge structure for the $k = 3$ minimal model explicitly. The complex vector space $V_{\mathbb{C}}$ is given by

$$\begin{aligned} V_{\mathbb{C}} &= \langle\, \rho(|0,-1,-1\rangle_{\mathcal{S}}) \,\rangle_{\mathbb{C}}^{\rho \in \mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})} \\ &= \langle\, |0,-1,-1\rangle_{\mathcal{S}} \, , |1,-2,-1\rangle_{\mathcal{S}} \, , |3,-4,-1\rangle_{\mathcal{S}} \, , |2,-3,-1\rangle_{\mathcal{S}} \,\rangle_{\mathbb{C}} \, . \end{aligned} \tag{4.3.67}$$

A compatible basis of boundary states that span the rational vector space $V_{\mathbb{Q}}$ is given by

$$V_{\mathbb{Q}} = \langle\, |B_{(0,0,0)}\rangle_{\mathcal{P}} \, , |B_{(3,-3,0)}\rangle_{\mathcal{P}} \, , |B_{(3,1,0)}\rangle_{\mathcal{P}} \, , |B_{(3,-1,0)}\rangle_{\mathcal{P}} \,\rangle_{\mathbb{Q}} \, , \tag{4.3.68}$$

with the intersection matrix

$$\Sigma_{(L,M,S),(L',M',S')} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} . \tag{4.3.69}$$

In the rational basis (4.3.68) the polarisation is given by

$$Q(\,|B_A\rangle_{\mathcal{P}}\,, |B_B\rangle_{\mathcal{P}}) = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{4.3.70}$$

The RR–ground states can be written as

$$|l,-(l+1),-1\rangle_{\mathcal{S}} \;=\; \sum_{k=0}^{3} \zeta_5^k \,|Q_k^{(l,-(l+1),-1)}\rangle \;\; \in V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_5)\;. \tag{4.3.71}$$

For the state $|0,-1,-1\rangle_{\mathcal{S}}$ the expansion reads

$$\begin{aligned}
|Q_0^{(0,-1,-1)}\rangle &= \tfrac{1}{10}\,|B_{(3,1,0)}\rangle_{\mathcal{P}} - \tfrac{1}{10}\,|B_{(3,-1,0)}\rangle_{\mathcal{P}} \\
|Q_1^{(0,-1,-1)}\rangle &= \tfrac{1}{10}\,|B_{(0,0,0)}\rangle_{\mathcal{P}} - \tfrac{1}{10}\,|B_{(3,-3,0)}\rangle_{\mathcal{P}} - \tfrac{1}{5}\,|B_{(3,-1,0)}\rangle_{\mathcal{P}} - \tfrac{1}{10}\,|B_{(3,1,0)}\rangle_{\mathcal{P}} \\
|Q_2^{(0,-1,-1)}\rangle &= -\tfrac{1}{10}\,|B_{(0,0,0)}\rangle_{\mathcal{P}} - \tfrac{1}{10}\,|B_{(3,-1,0)}\rangle_{\mathcal{P}} \\
|Q_3^{(0,-1,-1)}\rangle &= \tfrac{1}{10}\,|B_{(3,-3,0)}\rangle_{\mathcal{P}} - \tfrac{1}{10}\,|B_{(3,-1,0)}\rangle_{\mathcal{P}}\;.
\end{aligned} \tag{4.3.72}$$

If we pick $|Q_k^{(0,-1,-1)}\rangle$ as our basis vectors, we can write the basis for the vector space $V_{\mathbb{C}}$ as

$$\begin{aligned}
|0,-1,-1\rangle_{\mathcal{S}} \;&=\; \begin{pmatrix} 1 \\ \zeta_5 \\ \zeta_5^2 \\ \zeta_5^3 \end{pmatrix}, \qquad |1,-2,-1\rangle_{\mathcal{S}} \;=\; \begin{pmatrix} 1 \\ \zeta_5^2 \\ \zeta_5^4 \\ \zeta_5 \end{pmatrix}, \\
|3,-4,-1\rangle_{\mathcal{S}} \;&=\; \begin{pmatrix} 1 \\ \zeta_5^4 \\ \zeta_5^3 \\ \zeta_5^2 \end{pmatrix}, \qquad |2,-3,-1\rangle_{\mathcal{S}} \;=\; \begin{pmatrix} 1 \\ \zeta_5^3 \\ \zeta_5 \\ \zeta_5^4 \end{pmatrix}.
\end{aligned} \tag{4.3.73}$$

The matrix $\mathrm{T}(\zeta_5)$ representing the multiplication by ζ_5 in $\mathbb{Q}(\zeta_5)$ is given, in the basis $\{1,\zeta_5,\zeta_5^2,\zeta_5^3\}$, as

$$\mathrm{T}(\zeta_5) \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix} . \tag{4.3.74}$$

By the previous discussion $\mathrm{T}(\zeta_5)$ represents a Hodge endomorphism $\varphi_{\zeta_5} : V_{\mathbb{Q}} \to V_{\mathbb{Q}}$ and it acts on the basis (4.3.72) as

$$\begin{aligned} \mathrm{T}(\zeta_5)\,|0,-1,-1\rangle_{\mathcal{S}} &= \zeta_5\,|0,-1,-1\rangle_{\mathcal{S}} \ , & \mathrm{T}(\zeta_5)\,|1,-2,-1\rangle_{\mathcal{S}} &= \zeta_5^2\,|1,-2,-1\rangle_{\mathcal{S}} \ , \\ \mathrm{T}(\zeta_5)\,|3,-4,-1\rangle_{\mathcal{S}} &= \zeta_5^4\,|3,-4,-1\rangle_{\mathcal{S}} \ , & \mathrm{T}(\zeta_5)\,|2,-3,-1\rangle_{\mathcal{S}} &= \zeta_5^3\,|2,-3,-1\rangle_{\mathcal{S}} \ , \end{aligned} \tag{4.3.75}$$

so it follows that the Hodge type is indeed preserved by the complexification of this endomorphism. A simple computation also shows that $\mathrm{I}, \mathrm{T}(\zeta_5), \mathrm{T}(\zeta_5)^2$, and $\mathrm{T}(\zeta_5)^3$ are independent over $\mathbb{Q}$, and that the minimal polynomial of $\mathrm{T}(\zeta_5)$ is

$$\mathrm{I} + \mathrm{T}(\zeta_5) + \mathrm{T}(\zeta_5)^2 + \mathrm{T}(\zeta_5)^3 + \mathrm{T}(\zeta_5)^4 \ = \ 0 \ . \tag{4.3.76}$$

These powers of $\mathrm{T}(\zeta_5)$ obviously commute with each other, so $\mathrm{T}(\zeta_5)$ generates a commutative subalgebra E of the algebra of Hodge endomorphisms $\mathrm{End}_{\mathrm{Hdg}}(V_{\mathbb{Q}})$ with $\dim_{\mathbb{Q}} E = \dim_{\mathbb{Q}} V_{\mathbb{Q}}$. The algebra generated by the matrices $\mathrm{T}(\zeta_5)$ over $\mathbb{Q}$ is in fact isomorphic to $\mathbb{Q}(\zeta_5)$ with the isomorphism given by

$$\mathrm{T}(\zeta_5) \longmapsto \zeta_5 \ . \tag{4.3.77}$$

Example: $k = 13$

The following example illustrates a non-trivial splitting of the Hodge structure. The complex vector space is spanned by the RR–ground states,

$$\begin{aligned} V_{\mathbb{C}} = \langle\ & |0,-1,-1\rangle_{\mathcal{S}}\,,\ |1,-2,-1\rangle_{\mathcal{S}}\,,\ |2,-3,-1\rangle_{\mathcal{S}}\,,\ |3,-4,-1\rangle_{\mathcal{S}}\,,\ |4,-5,-1\rangle_{\mathcal{S}}\,,\ |5,-6,-1\rangle_{\mathcal{S}}\,, \\ & |6,-7,-1\rangle_{\mathcal{S}}\,,\ |7,-8,-1\rangle_{\mathcal{S}}\,,\ |8,-9,-1\rangle_{\mathcal{S}}\,,\ |9,-10,-1\rangle_{\mathcal{S}}\,,\ |10,-11,-1\rangle_{\mathcal{S}}\,,\ |11,-12,-1\rangle_{\mathcal{S}}\,, \\ & |12,-13,-1\rangle_{\mathcal{S}}\,,\ |13,-14,-1\rangle_{\mathcal{S}}\ \rangle_{\mathbb{C}}\ , \end{aligned} \tag{4.3.78}$$

with rational basis

$$\begin{aligned} V_{\mathbb{Q}} = \langle\ & |B_{(0,0,0)}\rangle_{\mathcal{P}}\,, |B_{(0,2,2)}\rangle_{\mathcal{P}}\,, |B_{(1,-1,0)}\rangle_{\mathcal{P}}\,, |B_{(1,3,2)}\rangle_{\mathcal{P}}\,, |B_{(2,-2,0)}\rangle_{\mathcal{P}}\,, |B_{(2,4,2)}\rangle_{\mathcal{P}}\,, |B_{(3,-3,0)}\rangle_{\mathcal{P}}\,, |B_{(3,5,2)}\rangle_{\mathcal{P}}\,, \\ & |B_{(4,-4,0)}\rangle_{\mathcal{P}}\,, |B_{(4,6,2)}\rangle_{\mathcal{P}}\,, |B_{(5,-5,0)}\rangle_{\mathcal{P}}\,, |B_{(5,7,2)}\rangle_{\mathcal{P}}\,, |B_{(6,-6,0)}\rangle_{\mathcal{P}}\,, |B_{(6,8,2)}\rangle_{\mathcal{P}}\ \rangle_{\mathbb{Q}}\,. \end{aligned} \tag{4.3.79}$$

Using eq. (4.3.52) we find the following splitting of the Hodge structure of weight 13,

$$\begin{aligned} W_{\mathbb{C}}^{(0,-1,-1)} &= \langle\ |0,-1,-1\rangle_{\mathcal{S}}\,,\ |1,-2,-1\rangle_{\mathcal{S}}\,,\ |3,-4,-1\rangle_{\mathcal{S}}\,,\ |6,-7,-1\rangle_{\mathcal{S}}\,, \\ &\qquad |7,-8,-1\rangle_{\mathcal{S}}\,,\ |10,-11,-1\rangle_{\mathcal{S}}\,,\ |12,-13,-1\rangle_{\mathcal{S}}\,,\ |13,-14,-1\rangle_{\mathcal{S}}\ \rangle_{\mathbb{C}}\,, \\ W_{\mathbb{C}}^{(2,-3,-1)} &= \langle\ |2,-3,-1\rangle_{\mathcal{S}}\,,\ |5,-6,-1\rangle_{\mathcal{S}}\,,\ |8,-9,-1\rangle_{\mathcal{S}}\,,\ |11,-12,-1\rangle_{\mathcal{S}}\ \rangle_{\mathbb{C}}\,, \\ W_{\mathbb{C}}^{(4,-5,-1)} &= \langle\ |4,-5,-1\rangle_{\mathcal{S}}\,,\ |9,-10,-1\rangle_{\mathcal{S}}\ \rangle_{\mathbb{C}}\,, \end{aligned} \tag{4.3.80}$$

with the field extensions

$$\begin{aligned}\mathbb{Q}(S_{(0,-1,-1)})^{\mathrm{Stab}((0,-1,-1))} &= \mathbb{Q}(\zeta_{15})\\ \mathbb{Q}(S_{(2,-3,-1)})^{\mathrm{Stab}((2,-3,-1))} &= \mathbb{Q}(\zeta_{5})\\ \mathbb{Q}(S_{(4,-5,-1)})^{\mathrm{Stab}((4,-5,-1))} &= \mathbb{Q}(\zeta_{3})\,.\end{aligned} \tag{4.3.81}$$

Example: k=4

For $\mathcal{N} = (2,2)$ minimal models with even k, a polarisation, if it exists, cannot be constructed as a twisted open string Witten index with twist operator $\mathcal{V}$ (4.2.51), as $l = k+2$ is even and hence the twist operator $\mathcal{V}$ is not central to the $\mathcal{N} = 2$ super Virasoro algebra. As we will illustrate now for $k = 4$, the Hodge structure for $\mathcal{N} = (2,2)$ minimal models with even k might still be polarisable. The Hodge structure for $k = 4$ is given by the complex vector space

$$V_{\mathbb{C}} \;=\; \langle\, (|l,-(l+1),-1\rangle_{\mathcal{S}})\,\rangle_{\mathbb{C}}^{l\in\{0,1,\ldots,4\}}\,, \tag{4.3.82}$$

with graded subspaces

$$\begin{aligned}V_{\mathbb{C}}^{0,4} &= \langle\, |0,-1,-1\rangle_{\mathcal{S}})\,\rangle_{\mathbb{C}}\\ V_{\mathbb{C}}^{1,3} &= \langle\, |1,-2,-1\rangle_{\mathcal{S}})\,\rangle_{\mathbb{C}}\\ V_{\mathbb{C}}^{2,2} &= \langle\, |2,-3,-1\rangle_{\mathcal{S}})\,\rangle_{\mathbb{C}}\\ V_{\mathbb{C}}^{3,1} &= \langle\, |3,-4,-1\rangle_{\mathcal{S}})\,\rangle_{\mathbb{C}}\\ V_{\mathbb{C}}^{4,0} &= \langle\, |4,-5,-1\rangle_{\mathcal{S}})\,\rangle_{\mathbb{C}}\,.\end{aligned} \tag{4.3.83}$$

The rational vector space $V_{\mathbb{Q}}$ with rational basis is given by

$$V_{\mathbb{Q}} = \langle\, |B_{(0,1,1)}\rangle_{\mathcal{P}}\,, |B_{(2,-1,-1)}\rangle_{\mathcal{P}}\,, |B_{(3,0,1)}\rangle_{\mathcal{P}}\,, |B_{(4,-1,1)}\rangle_{\mathcal{P}}\,, |B_{(4,3,1)}\rangle_{\mathcal{P}}\,\rangle_{\mathbb{Q}}\,, \tag{4.3.84}$$

where the rational-valued intersection matrix in the above basis reads

$$\Sigma_{(L,M,S),(L',M',S')} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ -1 & 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 1 & -1 & 0 & 1\end{pmatrix}\,. \tag{4.3.85}$$

To construct a polarisation, we consider the most general form in the basis of $V_{\mathbb{C}}$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & a\\ 0 & 0 & 0 & -b & 0\\ 0 & 0 & c & 0 & 0\\ 0 & -b & 0 & 0 & 0\\ a & 0 & 0 & 0 & 0\end{pmatrix}\,. \tag{4.3.86}$$

First, we check that this symmetric matrix is the complex-linear extension of a non-degenerate rational-valued bilinear form on the rational vector space $V_{\mathbb{Q}}$. To do so, we consider the matrix

$$B = \begin{pmatrix} \frac{(-1)^{1/3}}{\sqrt{6}} & -(-1)^{2/3}\sqrt{\frac{2}{3}} & \frac{i}{\sqrt{2}} & \frac{(-1)^{2/3}}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{(-1)^{1/6}}{\sqrt{2}\cdot 3^{1/4}} & 0 & -\frac{i}{\sqrt{2}\cdot 3^{1/4}} & -\frac{(-1)^{5/6}}{\sqrt{2}\cdot 3^{1/4}} & \frac{i}{\sqrt{2}\cdot 3^{1/4}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{(-1)^{5/6}}{\sqrt{2}\cdot 3^{1/4}} & 0 & \frac{i}{\sqrt{2}\cdot 3^{1/4}} & \frac{(-1)^{1/6}}{\sqrt{2}\cdot 3^{1/4}} & -\frac{i}{\sqrt{2}\cdot 3^{1/4}} \\ -\frac{(-1)^{2/3}}{\sqrt{6}} & (-1)^{1/3}\sqrt{\frac{2}{3}} & -\frac{i}{\sqrt{2}} & -\frac{(-1)^{1/3}}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \tag{4.3.87}$$

where the columns are the basis elements of $V_{\mathbb{Q}}$ written in terms of the basis of $V_{\mathbb{C}}$. Then we calculate

$$B^T Q B = \begin{pmatrix} \frac{1}{3}(a-\sqrt{3}b+c) & \frac{1}{3}(-a-c) & \frac{1}{6}(3a+\sqrt{3}b) & \frac{1}{6}(a-\sqrt{3}b-2c) & \frac{1}{6}(a-\sqrt{3}b-2c) \\ \frac{1}{3}(-a-c) & \frac{1}{3}(4a+c) & -a & \frac{1}{3}(-2a+c) & \frac{a+c}{3} \\ \frac{1}{6}(3a+\sqrt{3}b) & -a & a-\frac{b}{\sqrt{3}} & \frac{1}{6}(3a-\sqrt{3}b) & \frac{b}{\sqrt{3}} \\ \frac{1}{6}(a-\sqrt{3}b-2c) & \frac{1}{3}(-2a+c) & \frac{1}{6}(3a-\sqrt{3}b) & \frac{1}{3}(a-\sqrt{3}b+c) & \frac{1}{6}(-a+\sqrt{3}b+2c) \\ \frac{1}{6}(a-\sqrt{3}b-2c) & \frac{a+c}{3} & \frac{b}{\sqrt{3}} & \frac{1}{6}(-a+\sqrt{3}b+2c) & \frac{1}{3}(a-\sqrt{3}b+c) \end{pmatrix}, \tag{4.3.88}$$

with $\det(B^TQB) = 3a^2b^2c$. We thus see, that we must choose $a, c \in \mathbb{Q}$ and b to be a rational multiple of $\sqrt{3}$ or $\sqrt{3}^{-1}$, for B^TQB to correspond to a rational bilinear form on $V_{\mathbb{Q}}$. Furthermore, condition (4.2.5) and (4.2.6) are clearly satisfied by (4.3.86). The third condition (4.2.7) then constrains a, b, c to be positive. Thus, the Hodge structure for the $k = 4$ model is polarisable, with polarisation given by (4.3.86) with $b = \left(\sqrt{3}\right)^{\pm 1} b'$ and rational parameters $a, c, b' > 0$.

4.4 Example: Gepner Models

As a second class of examples of $\mathcal{N} = (2,2)$ rational superconformal field theories, we are going to discuss so-called Gepner models [185]. Gepner models play an important role in worldsheet constructions in superstring theory, as they solve the supersymmetric non-linear sigma model of the worldsheet of closed strings with total space $\mathbb{R}^{1,D-1} \times \mathcal{M}_{10-D}$ in type IIA string theory in the absence of background fluxes exactly. They can be formed as simple current extensions of tensor products of $\mathcal{N} = (2,2)$ minimal models that were already discussed in section 4.3, combined with a theory describing the D non-compact space-time dimensions and a ghost sector for gauging the $\mathcal{N} = 1$ worldsheet superconformal symmetry. The non-compact theory consists of D free fermions, D free bosons, and a system of ghosts with total central charge -26 and superghosts with total central charge 11.

Understanding the connection between Gepner models and supersymmetric non-linear sigma models with Calabi–Yau target spaces involves two main steps. The first step relies on the well-established correspondence between $\mathcal{N} = (2,2)$ minimal models and two-dimensional $\mathcal{N} = 2$ Landau-Ginzburg theories. As demonstrated in [145, 186–188], a Landau-Ginzburg theory constructed from a single chiral superfield Ψ with action

$$S = \int d^2z\, d^4\theta\, K(\Psi, \overline{\Psi}) + \left(\int d^2z\, d^2\theta\, W(\Psi) + \text{c.c.} \right) , \tag{4.4.1}$$

where $K(\Psi, \overline{\Psi})$ denotes the Kähler potential, and $W(\Psi)$ is the superpotential given by

$$W(\Psi) = \Psi^{P+2} , \tag{4.4.2}$$

flows, at its infrared (IR) fixed point, to an $\mathcal{N} = 2$ superconformal field theory with central charge

$$c_P = \frac{3P}{P+2} . \tag{4.4.3}$$

It is important to recall that the unitary $c < 3$, $\mathcal{N} = 2$ superconformal theories are precisely the $\mathcal{N} = (2,2)$ minimal models. Consequently, a Landau-Ginzburg theory defined by the above superpotential $W(\Psi) = \Psi^{P+2}$ flows in the infrared to the $\mathcal{N} = (2,2)$ minimal model with level $k = P$ (with A-modular invariant), provided that a conformally invariant IR fixed point exists.

The second step in understanding the connection to Calabi–Yau geometries was established in the seminal work [188]. There, it was shown that supersymmetric non-linear sigma models with Calabi–Yau target spaces and (orbifolds of) Landau-Ginzburg theories can be understood as distinct phases of a single, more general framework: the *gauged linear sigma model* (GLSM). The GLSM is an $\mathcal{N} = 2$ supersymmetric gauge theory that interpolates, for a suitably chosen superpotential, between these two descriptions depending on the region of the Kähler moduli space being probed. From this perspective, the conformal fixed point of a Landau-Ginzburg orbifold model, such as a Gepner model, can be interpreted as the analytic continuation of the conformal fixed point associated with the non-linear sigma model on a Calabi–Yau manifold, continued into a region of small Kähler class — the so-called *small volume regime*. In this regime, the perturbative expansion of the non-linear sigma model breaks down, and the Landau-Ginzburg orbifold (e.g., a Gepner model) becomes the appropriate field theory description.

We now present the Gepner model construction from the modern perspective of simple current extensions, using the notation of [189]. To discuss the relevant projections that yield consistent worldsheet theories in superstring theory at the level of simple current extensions, one uses the so-called *bosonic string map* [190–192]. This amounts to mapping the D free fermion theory combined with the superghost theory, which is a non-unitary conformal field theory, to a unitary conformal field theory which is described by the WZW model $\widehat{\mathfrak{so}}(D+6)_1$. We will denote the latter theory by $D_{D/2+3}$ with primary fields (o, s, v, c), and the tensor product of a number of r $\mathcal{N} = (2, 2)$ minimal model theories by C_{int}, with primary fields labeled by

$$(\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{s}) = \bigotimes_{i=1}^{r} (l_i, m_i, s_i) \,, \tag{4.4.4}$$

and conditions on the indices as in (4.3.6). We denote by $\mathcal{A}_{\text{int}}$ the tensor product of the coset algebras (4.3.1) of the $\mathcal{N} = (2, 2)$ minimal model theories. Requiring the absence of a total (super)conformal gauge anomaly constraints the central charge of the internal theory C_{int} to be $c_{\text{int}} = \frac{3}{2}(10 - D)$.

To restore $\mathcal{N} = 2$ worldsheet supersymmetry, we perform a simple current extension of the theory $D_{D/2+3} \otimes C_{\text{int}}$, with respect to the order-two simple currents

$$v_i := (v, v_{i,\text{int}}) \,, \tag{4.4.5}$$

where we denote by $v_{i,\text{int}}$ the representation

$$v_{i,\text{int}} = (0, 0, 0) \otimes \cdots \otimes \underbrace{(0, 0, 2)}_{i\text{th pos.}} \otimes \cdots \otimes (0, 0, 0) \,, \qquad i = 1, \ldots, r \,, \tag{4.4.6}$$

with $(0, 0, 2)$ being the representation in the ith $\mathcal{N} = (2, 2)$ minimal model factor that contains the $\mathcal{N} = 2$ supercurrents $G_i^{\pm}$. The simple current v in the $D_{D/2+3}$ theory satisfies $v^{d/2} = s^2$, with $d = D-2$. This simple current extension ensures that the total $\mathcal{N} = 2$ super Virasoro algebra consistently splits into a module containing the vacuum, and a module containing the supercurrents $G^{\pm} = \sum_{i=1}^{r} G_i^{\pm}$. Therefore, only those states in $D_{D/2+3} \otimes C_{\text{int}}$ survive the projection where all the tensor product factors are either in the R–sector or all in the NS–sector. For this reason, the simple current extension with respect to the v_i currents is typically referred to as *fermion alignment*.

In a second step, we perform another simple current extension with respect to the simple current

$$s_{\text{tot}} := (s, s_{\text{int}}) \,, \qquad s_{\text{int}} = \bigotimes_{i=1}^{r} (0, -1, -1) \,. \tag{4.4.7}$$

This projection leads to a (space-time) tachyon-free string theory with $\mathcal{N}=2$ space-time supersymmetry and is known as *GSO projection* [193–195]. The monodromy charge with respect to the simple current $s_{\rm tot}$ can be shown to be half of the U(1)-charge of a state in $D_{D/2+3}\otimes C_{\rm int}$ modulo $\mathbb{Z}$, see e.g., [15]. Note that the bosonic string map changes the projection from odd U(1)-charges to even U(1)-charges [189]. We denote by $G^{(\rm Gep)}$ the group generated by the simple currents v_i, $i=1,\dots,r$, and $s_{\rm tot}$.

In this section, we focus exclusively on the internal part of the Gepner model, namely the tensor product theory $k_1\otimes\cdots\otimes k_r$, with total central charge,

$$c=\frac{3}{2}(10-D)\,, \tag{4.4.8}$$

after performing the simple current extension. This theory has been shown to correspond to a conformal fixed point of a non-linear sigma model with target space a Calabi–Yau manifold X [152, 185, 188, 196]. As briefly reviewed in section 2.5, we therefore expect the elements of the (c,c)$_{q_L=q_R}$-ring to give rise to the geometric Hodge structure $H^3(X,\mathbb{C})$, with rational structure given by the A-boundary states [197]. Following the discussion in [189], we denote by $G^{(\rm CY)}$ the group generated by elements in $G^{(\rm Gep)}$ that act trivially on the non-compact space-time part. By using the fusion rules of the WZW model $\widehat{\mathfrak{so}}(D+6)_1$, we find

$$G^{(\rm CY)}=\mathbb{Z}_2^{r-1-\eta}\times\mathbb{Z}_{N_s/2}\ , \tag{4.4.9}$$

which is generated by the simple currents of order two and $N_s/2=\mathrm{lcm}(\gamma_i(k_i+2))$ respectively,

$$J_i=|0,0,2\rangle\otimes|0,0,0\rangle\otimes\cdots\otimes|0,0,0\rangle\otimes\overbrace{|0,0,2\rangle}^{i\text{th pos.}}\otimes|0,0,0\rangle\otimes\cdots\otimes|0,0,0\rangle \qquad i=2,\dots,r\ , \tag{4.4.10}$$

$$J\ =(|0,-2,-2+2^{d/2}\rangle)\otimes\cdots\otimes|0,-2,-2\rangle\ .\qquad D=d+2\ , \tag{4.4.11}$$

Here, $s=|0,-1,-1\rangle$ denotes the state of conformal weight $c/24=k/(8(k+2))$ corresponding to the spectral flow operator $\mathcal{U}$ in the respective $\mathcal{N}=(2,2)$ minimal model factor, and similarly $v=|0,0,2\rangle$ the primary of conformal weight 3/2 containing the two worldsheet supercurrents $G^{\pm}$. Furthermore we have that $\gamma_i=2$, $\eta=1$, for odd k_i and $\gamma_i=1$, $\eta=0$ for even k_i. Note that when $\gamma_i=1$ for some i, as will be discussed later, there exist states that are fixed under the action of a simple current in $G^{(\rm CY)}$. This modifies the standard Cardy construction of boundary states [189]. We denote the chiral algebra extended by the currents J_i, J by $\mathcal{A}_{\rm CY}$, and the chiral algebra extended by the currents J_i only by $\mathcal{A}_{\rm SUSY}$.

The first tensor factor in the simple current J results from combining the two-fold spectral flow operator s^2 with the order-two simple current $v^{\frac{d}{2}}$. To understand the factor $v^{\frac{d}{2}}$ note that the spectral flow operator from the NS–sector to the R-sector changes the U(1)-charge by $c_{\rm int}/6=2-d/4$. Hence, while in the NS–sector we always project onto integral U(1)-charges, in the R–sector we project onto integral U(1)-charges for the space-time dimension

$D = d + 2 = 6$, and half-integral U(1)-charges for $D = 4$ or $D = 8$. Since $Q^{(v_i)}_{(l_i,m_i,s_i)} = 0$ in the NS–sector and $Q^{(v_i)}_{(l_i,m_i,s_i)} = 1/2$ in the R–sector, the inclusion of the factor $v^{\frac{d}{2}}$ in the simple current J yields the correct projection behavior.

The construction of the (c,c)-ring in the case where we choose in each tensor product factor theory a diagonal modular invariant is a purely combinatorial problem and one combines the chiral states in each tensor product factor theory to obtain states with

$$\sum_{i=1}^{r} h_L(|l_i, m_i, s_i\rangle) = \frac{1}{2}\sum_{i=1}^{r} q_L(|l_i, m_i, s_i\rangle \ , \qquad \sum_{i=1}^{r} q_L(|l_i, m_i, s_i\rangle = \sum_{i=1}^{r} \frac{l_i}{k_i+2} \in \mathbb{Z} \, . \tag{4.4.12}$$

In theories where at least one k_i is even, one needs to resolve states that appear with some multiplicity in the partition function by a further quantum number, as they are indistinguishable as representations of $\mathcal{A}_{\text{SUSY}}$. For more details on resolving fixed points in simple current extensions, we refer to [57, 134, 189]. We will briefly discuss one example of this kind in the next section.

To construct the rational Hodge structure from Gepner model data, it is necessary to describe A-type BPS boundary states that preserve at least one copy of $\mathcal{A}_{\text{SUSY}}$ to ensure $\mathcal{N} = 1$ worldsheet supersymmetry on the boundary. The classification of all subalgebras $\mathcal{A}_\theta$ satisfying

$$\mathcal{A}_{\text{CY}} \supseteq \mathcal{A}_\theta \supseteq \mathcal{A}_{\text{SUSY}} \, , \tag{4.4.13}$$

for a general $\mathcal{N} = (2, 2)$ rational superconformal field theory remains an open problem. For Gepner models, however, a complete classification is known. In [189], the authors showed that, for Gepner models, the full set of A-boundary conditions preserving the subalgebra $\mathcal{A}_\theta$ with $\mathcal{A}_{\text{CY}} \supseteq \mathcal{A}_\theta \supseteq \mathcal{A}_{\text{SUSY}}$ is classified by subgroups $K_a \subseteq K$, where K is the group of automorphisms of $\mathcal{A}_{\text{CY}}$.

The subalgebra $\mathcal{A}_\theta$ preserved by a given boundary state is then identified with the subalgebra that is pointwise fixed by the subgroup K_a.

The boundary states can be expressed as linear combinations of generalised Ishibashi states,

$$|a\rangle_\theta = \sum_\alpha B^a_\alpha \|\alpha, \psi_\alpha\rangle\!\rangle_\theta, \tag{4.4.14}$$

where α labels the representations of $\mathcal{A}_{\text{SUSY}}$ with vanishing monodromy charge with respect to the simple current J, and ψ_α denotes a group character of the stabiliser subgroup of the full simple current group $G^{(\text{CY})}$ [10], i.e., the subgroup consisting of elements in $G^{(\text{CY})}$ that fix some representations in $\mathcal{A}_{\text{SUSY}}$ under the fusion product.

A particular boundary state $|a\rangle_\theta$ is labeled by orbits of irreducible representations of $\mathcal{A}_{\text{SUSY}}$ (dressed with an appropriate group character) under the action of J, with monodromy

[10] We restrict our discussion to theories with $\mathbb{Z}_2$ group character, for simplicity.

charge

$$\begin{aligned} Q_J(a) &= \theta/\pi \mod \mathbb{Z} \\ \theta &= 2\pi n/N_s', \quad n \in \{0, 1, \ldots, N_s' - 1\}\,, \end{aligned} \tag{4.4.15}$$

where $N_s'/2$ denotes the length of the image of the simple current J acting on representations of $\mathcal{A}_{\mathrm{SUSY}}$. In theories with no non-trivial stabiliser subgroups K_a, which happens for Gepner models with only odd minimal model levels k_i and A-type modular invariant in each tensor factor, the coefficient B^a_α is given by the product of S-matrices of the individual minimal model theories, following Cardy's construction [137, 198]. In the presence of non-trivial stabiliser subgroups (which corresponds to the appearance of multiplicities in the partition function), the expression for B^a_α gets adjusted following [189], for an explicit example see eq. (4.4.35). We will discuss two examples illustrating both situations in the class of Gepner models that are built from A-type $\mathcal{N} = (2, 2)$ minimal models.

The model $(3,3,3,3,3)$ Corresponding to the Fermat Quintic Threefold

Our first example is the Gepner model $(3,3,3,3,3)$ corresponding to the Fermat quintic threefold

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0, \qquad (x_1 : x_2 : x_3 : x_4 : x_5) \in \mathbb{CP}^4 \,. \tag{4.4.16}$$

Given the levels $k_i = 3$, it is a simple combinatorial exercise to enumerate the states in the (c,c)-ring, which provide the complexified vector space $V_{\mathbb{C}}$. We display the result in table 4.1.

State	$V_{\mathbb{C}}^{p,q}$	Number
$\|3,-3,0\rangle \otimes \|2,-2,0\rangle \otimes \|0,0,0\rangle^{\otimes 3}$	$V_{\mathbb{C}}^{1,2}$	20
$\|3,-3,0\rangle \otimes \|1,-1,0\rangle^{\otimes 2} \otimes \|0,0,0\rangle^{\otimes 2}$	$V_{\mathbb{C}}^{1,2}$	30
$\|2,-2,0\rangle^{\otimes 2} \otimes \|1,-1,0\rangle \otimes \|0,0,0\rangle^{\otimes 2}$	$V_{\mathbb{C}}^{1,2}$	30
$\|2,-2,0\rangle \otimes \|1,-1,0\rangle^{\otimes 3} \otimes \|0,0,0\rangle$	$V_{\mathbb{C}}^{1,2}$	20
$\|1,-1,0\rangle^{\otimes 5}$	$V_{\mathbb{C}}^{1,2}$	1
$\|3,-3,0\rangle^{\otimes 3} \otimes \|1,-1,0\rangle \otimes \|0,0,0\rangle$	$V_{\mathbb{C}}^{2,1}$	20
$\|3,-3,0\rangle^{\otimes 2} \otimes \|2,-2,0\rangle^{\otimes 2} \otimes \|0,0,0\rangle$	$V_{\mathbb{C}}^{2,1}$	30
$\|3,-3,0\rangle^{\otimes 2} \otimes \|2,-2,0\rangle \otimes \|1,-1,0\rangle^{\otimes 2}$	$V_{\mathbb{C}}^{2,1}$	30
$\|3,-3,0\rangle \otimes \|2,-2,0\rangle^{\otimes 3} \otimes \|1,-1,0\rangle$	$V_{\mathbb{C}}^{2,1}$	20
$\|2,-2,0\rangle^{\otimes 5}$	$V_{\mathbb{C}}^{2,1}$	1
$\|3,-3,0\rangle^{\otimes 5}$	$V_{\mathbb{C}}^{0,3}$	1
$\|0,0,0\rangle^{\otimes 5}$	$V_{\mathbb{C}}^{3,0}$	1

Table 4.1: *The (c,c)-states of the Gepner model $(3,3,3,3,3)$ corresponding to the Fermat quintic. The first column lists the state up to a permutation of the factors of the tensor product. The second column lists the Hodge type of the corresponding state, and the third column indicates how many distinct chiral states can be obtained by permutations of the displayed state.*

Let us first observe that constructing a sufficient number of boundary states from the highest-weight representations of the chiral algebra $\mathcal{A}_{\text{CY}}$, i.e., those boundary states that preserve the maximal chiral algebra $\mathcal{A}_{\text{CY}}$, is not possible. To see this, we note that the states $|3,-3,0\rangle^{\otimes 5}$ and $|0,0,0\rangle^{\otimes 5}$ belong to the same representation with respect to $\mathcal{A}_{\text{CY}}$, as do $|1,-1,0\rangle^{\otimes 5}$ and $|2,-2,0\rangle^{\otimes 5}$. In fact, it can be shown that there are only 202 highest-weight representations of $\mathcal{A}_{\text{CY}}$. Thus, the resulting set of boundary states is insufficient to construct the 204-dimensional Hodge structure that arises from the geometry of the Fermat quintic, since two representations contain two distinct elements of the (c,c)-ring.

However, when considering the subalgebra $\mathcal{A}_{\text{SUSY}}$ generated only by the order-two currents J_i, each state in the (c,c)-ring belongs to a distinct highest-weight representation. This allows the construction described in section 4.2.2 to proceed. In particular, we must verify that the action of the Galois groups $\text{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})$ on the modular S-matrix is compatible with the structure of the (c,c)-ring, as explained in eq. (4.2.65). Specifically, for any representation $\alpha \in \mathcal{R}$ corresponding to an RR–ground state $|\alpha\rangle_{\mathcal{S}}$, we require that for all $\rho \in \text{Gal}(\mathbb{Q}(S_\alpha)/\mathbb{Q})$,

$$\rho(\text{S}_{\alpha A}) \;=\; \epsilon_\rho(\alpha)\text{S}_{\varrho(\alpha)A} \;, \tag{4.4.17}$$

where $\varrho(\alpha) \in \mathcal{R}$.

The modular S-matrix of the chiral algebra $\mathcal{A}_{\text{SUSY}}$ can be obtained by a standard simple current extension of the tensor product of five $\mathcal{N} = (2,2)$ coset algebras. This extension is performed with respect to the simple current group generated by the order-two simple currents J_i. Representations of $\mathcal{A}_{\text{SUSY}}$ are labeled as

$$\begin{aligned}(\mathbf{l},\mathbf{m},\mathbf{s}) &:= (l_1,m_1,s_1)\otimes\cdots\otimes(l_5,m_5,s_5)\\ (\mathbf{L},\mathbf{M},\mathbf{S}) &:= (L_1,M_1,S_1)\otimes\cdots\otimes(L_5,M_5,S_5)\,.\end{aligned} \tag{4.4.18}$$

The modular S-matrix is then given by [150, 198]

$$\text{S}_{(\mathbf{l},\mathbf{m},\mathbf{s}),(\mathbf{L},\mathbf{M},\mathbf{S})} \;=\; 2^{r-1}\frac{1}{5^5}\prod_{i=1}^{5}\sin\left(\frac{\pi}{5}(l_i+1)(L_i+1)\right)\exp\left(\frac{\text{i}\pi}{5}m_iM_i\right)\exp\left(-\frac{\text{i}\pi}{2}s_iS_i\right)\,, \tag{4.4.19}$$

which is, up to an overall integer factor, the product of the S-matrices of the $\mathcal{N} = (2,2)$ minimal model factor theories with $k_i = 3$.

The labels above represent orbits of highest-weight representations of $\mathcal{A}_{\text{int}}$ with vanishing monodromy charge under the simple currents J_i. A compatible basis of boundary states $|B_{(\mathbf{L},\mathbf{M},\mathbf{S})}\rangle_{\mathcal{P}}$ for the rational vector space $V_\mathbb{Q}$ can be obtained by restricting to boundary states for which $|\mathbf{S}| := \sum_i S_i$ is either even or odd [150]. This distinction arises because these two classes of boundary states preserve different linear combinations of the left- and right-moving supercharges [97]. In fact, a direct calculation shows that the intersection matrix between boundary states with $|\mathbf{S}|$ even and those with $|\mathbf{S}|$ odd is not rational.

It is straightforward to verify that a 204-dimensional basis of the rational vector space $V_\mathbb{Q}$ can be constructed using projected boundary states $|B_{(\mathbf{L},\mathbf{M},\mathbf{S})}\rangle_{\mathcal{P}}$ with $|\mathbf{S}|$ even, ensuring that the resulting rational intersection matrix Σ is non-singular. The remainder of our argument does not depend on the specific choice of basis.

Since we are concerned with the S-matrix entries corresponding to RR–ground states (with

$s_i = -1$), we note that when $|\boldsymbol{S}|$ is even, the exponential factor in the matrix becomes

$$\prod_{i=1}^{5} \exp\left(-\frac{\mathrm{i}\pi}{2} s_i S_i\right) \;=\; \exp\left(\frac{\mathrm{i}\pi}{2}|\mathbf{S}|\right) \tag{4.4.20}$$

i.e., a sign. Therefore, the modular S-matrix can be written as

$$\mathrm{S}_{(\mathbf{l},\mathbf{m},\mathbf{s}),(\mathbf{L},\mathbf{M},\mathbf{S})} = 2^{r-1}\frac{1}{5^5}(-1)^{|\mathbf{S}|/2}\frac{\mathrm{i}}{2^5}\prod_{i=1}^{5}(-1)^{(l_i+1)(L_i+1)+m_i M_i}\widetilde{\mathrm{S}}_{(l_i,m_i,s_i),(L_i,M_i,S_i)} \;, \tag{4.4.21}$$

where $\zeta_5 = \mathrm{e}^{2\pi\mathrm{i}/5}$ is a primitive fifth root of unity, and

$$\widetilde{\mathrm{S}}_{(l_i,m_i,s_i),(L_i,M_i,S_i)} := \left(\zeta_5^{2(l_i+1)(L_i+1)} - \zeta_5^{3(l_i+1)(L_i+1)}\right)\zeta_5^{3m_i M_i} \;. \tag{4.4.22}$$

We observe that the number field $\mathbb{Q}(S_{(\mathbf{l},\mathbf{m},\mathbf{s})})$, with $(\mathbf{l},\mathbf{m},\mathbf{s}) \in \mathcal{R}$, generated by the S-matrix elements

$$\left\{\mathrm{S}_{(\mathbf{l},\mathbf{m},\mathbf{s}),(\mathbf{L},\mathbf{M},\mathbf{S})} \;\middle|\; |B_{(\mathbf{L},\mathbf{M},\mathbf{S})}\rangle_{\mathcal{P}} \;,\; (\mathbf{L},\mathbf{M},\mathbf{S}) \in \mathcal{B}\right\} \;, \tag{4.4.23}$$

is given by

$$\mathbb{Q}(S_{(\mathbf{l},\mathbf{m},\mathbf{s})}) \;=\; \mathbb{Q}(\mathrm{i},\zeta_5) \;=\; \mathbb{Q}(\zeta_{20}) \;, \tag{4.4.24}$$

where ζ_{20} is a 20th root of unity.

Recalling the discussion of the stabiliser group in $\mathcal{N} = (2,2)$ minimal models in section 4.3, we identify the fixed field as

$$\mathbb{Q}(S_{(\mathbf{l},\mathbf{m},\mathbf{s})}^{\mathrm{Stab}(\mathbf{l},\mathbf{m},\mathbf{s})}) \;=\; \mathbb{Q}(\zeta_5) \,. \tag{4.4.25}$$

The number field $\mathbb{Q}(\zeta_5)$ is a cyclotomic field, whose defining polynomial is

$$1 + x + x^2 + x^3 + x^4 \;, \tag{4.4.26}$$

which is the minimal polynomial of ζ_5 over $\mathbb{Q}$. It follows that the field extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}$ has degree $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = 4$, and its Galois group is the cyclic group $C_4 \simeq \mathbb{Z}_4$, which can be taken to be generated by $\rho : \zeta_5 \mapsto \zeta_5^2$.

All that remains in order to apply the proof from section 4.2.2 is to demonstrate that this indeed corresponds to a map ϱ which permutes the representations containing RR–ground states among themselves. Since the S-matrix in eq. (4.4.21) is a product of minimal model S-matrices, it is natural to view the action of ϱ as acting independently on each factor in the tensor product,

$$\varrho\left((\mathbf{l},\mathbf{m},\mathbf{s})\right) = \varrho((l_1,m_1,s_1)) \otimes \cdots \otimes \varrho((l_5,m_5,s_5)) \;. \tag{4.4.27}$$

Accordingly, we may consider the transformation behavior of the quantity $\widetilde{\mathrm{S}}_{(l_i,m_i,s_i),(L_i,M_i,S_i)}$. This is because, if we can show that each individual tensor factor representation (l_i, m_i, s_i) is mapped to a representations containing an RR–ground state under the action of ϱ, then the full tensor product representation $\varrho\,(\mathbf{l},\mathbf{m},\mathbf{s})$ will automatically correspond to an RR–ground state of the full GSO-projected theory, i.e., it will lie in the set $\mathcal{R}$. Recall that representations in the RR–sector survive the GSO-projection with respect to the simple current J if the total U(1)-charge is a half-integer. Choosing, for each tensor factor, a representation corresponding to an RR–ground state of the individual $\mathcal{N} = (2, 2)$ minimal model, the condition for the resulting tensor product to define an RR–ground state of the GSO-projected theory becomes

$$\sum_{i=1}^{5} \frac{l_i + 1}{k_i + 2} = \sum_{i=1}^{5} \frac{l_i + 1}{5} \in \mathbb{Z}\,, \tag{4.4.28}$$

or equivalently

$$\prod_{i=1}^{5} \zeta_5^{l_i+1} = 1\,, \tag{4.4.29}$$

where we used eq. (4.3.19) and eq. (4.4.12).

Let us now fix a representation $(\mathbf{l},\mathbf{m},\mathbf{s}) \in \mathcal{R}$ and consider the action of a Galois automorphism $\rho \in \mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$. If, under this action, each tensor factor is again mapped to a representation corresponding to an RR–ground state of the individual $\mathcal{N} = (2, 2)$ minimal model, then it follows immediately that $\varrho\,(\mathbf{l},\mathbf{m},\mathbf{s}) \in \mathcal{R}$. This is because the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ consists of field automorphisms of $\mathbb{Q}(\zeta_5)$ that fix $\mathbb{Q}$ and permute the fifth roots of unity while preserving all algebraic relations over $\mathbb{Q}$, including, for instance (4.4.29). Since we have already shown in the previous section that, for the $\mathcal{N} = (2, 2)$ minimal models, the action of any $\rho \in \mathrm{Gal}(\mathbb{Q}(S_{(l,m,s)}^{\mathrm{Stab}(l,m,s)})/\mathbb{Q})$ maps a representation corresponding to an RR–ground state to another such representation, the claim follows.

By the proof given in section 4.2.2, this is sufficient to establish that the Gepner model $(3, 3, 3, 3, 3)$ admits a Hodge structure of CM-type with CM–field $\mathbb{Q}(\zeta_5)$, and polarisation induced by the open string Witten index, as defined in eq. (4.2.46), with trivial twist operator $\mathcal{V}$. Since this Gepner model corresponds to the conformal fixed point of the non-linear sigma model on the Fermat quintic threefold [150], we have thus provided a physics-based derivation of the well-known mathematical result that the Fermat quintic admits complex multiplication, with CM–field $\mathbb{Q}(\zeta_5)$ (see, for example, [199]).

In particular, the action of ρ partitions the 204 RR–ground states into 51 distinct orbits of length four, with each orbit corresponding to an irreducible Hodge substructure of the full 204-dimensional rational Hodge structure of weight three. The rational Hodge endomorphism algebra, when restricted to each Hodge substructure, is isomorphic to $\mathbb{Q}(\zeta_5)$.

The Model $(6, 6, 2, 2, 2)$ Corresponding to $\mathbb{P}^4_{1,1,2,2,2}[8]$

As a second example, we consider the Gepner model $k = (6, 6, 2, 2, 2)$, which corresponds to the conformal fixed point of the non-linear sigma model on the degree eight hypersurface

$$x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 = 0, \qquad (x_1 : x_2 : x_3 : x_4 : x_5) \in \mathbb{P}^4_{1,1,2,2,2}\,. \tag{4.4.30}$$

A detailed analysis of the geometry and mirror symmetry associated with this model can be found in [200].

In this Gepner model, the group of simple currents $G^{(\mathrm{CY})}$ possesses a unique simple current of the form

$$L = |6,0,0\rangle \otimes |6,0,0\rangle \otimes |0,0,0\rangle \otimes |0,0,0\rangle \otimes |0,0,0\rangle\ , \tag{4.4.31}$$

which has fixed points. Using the fusion rules of $\mathcal{N} = (2, 2)$ minimal models, see eq. (4.3.11), one easily finds that the following states are fixed by the simple current L, where we restrict for simplicity to the (c,c)-states:

$$\begin{aligned} &|3,-3,0\rangle \otimes |3,-3,0\rangle \otimes |1,-1,0\rangle \otimes |0,0,0\rangle \otimes |0,0,0\rangle\ , \\ &|3,-3,0\rangle \otimes |3,-3,0\rangle \otimes |2,-1,0\rangle \otimes |2,-2,0\rangle \otimes |1,-1,0\rangle\ . \end{aligned} \tag{4.4.32}$$

Each state is a representative of a family of 3 states obtained by a non-trivial permutation of the last three factors. It follows that the Gepner model $(6, 6, 2, 2, 2)$ has multiple isomorphic highest-weight representations of the chiral algebra $\mathcal{A}_{\mathrm{SUSY}}$ appearing in the spectrum. Following the procedure of fixed-point resolutions in simple current extensions, see e.g., refs. [57, 134, 189], we therefore label the highest-weight representations as

$$(\mathbf{l},\mathbf{m},\mathbf{s})_\psi := (l_1, m_1, s_1) \otimes \cdots \otimes (l_5, m_5, s_5)_\psi\ , \tag{4.4.33}$$

where $(\mathbf{l},\mathbf{m},\mathbf{s}) = (l_1, m_1, s_1) \otimes \cdots \otimes (l_5, m_5, s_5) \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A}_{\mathrm{SUSY}})$ labels the distinct representations of $\mathcal{A}_{\mathrm{SUSY}}$, and $\psi \in \mathrm{Stab}_{\mathcal{A}_{\mathrm{SUSY}}}((\mathbf{l},\mathbf{m},\mathbf{s}))^*$ is a character[11] of the stabiliser group $\mathrm{Stab}_{\mathcal{A}_{\mathrm{SUSY}}}((\mathbf{l},\mathbf{m},\mathbf{s}))$ ($= \mathbb{Z}_2$ or 1) of the simple current algebra. The corresponding highest-weight state is denoted by $|(\mathbf{l},\mathbf{m},\mathbf{s})_\psi\rangle$.

Two representations differing only by the character $\psi \in \mathrm{Stab}_{\mathcal{A}_{\mathrm{SUSY}}}((\mathbf{l},\mathbf{m},\mathbf{s}))^*$ are isomorphic as representations of $\mathcal{A}_{\mathrm{SUSY}}$. For the model $(6, 6, 2, 2, 2)$, the only stabiliser groups that appear are the trivial group 1 or $\mathbb{Z}_2$, so we only need to consider the trivial character that we denote by 1 and the non-trivial character -1 which maps

$$L \mapsto -1\,. \tag{4.4.34}$$

The boundary states are constructed from the generalised Ishibashi states $|(\mathbf{l},\mathbf{m},\mathbf{s}),\psi\rangle\rangle$, which are labelled by those $(\mathbf{l},\mathbf{m},\mathbf{s}) \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A}_{\mathrm{SUSY}})$ that have vanishing monodromy charge with respect to the simple current group that extents $\mathcal{A}_{\mathrm{SUSY}}$ to $\mathcal{A}_{\mathrm{CY}}$ and a group

[11] That is, a group homomorphism $G \to \mathbb{C}^*$.

character $\psi \in \mathrm{Stab}_{\mathcal{A}_{\mathrm{SUSY}}}((\mathbf{l},\mathbf{m},\mathbf{s}))^*$. The boundary states are labelled by orbits of representations $\mathrm{HWR}_{\mathcal{H}}(\mathcal{A}_{\mathrm{SUSY}})$ with respect to the simple current J that extends $\mathcal{A}_{\mathrm{SUSY}}$ to $\mathcal{A}_{\mathrm{CY}}$, with no restriction on the monodromy charge, and a character $\Psi \in \mathrm{Stab}_{\mathcal{A}_{\mathrm{SUSY}}}((\mathbf{l},\mathbf{m},\mathbf{s}))^*$. These states can be written as [189, 201, 202],

$$|B_{(\mathbf{L},\mathbf{M},\mathbf{S})_\Psi}\rangle \;=\; \sum_{\substack{(\mathbf{l},\mathbf{m},\mathbf{s})\in\mathrm{HWR}_{\mathcal{H}}(\mathcal{A}_{\mathrm{SUSY}})\\ \psi\in\mathrm{Stab}_{\mathcal{A}_{\mathrm{SUSY}}}((\mathbf{l},\mathbf{m},\mathbf{s}))^*}} \frac{\mathrm{S}_{(\mathbf{L},\mathbf{M},\mathbf{S})_\Psi,(\mathbf{l},\mathbf{m},\mathbf{s})_\psi}}{\sqrt{\mathrm{S}_{(\mathbf{L},\mathbf{M},\mathbf{S})_\Psi,(\mathbf{0},\mathbf{0},\mathbf{0})_{\mathrm{I}}}}}\,|(\mathbf{l},\mathbf{m},\mathbf{s}),\psi\rangle\!\rangle \;, \tag{4.4.35}$$

where the sum runs over those $(\mathbf{l},\mathbf{m},\mathbf{s}) \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A}_{\mathrm{SUSY}})$ that have vanishing monodromy charge (i.e., representations corresponding to states that appear in the theory) and the 'generalised S-matrix' is given by [189, 201, 202]

$$\begin{aligned}
\mathrm{S}_{(\mathbf{L},\mathbf{M},\mathbf{S})_\Psi,(\mathbf{l},\mathbf{m},\mathbf{s})_\psi} :=\; & \frac{128}{\prod_{i=1}^r h_i}\prod_{r=1}^5 e^{\pi \mathrm{i}(m_iM_i/h_i - s_iS_i/2)}\prod_{i=3}^5 \sin\left(\pi\frac{(l_i+1)(L_i+1)}{h_i}\right)\\
& \Bigg[\left(1-\frac{3}{4}\prod_{i=1}^2\delta_{l_i,k_i/2}\delta_{L_i,k_i/2}\right)\prod_{i=1}^2\sin\left(\pi\frac{(l_i+1)(L_i+1)}{h_i}\right)\\
& +\frac{1}{2}(-1)^{\frac{k_1+k_2}{4}}\Psi(L)\psi(L)\prod_{i=1}^2\delta_{l_i,k_i/2}\delta_{L_i,k_i/2}\,h_i^{1/2}\Bigg]\\
=\; & 128\prod_{r=3}^5 \mathrm{S}_{(L_i,M_i,S_i),(l_i,m_i,s_i)}\\
& \Bigg[\left(1-\frac{3}{4}\prod_{i=1}^2\delta_{l_i,k_i/2}\delta_{L_i,k_i/2}\right)\prod_{i=1}^2 \mathrm{S}_{(L_i,M_i,S_i),(l_i,m_i,s_i)}\\
& +\left(\prod_{i=1}^2 e^{\pi \mathrm{i}(m_iM_i/h_i - s_iS_i/2)}\right)\frac{1}{2h_1h_2}(-1)^{\frac{k_1+k_2}{4}}\Psi(L)\psi(L)\prod_{i=1}^2\delta_{l_i,k_i/2}\delta_{L_i,k_i/2}\,h_i^{1/2}\Bigg]
\end{aligned}\;, \tag{4.4.36}$$

where $\mathrm{S}_{(L_i,M_i,S_i),(l_i,m_i,s_i)}$ denotes the S-matrix of the i-th minimal model factor, and we define $h_i := k_i + 2$.

The set $\mathcal{R} \ni \alpha_\psi$ of representations corresponding to RR–ground states is given by[12]

$$\mathcal{R} := \left\{(\mathbf{l},\mathbf{m},\mathbf{s})_\psi \;\middle|\; \mathcal{U}^{-1}\,|(\mathbf{l},\mathbf{m},\mathbf{s})_\psi\rangle \in (\mathrm{c},\mathrm{c})_{q_L=q_R}\right\} \;. \tag{4.4.37}$$

We normalise the RR–ground states $|\alpha_\psi\rangle_{\mathcal{N}}$, $\alpha_\psi \in \mathcal{R}$ so that they satisfy

$${}_{\mathcal{N}}\langle\alpha_\psi|\beta_\phi\rangle_{\mathcal{N}} \;=\; \delta_{\alpha\beta}\delta_{\psi\phi} \;. \tag{4.4.38}$$

[12] This is almost the same definition as in section 4.2.1, but now two distinct elements of $\mathcal{R}$ can correspond to isomorphic representations in $\mathrm{HWR}_{\mathcal{H}}(\mathcal{A}_{\mathrm{SUSY}})$, if they differ only by a character.

These states also satisfy the orthogonality condition ${}_{\mathcal{N}}\langle\alpha_\psi|i,\phi\rangle\rangle \;=\; \delta_{\alpha i}\delta_{\psi\phi}$ with the generalised Ishibashi states. This implies that the discussion in sections 4.2.1 and 4.2.2 goes through with only minor modifications: We define $\widetilde{\mathcal{B}}$ as the largest subset

$$\widetilde{\mathcal{B}} \subset \{(\mathbf{L},\mathbf{M},\mathbf{S})_\Psi \mid (\mathbf{L},\mathbf{M},\mathbf{S}) \in \mathrm{HWR}_{\mathcal{H}}(\mathcal{A}_{\mathrm{SUSY}})\ ,\quad \Psi \in \mathrm{Stab}_{\mathcal{A}_{\mathrm{SUSY}}}((\mathbf{L},\mathbf{M},\mathbf{S}))^*\}\ , \quad (4.4.39)$$

such that the intersection matrix Σ is $\mathbb{Q}$-valued, where Σ is defined again by (4.2.31):

$$\Sigma_{AB} := {}_{\mathcal{P}}\langle B_A|\,(-1)^{J_0}\,|B_B\rangle_{\mathcal{P}}\ , \qquad A,B \in \widetilde{\mathcal{B}}\ . \quad (4.4.40)$$

The rational vector space $V_{\mathbb{Q}}$ is defined as the $\mathbb{Q}$-span of the states $|B_A\rangle_{\mathcal{P}}$, $A \in \widetilde{\mathcal{B}}$, and we denote by $\mathcal{B} \subset \widetilde{\mathcal{B}}$ a subset corresponding to a set of projected boundary states forming a basis of $V_{\mathbb{Q}}$.

In analogy to the general discussion in section 4.2.2, we define the field extension $\mathbb{Q}(S)$ to be generated by the extended S-matrix elements (4.4.36). Similarly, we define $\mathbb{Q}(S_{\alpha_\psi})$ to be generated by the S-matrix elements $\mathrm{S}_{\alpha_\psi,A_\psi}$ with $A_\psi \in \mathcal{B}$.

The proof of the existence of the CM–property in section 4.2.2 will apply in this case as well, if we can show that the Galois group acts in such a way that for all $\alpha_\psi \in \mathcal{R}$, every $\rho \in \mathrm{Gal}(\mathbb{Q}(S_{\alpha_\psi})/\mathbb{Q})$ satisfies[13]

$$\rho(\mathrm{S}_{\alpha_\psi A_\Psi}) \;=\; \epsilon_\rho(\alpha_\psi)\mathrm{S}_{\varrho(\alpha_\psi)A_\Psi}\ , \qquad \text{where} \qquad \epsilon_\sigma(\alpha_\psi) \in \{\pm 1\}\ , \quad (4.4.41)$$

for all $A_\Psi \in \mathcal{B}$, and $\varrho : \mathcal{R} \to \mathcal{R}$.

Depending on the representation $\alpha_\psi \in \mathcal{R}$, the field extension $\mathbb{Q}(S_{\alpha_\psi})$, generated by the S-matrix elements S_{α_ψ,A_ψ} with $A_\psi \in \mathcal{B}$ is $\mathbb{Q}(\zeta_8)$ or $\mathbb{Q}(\mathrm{i})$:

$$\mathbb{Q}(S_{\alpha_\psi}) \;=\; \begin{cases} \mathbb{Q}(\mathrm{i}) & \text{if } l_1 \in \{1,3,5\} \text{ or } l_2 \in \{1,3,5\}\ , \\ \mathbb{Q}(\zeta_8) & \text{otherwise}\ . \end{cases} \quad (4.4.42)$$

The corresponding Galois groups are

$$\mathrm{Gal}(\mathbb{Q}(S_{\alpha_\psi})/\mathbb{Q}) \;\simeq\; \begin{cases} \mathbb{Z}_2 & \text{if } \mathbb{Q}(S_{\alpha_\psi}) \;=\; \mathbb{Q}(\mathrm{i})\ , \\ \mathbb{Z}_2\times\mathbb{Z}_2 & \text{if } \mathbb{Q}(S_{\alpha_\psi}) \;=\; \mathbb{Q}(\zeta_8)\ . \end{cases} \quad (4.4.43)$$

We will now show that the condition (4.4.41) is satisfied and discuss the resulting split of the Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}})$.

It is most convenient to consider the two cases separately.

[13] In this case, the proof of [57, 136] on the action of the Galois group does not apply, and hence we cannot deduce the form of the action from general principles. This is what sets the case of the Gepner model $(6,6,2,2,2)$ apart from the case where all boundary states are given by Cardy states.

Case 1: $\mathbb{Q}(\boldsymbol{S}_{\boldsymbol{\alpha}_\psi}) = \mathbb{Q}(\boldsymbol{\zeta}_8)$

The Galois group $\mathrm{Gal}(\mathbb{Q}(S_{\alpha_\psi})/\mathbb{Q})$ is generated by two elements ρ_1 and ρ_2 that act as

$$\rho_1(\zeta_8) \;=\; \zeta_8^3 \;, \qquad \rho_2(\zeta_8) \;=\; -\zeta_8 \;. \tag{4.4.44}$$

For representations with trivial stabiliser, i.e., $\mathrm{Stab}_{\mathcal{A}_{\mathrm{SUSY}}}((\mathbf{l},\mathbf{m},\mathbf{s})_\psi) = 1$, the generalised S-matrix (4.4.36) reduces to a product of minimal model S-matrices. As shown in section 4.3, the S-matrix for each minimal model factor can be expressed in terms of roots of unity of the form $\zeta_{k_i+2}^{l_i+1}$. In the present case, this corresponds to $\zeta_8^{l_i+1}$ in the first two tensor factors and $\zeta_4^{l_i+1}$ in the remaining three.

In particular, the Galois automorphism $\rho_1 \in \mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$, defined by $\rho_1(\zeta_8) = \zeta_8^3$, acts uniformly across all minimal model factors by raising the relevant root of unity to the third power.

Therefore, if the stabiliser is trivial, i.e., $\mathrm{Stab}_{\mathcal{A}_{\mathrm{SUSY}}}((\mathbf{l},\mathbf{m},\mathbf{s})_\psi) = 1$, the action of the Galois automorphism $\rho_1 \in \mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ on the representations in $\mathcal{R}$ is given by

$$\rho_1((\mathbf{l},\mathbf{m},\mathbf{s}))_1 \;=\; \varrho_1((l_1,m_1,s_1)) \otimes \cdots \otimes \varrho_1((l_5,m_5,s_5))_1 \;. \tag{4.4.45}$$

In particular, the action of ϱ_1 on each tensor factor can be determined using the same techniques described in section 4.3. For the first two factors, we find:

$$\begin{aligned} \zeta_8^{l+1} &\to \zeta_8^{3(l+1)} \\ 3(l+1) &= l'+1 \\ l' &= 3l+2 \mod 8\,, \end{aligned} \tag{4.4.46}$$

and for the remaining three factors:

$$\begin{aligned} \zeta_4^{l+1} &\to \zeta_4^{3(l+1)} \\ 3(l+1) &= l'+1 \\ l' &= 3l+2 \mod 4\,. \end{aligned} \tag{4.4.47}$$

Here, we used the fact that the canonical restriction morphism of a Galois automorphism $\rho_1 \in \mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ to $\mathrm{Gal}(\mathbb{Q}(\zeta_4)/\mathbb{Q})$ is given by reduction modulo 4.

In summary, we get:

$$\begin{aligned} \varrho_1((0,-1,-1)) \;&=\; (2,-3,-1)\;, \qquad \varrho_1((2,-3,-1)) \;=\; (0,-1,-1)\;, \\ \varrho_1((4,-5,-1)) \;&=\; (6,-7,-1)\;, \qquad \varrho_1((6,-7,-1)) \;=\; (4,-5,-1)\;, \end{aligned} \tag{4.4.48}$$

with the representation $(1,-2,-1)$ being invariant in the last three factors.

Let us now determine the action of $\rho_2 \in \mathrm{Gal}(\mathbb{Q}(S_{\alpha_\psi})/\mathbb{Q})$. First, observe that ρ_2 acts trivially on ζ_4, and thus on the last three factors in the product of S-matrices. Therefore, its action on the full representation has the form

$$\rho_2((\mathbf{l},\mathbf{m},\mathbf{s})_1) \;=\; \varrho_2((l_1,m_1,s_1))\otimes\varrho_2((l_2,m_2,s_2))\otimes(l_3,m_3,s_3)\cdots\otimes(l_5,m_5,s_5)_1\;. \tag{4.4.49}$$

Following the same steps as in eq. (4.4.46), we find that for representations with trivial stabiliser group $\mathrm{Stab}_{\mathcal{A}_{\mathrm{SUSY}}}((\mathbf{l},\mathbf{m},\mathbf{s})_\psi)$, the Galois automorphism ρ_2 acts via

$$l' = 5l+4 \mod 8\,, \tag{4.4.50}$$

leading to the transformations:

$$\begin{aligned} \varrho_2((0,-1,-1)) &= (4,-5,-1)\;, & \varrho_2((4,-5,-1)) &= (0,-1,-1)\;,\\ \varrho_2((2,-3,-1)) &= (6,-7,-1)\;, & \varrho_2((6,-7,-1)) &= (2,-3,-1)\;. \end{aligned} \tag{4.4.51}$$

Case 2: $\mathbb{Q}(S_{\alpha_\psi}) = \mathbb{Q}(i)$

Now, the Galois action reduces to complex conjugation. Repeating the same analysis as above, we determine its action on the representations. For the first two factors, we find:

$$\varrho((1,-2,-1)) \;=\; (5,-6,-1)\;,\qquad \varrho((5,-6,-1)) \;=\; (1,-2,-1)\;, \tag{4.4.52}$$

with the representation $(3,-4,-1)$ being invariant, while for the remaining three factors, we obtain:

$$\varrho((0,-1,-1)) \;=\; (2,-3,-1)\;,\qquad \varrho((2,-3,-1)) \;=\; (0,-1,-1)\;, \tag{4.4.53}$$

with the representation $(1,-2,-1)$ being invariant under this action.

If $\mathrm{Stab}_{\mathcal{A}_{\mathrm{SUSY}}}((\mathbf{l},\mathbf{m},\mathbf{s})_\psi)$ is non-trivial, we first observe that the generalised S-matrix (4.4.36) continues to reduce to an ordinary product of minimal model S-matrices, provided the boundary labels $(\mathbf{L},\mathbf{M},\mathbf{S})_\Psi$ do not correspond to fixed-point representations. The only derivation arises when considering boundary labels corresponding to fixed-point representations. As seen from the form of the generalised S-matrix (4.4.36), such cases introduce, in addition to the usual product of S-matrices from the first two minimal model factors, an extra term of the form

$$(\zeta_8)^{-2M_1}(\zeta_8)^{-2M_2}\Psi(L)\psi(L)\,, \tag{4.4.54}$$

where we have used the fact that fixed-point representations are characterised by the label $m_1 = m_2 = -4$, a property that also holds for the corresponding boundary labels M_1 and M_2.

Importantly, the extra term involving the $\mathbb{Z}_2$-characters remains invariant under all Galois automorphisms. It follows that the fixed-point representations transform under the Galois group in exactly the same way as the non-fixed-point representations, i.e., according to the rules in (4.4.53), with the $\mathbb{Z}_2$-character and the first two tensor factors remaining fixed.

Recalling the discussion around eq. (4.4.29), we note once more that any Galois automorphism in $\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ preserves the algebraic relation

$$\prod_{i=1}^{2} \zeta_8^{l_1+1} \prod_{j=3}^{5} \zeta_4^{l_j+1} = 1 \,, \tag{4.4.55}$$

which ensures that the resulting states are RR–ground states of the simple current extended theory.

This concludes the proof that the Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}})$ is of CM-type, with polarisation induced by the open string Witten index, as described in (4.2.46) with trivial twist operator $\mathcal{V}$. More explicitly, based on the previously determined Galois actions on the states, the Hodge structure splits into 27 Hodge substructures $W_{\mathbb{Q}}^{\alpha_\psi}$, each associated with a representation $\alpha_\psi \in \mathcal{R}$ for which the number field is $\mathbb{Q}(S_{\alpha_\psi}) = \mathbb{Q}(\zeta_8)$. In these cases, the Hodge endomorphism algebra restricted to each Hodge substructure is isomorphic to $\mathbb{Q}(\zeta_8)$. Additionally, there are 33 Hodge substructures corresponding to representations with $\mathbb{Q}(S_{\alpha_\psi}) = \mathbb{Q}(\mathrm{i})$, where the Hodge endomorphism algebra is isomorphic to $\mathbb{Q}(\mathrm{i})$.

Chapter 5

Conclusions

In this thesis, we have explored the deep interplay between two-dimensional rational conformal field theories, Hodge theory, complex multiplication (CM), and number theory. The work presented here has yielded two main results:

- A *number-theoretic interpretation of one-loop partition functions of toroidal rational conformal field theories*, showing that in the case of even-dimensional flat tori these can be expressed explicitly in terms of a finite sum of products of ray class theta functions associated with imaginary quadratic number fields. Concrete formulas are derived.
- A *generalisation of the established correspondence between $\mathcal{N} = (2,2)$ toroidal rational conformal field theories and CM-type Hodge structures* to a broader class of exactly solvable $\mathcal{N} = (2,2)$ rational superconformal field theories. Specifically, it is worked out in detail for which class of $\mathcal{N} = (2,2)$ rational conformal field theories one can explicitly associate a rational Hodge structure, and under what conditions the presence of a Galois group endows this rational Hodge structure with sufficiently many complex multiplications. This construction is exemplified by Gepner models and A-type $\mathcal{N} = (2,2)$ minimal models.

In what follows, we reflect on these results, discuss their broader significance, and identify several promising directions for future research.

Arithmetic Structures in Toroidal Rational Conformal Field Theories

We have derived an explicit expansion of the partition function of a toroidal rational conformal field theory in terms of a finite sum of products of generalised theta functions; see eqs. (3.3.36) and (3.3.10). Through the known relation between generalised theta functions and ray class theta functions associated with quadratic imaginary number fields (see eq. (3.5.22)), we have thereby provided a fully number-theoretic interpretation of the partition function of (even-dimensional) toroidal rational conformal field theories. This construction establishes an explicit dictionary between a consistent spectrum of a toroidal rational conformal field theory and a set of ray classes, together with a gluing prescription that determines how the corresponding ray class theta functions must be combined to ensure modular invariance. The exponents appearing in the ray class theta functions correspond to scaled norms of integral ideals in ray classes of the ring of integers of a fixed imaginary quadratic field with prescribed conductor. These are mapped by eq. (3.5.22) and eq. (3.3.36) to the conformal weights of the Virasoro primary fields in the spectrum of the toroidal rational conformal field theory. This construction not only gives a new arithmetic interpretation of rational toroidal partition functions but also raises several intriguing follow-up questions.

First, in this work, we have presented an explicit expansion of the partition function in terms of products of ray class theta functions associated with number fields of the lowest possible degree — namely, quadratic imaginary number fields of degree two. From a broader perspective, and guided by the connection between $\mathcal{N} = (2, 2)$ toroidal rational conformal field theories and Hodge structures of CM-type, it is natural to speculate that, for rational conformal field theories on tori of real dimension greater than two, the partition function can similarly be expanded in terms of ray class theta functions associated with number fields of higher degree. Such a generalisation would reflect the richer arithmetic structure expected in higher-dimensional cases. However, it must be emphasised that the relevant aspects of class field theory become significantly more intricate as the degree of the number field increases.

Nevertheless, given that the contribution to the partition function arising solely from the generators of the maximally extended chiral algebra always admits a representation in terms of theta functions, it would be of particular interest to systematically work out the ray class theta function expansion with respect to a number field of the highest possible degree. This might reveal a deep link between all possible subalgebras for which the toroidal rational conformal field theory is still rational, and subfields of a number field of maximal degree.

Finally, from a more general physics perspective, it would be intriguing to investigate whether the correspondence between ray class theta functions and partition functions, demonstrated here for toroidal rational conformal field theories, can be generalised to encompass a broader class of rational conformal field theories. One may also ask whether other relevant conformal field theoretic objects, such as conformal blocks, admit analogous expressions in terms of arithmetic functions. For a discussion of the conformal field theoretic interpretation of weight 2 modular forms and their associated L-functions in the context of $\mathcal{N} = (2, 2)$ superconformal field theories on elliptic curves, see for example [203, 204], and [205, 206] for a discussion of automorphic forms in special K3-models.

Complex Multiplication and $\mathcal{N} = (2,2)$ Rational Superconformal Field Theories

The second main result of this thesis is the extension of the correspondence between certain $\mathcal{N} = (2,2)$ rational superconformal field theories and Hodge structures of complex multiplication (CM) type beyond the class of toroidal theories. We have constructed, for any $\mathcal{N} = (2,2)$ conformal field theory, an associated real Hodge structure by identifying the U(1)-charges of Ramond–Ramond (RR) ground states with the Hodge grading. Under the assumption that a suitable collection of Cardy boundary states spans the RR–charge lattice, we demonstrated how this real Hodge structure acquires a rational structure, yielding a well-defined rational Hodge structure.

In theories with integral U(1)-charges, such as GSO-projected worldsheet theories relevant to superstring compactifications, we explicitly constructed a polarisation, using the open string Witten index. In theories with rational U(1)-charges, we introduced a twisted open string Witten index that often gives rise to a polarisation. Even in cases where this twisted index does not yield a polarisation, we have shown — using the example of the $\mathcal{N} = (2,2)$ minimal model at level $k = 4$ — that the associated Hodge structures may still admit a polarisation by other means.

A key component in our construction is the identification of a Galois symmetry acting on the modular S-matrix of the theory, which we reviewed in detail in section 2.3. Assuming this Galois action closes on the set of RR–ground states — see eq. (4.2.65) — we showed that the resulting rational Hodge structure admits sufficiently many Hodge endomorphisms to be of CM-type, provided it is polarisable. This framework was exemplified using the diagonal A-series of $\mathcal{N} = (2,2)$ minimal models, the Gepner model $(5,5,5,5,5)$ corresponding to the quintic Calabi–Yau threefold, and the Gepner model $(6,6,2,2,2)$ associated with a hypersurface in the weighted projective space $\mathbb{P}^4_{1,1,2,2,2}[8]$. In the latter example, where standard Cardy states do not suffice to span the RR–charge lattice, we employed a well-known generalised S-matrix formalism, demonstrating the continued validity of the construction under more general conditions.

While these results significantly broaden the class of rational theories for which the CM-type correspondence applies, several simplifying assumptions were made in the construction. Most notably, we focused primarily on theories with diagonal modular invariants and conventional Cardy boundary state constructions. This was motivated by the technical tractability and the well-understood structure of Galois symmetries in these cases. However, our treatment of the $(6,6,2,2,2)$ Gepner model indicates that these assumptions can be relaxed. For models with non-trivial simple current fixed-point groups, especially those beyond $\mathbb{Z}_2$, a more detailed analysis of boundary state resolutions is necessary, and in particular, how these alter the presented construction. Similarly, theories with non-diagonal modular invariants — which introduce additional complexity in the explicit realisation of boundary states — remain to be systematically addressed.

Our results demonstrate that one direction of the conjecture proposed in [94] holds for a large class of $\mathcal{N} = (2,2)$ rational superconformal field theories: namely, that $\mathcal{N} = (2,2)$ rational conformal field theories give rise to Hodge structures of CM-type, even when no geometric interpretation as a non-linear sigma model IR fixed point is known. This raises the question whether *all* $\mathcal{N} = (2,2)$ rational superconformal field theories possess associated A- and B-type Hodge structures of CM-type. We already observed in section 4.1 that the converse direction of the Gukov–Vafa conjecture fails in general. There exist non-rational $\mathcal{N} = (2,2)$ superconformal field theories associated with isogenous mirror pairs of CM-type abelian varieties. Recent advances in the study of T^4 targets [167, 168] have led to important refinements of the original conjecture. However, a comprehensive understanding of the interplay between CM-type A- and B-type Hodge structures remains elusive.

In the introduction, we motivated our investigation of the conjectural correspondence between $\mathcal{N} = (2,2)$ rational conformal field theories and Hodge structures of CM-type by posing the question of how rational theories are distributed within the conformal moduli space. From the preceding discussion and given the subtlety involved in formulating a precise version of the correspondence, no definitive conclusion about the distribution of rational theories can currently be drawn based on the distribution of Hodge structures of CM-type.

Nevertheless, it is worth noting that there exist significant mathematical results concerning the distribution of arithmetically special Hodge structures in moduli spaces. In particular, the André–Oort conjecture [207, 208], now a theorem due to the proof in [209, 210], asserts that every *special subvariety* of a *Shimura variety* contains a Zariski dense set of arithmetically *special points*. In particular, the converse also holds. A prototypical example of a Shimura variety is the Siegel moduli space of principally polarised abelian varieties. In this setting, the special subvarieties — which parameterise "non-generic" Hodge structures — contain a Zariski dense set of CM points, corresponding to abelian varieties of CM-type. For moduli spaces that are not of Shimura type, little is known, and it is generally expected that arithmetically special points occur only as isolated points.

Outlook

Future work should address the role of *non-diagonal boundary constructions*, as these may introduce additional symmetries or lead to modifications in the correspondence between $\mathcal{N} = (2,2)$ rational superconformal field theories and CM-type Hodge structures. In particular, it remains an open question whether the CM property of the associated Hodge structures persists — or how it might be altered — in the presence of more general boundary conditions. This is especially relevant in theories involving more intricate *fixed-point resolutions*, such as those arising from simple current extensions with fixed-point groups larger than $\mathbb{Z}_2$, or in theories with *exceptional modular invariants*.

Another promising direction concerns the arithmetic interpretation of $\mathcal{N} = (2,2)$ minimal models, and more general rational coset theories [211]. These models, which represent some of the most fundamental examples of $\mathcal{N} = (2,2)$ rational superconformal field theories, can be interpreted as elementary constituents in the construction of more intricate rational theories, such as Gepner models. From the perspective of arithmetic geometry, it is natural to speculate that $\mathcal{N} = (2, 2)$ minimal models (and their generalisations) may correspond to *motives*. This idea was first proposed in [203, 212–214]. The theory of motives, a still largely conjectural framework, aims to unify the various cohomological theories that can be associated with algebraic varieties. The category of pure motives over $\mathbb{Q}$ is envisioned as consisting of the universal building blocks for such algebraic varieties, providing a conceptual foundation for understanding the emergence of geometry from algebraic data. In this light, the appearance of motive-like structures in $\mathcal{N} = (2, 2)$ minimal models could offer insights into how space-time geometry may emerge from the worldsheet perspective in string theory. This also suggests a broader principle whereby the worldsheet conformal field theory encodes not just topological or geometric data, but potentially deep arithmetic structures as well.

Finally, it is worth highlighting that the notion of rationality in worldsheet theories in string theory and its relation to non-perturbative string dualities remains largely unexplored. As noted in [94], using S-duality in type IIB string theory, the roles of fundamental strings and D-strings are interchanged. However, the rationality property of the associated worldsheet theory appears to be independent of the background values of fields that couple to the D-strings, and therefore is not necessarily preserved under such dualities. In the context of heterotic–type IIA duality, a natural setting to explore these ideas is the duality between heterotic string theory on $K3 \times T^2$ and type IIA string theory on a $K3$-fibered Calabi–Yau threefold [215]. Understanding under which conditions both sides of such a duality possess rational worldsheet theories could lead to significant insights into the algebraic structure of string dualities. Moreover, it would be valuable to explore how the presence of extended chiral algebras in the worldsheet theory manifests as enhanced symmetries in the effective target space description.

Another intriguing extension is to study the rationality and potential arithmetic structure of $\mathcal{N} = 1$ rational conformal field theories, particularly in relation to compactifications on G_2 manifolds [216–218]. In such settings, where Hodge structures are no longer available, it remains an open question whether analogous arithmetic invariants can be defined and linked to rationality, or whether entirely new conceptual frameworks will be required.

In conclusion, this work provides indications of deep arithmetic structures underlying exactly solvable supersymmetric conformal field theories, suggesting a rich landscape for further investigation at the intersection of number theory, geometry, quantum field theory, and string theory.

Appendices

Appendix A

Elliptic Curves of Complex Multiplication Type

In the following we want to prove the claim made in section 3.3 that elliptic curves are of complex multiplication type iff the complex structure parameter τ is valued in an imaginary quadratic field extension $\mathbb{Q}(\sqrt{-D'})$, for some positive square-free integer D'. For an in-depth discussion of arithmetic aspects in the theory of elliptic curves, we refer to [219, 220].

Recall that any elliptic T^2_τ is isomorphic to the lattice quotient

$$T^2_\tau \cong \mathbb{C}/\Lambda, \tag{A.0.1}$$

with $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ and $\tau \in \mathbb{H}$. We denote by $\{A, B\}$ a symplectic basis of $H_1(T^2_\tau, \mathbb{Z})$ and by $\{a, b\}$ a basis of the Poincaré dual space $H^1(T^2_\tau, \mathbb{Z})$. The periods are defined as

$$\omega_1 := \int_A dz\,, \qquad \omega_2 := \int_B dz\,, \tag{A.0.2}$$

where z denotes coordinates on $\mathbb{C}$. From (A.0.2) it follows that $dz = \omega_1 a + \omega_2 b$. The complex structure parameter τ is given by

$$\tau = \frac{\omega_2}{\omega_1}\,. \tag{A.0.3}$$

We will now construct the rational Hodge endomorphism algebra. Any endomorphism on $H^1(T^2_\tau, \mathbb{Q})$ can be described by a $\mathbb{Q}$-valued matrix $M \in \mathrm{Mat}(2{\times}2, \mathbb{Q})$. Recall from (4.2.8) that an endomorphism is a Hodge endomorphism if it preserves the Hodge decomposition. Writing the holomorphic one-form dz/ω_1, which spans the space $H^{1,0}(T^2_\tau, \mathbb{C})$, as a vector in the basis $\{a, b\}$, the condition for M to preserve the Hodge decomposition reads

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix} = C \begin{pmatrix} 1 \\ \tau \end{pmatrix} \qquad C \in \mathbb{C}\,, \tag{A.0.4}$$

or equivalently

$$a' - C + b'\tau \;=\; 0 \;, \qquad c' - C\tau + d'\tau \;=\; 0 \;, \tag{A.0.5}$$

from which it follows that

$$b'\tau^2 + (a' - d')\tau - c' = 0 \,, \tag{A.0.6}$$

which implies for the complex structure parameter τ

$$\tau = \frac{d' - a' \pm \sqrt{D}}{2b'} \,, \qquad D = (a' - d')^2 + 4b'c' \,. \tag{A.0.7}$$

Since $\tau \in \mathbb{H}$, the complex structure parameter has strictly positive imaginary part, and hence for non-trivial Hodge endomorphisms to exist, τ must be valued in an imaginary quadratic field $\mathbb{Q}(\sqrt{D})$, with D a negative integer.

Let us now study the structure of the rational Hodge endomorphism algebra. Since the rational endomorphism algebra of an abelian variety is preserved under isogenies, we can assume without loss of generality that $\tau = \mathrm{i}\sqrt{D'}$ for some positive square-free integer D'. Then, from (A.0.5) it directly follows,

$$\mathrm{M} \in \left\{ \begin{pmatrix} a' & b' \\ -D'b' & a' \end{pmatrix} \,\middle|\, a', b' \in \mathbb{Q} \right\} \;\cong\; \mathrm{End}_{\mathrm{Hdg}}(H^1(T^2_\tau, \mathbb{Q})) \;, \tag{A.0.8}$$

with the algebra isomorphism

$$\mathrm{End}_{\mathrm{Hdg}}(H^1(T^2_\tau, \mathbb{Q})) \cong \mathbb{Q}(\sqrt{-D'}) \;, \qquad \begin{pmatrix} a' & b' \\ -D'b' & a' \end{pmatrix} \mapsto a' + b'\sqrt{-D'} \;. \tag{A.0.9}$$

A polarisation for the Hodge structure $H^1(T^2_\tau, \mathbb{Q})$ that satisfies conditions (4.2.5) – (4.2.7), is given by

$$Q(v, w) \;:=\; \mathrm{i} \int_{T^2_\tau} v \wedge w \;. \tag{A.0.10}$$

Hence, by the definition given in section 4.2, we have shown that $H^1(T^2_\tau, \mathbb{Q})$ is of CM-type iff the complex structure parameter τ is valued in some quadratic imaginary number field $\mathbb{Q}(-D')$ for some positive square-free integer D'. The Hodge endomorphism algebra in that case is isomorphic to $\mathbb{Q}(-D')$, with the isomorphism given by (A.0.9).

Appendix B

Rational Basis for the Gepner Model $(3, 3, 3, 3, 3)$

In the following table we list a basis of boundary states for the rational vector space $V_{\mathbb{Q}}$ for the Gepner model $(3, 3, 3, 3, 3)$, corresponding to the Fermat quintic threefold.

Table B.1: Rational basis of boundary states for the Gepner model (3, 3, 3, 3, 3)

	Boundary state				
1.	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
2.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|3,-1,0\rangle$	$\|3,1,0\rangle$
3.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-1,0\rangle$	$\|3,-3,0\rangle$	$\|3,-1,0\rangle$
4.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$
5.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$
6.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|3,-1,0\rangle$	$\|3,-3,0\rangle$
7.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|3,-1,0\rangle$	$\|0,0,0\rangle$
8.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|3,1,0\rangle$
9.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-1,0\rangle$	$\|3,-3,0\rangle$	$\|3,3,0\rangle$
10.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$
11.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|3,3,0\rangle$	$\|0,0,0\rangle$
12.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-1,0\rangle$	$\|1,-1,0\rangle$	$\|3,-1,0\rangle$
13.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$
14.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,1,0\rangle$	$\|3,-1,0\rangle$	$\|1,-1,0\rangle$
15.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$
16.	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$
17.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|3,1,0\rangle$	$\|3,1,0\rangle$
18.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$
19.	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
20.	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$
21.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$
22.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,1,0\rangle$	$\|3,-3,0\rangle$	$\|3,-1,0\rangle$
23.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|3,-1,0\rangle$	$\|3,1,0\rangle$
24.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$
25.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$
26.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$
27.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$
28.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$
29.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|3,1,0\rangle$
30.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|3,-1,0\rangle$	$\|3,-3,0\rangle$
31.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,3,0\rangle$	$\|0,0,0\rangle$	$\|3,3,0\rangle$
32.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,3,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
33.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|3,-1,0\rangle$	$\|0,0,0\rangle$
34.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|3,-1,0\rangle$	$\|0,0,0\rangle$
35.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|3,-1,0\rangle$	$\|0,0,0\rangle$
36.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|3,1,0\rangle$
37.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-1,0\rangle$	$\|3,3,0\rangle$	$\|1,-1,0\rangle$
38.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|3,1,0\rangle$	$\|3,-3,0\rangle$
39.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|3,1,0\rangle$	$\|3,-3,0\rangle$
40.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$	$\|3,1,0\rangle$	$\|3,-3,0\rangle$

	Boundary state				
41.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|3,1,0\rangle$	$\|3,-3,0\rangle$
42.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|3,1,0\rangle$	$\|3,-3,0\rangle$
43.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|3,3,0\rangle$
44.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-1,0\rangle$	$\|2,2,0\rangle$	$\|3,1,0\rangle$
45.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,1,0\rangle$
46.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-1,0\rangle$	$\|1,-1,0\rangle$	$\|3,3,0\rangle$
47.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
48.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|3,-3,0\rangle$	$\|3,1,0\rangle$
49.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-1,0\rangle$	$\|3,-1,0\rangle$	$\|2,2,0\rangle$
50.	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$
51.	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$
52.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$
53.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$
54.	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
55.	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
56.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
57.	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
58.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|3,3,0\rangle$	$\|0,0,0\rangle$
59.	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$
60.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
61.	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
62.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
63.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$
64.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$
65.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$
66.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|3,-1,0\rangle$
67.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|3,-1,0\rangle$	$\|1,-1,0\rangle$
68.	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$
69.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$
70.	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$
71.	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$
72.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|1,-1,0\rangle$	$\|3,1,0\rangle$
73.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$	$\|3,1,0\rangle$
74.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$
75.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$
76.	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$
77.	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$
78.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$
79.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$
80.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$

	Boundary state				
81.	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$
82.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$
83.	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$
84.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$
85.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-1,0\rangle$	$\|2,-2,0\rangle$	$\|1,1,0\rangle$
86.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|3,1,0\rangle$
87.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|3,1,0\rangle$
88.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|3,1,0\rangle$
89.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|3,1,0\rangle$	$\|1,-1,0\rangle$
90.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-1,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$
91.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$
92.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$
93.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$	$\|3,-1,0\rangle$	$\|2,-2,0\rangle$
94.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|3,-1,0\rangle$	$\|2,-2,0\rangle$
95.	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
96.	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
97.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$
98.	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$
99.	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$
100.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|1,1,0\rangle$	$\|3,-1,0\rangle$	$\|0,0,0\rangle$
101.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|3,-1,0\rangle$	$\|0,0,0\rangle$
102.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$
103.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$	$\|3,1,0\rangle$
104.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,3,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$
105.	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$
106.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$
107.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$
108.	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$
109.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
110.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
111.	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
112.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
113.	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
114.	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
115.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
116.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,3,0\rangle$	$\|3,-3,0\rangle$	$\|1,1,0\rangle$
117.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|3,1,0\rangle$
118.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$	$\|2,-2,0\rangle$
119.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$
120.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$

	Boundary state				
121.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$
122.	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$
123.	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$
124.	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$
125.	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$
126.	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$
127.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-3,0\rangle$	$\|1,1,0\rangle$	$\|2,-2,0\rangle$
128.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|3,-1,0\rangle$	$\|2,0,0\rangle$	$\|2,2,0\rangle$
129.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$	$\|3,3,0\rangle$	$\|2,0,0\rangle$
130.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|3,3,0\rangle$	$\|0,0,0\rangle$
131.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|1,1,0\rangle$	$\|3,3,0\rangle$	$\|0,0,0\rangle$
132.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
133.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
134.	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
135.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
136.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
137.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,-2,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$
138.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$	$\|3,3,0\rangle$	$\|2,-2,0\rangle$
139.	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
140.	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
141.	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$
142.	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$
143.	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$
144.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$
145.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$
146.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$
147.	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$
148.	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$
149.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|2,-2,0\rangle$	$\|3,1,0\rangle$
150.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|1,1,0\rangle$	$\|2,-2,0\rangle$	$\|3,1,0\rangle$
151.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|3,1,0\rangle$	$\|1,-1,0\rangle$
152.	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
153.	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
154.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
155.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,1,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$
156.	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
157.	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
158.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$
159.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,2,0\rangle$	$\|3,3,0\rangle$	$\|2,2,0\rangle$
160.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|2,2,0\rangle$	$\|2,-2,0\rangle$

	Boundary state				
161.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,3,0\rangle$	$\|1,1,0\rangle$	$\|2,2,0\rangle$
162.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$
163.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$
164.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$
165.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,2,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$
166.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$
167.	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$
168.	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$
169.	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
170.	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
171.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-3,0\rangle$	$\|2,0,0\rangle$	$\|2,-2,0\rangle$
172.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,0,0\rangle$	$\|2,-2,0\rangle$
173.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|2,0,0\rangle$	$\|2,-2,0\rangle$
174.	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
175.	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
176.	$\|3,1,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|1,1,0\rangle$	$\|2,-2,0\rangle$
177.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|3,-1,0\rangle$	$\|1,1,0\rangle$	$\|2,-2,0\rangle$
178.	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
179.	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
180.	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
181.	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
182.	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$
183.	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$
184.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|1,1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
185.	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
186.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
187.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$	$\|1,1,0\rangle$
188.	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$
189.	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$
190.	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
191.	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
192.	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
193.	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
194.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|1,1,0\rangle$	$\|2,2,0\rangle$	$\|2,-2,0\rangle$
195.	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
196.	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
197.	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$
198.	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$
199.	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|3,-3,0\rangle$
200.	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
201.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$	$\|2,-2,0\rangle$
202.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$	$\|2,-2,0\rangle$
203.	$\|3,1,0\rangle$	$\|2,2,0\rangle$	$\|2,-2,0\rangle$	$\|1,1,0\rangle$	$\|2,-2,0\rangle$
204.	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$

Appendix C

Chiral States and Rational Basis for the Gepner Model $(6, 6, 2, 2, 2)$

In the following tables, we list the chiral states for the Gepner model $(2, 2, 2, 6, 6)$, and a basis of boundary states for the rational vector space $V_{\mathbb{Q}}$. The first column of the first table gives the chiral state up to a permutation of the factors of the tensor product. The second column lists the Hodge type of the corresponding state, and the third column indicates how many distinct chiral states can be obtained by permutations of the displayed state. Each resolved fixed-point state is labeled by a $\mathbb{Z}_2$ group character of the stabiliser subgroup associated with the simple current group, taking values $+1$ and -1. The second table lists explicitly a basis of boundary states.

Chiral State	$V_{\mathbb{C}}^{p,q}$	Number
$\|0,0,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{12}$	3
$\|0,0,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{12}$	3
$\|0,0,0\rangle\ \|2,-2,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{12}$	12
$\|0,0,0\rangle\ \|2,-2,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle$	$V_{\mathbb{C}}^{12}$	2
$\|0,0,0\rangle\ \|4,-4,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{12}$	6
$\|0,0,0\rangle\ \|4,-4,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle$	$V_{\mathbb{C}}^{12}$	6
$\|0,0,0\rangle\ \|6,-6,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle$	$V_{\mathbb{C}}^{12}$	6
$\|1,-1,0\rangle\ \|1,-1,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{12}$	6
$\|1,-1,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle$	$V_{\mathbb{C}}^{12}$	1
$\|1,-1,0\rangle\ \|3,-3,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{12}$	6
$\|1,-1,0\rangle\ \|3,-3,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle$	$V_{\mathbb{C}}^{12}$	6
$\|1,-1,0\rangle\ \|5,-5,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle$	$V_{\mathbb{C}}^{12}$	6
$\|2,-2,0\rangle\ \|2,-2,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{12}$	3
$\|2,-2,0\rangle\ \|2,-2,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle$	$V_{\mathbb{C}}^{12}$	3
$\|2,-2,0\rangle\ \|4,-4,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle$	$V_{\mathbb{C}}^{12}$	6
$\|2,-2,0\rangle\ \|6,-6,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle$	$V_{\mathbb{C}}^{12}$	2
$\|3,-3,0\rangle\ \|3,-3,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle_+$	$V_{\mathbb{C}}^{12}$	3
$\|3,-3,0\rangle\ \|3,-3,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle_-$	$V_{\mathbb{C}}^{12}$	3
$\|3,-3,0\rangle\ \|5,-5,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle$	$V_{\mathbb{C}}^{12}$	2
$\|4,-4,0\rangle\ \|4,-4,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle$	$V_{\mathbb{C}}^{12}$	1
$\|0,0,0\rangle\ \|4,-4,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	2
$\|0,0,0\rangle\ \|6,-6,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	6
$\|1,-1,0\rangle\ \|3,-3,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	2
$\|1,-1,0\rangle\ \|5,-5,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	6
$\|2,-2,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	1
$\|2,-2,0\rangle\ \|4,-4,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	6
$\|2,-2,0\rangle\ \|6,-6,0\rangle\ \|0,0,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	6
$\|2,-2,0\rangle\ \|6,-6,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	6
$\|3,-3,0\rangle\ \|3,-3,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle_+$	$V_{\mathbb{C}}^{21}$	3
$\|3,-3,0\rangle\ \|3,-3,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle_-$	$V_{\mathbb{C}}^{21}$	3
$\|3,-3,0\rangle\ \|5,-5,0\rangle\ \|0,0,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	6
$\|3,-3,0\rangle\ \|5,-5,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	6
$\|4,-4,0\rangle\ \|4,-4,0\rangle\ \|0,0,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	3
$\|4,-4,0\rangle\ \|4,-4,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	3
$\|4,-4,0\rangle\ \|6,-6,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	12
$\|4,-4,0\rangle\ \|6,-6,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle$	$V_{\mathbb{C}}^{21}$	2
$\|5,-5,0\rangle\ \|5,-5,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	6
$\|5,-5,0\rangle\ \|5,-5,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle$	$V_{\mathbb{C}}^{21}$	1
$\|6,-6,0\rangle\ \|6,-6,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{21}$	3
$\|6,-6,0\rangle\ \|6,-6,0\rangle\ \|0,0,0\rangle\ \|1,-1,0\rangle\ \|1,-1,0\rangle$	$V_{\mathbb{C}}^{21}$	3
$\|6,-6,0\rangle\ \|6,-6,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle\ \|2,-2,0\rangle$	$V_{\mathbb{C}}^{03}$	1
$\|0,0,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle\ \|0,0,0\rangle$	$V_{\mathbb{C}}^{30}$	1

Table C.1: *The chiral states of the Gepner model* $(6,6,2,2,2)$.

Table C.2: Rational basis of boundary states for the Gepner model $(6, 6, 2, 2, 2)$

	Boundary state				
1.	$\|3,-3,0\rangle$	$\|5,-5,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
2.	$\|5,-5,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
3.	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
4.	$\|5,-5,0\rangle$	$\|5,-5,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
5.	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
6.	$\|5,-5,0\rangle$	$\|5,-5,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
7.	$\|3,-3,0\rangle$	$\|5,-5,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
8.	$\|5,-5,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
9.	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
10.	$\|5,-5,0\rangle$	$\|5,-5,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
11.	$\|5,-5,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
12.	$\|5,-5,0\rangle$	$\|5,-5,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
13.	$\|5,-5,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
14.	$\|5,-5,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
15.	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
16.	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
17.	$\|3,-3,0\rangle$	$\|5,-5,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
18.	$\|5,-5,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
19.	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
20.	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
21.	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
22.	$\|1,-1,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
23.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle_+$
24.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle_+$
25.	$\|3,-3,0\rangle$	$\|5,-5,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
26.	$\|5,-5,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
27.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle_+$
28.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle_+$
29.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle_+$
30.	$\|3,-3,0\rangle$	$\|3,-3,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle_+$
31.	$\|4,-4,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
32.	$\|2,-2,0\rangle$	$\|4,-4,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
33.	$\|2,-2,0\rangle$	$\|4,-4,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
34.	$\|4,-4,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$
35.	$\|2,-2,0\rangle$	$\|4,-4,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
36.	$\|4,-4,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$
37.	$\|4,-4,0\rangle$	$\|4,-4,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
38.	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
39.	$\|4,-4,0\rangle$	$\|4,-4,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$
40.	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$

	Boundary state				
41.	$\|4,-4,0\rangle$	$\|4,-4,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
42.	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
43.	$\|4,-4,0\rangle$	$\|4,-4,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
44.	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
45.	$\|4,-4,0\rangle$	$\|6,-6,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
46.	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
47.	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
48.	$\|4,-4,0\rangle$	$\|6,-6,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
49.	$\|4,-4,0\rangle$	$\|6,-6,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
50.	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
51.	$\|4,-4,0\rangle$	$\|6,-6,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
52.	$\|6,-6,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
53.	$\|6,-6,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
54.	$\|4,-4,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
55.	$\|6,-6,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$
56.	$\|4,-4,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$
57.	$\|6,-6,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
58.	$\|4,-4,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
59.	$\|6,-6,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
60.	$\|4,-4,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
61.	$\|6,-6,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$
62.	$\|6,-6,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
63.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
64.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$
65.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|1,-1,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$
66.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|2,0,0\rangle$
67.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|2,0,0\rangle$
68.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$
69.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|2,-2,0\rangle$	$\|2,0,0\rangle$
70.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|1,1,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$
71.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|1,1,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$
72.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|1,1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
73.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
74.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$	$\|2,0,0\rangle$
75.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,0,0\rangle$
76.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
77.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,0,0\rangle$
78.	$\|6,-6,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
79.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
80.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$

	Boundary state				
81.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
82.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,0,0\rangle$
83.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
84.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
85.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$
86.	$\|0,0,0\rangle$	$\|6,-6,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
87.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
88.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
89.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,-2,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$
90.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,-2,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$
91.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,-2,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$
92.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,0,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$
93.	$\|6,-6,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
94.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$
95.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$
96.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,2,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$
97.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,0,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$
98.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$
99.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$
100.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,0,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$
101.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$
102.	$\|6,-6,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$
103.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$
104.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$
105.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$
106.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$
107.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,0,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$
108.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$
109.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
110.	$\|6,-6,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$
111.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
112.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
113.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
114.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
115.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|2,0,0\rangle$
116.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,2,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$
117.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,-2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
118.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,0,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$
119.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,0,0\rangle$
120.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,0,0\rangle$

	Boundary state				
121.	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$
122.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$
123.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$
124.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
125.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,2,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$
126.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$
127.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,-2,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$
128.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$
129.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,-2,0\rangle$	$\|2,2,0\rangle$	$\|0,0,0\rangle$
130.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
131.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$
132.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$
133.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
134.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$
135.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$
136.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
137.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|2,-2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$
138.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$
139.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,2,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$
140.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,-2,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$
141.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,-2,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$
142.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,-2,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$
143.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$
144.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$
145.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|2,0,0\rangle$
146.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$	$\|0,0,0\rangle$
147.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$
148.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$	$\|2,0,0\rangle$
149.	$\|6,-6,0\rangle$	$\|6,-6,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
150.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
151.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$
152.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
153.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$
154.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|0,0,0\rangle$
155.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|1,1,0\rangle$	$\|1,-1,0\rangle$	$\|2,0,0\rangle$
156.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$
157.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$
158.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$
159.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$	$\|1,-1,0\rangle$
160.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$	$\|1,-1,0\rangle$

	Boundary state				
161.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|1,-1,0\rangle$	$\|2,2,0\rangle$	$\|1,-1,0\rangle$
162.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$
163.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|1,-1,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$
164.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$
165.	$\|6,2,0\rangle$	$\|6,2,0\rangle$	$\|1,-1,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$
166.	$\|0,0,0\rangle$	$\|0,0,0\rangle$	$\|1,-1,0\rangle$	$\|1,-1,0\rangle$	$\|2,-2,0\rangle$
167.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$
168.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|1,1,0\rangle$	$\|1,1,0\rangle$	$\|2,0,0\rangle$
169.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|1,1,0\rangle$	$\|1,1,0\rangle$	$\|0,0,0\rangle$
170.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,0,0\rangle$	$\|1,1,0\rangle$	$\|1,-1,0\rangle$
171.	$\|6,2,0\rangle$	$\|6,4,0\rangle$	$\|2,0,0\rangle$	$\|1,1,0\rangle$	$\|1,-1,0\rangle$
172.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,-2,0\rangle$	$\|1,1,0\rangle$	$\|1,-1,0\rangle$
173.	$\|6,2,0\rangle$	$\|6,6,0\rangle$	$\|2,2,0\rangle$	$\|1,1,0\rangle$	$\|1,-1,0\rangle$
174.	$\|6,2,0\rangle$	$\|0,0,0\rangle$	$\|2,2,0\rangle$	$\|1,1,0\rangle$	$\|1,-1,0\rangle$

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Declaration

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