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PAPER

Speed limits of the trace distance for open quantum system

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Abstract

We investigate the speed limit of the state transformation in open quantum systems described by the Lindblad type quantum master equation. We obtain universal bounds of the total entropy production described by the trace distance between the initial and final states in the interaction picture. Our bounds can be tighter than the bound of Vu and Hasegawa (2021 *Phys. Rev. Lett.* **126** 010601) which measures the distance by the eigenvalues of the initial and final states: this distance is less than or equal to the trace distance. For this reason, our results can significantly improve Vu–Hasegawa's bound. The trace distance in the Schrödinger picture is bounded by a sum of the trace distance in the interaction picture and the trace distance for unitary dynamics described by only the Hamiltonian in the quantum master equation.

1. Introduction

In recent years, studies of time-dependent open systems have been active [1]. These studies relate to quantum pumps [2, 3], excess entropy production [4–6], the information geometric approach [7, 8], the efficiency and power of heat engines [9–11], shortcuts to adiabaticity [12–16], and speed limits [16–28]. Obtaining a fundamental bound on the speed of state transformation is an important issue relevant to broad research fields including quantum control theory [29] and foundations of nonequilibrium statistical mechanics [30]. Speed limits for time-dependent closed quantum systems have been studied for more than a half-century [1]. Since 1945, the Mandelstam–Tamm relation [31] $\mathcal{L}/\int_0^\tau \mathrm{d}t\Delta E \leqslant 1$ has been known (appendix A. In this paper, we set $\hbar = 1$). Here, $\mathcal{L}$ is the distance between the initial and final states (the Bures angle, see appendix A), and ΔE is the energy fluctuation.

About a decade ago, speed limits of open quantum systems had been intensively studied by adopting various distance measures between two quantum states [17–20]. Reference [17] derived the upper bound of the Bures angle expressed by the quantum Fisher information. For systems described by the quantum master equation $\mathrm{d}\rho/\mathrm{d}t = \hat{K}(\rho)$ [$\rho(t)$ is the density operator (the state) of the system], reference [18] provided the upper bound of the relative purity $\mathrm{tr}[\rho(t)\rho(0)]/\mathrm{tr}[\rho(0)^2]$ expressed with the adjoint of the generator of the dynamical map $\hat{K}$. Reference [19] estimated Margolus–Levitin-type and Mandelstam–Tamm-type bounds by using the Bures angle for pure initial state and several norms of $\hat{K}(\rho)$. A review of quantum speed limits for closed and open systems until around 2017 is given by reference [20]. More recently, the speed limits on observables of open quantum systems are discussed [28].

Recently, even in classical systems, it turns out that there exist speed limits expressed in terms of the distance between states [21]. Remarkably, the classical speed limits connect the distance and the thermodynamic entropy. Shiraishi *et al* [21] demonstrated that

$$\sigma \geqslant \frac{l^2}{\int_0^\tau \mathrm{d}t\, 2A_\mathrm{c}(t)} \tag{1}$$

for a system described by a classical master equation $\frac{\mathrm{d}}{\mathrm{d}t}p_n(t) = \sum_m W_{nm}p_m(t)$. W_{nm} is the transition matrix satisfying the local detailed balance condition [32] and $p_n(t)$ is the probability of state n at time t. σ is the total entropy production, the distance $l := \sum_n |p_n(\tau) - p_n(0)|$ is the L^1 norm, and $A_c(t) := \sum_{n\neq m} W_{nm} p_m(t)$ is the activity (total number of transitions per unit time).

The speed limits in terms of the entropy production for the open quantum systems described by the Lindblad type quantum master equation (the Gorini–Kossakowski–Sudarshan–Lindblad equation),

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -i[H(t), \rho(t)] + \mathcal{D}(\rho(t)), \tag{2}$$

have been researched actively in recent years. Here, $H(t) := H_S(t) + H_L(t)$, where H_S is the system Hamiltonian and H_L is the Lamb shift Hamiltonian, which satisfies $[H_L(t), H_S(t)] = 0$. $\mathcal{D}(\rho)$ represents dissipation and is given by $\mathcal{D}(\rho) = \sum_k \gamma_k \hat{D}[L_k](\rho)$ with $\hat{D}[X](Y) := \left(XYX^\dagger - \frac{1}{2}X^\dagger XY - \frac{1}{2}YX^\dagger X\right)$. In this paper, X and Y denote linear operators of the system. γ_k are non-negative real numbers which describe the strength of the dissipation. The label k is a tuple (b, a, ω) where b is the label of the bath. The jump operators $L_{b,a,\omega}$ satisfy

$$[L_{b,a,\omega}, H_S] = \omega L_{b,a,\omega}\,, \quad L_{b,a,-\omega} = L^\dagger_{b,a,\omega}. \tag{3}$$

We assume the local detailed balance condition

$$\gamma_{b,a,-\omega} = e^{-\beta_b \omega}\gamma_{b,a,\omega}, \tag{4}$$

where β_b is the inverse temperature of the bath b. Note that L_k, ω, γ_k and β_b can depend on time. The total entropy production is given by $\sigma := \int_0^\tau \mathrm{d}t\dot{\sigma}$ where

$$\dot{\sigma} := -\,\mathrm{tr}\left[\frac{\mathrm{d}\rho}{\mathrm{d}t}\ \ln\,\rho\right] - \sum_b \beta_b\ \mathrm{tr}[\mathcal{D}_b(\rho)H_S] \tag{5}$$

is the entropy production rate. Here, $\mathcal{D}_b(\rho)$ denotes the contribution from the bath b of $\mathcal{D}(\rho)$.

For the system described by (2), there are two approaches to speed limits. The first approach is Funo *et al*'s approach [22], which treats the first and second terms of the right-hand side of (2) equally. Funo *et al* [22] demonstrated that

$$||\rho(\tau) - \rho(0)||_1 \leqslant c_1 + c_2 + c_3, \tag{6}$$

with

$$c_1 \leqslant 2\int_0^\tau \mathrm{d}t\Delta E, \tag{7}$$

$$c_3 \leqslant \sqrt{2\sigma\int_0^\tau \mathrm{d}t\ \mathcal{A}(t)}. \tag{8}$$

Here, $||\rho(\tau) - \rho(0)||_1$ is the trace distance and $||X||_1 := \mathrm{tr}\ \sqrt{X^\dagger X}$ is the trace norm. c_1 corresponds to the contribution from the first term of the right-hand side of (2), c_2 and c_3 correspond to the contribution from the second term of the right-hand side of (2) [33]. $\Delta E := \sqrt{\mathrm{tr}(\rho(t)H(t)^2) - [\mathrm{tr}(\rho(t)H(t))]^2}$ is the energy fluctuation. $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t) := \sum_{n\neq m} \mathcal{W}_{mn} p_n(t) \tag{9}$$

with $\mathcal{W}_{mn} := \sum_k \gamma_k |\langle m(t)|L_k|n(t)\rangle|^2$. Here, we used the spectral decomposition of $\rho(t)$:

$$\rho(t) = \sum_n p_n(t)|n(t)\rangle\langle n(t)|. \tag{10}$$

If the quantum master equation reduces to the classical master equation [34], (6) reduces to (1) because $c_1 = c_2 = 0$. For no dissipation limit $\gamma_k = 0$, (6) becomes a Mandelstam–Tamm type relation because of $c_2 = c_3 = 0$.

The second approach is Vu's approach [24, 25], which focuses on the second term of the right-hand side of (2). Vu and Hasegawa [24] demonstrated that

$$\sigma \geqslant \sigma_{\mathrm{V1}} := \frac{d_T(\rho(\tau), \rho(0))^2}{\int_0^\tau \mathrm{d}t\ 2B(t)}. \tag{11}$$

Here,

$$B(t) := \mathrm{tr}\left[\rho(t)\sum_k \gamma_k L_k^\dagger L_k\right] = \mathcal{A}(t) + \sum_n \sum_k p_n(t)\gamma_k |\langle n(t)|L_k|n(t)\rangle|^2 \tag{12}$$

corresponds to the activity [35] (a similar quantity appears in references [36, 37] in the context of decoherence times). d_T is defined by $d_T(\rho(\tau), \rho(0)) := \sum_n |b_n - a_n|$ where $\{a_n\}$ and $\{b_n\}$ are increasing eigenvalues of $\rho(0)$ and $\rho(\tau)$. For no dissipation limit, (11) is consistent because $d_T(\rho(\tau), \rho(0)) = 0$ holds with $B(t) = 0$ and $\sigma = 0$. (11) is improved as [26]

$$\sigma \geqslant \sigma_{\mathrm{V0}} := \frac{d_T(\rho(\tau), \rho(0))^2}{\int_0^\tau \mathrm{d}t\, 2M(t)} \tag{13}$$

with

$$M(t) := \sum_k \sum_{m\neq n} \Psi(a_{mn}^{(k)}, a_{nm}^{(-k)}), \tag{14}$$

$$a_{mn}^{(k)} := \gamma_k |\langle m(t)|L_k|n(t)\rangle|^2 p_n(t). \tag{15}$$

Here, we used (10) and $-k := (b, a, -\omega)$. $\Psi(\alpha, \beta)$ is the logarithmic mean of α and β given by $\Psi(\alpha, \beta) := (\beta - \alpha)/(\ln \beta/\alpha)$ $(\alpha \neq \beta)$ and $\Psi(\alpha, \alpha) := \alpha$. The relation $\Psi(\alpha, \beta) \leqslant (\alpha + \beta)/2$ leads to $M(t) \leqslant \mathcal{A}(t) \leqslant B(t)$ and thus $\sigma_{\mathrm{V0}} \geqslant \sigma_{\mathrm{V1}}$.

For a system of which Hilbert space is d-dimensional, Vu and Saito [25] demonstrated that

$$\sigma \geqslant \frac{||\rho(\tau) - \rho_0||_1^2}{\int_0^\tau \mathrm{d}t\, 2B(t)} \tag{16}$$

under the condition that the initial state is completely mixed as $\rho(0) = \rho_0 := 1/d$. For no dissipation limit, (16) is also consistent because of $||\rho(\tau) - \rho_0||_1 = 0$.

We consider (11) and (16) possess the following shortcomings, which we would like to improve in the present paper. (i) d_T can be zero between different states: when there is an unitary operator U_τ such that $\rho(\tau) = U_\tau \rho(0) U_\tau^\dagger$, d_T becomes zero and thus cannot distinguish between the two states. (ii) Even in the classical master equation limit [34], (11) does not lead to (1): $d_T(\rho(\tau), \rho(0))$ does not become l [38] and $B(t) > A_{\mathrm{c}}(t)$ in general. (iii) In (16), we can not replace ρ_0 by an any initial state $\rho(0)$. In fact, in the weak dissipation limit $\gamma_k \to 0$, although σ and $B(t)$ vanish, $||\rho(\tau) - \rho(0)||_1$ remains.

The structure of the paper is as follows. First, we summarize our main results (section 2). Next, we explain derivations (section 3). We apply our speed limits to a general system of which Hilbert space is two-dimensional (section 4.1): it includes a spinless quantum dot coupled to a single lead (section 4.2) and a qubit system (section 4.3). In section 5, we summarize this paper. In appendix A, we derive the Mandelstam–Tamm relation for mixed state. Appendix B is for the detailed calculations for section 3. In appendix C, we derive a bound for the trace distance in the Schrödinger picture. We prove $d_V \geqslant d_T$ in appendix D. Appendix E is for the detailed calculations for section 4.3.

2. Main results

The main results of this paper are

$$\sigma \geqslant \sigma_0 \geqslant \sigma_1 \geqslant \sigma_2, \tag{17}$$

$$\sigma_0 := \frac{||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1^2}{\int_0^\tau \mathrm{d}t\, 2A_\varphi(t)}, \tag{18}$$

$$\sigma_1 := \frac{||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1^2}{\int_0^\tau \mathrm{d}t [B(t) + B'(t)]}, \tag{19}$$

$$\sigma_2 := \frac{||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1^2}{\int_0^\tau \mathrm{d}t [B(t) + B_\infty(t)]}, \tag{20}$$

where $\tilde{\rho}(t) := U^\dagger(t)\rho(t)U(t)$ denotes the interaction picture. Here, $U(t)$ is defined by $\frac{\mathrm{d}}{\mathrm{d}t}U(t) = -iH(t)U(t)$ and $U(0) = 1$. A_φ, B, B', and B_∞ are quantum extensions of the activity. $A_\varphi(t)$ is given by

$$A_\varphi(t) := \mathrm{tr}\left(\tilde{\rho}(t)\frac{1}{4}\sum_k \gamma_k[\varphi, \tilde{L}_k]^\dagger[\varphi, \tilde{L}_k]\right). \tag{21}$$

$\varphi(t)$ is defined by

$$\varphi(t) := \Phi(\tilde{\rho}(t) - \tilde{\rho}(0)). \tag{22}$$

Here, Φ maps a self-adjoint operator X to a self-adjoint operator as $\Phi(X) := \sum_n \mathrm{sign}(x_n)|n\rangle\langle n|$, where the spectral decomposition of X is $X = \sum_n x_n|n\rangle\langle n|$. $\mathrm{sign}(x)$ is the sign of x. $B'(t)$ and $B_\infty(t)$ are defined by

$$B'(t) := \mathrm{tr}\left(\varphi\tilde{\rho}\varphi\sum_k \gamma_k\tilde{L}_k^\dagger\tilde{L}_k\right), \tag{23}$$

$$B_\infty(t) := \sum_k \gamma_k||L_k||_\infty^2 \geqslant B'(t), \tag{24}$$

where $||L_k||_\infty^2$ equals to the maximum eigenvalues of $L_k^\dagger L_k$. $||Y||_\infty$ is called the spectral norm.

The second inequality of (17) $\sigma \geqslant \sigma_1$ leads to (16) for $\rho(0) = \rho_0$ because $B'(t) = B(t)$ and $||\tilde{\rho}(\tau) - \tilde{\rho}_0||_1 = ||\rho(\tau) - \rho_0||_1$ hold and thus $\sigma_1 = \sigma_{\mathrm{V}1}$. In the classical master equation limit, the first inequality of (17) $\sigma \geqslant \sigma_0$ leads to (1): in this limit, $||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1 = l$ and $A_\varphi \leqslant A_c$ hold (appendix B.3) and thus σ_0 is larger than the right-hand side of (1). Even for no dissipation limit, (17) is consistent because $||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1 = 0$ holds with $\sigma = 0$ and $A_\varphi(t) = B(t) = B'(t) = B_\infty(t) = 0$.

We notice that

$$d_T(\tilde{\rho}(\tau), \tilde{\rho}(0)) \leqslant ||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1 \tag{25}$$

for finite dimensional Hilbert space (p 512 in reference [39]) and $d_T(\tilde{\rho}(\tau), \tilde{\rho}(0)) = d_T(\rho(\tau), \rho(0))$. (25) can be also derived from [26]

$$d_T(\rho_2, \rho_1) = \min_{V\in\{U|U^\dagger U=1\}} ||V\rho_2V^\dagger - \rho_1||_1. \tag{26}$$

(25) indicates that our bounds σ_k $(k = 0, 1, 2)$ can be better than (11).

Similarly to (6), we can separate contributions from unitary dynamics and dissipation. By the triangle inequality, the trace distance in the Schrödinger picture is bounded as [40]:

$$\begin{aligned}||\rho(\tau) - \rho(0)||_1 &\leqslant ||\check{\rho}(\tau) - \rho(0)||_1 + ||\rho(\tau) - \check{\rho}(\tau)||_1 \\ &= ||\check{\rho}(\tau) - \rho(0)||_1 + ||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1.\end{aligned} \tag{27}$$

Here, $\check{\rho}(t) := U(t)\rho(0)U^\dagger(t)$. The first term of the right-hand side of (27) is related to the unitary time evolution and is bounded by the Mandelstam–Tamm type relation (appendix A, [22])

$$||\check{\rho}(\tau) - \rho(0)||_1 \leqslant 2\int_0^\tau \mathrm{d}t\Delta\check{E}. \tag{28}$$

Here, $\Delta\check{E} := \sqrt{\mathrm{tr}(\check{\rho}(t)H(t)^2) - [\mathrm{tr}(\check{\rho}(t)H(t))]^2}$. Using (17) and (18), the second term of the right-hand side of (27) is bounded as

$$||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1 \leqslant \sqrt{2\sigma\int_0^\tau \mathrm{d}t\,A_\varphi(t)}. \tag{29}$$

In appendix C, we explain an alternative bound for the trace distance in the Schrödinger picture which does not refer to the virtual isolated system.

We discuss the meaning of the two distances when the Hilbert space is two-dimensional. In this case, the state of the system can be written as $\tilde{\rho}(t) = \frac{1}{2}(1 + \boldsymbol{r}(t)\cdot\boldsymbol{\tau})$. Here, $\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$, τ_i is the Pauli matrix, and $\boldsymbol{r}(t)$ is the Bloch vector. The trace distance and d_T are given by

$$||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1 = |\boldsymbol{r}(\tau) - \boldsymbol{r}(0)|, \tag{30}$$

$$d_T(\tilde{\rho}(\tau), \tilde{\rho}(0)) = ||\boldsymbol{r}(\tau)| - |\boldsymbol{r}(0)||, \tag{31}$$

with $|\boldsymbol{x}| := \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$. d_T measures the difference between the length of the two Bloch vectors and does not quantify the coherence.

3. Derivation of the main results

Our key idea is the use of the trace distance in the interaction picture within Vu's framework [24, 25]. We introduce a semi-inner product by

$$\langle\langle X, Y\rangle\rangle_{\tilde{\rho},k} := \mathrm{tr}\left[X^{\dagger}\{\tilde{\rho}\}_k(Y)\right] \tag{32}$$

with

$$\{\rho\}_k(X) := \int_0^1 \mathrm{d}s(\gamma_{-k}\rho)^s X(\gamma_k\rho)^{1-s}. \tag{33}$$

Here, $\gamma_{-k} = \gamma_{b,a,-\omega}$. $\{\rho\}_k(X)$ is the logarithmic mean transformation [41]. The semi-inner product satisfies $\left(\langle\langle X, Y\rangle\rangle_{\tilde{\rho},k}\right)^* = \langle\langle Y, X\rangle\rangle_{\tilde{\rho},k}$ and $||X||^2_{\tilde{\rho},k} := \langle\langle X, X\rangle\rangle_{\tilde{\rho},k} \geqslant 0$. This semi-inner product differs from Vu's semi-inner product $\mathrm{tr}[X^{\dagger}\hat{O}^{(b)}(Y)]$ where $\hat{O}^{(b)}(X) := \frac{1}{2}\sum_{a,\omega}[\tilde{L}_k^{\dagger}, \{\tilde{\rho}\}_k([\tilde{L}_k, X])]$ [24]. The super-operator $\hat{O}^{(b)}$ corresponds to the Laplacian of a weighted graph [42] (see (35) and appendix B.2). We consider the semi-inner product (32) provides more transparent descriptions (appendix B).

Using the semi-inner product, we can obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}||\tilde{\rho}(t) - \tilde{\rho}(0)||_1 &= \mathrm{tr}\left[\varphi(t)\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t}\right] \\ &= \frac{1}{2}\sum_k \langle\langle[\tilde{L}_k, \varphi(t)], [\tilde{L}_k, -\ln\tilde{\rho} - \beta_b\tilde{H}_S]\rangle\rangle_{\tilde{\rho},k}.\end{aligned} \tag{34}$$

We used $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{tr}[f(X(t))] = \mathrm{tr}\left[f'(X(t))\frac{\mathrm{d}X(t)}{\mathrm{d}t}\right]$ for a self-adjoint operator $X(t)$ and a differentiable function $f(x)$ in the first line. In the second line, we used the quantum master equation in the interaction picture (appendix B.2)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}(t) &= \sum_k \gamma_k\hat{D}[\tilde{L}_k](\tilde{\rho}) \\ &= \frac{1}{2}\sum_k [\tilde{L}_k^{\dagger}, \{\tilde{\rho}\}_k([\tilde{L}_k, -\ln\tilde{\rho} - \beta_b\tilde{H}_S])].\end{aligned} \tag{35}$$

Further, (34) leads to

$$\begin{aligned}||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1 &= \frac{1}{2}\sum_k \int_0^{\tau}\mathrm{d}t\ \langle\langle[\tilde{L}_k, \varphi(t)], [\tilde{L}_k, -\ln\tilde{\rho} - \beta_b\tilde{H}_S]\rangle\rangle_{\tilde{\rho},k} \\ &\leqslant \frac{1}{2}\sum_k \int_0^{\tau}\mathrm{d}t\ \sqrt{||[\tilde{L}_k, \varphi]||^2_{\tilde{\rho},k}||[\tilde{L}_k, -\ln\tilde{\rho} - \beta_b\tilde{H}_S]||^2_{\tilde{\rho},k}} \\ &\leqslant \sqrt{\int_0^{\tau}\mathrm{d}t\ \frac{1}{2}\sum_k ||[\tilde{L}_k, \varphi]||^2_{\tilde{\rho},k}}\cdot\sqrt{\sigma}.\end{aligned} \tag{36}$$

Here, we used the Cauchy–Schwarz inequalities

$$|\langle\langle X, Y\rangle\rangle_{\tilde{\rho},k}| \leqslant \sqrt{||X||^2_{\tilde{\rho},k}||Y||^2_{\tilde{\rho},k}} \tag{37}$$

and

$$\sum_k \int_0^{\tau}\mathrm{d}t\sqrt{\alpha_k(t)\beta_k(t)} \leqslant \sqrt{\sum_k\int_0^{\tau}\mathrm{d}t\ \alpha_k(t)}\sqrt{\sum_k\int_0^{\tau}\mathrm{d}t\ \beta_k(t)}. \tag{38}$$

Here, $\alpha_k(t)$ and $\beta_k(t)$ are non-negative real numbers. The entropy production rate can be written by using the semi-inner product [24]:

$$\dot{\sigma} = \frac{1}{2}\sum_k ||[\tilde{L}_k, -\ln\tilde{\rho} - \beta_b\tilde{H}_S]||^2_{\tilde{\rho},k}. \tag{39}$$

In (36), we can demonstrate (appendix B.3)

$$\frac{1}{2}\sum_k ||[\tilde{L}_k, \varphi]||^2_{\tilde{\rho},k} \leqslant 2A_{\varphi}(t), \tag{40}$$

which leads to the tightest inequality of (17). We can demonstrate

$$2A_{\varphi}(t) \leqslant B(t) + B'(t) \leqslant B(t) + B_{\infty}(t) \tag{41}$$

using $\mathrm{tr}\left(\tilde{\rho}(t)\{\varphi, \tilde{L}_k\}^{\dagger}\{\varphi, \tilde{L}_k\}\right) \geqslant 0$, $\varphi(t)^2 = 1$, and $B_{\infty}(t) \geqslant B'(t)$. Here, $\{X, Y\} = XY + YX$. Then, we obtain the other inequalities of (17).

We compare the derivations of (11) and (17). (11) can be derived as follows [24, 25]. For the spectral decomposition $\tilde{\rho}(t) = \sum_n p_n(t)|\tilde{n}(t)\rangle\langle\tilde{n}(t)|$, we put

$$\tilde{\phi}(t) := \sum_n c_n(t)|\tilde{n}(t)\rangle\langle\tilde{n}(t)|, \quad c_n(t)^2 = 1. \tag{42}$$

Then, $[\tilde{\rho}(t), \tilde{\phi}(t)] = 0$ and $\tilde{\phi}(t)^2 = 1$ hold. For $c_n(t) = \mathrm{sign}(p_n(t) - p_n(0))$ or $c_n(t) = \mathrm{sign}(p_n(\tau) - p_n(0))$,

$$d_V(\tilde{\rho}(\tau), \tilde{\rho}(0)) := \sum_n |p_n(\tau) - p_n(0)| = \mathrm{tr}\int_0^{\tau} \mathrm{d}t\, \tilde{\phi}(t)\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}(t) \tag{43}$$

holds [43]. By repeating similar calculations from (36) and by exploiting $d_V(\tilde{\rho}(\tau), \tilde{\rho}(0)) \geqslant d_T(\tilde{\rho}(\tau), \tilde{\rho}(0))$ (appendix D) and $d_V(\tilde{\rho}(\tau), \tilde{\rho}(0)) = d_V(\rho(\tau), \rho(0))$, we derive (11) (appendices B.3 and B.4). Note that $\tilde{\phi}(t) \neq \varphi$ for any $c_n(t)$ in general.

4. Application

4.1. General two-dimensional system

In this subsection, we consider a general system of which Hilbert space is two-dimensional. In general, the jump operators are written as [44]

$$\tilde{L}_k = (\boldsymbol{R}_k + i\boldsymbol{I}_k) \cdot \boldsymbol{\tau}. \tag{44}$$

Here, the components of $\boldsymbol{R}_k$ and $\boldsymbol{I}_k$ are real numbers. The equation of the motion of the Bloch vector of $\tilde{\rho}$ is given by

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = 2\sum_k \gamma_k[-(\boldsymbol{R}_k^2 + \boldsymbol{I}_k^2)\boldsymbol{r} + (\boldsymbol{R}_k \cdot \boldsymbol{r})\boldsymbol{R}_k + (\boldsymbol{I}_k \cdot \boldsymbol{r})\boldsymbol{I}_k + 2\boldsymbol{R}_k \times \boldsymbol{I}_k]. \tag{45}$$

The activities are given by

$$A_{\varphi} = \sum_k \gamma_k\left[\boldsymbol{R}_k^2 - (\boldsymbol{\varphi} \cdot \boldsymbol{R}_k)^2 + \boldsymbol{I}_k^2 - (\boldsymbol{\varphi} \cdot \boldsymbol{I}_k)^2 - 2[\boldsymbol{\varphi} \cdot (\boldsymbol{R}_k \times \boldsymbol{I}_k)]\boldsymbol{\varphi} \cdot \boldsymbol{r}\right], \tag{46}$$

$$B = \sum_k \gamma_k\left[\boldsymbol{R}_k^2 + \boldsymbol{I}_k^2 - 2(\boldsymbol{R}_k \times \boldsymbol{I}_k) \cdot \boldsymbol{r}\right], \tag{47}$$

$$B' = \sum_k \gamma_k\left[\boldsymbol{R}_k^2 + \boldsymbol{I}_k^2 - 2(\boldsymbol{R}_k \times \boldsymbol{I}_k) \cdot \boldsymbol{r}'\right], \tag{48}$$

$$B_{\infty} = \sum_k \gamma_k\left[\boldsymbol{R}_k^2 + \boldsymbol{I}_k^2 + 2|\boldsymbol{R}_k \times \boldsymbol{I}_k|\right]. \tag{49}$$

Here, we expanded φ as $\varphi = \boldsymbol{\varphi} \cdot \boldsymbol{\tau}$ with $\boldsymbol{\varphi} := \frac{1}{|\boldsymbol{r}-\boldsymbol{r}(0)|}[\boldsymbol{r} - \boldsymbol{r}(0)]$. $\boldsymbol{r}' = (x'(t), y'(t), z'(t))$ is the Bloch vector of $\varphi(t)\tilde{\rho}\varphi(t)$:

$$\boldsymbol{r}' = -\boldsymbol{r} + \frac{2([\boldsymbol{r} - \boldsymbol{r}(0)] \cdot \boldsymbol{r})[\boldsymbol{r} - \boldsymbol{r}(0)]}{|\boldsymbol{r} - \boldsymbol{r}(0)|^2}. \tag{50}$$

(46) is simplified as

$$A_{\varphi} = \frac{B + B'}{2} - \sum_k \gamma_k\left[(\boldsymbol{\varphi} \cdot \boldsymbol{R}_k)^2 + (\boldsymbol{\varphi} \cdot \boldsymbol{I}_k)^2\right]. \tag{51}$$

Because $\boldsymbol{\varphi}$ depends on the initial state, A_{φ} and B' depend on it. One can check for $\rho(0) = \rho_0$, i.e., $\boldsymbol{r}(0) = 0$, $B = B'$, and $||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1 = d_T(\rho(\tau), \rho(0))$, and thus $\sigma_1 = \sigma_{\mathrm{V}1}$.

4.2. Quantum dot

We analyze our inequality (17) for a spinless quantum dot coupled to a single lead [2, 6]. The quantum mater equation is given by

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H_S, \rho] + \gamma[1 - f(\varepsilon)]\hat{D}[a](\rho) + \gamma f(\varepsilon)\hat{D}[a^{\dagger}](\rho) \tag{52}$$

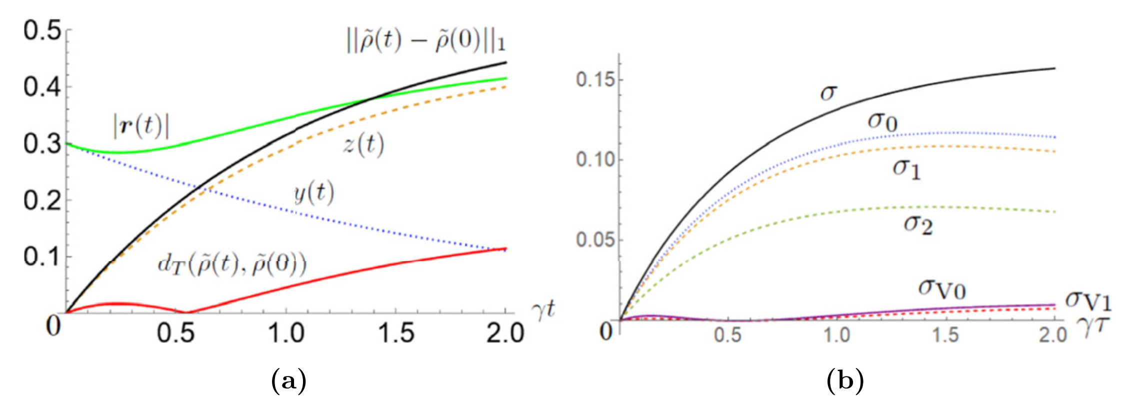


Figure 1. (a) $||\tilde{\rho}(t) - \tilde{\rho}(0)||_1$, $d_T(\tilde{\rho}(t), \tilde{\rho}(0))$, $y(t)$, $z(t)$, and $|\boldsymbol{r}(t)|$. (b) The total entropy production σ and its bounds σ_0, σ_1, σ_2, σ_{V0}, and σ_{V1}. The horizontal axes are time. $x(t)$ equals zero identically. $\beta\varepsilon = 1$ and $\rho(0) = \frac{1}{2}(1 + 0.3\tau_y)$.

with $H_S = \varepsilon a^\dagger a$. Here, a is the annihilation operator of the electron of the system, ε is the energy level of the system, $f(\varepsilon) = \frac{1}{e^{\beta\varepsilon}+1}$ is the Fermi distribution, β is the inverse temperature of the lead, and γ is the coupling strength. From (52), we obtain $L_1 = 2a$, $\gamma_1 = \frac{1}{4}\gamma[1 - f(\varepsilon)]$, $L_2 = 2a^\dagger$, and $\gamma_2 = \frac{1}{4}\gamma f(\varepsilon)$. These lead to $\boldsymbol{R}_1 = \boldsymbol{R}_2 = (0, 0, 1)$, and $\boldsymbol{I}_1 = -\boldsymbol{I}_2 = (0, 1, 0)$. The equation of the motion of the Bloch vector $\boldsymbol{r} = (x, y, z)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x = -\frac{1}{2}\gamma x, \quad \frac{\mathrm{d}}{\mathrm{d}t}y = -\frac{1}{2}\gamma y, \quad \frac{\mathrm{d}}{\mathrm{d}t}z = -\gamma(z - [1 - 2f(\varepsilon)]). \tag{53}$$

We calculate the trace distance and d_T by (30) and (31). The activity $B(t)$ and its upper limit $B_\infty(t)$ are given by using $B(t) = \gamma\big(1 + [2f(\varepsilon) - 1]z(t)\big)/2$ and $B_\infty(t) = \gamma$. $B'(t)$ is obtained from $B(t)$ by replacing $z(t)$ with $z'(t)$. $A_\varphi(t)$ is given by

$$A_\varphi(t) = \frac{\gamma}{4}\left(1 + \frac{[z - z(0)]^2}{|\boldsymbol{r} - \boldsymbol{r}(0)|^2} + 2[2f(\varepsilon) - 1]\frac{[z - z(0)][\boldsymbol{r} - \boldsymbol{r}(0)]\cdot\boldsymbol{r}}{|\boldsymbol{r} - \boldsymbol{r}(0)|^2}\right). \tag{54}$$

The entropy production is calculated as

$$\sigma = H_2\left(\frac{1 + |\boldsymbol{r}(\tau)|}{2}\right) - H_2\left(\frac{1 + |\boldsymbol{r}(0)|}{2}\right) + \int_0^\tau \mathrm{d}t\, \beta J(t). \tag{55}$$

Here, $H_2(p) := -p\ln p - (1 - p)\ln(1 - p)$ is the binary entropy. The heat current $J(t) := -\mathrm{tr}[H_S\mathcal{D}(\rho)]$ is given by

$$J(t) = \frac{1}{2}\gamma\varepsilon\big([1 - 2f(\varepsilon)] - z(t)\big). \tag{56}$$

For $\gamma t \gg 1$ and $\beta\varepsilon(t) \gg 1$ region, $B(t) \approx 0$ holds, however, $A_\varphi(t)$ and $B'(t)$ remain finite in general.

Figure 1(a) shows that the direction of the Bloch vector changes from the y-direction to the z-direction. In this process, the norm of the Bloch vector and then $d_T(\tilde{\rho}(t), \tilde{\rho}(0))$ change only slightly, see (31). At $\gamma t = 0.547\ldots$, although $\tilde{\rho}(t) \neq \tilde{\rho}(0)$, the distance $d_T(\tilde{\rho}(t), \tilde{\rho}(0))$ becomes zero. On the other hand, $||\tilde{\rho}(t) - \tilde{\rho}(0)||_1^2$ is much larger than $d_T(\tilde{\rho}(t), \tilde{\rho}(0))^2$. Figure 1(b) shows that our bounds σ_0, σ_1 and σ_2 are superior to Vu–Hasegawa's bound σ_{V1}. The Vu–Saito relation (16) is not applicable in this case.

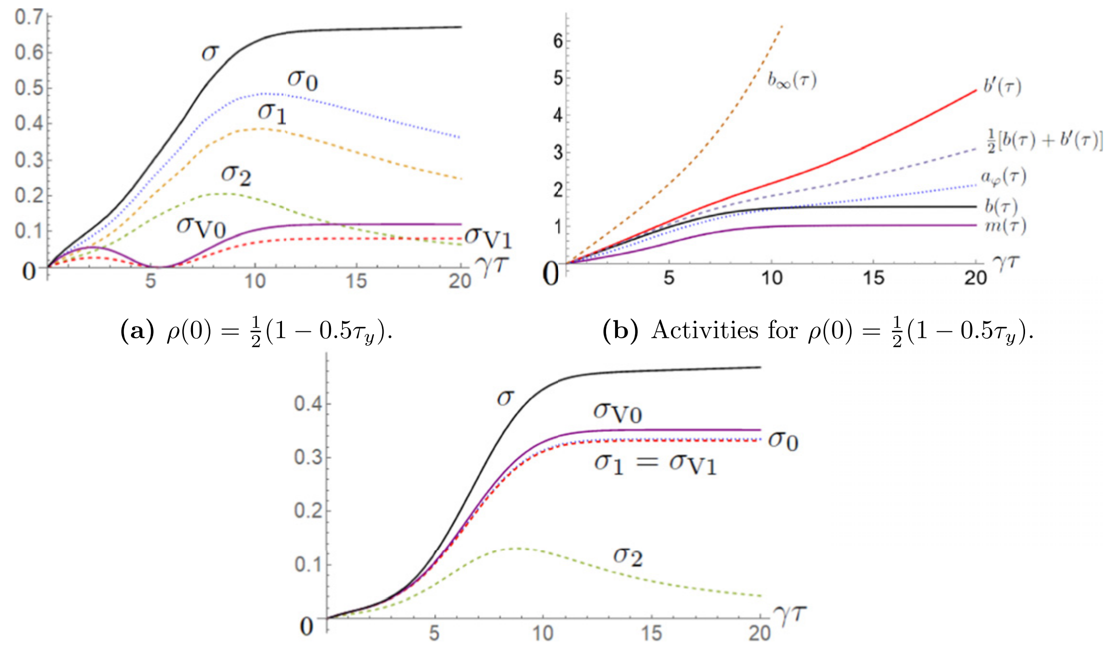
4.3. Qubit

We compare our bounds and Vu–Hasegawa's bound in the system studied in reference [25]. We consider the qubit system described by the quantum master equation (see appendix E for detailed calculation)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H_S(t), \rho] + \alpha\gamma\varepsilon(t)n(\varepsilon(t))\hat{D}[\tau_+(\theta(t))](\rho) + \alpha\gamma\varepsilon(t)[n(\varepsilon(t)) + 1]\hat{D}[\tau_-(\theta(t))](\rho) \tag{57}$$

with

$$H_S(t) = \frac{1}{2}\varepsilon(t)\tau_z(\theta(t)). \tag{58}$$



(a) $\rho(0) = \frac{1}{2}(1 - 0.5\tau_y)$. **(b)** Activities for $\rho(0) = \frac{1}{2}(1 - 0.5\tau_y)$.

(c) $\rho(0) = \frac{1}{2}$.

Figure 2. (a) σ and its bounds for $\rho(0) = \frac{1}{2}(1 - 0.5\tau_y)$. (b) $m(\tau) := \int_0^\tau \mathrm{d}t\, M(t)$, $b(\tau) := \int_0^\tau \mathrm{d}t\, B(t)$, $a_\varphi(\tau) := \int_0^\tau \mathrm{d}t\, A_\varphi(t)$, $b'(\tau) := \int_0^\tau \mathrm{d}t\, B'(t)$, $b_\infty(\tau) := \int_0^\tau \mathrm{d}t\, B_\infty(t)$, and $\frac{1}{2}[b(\tau) + b'(\tau)]$ for $\rho(0) = \frac{1}{2}(1 - 0.5\tau_y)$. (c) σ and its bounds for $\rho(0) = \frac{1}{2}$. The horizontal axes are time. $\alpha = 0.2\beta$, $\beta\gamma = 1$, $\beta\varepsilon_0 = 0.4$, $\beta\varepsilon_1 = 10$, $\gamma T = 20$. In (a), for $\gamma\tau < 0.15$, σ_{V0} is larger than σ_0. In (c), σ_0 is slightly larger than σ_1.

$n(\varepsilon) = \frac{1}{e^{\beta\varepsilon} - 1}$ is the Bose distribution. Here,

$$\tau_\pm(\theta) := \frac{1}{2}(\tau_x(\theta) \pm i\tau_y(\theta)) \tag{59}$$

with

$$\tau_x(\theta) := \tau_x \cos\theta - \tau_z \sin\theta, \quad \tau_y(\theta) := \tau_y, \quad \tau_z(\theta) := \tau_z \cos\theta + \tau_x \sin\theta. \tag{60}$$

We consider the 'nonoptimal protocol' in reference [25]:

$$\theta(t) = \pi\left(\frac{t}{T} - 1\right), \qquad \varepsilon(t) = \varepsilon_0 + (\varepsilon_1 - \varepsilon_0)\sin^2\frac{\pi t}{T}. \tag{61}$$

Figure 2 shows σ and its bounds σ_0, σ_1, σ_2, σ_{V0}, and σ_{V1} for (a) $\rho(0) = \frac{1}{2}(1 - 0.5\tau_y)$ and (c) $\rho(0) = \frac{1}{2}$ and (b) the integrals of the activities for $\rho(0) = \frac{1}{2}(1 - 0.5\tau_y)$. Figure 2(a) shows that our bounds σ_0 and σ_1 are superior in a wide range. In figure 2(b), for $\gamma t > 12$, $M(t) \approx 0$ and $B(t) \approx 0$ hold, however, $A_\varphi(t)$ and $B'(t)$ remain finite because they depend on the initial state. If we fix $\varepsilon(t)$ at ε_1 after $t > T$ while $\theta(t) = \pi\left(\frac{t}{T} - 1\right)$, σ_0 cross to σ_{V1} at $\gamma\tau \approx 120$. In this protocol, if $\boldsymbol{r}(0) \neq 0$ and $\beta\varepsilon_1 \gg 1$, our bounds get worse over time in general. For small $\varepsilon(t)$, whether our bounds σ_0 and σ_1 or Vu–Hasegawa's bound σ_{V1} decreases faster depends on the initial condition $\boldsymbol{r}(0)$. Figure 2(c) corresponds to the figure 1(c) in reference [25]. In this case, σ_{V0} is superior to σ_0. In figure 2(c), not only $M(t)$ and $B(t)$ but also $A_\varphi(t)$ and $B'(t)$ almost vanish for $\gamma t > 12$.

5. Summary

In open quantum systems described by the Lindblad type quantum master equation, we obtained universal bounds of the total entropy production described by the trace distance between the initial and final states. We considered the trace distance in the interaction picture instead of the distance of reference [24] (measured by the eigenvalues of the initial and final states) and trace distance in the Schrödinger picture [25]. Our bounds can be tighter than the bound of Vu and Hasegawa [24]. Our results are applicable to an arbitrary initial state, beyond Vu–Saito's bound [25] applicable only to the completely mixed initial state. In the classical master equation limit, our tightest inequality leads to the inequality by Shiraishi *et al* [21]. The trace distance in the Schrödinger picture is bounded by a sum of the trace distance for unitary dynamics described by the system Hamiltonian and the trace distance in the interaction picture.

Acknowledgments

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Mandelstam–Tamm relation

In this section, we demonstrate that the Mandelstam–Tamm relation for mixed state [18]

$$\mathcal{L}(\rho(\tau), \rho(0)) \leqslant \int_0^\tau \mathrm{d}t \Delta E \tag{A.1}$$

and

$$\frac{1}{2}||\rho(\tau) - \rho(0)||_1 \leqslant \int_0^\tau \mathrm{d}t \Delta E \tag{A.2}$$

for a closed quantum system S (described by $\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H(t), \rho]$). Here, $\mathcal{L}(\rho, \sigma) := \cos^{-1} F(\rho, \sigma)$ is the Bures angle, $F(\rho, \sigma) :=$ tr $\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$ is the fidelity [39], and $\Delta E := \sqrt{\mathrm{tr}(\rho H^2) - [\mathrm{tr}(\rho H)]^2}$. First, we demonstrate (A.1). For the spectral decomposition

$$\rho(0) = \sum_i p_i |\phi_i\rangle_S \langle\phi_i|, \tag{A.3}$$

we define a state

$$|\rho(t)\rangle\rangle := U(t) \sum_i \sqrt{p_i} |\phi_i\rangle_S \otimes |i\rangle_A. \tag{A.4}$$

Here, $U(t)$ is given by $\mathrm{d}U(t)/\mathrm{d}t = -iH(t)U(t)$ and $U(0) = 1$. $\{|i\rangle_A\}$ is an orthonormal basis of the ancilla system A. Because $\mathrm{tr}_A(|\rho(t)\rangle\rangle\langle\langle\rho(t)|) = \rho(t)$ holds, $|\rho(t)\rangle\rangle$ is a purification of $\rho(t)$. We consider the Robertson inequality

$$\Delta X \Delta E \geqslant \frac{1}{2} |\langle i[H, X]\rangle_t| \tag{A.5}$$

with $X = |\rho(0)\rangle\rangle\langle\langle\rho(0)|$. Here, $\Delta Y := \sqrt{\langle Y^2\rangle_t - \langle Y\rangle_t^2}$ and $\langle Y\rangle_t := \langle\langle\rho(t)|Y|\rho(t)\rangle\rangle$. For a operator Y_S of system S, $\langle Y_S\rangle_t = \mathrm{tr}_S(\rho(t)Y_S)$ holds. Using $\mathrm{d}|\rho(t)\rangle\rangle/\mathrm{d}t = -iH|\rho(t)\rangle\rangle$ and (A.5), we obtain

$$\Delta E \geqslant \left| \frac{\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\langle X\rangle_t}}{\sqrt{1 - \langle X\rangle_t}} \right|. \tag{A.6}$$

Then,

$$L := \left| \int_0^\tau \mathrm{d}t \frac{\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\langle X\rangle_t}}{\sqrt{1 - \langle X\rangle_t}} \right| \leqslant \int_0^\tau \mathrm{d}t \left| \frac{\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\langle X\rangle_t}}{\sqrt{1 - \langle X\rangle_t}} \right| \leqslant \int_0^\tau \mathrm{d}t \Delta E \tag{A.7}$$

holds. Here,

$$L = |\cos^{-1}\sqrt{\langle X\rangle_\tau} - \cos^{-1}\sqrt{\langle X\rangle_0}| = \cos^{-1}|\langle\langle\rho(\tau)|\rho(0)\rangle\rangle|. \tag{A.8}$$

Because $|\langle\langle\rho(t)|\rho(0)\rangle\rangle| \leqslant F(\rho(t), \rho(0))$ (theorem 9.4 in reference [39]), we obtain the Mandelstam–Tamm relation

$$\mathcal{L}(\rho(\tau), \rho(0)) \leqslant \cos^{-1}|\langle\langle\rho(\tau)|\rho(0)\rangle\rangle| \leqslant \int_0^\tau \mathrm{d}t \Delta E. \tag{A.9}$$

Next, we demonstrate (A.2). We utilize the following inequality ((9.110) in reference [39])

$$\frac{1}{2}||\rho - \sigma||_1 \leqslant \sqrt{1 - [F(\rho, \sigma)]^2}. \tag{A.10}$$

This leads to

$$\frac{1}{2}||\rho - \sigma||_1 \leqslant \sin \mathcal{L}(\rho, \sigma) \leqslant \mathcal{L}(\rho, \sigma). \tag{A.11}$$

This relation and (A.9) lead to (A.2).

Appendix B. Total entropy production rate and activities

B.1. Semi-inner product

If we diagonalize $\tilde{\rho}(t)$ as

$$\tilde{\rho}(t)=\sum_{n}p_n(t)|n(t)\rangle\langle n(t)|, \tag{B.1}$$

we obtain

$$\begin{aligned}\langle\langle X,Y\rangle\rangle_{\tilde{\rho},k}&=\mathrm{tr}\left[X^{\dagger}\int_0^1\mathrm{d}s\ (\gamma_{-k}\tilde{\rho})^s\,Y(\gamma_k\tilde{\rho})^{1-s}\right]\\&=\sum_{n,m}\langle n|X^{\dagger}|m\rangle\langle m|Y|n\rangle\int_0^1\mathrm{d}s\ (\gamma_{-k}p_m)^s(\gamma_k p_n)^{1-s}\\&=\sum_{n,m}M_k(m,n)\langle m|X|n\rangle^*\langle m|Y|n\rangle.\end{aligned}\tag{B.2}$$

Here, $M_k(m,n):=\Psi(\gamma_{-k}p_m,\gamma_k p_n)$ is a weight and $\Psi(a,b)$ is the logarithmic mean. Then, the semi-inner product corresponds to a weighted inner product introduced for the master equation in reference [42]. Note that $||X||^2_{\tilde{\rho},k}=0$ does not lead to $X=0$. Even if $||X||^2_{\tilde{\rho},k}=0$, $\langle m|X|n\rangle$ can remain for (m,n) such that $M_k(m,n)=0$.

B.2. Total entropy production rate

In this subsection, we demonstrate that

$$\dot{\sigma}=\frac{1}{2}\sum_k\left|\left|[\tilde{L}_k,-\ln\,\tilde{\rho}-\beta_b\tilde{H}_S]\right|\right|^2_{\tilde{\rho},k}. \tag{B.3}$$

As we will prove in the end of this section,

$$\{\rho\}_k([X,\ln\,\rho]+\beta_b\omega X)=\gamma_k X\rho-\gamma_{-k}\rho X \tag{B.4}$$

holds. Using (B.4), we obtain

$$\begin{aligned}\gamma_k\tilde{L}_k\tilde{\rho}-\gamma_{-k}\tilde{\rho}\tilde{L}_k&=\{\tilde{\rho}\}_k([\tilde{L}_k,\ln\,\tilde{\rho}]+\beta_b\omega\tilde{L}_k)\\&=\{\tilde{\rho}\}_k([\tilde{L}_k,\ln\,\tilde{\rho}+\beta_b\tilde{H}_S]).\end{aligned}\tag{B.5}$$

Here, we used (3) in the second line. Using this, we obtain

$$\begin{aligned}\frac{1}{2}\sum_{a,\omega}[\tilde{L}_k^{\dagger},\{\tilde{\rho}\}_k([\tilde{L}_k,-\ln\,\tilde{\rho}-\beta_b\tilde{H}_S])]&=\frac{1}{2}\sum_{a,\omega}[\tilde{L}_k^{\dagger},-\gamma_k\tilde{L}_k\tilde{\rho}+\gamma_{-k}\tilde{\rho}\tilde{L}_k]\\&=\frac{1}{2}\sum_{a,\omega}\left(-\gamma_k\{\tilde{L}_k^{\dagger}\tilde{L}_k\tilde{\rho}-\tilde{L}_k\tilde{\rho}\tilde{L}_k^{\dagger}\}+\gamma_{-k}\{\tilde{L}_k^{\dagger}\tilde{\rho}\tilde{L}_k-\tilde{\rho}\tilde{L}_k\tilde{L}_k^{\dagger}\}\right)\\&=\frac{1}{2}\sum_{a,\omega}\gamma_k\left(-\tilde{L}_k^{\dagger}\tilde{L}_k\tilde{\rho}+\tilde{L}_k\tilde{\rho}\tilde{L}_k^{\dagger}+\tilde{L}_k\tilde{\rho}\tilde{L}_k^{\dagger}-\tilde{\rho}\tilde{L}_k^{\dagger}\tilde{L}_k\}\right)\\&=\tilde{\mathcal{D}}_b(\tilde{\rho}).\end{aligned}\tag{B.6}$$

Here, we used (3) in the third line, and $\tilde{\mathcal{D}}_b(\tilde{\rho}):=\sum_{a,\omega}\gamma_k\hat{D}[\tilde{L}_k](\tilde{\rho})$. (B.6) demonstrates that the Laplacian $\hat{O}^{(b)}(X)=\frac{1}{2}\sum_{a,\omega}[\tilde{L}_k^{\dagger},\{\tilde{\rho}\}_k([\tilde{L}_k,X])]$ acting on the operator of thermodynamic force $(X=-\ln\,\tilde{\rho}-\beta_b\tilde{H}_S)$ becomes the dissipator [42]. Eventually, we relate the entropy production rate with the norms of the commutators:

$$\begin{aligned}\dot{\sigma} &= \sum_b \mathrm{tr}\{(-\ln\rho - \beta_b H_S)\mathcal{D}_b(\rho)\}\\ &= \sum_b \mathrm{tr}\{(-\ln\tilde{\rho} - \beta_b \tilde{H}_S)\tilde{\mathcal{D}}_b(\tilde{\rho})\}\\ &= \sum_b \mathrm{tr}\left[(-\ln\tilde{\rho} - \beta_b\tilde{H}_S)\frac{1}{2}\sum_{a,\omega}[\tilde{L}_k^\dagger, \{\tilde{\rho}\}_k([\tilde{L}_k, -\ln\tilde{\rho} - \beta_b\tilde{H}_S])]\right]\\ &= \frac{1}{2}\sum_k \left|\left|[\tilde{L}_k, -\ln\tilde{\rho} - \beta_b\tilde{H}_S]\right|\right|^2_{\tilde{\rho},k}.\end{aligned} \tag{B.7}$$

The equality (B.4) is derived as follows:

$$\begin{aligned}\{\rho\}_k([X,\ln\rho] + \beta_b\omega X) &= \int_0^1 \mathrm{d}s(\gamma_{-k}\rho)^s(X\ln\rho - \ln\rho X - \beta_b\omega X)(\gamma_k\rho)^{1-s}\\ &= -\gamma_k\int_0^1 \mathrm{d}s\,\frac{\mathrm{d}}{\mathrm{d}s}[e^{-s\beta_b\omega}e^{s\ln\rho}Xe^{(1-s)\ln\rho}]\\ &= \gamma_k X\rho - \gamma_{-k}\rho X.\end{aligned} \tag{B.8}$$

Here, we used (4). (B.8) corresponds to (S5e) of reference [24].

B.3. Activity

In this subsection, we demonstrate that

$$\mathcal{B}(t) := \frac{1}{2}\sum_k \left|\left|[\tilde{L}_k, \varphi]\right|\right|^2_{\tilde{\rho},k} \leqslant 2A_\varphi(t). \tag{B.9}$$

Using (B.2) and $\Psi(a,b) \leqslant \frac{a+b}{2}$, we obtain [24, 25]

$$\begin{aligned}||X||^2_{\tilde{\rho},k} &\leqslant \frac{1}{2}\sum_{n,m}\left(\gamma_{-k}p_m\langle n|X^\dagger|m\rangle\langle m|X|n\rangle + \gamma_k p_n\langle n|X^\dagger|m\rangle\langle m|X|n\rangle\right)\\ &= \frac{1}{2}\left[\gamma_{-k}\,\mathrm{tr}\left(\tilde{\rho}XX^\dagger\right) + \gamma_k\,\mathrm{tr}\left(\tilde{\rho}X^\dagger X\right)\right].\end{aligned} \tag{B.10}$$

Then, for $X_k := [\tilde{L}_k, \varphi]$,

$$\begin{aligned}\mathcal{B}(t) &\leqslant \frac{1}{4}\sum_k \left[\gamma_k\,\mathrm{tr}(\tilde{\rho}X_k^\dagger X_k) + \gamma_{-k}\,\mathrm{tr}(\tilde{\rho}X_k X_k^\dagger)\right]\\ &= \frac{1}{2}\sum_k \gamma_k\,\mathrm{tr}(\tilde{\rho}X_k^\dagger X_k)\\ &= 2A_\varphi(t)\end{aligned} \tag{B.11}$$

holds. Here, we used (3) in the second line of (B.11). Further by using

$$\mathrm{tr}\left(\tilde{\rho}(t)\{\varphi, \tilde{L}_k\}^\dagger\{\varphi, \tilde{L}_k\}\right) \geqslant 0, \tag{B.12}$$

$\varphi(t)^2 = 1$, and $B_\infty(t) \geqslant B'(t)$, we obtain (41).

B.4. Partial activity

Because of (43),

$$\sigma \geqslant \frac{d_V(\rho(\tau), \rho(0))^2}{\int_0^\tau \mathrm{d}t\,2A_\phi(t)} \tag{B.13}$$

can be derived in the same way as (36) and appendix B.3. $d_V(\rho(\tau), \rho(0))$ is defined by

$$d_V(\rho(\tau), \rho(0)) := \sum_n |p_n(\tau) - p_n(0)| \tag{B.14}$$

using the spectral decomposition

$$\rho(t) = \sum_n p_n(t)|n(t)\rangle\langle n(t)| \tag{B.15}$$

with differentiable eigenstates $\{|n(t)\rangle\}$. $A_\phi(t)$ is given by (21) replacing φ with $\tilde{\phi}$:

$$A_\phi(t) = \mathrm{tr}\left(\rho(t)\frac{1}{4}\sum_k \gamma_k[\phi, L_k]^\dagger[\phi, L_k]\right). \tag{B.16}$$

Here,

$$\phi(t) := \sum_n c_n(t)|n(t)\rangle\langle n(t)|, \tag{B.17}$$

and $c_n(t) = \mathrm{sign}(p_n(t) - p_n(0))$ or $c_n(t) = \mathrm{sign}(p_n(\tau) - p_n(0))$. We obtain

$$\begin{aligned} A_\phi(t) &= \sum_{n,m}\frac{1}{4}\{c_m(t) - c_n(t)\}^2 p_n(t)\sum_k \gamma_k|\langle m(t)|L_k|n(t)\rangle|^2 \\ &= \sum_{c_n(t)\neq c_m(t)} p_n(t)\sum_k \gamma_k|\langle m(t)|L_k|n(t)\rangle|^2 \\ &\leqslant \sum_{n\neq m} p_n(t)\sum_k \gamma_k|\langle m(t)|L_k|n(t)\rangle|^2 = \mathcal{A}(t). \end{aligned} \tag{B.18}$$

Here, $\mathcal{A}(t)$ is given in (9). $A_\phi(t)$ corresponds to the partial activity [24]. In the classical master equation limit, $A_\varphi(t)$ becomes $A_\phi(t)$ with $c_n(t) = \mathrm{sign}(p_n(t) - p_n(0))$.

Appendix C. Supplement for (27)

We derive a bound for the trace distance in the Schrödinger picture. This bound does not refer to the virtual isolated system, as opposed to the first term of the right-hand side of (27) referencing it.

From the quantum master equation (2) and the triangle inequality, we obtain [16, 22]

$$\begin{aligned} ||\rho(\tau) - \rho(0)||_1 &\leqslant \int_0^\tau \mathrm{d}t \left\|\frac{\mathrm{d}\rho}{\mathrm{d}t}\right\|_1 \\ &\leqslant \int_0^\tau \mathrm{d}t \left(||-i[H_S, \rho]||_1 + ||\mathcal{D}(\rho)||_1\right). \end{aligned} \tag{C.1}$$

For simplicity, in this section, we suppose that the dimension of the Hilbert space of the system is finite. The polar decomposition [39] of $\mathcal{D}(\rho)$ is given by $\mathcal{D}(\rho) = V^\dagger\sqrt{[\mathcal{D}(\rho)]^\dagger\mathcal{D}(\rho)}$ $(V^\dagger V = 1)$. This decomposition leads to

$$||\mathcal{D}(\rho)||_1 = \mathrm{tr}(V\mathcal{D}(\rho)). \tag{C.2}$$

By repeating similar calculations from (34), we obtain

$$\mathrm{tr}(V\mathcal{D}(\rho)) \leqslant \sqrt{\dot{\sigma}}\sqrt{\frac{1}{2}\sum_k ||[L_k, V]||^2_{\rho,k}} \tag{C.3}$$

with

$$\frac{1}{2}\sum_k ||[L_k, V]||^2_{\rho,k} \leqslant 2A_V(t) \leqslant B(t) + B_V(t) \leqslant B(t) + B_\infty(t) \tag{C.4}$$

where

$$A_V(t) := \mathrm{tr}\left(\rho\sum_k \gamma_k\frac{1}{4}[V, L_k]^\dagger[V, L_k]\right) \tag{C.5}$$

and $B_V(t) := \mathrm{tr}\left(V\rho V^\dagger\sum_k\gamma_k L_k^\dagger L_k\right)$. Thus, we obtain a bound for the trace distance in the Schrödinger picture:

$$||\rho(\tau) - \rho(0)||_1 \leqslant \int_0^\tau \mathrm{d}t||-i[H_S, \rho]||_1 + \sqrt{\sigma}\sqrt{\int_0^\tau \mathrm{d}t\, 2A_V(t)}. \tag{C.6}$$

Because $||-i[H_S, \rho]||_1 \leqslant 2\Delta E$ [22], the first term of the right-hand side is bounded by the Mandelstam–Tamm type term. The second term of the right-hand side corresponds to Shiraishi *et al*'s relation (1) and $c_2 + c_3$ in (6). In the classical master equation limit, V becomes a diagonal matrix of which components are $\mathrm{sgn}(\frac{\mathrm{d}p_n}{\mathrm{d}t}) = \pm 1$. In this case, the first term of the right-hand side of (C.6) vanishes and $A_V \leqslant A_c$.

Appendix D. Distances d_T and d_V

The distance d_T is defined by

$$d_T(\rho(\tau), \rho(0)) := \sum_n |b_n - a_n| \tag{D.1}$$

where $\{a_n\}$ and $\{b_n\}$ are increasing eigenvalues of $\rho(0)$ and $\rho(\tau)$. $d_V(\rho(\tau), \rho(0))$ is defined by (B.14) and can be rewritten as

$$d_V(\rho(\tau), \rho(0)) = \sum_n |b_{\chi(n)} - a_n|. \tag{D.2}$$

Here, χ is a permutation. For any two increasing sequences $\{x_n\}$ and $\{y_n\}$, we can demonstrate that

$$\sum_n |y_{\sigma(n)} - x_n| \geqslant \sum_n |y_n - x_n| \tag{D.3}$$

for an arbitrary permutation σ. Then, we obtain

$$d_V(\rho(\tau), \rho(0)) \geqslant d_T(\rho(\tau), \rho(0)). \tag{D.4}$$

If the eigenvalues of $\rho(t)$ do not intersect, $d_V(\rho(\tau), \rho(0)) = d_T(\rho(\tau), \rho(0))$ holds.

We prove (D.3). For $i < j$ with $\sigma(i) > \sigma(j)$, we can prove that

$$|y_{\sigma(i)} - x_i| + |y_{\sigma(j)} - x_j| \geqslant |y_{\sigma(j)} - x_i| + |y_{\sigma(i)} - x_j|. \tag{D.5}$$

The above equation leads to (D.3) because $\{y_{\sigma(n)}\}$ becomes the increasing sequence $\{y_n\}$ by repeating this type of exchanging.

Appendix E. Detailed calculation for section 4.3

From (57), we obtain $L_1 = 2\sigma_+(\theta(t))$, $\gamma_1 = \frac{1}{4}\alpha\gamma\varepsilon(t)n(\varepsilon(t))$, $L_2 = 2\sigma_-(\theta(t))$, and $\gamma_2 = \frac{1}{4}\alpha\gamma\varepsilon(t)[n(\varepsilon(t)) + 1]$. Denoting $U^\dagger(t)\tau_i(\theta(t))U(t)$ by $\sum_k T_{ik}(t)\tau_k$, we obtain $(\boldsymbol{R}_1)_i = (\boldsymbol{R}_2)_i = T_{1i}$, $(\boldsymbol{I}_1)_i = -(\boldsymbol{I}_2)_i = T_{2i}$, and $(\boldsymbol{R}_1 \times \boldsymbol{I}_1)_i = T_{3i}$. The equations of motion of T_{ik} are given by

$$\frac{\mathrm{d}T_{1j}}{\mathrm{d}t} = -\varepsilon(t)T_{2j} - T_{3j}\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}, \tag{E.1}$$

$$\frac{\mathrm{d}T_{2j}}{\mathrm{d}t} = \varepsilon(t)T_{1j}, \tag{E.2}$$

$$\frac{\mathrm{d}T_{3j}}{\mathrm{d}t} = T_{1j}\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}. \tag{E.3}$$

The equation of motion of the Bloch vector is given by

$$\frac{\mathrm{d}r_i}{\mathrm{d}t} = -\alpha\gamma\varepsilon(t)\left([2n(\varepsilon(t)) + 1]\left[r_i - \sum_k \frac{1}{2}\{T_{1k}T_{1i} + T_{2k}T_{2i}\}r_k\right] + T_{3i}\right). \tag{E.4}$$

The activities $A_\varphi(t)$, $B(t)$, $B'(t)$, and $B_\infty(t)$ are given by

$$A_\varphi(t) = \frac{\alpha\gamma\varepsilon(t)}{4}\big([2n(\varepsilon(t)) + 1]\big[1 + (\boldsymbol{\varphi}\cdot\boldsymbol{T}_3)^2\big] + 2(\boldsymbol{\varphi}\cdot\boldsymbol{T}_3)(\boldsymbol{\varphi}\cdot\boldsymbol{r})\big), \tag{E.5}$$

$$B(t) = \frac{\alpha\gamma\varepsilon(t)}{2}[2n(\varepsilon(t)) + 1 + x_3], \tag{E.6}$$

$$B'(t) = \frac{\alpha\gamma\varepsilon(t)}{2}[2n(\varepsilon(t)) + 1 - x_3 + 2(\boldsymbol{\varphi}\cdot\boldsymbol{T}_3)(\boldsymbol{\varphi}\cdot\boldsymbol{r})], \tag{E.7}$$

and $B_\infty(t) = \Gamma(t) := \alpha\gamma\varepsilon(t)[2n(\varepsilon(t)) + 1]$. Here, $(\boldsymbol{T}_3)_i := T_{3i}$ and $x_i := \sum_k T_{ik}r_k$. The equations of motion of $x_i(t)$ are given by

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -\varepsilon(t)x_2 - \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}x_3 - \frac{1}{2}\Gamma(t)x_1, \tag{E.8}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \varepsilon(t)x_1 - \frac{1}{2}\Gamma(t)x_2, \tag{E.9}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}x_1 - \Gamma(t)\left[x_3 + \frac{1}{2n(\varepsilon(t))+1}\right]. \tag{E.10}$$

The heat current is given by

$$\begin{aligned} J(t) &= -\frac{\varepsilon(t)}{2}\sum_k T_{3k}\frac{\mathrm{d}r_k}{\mathrm{d}t} \\ &= \frac{\varepsilon(t)}{2}\alpha\gamma\varepsilon(t)([2n(\varepsilon(t))+1]x_3+1). \end{aligned} \tag{E.11}$$

The entropy production is calculated by (55).

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[32] W_{nm} is given by $W_{nm} = \sum_b W_{nm}^{(b)}$ where $W_{nm}^{(b)}$ is the contribution from the bath b. The local detailed balance condition is $W_{nm}^{(b)} e^{-\beta_b E_m} = W_{nm}^{(b)} e^{-\beta_b E_n}$ where E_n is the energy of the state n and β_b is the inverse temperature of the bath b.
[33] $c_1 := \int_0^\tau dt \, \| - i[H, \rho]||_1$, $c_2 := \int_0^\tau dt \, \|\mathcal{D}_{\mathrm{nd}}(\rho)||_1$, and $c_3 := \int_0^\tau dt \, \|\mathcal{D}_{\mathrm{d}}(\rho)\|_1$ where $\mathcal{D}_{\mathrm{d}}(\rho) := \sum |n\rangle\langle n|\mathcal{D}(\rho)|n\rangle\langle n|$ and $\mathcal{D}_{\mathrm{nd}}(\rho) := \sum_{n\neq m} |n\rangle\langle n|\mathcal{D}[\rho]|m\rangle\langle m|$. Here, $\{|n\rangle\}$ is defined by (10). Because of $||-i[H, \rho]||_1 \leqslant 2\Delta E$ [22], (7) holds.
[34] If (i) the eigenstate $|E_n\rangle$ of H_S are time-independent, (ii) H_S is non-degenerate, (iii) the system state can be represented as $\rho(t) = \sum_n p_n(t)|E_n\rangle\langle E_n|$, the quantum master equation reduces to the classical master equation. In this case, corresponds to W_{nm}.
[35] Precisely, in reference [24], σ_{V1} is $\frac{d_T(\rho(\tau),\rho(0))^2}{\int_0^\tau \mathrm{d}t \, 2B_\infty(t)}$ where $B_\infty(\mathrm{t}) = \sum_k \gamma_k ||L_k||_\infty^2 \geqslant B(t)$. However, $B_\infty(t)$ can be replaced by $B(t)$ as is done in reference [25].
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[38] d_T can be replaced with d_V defined by $d_V(\rho(\tau), \rho(0)) := \sum_n |p_n(\tau) - p_n(0)|$ using (10) with differentiable eigenstates $\{|n(t)\rangle\}$. $d_V(\rho(\tau), \rho(0)) \geqslant d_T(\rho(\tau), \rho(0))$ holds (appendix D). In the classical master equation limit, $d_V(\rho(\tau), \rho(0))$ becomes l. However, if the eigenvalues of $\rho(t)$ intersect as functions of time, it is difficult to find differentiable eigenstates or corresponding labeling.
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[40] By the triangle inequality, we also obtain $|D| \leqslant ||\rho(\tau) - \rho(0)||_1$ with $D := ||\tilde{\rho}(\tau) - \rho(0)||_1 - ||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1$. A similar idea used to derive this equation and (27) has also been used in reference [27].
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[43] In reference [24], $c_n(t) = \mathrm{sign}(p_n(\tau) - p_n(0))$ has been chosen.
[44] From the first equation of (3) $[L_k, H_S] = \omega L_k$, $\{L_k\}_{\omega\neq 0}$ are traceless. For $\{L_k\}_{\omega=0}$, if L_k satisfy $[L_k, H_S] = 0$, $L_k' := L_k + l_k$ (l_k are c-number satisfying $l_{-k} = l_k^*$) also satisfy they. And ${}_{\gamma k}\hat{D}[L_k'](Y) + {}_{\gamma-k}\hat{D}[L_{-k}'](Y) = {}_{\gamma k}\hat{D}[L_k](Y) + {}_{\gamma-k}\hat{D}[L_{-k}](Y)$ holds. Then, we can assume that $\{L_k\}_{\omega=0}$ are also traceless.