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THE $m = 2$ AMPLITUHEDRON AND THE HYPERSIMPLEX: SIGNS, CLUSTERS, TILINGS, EULERIAN NUMBERS

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ABSTRACT. The hypersimplex $\Delta_{k+1,n}$ is the image of the positive Grassmannian $\mathrm{Gr}_{k+1,n}^{\geq 0}$ under the moment map. It is a polytope of dimension $n-1$ in $\mathbb{R}^n$. Meanwhile, the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ is the projection of the positive Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ into the Grassmannian $\mathrm{Gr}_{k,k+2}$ under a map $\tilde{Z}$ induced by a positive matrix $Z \in \mathrm{Mat}_{n,k+2}^{>0}$. Introduced in the context of *scattering amplitudes*, it is not a polytope, and has full dimension $2k$ inside $\mathrm{Gr}_{k,k+2}$. Nevertheless, there seem to be remarkable connections between these two objects via *T-duality*, as conjectured by Łukowski, Parisi, and Williams [Int. Math. Res. Not. (2023)]. In this paper we use ideas from oriented matroid theory, total positivity, and the geometry of the hypersimplex and positroid polytopes to obtain a deeper understanding of the amplituhedron. We show that the inequalities cutting out *positroid polytopes*—images of positroid cells of $\mathrm{Gr}_{k+1,n}^{\geq 0}$ under the moment map—translate into sign conditions characterizing the T-dual *Grasstopes*—images of positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ under $\tilde{Z}$. Moreover, we subdivide the amplituhedron into *chambers*, just as the hypersimplex can be subdivided into simplices, with both chambers and simplices enumerated by the Eulerian numbers. We use these properties to prove the main conjecture of Łukowski, Parisi, and Williams [Int. Math. Res. Not. (2023)]: a collection of positroid polytopes is a tiling of the hypersimplex if and only if the collection of T-dual Grasstopes is a tiling of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ for all Z. Moreover, we prove Arkani-Hamed–Thomas–Trnka's conjectural sign-flip characterization of $\mathcal{A}_{n,k,2}$, and Łukowski–Parisi–Spradlin–Volovich's conjectures on $m = 2$ *cluster adjacency* and on *positroid tiles* for $\mathcal{A}_{n,k,2}$ (images of $2k$-dimensional positroid cells which map injectively into $\mathcal{A}_{n,k,2}$). Finally, we introduce new cluster structures in the amplituhedron.

CONTENTS

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1. Introduction

1.1. **Context.** This article concerns the interaction between algebraic combinatorics and high energy physics, particularly *scattering amplitudes*. In a quantum field theory, scattering amplitudes are the probability amplitudes for fundamental particles to interact in a scattering process. They are central to understanding both salient features of the physical theory and experimental data from particle colliders. In a seminal work [AHT14], physicists Arkani-Hamed and Trnka introduced the *amplituhedron* $\mathcal{A}_{n,k,m}$, which is a subset of the real Grassmannian $\mathrm{Gr}_{k,k+m}$ of k-planes in $\mathbb{R}^{k+m}$. The 'volume' of the $m = 4$ amplituhedron encodes the scattering amplitudes of *maximally supersymmetric Yang-Mills* ($\mathcal{N} = 4$ SYM), a close cousin of the theory of strong interactions of quarks and gluons. The $m = 2$ amplituhedron is also connected to scattering amplitudes (at the subleading order in perturbation theory) and to correlation functions in $\mathcal{N} = 4$ SYM theory [KL20, CHCM23]. A novel way to compute $\mathcal{N} = 4$ SYM scattering amplitudes is by 'tiling' $\mathcal{A}_{n,k,m}$—that is, decomposing the amplituhedron into smaller 'tiles'—and summing the 'volumes' of the tiles.

While its motivation comes from physics, the amplituhedron $\mathcal{A}_{n,k,m}$ is mathematically very rich: it interpolates between cyclic polytopes (when $k = 1$) on the one hand, and the positive Grassmannian (when $k = n - m$) on the other. Cyclic polytopes and their triangulations have been extensively studied in polyhedral geometry going back to Carathéodory [Car11] (see also [Ram97]). Meanwhile, the positive Grassmannian is a prototypical example of the 'positive part' of a *cluster variety* [Pos06, Sco06, FZ02]. In this paper we focus on the $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}$, exploring both its 'tilings' (the appropriate generalization of triangulations) and its connection to cluster algebras. In particular, we prove that tilings of the amplituhedron $\mathcal{A}_{n,k,2}$ are in bijection with tilings of the *hypersimplex* $\Delta_{k+1,n}$ by positroid polytopes; this result together with [ŁPW20] suggests that the positive tropical Grassmannian [SW05] plays the role of 'secondary polytope' in governing the tilings of $\mathcal{A}_{n,k,2}$. Our result also connects the amplituhedron to a beautiful body of work on matroid subdivisions of the hypersimplex that dates back to Gel′fand-Goresky-MacPherson-Serganova [GGMS87], see also [Laf03, Spe08]. In a different direction, we associate a *cluster algebra* [FZ02] to each tile for $\mathcal{A}_{n,k,2}$, and show that the tile can be viewed as the positive part of a *cluster variety*. This proves the *cluster adjacency* conjecture for $\mathcal{A}_{n,k,2}$ and provides a novel connection between the amplituhedron, total positivity, and cluster algebras.

1.2. **Results.** The *positive Grassmannian*[1] $\mathrm{Gr}_{k,n}^{\geq 0}$ is the subset of the real Grassmannian $\mathrm{Gr}_{k,n}$ where all Plücker coordinates are nonnegative [Pos06, Rie98, Lus94]. This is a remarkable space with connections to cluster algebras, integrable systems, and

[1]More formally, the *totally nonnegative Grassmannian*.

high energy physics [FZ02, Sco06, KW14, AHBC$^+$16], and it has a beautiful CW decomposition into *positroid cells* S_π, which are indexed by various combinatorial objects including *decorated permutations* π [Pos06].

There are several interesting maps which one can apply to the positive Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ and its cells. The first map is the *moment map* μ, initially studied by Gel′fand-Goresky-MacPherson-Serganova [GGMS87] in the context of the Grassmannian and its torus orbits, who showed that the image of the Grassmannian is the *hypersimplex* $\Delta_{k,n} \subset \mathbb{R}^n$, a polytope of dimension $n-1$. When one restricts μ to $\mathrm{Gr}_{k,n}^{\geq 0}$, the image is still the hypersimplex [TW15].

The second map is the *amplituhedron map*, introduced by Arkani-Hamed and Trnka [AHT14] in the context of *scattering amplitudes* in $\mathcal{N} = 4$ SYM. In particular, any $n \times (k+m)$ matrix Z with maximal minors positive induces a map $\tilde{Z}$ from $\mathrm{Gr}_{k,n}^{\geq 0}$ to the Grassmannian $\mathrm{Gr}_{k,k+m}$, whose image has full dimension km and is called the *amplituhedron* $\mathcal{A}_{n,k,m}(Z)$.

Given any surjective map $\phi : \mathrm{Gr}_{k,n}^{\geq 0} \to X$ where $\dim X = d$, it is natural to try to decompose X using images of positroid cells under ϕ. This leads to Definition 1.1.[2]

Definition 1.1. Let $\phi : \mathrm{Gr}_{k,n}^{\geq 0} \to X$ be a continuous surjective map where $\dim X = d$. A *positroid tiling* of X (with respect to ϕ) is a collection $\{\overline{\phi(S_\pi)}\}$ of images of d-dimensional positroid cells such that

- ϕ is injective on each S_π from the collection
- pairs of distinct images $\phi(S_\pi)$ and $\phi(S_{\pi'})$ are disjoint
- $\cup \overline{\phi(S_\pi)} = X$.

When ϕ is the moment map, the (closures of the) images of the positroid cells S_π are the *positroid polytopes* Γ_π [TW15], so a positroid tiling of the hypersimplex is a decomposition into positroid polytopes. When ϕ is the amplituhedron map $\tilde{Z}$, the (closures of the) images of the positroid cells S_π are *Grasstopes* Z_π, which were first studied in [AHT14] as the building blocks of conjectural tilings of the amplituhedron. Note that neither the amplituhedron nor the Grasstopes are polytopes.

At first glance, the $(n-1)$-dimensional hypersimplex $\Delta_{k+1,n} \subset \mathbb{R}^n$ doesn't seem to have any relation to the $2k$-dimensional amplituhedron $\mathcal{A}_{n,k,2}(Z) \subset \mathrm{Gr}_{k,k+2}$. Nevertheless, the recent paper [ŁPW20] showed that there are surprising parallels between them. In particular, they showed that *T-duality* gives a bijection between loopless cells S_π of $\mathrm{Gr}_{k+1,n}^{\geq 0}$ and coloopless cells $S_{\hat{\pi}}$ of $\mathrm{Gr}_{k,n}^{\geq 0}$, and conjectured that T-duality gives a bijection between positroid tilings $\{\Gamma_\pi\}$ of the hypersimplex $\Delta_{k+1,n}$ and positroid tilings $\{Z_{\hat{\pi}}\}$ of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$. [ŁPW20] proved this conjecture for infinitely many tilings—specifically, the positroid tilings of $\mathcal{A}_{n,k,2}(Z)$ obtained from a BCFW-like recurrence [BH19].

In this paper we use *twistor coordinates* and the geometry of the hypersimplex and positroid polytopes to obtain a deeper understanding of the amplituhedron. We prove the conjecture of Łukowski–Parisi–Spradlin–Volovich [ŁPSV19] classifying *positroid*

[2]There are many reasonable variations of Definition 1.1. One might want to relax the injectivity assumption, or to impose further restrictions on how boundaries of the images of cells should overlap. Note that in the literature, positroid tilings are sometimes called (*positroid*) *triangulations*. We avoid this terminology in order to avoid confusion with the notion of e.g. polytopal triangulations.

tiles, full-dimensional images of positroid cells which map injectively into the amplituhedron $\mathcal{A}_{n,k,2}(Z)$. We then give a new characterization of them in terms of the signs of their twistor coordinates. We use this result to prove a conjecture of Arkani-Hamed–Thomas–Trnka that $\mathcal{A}_{n,k,2}(Z)$ can be characterized using sign flips of twistor coordinates. And we prove two results relating the amplituhedron to cluster algebras. First, we prove the *cluster adjacency* conjecture [ŁPSV19] for $\mathcal{A}_{n,k,2}(Z)$, which says that the Plücker coordinates labeling facets of a given positroid tile consist of pairwise compatible cluster variables. We also state and prove a generalization of this conjecture by showing that twistor coordinates of a positroid tile associated to Plücker coordinates compatible with the ones labeling its facets have constant sign. Second, we associate a cluster variety to each positroid tile in $\mathcal{A}_{n,k,2}(Z) \subset \mathrm{Gr}_{k,k+2}$, and show that the positroid tile is the totally positive part of that cluster variety. We then have the novel phenomenon that the $2k$-dimensional amplituhedron $\mathcal{A}_{n,k,2}(Z)$ can be decomposed into $\binom{n-2}{k}$ $2k$-dimensional positroid tiles, each of which is the totally positive part of a cluster variety. (Moreover, there are many such decompositions.)

Additionally, we draw striking parallels between $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$, some of which are illustrated in Table 1. We find that the inequalities describing positroid polytopes translate into sign conditions on twistor coordinates characterizing the corresponding Grasstopes. And we show that the sign patterns on twistor coordinates naturally subdivide the amplituhedron into *chambers*. We prove that the ones which are *realizable* are exactly enumerated by the Eulerian numbers $E_{k,n-1}$, just as the hypersimplex can be subdivided into simplices enumerated by $E_{k,n-1}$. We use these properties to prove the main conjecture of [ŁPW20]: a collection of positroid polytopes is a positroid tiling of $\Delta_{k+1,n}$ if and only if the collection of T-dual Grasstopes is a positroid tiling of $\mathcal{A}_{n,k,2}(Z)$ for all Z.

TABLE 1. Table of correspondences via T-duality: The hypersimplex vs the amplituhedron

The hypersimplex $\Delta_{k+1,n}$	VS	The amplituhedron $\mathcal{A}_{n,k,2}$
$\Delta_{k+1,n} = \mu(\mathrm{Gr}^{\geq 0}_{k+1,n})$ (moment map)		(amplituhedron map) $\mathcal{A}_{n,k,2} = \tilde{Z}(\mathrm{Gr}^{\geq 0}_{k,n})$
$\dim(\Delta_{k+1,n}) = n-1$ in $\mathbb{R}^n$		$\dim(\mathcal{A}_{n,k,2}) = 2k$ in $\mathrm{Gr}_{k,k+2}$
	POSITROID TILES (PT)	
$\Gamma_{G(\mathcal{T})}$ (positroid polytope)	bicolored subdivision $\mathcal{T}$ of type (k,n)	(Grasstope) $Z_{\hat{G}(\mathcal{T})}$
$x_{[h,j-1]} \geq \mathrm{area}_{\mathcal{T}}(h \to j)$	compatible arc $h \to j$	$\mathrm{sgn}\langle Yhj\rangle = (-1)^{\mathrm{area}_{\mathcal{T}}(h\to j)}$
$x_{[h,j-1]} = \mathrm{area}_{\mathcal{T}}(h \to j)$	facet defining arc $h \to j$	$\langle Yhj\rangle = 0$
	w-SIMPLICES and w-CHAMBERS	
w-simplex $\Delta_w \subset \Delta_{k+1,n}$ such that:	$w \in S_n$: $w_n = n$, $\#\,\mathrm{Des}_{\mathrm{L}}(w) = k$	w-chamber $\hat{\Delta}_w(Z) \subset \mathcal{A}_{n,k,2}$ such that:
$\Delta_w = \mathrm{conv}\{e_{I_1}, \dots, e_{I_n}\}$	$I_a = I_a(w) := \mathrm{cDes}_{\mathrm{L}}(w^{(a-1)})$	$\mathrm{Flip}(\langle Ya\hat{1}\rangle, \langle Ya\hat{2}\rangle, \dots, \langle Yan\rangle) = I_a \setminus \{a\}$
Hypersimplex $\Delta_{k+1,n} = \bigcup_w \Delta_w$		Amplituhedron $\mathcal{A}_{n,k,2}(Z) = \bigcup_w \hat{\Delta}_w(Z)$
PT $\Gamma_\pi = \bigcup_{\Delta_w \cap \Gamma^\circ_\pi \neq \emptyset} \Delta_w$		PT $Z_{\hat{\pi}} = \bigcup_{\hat{\Delta}_w(Z) \cap Z^\circ_{\hat{\pi}} \neq \emptyset} \hat{\Delta}_w(Z)$
$\Delta_w \subset \Gamma_\pi$	$\Leftrightarrow$	$\hat{\Delta}_w(Z) \subset Z_{\hat{\pi}}$
Hypercube $\square_{n-1} = \bigcup_k \Delta_{k+1,n} = \bigcup_w \Delta_w$	$w \in S_n : w_n = n$	Total amplituhedron $\mathcal{G}^{(2)}_n = \bigcup_k \mathcal{G}_{n,k,2} = \bigcup_w \hat{\Delta}_w(\mathcal{G})$
	POSITROID TILINGS	
$\{\Gamma_\pi\}$ tiles $\Delta_{k+1,n}$	$\Leftrightarrow$	$\{Z_{\hat{\pi}}\}$ tiles $\mathcal{A}_{n,k,2}(Z)$, for all Z
BCFW tiling $\{\Gamma_\pi\}$	BCFW recurrences	BCFW tiling $\{Z_{\hat{\pi}}\}$
Regular tiling $\{\Gamma_\pi\}$	max cones of $\mathrm{Trop}^+\,\mathrm{Gr}_{k+1,n}$	Regular tiling $\{Z_{\hat{\pi}}\}$
Catalan tiling $\{\Gamma_\pi\}$	Coloring vertices of a fixed planar tree	Catalan tiling $\{Z_{\hat{\pi}}\}$
Descent tiling $\{\Gamma_\pi\}$	positions of descents/sign flips	Sign-flip tiling $\{Z_{\hat{\pi}}\}$

1.3. **Connections to physics.** Let us now explain how the various geometric objects in our story are related to scattering amplitudes. In the last fifteen years, it was gradually realized that the Grassmannian, and in particular, the positive Grassmannian, can be used to encode most of the physical properties of scattering amplitudes in planar $\mathcal{N} = 4$ super Yang-Mills [AHCCK10, BMS10, AHBC$^+$16]. Building on these developments and on Hodges' idea that the amplitude should be the 'volume' of some 'polytope' [Hod13], Arkani-Hamed and Trnka defined the *amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ [AHT14].

The object most relevant to physics is the $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$: in this case, the amplituhedron can be tiled by 'BCFW cells' [EZLT21], which implies that the amplituhedron recovers the Britto-Cachazo-Feng-Witten recurrence [BCFW05] for computing scattering amplitudes. Meanwhile, the $m = 2$ amplituhedron governs the geometry of planar $\mathcal{N} = 4$ SYM amplitudes at the subleading order in perturbation theory ('one-loop') of some sectors of the theory, specifically the 'MHV' and 'NMHV' sector [KL20]. It also encodes scattering amplitudes for a Gaussian model on a superline [CHCM23], and it is related to correlation functions of determinant operators in $\mathcal{N} = 4$ SYM.

Scattering amplitudes in planar $\mathcal{N} = 4$ SYM enjoy a remarkable duality called 'Amplitude/Wilson loop duality' [AR08], which was shown to arise from a more fundamental duality in String Theory called 'T-duality' [BM08]. The geometric counterpart of this fact is a conjectural duality between collections of $4k$-dimensional 'BCFW' cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ which give positroid tilings of the $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$, and corresponding collections of $(2n - 4)$-dimensional cells of $\mathrm{Gr}_{k+2,n}^{\geq 0}$ which give positroid tilings of the *momentum amplituhedron* $\mathcal{M}_{n,k,4}$ [DFŁP19, ŁPW20]. This duality was evocatively called *T-duality* in [ŁPW20] and conjectured to generalize for any (even) m. The present paper explores T-duality for $m = 2$, showing that questions about the $m = 2$ amplituhedron can be reduced to questions about the hypersimplex.

One recent trend in physics is the connection between analytic properties of scattering amplitudes and *cluster algebras* [FZ02]; these connections have led to both computational and theoretical advances [GGS$^+$14, DFG18, DFG19, ŁPSV19, GP20, MSSV20, HL21]. In this paper, we use twistor coordinates to prove (and generalize) the conjecture of Łukowski–Parisi–Spradlin–Volovich [ŁPSV19] about $m = 2$ cluster adjacency and probe new cluster structures in the amplituhedron.

Our result that Eulerian numbers count sign chambers of the $m = 2$ amplituhedron is intriguing because Eulerian numbers have also come up in the context of *scattering equations* [CHY13]. Scattering equations connect the singularity structure of scattering amplitudes of n-particles to that of the boundaries of the moduli space of *Riemann spheres with n punctures*. For $\mathcal{N} = 4$ SYM, the number of solutions of the 'N^kMHV' sector of the theory is exactly the Eulerian number $E_{k,n-3}$ [SV09, CHY13]. Moreover, [CMZ17] provided an explicit bijection between such solutions and permutations on $[n-3]$ with k descents. Finally, in the case of certain scalar quantum field theories, the authors of [CEGM19] formulated a generalization of scattering equations. By studying 'arrays of Feynman diagrams', they made connections to the positive tropical Grassmannian, and, by results of [ŁPW20], to the hypersimplex. It would be fascinating to explore possible relations between (generalized) scattering equations, simplices of the hypersimplex, and chambers of the amplituhedron.

We note that some of the ideas used in this paper can be applied to amplituhedra for other m, and to the momentum amplituhedron; we will pursue this in a separate work.

1.4. **Structure of the paper.** The structure of this paper is as follows. In Section 2 we give background on the positive Grassmannian and the amplituhedron. In Section 3 we define twistor coordinates for the amplituhedron, and define the *sign stratification* of $\mathcal{A}_{n,k,m}(Z)$, which is analogous to the oriented matroid stratification of the Grassmannian. In Section 4 we study positroid tiles of $\mathcal{A}_{n,k,2}(Z)$: we prove a conjecture of Łukowski–Parisi–Spradlin–Volovich characterizing tiles in terms of *bicolored subdivisions* of a polygon, and we give an inequality description of tiles in terms of signs of twistor coordinates. In Section 5 we prove Arkani-Hamed–Thomas–Trnka's conjectural description of $\mathcal{A}_{n,k,2}(Z)$ in terms of sign flips of twistor coordinates. In Section 6 we introduce a generalization of the $m = 2$ cluster adjacency conjecture of [ŁPSV19] and define a cluster variety for each positroid tile. In Section 7 we give background on the hypersimplex, T-duality, and positroid tilings of the hypersimplex. In Section 8, we describe T-duality as a map on decorated permutations and plabic graphs. In Section 9 we discuss the close parallel between the inequality descriptions and facets of positroid tiles in $\Delta_{k+1,n}$ and the T-dual positroid tiles in $\mathcal{A}_{n,k,2}(Z)$. We also prove (a generalization of) the $m = 2$ cluster adjacency conjecture. In Section 10 we show how the subdivision of $\Delta_{k+1,n}$ into w-simplices corresponds to the decomposition of $\mathcal{A}_{n,k,2}(Z)$ into w-chambers, where in both cases w ranges over a set of permutations enumerated by the Eulerian number. In Section 11 we use this correspondence to prove the main conjecture of [ŁPW20] about positroid tilings. We also present algorithms to find tilings of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ based on w-simplices and w-chambers, and we show some examples. We also explain other combinatorial manifestations of this correspondence. In Section 12 we prove that positroid tiles are enumerated by a refinement of Schröder numbers via a bijection with *separable* permutations. Appendix A gives background on the combinatorics of the positroid cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$.

2. The positive Grassmannian and the amplituhedron

2.1. **The Grassmannian and positive Grassmannian.** The (*real*) *Grassmannian* $\mathrm{Gr}_{k,n}$ is the space of all k-dimensional subspaces of $\mathbb{R}^n$, for $0 \leq k \leq n$. An element of $\mathrm{Gr}_{k,n}$ can be viewed as a $k \times n$ matrix of rank k, modulo left multiplication by invertible $k \times k$ matrices. That is, two $k \times n$ matrices of rank k represent the same point in $\mathrm{Gr}_{k,n}$ if and only if they can be obtained from each other by invertible row operations. For C a full-rank $k \times n$ matrix, we will often abuse notation and write $C \in \mathrm{Gr}_{k,n}$, identifying C with its rowspan.

Let $[n]$ denote $\{1, \dots, n\}$, and $\binom{[n]}{k}$ the set of all k-element subsets of $[n]$. We embed $\mathrm{Gr}_{k,n}$ into projective space $\mathbb{P}(\wedge^k \mathbb{R}^n)$ in the usual way. That is, choose $V \in \mathrm{Gr}_{k,n}$ and any representative matrix C with rows $C_1, \dots, C_k$. We map V to the equivalence class of $C_1 \wedge \cdots \wedge C_k$ in $\mathbb{P}(\wedge^k \mathbb{R}^n)$. This equivalence class depends only on V, not on the choice of C.

The embedding $V \mapsto C_1 \wedge \cdots \wedge C_k$ gives a natural choice of coordinates for the Grassmannian. Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$, and for $I = \{i_1 < i_2 < \cdots <$

$i_k\} \subset \binom{[n]}{k}$, let $E_I := e_{i_1} \wedge \cdots \wedge e_{i_k}$. Writing $C_1 \wedge \cdots \wedge C_k$ in terms of the E_I, we obtain

$$C_1 \wedge \cdots \wedge C_k = \sum_{I \in \binom{[n]}{k}} p_I(V) E_I \in \wedge^k(\mathbb{R}^n), \tag{2.1}$$

where $p_I(V)$ is the maximal minor of C located in column set I. The $p_I(V)$ are the *Plücker coordinates* of V, and are independent of C (up to simultaneous rescaling by a constant).

We will also use the notation $\langle C_1, \ldots, C_k \rangle$ for $C_1 \wedge \cdots \wedge C_k$.

Definition 2.2 ([Pos06, Section 3]). We say that $C \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if $p_I(C) \geq 0$ for all $I \in \binom{[n]}{k}$, and *totally positive* if $p_I(C) > 0$ for all $I \in \binom{[n]}{k}$. The set of all totally nonnegative $C \in \mathrm{Gr}_{k,n}$ is the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$, and the set of all totally positive C is the *totally positive Grassmannian* $\mathrm{Gr}_{k,n}^{>0}$. For $\mathcal{M} \subseteq \binom{[n]}{k}$, the *positroid cell* $S_{\mathcal{M}}$ is the set of $C \in \mathrm{Gr}_{k,n}^{\geq 0}$ such that $p_I(C) > 0$ for all $I \in \mathcal{M}$, and $p_J(C) = 0$ for all $J \in \binom{[n]}{k} \setminus \mathcal{M}$. We call $\mathcal{M}$ a *positroid* if $S_{\mathcal{M}}$ is nonempty. We let $Q(k,n)$ denote the poset on the cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ defined by $S_{\mathcal{M}} \leq S_{\mathcal{M}'}$ if and only if[3] $S_{\mathcal{M}} \subseteq \overline{S_{\mathcal{M}'}}$.

Remark 2.3. The positive and nonnegative part of a flag variety G/P was first introduced by Lusztig [Lus94] (who gave a Lie-theoretic definition of $(G/P)_{>0}$ and defined $(G/P)_{\geq 0} := \overline{(G/P)_{>0}}$), and proved to have a cell decomposition by Rietsch [Rie98]. Postnikov [Pos06] subsequently defined the nonnegative part of the Grassmannian as in Definition 2.2, and independently gave the above decomposition into cells. From the beginning it was believed by experts that Postnikov's definition of $\mathrm{Gr}_{k,n}^{\geq 0}$ should agree with Lusztig's (in the case G/P is the Grassmannian); this was first proved by Rietsch [Rie], and reproved in [TW13, Corollary 1.2], where the authors additionally proved that the two cell decompositions coincide. Two subsequent proofs that the two definitions of $\mathrm{Gr}_{k,n}^{\geq 0}$ coincide were given in [Lam16b, Lus19].

There are many ways to index the positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ [Pos06], including *decorated permutations* π, *affine permutations* f, and *plabic graphs* G. We will refer to the corresponding positroid cells using the notation S_π, S_f, S_G. For background, see Appendix A.

2.2. **The amplituhedron.** Building on [AHBC+16], Arkani-Hamed and Trnka [AHT14] introduced a new mathematical object called the (*tree*) *amplituhedron*, which is the image of the totally nonnegative Grassmannian under a particular map. In what follows, we let $\mathrm{Mat}_{n,p}^{>0}$ denote the set of $n \times p$ matrices whose maximal minors are positive.

Definition 2.4. Choose positive integers $k < n$ and m such that $k + m \leq n$, and let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. Then Z induces a map $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ defined by

$$\tilde{Z}(\langle c_1, \ldots, c_k \rangle) := \langle Z(c_1), \ldots, Z(c_k) \rangle.$$

Equivalently, if C is a matrix representing an element of $\mathrm{Gr}_{k,n}^{\geq 0}$, then $\tilde{Z}(C)$ is defined to be the element of $\mathrm{Gr}_{k,k+m}$ represented by the matrix CZ. The (*tree*) *amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is defined to be the image $\tilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$ inside $\mathrm{Gr}_{k,k+m}$.

[3]Here, and in what follows, we use closure in the Hausdorff topology.

The fact that Z has positive maximal minors ensures that $\tilde{Z}$ is well-defined [AHT14]. See [Kar17, Theorem 4.2] for a necessary and sufficient condition (in terms of sign-variation) for a matrix Z to give rise to a well-defined map $\tilde{Z}$. The amplituhedron $\mathcal{A}_{n,k,m}(Z)$ has full dimension km inside $\mathrm{Gr}_{k,k+m}$.

In special cases the amplituhedron recovers familiar objects. If Z is a square matrix, i.e. $k+m=n$, then $\mathcal{A}_{n,k,m}(Z)$ is isomorphic to the totally nonnegative Grassmannian. If $k=1$, $\mathcal{A}_{n,1,m}(Z)$ is a *cyclic polytope* in projective space $\mathbb{P}^m$ [Stu88]. If $m=1$, then $\mathcal{A}_{n,k,1}(Z)$ can be identified with the complex of bounded faces of a cyclic hyperplane arrangement [KW19].

We will consider the restriction of the $\tilde{Z}$-map to positroid cells in $\mathrm{Gr}_{k,n}^{\geq 0}$.

Definition 2.5. Fix k, n, m with $k+m \leq n$ and choose $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. Given a positroid cell S_π of $\mathrm{Gr}_{k,n}^{\geq 0}$, we let $Z_\pi^\circ = \tilde{Z}(S_\pi)$ and $Z_\pi = \overline{\tilde{Z}(S_\pi)} = \tilde{Z}(\overline{S_\pi})$, and we refer to Z_π° and Z_π as *open Grasstopes* and *Grasstopes*, respectively. We call Z_π and Z_π° a *positroid tile* and an *open positroid tile* for $\mathcal{A}_{n,k,m}(Z)$ if $\dim(S_\pi) = km$ and $\tilde{Z}$ is injective on S_π.

Definition 2.6. Let Z_π be a Grasstope of $\mathcal{A}_{n,k,m}(Z)$. We say that $Z_{\pi'}$ is a *facet* of Z_π if it is maximal by inclusion among the Grasstopes satisfying the following three properties:

- the cell $S_{\pi'}$ is contained in $\overline{S_\pi}$
- $Z_{\pi'}$ is contained in the boundary ∂Z_π
- $Z_{\pi'}$ has codimension 1 in Z_π.

Remark 2.7. By [Lam16b, Proposition 15.2], $\tilde{Z}(\overline{S_\pi}) = \overline{\tilde{Z}(S_\pi)}$.

If $k=1$ and $m=2$, the amplituhedron $\mathcal{A}_{n,1,2}(Z)$ is a convex n-gon in $\mathbb{P}^2$. The positroid tiles are exactly the triangles on vertices of the polygon.

Images of positroid cells under the map $\tilde{Z}$ have been studied since the introduction of the amplituhedron. In particular, Arkani-Hamed and Trnka [AHT14] conjectured that the images of certain *BCFW* collections of $4k$-dimensional cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ give a positroid tiling of the amplituhedron $\mathcal{A}_{n,k,4}(Z)$. Positroid tiles were called *generalized triangles* in [ŁPSV19]. The terminology of Grassmann polytopes to describe images of positroid cells in the amplituhedron was used in [Lam16b]. For brevity, we prefer the term *Grasstopes*.

Remark 2.8. While the definition of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ depends on a choice of $Z \in \mathrm{Mat}_{n,k+m}^{>0}$, it is believed that many of its combinatorial properties do not depend on this choice. For example, whether or not $\overline{\tilde{Z}(S_\pi)}$ is a positroid tile should be independent of the choice of Z; we will see that this is true in Theorem 4.25 in the case that $m=2$. It is also believed that whether or not a collection of cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ gives a positroid tiling of $\mathcal{A}_{n,k,m}(Z)$ should be independent of Z.

Remark 2.9. We note that matrices whose maximal minors are positive (or nonnegative) have a *twisted cyclic symmetry*. If $Z \in \mathrm{Mat}_{n,p}^{>0}$ with $n \geq p$ has rows $Z_1, Z_2, \dots, Z_n$, and if we let $\hat{Z}_i$ denote $(-1)^{p-1} Z_i$, then the matrix with rows $Z_2, \dots, Z_n, \hat{Z}_1$ also lies in $\mathrm{Mat}_{n,p}^{>0}$. Similarly for the matrix with rows $Z_3, \dots, Z_n, \hat{Z}_1, \hat{Z}_2$, etc.[4]

[4]We will use the 'hat' notation ˆ also in the context of T-duality with a different meaning. It will be always clear from context which one we mean.

2.3. **Previous work on the $m = 2$ amplituhedron.** The original paper [AHT14] gave a conjectural positroid tiling $\{Z_\pi\}$ of $\mathcal{A}_{n,k,2}(Z)$. [KWZ20] proved that the above collection consists of positroid tiles, that is, $\tilde{Z}$ is injective on the corresponding positroid cells. A BCFW-style recursion for positroid tilings of $\mathcal{A}_{n,k,2}(Z)$ was also conjectured in [KWZ20]; the fact that this recursion indeed produces positroid tilings was proved in [BH19]. A conjectural classification of $m = 2$ positroid tiles was given in [ŁPSV19].

Meanwhile, [AHTT18] gave a conjectural alternative description of $\mathcal{A}_{n,k,2}(Z)$ in terms of sign flips of twistor coordinates; they gave a proof sketch of one direction of the conjecture, and an independent proof of the same direction was given in [KW19]. In a different direction, [Łuk19] gave a conjectural description of the boundaries of the $m = 2$ amplituhedron. Finally, [ŁPW20] discovered a link between the $m = 2$ amplituhedron and the hypersimplex via T-duality and the *tropical positive Grassmannian*, which inspired the present paper.

3. The sign stratification of the amplituhedron

In this section we introduce *twistor coordinates* for the amplituhedron $\mathcal{A}_{n,k,m}(Z)$, and we use them to define the *sign stratification* of the amplituhedron. We also introduce terminology for sign variation and sign flips. We will subsequently use twistor coordinates to prove a sign flip description of $\mathcal{A}_{n,k,2}$ in Theorem 5.1, to characterize positroid tiles, and to describe Grasstopes.

The definitions and results in this section hold for any positive m. The subsequent sections of the paper are mostly concerned with $m = 2$. However, many of our techniques can be applied to other m, in particular $m = 4$; we plan to investigate this in a separate paper.

Twistor coordinates were first considered in [AHT14], and subsequently used in [AHTT18] to give a conjectural 'sign flip' description of the amplituhedron. In the case $m = 1$, [KW19, Corollary 3.19] studied the sign stratification and proved a sign flip description of $\mathcal{A}_{n,k,1}(Z)$.

3.1. Twistor coordinates for $\mathcal{A}_{n,k,m}$.

Definition 3.1. Fix positive $k < n$ and m such that $k + m \leq n$. Choose $Z \in \mathrm{Mat}^{>0}_{n,k+m}$ and denote its rows by $Z_1, \ldots, Z_n \in \mathbb{R}^{k+m}$. Given a matrix Y with rows $y_1, \ldots, y_k$ representing an element of $Gr_{k,k+m}$, and $i_1, \ldots, i_m$ a sequence of elements of $[n]$, we let

$$\langle Y Z_{i_1} Z_{i_2} \ldots Z_{i_m} \rangle = \langle y_1, \ldots, y_k, Z_{i_1}, \ldots, Z_{i_m} \rangle$$

denote the determinant of the $(k+m) \times (k+m)$ matrix whose rows are $y_1, \ldots, y_k, Z_{i_1}, \ldots, Z_{i_m}$. We call $\langle Y Z_{i_1} Z_{i_2} \ldots Z_{i_m} \rangle$ a *twistor coordinate*. We abbreviate $\langle Y Z_{i_1} Z_{i_2} \ldots Z_{i_m} \rangle$ by writing $\langle Y i_1 i_2 \ldots i_m \rangle$, when Z is understood.

Note that the twistor coordinates are a subset of the Plücker coordinates of the $(k + m) \times (k + n)$ matrix whose columns are $y_1, \ldots, y_k, Z_1, \ldots, Z_n$. There is also an interpretation of the twistor coordinates as Plücker coordinates in $\mathrm{Gr}_{m,n}$, as we explain in Proposition 3.3. In the context of scattering amplitudes of n particles in SYM theory, $\mathrm{Gr}_{m,n}$ is the space of *momentum twistors*[5] for $m = 4$, which is why we call

[5]Momentum twistors, introduced by Hodges in [Hod13], are points $z_1, \ldots, z_n$ in $\mathbb{P}^3$ encoding the kinematic data of scattering particles. Due to *dual conformal symmetry* of scattering amplitudes in SYM theory,

the coordinates from Definition 3.1 *twistor coordinates*. Remarkable connections between scattering amplitudes and the cluster algebra associated to the Grassmannian $Gr_{4,n}$ were discovered in these coordinates [GGS+14].

The fact that the twistor coordinates uniquely determine points of the amplituhedron can be deduced from some results of [KW19].

Definition 3.2 ([KW19, Definition 3.8]). Given $W \in \mathrm{Gr}_{k+m,n}^{>0}$, we define the $\mathcal{B}$-*amplituhedron*

$$\mathcal{B}_{n,k,m}(W) := \{V^\perp \cap W \mid V \in \mathrm{Gr}_{k,n}^{\geq 0}\} \subseteq \mathrm{Gr}_m(W),$$

where $V^\perp \in \mathrm{Gr}_{n-k,n}$ denotes the orthogonal complement of V in $\mathbb{R}^n$ and $\mathrm{Gr}_m(W) \subseteq \mathrm{Gr}_{m,n}$ denotes the subset of $\mathrm{Gr}_{m,n}$ of elements $X \in \mathrm{Gr}_{m,n}$ with $X \subseteq W$.

Proposition 3.3 ([KW19, Lemma 3.10, Proposition 3.12]). *Fix k, n, m and Z as in Definition* 3.1*, and let $W \in \mathrm{Gr}_{k+m,n}^{>0}$ be the column span of Z. Then the map*

$$\begin{aligned} f_Z : \mathrm{Gr}_m(W) &\to \mathrm{Gr}_{k,k+m}, \\ X &\mapsto Z(X^\perp) = \{Z(x) \mid x \in X^\perp\} = \mathrm{rowspan}(X^\perp Z) =: Y \end{aligned}$$

is an isomorphism. Here $X^\perp \in \mathrm{Gr}_{n-m,n}$ denotes the orthogonal complement of X in $\mathbb{R}^n$.

Moreover, for $X \in \mathrm{Gr}_m(W)$, $Y := f_Z(X)$, and $I = \{i_1 < \cdots < i_m\} \subseteq [n]$, we have

$$p_I(X) = \langle YZ_{i_1} \dots Z_{i_m} \rangle \tag{3.4}$$

(*where we view Plücker and twistor coordinates as coordinates on points in projective space*).

Finally, $f_Z : \mathcal{B}_{n,k,m}(W) \to \mathcal{A}_{n,k,m}(Z)$ is a homeomorphism sending $V^\perp \cap W \mapsto \tilde{Z}(V)$.

From (3.4) we see that $Y \in \mathrm{Gr}_{k,k+m}$ is uniquely determined by its twistor coordinates.

Remark 3.5. As an alternative to Proposition 3.3 we can consider the injective map ψ_Z

$$\begin{aligned} \psi_Z : \mathrm{Gr}_{k,k+m} &\to \mathrm{Gr}_{m,n}, \\ Y &\mapsto Y^\perp Z^T =: z, \end{aligned}$$

where $Y^\perp$ is any matrix representing the orthogonal complement of Y. Then it's not hard to see that for $I = \{i_1 < \cdots < i_m\} \subseteq [n]$, $p_I(z) = \langle YZ_{i_1} \dots Z_{i_m} \rangle$ (viewing both Plücker and twistor coordinates as coordinates on points in projective space).

The following expansion formula (3.7) will be useful in our proofs on positroid tiles.

Lemma 3.6. *Use the notation of Definition* 3.1*. If we write $Y \in \mathrm{Gr}_{k,k+m}$ as $Y = CZ$ with $C \in \mathrm{Gr}_{k,k+n}$, we can write the twistor coordinates in the form*

$$\langle CZ, Z_{i_1}, \dots, Z_{i_m} \rangle = \sum_{\{j_1 < \cdots < j_k\} \in \binom{[n]}{k}} p_J(C) \langle Z_{j_1}, \dots, Z_{j_k}, Z_{i_1}, \dots, Z_{i_m} \rangle. \tag{3.7}$$

Proof. Identifying the $k \times (k+m)$ matrix CZ with the corresponding element $\langle CZ \rangle$ of $\wedge^k(\mathbb{C}^{k+m})$, we have

$$\langle CZ \rangle = \sum_{\{j_1 < \cdots < j_k\} \in \binom{[n]}{k}} p_J(C) \langle Z_{j_1}, \dots, Z_{j_k} \rangle.$$

these are defined up to a PGL_4 transformation on $\mathbb{P}^3$. Therefore, momentum twistors can be embedded in $Gr_{4,n}/(\mathbb{C}^*)^{n-1}$ and scattering amplitudes are functions of Plücker coordinates in $\mathrm{Gr}_{4,n}$. See [GGS+14].

This implies the result. □

We will give a description of positroid tiles in $\mathcal{A}_{n,k,2}$ using signs of twistor coordinates. One ingredient in our proofs is the following easy sufficient condition for a twistor coordinate to have constant sign on a Grasstope, which follows directly from (3.7).

Lemma 3.8. *Fix positive $k < n$ and m such that $k + m \leq n$. Let $S_{\mathcal{M}}$ be a cell of $\mathrm{Gr}_{k,n}^{\geq 0}$. Fix $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ and as usual let $Z_1, \dots, Z_n$ denote the row vectors of Z. Choose an m-element subset $1 \leq i_1 < i_2 < \cdots < i_m \leq n$.*

- *If $\langle Z_{j_1}, \dots, Z_{j_k}, Z_{i_1}, \dots, Z_{i_m} \rangle \geq 0$ for each $J = \{j_1 < \cdots < j_k\} \in \mathcal{M}$, then $\langle CZ, Z_{i_1}, \dots, Z_{i_m} \rangle \geq 0$ for each $C \in S_{\mathcal{M}}$.*
- *If in addition $\langle Z_{j_1}, \dots, Z_{j_k}, Z_{i_1}, \dots, Z_{i_m} \rangle > 0$ for some $J = \{j_1 < \cdots < j_k\} \in \mathcal{M}$ then $\langle CZ, Z_{i_1}, \dots, Z_{i_m} \rangle > 0$ for each $C \in S_{\mathcal{M}}$.*

3.2. **The sign stratification of $\mathcal{A}_{n,k,m}$.** Since $Y \in \mathrm{Gr}_{k,k+m}$ is uniquely determined by its twistor coordinates, it makes sense to stratify $\mathcal{A}_{n,k,m}(Z) \subset \mathrm{Gr}_{k,k+m}$ by the signs of the twistor coordinates. This was done in [KW19] in the case that $m = 1$. Moreover, this sign stratification is closely related to the *oriented matroid stratification* on the Grassmannian, which partitions elements of the real Grassmannian into strata based on the signs of the Plücker coordinates. By Proposition 3.3, the twistor coordinates of $Y \in \mathcal{A}_{n,k,m}(Z)$ are Plücker coordinates on the corresponding element of the *B-amplituhedron* [KW19] or *amplituhedron in momentum twistor space* [AHTT18], so this sign stratification reduces to the oriented matroid stratification in momentum twistor space.

Definition 3.9 (Amplituhedron chambers). Fix positive $k < n$ and m such that $k+m \leq n$. Let $\sigma = (\sigma_{i_1,\dots,i_m}) \in \{0,+,-\}^{\binom{n}{m}}$ be a nonzero sign vector, considered[6] *modulo multiplication by* ± 1. Set

$$\mathcal{A}_{n,k,m}^{\sigma}(Z) := \{Y \in \mathcal{A}_{n,k,m}(Z) \mid \mathrm{sign}\langle Y Z_{i_1} \dots Z_{i_m} \rangle = \sigma_{i_1,\dots,i_m}\}.$$

We call $\mathcal{A}_{n,k,m}^{\sigma}(Z)$ an (*amplituhedron*) *sign stratum*. Clearly

$$\mathcal{A}_{n,k,m}(Z) = \sqcup_{\sigma} \mathcal{A}_{n,k,m}^{\sigma}(Z).$$

If $\sigma \in \{+,-\}^{\binom{n}{m}}$, we call $\mathcal{A}_{n,k,m}^{\sigma}(Z)$ an open (*amplituhedron*) *chamber*.[7]

For $m = 1$, all strata are nonempty [KW19, Definition 5.2], but this is not true for $m > 1$. Moreover, whether or not $\mathcal{A}_{n,k,m}^{\sigma}(Z)$ is empty depends on Z, see Section 11.

Definition 3.10. We say that a sign vector σ (or sign stratum $\mathcal{A}_{n,k,m}^{\sigma}$) is *realizable* for $\mathcal{A}_{n,k,m}$ if $\mathcal{A}_{n,k,m}^{\sigma}(Z)$ is nonempty for some Z.

3.3. **Sign variation and sign flips.** Signs and sign flips will be important to our description of the amplituhedron, so we introduce some useful terminology here.

Definition 3.11. Given $v \in \mathbb{R}^n$, let $\mathrm{var}(v)$ be the number of times v changes sign when we read the components from left to right and ignore any zeros. If $v \in \{0,+,-\}^n$, we define $\mathrm{var}(v)$ in the obvious way.

[6] Plücker and twistor coordinates are defined only up to multiplication by a common scalar.

[7] We borrow the word "chamber" from the theory of hyperplane arrangements.

For example, if $v := (4, -1, 0, -2) \in \mathbb{R}^4$ then $\mathrm{var}(v) = 1$.

Definition 3.12. If $v \in \mathbb{R}^n$, or $v \in \{+, -, 0\}^n$, we say that v has a *sign flip in position* i if $v_i, v_{i+1} \neq 0$ and they have different signs, where indices are considered modulo n. We define

$$\mathrm{Flip}(v) = \mathrm{Flip}(v_1, \dots, v_n) := \{i \mid v \text{ has a sign flip in position } i\} \subseteq [n].$$

Remark 3.13. We caution the reader that $|\mathrm{Flip}(v)|$ may not equal $\mathrm{var}(v)$. For example, the sequence $(+, 0, -, 0, +, +, -) \in \{+, -, 0\}^7$ has sign flips in positions $\{6, 7\}$, but $\mathrm{var}(+, 0, -, 0, +, +, -)$ is 3.

4. POSITROID TILES OF $\mathcal{A}_{n,k,2}$

Recall that a *positroid tile* of $\mathcal{A}_{n,k,m}(Z)$ is the full-dimensional image of a positroid cell on which $\tilde{Z}$ is injective. In this section, we will obtain a detailed description of the positroid tiles of $\mathcal{A}_{n,k,2}(Z)$. The main results of this section are the following:

- In Theorem 4.25 we classify the positroid tiles of $\mathcal{A}_{n,k,2}(Z)$, describing them as the Grasstopes $Z_{\hat{G}(\overline{\mathcal{T}})}$ obtained from the $2k$-dimensional positroid cells $S_{\hat{G}(\overline{\mathcal{T}})}$ associated to bicolored subdivisions of polygons, proving a conjecture of [ŁPSV19]. This implies that whether or not $\tilde{Z}(S_\pi)$ is a positroid tile is independent of the choice of Z.
- In Theorem 4.28 we characterize each (open) positroid tile $Z^\circ_{\hat{G}(\mathcal{T})}$ as the subset of $\mathrm{Gr}_{k,k+2}$ where certain twistor coordinates have a fixed sign; this shows that each positroid tile is a union of (closures of) amplituhedron chambers.
- In Theorem 4.19 we solve a kind of "inverse problem" for positroid tiles: given an element $Y \in \mathrm{Gr}_{k,k+2}$ which lies in an open positroid tile $Z^\circ_{\hat{G}(\mathcal{T})}$, we explicitly construct an element $C \in \mathrm{Gr}_{k,n}$ whose image in $\mathcal{A}_{n,k,2}(Z)$ is Y, i.e. $CZ = Y$; the entries of C are in fact twistor coordinates.

We note that the techniques that we use in this section can be extended to give a *cell decomposition* of $\mathcal{A}_{n,k,2}(Z)$. This will be explored in a separate paper.

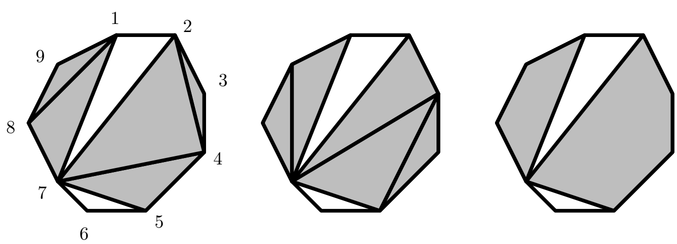


FIGURE 1. Two equivalent bicolored triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of type $(5, 9)$, and the corresponding bicolored subdivision $\overline{\mathcal{T}}_1 = \overline{\mathcal{T}}_2$ of type $(5, 9)$

Definition 4.1 (Bicolored triangulations and subdivisions)**.** Let $\mathbf{P}_n$ be a convex n-gon with vertices labeled from 1 to n in clockwise order. A *bicolored triangulation of type* (k, n) is a triangulation of $\mathbf{P}_n$ where k triangles are colored black and the rest are colored white. Two bicolored triangulations are *equivalent* if the union of the black triangles of one is equal to the union of the black triangles of the other. We represent the equivalence class of a bicolored triangulation $\mathcal{T}$ by erasing the diagonals that separate pairs of triangles of the same color. The resulting object $\overline{\mathcal{T}}$ is a subdivision of $\mathbf{P}_n$ into white and black polygons, and is called a *bicolored subdivision of type* (k,n). See Figure 1.

Note that in a bicolored subdivision, as defined above, no two polygons of the same color share an edge. Bicolored triangulations of type (k,n) were called *k nonintersecting triangles in a convex n-gon* in [ŁPSV19]. We will see later (cf. Remark 8.11) that bicolored triangulations and subdivisions are special cases of the *plabic tilings* of [OPS15].

Given $\mathcal{T}$ a bicolored triangulation of type (k,n), we build a corresponding bipartite graph $\hat{G}(\mathcal{T})$ as in Figure 2, then use the recipe from Theorem A.7 and Remark A.8 to construct all points of the $2k$-dimensional cell $S_{\hat{G}(\mathcal{T})}$ of $\mathrm{Gr}_{k,n}^{\geq 0}$.

Definition 4.2. Given $\mathcal{T}$ a bicolored triangulation of type (k,n), we build a labeled bipartite graph $\hat{G}(\mathcal{T})$ by placing black boundary vertices labeled $B_1, B_2, \dots, B_n$ in clockwise order at the n vertices of the n-gon, and placing a trivalent white vertex in the middle of each black triangle, connecting it to the three vertices of the triangle. We label the k white vertices by $W_1, \dots, W_k$; we will usually label them in the order specified by Remark 4.5.

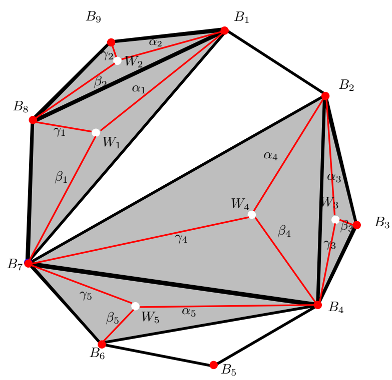


FIGURE 2. The planar bipartite graph $\hat{G}(\mathcal{T}_1)$ together with its edge-weighting

Remark 4.3. We can think of $\hat{G}(\mathcal{T})$ as a *plabic graph* (see Definition A.2) if we enclose it in a slightly larger disk and add n edges connecting each B_i to the boundary of the disk. We will often abuse terminology and refer to $\hat{G}(\mathcal{T})$ as a plabic graph. Note that $\hat{G}(\mathcal{T})$ does not depend on the triangulation of the white polygons of $\overline{\mathcal{T}}$.

Lemma 4.4. *If two bicolored triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent, then the plabic graphs $\hat{G}(\mathcal{T}_1)$ and $\hat{G}(\mathcal{T}_2)$ are move-equivalent* (*see Definition* A.3)*. In other words, these two plabic graphs represent the same cell of* $\mathrm{Gr}_{k,n}^{\geq 0}$.

Proof. The fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ are equivalent means that we can get from $\mathcal{T}_1$ to $\mathcal{T}_2$ by flipping diagonals inside the black and white polygons of $\overline{\mathcal{T}_1}$. A flip inside a white polygon does not change the plabic graph, while a flip inside a black polygon corresponds to performing a square move on the plabic graph. So $\hat{G}(\mathcal{T}_1)$ and $\hat{G}(\mathcal{T}_2)$ are move-equivalent. □

In light of Lemma 4.4, we let $S_{\hat{G}(\overline{\mathcal{T}})}$ denote the cell specified by any triangulation of $\overline{\mathcal{T}}$.

Remark 4.5. We identify each black triangle T in a bicolored triangulation $\mathcal{T}$ with its three vertices $a < b < c$ listed in increasing order. We list the k black triangles

$$(T_1, \dots, T_k) = (\{a_1 < b_1 < c_1\}, \dots, \{a_k < b_k < c_k\})$$

in lexicographically increasing order, and label the white vertex inside of T_i by W_i.

For example, we list the five black triangles of the bicolored triangulation $\mathcal{T}_1$ from Figure 1 in the order

$$(\{1 < 7 < 8\}, \{1 < 8 < 9\}, \{2 < 3 < 4\}, \{2 < 4 < 7\}, \{4 < 6 < 7\}).$$

We label the white vertices of $\hat{G}(\mathcal{T}_1)$ in Figure 2 so as to reflect this ordering on black triangles.

Definition 4.6 (Statistics of bicolored triangulations)**.** Given a bicolored triangulation $\mathcal{T}$ of type (k, n) and a pair of vertices h, j of $\mathbf{P}_n$, we say that the arc $h \to j$ is:

- *compatible* with $\mathcal{T}$ if the arc does not cross any arcs of the underlying bicolored subdivision $\overline{\mathcal{T}}$, i.e. it either bounds a polygon of $\overline{\mathcal{T}}$ or it lies entirely inside a black or white polygon;
- a *black arc* of $\mathcal{T}$ if it bounds a black triangle of $\mathcal{T}$;
- *facet-defining* if it bounds a black polygon of $\overline{\mathcal{T}}$ on its left.

In particular, each black arc of $\mathcal{T}$ is compatible with $\mathcal{T}$.

When $h \to j$ is compatible with $\mathcal{T}$, we let $\text{area}(h \to j) = \text{area}_{\mathcal{T}}(h \to j)$ denote the number of black triangles to the left of $h \to j$ in any triangulation of $\overline{\mathcal{T}}$ which uses $h \to j$.

For example, the arcs $1 \to 8$, $1 \to 7$ and $2 \to 6$ are compatible with the bicolored triangulation $\mathcal{T}$ from Figure 2, and we have $\text{area}(1 \to 8) = 4$, $\text{area}(1 \to 7) = 3$, $\text{area}(2 \to 6) = 2$. However, the arcs $2 \to 8$ and $3 \to 8$ are not compatible with $\mathcal{T}$.

We can easily write down representative matrices for points in $S_{\hat{G}(\mathcal{T})}$ using the theory of Kasteleyn matrices. Note that matrices with the same pattern of zero/nonzero entries appeared in [ŁPSV19] (though the authors did not prove Proposition 4.7 there).

Proposition 4.7. *Let $\mathcal{T}$ be a bicolored triangulation of type (k, n). We let*

$$(\{a_1 < b_1 < c_1\}, \dots, \{a_k < b_k < c_k\})$$

denote the list of k black triangles of $\mathcal{T}$, written in lexicographically increasing order, as in Remark 4.5. *Choose a set of edge-weights for the graph $\hat{G}(\mathcal{T})$, which we write as*

$$(\alpha, \beta, \gamma) = ((\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2), \dots, (\alpha_k, \beta_k, \gamma_k)) \in (\mathbb{R}_{>0})^{3k},$$

with $\alpha_i, \beta_i, \gamma_i$ denoting the weights on the edges from W_i to B_{a_i}, B_{b_i} and B_{c_i}, respectively.

Let $M_{\mathcal{T}}(\alpha, \beta, \gamma) = (M_{i,j})$ be the $k \times n$ matrix with precisely 3 *nonzero entries in each row:*

$$M_{i,a_i} = \alpha_i, \quad M_{i,b_i} = (-1)^{\text{area}(a_i \to b_i)} \beta_i, \quad M_{i,c_i} = (-1)^{\text{area}(a_i \to b_i) + \text{area}(b_i \to c_i)} \gamma_i. \tag{4.8}$$

Then the cell $S_{\hat{G}(\mathcal{T})}$ is the image of the map $(\mathbb{R}_{>0})^{3k} \to \text{Gr}_{k,n}^{\geq 0}$ sending $(\alpha, \beta, \gamma) \mapsto M_{\mathcal{T}}(\alpha, \beta, \gamma)$.

Note that $M_{\mathcal{T}}(\alpha,\beta,\gamma)$ has rows and columns indexed by the white and black vertices of $\hat{G}(\mathcal{T})$. The ij-entry is nonzero if and only if there is an edge e in $\hat{G}(\mathcal{T})$ between W_i and B_j, and in that case is (up to a sign) equal to the weight of e.

Remark 4.9. Clearly the image of the map $(\alpha,\beta,\gamma)\mapsto M_{\mathcal{T}}(\alpha,\beta,\gamma)$ is unchanged if we rescale each row of the matrix so that the leftmost nonzero entry is 1, i.e. set each $\alpha_i=1$. This map is then a homeomorphism from $(\mathbb{R}_{>0})^{2k}$ to the positroid cell $S_{\hat{G}(\mathcal{T})}$.

Proof of Proposition 4.7. This follows from Theorem A.7 and Remark A.8. For completeness, we sketch why the choice of signs of entries is correct. For a black triangle $T_i=\{a<b<c\}$ of $\mathcal{T}$, define

$$\epsilon_{i,a} := (-1)^{\#\{j<i:a_i=a_j\}},\quad \epsilon_{i,b} := \epsilon_{i,a}\cdot(-1)^{\mathrm{area}(a_i\to b_i)},\quad \epsilon_{i,c} := \epsilon_{i,a}\cdot(-1)^{\mathrm{area}(a_i\to c_i)+1}.$$

Let $(d_1,\dots,d_k)$ be a tuple of distinct vertices of black triangles of $\mathcal{T}$ such that $d_i\in T_i$. The sign of the permutation σ such that $d_{\sigma(1)}<\dots<d_{\sigma(k)}$ is the product $\epsilon_{1,d_1}\cdots\epsilon_{k,d_k}$. Then:

$$p_I(M)E_I=\sum_{(d_1,\dots,d_k)} M_{1,d_1}\cdots M_{k,d_k}\langle e_{d_1},\dots,e_{d_k}\rangle=\sum_{(d_1,\dots,d_k)}(\epsilon_{1,d_1}M_{1,d_1})\cdots(\epsilon_{k,d_k}M_{k,d_k})\,E_I,$$

where the sum is over the collections defined above satisfying $\{d_1,\dots,d_k\}=I$. A sufficient condition for $p_I(M)\ge 0$ is that $\operatorname{sgn} M_{i,d_i}=\epsilon_{i,d_i}$. Up to rescaling the row i of M by ϵ_{i,a_i}, this is true, as $\mathrm{area}(a_i\to c_i)=\mathrm{area}(a_i\to b_i)+\mathrm{area}(b_i\to c_i)+1$. □

Example 4.10. For example, the matrix $M_{\mathcal{T}_1}$ corresponding to the bicolored triangulation $\mathcal{T}_1$ from Figure 2 is

$$\begin{pmatrix}\alpha_1&0&0&0&0&0&-\beta_1&-\gamma_1&0\\ \alpha_2&0&0&0&0&0&0&\beta_2&\gamma_2\\ 0&\alpha_3&\beta_3&\gamma_3&0&0&0&0&0\\ 0&\alpha_4&0&-\beta_4&0&0&\gamma_4&0&0\\ 0&0&0&\alpha_5&0&\beta_5&\gamma_5&0&0\end{pmatrix}. \tag{4.11}$$

Proposition 4.7 says that if we let the parameters $((\alpha_1,\beta_1,\gamma_1),\dots,(\alpha_5,\beta_5,\gamma_5))$ range over all elements of $(\mathbb{R}_{>0})^{15}$, the matrices (4.11) will sweep out all points of the cell $S_{\hat{G}(\mathcal{T}_1)}$.

Remark 4.12. The matrices constructed in Proposition 4.7 may have nonpositive maximal minors rather than nonnegative maximal minors. To obtain a matrix which has nonnegative maximal minors, multiply row j by $(-1)^{\#\{i<j:a_i=a_j\}}$.

Lemma 4.13. *Let $\mathcal{T}=\{T_1,\dots,T_k\}$ be a bicolored triangulation of type (k,n). Then $P_I\neq 0$ on the positroid cell $S_{\hat{G}(\mathcal{T})}$ if and only if there is a bijection $\phi : I=\{i_1,\dots,i_k\}\to\{T_1,\dots,T_k\}$ with i a vertex of $\phi(i)$ for all i.*

Proof. It suffices to show that the Plücker coordinate P_I is nonzero on $S_{\hat{G}(\mathcal{T})}$ if and only if there is a bijection $\phi : I=\{i_1,\dots,i_k\}\to\{T_1,\dots,T_k\}$ with i a vertex of $\phi(I)$ for all i.

By Theorem A.7, $p_I\neq 0$ on $S_{\hat{G}(\mathcal{T})}$ if and only if there is a matching M of $\hat{G}(\mathcal{T})$ such that $\partial M=I$. Note that any matching M of $\hat{G}(\mathcal{T})$ consists of k edges, obtained by pairing each white vertex W_j with one of its three incident black vertices $\{B_{a_j},B_{b_j},B_{c_j}\}$. The k black vertices $\{B_{i_1},\dots,B_{i_k}\}$ obtained in this way must be distinct (since M is a matching), so we get a bijection between $I:=\{i_1,\dots,i_k\}$ and the black triangles $T_1,\dots,T_k$. Moreover $\partial M=I$. □

Now, we turn to the open Grasstopes $Z^{\circ}_{\hat{G}(\mathcal{T})}$ and their properties.

Theorem 4.14 (Definite signs of twistor coordinates). *Let $\mathcal{T}$ be a bicolored triangulation of type (k, n) and let $Y := CZ \in \mathrm{Gr}_{k,k+2}$, where C is a matrix representing a point of the cell $S_{\hat{G}(\mathcal{T})}$. Choose $h < j$ such that the chord $h \to j$ is compatible with $\mathcal{T}$. Then*

$$\mathrm{sgn}\langle YZ_hZ_j\rangle = (-1)^{\mathrm{area}(h\to j)}, \text{ or equivalently, } (-1)^{\mathrm{area}(h\to j)}\langle YZ_hZ_j\rangle > 0. \tag{4.15}$$

In other words, we have that

$$Z^{\circ}_{\hat{G}(\mathcal{T})} \subseteq \{Y \in \mathrm{Gr}_{k,k+2} \mid (4.15) \text{ holds for all arcs } h \to j \text{ compatible with } \mathcal{T}\}. \tag{4.16}$$

Proof. We start by choosing a bicolored triangulation $\mathcal{T}_1$ such that $\overline{\mathcal{T}_1} = \mathcal{T}$, and such that the chord $h \to j$ is one of the diagonals of $\mathcal{T}_1$. By Lemma 4.4, the choice of $\mathcal{T}_1$ does not affect the corresponding positroid cell. By Lemma 3.8, it suffices to verify (4.15) for each $Y := \langle Z_{i_1}, \dots, Z_{i_k}\rangle$ indexed by $\{i_1 < \cdots < i_k\} = I$ such that $p_I \neq 0$ on the cell $S_{\hat{G}(\mathcal{T})}$. And by Lemma 4.13, $p_I \neq 0$ on $S_{\hat{G}(\mathcal{T})}$ if and only if there is a bijection $\phi : I = \{i_1, \dots, i_k\} \to \{T_1, \dots, T_k\}$ with i a vertex of $\phi(i)$ for all i.

Towards this end, choose $I = \{i_1 < \cdots < i_k\}$ such that $p_I \neq 0$ on the cell $S_{\hat{G}(\mathcal{T})}$. We need to calculate $\mathrm{sgn}\langle Z_{i_1}, \dots, Z_{i_k}, Z_h, Z_j\rangle$.

If $h \in I$ or $j \in I$, $\langle Z_{i_1}, \dots, Z_{i_k}, Z_h, Z_j\rangle = 0$. So without loss of generality, we can assume that h and j are not elements of I. Recall that maximal minors of Z are positive: this means that for any ordered sequence $\ell_1 < \cdots < \ell_{k+2}$, we have $\mathrm{sgn}\langle Z_{\ell_1}, \dots, Z_{\ell_{k+2}}\rangle = 1$. To determine $\mathrm{sgn}\langle Z_{i_1}, \dots, Z_{i_k}, Z_h, Z_j\rangle$, we need to know how many swaps are required to put the sequence $(i_1, \dots, i_k, h, j)$ in order. Any i_ℓ which is greater than both h and j needs to get swapped past both of them, which has no effect on the sign of the determinant. Any i_ℓ which is less than both h and j does not need to get swapped past either. Each i_ℓ such that $h < i_\ell < j$ needs to get swapped past h (but not j). Therefore the parity of the number of swaps required to put the sequence $(i_1, \dots, i_k, h, j)$ in order is the same as the parity of $\#\{i_\ell \in I : h < i_\ell < j\}$. It follows that $\mathrm{sgn}\langle Z_{i_1}, \dots, Z_{i_k}, Z_h, Z_j\rangle = (-1)^{\#\{i_\ell \in I : h < i_\ell < j\}}$. Finally, the existence of the bijection ϕ means that $\#\{i_\ell \in I : h < i_\ell < j\}$ is the number of black triangles of $\mathcal{T}_1$ which are to the left of $h \to j$.

To complete the proof, we must show that there is some $I \in \binom{[n]}{k}$ containing neither h nor j such that p_I is nonzero. Equivalently, we must find a matching of $\hat{G}(\mathcal{T}_1)$ which does not have h or j in its boundary. We do so by induction on the number of black triangles of $\mathcal{T}_1$. Clearly there is such a matching if $\hat{G}(\mathcal{T}_1)$ has a single black triangle. If $h \to j$ is contained in a white polygon of $\mathcal{T}_1$, we cut along $h \to j$ to obtain two smaller bicolored triangulations $\mathcal{T}_2$ and $\mathcal{T}_3$. By induction, we can find matchings of $\hat{G}(\mathcal{T}_2)$ and $\hat{G}(\mathcal{T}_3)$ avoiding h and j; their union gives the desired matching of $\hat{G}(\mathcal{T}_1)$. Otherwise, $h \to j$ is the boundary of a black triangle T_r of $\mathcal{T}_1$. Let c be the third vertex of this triangle. Cut $\mathcal{T}_1$ along $h \to j$, $j \to c$ and $c \to h$ to obtain bicolored triangulations of smaller polygons. The $\hat{G}$ plabic graphs of these bicolored triangulations have matchings avoiding h, j, c by induction, since each smaller polygon contains exactly two of these vertices. The union of these matchings, together with the edge from B_c to W_r, gives the desired matching of $\mathcal{T}_1$. □

The following result solves a kind of 'inverse problem:' given $Y \in Z^{\circ}_{\hat{G}(\mathcal{T})}$, we can construct a particular matrix representative $C^{\mathrm{tw}}_{\mathcal{T}}(Y)$ of $\mathrm{Gr}_{k,n}$ whose image in $\mathcal{A}_{n,k,2}(Z)$ is Y.

Definition 4.17 (Twistor coordinate matrix). Let $Y \in \mathrm{Gr}_{k,k+2}$ and let $\mathcal{T}$ be a bicolored triangulation of type (k, n) with black triangles $T_1, \dots, T_k$ labeled as in Remark 4.5. The *twistor coordinate matrix* of Y is the $k \times n$ matrix $C^{\mathrm{tw}}_{\mathcal{T}}(Y) = (C_{i,j})$ with precisely 3 nonzero entries in each row:

$$C_{i,a_i} = \langle Y b_i c_i \rangle, \quad C_{i,b_i} = -\langle Y a_i c_i \rangle, \quad C_{i,c_i} = \langle Y a_i b_i \rangle. \tag{4.18}$$

(Recall that e.g. $\langle Y b_i c_i \rangle$ is short-hand for $\langle Y Z_{b_i} Z_{c_i} \rangle$.)

Theorem 4.19 (Inverse problem). *Let $\mathcal{T}$ be a bicolored triangulation of type (k, n) with black triangles $T_1, \dots, T_k$ labeled as in Remark 4.5. Let $Y \in Z^{\circ}_{\hat{G}(\mathcal{T})}$, i.e. $Y := \tilde{Z}(V)$ for some $V \in S_{\hat{G}(\mathcal{T})}$. Then V is the row span of the twistor coordinate matrix $C' := C^{\mathrm{tw}}_{\mathcal{T}}(Y)$.*

In other words, if we let $Y' = C'Z$, then there is a global scalar λ (a polynomial in $\langle Yab \rangle$'s) such that

$$\langle Y' i j \rangle = \lambda \langle Y i j \rangle \text{ for all } i, j.$$

Example 4.20. Let $\mathcal{T}_1$ be the bicolored triangulation from Figure 2. Theorem 4.19 says that if $V \in S_{\hat{G}(\mathcal{T}_1)}$ and $Y := \tilde{Z}(V)$ is the image of V in $\mathcal{A}_{n,k,2}(Z)$, then V is the row span of the following matrix:

$$\begin{pmatrix} \langle YZ_7Z_8 \rangle & 0 & 0 & 0 & 0 & 0 & -\langle YZ_1Z_8 \rangle & \langle YZ_1Z_7 \rangle & 0 \\ \langle YZ_8Z_9 \rangle & 0 & 0 & 0 & 0 & 0 & 0 & -\langle YZ_1Z_9 \rangle & \langle YZ_1Z_8 \rangle \\ 0 & \langle YZ_3Z_4 \rangle & -\langle YZ_2Z_4 \rangle & \langle YZ_2Z_3 \rangle & 0 & 0 & 0 & 0 & 0 \\ 0 & \langle YZ_4Z_7 \rangle & 0 & -\langle YZ_2Z_7 \rangle & 0 & 0 & \langle YZ_2Z_4 \rangle & 0 & 0 \\ 0 & 0 & 0 & \langle YZ_6Z_7 \rangle & 0 & -\langle YZ_4Z_7 \rangle & \langle YZ_4Z_6 \rangle & 0 & 0 \end{pmatrix}.$$

Proof. Choose a weight vector (α, β, γ) so that the matrix $C := M_{\mathcal{T}}(\alpha, \beta, \gamma)$ from Proposition 4.7 represents V.

Consider a black triangle $\{a < b < c\}$ of $\mathcal{T}$. Let W be the white vertex of $\hat{G}(\mathcal{T})$ in the middle of this triangle and let the edges from W to B_a, B_b, and B_c, respectively, be denoted e_a, e_b, and e_c. Say the weights of these edges are α, β, and γ, respectively.

Choose $J \in \binom{[n]}{k-1}$ which does not contain a, b, or c. Then

$$\frac{1}{\alpha} p_{J \cup \{a\}}(C) = \frac{1}{\beta} p_{J \cup \{b\}}(C) = \frac{1}{\gamma} p_{J \cup \{c\}}(C). \tag{4.21}$$

Indeed, each Plücker coordinate is a sum of weights of matchings. Any matching M_a contributing to $p_{J \cup \{a\}}(C)$ must include an edge covering the white vertex W. Since $b, c \notin J \cup \{a\}$, this edge must be e_a. Now, $M_b := M_a \setminus \{e_a\} \cup \{e_b\}$ is a valid matching because M_a does not include any edges covering B_b. Moreover, the boundary of M_b is $J \cup \{b\}$. This is easily seen to be a bijection between matchings with boundary $J \cup \{a\}$ and matchings with boundary $J \cup \{b\}$. It is also easy to see that $\mathrm{wt}(M_a)/\alpha = \mathrm{wt}(M_b)/\beta$, so the first equality above holds. The second equality is similar.

Now, we consider the twistor coordinate

$$\langle Ybc \rangle = \sum_{I \in \binom{[n]}{k}} p_I(C) \langle Z_{i_1}, Z_{i_2}, \dots, Z_{i_k}, Z_b, Z_c \rangle$$

which is nonzero by Theorem 4.14.

Notice that the terms in this sum indexed by I containing b or c are zero. Further, for $I \cap \{b, c\} = \emptyset$, $p_I(C)$ is zero if I does not contain a. So we can rewrite $\langle Ybc \rangle$ as

$$\langle Ybc \rangle = \alpha \cdot \sum_{\substack{J \in \binom{[n]}{k-1}: \\ \{a,b,c\} \cap J = \emptyset}} \frac{1}{\alpha} p_{J \cup a}(C) \langle Z_{j_1}, Z_{j_2}, \dots, Z_a, \dots Z_{j_{k-1}}, Z_b, Z_c \rangle, \tag{4.22}$$

where $Z_{j_1}, \dots, Z_a, \dots, Z_{j_{k-1}}$ are ordered so the indices are increasing.

Similarly, we can write

$$\langle Yac \rangle = \beta \cdot \sum_{\substack{J \in \binom{[n]}{k-1}: \\ \{a,b,c\} \cap J = \emptyset}} \frac{1}{\beta} \, p_{J \cup b}(C) \langle Z_{j_1}, Z_{j_2}, \dots, Z_b, \dots Z_{j_{k-1}}, Z_a, Z_c \rangle, \tag{4.23}$$

$$\langle Yab \rangle = \gamma \cdot \sum_{\substack{J \in \binom{[n]}{k-1}: \\ \{a,b,c\} \cap J = \emptyset}} \frac{1}{\gamma} \, p_{J \cup c}(C) \langle Z_{j_1}, Z_{j_2}, \dots, Z_c, \dots Z_{j_{k-1}}, Z_a, Z_b \rangle. \tag{4.24}$$

Consider a nonzero term in (4.22), which is indexed by J such that $p_{J \cup a}(C)$ is nonzero. The corresponding term in (4.23) is also nonzero. Because of the first equality in (4.21), these two terms differ only by the sign $(-1)^s$, where

$$\langle Z_{j_1}, Z_{j_2}, \dots, Z_a, \dots Z_{j_{k-1}}, Z_b, Z_c \rangle = (-1)^s \langle Z_{j_1}, Z_{j_2}, \dots, Z_b, \dots Z_{j_{k-1}}, Z_a, Z_c \rangle.$$

In other words, $s = |J \cap [a+1, b-1]| + 1 = |(J \cup a) \cap [a+1, b-1]| + 1$. Because $J \cup a$ is the boundary of some matching, the size of $(J \cup a) \cap [a+1, b-1]$ is exactly $\text{area}(a \to b)$, and in particular does not depend on J.

Similarly, consider the term of (4.24) indexed by J. The sign difference between this term and the corresponding one in (4.23) is $(-1)^s$, where

$$\langle Z_{j_1}, Z_{j_2}, \dots, Z_b, \dots Z_{j_{k-1}}, Z_a, Z_c \rangle = (-1)^s \langle Z_{j_1}, Z_{j_2}, \dots, Z_c, \dots Z_{j_{k-1}}, Z_a, Z_b \rangle.$$

It is not hard to see that $s = \text{area}(b \to c) + 1$.

Altogether, we have

$$\begin{aligned} \langle Ybc \rangle &= \alpha \cdot Q, \\ \langle Yac \rangle &= (-1)^{\text{area}(a \to b)+1} \beta \cdot Q, \\ \langle Yab \rangle &= (-1)^{\text{area}(a \to b)+\text{area}(b \to c)} \gamma \cdot Q, \end{aligned}$$

where Q is a nonzero scalar. Notice that up to the factor of Q, these three twistor coordinates recover the entries of C corresponding to the edges e_a, e_b, and e_c. This means that the matrix C' with nonzero entries

$$C'_{j,a_j} = \langle Y b_j c_j \rangle, \quad C'_{j,b_j} = -\langle Y a_j c_j \rangle, \quad C'_{j,c_j} = \langle Y a_j b_j \rangle$$

is related to $M_{\hat{G}(\mathcal{T})}(\alpha, \beta, \gamma)$ by rescaling rows, and so also represents the subspace V. □

Using Theorem 4.19, we can show that $\tilde{Z}$ is injective on $S_{\hat{G}(\mathcal{T})}$, and moreover prove that $\tilde{Z}$ is not injective on any other $2k$-dimensional positroid cells. This will prove the conjectural characterization of positroid tiles from [ŁPSV19]. We note that the injectivity of $\tilde{Z}$ on $S_{\hat{G}(\mathcal{T})}$ was also proved rather indirectly in [ŁPW20, Proposition 6.4] using results of [BH19].

Theorem 4.25 (Characterization of positroid tiles). *Fix $k < n$ and $Z \in \text{Mat}^{>0}_{n,k+2}$. Then $\tilde{Z}$ is injective on the $2k$-dimensional cell $S_{\mathcal{M}}$ if and only if $S_{\mathcal{M}} = S_{\hat{G}(\overline{\mathcal{T}})}$ for some bicolored subdivision $\overline{\mathcal{T}}$ of type (k, n). That is, the positroid tiles for $\mathcal{A}_{n,k,2}(Z)$ are exactly the Grasstopes $Z_{\hat{G}(\overline{\mathcal{T}})}$, where $\overline{\mathcal{T}}$ is a bicolored subdivision of type (k, n).*

Corollary 4.26. *Whether or not $\overline{\tilde{Z}(S_\pi)}$ is a positroid tile is independent of Z.*

Proof of Theorem 4.25. This proof uses some facts from Section 8. We first show that all cells $S_{\hat{G}(\overline{\mathcal{T}})}$ are positroid tiles. The cell $S_{\hat{G}(\overline{\mathcal{T}})} \subset \mathrm{Gr}_{k,n}^{\geq 0}$ is $2k$-dimensional because it is T-dual to an $(n-1)$-dimensional cell in $\mathrm{Gr}_{k+1,n}^{\geq 0}$ (see Remark 8.13) and T-duality preserves codimension (see Proposition 8.1). Say $V, V' \in S_{\hat{G}(\mathcal{T})}$ are represented by matrices C, C', and suppose $Y := CZ, Y' := C'Z$ represent the same subspace. Then by Theorem 4.19 V and V' are represented by the twistor coordinate matrices N and N' of Y and Y', respectively. But the twistor coordinates of Y and Y' are the same up to a global scalar, so $V = V'$.

Now, suppose a $2k$-dimensional cell $S_{\mathcal{M}}$ is not equal to $S_{\hat{G}(\overline{\mathcal{T}})}$ for any $\overline{\mathcal{T}}$. We will show $\tilde{Z}$ is not injective on $S_{\mathcal{M}}$.

First, suppose M has a *coloop* c; that is, p_I is identically 0 on $S_{\mathcal{M}}$ for all $I \in \binom{[n]}{k}$ that do not contain c. Then the twistor coordinate $\langle Ycj\rangle$ is identically zero on $Z^\circ_{\mathcal{M}}$ for all j. Indeed, in the sum

$$\langle Ycj\rangle = \sum_{I\in\binom{[n]}{k}} p_I(C)\langle Z_{i_1}\dots Z_{i_k} Z_c Z_j\rangle,$$

$p_I(C)$ is zero for $c \notin I$ and $\langle Z_{i_1}\dots Z_{i_k} Z_c Z_j\rangle$ is zero for $c \in I$. In particular, $Z^\circ_{\mathcal{M}}$ is contained in the hypersurface $\{Y \in \mathrm{Gr}_{k,k+2} : \langle Yc(c+1)\rangle = 0\}$, and so has dimension at most $2k-1$. So $\tilde{Z}$ is not injective on $S_{\mathcal{M}}$.

Now, if $\mathcal{M}$ does not have a coloop, then $S_{\mathcal{M}}$ is T-dual to an $(n-1)$-dimensional cell S_π of $\mathrm{Gr}_{k+1,n}^{\geq 0}$ by Proposition 8.1. Because $S_{\mathcal{M}}$ is not of the form $S_{\hat{G}(\mathcal{T})}$, a plabic graph G with trip permutation π is not a tree and so has at least one internal face. Since G has n faces total, G is not connected.

Let G be a plabic graph with trip permutation π, and say $[i, j-1]$, $[j, l]$ are the boundary vertex sets of two connected components of G. There is a single boundary face f which is adjacent to $i-1$, i, $j-1$ and j. In the plabic graph $\hat{G}$ for $S_{\mathcal{M}}$ (constructed in Proposition 8.8), notice that i and j are adjacent to the same black vertex, $\hat{b}(f)$. After adding bivalent white vertices to $\hat{G}$ so that every boundary vertex is adjacent to a white vertex, it is clear that all matchings of $\hat{G}$ have either i or j in the boundary. This means that if I contains neither i nor j, then p_I is identically zero on $S_{\mathcal{M}}$. Just as in the coloop case, $\langle Yij\rangle$ is identically zero on $Z^\circ_{\mathcal{M}}$, because all terms of

$$\sum_{I\in\binom{[n]}{k}} p_I(C)\langle Z_{i_1}\dots Z_{i_k} Z_i Z_j\rangle$$

vanish for $C \in S_{\mathcal{M}}$. So $Z^\circ_{\mathcal{M}}$ is contained in a hypersurface and hence $\dim Z^\circ_{\mathcal{M}} \leq 2k - 1$. □

Remark 4.27. As conjectured in [ŁPSV19], the number of positroid tiles for $\mathcal{A}_{n,k,2}$ is sequence A175124 in the OEIS [S$^+$], a refinement of the *large Schröder numbers* (see Section 12).

Refining (4.16), we will now give an explicit description of each open positroid tile as a subset of $\mathrm{Gr}_{k,k+2}$ where certain twistor coordinates have a definite sign. In fact, since there are generally multiple bicolored triangulations represented by of one bicolored subdivision $\overline{\mathcal{T}}$, Theorem 4.28 gives multiple descriptions of each open positroid tile – one for each bicolored triangulation represented by $\overline{\mathcal{T}}$.

Theorem 4.28 (Sign characterization of positroid tiles). *Fix $k < n$, $m = 2$, and $Z \in \mathrm{Mat}^{>0}_{n,k+2}$. Let $\mathcal{T}$ be a bicolored triangulation of type (k, n). Then we have*

$$Z^{\circ}_{\hat{G}(\overline{\mathcal{T}})} = \{Y \in \mathrm{Gr}_{k,k+2} \mid \mathrm{sgn}\langle Y i j\rangle = (-1)^{\mathrm{area}(i\to j)} \textit{ for all black arcs } i \to j \textit{ of } \mathcal{T} \textit{ with } i < j\}.$$

Moreover, if $Y \in Z^{\circ}_{\hat{G}(\mathcal{T})}$, then $C' := C^{\mathrm{tw}}_{\mathcal{T}}(Y)$ (cf. Definition 4.17) lies in the positroid cell $S_{\hat{G}(\mathcal{T})}$, and Y and $C'Z$ represent the same element of $\mathrm{Gr}_{k,k+2}$.

In the proof of Theorem 4.28, we use the notation $N_i(A) := \#\{a \in A : a < i\}$ and $N_{i,j}(A) := \#\{a \in A : i < a < j\}$. We will need the following lemmas.

Lemma 4.29. *Let $S \in \binom{[n]}{k+3}$, and define $\omega^S \in \mathbb{R}^n$ as*

$$\omega^S_i = \begin{cases} (-1)^{N_i(S)}\langle Z_{S\setminus\{i\}}\rangle & \textit{if } i \in S,\\ 0 & \textit{else}. \end{cases}$$

Then ω^S is in the left kernel of Z.

Proof. We have that

$$(\omega^S)^T \cdot Z = \sum_{i=1}^{n} Z_i \omega^S_i = \sum_{i\in S} (-1)^{N_i(S)} Z_i \langle Z_{S\setminus\{i\}}\rangle = \sum_{i\in S} \epsilon_{\{i\},S\setminus\{i\}} Z_i \langle Z_{S\setminus\{i\}}\rangle.$$

From the rightmost expression, one can see that the jth coordinate of $\omega^S \cdot Z$ is the determinant of the submatrix of Z using rows S and columns $1, \dots, j, j, \dots, k+2$, written using Laplace expansion along column j. Therefore it is zero. □

Proposition 4.30. *Let $Y \in \mathrm{Gr}_{k,k+2}$ and let $\mathcal{T}$ be a bicolored triangulation of type (k, n). Let $C^{\mathrm{tw}}_{\mathcal{T}} = C^{\mathrm{tw}}_{\mathcal{T}}(Y)$ be the twistor coordinate matrix of Y and let $Y' := C^{\mathrm{tw}}_{\mathcal{T}} Z$. Then*

$$\mathrm{rowspan}(Y') \subseteq \mathrm{rowspan}(Y).$$

Proof. We start by writing $Y = CZ$, where C is a full-rank $k \times n$ matrix (we can always do this because the linear map $Z : \mathbb{R}^n \to \mathbb{R}^{k+2}$ is surjective).

Let $C_1, \dots, C_k$ be the rows of the matrix C. We will replace each row C_i with a linear combination of the rows of C *and* a linear combination of elements of $\ker(Z)$ to obtain a new matrix C'; by construction, the rowspan of $C'Z$ is contained in the rowspan of CZ. We will show that this new matrix C' is equal to $C^{\mathrm{tw}}_{\mathcal{T}}$.

Specifically, let $T_i = \{a < b < c\}$ be a black triangle in $\mathcal{T}$. The ith row C'_i of C' is

$$C'_i := \sum_{j=1}^{k} \lambda_j C_j + \sum_{S\in\binom{[n]}{k+3}} \rho_S \omega^S, \text{ where} \tag{4.31}$$

$$\lambda_j = \sum_{J\in\binom{[n]}{k}} (-1)^{j+1+N_a(J)+N_{b,c}(J)} p_{J\setminus\{a\}}(C_{\hat{j}})\langle Z_{J\cup\{b,c\}}\rangle$$

$$\text{and } \rho_S = (-1)^{N_a(S)+N_{b,c}(S)} p_{S\setminus\{a,b,c\}}(C),$$

where $C_{\hat{j}}$ denotes the matrix obtained from C by removing row j, and we make the convention that $p_{A\setminus B}(C) = 0$ if B is not contained in A, and $\langle Z_{A\cup B}\rangle = 0$ if A intersects B.

Step 1. We first show that $\langle Ybc\rangle = C'_{ia}$. By (4.31), we have

$$C'_{ia} := \sum_{j=1}^{k} \lambda_j C_{ja} + \sum_{S\in\binom{[n]}{k+3}} \rho_S \omega_a^S. \tag{4.32}$$

Let us expand $\langle Ybc\rangle$ as:

$$\langle Ybc\rangle = \sum_{J\in\binom{[n]}{k}} (-1)^{N_{b,c}(J)} p_J(C)\langle Z_{J\cup\{b,c\}}\rangle. \tag{4.33}$$

Call the terms in this sum with $a \in J$ "type A" and the other terms "type B."

When $a \in J$, we can compute $p_J(C)$ by Laplace expansion around column a:

$$p_J(C) = \sum_{j=1}^{k}(-1)^{j+1+N_a(J)} p_{J\setminus\{a\}}(C_{\hat{j}})\, C_{ja}.$$

Inserting this into the type A terms and summing over J, we obtain the first term in the right hand side of (4.32).

For the type B terms, we can change the summation index in (4.33) from J to $S = J \cup \{a, b, c\}$, obtaining:

$$\sum_{S\in\binom{[n]}{k+3}} (-1)^{N_{b,c}(S\setminus\{a,b,c\})} p_{S\setminus\{a,b,c\}}(C)\langle Z_{S\setminus\{a\}}\rangle.$$

Since $a < b < c$, we have $N_{b,c}(S \setminus \{a, b, c\}) = N_{b,c}(S)$. This gives the second term in the right hand side of (4.32). Hence, summing the terms of type A and type B we get exactly C'_{ia}.

Step 2. We will show that $\langle Yac\rangle = -C'_{ib}$.

Let us consider the first term ('type A') in the right hand side of (4.31). We observe that:

$$\sum_{j=1}^{k}(-1)^{j+1+N_a(J)} p_{J\setminus\{a\}}(C_{\hat{j}})C_{jb} = p_J(C^{a\to b}) = (-1)^{N_{a,b}(J)} p_{J\setminus\{a\}\cup\{b\}}(C),$$

where $C^{a\to b}$ is the matrix C with column a substituted with column b. Noting that $N_{a,b}(J) + N_{b,c}(J) = N_{a,c}(J)$ as terms with $b \in J$ do not contribute, type A reads:

$$\sum_{J\in\binom{[n]}{k}} (-1)^{N_{a,c}(J)} p_{J\setminus\{a\}\cup\{b\}}(C)\langle Z_{J\cup\{b,c\}}\rangle.$$

Finally, we change summation index into $J' = J \setminus \{a\} \cup \{b\}$ and use $N_{a,c}(J') = N_{a,c}(J' \setminus \{b\} \cup \{a\}) + 1$, as $b \in J'$ and $a < b < c$, to obtain:

$$-\sum_{J'\in\binom{[n]}{k}:\, b\in J'} (-1)^{N_{a,c}(J')} p_{J'}(C) Z_{J'\cup\{a,c\}}. \tag{4.34}$$

Let us consider the second term ('type B') in the right hand side of (4.31). Using $N_b(S) - N_a(S) = N_{a,b}(S) - 1$ and $N_{a,b}(S) + N_{b,c}(S) = N_{a,c}(S) - 1$, as $a, b \in S$, type B reads:

$$\sum_{S\in\binom{[n]}{k+3}} (-1)^{N_{a,c}(S)} p_{S\setminus\{a,b,c\}}(C)\langle Z_{S\setminus\{b\}}\rangle.$$

Finally, we perform the change of summation index into $J = S \setminus \{a,b,c\}$ and note that $N_{a,c}(S) = N_{a,c}(J \cup \{a,b,c\}) = N_{a,c}(J) + 1$, as $b \notin J$ and $a < b < c$. We obtain:

$$-\sum_{J \in \binom{[n]}{k}:\, b \notin J} (-1)^{N_{a,c}(J)} p_J(C) \langle Z_{J \cup \{a,c\}} \rangle. \tag{4.35}$$

Hence adding together (4.34) and (4.35) we immediately get $-\langle Yac \rangle = C'_{ib}$.

Step 3. Showing that $\langle Yab \rangle = C'_{ic}$ is similar to the previous case.

Step 4. We will show that $C'_{i\ell} = 0$ for $\ell \notin \{a,b,c\}$.

Let us consider the first term ('type A') in the right hand side of (4.31). We observe that:

$$\sum_{j=1}^{k} (-1)^{j+1+N_a(J)} p_{J \setminus \{a\}}(C_{\hat{j}}) C_{j\ell} = p_J(C^{a \to \ell}) = (-1)^{\tilde{N}_{a,\ell}(J)} p_{J \setminus \{a\} \cup \{\ell\}}(C),$$

where $\tilde{N}_{a,\ell}(J)$ is defined as $N_{a,\ell}(J)$ if $a < \ell$ and $N_{\ell,a}(J)$ if $\ell < a$. Then type A reads:

$$\sum_{J \in \binom{[n]}{k}} (-1)^{N_{b,c}(J) + \tilde{N}_{a,\ell}(J)} p_{J \setminus \{a\} \cup \{\ell\}}(C) Z_{J \cup \{b,c\}}.$$

By changing the summation index into $J' = J \setminus \{a\} \cup \{\ell\}$ and noting that $N_{b,c}(J' \setminus \{\ell\} \cup \{a\}) = N_{b,c}(J' \setminus \{\ell\})$ and $\tilde{N}_{a,\ell}(J' \setminus \{\ell\} \cup \{a\}) = \tilde{N}_{a,\ell}(J')$, we obtain:

$$\sum_{J' \in \binom{[n]}{k}} (-1)^{N_{b,c}(J' \setminus \{\ell\}) + \tilde{N}_{a,\ell}(J')} p_{J'}(C) \langle Z_{J' \setminus \{\ell\} \cup \{a,b,c\}} \rangle. \tag{4.36}$$

The Type B term can be rewritten as:

$$-\sum_{S \in \binom{[n]}{k+3}} (-1)^{N_{b,c}(S) + \tilde{N}_{a,\ell}(S)} p_{S \setminus \{a,b,c\}}(C) \langle Z_{S \setminus \{\ell\}} \rangle$$

using $(-1)^{N_a(S) + N_\ell(S)} = (-1)^{\tilde{N}_{a,\ell}(S)+1}$. Indeed, $N_a(S) - N_\ell(S) = N_{\ell,a}(S) + 1$ if $\ell < a$ and $N_\ell(S) - N_a(S) = N_{a,\ell}(S) + 1$ if $\ell > a$, since $\ell, a \in S$. Finally, we perform the change of variables $J' = S \setminus \{a,b,c\}$ and note that $N_{b,c}(J' \cup \{a,b,c\}) = N_{b,c}(J')$ and $\tilde{N}_{a,\ell}(J' \cup \{a,b,c\}) = \tilde{N}_{a,\ell}(J' \cup \{b,c\})$, as $a < b < c$, obtaining:

$$-\sum_{J' \in \binom{[n]}{k}} (-1)^{N_{b,c}(J') + \tilde{N}_{a,\ell}(J' \cup \{b,c\})} p_{J'}(C) \langle Z_{J' \setminus \{\ell\} \cup \{a,b,c\}} \rangle. \tag{4.37}$$

In order to complete the proof, we need to show that the sum of type A in (4.36) with type B in (4.37) is zero. Therefore it is enough to show that:

$$(-1)^{N_{b,c}(J') + \tilde{N}_{a,\ell}(J' \cup \{b,c\})} = (-1)^{N_{b,c}(J' \setminus \{\ell\}) + \tilde{N}_{a,\ell}(J')}, \tag{4.38}$$

recalling that the only terms contributing have $\ell \in J'$ and $a, b, c \notin J'$. If $\ell < b$, then $\tilde{N}_{a,\ell}(J' \cup \{b,c\}) = \tilde{N}_{a,\ell}(J')$ and $N_{b,c}(J' \setminus \{\ell\}) = N_{b,c}(J')$. If $b < \ell < c$, then $\tilde{N}_{a,\ell}(J' \cup \{b,c\}) = \tilde{N}_{a,\ell}(J') + 1$ and $N_{b,c}(J' \setminus \{\ell\}) = N_{b,c}(J') - 1$. Finally, if $\ell > c$ then $\tilde{N}_{a,\ell}(J' \cup \{b,c\}) = \tilde{N}_{a,\ell}(J') + 2$ and $N_{b,c}(J' \setminus \{\ell\}) = N_{b,c}(J')$. Therefore (4.38) holds for all three cases and the proof that $C'_{i\ell} = 0$ when $\ell \notin \{a,b,c\}$ is complete. □

Proof of Theorem 4.28. By (4.16), we just need to show the inclusion

$$Z^{\circ}_{\hat{G}(\mathcal{T})} \supseteq \{ Y \in \mathrm{Gr}_{k,k+2} \mid \text{(4.15) holds for all black chords } h \to j \text{ of } \mathcal{T} \}.$$

We will do this using the twistor coordinate matrix $C' := C^{\mathrm{tw}}_{\mathcal{T}}(Y)$ for Y in the right-hand set. First, we show that $C' \in S_{\hat{G}(\mathcal{T})}$. The nonzero entries of C' correspond to the edges of $\hat{G}(\mathcal{T})$. By Theorem A.7 and Remark A.8, whether or not C' represents an element of $S_{\hat{G}(\mathcal{T})}$ is just a question of whether or not the nonzero entries have the correct signs. By assumption, the nonzero entries of C' have the same signs as the nonzero entries of the matrix C'' from Theorem 4.19. Since the matrix C'' represents an element of $S_{\hat{G}(\mathcal{T})}$, so does C'.

Now, let $Y' := C'Z$. By Proposition 4.30, rowspan $Y' \subseteq$ rowspan Y. Because C' is an element of $\mathrm{Gr}^{\geq 0}_{k,n}$, in fact Y' has rank k, so the two rowspans are equal. Thus, $Y = C'Z$, which shows $Y \in Z^{\circ}_{\hat{G}(\mathcal{T})}$. □

Remark 4.39. In the previous proof, the only place that we used the fact that the twistor coordinates $\langle Yhj\rangle$ associated to black arcs had particular signs was in showing that the matrix C' that we constructed has maximal minors all nonnegative (or all nonpositive). We will use this observation in Section 6.2, when we show that each positroid tile is the totally positive part of a *cluster variety*.

Corollary 4.40. *Let $\mathcal{T}$ be a bicolored triangulation of type (k,n). The map sending the $k \times n$ matrix $M := M_{\mathcal{T}}(\alpha,\beta,\gamma)$ from* (4.8) *representing a point of $S_{\hat{G}(\mathcal{T})} \cong (\mathbb{R}_{>0})^{2k}$ to $Y := MZ \in Z^{\circ}_{\hat{G}(\mathcal{T})}$ is a bijection from $S_{\hat{G}(\mathcal{T})} \cong (\mathbb{R}_{>0})^{2k}$ to $Z^{\circ}_{\hat{G}(\mathcal{T})}$, and we have*

$$\frac{M_{i,b_i}}{M_{i,a_i}} = -\frac{\langle Ya_ic_i\rangle}{\langle Yb_ic_i\rangle} \quad \textit{and} \quad \frac{M_{i,c_i}}{M_{i,a_i}} = \frac{\langle Ya_ib_i\rangle}{\langle Yb_ic_i\rangle} \tag{4.41}$$

for all black triangles $\{a_i,b_i,c_i\}$ of $\mathcal{T}$. In particular, the $2k$ ratios of twistor coordinates $\{\frac{\langle Ya_ic_i\rangle}{\langle Yb_ic_i\rangle}, \frac{\langle Ya_ib_i\rangle}{\langle Yb_ic_i\rangle}\}$ are algebraically independent.

Proof. Injectivity follows from Theorem 4.25. Surjectivity follows from Theorem 4.28. Finally, (4.41) follows from Proposition 4.7 and Theorem 4.19. □

5. The equivalence of the two definitions of the amplituhedron

In this section we will give an alternative description of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ in terms of sign flips of twistor coordinates; this description was conjectured by Arkani-Hamed–Thomas–Trnka [AHTT18, (5.6)]. In [AHTT18, Section 5.4], they sketched an argument that all elements of $\mathcal{A}_{n,k,m}(Z)$ satisfy the sign flip description; a proof using a different argument was independently given in [KW19, Corollary 3.21]. However, the opposite inclusion remained open. We will complete the proof for $m = 2$ using the results of the previous section. Finally, we will translate the sign-flip characterization of $\mathcal{A}_{n,k,2}(Z)$ into a sign-flip characterization of the $\mathcal{B}$-amplituhedron $\mathcal{B}_{n,k,2}(W)$.

Recall the definition of $\hat{Z}_i$ from Remark 2.9.

Theorem 5.1 (Sign-flip characterization of $\mathcal{A}_{n,k,2}$). *Fix $k < n$ and $Z \in \mathrm{Mat}^{>0}_{n,k+2}$. Let*

$$\mathcal{F}^{\circ}_{n,k,2}(Z) := \{Y \in Gr_{k,k+2} \mid \langle YZ_iZ_{i+1}\rangle > 0 \textit{ for } 1 \leq i \leq n-1, \textit{ and } \langle YZ_n\hat{Z}_1\rangle > 0, \\ \textit{and } \mathrm{var}(\langle YZ_1Z_2\rangle, \langle YZ_1Z_3\rangle, \dots \langle YZ_1Z_n\rangle) = k\}.$$

Then $\mathcal{A}_{n,k,2}(Z) = \overline{\mathcal{F}^{\circ}_{n,k,2}(Z)}$.

Proof. Let $\mathcal{A}^{\circ}_{n,k,2}(Z) := \tilde{Z}(\mathrm{Gr}^{>0}_{k,n})$. By Remark 2.3 and Remark 2.7, $\mathcal{A}_{n,k,2}(Z) = \overline{\mathcal{A}^{\circ}_{n,k,2}(Z)}$.

We first show that $\mathcal{A}^\circ_{n,k,2} \subseteq \mathcal{F}^\circ_{n,k,2}(Z)$. Suppose that $C \in \mathrm{Gr}^{>0}_{k,n}$ and let $Y := CZ$. Choose $1 \leq i \leq n-1$, and consider any $J = \{j_1 < \cdots < j_k\} \in \binom{[n]}{k}$. Since Z has maximal minors positive, the sign of $\langle Z_{j_1}, \dots, Z_{j_k}, Z_i, Z_{i+1}\rangle$ is determined by the parity of the number of swaps needed to put the sequence $\{j_1, \dots, j_k, i, i+1\}$ into increasing order. Clearly this number is even, so $\langle Z_{j_1}, \dots, Z_{j_k}, Z_i, Z_{i+1}\rangle \geq 0$ (with equality if $J \cap \{i, i+1\} \neq \emptyset$). Therefore by Lemma 3.8, $\langle YZ_iZ_{i+1}\rangle > 0$. The argument that $\langle YZ_n\hat{Z}_1\rangle > 0$ is similar, using the fact that the matrix with rows $Z_2, \dots, Z_n, \hat{Z}_1$ has maximal minors positive. To see that Y satisfies the sign variation condition, see the proof sketch in [AHTT18, Section 5.4] or [KW19, Corollary 3.21]. This implies that $\mathcal{A}^\circ_{n,k,2} \subseteq \mathcal{F}^\circ_{n,k,2}(Z)$ and hence $\mathcal{A}_{n,k,2} \subseteq \overline{\mathcal{F}^\circ_{n,k,2}(Z)}$.

For the other direction, we will show that $\mathcal{F}^\circ_{n,k,2}(Z) \subseteq \mathcal{A}_{n,k,2}(Z)$. Suppose $Y \in \mathcal{F}^\circ_{n,k,2}(Z)$. We want to show that we can write $\mathrm{rowspan}\, Y = \mathrm{rowspan}\, CZ$ for some $C \in \mathrm{Gr}^{\geq 0}_{k,n}$.

Since $\langle YZ_1Z_2\rangle > 0$, and $\mathrm{var}(\langle YZ_1Z_2\rangle, \langle YZ_1Z_3\rangle, \dots \langle YZ_1Z_n\rangle) = k$, we can find a sequence $1 = i_0 < i_1 < \cdots < i_k \leq n-1$ such that $\mathrm{sgn}\langle YZ_1Z_{i_\ell+1}\rangle = (-1)^\ell$ for all ℓ; choose the lexicographically minimal such sequence. Let $\mathcal{T}$ be the bicolored triangulation of type (k, n) whose k black triangles have vertices $\{1, i_\ell, i_\ell + 1\}$ for $1 \leq \ell \leq k$. By Proposition 4.30, if we let $C^{\mathrm{tw}}_{\mathcal{T}} = C^{\mathrm{tw}}_{\mathcal{T}}(Y)$ be the twistor coordinate matrix of Y, and $Y' := C^{\mathrm{tw}}_{\mathcal{T}}Z$, then $\mathrm{rowspan}(Y') \subseteq \mathrm{rowspan}(Y)$. To complete the proof, we need to show that $C^{\mathrm{tw}}_{\mathcal{T}} \in \mathrm{Gr}^{\geq 0}_{k,n}$, and Y' has full rank.

Using Theorem A.7 (as in the proof of Theorem 4.28), $C^{\mathrm{tw}}_{\mathcal{T}}(Y)$ is a Kasteleyn matrix associated to the bipartite graph obtained from $\hat{G}(\mathcal{T})$, as in Figure 2. (Some of the twistor coordinates $\langle YZ_1Z_i\rangle$ of Y may vanish, in which case we just erase some of the edges of the bipartite graph.) If none of the twistor coordinates vanish, Theorem 4.28 implies that all nonzero minors of $C^{\mathrm{tw}}_{\mathcal{T}}(Y)$ have the same sign. Erasing some of the edges of the bipartite graph preserves this property. We now claim that $C^{\mathrm{tw}}_{\mathcal{T}}$ has full-rank. To see this, note that if we let $I := \{i_1, \dots, i_k\}$, then $p_I(C^{\mathrm{tw}}_{\mathcal{T}}) \neq 0$. This is because when we restrict to columns $i_1, \dots, i_k$, the only nonzero entry in column i_ℓ (for $1 \leq \ell \leq k$) is the entry $\langle YZ_1Z_{i_\ell+1}\rangle$ in row ℓ, which has sign $(-1)^\ell$. Therefore $C^{\mathrm{tw}}_{\mathcal{T}} \in \mathrm{Gr}^{\geq 0}_{k,n}$, so $Y' = C^{\mathrm{tw}}_{\mathcal{T}}Z$ has full rank. □

Corollary 5.2. *Fix $k < n$, $m = 2$, and $Z \in \mathrm{Mat}^{>0}_{n,k+2}$. For any a with $1 \leq a \leq n$, we define*

$$\mathcal{F}^{\circ,a}_{n,k,2}(Z) = \{Y \in Gr_{k,k+2} \mid \langle YZ_iZ_{i+1}\rangle > 0 \text{ for } 1 \leq i \leq n-1, \text{ and } \langle YZ_n\hat{Z}_1\rangle > 0, \text{ and } \mathrm{var}(\langle YZ_aZ_{a+1}\rangle, \dots \langle YZ_aZ_n\rangle, \langle YZ_a\hat{Z}_1\rangle, \dots, \langle YZ_a\hat{Z}_{a-1}\rangle) = k\}.$$

We have $\mathcal{A}_{n,k,2}(Z) = \overline{\mathcal{F}^{\circ,a}_{n,k,2}(Z)} = \overline{\mathcal{F}^\circ_{n,k,2}(Z)}$.

Proof. The proof is nearly the same as the one for Theorem 5.1. To adapt it, in the second paragraph of that proof, we choose the sequence $i_0 < i_1 < \cdots < i_k \leq n-1$ based on examining the signs of the sequence $(\langle YZ_aZ_{a+1}\rangle, \dots \langle YZ_aZ_n\rangle, \langle YZ_a\hat{Z}_1\rangle, \dots, \langle YZ_a\hat{Z}_{a-1}\rangle)$. We then use the bicolored triangulation whose k black triangles have vertices $\{a, i_\ell, i_\ell + 1\}$ for $1 \leq \ell \leq k$. □

By combining Proposition 3.3 with Theorem 5.1 (or Corollary 5.2), we can obtain a sign-flip characterization of the $\mathcal{B}$-amplituhedron $\mathcal{B}_{n,k,2}(W)$ (see Definition 3.2).

Corollary 5.3. *Fix $k < n$ and $W \in \mathrm{Gr}_{k+2,n}^{>0}$. Let*

$$\mathcal{G}^{\circ}_{n,k,2}(W) := \{X \in \mathrm{Gr}_2(W) \mid p_{i,i+1}(X) > 0 \text{ for } 1 \leq i \leq n-1, \text{ and } p_{n,\hat{1}}(X) > 0, \\ \text{and } \mathrm{var}((p_{12}(X), p_{13}(X), \dots p_{1n}(X)) = k\},$$

where for $i < j$, $p_{j\hat{i}}(X) := (-1)^k p_{ij}(X)$. Then $\mathcal{B}_{n,k,2}(W) = \overline{\mathcal{G}^{\circ}_{n,k,2}(W)}$.

The set $\mathcal{G}^{\circ}_{n,k,2}(W)$ should agree with the set $\mathcal{G}$ from [KW19, Prop 3.20] when $m = 2$.

6. Cluster algebras and the amplituhedron

In this section, we discuss two aspects of how amplituhedra and their positroid tiles are related to *cluster algebras* [FZ02]. We assume the reader has some familiarity with the basics of cluster algebras and cluster varieties, as in [FWZ16, GHK15].

In Section 6.1, we will discuss the *cluster adjacency* conjecture, which says that facets of a positroid tile for $\mathcal{A}_{n,k,m}$ should be naturally associated to a collection of compatible cluster variables in $\mathrm{Gr}_{m,n}$. We will prove this conjecture for $m = 2$ in Theorem 9.12.

In Section 6.2, we will prove a related but more geometric statement, which illustrates a new phenomenon in the setting of amplituhedra: we will associate a cluster variety to each positroid tile of $\mathcal{A}_{n,k,2}(Z) \subset \mathrm{Gr}_{k,k+2}$, and we will show that the positroid tile is the totally positive part of that cluster variety. We then have the strange phenomenon that the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ can be subdivided into $\binom{n-2}{k}$ $2k$-dimensional positroid tiles, each of which is the totally positive part of a cluster variety. (In contrast, most other geometric objects with a cluster structure have a unique top-dimensional stratum which is the totally positive part of a cluster variety.)

6.1. Cluster adjacency.

In 2013, Golden–Goncharov–Spradlin–Vergu–Volovich [GGS+14] established that singularities of scattering amplitudes of planar $\mathcal{N} = 4$ SYM at loop level can be described using cluster algebras. In particular, a large class of loop amplitudes can be expressed in terms of *multiple polylogarithms* whose branch points are encoded in the so-called *symbol alphabet*. Remarkably, elements of this alphabet were observed to be $\mathcal{X}$-cluster variables for $\mathrm{Gr}_{4,n}$. This enabled the powerful program of *cluster bootstrap* which pushed both the computation and the understanding of the mathematical structure of scattering amplitudes beyond the frontiers, see [CHDD+20] for a recent review. In 2017 Drummond–Foster–Gürdoğan [DFG18] enhanced the connection with cluster algebras by observing phenomena they called *cluster adjacencies*, related to compatibility of cluster variables. Shortly thereafter, they conjectured that the terms in tree-level $\mathcal{N} = 4$ SYM amplitudes coming from the BCFW recursions are rational functions whose poles correspond to compatible cluster variables of the cluster algebra associated to $\mathrm{Gr}_{4,n}$ [DFG19]. In [MSSV19], this conjecture was extended to all (rational) *Yangian invariants*, i.e. the 'building blocks' of tree-level amplitudes and leading singularities of planar $\mathcal{N} = 4$ SYM.

These conjectures can be reformulated in terms of the geometry of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ and the facets of its positroid tiles. This version of cluster adjacency for the $m = 2$ amplituhedron was studied in [ŁPSV19], and for the $m = 4$ amplituhedron in [GP20], where the authors made connections with *leading* and *Landau singularities*.

For each positroid tile $Z_{\hat{G}(\mathcal{T})}$ of $\mathcal{A}_{n,k,2}(Z)$, the corresponding Yangian invariant is a rational function[8] in the twistor coordinates. A defining property of this function is

[8]Within the framework of *positive geometries*, this is the *canonical function* of $Z_{\hat{G}(\mathcal{T})}$ [AHBL17].

that it has a simple pole at $\langle Yij\rangle = 0$ if and only if there is a facet of $Z_{\hat{G}(\mathcal{T})}$ lying on the hypersurface $\{\langle Yij\rangle = 0\}$. Let us consider the collection $\{\langle Yij\rangle\}_{\hat{G}(\mathcal{T})}$ of twistor coordinates corresponding to such poles, and identify it via Proposition 3.3 with a collection of Plücker coordinates $\{p_{ij}(X)\}_{\hat{G}(\mathcal{T})}$ in the Grassmannian $\mathrm{Gr}_{2,n}(\mathbb{C})$ (with Y the row span of $X^{\perp}Z$). These Plücker coordinates are cluster variables of the type A_{n-3} cluster algebra associated to $\mathrm{Gr}_{2,n}(\mathbb{C})$ [FZ03]. In this cluster algebra, p_{ab} and p_{cd} are *compatible* cluster variables if the arcs $a \to b$ and $c \to d$ in the polygon $\mathbf{P}_n$ do not cross. The $m = 2$ *cluster adjacency conjecture* of Łukowski–Parisi–Spradlin–Volovich [ŁPSV19] says that the cluster variables of $\mathrm{Gr}_{2,n}(\mathbb{C})$ associated to the facets of a positroid tile of $\mathcal{A}_{n,k,2}$ are compatible. We generalize this conjecture as follows.

Conjecture 6.1. *Let $Z_{\hat{G}(\mathcal{T})}$ be a positroid tile of $\mathcal{A}_{n,k,2}(Z)$. Each facet lies on a hypersurface $\langle Yij\rangle = 0$, and the collection of Plücker coordinates $\{p_{ij}\}_{\hat{G}(\mathcal{T})}$ corresponding to facets is a collection of compatible cluster variables for* $\mathrm{Gr}_{2,n}(\mathbb{C})$.

Moreover, if p_{hl} is compatible with $\{p_{ij}\}_{\hat{G}(\mathcal{T})}$, then $\langle Yhl\rangle$ has a fixed sign on $Z^{\circ}_{\hat{G}(\mathcal{T})}$.

We will prove Conjecture 6.1 in Theorem 9.12.

We now generalize Conjecture 6.1 for other m. The relevant cluster algebra is the homogeneous coordinate ring of $\mathrm{Gr}_{m,n}(\mathbb{C})$ [Sco06]. Each cluster variable is a polynomial $Q(p_I)$ in the $\binom{n}{m}$ Plücker coordinates. Each facet of a positroid tile Z_π of $\mathcal{A}_{n,k,m}$ lies on a hypersurface defined by the vanishing of some (often nonlinear) polynomial $Q(\langle YZ_I\rangle)$ in the $\binom{n}{m}$ twistor coordinates, where we write $\langle YZ_I\rangle$ for $\langle YZ_{i_1}\dots Z_{i_m}\rangle$.

Conjecture 6.2 (Cluster adjacency for $\mathcal{A}_{n,k,m}$). *Let Z_π be a positroid tile of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ and let*

$$\mathrm{Facet}(Z_\pi) := \{Q(p_I) \mid \textit{a facet of } Z_\pi \textit{ lies on the hypersurface } Q(\langle YZ_I\rangle) = 0\},$$

where Q is a polynomial in the $\binom{n}{m}$ Plücker coordinates. Then

(1) *Each $Q \in$ Facet(Z_π) is a cluster variable for* $\mathrm{Gr}_{m,n}(\mathbb{C})$.
(2) Facet(Z_π) *consists of compatible cluster variables.*
(3) *If $\tilde{Q}$ is a cluster variable compatible with* Facet(Z_π)*, the polynomial $\tilde{Q}(\langle YZ_I\rangle)$ in twistor coordinates has a fixed sign on Z°_π.*

Positroid tiles for $m = 4$ are not yet characterized.[9] In general, the polynomials appearing in the sets Facet(Z_π) are unknown. Moreover, for $n \geq 8$, there is no classification of the cluster variables of $\mathrm{Gr}_{4,n}$. Also note that Part (1) of Conjecture 6.2 is in a similar spirit to [Lam16b, Conjecture 19.8].

6.2. **Positroid tiles are totally positive parts of cluster varieties.** In this subsection, we build a cluster variety $\mathcal{V}_{\overline{\mathcal{T}}}$ in $\mathrm{Gr}_{k,k+2}(\mathbb{C})$ for each positroid tile $Z_{\hat{G}(\overline{\mathcal{T}})}$ of $\mathcal{A}_{n,k,2}$. Each bicolored triangulation represented by $\overline{\mathcal{T}}$ gives a *seed torus* of $\mathcal{V}_{\overline{\mathcal{T}}}$. We will show that the positroid tile $Z^{\circ}_{\hat{G}(\overline{\mathcal{T}})}$ is exactly the *totally positive part* of $\mathcal{V}_{\overline{\mathcal{T}}}$.

Fix a bicolored subdivision $\overline{\mathcal{T}}$ of type (k, n), with black polygons $P_1, \dots, P_r$. For each black polygon P_i, fix an arc $h_i \to j_i$ with $h_i < j_i$ in the boundary of P_i. We call this the *distinguished boundary arc* of P_i. We will build $\mathcal{V}_{\overline{\mathcal{T}}}$ by defining seeds in the field of rational functions on $\mathrm{Gr}_{k,k+2}(\mathbb{C})$.

[9] Conjecturally they are images of positroid cells with *intersection number* one [GP20], which correspond to 'rational' Yangian invariants [MSSV19].

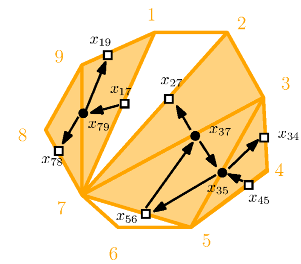


FIGURE 3. In orange, a bicolored triangulation $\mathcal{T}$. In black, the seed $\Sigma_{\mathcal{T}}$. The distinguished boundary arcs are $2 \to 3$ and $8 \to 9$.

Definition 6.3 (Cluster variables)**.** Let $a \to b$ with $a < b$ be an arc which is contained in a black polygon P_i and is not the distinguished boundary arc $h_i \to j_i$. We define

$$x_{ab} := \frac{(-1)^{\text{area}(a\to b)}\langle Yab\rangle}{(-1)^{\text{area}(h_i\to j_i)}\langle Yh_ij_i\rangle}.$$

This is a rational function on $\text{Gr}_{k,k+2}(\mathbb{C})$ and is regular away from the hypersurface $\{\langle Yh_ij_i\rangle = 0\}$.

Definition 6.4 (Seeds)**.** Let $\mathcal{T}$ be a bicolored triangulation represented by $\overline{\mathcal{T}}$. The *quiver* $Q_{\mathcal{T}}$ is obtained as follows:

- Place a frozen vertex on each boundary arc of $P_1, \dots, P_r$ and a mutable vertex on every other black arc of $\mathcal{T}$.
- If arcs $a \to b$, $b \to c$, $c \to a$ form a triangle, put arrows between the corresponding vertices, going clockwise around the triangle. Then delete the frozen vertex on the distinguished boundary arc (and all arrows involving this vertex) and arrows connecting two frozen vertices.

We label the vertex of $Q_{\mathcal{T}}$ on arc $a \to b$ of $\mathcal{T}$ with the function x_{ab}. The collection of vertex labels is the (*extended*) *cluster* $\mathbf{x}_{\mathcal{T}}$. The pair $(Q_{\mathcal{T}}, \mathbf{x}_{\mathcal{T}})$ is the *seed* $\Sigma_{\mathcal{T}}$.

Note that there are no frozen variables corresponding to the distinguished boundary arcs, and the cluster $\mathbf{x}_{\mathcal{T}}$ has size $2k$. Note also that $\Sigma_{\mathcal{T}}$ does not depend on the triangulation of the white polygons of $\overline{\mathcal{T}}$. See Figure 3 for an example.

Now we show that each seed gives a *seed torus* in $\text{Gr}_{k,k+2}(\mathbb{C})$.

Proposition 6.5. *Let $\mathcal{T}$ be a bicolored triangulation represented by $\overline{\mathcal{T}}$. Consider the Zariski-open subset*

$$\mathcal{V}_{\mathcal{T}} := \left\{ Y \in \text{Gr}_{k,k+2}(\mathbb{C}) : \prod_{a\to b \text{ black arc of } \mathcal{T}} \langle Yab\rangle \neq 0 \right\}.$$

This is birational to an algebraic torus of dimension $2k$, with field of rational functions $\mathbb{C}(\mathbf{x}_{\mathcal{T}})$, the field of rational functions in the cluster $\mathbf{x}_{\mathcal{T}}$.

Proof. The main idea is that Corollary 4.40—which gave a bijection between

$$Z^{\circ}_{\hat{G}(\overline{\mathcal{T}})} = \{Y \in \text{Gr}_{k,k+2}(\mathbb{R}) : \text{for all arcs } i \to j \text{ of } \mathcal{T} \text{ with } i < j, (-1)^{\text{area}(i\to j)}\langle Yij\rangle > 0\}$$

and $(\mathbb{R}_{>0})^{2k}$—extends directly to give a birational morphism from $\mathcal{V}_{\mathcal{T}}$ to $(\mathbb{C}^*)^{2k}$. When we let the edge weights $\alpha_i, \beta_i, \gamma_i$ (used to define matrix M in (4.8)) range over all nonzero complex numbers, the set of $k \times n$ matrices we get sweeps out the *open Deodhar stratum*[10] $D_{\mathcal{T}}$ as opposed to the positroid cell $S_{\hat{G}(\mathcal{T})}$. That is, the stratum $D_{\mathcal{T}} \subset \mathrm{Gr}_{k,n}(\mathbb{C})$ consists of subspaces represented by the matrices $M_{\mathcal{T}}(\alpha, \beta, \gamma)$ of (4.8), where (α, β, γ) vary over $(\mathbb{C}^*)^{3k}$ rather than $(R_{>0})^{3k}$.

Let us define the map

$$\mathcal{V}_{\mathcal{T}} \to (\mathbb{C}^*)^{2k},$$
$$Y \mapsto \mathbf{x}_{\mathcal{T}}(Y).$$

To see that the map is surjective onto a Zariski-open subset of $(\mathbb{C}^*)^{2k}$, consider some $2k$-tuple of nonzero complex numbers $\mathbf{q}_{\mathcal{T}}$. Define a weight vector (α, β, γ) for $\hat{G}(\mathcal{T})$, where for a triangle $\{a_i, b_i, c_i\}$, the weights are

$$\alpha_i = q_{b_i c_i}, \quad \beta_i = q_{a_i c_i}, \quad \gamma_i = q_{a_i b_i}.$$

(As usual, if $a \to b$ is a distinguished boundary arc, we take $q_{ab} = 1$.) Let $C := M_{\mathcal{T}}(\alpha, \beta, \gamma)$.

The matrix C lies in the Deodhar stratum $D_{\mathcal{T}}$ and so has full rank. Let $Y := CZ$. Consider an arc $a \to b$ of $\mathcal{T}$ which is in a black polygon P. From the proof of Theorem 4.19, we have

$$\langle Yab \rangle = (-1)^{\mathrm{area}(a \to b)} q_{ab} \cdot \mathcal{Q}_P,$$

where $\mathcal{Q}_P$ is a polynomial with positive coefficients in the q_{ij}'s and the minors of Z, and depends only on the polygon P. $\mathcal{Q}_P$ is generically nonzero, in which case it is easy to check that $x_{ab}(Y) = q_{ab}$. Moreover, in this case, Y is a full-rank matrix, since it has at least one nonzero twistor coordinate.

Now, suppose $\mathbf{q}_{\mathcal{T}}$ lies in the open subset O of $(\mathbb{C}^*)^{2k}$ where the polynomials $\mathcal{Q}_P$ are nonzero for all polygons P. Then Y, as defined above, lies in $\mathcal{V}_{\mathcal{T}}$ and maps to $\mathbf{q}_{\mathcal{T}}$.

The map is injective on the preimage of O. Indeed, pick $Y, Y' \in \mathcal{V}_{\mathcal{T}}$ which map to $\mathbf{q}_{\mathcal{T}} \in O$. Consider the twistor coordinate matrices $C := C^{\mathrm{tw}}_{\mathcal{T}}(Y)$ and $C' := C^{\mathrm{tw}}_{\mathcal{T}}(Y')$. Proposition 4.30 works equally well for matrices with complex entries, so the rowspans of CZ and $C'Z$ are contained in Y and Y', respectively. On the other hand, the rows of C and C' can both be rescaled to obtain the matrix $M_{\mathcal{T}}(\alpha, \beta, \gamma)$ defined above, so the rowspans of CZ and $C'Z$ are the same. Finally, because of the assumption $\mathbf{q}_{\mathcal{T}} \in O$, the matrix CZ has some nonzero twistor coordinate and so in particular is full rank. This shows the rowspan of CZ is equal to Y and to Y'. ☐

Next, we verify that the seeds given by different bicolored triangulations are related by mutation.

Proposition 6.6. *Let $\mathcal{T}$ be a bicolored triangulation represented by $\overline{\mathcal{T}}$ and let $a \to b$ correspond to a mutable vertex of $\Sigma_{\mathcal{T}}$. Let $\mathcal{T}'$ be related to $\mathcal{T}$ by flipping the arc $a \to b$. Then $\Sigma_{\mathcal{T}}$ and $\Sigma_{\mathcal{T}'}$ are related by mutation at x_{ab}.*

The seeds which can be obtained from $\Sigma_{\mathcal{T}}$ by an arbitrary sequence of mutations are exactly the seeds $\Sigma_{\mathcal{T}'}$ where $\mathcal{T}'$ is represented by $\overline{\mathcal{T}}$.

In light of Proposition 6.6, we can make Definition 6.7.

[10] Parameterizations of Deodhar strata in flag varieties are given in [MR04]; in the Grassmannian, these can be equivalently parameterized using weighted networks, as shown in [TW13].

Definition 6.7. Let $\mathcal{T}$ be a bicolored triangulation and $\overline{\mathcal{T}}$ the corresponding bicolored subdivision. We let $\mathcal{A}(\overline{\mathcal{T}})$ denote the cluster algebra $\mathcal{A}(Q_{\mathcal{T}}, \mathbf{x}_{\mathcal{T}})$.

Proof of Proposition 6.6. On the level of quivers, the first statement follows immediately from the well-known combinatorics of type A cluster algebras.

Say the arc $a \to b$ is in triangles $\{a < u < b\}$ and $\{a < b < v\}$ in $\mathcal{T}$, so $a \to b$ is flipped to $u \to v$ (the argument is analogous if instead $v < a$). We need to check that, in the field of rational functions on $\mathrm{Gr}_{k,k+2}(\mathbb{C})$, we have

$$x_{ab}x_{uv} = x_{au}x_{bv} + x_{av}x_{ub}$$

(where $x_{h_i j_i}$ is defined to be 1). This follows easily from the 3-term Plücker relations for the corresponding twistor coordinates.

The second statement follows from the fact that triangulations are flip-connected. □

Together, Proposition 6.5 and Proposition 6.6 tell us that the union of the seed tori is a cluster variety in $\mathrm{Gr}_{k,k+2}(\mathbb{C})$.

Theorem 6.8. *Let $\overline{\mathcal{T}}$ be a bicolored subdivision of type (k, n). Then*

$$\mathcal{V}_{\overline{\mathcal{T}}} := \bigcup_{\mathcal{T}} \mathcal{V}_{\mathcal{T}}$$

is a cluster variety in $\mathrm{Gr}_{k,k+2}(\mathbb{C})$*, where the union is over bicolored triangulations represented by $\overline{\mathcal{T}}$. We call $\mathcal{V}_{\overline{\mathcal{T}}}$ the* amplituhedron (cluster) variety[11] *of* $Z_{\hat{G}(\mathcal{T})}$.

Moreover, the positive part

$$\mathcal{V}_{\overline{\mathcal{T}}}^{\geq 0} := \{Y \in \mathcal{V}_{\overline{\mathcal{T}}} : x_{ab}(Y) > 0 \text{ for all cluster variables } x_{ab}\}$$

is equal to the positroid tile $Z^{\circ}_{\hat{G}(\overline{\mathcal{T}})}$.

Proof. The first assertion follows from the definition of cluster variety and Propositions 6.5 and 6.6.

For the second statement, note that by Theorem 4.28, points of $Z^{\circ}_{\hat{G}(\overline{\mathcal{T}})}$ are in the positive part $\mathcal{V}_{\overline{\mathcal{T}}}^{\geq 0}$. To see the opposite inclusion, take a point Y in the positive part and choose a bicolored triangulation $\mathcal{T}$ represented by $\overline{\mathcal{T}}$. Let $C := C^{\mathrm{tw}}_{\mathcal{T}}(Y)$ be the twistor coordinate matrix of Y.

If row i of C corresponds to a triangle in $\mathcal{T}$ lying in polygon P_i, rescale row i by $(-1)^{\mathrm{area}(h_i \to j_i)}/\langle Y h_i j_i \rangle$. Call the resulting matrix C'. Because $x_{ab} > 0$ for all arcs $a \to b$ of $\mathcal{T}$, the entry $\langle Yab \rangle$ of C has been rescaled to a real number with sign $(-1)^{\mathrm{area}(a \to b)}$. By the same argument as the last paragraph of Theorem 4.28, C' (and thus C) represents an element of $S_{\hat{G}(\mathcal{T})}$. This, together with Proposition 4.30, implies that $Y \in Z^{\circ}_{\hat{G}(\overline{\mathcal{T}})}$. □

Theorem 6.9. *The cluster algebra $\mathcal{A}(\overline{\mathcal{T}})$ equals the upper cluster algebra $\overline{\mathcal{A}}(\overline{\mathcal{T}})$. If the bicolored subdivision $\overline{\mathcal{T}}$ has black polygons $P_1, \dots, P_r$, where P_i has n_i vertices, then $\mathcal{A}(\overline{\mathcal{T}})$ is a finite type cluster algebra of Cartan-Killing type $A_{n_1-2} \times \cdots \times A_{n_r-2}$.*

Proof. The quiver we are associating to each bicolored triangulation is a disjoint union quiver associated to a triangulated n_i-gon, or equivalently to $\mathbb{C}[\mathrm{Gr}_{2,n_i-2}]$. Notice that

[11] This is closely related to the amplituhedron variety defined in [Lam16a].

for each one of these quivers, the corresponding exchange matrix has full $\mathbb{Z}$-rank (the argument is very similar to the one in [FWZ17, Proof of Theorem 5.3.2]).

It is well known that the quiver associated to a triangulated r-gon has Cartan-Killing type A_{r-2} [FZ02]. This implies that $\mathcal{A}(\overline{\mathcal{T}}) = \mathcal{A}(Q_{\mathcal{T}}, \mathbf{x}_{\mathcal{T}})$ has type $A_{n_1-2} \times \cdots \times A_{n_r-2}$.

Because our quiver is just a disjoint union of type A quivers (one from each P_i), our cluster algebra has an acyclic seed. Moreover, since the exchange matrix corresponding to each of these type A quivers is full rank, $\mathcal{A}(\overline{\mathcal{T}})$ also has a full $\mathbb{Z}$-rank exchange matrix.

Using [BFZ05, Proposition 1.8 and Remark 1.22], the fact that $\mathcal{A}(\overline{\mathcal{T}})$ has an acyclic seed and also has a full rank exchange matrix implies that the upper cluster algebra $\overline{\mathcal{A}}(\mathcal{T})$ equals the cluster algebra $\mathcal{A}(\overline{\mathcal{T}})$. □

Remark 6.10. Given what we've proved, one can make an argument as in the proof of [BFZ05, Theorem 2.10] that $\mathcal{A}(\overline{\mathcal{T}})$ is the coordinate ring of the amplituhedron variety $\mathcal{V}_{\overline{\mathcal{T}}}$ and also the closely related variety

$$V_{\overline{\mathcal{T}}} := \{Y \in \mathrm{Gr}_{k,k+2}(\mathbb{C}) \mid \langle Yij \rangle \neq 0 \text{ for } h \to j \text{ a boundary arc of a black polygon of } \overline{\mathcal{T}}\}.$$

7. Background on the hypersimplex, T-duality, and positroid tilings

In [ŁPW20], a surprising parallel was found between the amplituhedron map $\tilde{Z}$ on $\mathrm{Gr}_{k,n}^{\geq 0}$ and the moment map μ on $\mathrm{Gr}_{k+1,n}^{\geq 0}$. A correspondence called *T-duality* was used to relate Grasstopes in the amplituhedron $\mathcal{A}_{n,k,2}$ to positroid polytopes in the hypersimplex $\Delta_{k+1,n}$. In the second part of this paper, we further explore this relationship and prove some of the conjectures of [ŁPW20]. We present relevant background here.[12]

7.1. The hypersimplex $\Delta_{k+1,n}$ and positroid polytopes.

Throughout, for $x \in \mathbb{R}^n$ and $I \subset [n]$, we use the notation $x_I := \sum_{i \in I} x_i$.

Definition 7.1 (The hypersimplex). Let $e_I := \sum_{i \in I} e_i \in \mathbb{R}^n$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. The $(k+1, n)$*-hypersimplex* $\Delta_{k+1,n}$ is the convex hull of the points e_I where I runs over $\binom{[n]}{k+1}$.

Remark 7.2. The hypersimplex $\Delta_{k+1,n}$ is obtained by intersecting the unit hypercube $□_n$ with the hyperplane $x_{[n]} = k+1$. Alternatively, under the projection $P : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$, $\Delta_{k+1,n}$ is linearly equivalent to

$$\tilde{\Delta}_{k+1,n} := \{(x_1, \ldots, x_{n-1}) \mid 0 \leq x_i \leq 1; k \leq x_{[n-1]} \leq k+1\} \subset \mathbb{R}^{n-1}.$$

That is, $\tilde{\Delta}_{k+1,n}$ is the slice of $□_{n-1}$ between the hyperplanes $x_{[n-1]} = k$ and $x_{[n-1]} = k+1$.

The torus $T = (\mathbb{C}^*)^n$ acts on $\mathrm{Gr}_{k+1,n}$ by scaling the columns of a matrix representative A. (This is really an $(n-1)$-dimensional torus since the Grassmannian is a projective variety.) We let TA denote the orbit of A under the action of T, and $\overline{TA}$ its closure.

The *moment map* from the Grassmannian $\mathrm{Gr}_{k+1,n}$ to $\mathbb{R}^n$ is defined as follows.

Definition 7.3 (The moment map). Let A be a $(k+1) \times n$ matrix representing a point of $\mathrm{Gr}_{k+1,n}$. The *moment map* $\mu : \mathrm{Gr}_{k+1,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}.$$

[12] We use '$k+1$' instead of 'k' here in order to match conventions of later sections.

It is well known that the image of the Grassmannian $\mathrm{Gr}_{k+1,n}$ under the moment map is the hypersimplex $\Delta_{k+1,n}$. If one restricts the moment map to $\mathrm{Gr}_{k+1,n}^{\geq 0}$ then the image is again the hypersimplex $\Delta_{k+1,n}$ [TW15, Proposition 7.10].

In general, it follows from classical work of Atiyah [Ati82] and Guillemin-Sternberg [GS82] that the image $\mu(\overline{TA})$ is a convex polytope, whose vertices are the images of the torus-fixed points, i.e. the vertices are the points e_I such that $p_I(A) \neq 0$ and $p_J(A) = 0$ for $J \neq I$. This motivates the notion of *matroid polytope*. Recall that any full rank $(k+1)\times n$ matrix A gives rise to a matroid $\mathcal{M}(A) = ([n], \mathcal{B})$, where $\mathcal{B} = \{I \in \binom{[n]}{k+1} \mid p_I(A) \neq 0\}$.

Definition 7.4. Given a matroid $\mathcal{M} = ([n], \mathcal{B})$, the (basis) *matroid polytope* $\Gamma_{\mathcal{M}}$ of $\mathcal{M}$ is the convex hull of the indicator vectors of the bases of $\mathcal{M}$:

$$\Gamma_{\mathcal{M}} \coloneqq \mathrm{convex}\{e_B \,:\, B \in \mathcal{B}\} \subset \mathbb{R}^n.$$

Matroid polytopes also have a straightforward description in terms of inequalities.

Proposition 7.5 ([Wel76]). *Let $\mathcal{M} = ([n], \mathcal{B})$ be any matroid of rank $k+1$, and let $r_{\mathcal{M}} : 2^{[n]} \to \mathbb{Z}_{\geq 0}$ be its rank function. Then the matroid polytope $\Gamma_{\mathcal{M}}$ can be described as*

$$\Gamma_{\mathcal{M}} = \{\mathbf{x} \in \mathbb{R}^n \,:\, x_{[n]} = k+1,\ x_A \leq r_{\mathcal{M}}(A) \text{ for all } A \subset [n]\},$$
$$\Gamma_{\mathcal{M}} = \{\mathbf{x} \in \mathbb{R}^n \,:\, x_{[n]} = k+1,\ x_A \geq k+1-r_{\mathcal{M}}([n] \setminus A) \text{ for all } A \subset [n]\}.$$

Here, we are interested in *positroid polytopes*, that is, matroid polytopes $\Gamma_{\mathcal{M}}$ where $\mathcal{M}$ is a positroid. They arise as $\mu(\overline{TA})$ where A is a totally nonnegative matrix. Of more interest to us, they can also be obtained as moment map images of positroid cells.

Proposition 7.6 ([TW15, Proposition 7.10]). *Let $\mathcal{M}$ be the positroid associated to the positroid cell S_π. Then $\Gamma_{\mathcal{M}} = \mu(\overline{S_\pi}) = \overline{\mu(S_\pi)}$.*

We will be particularly interested in the cells on which the moment map is injective.

Definition 7.7 (Positroid polytopes). Given a positroid cell S_π of $\mathrm{Gr}_{k+1,n}^{\geq 0}$, we let $\Gamma_\pi^\circ = \mu(S_\pi)$ and $\Gamma_\pi = \overline{\mu(S_\pi)}$, and we refer to Γ° and Γ_π as *open positroid polytopes* and *positroid polytopes*, respectively. We call Γ_π a *positroid tile* for $\Delta_{k+1,n}$ if $\dim(S_\pi) = n-1$, and μ is injective on S_π.

Theorem 7.8 (Characterization of positroid tiles of $\Delta_{k+1,n}$ [ŁPW20, Propositions 3.15, 3.16]). *Consider a positroid cell $S_G \subset \mathrm{Gr}_{k+1,n}^{\geq 0}$, with G a reduced plabic graph. Then the moment map is injective on S_G if and only if G is a forest. When G is a forest, μ is moreover a stratification-preserving homeomorphism from $\overline{S_G}$ to the polytope $\Gamma_G \subset \mathbb{R}^n$. We have $\dim S_G = \dim \Gamma_G = n - c$, where c is the number of connected components of G.*

In particular, given an $(n-1)$-dimensional cell $S_G \subset \mathrm{Gr}_{k+1,n}^{\geq 0}$, Γ_G is a positroid tile for $\Delta_{k+1,n}$ if and only if G is a tree.

7.2. **T-duality and positroid tilings.** Recall the definition of positroid tiling from Definition 1.1. Specializing to $\mathcal{A}_{n,k,2}(Z)$, we get the following.

Definition 7.9 (Positroid tilings of $\mathcal{A}_{n,k,2}$). Let $\mathcal{C} = \{Z_\pi\}$ be a collection of Grasstopes, with $\{S_\pi\}$ positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. We say that $\mathcal{C}$ is a *positroid tiling* of $\mathcal{A}_{n,k,2}(Z)$ if:

- each Grasstope Z_π is a positroid tile (i.e. $\widetilde{Z}$ is injective on S_π and $\dim Z_\pi = 2k$);
- pairs of distinct open Grasstopes Z_π° and $Z_{\pi'}^\circ$ in the collection are disjoint;

- $\cup_\pi Z_\pi = \mathcal{A}_{n,k,2}(Z)$.

Remark 7.10. Alternatively, one could define a positroid tiling as coming from a collection $\{S_\pi\}$ of cells such that $\{Z_\pi\}$ is a positroid tiling (as above) for *all* choices of Z. We use Definition 7.9 here since some objects we define will be sensitive to the choice of Z.

In the case of the hypersimplex, a positroid tiling is as follows.

Definition 7.11 (Positroid tilings of $\Delta_{k+1,n}$). Let $\mathcal{C} = \{\Gamma_\pi\}$ be a collection of positroid polytopes, with $\{S_\pi\}$ positroid cells of $\mathrm{Gr}^{\geq 0}_{k+1,n}$. We say that $\mathcal{C}$ is a *positroid tiling* of $\Delta_{k+1,n}$ if:

- each Γ_π is a positroid tile (μ is injective on S_π, and $\dim S_\pi = n - 1$);
- pairs of distinct open positroid polytopes Γ°_π and $\Gamma^\circ_{\pi'}$ in the collection are disjoint;
- $\cup_\pi \Gamma_\pi = \Delta_{k+1,n}$.

Remark 7.12. "Positroid tiling" differs slightly from "positroid triangulation" in [ŁPW20].

By Theorem 7.8, the positroid tiles of $\Delta_{k+1,n}$ are the positroid polytopes Γ_G where G is a plabic tree. And by Theorem 4.25, the positroid tiles of $\mathcal{A}_{n,k,2}(Z)$ are the Grasstopes $Z_{\hat{G}(\mathcal{T})}$ for $\mathcal{T}$ a bicolored subdivision of type (k, n). In [ŁPW20], it was conjectured that positroid tiles and the two notions of positroid tiling are related by a very simple correspondence, called *T-duality*.

Definition 7.13 (T-duality on decorated permutations). Let $\pi = a_1 a_2 \dots a_n$ be a loopless decorated permutation (written in one-line notation). The *T-dual* decorated permutation is $\hat{\pi} : i \mapsto \pi(i-1)$, so that $\hat{\pi} = a_n a_1 a_2 \dots a_{n-1}$. Any fixed points in $\hat{\pi}$ are declared to be loops.

Remark 7.14. This map was previously defined in [KWZ20, Definition 4.5] and was studied in [ŁPW20], where it was used to draw parallels between the hypersimplex $\Delta_{k+1,n}$ and the $m = 2$ amplituhedron $\mathcal{A}_{n,k,2}(Z)$. The T-duality map was also studied in [BCTJ22] in relation to quotients of positroids, and in [Gal21], in relation to critical varieties and the Ising model. The map $\pi \to \hat{\pi}$ is an $m = 2$ version of a map that appeared in [AHBC$^+$16] for the case $m = 4$.

Lemma 7.15 ([ŁPW20, Lemma 5.2]). *The T-duality map $\pi \mapsto \hat{\pi}$ is a bijection from loopless decorated permutations of type $(k+1, n)$ to coloopless decorated permutations of type (k, n). That is, the map $S_\pi \mapsto S_{\hat{\pi}}$ is a bijection from the set of loopless cells in $\mathrm{Gr}^{\geq 0}_{k+1,n}$ to the set of coloopless cells in $\mathrm{Gr}^{\geq 0}_{k,n}$.*

The philosophy of [ŁPW20] is that if the moment map behaves well on S_π, then the $\tilde{Z}$-map behaves well on $S_{\hat{\pi}}$. For example, if the image of S_π is a positroid tile for $\Delta_{k+1,n}$, then the image of $S_{\hat{\pi}}$ is a positroid tile for $\mathcal{A}_{n,k,2}(Z)$ [ŁPW20, Proposition 6.6.]. Moreover, there is a main conjecture involving positroid tilings:

Conjecture 7.16 ([ŁPW20, Conjecture 6.9]). *A collection $\{\Gamma_\pi\}$ of positroid polytopes in $\Delta_{k+1,n}$ gives a positroid tiling of $\Delta_{k+1,n}$ if and only if for all $Z \in \mathrm{Mat}^{>0}_{n,k+2}$, the collection $\{Z_{\hat{\pi}}\}$ of Grasstopes gives a positroid tiling of $\mathcal{A}_{n,k,2}(Z)$.*

In Section 8.2, we will prove a number of additional results on T-duality, upgrading it to a map on plabic graphs. We will also prove Conjecture 7.16 in Theorem 11.6.

8. T-duality on decorated permutations and plabic graphs

In this section we prove that T-duality is a poset isomorphism and can be extended to a map on plabic graphs and plabic tilings.

We refer the reader to Appendix A for the definition of decorated permutations, their affinizations, loops, coloops, etc., as well as details on plabic graphs and trips.

8.1. **T-duality as a poset isomorphism.** Here we show that the bijection from Lemma 7.15 is a poset isomorphism. Abusing notation, in this subsection we use π, ν to denote bounded affine permutations rather than decorated permutations.

Proposition 8.1 (T-duality as a poset isomorphism)**.** *T-duality is a codimension-preserving poset isomorphism between loopless cells of* $\mathrm{Gr}_{k+1,n}^{\geq 0}$ *and coloopless cells of* $\mathrm{Gr}_{k,n}^{\geq 0}$*. That is, for* π, ν *loopless decorated permutations of type* $(k+1, n)$*,* $S_\nu \subset \overline{S_\pi}$ *if and only if* $S_{\hat{\nu}} \subset \overline{S_{\hat{\pi}}}$*. Furthermore,* $\operatorname{codim} S_\nu = \operatorname{codim} S_{\hat{\nu}}$.

Proof. We will work with the poset $\mathrm{Bound}(k, n)$ of bounded affine permutations with respect to the Bruhat order [KLS13], which is dual to the poset $Q(k, n)$ (see Definition 2.2). In $\mathrm{Bound}(k, n)$, $\pi \gtrdot \nu$ if $\pi = \tau \circ \nu$ for some transposition τ and $\mathrm{inv}(\pi) = \mathrm{inv}(\nu) + 1$.

Let $\delta : \mathbb{Z} \to \mathbb{Z}$ be the map $i \mapsto i - 1$. For loopless $\pi \in \mathrm{Bound}(k+1, n)$, the T-dual of π is $\hat{\pi} = \pi \circ \delta$. Fix loopless $\pi, \nu \in \mathrm{Bound}(k+1, n)$. Note that π and $\pi \circ \delta$ have the same length. Further, $\widehat{\tau \circ \nu} = \tau \circ \nu \circ \delta = \tau \circ \hat{\nu}$. So $\pi \gtrdot \nu$ if and only if $\hat{\pi} \gtrdot \hat{\nu}$.

To extend this beyond cover relations, notice that if $\nu \in \mathrm{Bound}(k+1, n)$ has $\nu(i) = i$ or $\nu(i) = i + n$, then for all $\pi \in \mathrm{Bound}(k+1, n)$ with $\pi > \nu$, we have $\pi(i) = \nu(i)$. In matroidal terms, if the positroid $\mathcal{M}_\nu$ has a loop (resp. coloop) at i, then so does $\mathcal{M}_\pi$ for all $\mathcal{M}_\pi \subset \mathcal{M}_\nu$.

Now, $\pi \geq \nu$ if and only if there exists a maximal chain $\pi \gtrdot \pi_1 \gtrdot \cdots \gtrdot \pi_r \gtrdot \nu$. Since π is loopless, the observation in the previous paragraph shows that π_i is loopless for $i = 1, \dots, r$. Since T-duality and its inverse preserve cover relations, we have such a chain if and only if we have the chain $\hat{\pi} \gtrdot \hat{\pi}_1 \gtrdot \cdots \gtrdot \hat{\pi}_r \gtrdot \hat{\nu}$ in $\mathrm{Bound}(k, n)$, which is equivalent to $\hat{\pi} \geq \hat{\nu}$.

The codimension statement follows from the fact that the codimension of S_π in $\overline{S_\nu}$ is the length of any maximal chain from π to ν in $\mathrm{Bound}(k, n)$. □

T-duality can also be defined for arbitrary even m, as in [ŁPW20, Equation 5.13], and is also of interest for understanding the $m = 4$ amplituhedron. As is clear from [ŁPW20, Equation 5.13], the T-duality map for even m is a composition of the "$m = 2$" T-duality map $m/2$ times. Proposition 8.1 also gives us information about this composition.

Definition 8.2. Let $L^r \mathrm{Gr}_{k,n}^{\geq 0}$ be the set of cells $S_\pi \subset \mathrm{Gr}_{k,n}^{\geq 0}$ such that $\pi(i) \geq i + r$ for all i. Analogously, let us define $CL^{-r} \mathrm{Gr}_{k,n}^{\geq 0}$ to be the set of cells $S_\nu \subset \mathrm{Gr}_{k,n}^{\geq 0}$ such that $\nu(i) \leq i + n - r$ for all i. Each is ordered by inclusion on the closures of cells.

Remark 8.3. The composition of T-duality r times is a well-defined map from $L^r \mathrm{Gr}_{k+r,n}^{\geq 0}$ to $CL^{-r} \mathrm{Gr}_{k,n}^{\geq 0}$. Indeed, if $\pi(i) \geq i + r$, then applying T-duality s times gives a loopless bounded affine permutation for $s = 1, \dots, r-1$. Moreover, it is easy to see that applying T-duality r times to such a π gives a bounded affine permutation ν with $\nu(i) \leq i + n - r$.

Remark 8.4. The bounded affine permutations labeling cells in $L^r\,\mathrm{Gr}_{k,n}^{\geq 0}$ ($CL^{-r}\,\mathrm{Gr}_{k,n}^{\geq 0}$) can be equivalently described in terms of the sets $\tilde{S}(-a,b)$ defined in [GL20, Section 2].

From Proposition 8.1 we immediately have the following:

Proposition 8.5. *The composition of T-duality r times gives a poset isomorphism between* $L^r\,\mathrm{Gr}_{k+r,n}^{\geq 0}$ *and* $CL^{-r}\,\mathrm{Gr}_{k,n}^{\geq 0}$.

8.2. **T-duality as a map on plabic graphs.** T-duality extends to an operation on particular plabic graphs.

Definition 8.6. A reduced plabic graph is called *black-trivalent* (resp. *white-trivalent*) if all of its interior black (resp. white) vertices are trivalent.

Note that in particular, black-trivalent (white-trivalent) graphs have no black (white) lollipops, so their trip permutations are loopless (coloopless).

Starting from a black-trivalent graph G with trip permutation π, we now give an explicit construction of a white-trivalent graph $\hat{G}$ with trip permutation $\hat{\pi}$. This construction streamlines the bijection of [GPW19, Proposition 7.15] and [Gal21, Proposition 8.3], and phrases the bijection entirely in terms of plabic graphs, rather than plabic and zonotopal tilings.

Definition 8.7 (T-duality on plabic graphs)**.** Let G be a reduced black-trivalent plabic graph. The *T-dual* of G, denoted $\hat{G}$, is the graph obtained as follows:

(1) In each face f of G, place a black vertex $\hat{b}(f)$.
(2) “On top of” each black vertex b of G, place a white vertex $\hat{w}(b)$;
(3) For each black vertex b of G in face f, put an edge $\hat{e}$ connecting $\hat{w}(b)$ and $\hat{b}(f)$;
(4) Put $\hat{i}$ on the boundary of G between vertices $i-1$ and i and draw an edge from $\hat{i}$ to $\hat{b}(f)$, where f is the adjacent boundary face.

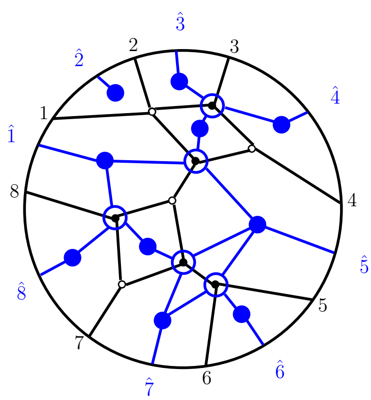


FIGURE 4. In black: A plabic graph G of type $(4,8)$ with trip permutation $(2,4,7,1,8,5,3,6)$. In blue: The T-dual plabic graph $\hat{G}$ of type $(3,8)$ with trip permutation $(6,2,4,7,1,8,5,3)$, which is built using Definition 8.7.

Proposition 8.8. *Let G be a reduced black-trivalent plabic graph with trip permutation π. Then $\hat{G}$ is a reduced white-trivalent plabic graph with trip permutation $\hat{\pi}$.*

Proof. First observe that since G is black-trivalent, $\hat{G}$ is white-trivalent (see Figure 5).

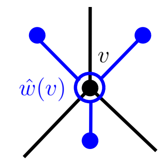


FIGURE 5. Black trivalent vertices of G correspond to white trivalent vertices of $\hat{G}$

We now show that if G has the trip $\gamma : i \to \pi(i)$, then $\hat{G}$ has the trip $\hat{\gamma} : \widehat{i+1} \to \widehat{\pi(i)}$.

Say γ starts at i. Let v be the first black vertex γ meets. By the rules of the road, there is one edge e attached to v at the left of γ (as G is black-trivalent). Note that vertex v is in the boundary face f containing boundary vertices i and $i+1$. This is because before meeting v, γ meets only white vertices, and by the rules of the road there are no edges involving these vertices lying to the left of γ. So $\hat{w}(v)$ is also connected to $\hat{b}(f)$. And by definition, $\widehat{i+1}$ is connected to $\hat{b}(f)$. Note that at the vertex $\hat{b}(f)$, if we start at the edge to $\widehat{i+1}$ and go counterclockwise, we see the edge to $\hat{w}(v)$. This means $\hat{\gamma}$ starts at $\widehat{i+1}$, goes to $\hat{b}(f)$, then to $\hat{w}(v)$ (see Figure 6). Now, let g be the face of G which contains e and the edge of γ following v. Clearly $\hat{b}(g)$ is connected to $\hat{w}(v)$. At the vertex $\hat{w}(v)$, if we start at the edge to $\hat{b}(f)$ and go clockwise, we see the edge to $\hat{b}(g)$. This means that γ goes from $\hat{w}(v)$ to $\hat{b}(g)$.

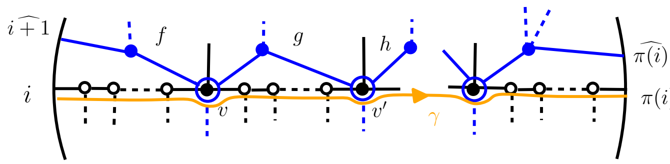


FIGURE 6. Black edges and vertices are in G; blue are $\hat{G}$. In orange, the trip $\gamma : i \to \pi(i)$ in G. The trip $\hat{\gamma}$ follows the solid blue edges.

Now, let v' be the next black vertex γ meets. Again, the edges involving any white vertices on γ between v, v' must lie to the right of γ, and there is exactly one edge e' at v' to the left of γ. So the face g also contains v'. Let h be the face of G which contains e' and the edge of γ following v'. Then $\hat{\gamma}$ goes from $\hat{b}(g)$ to $\hat{w}(v')$ to $\hat{b}(h)$ (see Figure 6). Continuing in this way, we see that if γ passes through a black vertex v, then $\hat{\gamma}$ passes through $\hat{w}(v)$ and then goes to $\hat{b}(f)$, where f is the face to the left of γ containing v and the edge of γ following v. If v is the last black vertex on γ, then f is the boundary face touching $\pi(i) - 1$ and $\pi(i)$. Note that at the vertex $\hat{b}(f)$, if we start at the edge to $\hat{w}(v)$ and go counterclockwise, we see the edge to $\widehat{\pi(i)}$. So γ will turn maximally right at $\hat{b}(f)$ to go to $\widehat{\pi(i)}$.

If γ meets no black vertices, there are no edges of G at the left of γ. This means $\pi(i) = i + 1$. The boundary face f between i and $\pi(i)$ contains only white vertices, so there will be a loop in $\hat{G}$ at boundary vertex $\widehat{i+1}$. Clearly $\hat{\pi}(i+1) = i+1 = \pi(i)$ as desired.

To show that $\hat{G}$ is reduced, it suffices to show $\hat{G}$ has $\dim(S_{\hat{\pi}}) + 1$ faces [FWZ21, Corollary 7.4.26 and Corollary 7.10.5]. Note that Definition 8.7 does not depend on the white vertices of G, so we may assume that G is bipartite and has a white vertex adjacent to every boundary vertex. With this assumption, it is not hard to see that the faces of $\hat{G}$ are in bijection with white vertices of G.

Let B, W, F, E denote the number of white vertices, black vertices, faces (excluding the infinite face), and edges (excluding edges between two boundary vertices) of G. Say that G is of type $(k+1, n)$. Since T-duality preserves codimension, we have

$$\dim(S_{\hat{\pi}}) = \dim(S_{\pi}) - n + 2k + 1.$$

As G is reduced, $F = \dim(S_{\pi}) + 1$. So to show $W = \dim(S_{\hat{\pi}}) + 1$, it suffices to show that $W = F - n + 2k + 1$. This follows immediately from

$$E = 3B + n, \quad F = 1 - (W + B) + E, \quad W - B = k + 1.$$

The first equation holds because every edge between two internal vertices contains a unique black vertex, and all black vertices are trivalent. The second equation follows from Euler's formula for planar graphs. The third holds because G is type $(k+1, n)$. □

Remark 8.9. It is straightforward to check that exchanging the roles of black and white vertices in Definition 8.7 gives a map from white-trivalent plabic graphs to black-trivalent graphs. This shows that T-duality is a bijection between black-trivalent graphs of type $(k+1, n)$ and white trivalent graphs of type (k, n) (where we consider both sets of graphs up to edge contraction and bivalent vertex addition/removal).

The map $G \to \hat{G}$ can also be phrased in terms of plabic tilings[13] [OPS15], which are dual to plabic graphs. Our notion of plabic tiling is slightly looser than that in [OPS15].

Definition 8.10 (Plabic tilings)**.** Let G be any connected reduced plabic graph with n boundary vertices, and let $\mathbf{P}_n$ be a convex n-gon, whose vertices are labeled from 1 to n in clockwise order. The *plabic tiling* $\mathcal{T}(G)$ dual to G is a tiling of $\mathbf{P}_n$ by colored polygons (bigons allowed) such that: (i) it is the planar dual of G; (ii) each black (white) vertex of G is dual to a black (white) polygon in $\mathcal{T}(G)$; (iii) vertex i of $\mathbf{P}_n$ is dual to the face of G touching boundary vertices $i-1$ and i. We consider two plabic tilings $\mathcal{T}(G)$ and $\mathcal{T}'(G')$ *equivalent* if G and G' are move-equivalent.

Conversely, if $\mathcal{T}$ is a plabic tiling, the dual plabic graph $G(\mathcal{T})$ is obtained from $\mathcal{T}$ by placing a black vertex in each black polygon, a white vertex in each white polygon, and connecting two vertices whenever they correspond to two polygons which share an edge.

Figure 7 shows three move-equivalent plabic graphs and the corresponding plabic tilings.

Remark 8.11. A bicolored subdivision or triangulation $\mathcal{T}$ of type (k, n) is a plabic tiling whose dual plabic graph $G(\mathcal{T})$ is a tree plabic graph of type $(k+1, n)$. All tree plabic graphs of type $(k+1, n)$ arise in this way.

[13] We caution the reader that plabic tilings and positroid tilings are very different objects, despite having a word in common.

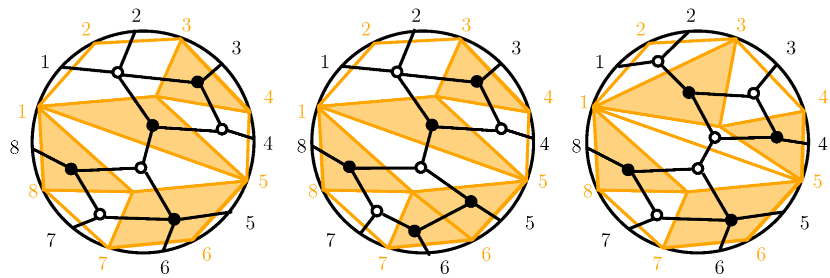


FIGURE 7. Three equivalent plabic tilings $\mathcal{T}$ (in orange), and the corresponding dual plabic graphs $G(\mathcal{T})$ (in black). The center plabic tiling is dual to a black-trivalent plabic graph.

The construction of $\hat{G}$ from G of Proposition 8.8 can also be phrased in terms of plabic tilings as follows. (This is equivalent to the construction in the proof of [Gal21, Proposition 8.3], though the description there uses horizontal sections of fine zonotopal tilings.) Figure 8 illustrates the following construction.

Proposition 8.12 (T-duality and plabic graphs)**.** *Let G be a connected reduced black-trivalent plabic graph and let $\mathcal{T} = \mathcal{T}(G)$ be the dual plabic tiling. Then the T-dual plabic graph $\hat{G} = \hat{G}(\mathcal{T})$ is obtained as follows:*

(1) *Place a black vertex at each vertex of each black triangle in $\mathcal{T}$.*
(2) *Place a white vertex in the middle of each black triangle of $\mathcal{T}$ and connect it to the vertices of the triangle.*
(3) *Add an edge of $\hat{G}$ from boundary vertex i on the disc to the black vertex on boundary vertex i of $\mathcal{T}$.*

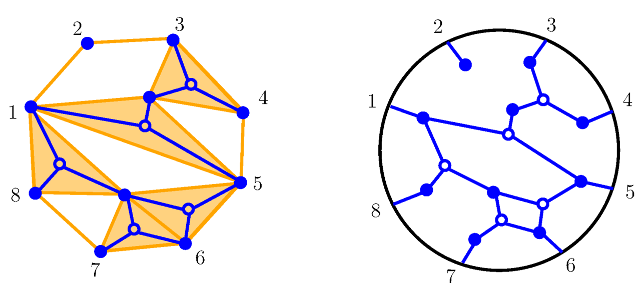


FIGURE 8. Left: In orange, the plabic tiling $\mathcal{T}$ dual to the black-trivalent graph in the center of Figure 7. In blue, the result of operations (1), (2) of Proposition 8.12. Right: the graph $\hat{G}(\mathcal{T})$.

Remark 8.13. The construction $\hat{G}(\mathcal{T})$ from Proposition 8.12 generalizes the construction from Definition 4.2 (viewing a bicolored triangulation as a special case of a plabic tiling). So Proposition 8.12 shows that the plabic graph $\hat{G}(\mathcal{T})$ from Definition 4.2 is T-dual to the plabic tree $G(\mathcal{T})$.

9. T-DUALITY, POSITROID TILES AND CLUSTER ADJACENCY

In this section, we show T-duality gives a bijection between positroid tiles of $\Delta_{k+1,n}$ and positroid tiles for $\mathcal{A}_{n,k,2}(Z)$ (Corollary 9.1). We then investigate parallels between the inequalities cutting out positroid polytopes Γ_π and the T-dual Grasstopes $Z^\circ_{\hat{\pi}}$; for positroid tiles, both pieces of data are encoded by the same bicolored subdivision (Theorem 9.2). We establish a similar parallel for facets of positroid tiles (Theorem 9.10), and use this to prove the $m = 2$ *cluster adjacency conjecture* of [ŁPSV19] in Theorem 9.12.

9.1. **T-duality, inequalities and signs.** In this subsection, we will see how bicolored triangulations encode positroid tiles of both $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$.

Theorem 4.25 and Theorem 7.8 characterize positroid tiles of $\mathcal{A}_{n,k,2}(Z)$ and $\Delta_{k+1,n}$ in terms of bicolored subdivisions and tree plabic graphs, respectively. These results with Remark 8.13 imply that positroid tiles of $\mathcal{A}_{n,k,2}(Z)$ and $\Delta_{k+1,n}$ are in bijection, and that both can be read off easily from bicolored subdivisions of type (k, n) (see Figure 9).

Corollary 9.1. *A positroid polytope Γ_G is a positroid tile of $\Delta_{k+1,n}$ if and only if the T-dual Grasstope $Z_{\hat{G}}$ is a positroid tile of $\mathcal{A}_{n,k,2}(Z)$. We read Γ_G and $Z_{\hat{G}}$ off of the same bicolored subdivision $\overline{\mathcal{T}}$ as follows:*

- *Choose any triangulation $\mathcal{T}$ of $\overline{\mathcal{T}}$.*
- *We let $G := G(\mathcal{T})$ be the dual plabic tree, as in Definition* 8.10.
- *We let $\hat{G} := \hat{G}(\mathcal{T})$ be the graph from Definition* 4.2 (*equivalently, in Proposition* 8.12).

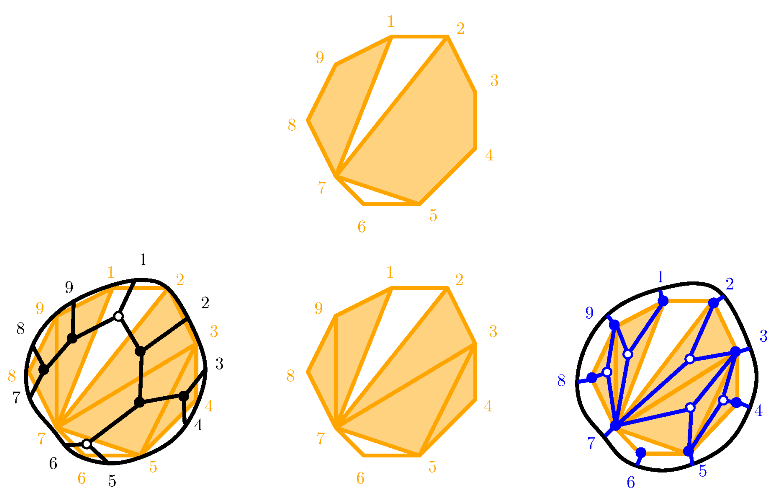


FIGURE 9. In the top row: A bicolored subdivision of type $(5,9)$ $\overline{\mathcal{T}}$. In the bottom row: A bicolored triangulation $\mathcal{T}$ obtained by triangulating $\overline{\mathcal{T}}$, with the dual graph $G(\mathcal{T})$ to its left, and the T-dual graph $\hat{G}(\mathcal{T})$ to its right.

From a bicolored subdivision $\overline{\mathcal{T}}$, we can obtain inequality descriptions of the positroid tile $\Gamma_{G(\mathcal{T})} \subset \Delta_{k+1,n}$ and the T-dual positroid tile $Z^\circ_{\hat{G}(\mathcal{T})} \subset \mathcal{A}_{n,k,2}(Z)$.

Given two positive numbers $a, b \in [n]$, the *cyclic interval* $[a, b]$ is defined to be

$$[a,b] := \begin{cases} \{a, a+1, \dots, b-1, b\} & \text{if } a \le b, \\ \{a, a+1, \dots, n, 1, \dots, b\} & \text{otherwise.} \end{cases}$$

Theorem 9.2 (Inequalities and signs via T-duality)**.** *Let $\overline{\mathcal{T}}$ be a bicolored subdivision and let $h \to j$ be a compatible arc, with $h < j$. Let $G(\mathcal{T})$ denote the tree plabic graph dual to $\mathcal{T}$, and $\hat{G}(\mathcal{T})$ the T-dual. Then:*

$$\text{area}(h \to j) + 1 > x_{[h,j-1]} > \text{area}(h \to j) \quad \textit{for } x \in \Gamma^{\circ}_{G(\mathcal{T})}, \tag{1}$$

$$\text{sgn}\langle Yhj\rangle = (-1)^{\text{area}(h\to j)} \quad \textit{for } Y \in Z^{\circ}_{\hat{G}(\mathcal{T})}. \tag{2}$$

The inequalities given by the arcs of any triangulation $\mathcal{T}'$ of $\mathcal{T}$ cut out $\Gamma^{\circ}_{G(\mathcal{T})}$ and $Z^{\circ}_{\hat{G}(\mathcal{T})}$.

Example 9.3. Consider the bicolored subdivision $\overline{\mathcal{T}}$ in Figure 9. We have

$$\begin{array}{llll} 5 > x_{[1,7]} > 4, & 4 > x_{[1,6]} > 3, & 3 > x_{[2,5]} > 2, & \text{for } x \in \Gamma^{\circ}_{G(\mathcal{T})}, \\ \langle Y18\rangle > 0, & \langle Y17\rangle < 0, & \langle Y26\rangle > 0, & \text{for } Y \in Z^{\circ}_{\hat{G}(\mathcal{T})}. \end{array}$$

To prove Theorem 9.2, we need a few results on positroid polytopes Γ_G.

Lemma 9.4. *Let G be a bipartite plabic graph and let $\mathcal{T}$ be the dual plabic tiling. Let $W(G)$ and $B(G)$ denote the set of white and black vertices of G, respectively. Then*

$$|W(G)| - |B(G)| + |\{\textit{bdry vt of } G \textit{ adjacent to a black vt}\}| = \text{area}(\mathcal{T}) - \text{punc}(\mathcal{T}) + 1,$$

where $\text{area}(\mathcal{T})$ *is the number of black triangles in any triangulation of $\mathcal{T}$ and* $\text{punc}(\mathcal{T})$ *is the number of internal vertices of $\mathcal{T}$.*

Proof. Let E denote the edges of G involving at least one internal vertex. Each black vertex of G is dual to a black polygon of $\mathcal{T}$ with $\deg(v)$ many sides, so we have

$$\begin{aligned} \text{area}(\mathcal{T}) &= \sum_{v \in B(G)} (\deg(v) - 2) = \sum_{v\in B(G)} \deg(v) - 2|B(G)| \\ &= |E(G)| - |\{\text{bdry vt adjacent to a white vt}\}| - 2|B(G)|, \end{aligned}$$

where the last equality follows from the fact that every edge of G contains a unique black vertex, except edges between a boundary vertex and a white vertex. The claim follows from this formula together with Euler's formula for planar graphs. □

Proposition 9.5. *Let $\mathcal{T}$ be a bicolored subdivision and $G(\mathcal{T})$ the dual bipartite tree plabic graph. For all arcs $h \to j$ compatible with $\mathcal{T}$, points of $\Gamma_{G(\mathcal{T})}$ satisfy*

$$\text{area}(h \to j) + 1 \ge x_{[h,j-1]} \ge \text{area}(h \to j).$$

Proof. Let G be the graph obtained from $G(\mathcal{T})$ by adding bivalent white vertices so that every boundary vertex is adjacent to a white vertex. Note that G is bipartite and represents the same positroid $\mathcal{M}$ as $G(\mathcal{T})$. In particular, the boundaries of matchings of G give the bases of $\mathcal{M}$. Let $W(G)$ and $B(G)$ denote the sets of white and black vertices of G, respectively.

Note that if $j = h + 1$, the inequality is clear.

We first deal with the case where $h \to j$ is an internal arc of $\mathcal{T}$. Let e be the edge of G which is dual to $h \to j$, and say the vertices of e are a white vertex w and black vertex b. If we remove the edge e, $G \setminus e$ has two connected components, G^w containing w and G^b

containing b. Notice that both connected components are again bipartite plabic trees. Let I^w and I^b denote the boundary vertices of G^w and G^b, respectively. Because vertex i of $\mathcal{T}$ lies between boundary vertices $i-1, i$ of G, $\{I^w, I^b\} = \{[h, j-1], [j, h-1]\}$.

Now, we would like to compute the ranks of I_w, I_b. That is, for a matching M of G, we need to compute the maximum size of $\partial M \cap I^w$ and $\partial M \cap I^b$.

Let M be a matching of G. If M does not contain e, then M restricts to a matching of G^w and G^b. It is easy to see that

$$\begin{aligned} |\partial M \cap I^w| &= |W(G^w)| - |B(G^w)|, \\ |\partial M \cap I^b| &= |W(G^b)| - |B(G^b)|. \end{aligned}$$

If M does contain e, then choose a path P from boundary to boundary which uses e and alternates between edges in M and edges not in M. Such a path can be constructed greedily because G is a tree. Orient P so it sees first w and then b. The edges of P in M are exactly the ones oriented from a white vertex to a black vertex. The first edge of P touches a boundary vertex in I^w and is oriented to a white vertex, so is not in M. The last edge of P touches a boundary vertex in I^b and is in M. Define a new matching N of G by $N := (M \setminus P) \cup (P \setminus M)$. The boundary ∂N contains one more element of I^w than ∂M, and one fewer element of I^b. The matching N does not contain e, so using the previous computation, we see that

$$\begin{aligned} |\partial M \cap I^w| &= |W(G^w)| - |B(G^w)| - 1, \\ |\partial M \cap I^b| &= |W(G^b)| - |B(G^b)| + 1. \end{aligned}$$

We conclude that $\operatorname{rank}(I^w) = |W(G^w)| - |B(G^w)|$ and $\operatorname{rank}(I^b) = |W(G^b)| - |B(G^b)| + 1$.

From Proposition 7.5 and the fact that the rank of $\mathcal{M}$ is $|W(G)| - |B(G)| = |W(G^w)| - |B(G^w)| + |W(G^b)| - |B(G^b)|$, we see that the points of $\Gamma_{G(\mathcal{T})}$ satisfy

$$\begin{aligned} |W(G^w)| - |B(G^w)| - 1 \le{} & x_{I^w} \le |W(G^w)| - |B(G^w)|, \\ |W(G^b)| - |B(G^b)| \le{} & x_{I^b} \le |W(G^b)| - |B(G^b)| + 1. \end{aligned}$$

All that remains is to rewrite the right hand sides of these inequalities in terms of area. Cut $\mathcal{T}$ along the arc $h \to j$, to get two smaller bicolored subdivisions $\mathcal{T}^w$ and $\mathcal{T}^b$ containing the polygons dual to w and b, respectively. Notice that the graph $G(\mathcal{T}^w)$ dual to $\mathcal{T}^w$ can be obtained from G^w by adding a boundary vertex adjacent to w. Similarly, $G(\mathcal{T}^b)$ is obtained from G^b by adding a boundary vertex adjacent to b. So, using Lemma 9.4,

$$\begin{aligned} |W(G^w)| - |B(G^w)| &= \operatorname{area}(\mathcal{T}^w) + 1, \\ |W(G^b)| - |B(G^b)| + 1 &= \operatorname{area}(\mathcal{T}^b) + 1. \end{aligned}$$

Now, choose $v \in \{b, w\}$ so that $I^v = [h, j-1]$. Since $\mathcal{T}^v$ is exactly the part of $\mathcal{T}$ to the left of $h \to j$, the proposition now follows.

We now consider the case where $h \to j$ is not an arc of $\mathcal{T}$. In this case, let $\mathcal{T}'$ be the plabic tiling obtained from $\mathcal{T}$ by adding the arc $h \to j$. Let G be the tree plabic graph dual to $\mathcal{T}'$, which we make bipartite by adding an appropriately colored bivalent vertex v to the edge dual to $h \to j$. We also add bivalent white vertices to G to make all boundary vertices adjacent to a white vertex. Let e and f denote the edges containing v, and say e is to the left of $h \to j$. Similar to the first case, removing edges e, f and vertex v from G gives a graph with two connected components G^e, G^f which contain vertices adjacent to e and f, respectively. Notice that the boundary vertices I^e of G^e are exactly $[h, j-1]$. The rest of the argument is very similar to the first case. □

Recall that an arc $h \to j$ of a bicolored subdivision is facet-defining if it bounds a black polygon on its left.

Proposition 9.6. *Let $\mathcal{T}$ be a bicolored subdivision, and let G be the dual plabic tree of type $(k+1, n)$. Then Γ_G is cut out of $\mathbb{R}^n$ by the equality $x_{[n]} = k+1$ and the following inequalities, each of which defines a facet:*

(1) *$x_i \geq 0$ for i a boundary vertex adjacent to a white vertex*
(2) *$x_{[h,j-1]} \geq \text{area}(h \to j)$ for $h \to j$ a facet-defining arc of $\mathcal{T}$.*

Proof. Recall from Theorem 7.8 that the moment map μ is a stratification-preserving homeomorphism on the closure of S_G. So the facets of Γ_G are exactly the positroid polytopes $\Gamma_{G'}$ where $S_{G'}$ is a positroid cell contained in $\overline{S_G}$ with codimension 1. From [Pos06, Corollary 18.10], each such cell is indexed by a reduced plabic graph G' obtained from G by removing a single edge (if the edge removed is between a boundary vertex and an internal vertex v, we also add a lollipop which is the opposite color of v).

Because G is a tree, $G' = G \setminus e$ is reduced for all edges e. If e is between a boundary vertex i and a white vertex, then G' has a black lollipop at i. Thus $S_{G'}$ has a loop at i, and $\Gamma_{G'}$ is contained in the hyperplane $x_i = 0$. Clearly Γ_G lies on the positive side of this hyperplane, which explains the facet inequalities of type 1.

If e is an edge between a boundary vertex i and a black vertex, then e is dual to the arc $(i+1) \to i$ of $\mathcal{T}$, which is a facet-defining arc. Then G' has a white lollipop at i, so $\Gamma_{G'}$ is contained in the hyperplane $x_i = 1$. Since we also have $x_{[n]} = k+1$, $\Gamma_{G'}$ is also contained in the hyperplane $x_{[i+1,i-1]} = k = \text{area}(i+1 \to i)$.

Now, consider the case when e is an edge between two internal vertices of G. The edge e is dual to the arc $h \to j$ of $\mathcal{T}$, which bounds a black polygon on the left. The proof of Proposition 9.5 shows that $\Gamma_{G'}$ is contained in the hyperplane $x_{[h,j-1]} = \text{area}(h \to j)$. This covers all edges of G, so we have described all facets. The directions of the facet inequalities follow immediately from Proposition 9.5. □

We can now prove Theorem 9.2.

Proof of Theorem 9.2. (1) follows from Proposition 9.5 and (2) follows from Theorem 4.14. The statement about inequalities cutting out $Z^{\circ}_{\hat{G}(\mathcal{T})}$ and $\Gamma^{\circ}_{G(\mathcal{T})}$ follows from Theorem 4.28, Proposition 9.6 and the fact that $x_{[n]} = \text{area}(h \to j) + \text{area}(j \to h) + 1$. □

We next generalize Theorem 9.2 by providing inequalities for full-dimensional positroid polytopes and Grasstopes from statistics of plabic tilings. We first generalize the definition of compatible arcs from Definition 4.6.

Definition 9.7 (Statistics of plabic tilings)**.** Let $\mathcal{T}$ be a plabic tiling in a convex n-gon $\mathbf{P}_n$ and h, j a pair of vertices of $\mathbf{P}_n$. We say that the arc $h \to j$ is *compatible* with $\mathcal{T}$ if the arc either bounds or lies entirely inside a single polygon of $\mathcal{T}$. When $h \to j$ is compatible with $\mathcal{T}$, we let $\text{area}(h \to j) = \text{area}_{\mathcal{T}}(h \to j)$ denote the number of black triangles to the left of $h \to j$ in any triangulation of $\mathcal{T}$. We call the internal vertices of $\mathcal{T}$ *punctures*. We also let $\text{punc}(h \to j) = \text{punc}_{\mathcal{T}}(h \to j)$ denote the number of punctures of $\mathcal{T}$ to the left of the arc $h \to j$. Note that black bigons do not contribute to the area.

For example, the tiling $\mathcal{T}$ in Figure 7 has two punctures, and $1 \to 3, 1 \to 5, 5 \to 7$ are compatible arcs. We have $\text{area}(1 \to 3) = 0$, $\text{area}(1 \to 5) = 2$, $\text{area}(5 \to 7) = 1$, $\text{punc}(1 \to 3) = \text{punc}(5 \to 7) = 0$, $\text{punc}(1 \to 5) = 1$.

Theorem 9.8. *Let $\mathcal{T}$ be a plabic tiling and let $h \to j$ be a compatible arc, with $h < j$. Let $G(\mathcal{T})$ denote the plabic graph dual to $\mathcal{T}$, and $\hat{G}(\mathcal{T})$ the T-dual. Then:*

(1) $\text{area}(h \to j) - \text{punc}(h \to j) + 1 > x_{[h,j-1]} > \text{area}(h \to j) - \text{punc}(h \to j)$ *for* $x \in \Gamma^{\circ}_{G(\mathcal{T})}$

(2) $\text{sgn}\langle Yhj\rangle = (-1)^{\text{area}(h\to j)-\text{punc}(h\to j)}$ *for* $Y \in Z^{\circ}_{\hat{G}(\mathcal{T})}$.

Note that compatible arcs depend on the tiling $\mathcal{T}$, while $\Gamma^{\circ}_{G(\mathcal{T})}$ and $Z^{\circ}_{\hat{G}(\mathcal{T})}$ depend only on $\overline{\mathcal{T}}$. Any arc compatible with any tiling equivalent to $\mathcal{T}$ gives inequalities for $\Gamma^{\circ}_{G(\mathcal{T})}$ and $Z^{\circ}_{\hat{G}(\mathcal{T})}$ via Theorem 9.8.

Proof. The proof of (1) proceeds similarly as in Proposition 9.5, where we compute the rank of $[h, j-1]$. The arc $h \to j$ is dual to an edge e of $G(\mathcal{T})$ (or a graph which differs from $G(\mathcal{T})$ only by uncontracting an edge and adding a bivalent vertex). Removing e gives two connected components, the boundary vertices of which are $[h, j-1]$ and $[j, h-1]$. Again, any matching of $G(\mathcal{T})$ will either use e or differ from a matching using e by a “swivel” (see [MS17, Appendix B]) along one of the two boundary faces containing e (which changes the boundary’s intersection with $[h, j-1]$ by precisely 1) followed by swivels at faces contained in one of the connected components (which do not change the boundary’s intersection with $[h, j-1]$). In this way we compute the rank of $[h, j-1]$ and $[j, h-1]$. One must apply Lemma 9.4 to obtain the ranks in terms of area and punc.

The proof of (2) proceeds similarly as in Theorem 4.14. Any almost-perfect matching M of $\hat{G}(\mathcal{T})$ which does not have h or j in ∂M will have $|\partial M \cap [h+1, j-1]| = \text{area}(h \to j) - \text{punc}(h \to j)$. Indeed, there are exactly $\text{area}(h \to j)$ internal white vertices to the left of $h \to j$, which must be covered by an edge of M, and exactly $\text{punc}(h \to j)$ many internal black vertices, which also must be covered. This leaves $\text{area}(h \to j) - \text{punc}(h \to j)$ edges of M which cover a boundary vertex. □

Example 9.9. For the plabic tiling $\mathcal{T}$ in Figure 7, Theorem 9.8 tells us that

$1 > x_{[1,2]} > 0,$	$2 > x_{[1,4]} > 1,$	$2 > x_{[5,6]} > 1,$	for $x \in \Gamma_{G(\mathcal{T})}$;
$\langle Y13\rangle > 0,$	$\langle Y15\rangle < 0,$	$\langle Y57\rangle < 0,$	for $Y \in Z^{\circ}_{\hat{G}(\mathcal{T})}$.

9.2. **T-duality, facets and cluster adjacency.** From a bicolored subdivision $\mathcal{T}$, we can also read off the facets of both $\Gamma_{G(\mathcal{T})}$ and $Z_{\hat{G}(\mathcal{T})}$ (see Definition 2.6).

Theorem 9.10 (Facets via T-duality). *Let $\mathcal{T}$ be a bicolored triangulation and let $h \to j$ be a facet-defining arc of $\mathcal{T}$. Let $G := G(\mathcal{T})$ be the plabic tree dual to $\mathcal{T}$ and let G' be the plabic forest obtained from G by deleting the edge dual to $h \to j$. Let $\hat{G}$ and $\hat{G}'$ denote their T-duals.*

(1) *The positroid polytope $\Gamma_{G'}$ is a facet of Γ_G, and lies on the hyperplane*

$$x_{[h,j-1]} = \text{area}(h \to j).$$

(2) *The Grasstope $Z_{\hat{G}'}$ is a facet of $Z_{\hat{G}}$, and lies on the hypersurface*

$$\langle Yhj\rangle = 0.$$

Moreover, if we let $h \to j$ range over the facet-defining arcs of $\mathcal{T}$ which are not on the boundary of $\mathbf{P}_n$, we obtain all facets of Γ_G and $Z_{\hat{G}}$ in the interior of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$.

Proof. (1) follows immediately from Proposition 9.6 and its proof.

For (2), we first show that $Z_{\hat{G}'}$ is contained in the hypersurface $\{\langle Yhj\rangle = 0\}$. The arc $h \to j$ is in a unique triangle T_r of $\mathcal{T}$; say its third vertex is i. Using Proposition 8.8, it is not hard to see that $\hat{G}'$ is obtained from $\hat{G}$ by deleting the edge e from B_i to W_r. This means that every almost perfect matching of $\hat{G}'$ must use either the edge from B_h to W_r or the edge from B_j to W_r, so h or j is in the boundary. From Lemma 3.6, we immediately conclude that $\langle Yhj\rangle$ is identically zero on $Z^\circ_{\hat{G}'}$ and thus on $Z_{\hat{G}'}$.

Now, we show that $\tilde{Z}$ is injective on $Z_{\hat{G}'}$, by showing Theorem 4.19 holds for $Z_{\hat{G}'}$ and then applying the first paragraph in the proof of Theorem 4.25. Consider $Y \in Z_{\hat{G}'}$, let $C := C^{\mathrm{tw}}_{\mathcal{T}}(Y)$ be the twistor coordinate matrix of Y and let $Y' := CZ$. We would like to show that $C \in S_{\hat{G}'}$ and that $\operatorname{rowspan} Y' = \operatorname{rowspan} Y$; by Proposition 4.30, it suffices to show the former. Note that the Kasteleyn matrix K' for $\hat{G}'$ is obtained from the Kasteleyn matrix K for $\hat{G}$ by setting the parameter in row r and column i to 0. So we only need to show that for all arcs $a \to b$ of $\mathcal{T}$ with $\{a, b\} \neq \{h, j\}$, $\langle Y'ab\rangle$ is nonzero.

Pick such an arc $a \to b$ of $\mathcal{T}$. It suffices to show that there is a matching of $\hat{G}$ which does not use e and does not have a or b in its boundary. We will argue by induction on the number of black triangles of $\mathcal{T}$. The base case, with 1 triangle, is clear by inspection. The arc $a \to b$ bounds some black triangle T_s of $\mathcal{T}$, with third vertex c. Cut $\mathcal{T}$ along the arcs $a \to b$, $b \to c$ and $c \to a$ to obtain bicolored triangulations $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ of smaller polygons (one of which may be empty), which each contain a single edge of T_s. One will have $h \to j$ as a facet-defining arc. By induction, $\hat{G}(\mathcal{T}_i)$ has an almost-perfect matching M_i whose boundary avoids the appropriate vertices of T_s and does not use the edge e. Take the union of these matchings, together with the edge f from W_s to B_c. This gives a matching of $\hat{G}$ whose boundary avoids a, b. Note that since $\{a, b\} \neq \{h, j\}$, the edge f is different from e, so this matching does not use e.

Now we check that $Z_{\hat{G}'}$ is a facet of $Z_{\hat{G}}$. We first show that the hypersurface $H := \{\langle Yhj\rangle = 0\}$ intersects $Z_{\hat{G}}$ only on the boundary of $Z_{\hat{G}}$, which shows $Z_{\hat{G}'}$ is contained in the boundary as well. Recall that the open positroid tile $Z^\circ_{\hat{G}}$ is dense in $Z_{\hat{G}}$ and moreover, $(-1)^{\mathrm{area}(h\to j)}\langle Yhj\rangle$ is positive on $Z^\circ_{\hat{G}}$ (Theorem 4.14). This implies that $(-1)^{\mathrm{area}(h\to j)}\langle Yhj\rangle$ is positive on the interior of $Z_{\hat{G}}$. Indeed, if the hypersurface H intersected the interior of $Z_{\hat{G}}$, one could find an open set in the interior where $(-1)^{\mathrm{area}(h\to j)}\langle Yhj\rangle$ is negative. (This is because $\langle Yhj\rangle$ is linear in the Plücker coordinates, so $\langle Yhj\rangle$ takes both positive and negative values on any open set in $\mathrm{Gr}_{k,k+2}$ containing a point of H). But such a set cannot be in $\overline{Z^\circ_{\hat{G}}}$.

Now we verify that $Z_{\hat{G}'}$ has the correct codimension. From the proof of Proposition 9.6, $S_{G'}$ is codimension 1 in $\overline{S_G}$. Since T-duality is a rank-preserving poset isomorphism, we also have that $S_{\hat{G}'}$ is contained in $\overline{S_{\hat{G}'}}$ and has codimension 1; that is, $S_{\hat{G}'}$ has dimension $2k-1$. Because $\tilde{Z}$ is injective on $S_{\hat{G}'}$, $Z^\circ_{\hat{G}'}$ and $Z_{\hat{G}'}$ also have dimension $2k-1$.

To see the last statement of the proposition, note that any codimension 1 cell $S_H \subset \overline{S_{\hat{G}}}$ with a coloop q will have Z_H contained in the hypersurface $\{\langle Yq(q+1)\rangle = 0\}$. So the facets Z_H avoiding the amplituhedron boundaries must come from coloopless cells S_H. These coloopless cells are T-dual to loopless codimension 1 cells contained in $\overline{S_G}$.

As $h \to j$ varies over all facet-defining arcs of $\mathcal{T}$, $S_{G'}$ varies over all such loopless cells, by Proposition 9.6. So the facets Z_H avoiding the amplituhedron boundary are of the form $Z_{\hat{G}'}$ for some arc $h \to j$. From the proof above, we see that $Z_{\hat{G}'}$ is not contained in an amplituhedron boundary precisely when $h \to j$ is not a boundary arc of $\mathcal{T}$. □

Example 9.11. Consider the bicolored subdivision $\mathcal{T}$ in Figure 1. The facet-defining arcs not on the boundary of $\mathbf{P}_9$ are $1 \to 7$, $2 \to 7$ and $4 \to 6$, with $\text{area}(1 \to 7) = \text{area}(2 \to 7) = 3$, $\text{area}(4 \to 6) = 0$. The corresponding internal facets lie on the following hyperplanes:

$$\langle Y17\rangle = 0, \qquad \langle Y27\rangle = 0, \qquad \langle Y46\rangle = 0, \qquad \text{for } Z_{\hat{G}(\mathcal{T})};$$
$$x_{[1,6]} = 3, \qquad x_{[2,6]} = 3, \qquad x_{[4,5]} = 0, \qquad \text{for } \Gamma_{G(\mathcal{T})}.$$

One facet-defining arc at the boundary of $\mathbf{P}_9$ is $2 \to 3$, with $\text{area}(2 \to 3) = 0$. This gives an external facet lying on $\langle Y23\rangle = 0$ for $Z_{\hat{G}(\mathcal{T})}$ and $x_2 = 0$ for $\Gamma_{G(\mathcal{T})}$.

We can now prove Conjecture 6.1, which extends the $m = 2$ *cluster adjacency conjecture* of Łukowski–Parisi–Spradlin–Volovich [ŁPSV19].

Theorem 9.12 (Cluster adjacency for $\mathcal{A}_{n,k,2}$). *Let $Z_{\hat{G}(\mathcal{T})}$ be a positroid tile of $\mathcal{A}_{n,k,2}(Z)$. Set* $\text{Facet}(Z_{\hat{G}(\mathcal{T})}) := \{p_{ij} \mid$ *there is a facet of $Z_{\hat{G}(\mathcal{T})}$ on the hypersurface* $\langle Yij\rangle = 0\}$. *Then:*

(1) $\text{Facet}(Z_{\hat{G}(\mathcal{T})})$ *consists of compatible cluster variables for* $\text{Gr}_{2,n}$.
(2) *If $p_{h\ell}$ is compatible with* $\text{Facet}(Z_{\hat{G}(\mathcal{T})})$*, then $\langle Yh\ell\rangle$ has a fixed sign on $Z^{\circ}_{\hat{G}(\mathcal{T})}$.*

Proof. The first part follows directly from Theorem 9.10 as $Z_{\hat{G}(\mathcal{T})}$ has a facet on $\{\langle Yij\rangle = 0\}$ if and only if $i \to j$ is a facet defining arc in $\mathcal{T}$, and the facet-defining arcs do not cross. The second part follows from Theorem 9.2. □

Using Theorems 9.2 and 9.10, we can translate the cluster adjacency theorem for the $m = 2$ amplituhedron into a cluster adjacency theorem for the hypersimplex.

Theorem 9.13 (Cluster adjacency for $\Delta_{k+1,n}$). *Let $\Gamma_{G(\mathcal{T})}$ be a positroid tile of $\Delta_{k+1,n}$.*

Set $\text{Facet}(\Gamma_{G(\mathcal{T})}) := \{p_{ij} \mid$ *there is a facet of $\Gamma_{G(\mathcal{T})}$ on the hyperplane* $x_{[i,j-1]} = a_{i,j}\}$,

where $a_{i,j}$ are some nonnegative integers. Then:

(1) $\text{Facet}(\Gamma_{G(\mathcal{T})})$ *consists of compatible cluster variables for* $\text{Gr}_{2,n}$.
(2) *If $p_{h\ell}$ is compatible with* $\text{Facet}(\Gamma_{G(\mathcal{T})})$*, then* $x_{[h,\ell-1]} > \text{area}(h \to \ell)$ *in $\Gamma^{\circ}_{G(\mathcal{T})}$.*

10. Eulerian numbers: w-simplices in $\Delta_{k+1,n}$ and w-chambers in $\mathcal{A}_{n,k,2}$

In this section we study the amplituhedron chambers of $\mathcal{A}_{n,k,2}(Z)$. Because positroid tiles in $\mathcal{A}_{n,k,2}(Z)$ are defined by sign conditions, the decomposition of $\mathcal{A}_{n,k,2}(Z)$ into chambers refines every positroid tiling. Separately, the hypersimplex $\Delta_{k+1,n}$ has a well-known decomposition into simplices which refines every positroid tiling. Both decompositions have chambers/maximal simplices which are naturally indexed by permutations of $n-1$ with k descents. We use this correspondence in Section 11 to establish results on tilings.

We begin by reviewing the decomposition of the hypersimplex $\Delta_{k+1,n}$. It is well known that the volume of the hypersimplex $\Delta_{k+1,n}$ is the *Eulerian number* $E_{k,n-1}$ [Sta12], which counts the permutations on $n-1$ letters with k descents. A triangulation of $\Delta_{k+1,n}$ into unit simplices indexed by such permutations was first discovered by Stanley [Sta77]. Sturmfels [Stu96] later gave an *a priori* different triangulation of $\Delta_{k+1,n}$. Lam and Postnikov [LP07] then gave two other triangulations, and showed that all four triangulations coincide. After defining some permutation statistics, we will define this triangulation.

Definition 10.1. Let $w \in S_n$. We call a letter $i \geq 2$ in w a *left descent* (or a *left descent top*) if i occurs to the left of $i-1$ in w. In other words, $w^{-1}(i) < w^{-1}(i-1)$. And we say that $i \in [n]$ in w is a *cyclic left descent* if either $i \geq 2$ is a left descent of w or if $i = 1$ and 1 occurs to the left of n in w, that is, $w^{-1}(1) < w^{-1}(n)$. We let $\mathrm{cDes_L}(w)$ denote the set of cyclic left descents of w, and $\mathrm{Des_L}(w)$ the set of left descents. We frequently refer to cyclic left descents as simply *cyclic descents*.

Remark 10.2. Left and right descents and descent sets are discussed extensively in [BB05, Chapter 1]. Left descents are sometimes called *recoils* in the literature.

Let $D_{k+1,n}$ be the set of permutations $w \in S_n$ with $k+1$ cyclic descents and $w_n = n$. Note that $|D_{k+1,n}| = E_{k+1,n-1}$.

Definition 10.3 (w-Simplices)**.** For $w \in D_{k+1,n}$, let $w^{(a)}$ denote the cyclic rotation of w ending at a. We define

$$I_r = I_r(w) := \mathrm{cDes_L}(w^{(r-1)}).$$

The *w-simplex* $\Delta_w \subseteq \Delta_{k+1,n}$ is the simplex with vertices $e_{I_1}, \dots, e_{I_n}$.

Example 10.4. Let $w = 324156$ in one-line notation. Then w has cyclic descents $\{1,2,3\} = I_1$. The rotation of w ending at 1 is 563241, which has cyclic descents $I_2 = \{2,3,5\}$. The rotation of w ending at 2 is 415632, which has cyclic descents $I_3 = \{1,3,4\}$.

Notice that r is always in I_r and $r-1$ is never in I_r.

The following triangulation of the hypersimplex first appeared in [Sta77], though the description there was slightly different.

Proposition 10.5 ([Sta77])**.** *The w-simplices $\{\Delta_w : w \in D_{k+1,n}\}$ are the maximal simplices of a triangulation of the hypersimplex $\Delta_{k+1,n}$. Moreover, projecting $\{\Delta_w : w \in S_n\}$ into $\mathbb{R}^{n-1}$ (see Remark 7.2), we obtain the maximal simplices in a triangulation of the hypercube $\square_{n-1}$ which refines the subdivision of the hypercube into hypersimplices.*

Remark 10.6. The w-simplex Δ_w as defined above agrees with the simplex denoted $\Delta_{(w)}$ in [LP07, Section 2.4]. In particular, the directed circuit the authors use to define $\Delta_{(w)}$ is given by $e_{I_1} \to e_{I_{w_1+1}} \to e_{I_{w_2+1}} \to \dots \to e_{I_{w_{n-1}+1}} \to e_{I_1}$. Another way to say this is I_{w_i+1} is equal to $(I_{w_{i-1}+1} \setminus \{w_i\}) \cup \{w_i+1\}$.

It follows from the results of [LP07] that every full-dimensional positroid polytope also has a triangulation into w-simplices. Indeed, the triangulation of $\Delta_{k+1,n}$ given by w-simplices is the simultaneous refinement of all positroid subdivisions of $\Delta_{k+1,n}$.

We now turn to the amplituhedron side. We define some special chambers in $\mathcal{A}_{n,k,2}(Z)$ whose sign vectors are obtained from cyclic descents of permutations. We later will show that these are precisely the realizable sign chambers (Theorem 10.10, Theorem 11.5).

Recall that for $v \in \mathbb{R}^n$, $\mathrm{Flip}(v)$ records where coordinates of v change sign (Definition 3.12).

Definition 10.7 (w-Chambers)**.** Let $w \in D_{k+1,n}$ and let the vertices of Δ_w be $e_{I_1}, \dots, e_{I_n}$, as in Definition 10.3. Then the *open amplituhedron w-chamber* $\hat{\Delta}^\circ_w(Z) := \mathcal{A}^w_{n,k,2}(Z)$ consists of $Y \in \mathcal{A}_{n,k,2}(Z)$ such that $\langle Yij \rangle \neq 0$ for $i \neq j$ and for $a = 1, \dots, n$,

$$\mathrm{Flip}(\langle Ya\hat{1}\rangle, \langle Ya\hat{2}\rangle, \dots, \langle Ya\widehat{a-1}\rangle, \langle Yaa\rangle, \langle Yaa+1\rangle, \dots, \langle Yan\rangle) = I_a \setminus \{a\}.$$

Equivalently, $\mathcal{A}^w_{n,k,2}(Z)$ consists of $Y \in \mathrm{Gr}_{k,k+2}$ such that

$$\begin{aligned} \mathrm{sgn}\langle Yaj\rangle &= (-1)^{|I_a \cap [a,j-1]|-1} \quad \text{for } j > a,\\ \mathrm{sgn}\langle Ya\hat{j}\rangle &= (-1)^{|I_a \cap [a,j-1]|-1} \quad \text{for } j < a. \end{aligned}$$

The *closed amplituhedron w-chamber* is the closure $\hat{\Delta}_w(Z) := \overline{\mathcal{A}^w_{n,k,2}(Z)}$. Abusing notation, we will often refer to closed amplituhedron w-chambers as simply *w-chambers*.

Remark 10.8. One might hope that the structure of $\hat{\Delta}_w(Z)$ does not depend on the choice of $Z \in \mathrm{Mat}^{>0}_{n,k+2}$. However, even the property that $\hat{\Delta}_w(Z)$ is nonempty depends on Z. More precisely, while we know that each $\hat{\Delta}_w(Z)$ is nonempty for some choice of Z (Theorem 11.5), it may be empty for other choices of Z (see Section 11.3).

Because the positroid tiles of $\mathcal{A}_{n,k,2}(Z)$ can be described entirely in terms of signs of twistor coordinates and the signs of twistor coordinates in $\hat{\Delta}_w(Z)$ are constant, we have Lemma 10.9. It is the analogue of the fact that for a tree positroid polytope Γ_π, either $\Delta_w \cap \Gamma^\circ_\pi = \emptyset$ or $\Delta_w \subseteq \Gamma_\pi$.

Lemma 10.9. *Let Z°_π be a positroid tile for $\mathcal{A}_{n,k,2}(Z)$ and let $\hat{\Delta}_w(Z)$ be a nonempty w-chamber. Then either $\hat{\Delta}_w(Z) \cap Z^\circ_\pi = \emptyset$ or $\hat{\Delta}_w(Z) \subset Z_\pi$.*

Despite the subtleties regarding the nonemptiness of $\hat{\Delta}_w(Z)$, the closed w-chambers always cover $\mathcal{A}_{n,k,2}(Z)$, in direct analogy to w-simplices in $\Delta_{k+1,n}$.

Theorem 10.10 ($\mathcal{A}_{n,k,2}$ is the union of w-chambers)**.** *Fix $k < n$ and $Z \in \mathrm{Mat}^{>0}_{n,k+2}$. Then*

$$\mathcal{A}_{n,k,2}(Z) = \bigcup_{w \in D_{k+1,n}} \hat{\Delta}_w(Z).$$

To prove Theorem 10.10, we use a characterization of the simplices Δ_w given by Sturmfels [Stu96], involving sorted collections. We follow the presentation of [LP07, Section 2.2].

Definition 10.11. Let $(J_1, \dots, J_t)$ be a tuple of distinct elements of $\binom{[n]}{k+1}$, where we write $J_s = \{j_{s1} < j_{s2} < \cdots < j_{s(k+1)}\}$. We call $(J_1, \dots, J_t)$ a *sorted collection* if $j_{11} \leq j_{21} \leq \cdots \leq j_{t1} \leq j_{12} \leq j_{22} \leq \cdots \leq j_{t(k+1)}$. If (J_1, J_2) is a sorted collection, we call them a *sorted pair*.

The w-simplices of $\Delta_{k+1,n}$ are exactly the simplices with vertices $e_{J_1}, \dots, e_{J_n}$ for $(J_1, \dots, J_n)$ a sorted collection. To see if a collection is sorted, one need only check pairs of elements.

Lemma 10.12. *Given $\{J_1, \dots, J_t\} \subset \binom{[n]}{k+1}$, suppose that for all $a \neq b$, either (J_a, J_b) or (J_b, J_a) is a sorted pair. Then $J_1, \dots, J_t$ can be ordered to give a sorted collection.*

Proof. First, notice that if (J_a, J_b) is a sorted pair and (J_b, J_c) is a sorted pair, then (J_a, J_c) is a sorted pair. Indeed, if $a \neq b$, then there exists i such that $j_{ai} < j_{bi} \leq j_{ci}$. It follows that (J_c, J_a) is not a sorted pair, so (J_a, J_c) must be. So on $\{J_1, \dots, J_t\}$, the property of being a sorted pair is reflexive, antisymmetric, and transitive, which means it is a partial order. We've assumed every pair is comparable, so we have a total order. The result follows. □

Proof of Theorem 10.10. Let $Y \in \mathcal{A}_{n,k,2}$ be a point whose twistor coordinates are all nonzero. We will show that Y lies in $\hat{\Delta}_w(Z)$ for some w. The points with nonzero twistor coordinates form a dense subset of $\mathcal{A}_{n,k,2}$ (their complement, a union of hypersurfaces, has codimension 1), so this will show the desired equality.

$$\text{Set } L_a := \text{Flip}(\langle Ya\hat{1}\rangle, \langle Ya\hat{2}\rangle, \dots, \langle Ya\,\widehat{a-1}\rangle, \langle Ya\;a\rangle, \langle Ya\;a+1\rangle, \dots, \langle Ya\;n\rangle).$$

By Corollary 5.2, we have $|L_a| = k$. Choose $a < b$. We will show that $I_a := L_a \cup \{a\}$ and $I_b := L_b \cup \{b\}$ are distinct and, for some ordering, form a sorted pair. We temporarily abuse notation by omitting the Y's and hats from our notation; if $a > i$, we write $\langle ai\rangle$ for $\langle Ya\hat{\imath}\rangle$.

Certain 3-term Plücker relations constrain sign flips, as noted in [AHTT18, Section 5]. For $j \in [a-2] \cup [b+1, n]$, we have the relation

$$\langle j\;j+1\rangle\langle a\;b\rangle = \langle a\;j\rangle\langle b\;j+1\rangle - \langle b\;j\rangle\langle a\;j+1\rangle \tag{10.13}$$

and for $j \in [a+1, b-2]$ we have

$$\langle j\;j+1\rangle\langle b\;a\rangle = \langle b\;j\rangle\langle a\;j+1\rangle - \langle a\;j\rangle\langle b\;j+1\rangle. \tag{10.14}$$

Because $\text{sgn}\langle j\;j+1\rangle = +$ for all j, the sign of the left hand sides of Equations (10.13) and (10.14) does not depend on j. This means that if $\text{sgn}\langle a\;b\rangle = +$, then for $j \in [a-2] \cup [b+1, n]$

$$\begin{pmatrix} \text{sgn}\langle a\;j\rangle & \text{sgn}\langle a\;j+1\rangle \\ \text{sgn}\langle b\;j\rangle & \text{sgn}\langle b\;j+1\rangle \end{pmatrix} \neq \begin{pmatrix} \delta & \epsilon \\ \epsilon & -\delta \end{pmatrix} \tag{10.15}$$

for any $\delta, \epsilon \in \{+, -\}$. Similarly, if $\text{sgn}\langle a\;b\rangle = -$, then for $j \in [a-2] \cup [b+1, n]$

$$\begin{pmatrix} \text{sgn}\langle a\;j\rangle & \text{sgn}\langle a\;j+1\rangle \\ \text{sgn}\langle b\;j\rangle & \text{sgn}\langle b\;j+1\rangle \end{pmatrix} \neq \begin{pmatrix} \delta & -\epsilon \\ \epsilon & \delta \end{pmatrix}. \tag{10.16}$$

If $\text{sgn}\langle b\;a\rangle = +$ (respectively, $-$), then for $j \in [a+1, b-1]$, the sign pattern in Equation (10.16) (respectively, Equation (10.15)) never occurs.

Suppose j is a value where the sign pattern in Equation (10.15) is forbidden. If there is a sign flip after $\langle a\;j\rangle$ and not after $\langle b\;j\rangle$, then $\text{sgn}\langle a\;j\rangle = \text{sgn}\langle b\;j\rangle$; if there is a sign flip after $\langle b\;j\rangle$ and not after $\langle a\;j\rangle$, then $\text{sgn}\langle a\;j\rangle \neq \text{sgn}\langle b\;j\rangle$. When the sign pattern in Equation (10.16) is forbidden, there are analogous statements with conclusions swapped. This means that for any interval $I \subset [n]$ where one of the patterns is forbidden, $|L_a \cap I|$ and $|L_b \cap I|$ differ by at most one and either $(L_a \cap I, L_b \cap I)$ or $(L_b \cap I, L_a \cap I)$ is sorted.[14]

Which one of $(L_a \cap I, L_b \cap I)$ and $(L_b \cap I, L_a \cap I)$ is sorted gives us additional information.

Let $P := L_a \cap [a-2]$, $Q := L_b \cap [a-2]$. Suppose $P \neq Q$, and consider the smallest j so that there is a sign flip after exactly one of $\langle a\;j\rangle$ and $\langle b\;j\rangle$. Clearly, (P, Q) is sorted if and only if there is a sign flip after $\langle a\;j\rangle$ and not after $\langle b\;j\rangle$. If the latter occurs, then $\text{sgn}\langle ab\rangle = +$ and $\text{sgn}\langle a\;1\rangle = \text{sgn}\langle b\;1\rangle$, or $\text{sgn}\langle ab\rangle = -$ and $\text{sgn}\langle a\;1\rangle \neq \text{sgn}\langle b\;1\rangle$; in short,

[14] We extend the definition of sorted in the obvious way to sets whose sizes differ by at most one. In particular, if (I, J) are sorted, we must have $|I| \geq |J|$.

$\operatorname{sgn}\langle a\, b\rangle \cdot \operatorname{sgn}\langle a\, 1\rangle = \operatorname{sgn}\langle b\, 1\rangle$. Analogously, if (Q, P) is sorted, then $\operatorname{sgn}\langle a\, b\rangle \cdot \operatorname{sgn}\langle a\, 1\rangle \neq \operatorname{sgn}\langle b\, 1\rangle$.

Let $T \coloneqq L_a \cap [a+1, b-2]$ and $U = L_b \cap [a+1, b-2]$, and suppose $T \neq U$. By essentially identical reasoning as in the previous paragraph, if (T, U) is sorted, then $\operatorname{sgn}\langle b\, a\rangle \cdot \operatorname{sgn}\langle a\, a+1\rangle \neq \operatorname{sgn}\langle b\, a+1\rangle$. Since $\langle a\, a+1\rangle > 0$ by assumption, the latter condition implies we have a sign flip between $\langle b\, a\rangle$ and $\langle b\, a+1\rangle$, so $a \in L_b$. Similar reasoning gives that if (U, T) is sorted, then $a \notin L_b$.

Let $V \coloneqq L_a \cap [b+1, n]$ and $W \coloneqq L_b \cap [b+1, n]$ and suppose that $V \neq W$. Repeating the arguments of the previous paragraphs gives that if (V, W) is sorted, then $b \notin L_a$, and if (W, V) is sorted, then $b \in L_a$.

Now, there are two cases: $|L_a \cap [b, n]|$ and $|L_b \cap [b, n]|$ have the same parity or they have opposite parity. They are similar, so we will assume we are in the first case, and leave the second to the reader.

Suppose $|L_a \cap [b, n]|$ and $|L_b \cap [b, n]|$ have the same parity. Note that $(-1)^{|L_i \cap [i+1, j-1]|}$ is $\operatorname{sgn}\langle i\, j\rangle$, and, since $b \notin L_b$, $L_b \cap [b, n]$ and $L_b \cap [b+1, n]$ are equal. So

$$\begin{aligned}\operatorname{sgn}\langle b\, n\rangle &= (-1)^{|L_a \cap [b,n]|}\\ &= (-1)^{|L_a \cap [a+1, b-1]|}(-1)^{|L_a \cap [a+1, n]|}\\ &= \operatorname{sgn}\langle a\, b\rangle \cdot \operatorname{sgn}\langle a\, 1\rangle\end{aligned}$$

and thus (P, Q) is sorted. We will show that (I_a, I_b) is sorted and $I_a \neq I_b$.

If $b \in L_a$, then $|V|$ and $|W|$ have different parity. In particular, V and W are not equal, so (W, V) is sorted and $|W| = |V| + 1$. The two sets interweave like

$$w_1 \leq v_1 \leq w_2 \leq v_2 \leq \cdots \leq w_r \leq v_r \leq w_{r+1}.$$

Since $V = I_a \cap [b+1, n]$ and $W = I_b \cap [b+1, n]$, we also have that I_a and I_b are distinct. Note that $I_b \cap [b, n] = W \cup \{b\}$ and $I_a \cap [b, n] = V \cup \{b\}$ and we have

$$b \leq b < w_1 \leq v_1 \leq w_2 \leq v_2 \leq \cdots \leq w_r \leq v_r \leq w_{r+1},$$

so $(I_b \cap [b, n], I_a \cap [b, n])$ form a sorted pair. If $b \notin L_a$, then $|V|$ and $|W|$ have the same parity. The pair (V, W) is sorted, and $I_a \cap [b, n] = V$, while $I_b \cap [b, n] = W \cup \{b\}$. So $(I_b \cap [b, n], I_a \cap [b, n])$ are a sorted pair in this case as well, since we have

$$b < v_1 \leq w_1 \leq v_2 \leq w_2 \leq \cdots \leq v_r \leq w_r.$$

Note that b is in I_b but not in I_a, so we also have that I_a and I_b are distinct in this case.

Now we turn to the sets T and U. Because $\operatorname{sgn}\langle a\, b\rangle = (-1)^k \operatorname{sgn}\langle b\, a\rangle$, we have

$$\begin{aligned}(-1)^{|L_a \cap [a, b-1]|} &= (-1)^k (-1)^{|L_b \cap [b, a-1]|}\\ &= (-1)^{|L_b \cap [1,n]| + |L_b \cap [b, a-1]|}\\ &= (-1)^{|L_b \cap [a, b-1]|}.\end{aligned}$$

Note that $|T| \leq |L_a \cap [a, b-1]| \leq 1 + |T|$, since $a \notin L_a$. If $a \in L_b$, then (T, U) is sorted and $|L_b \cap [a, b-1]| = 1 + |U|$, since $b-1 \notin L_b$. If T and U have the same cardinality, then $|L_a \cap [a, b-1]|$ must be equal to $1 + |T|$ in order to have the same parity as $1 + |U|$. Thus $b - 1 \in L_a$. This means that $(I_a \cap [a, n], I_b \cap [a, n])$ is a sorted pair, as we have

$$a \leq a < t_1 \leq u_1 \leq \cdots \leq t_j \leq u_j < b-1 < b \leq \cdots \leq v_r \leq w_r.$$

If $|T| = |U| + 1$, we conclude by similar reasoning that $b - 1 \notin L_a$, and again $(I_a \cap [a, n], I_b \cap [a, n])$ is a sorted pair, as we have

$$a \leq a < t_1 \leq u_1 \leq \cdots \leq t_j \leq u_j \leq t_{q+1} < b \leq \cdots \leq v_r \leq w_r.$$

If $a \notin L_b$, then (U, T) is sorted and $L_b \cap [a, b-1] = U$, since a and $b-1$ are not in L_b. A parity argument as in the last paragraph shows that if $|U| = |T|$, then $b-1 \notin L_a$; if $|U| = |T| + 1$, then $b - 1 \in L_a$. Either way, $(I_a \cap [a, n], I_b \cap [a, n])$ is a sorted pair; we see

$$a \leq u_1 \leq t_1 \leq \cdots \leq u_j \leq t_j < b \leq \cdots \leq v_r \leq w_r$$

in the first case and

$$a \leq u_1 \leq t_1 \leq \cdots \leq u_j \leq t_j \leq u_{q+1} < b - 1 < b \leq \cdots \leq v_r \leq w_r$$

in the second.

Finally, we deal with P and Q. Recall that (P, Q) are sorted. Since $|L_a \cap [b, n]|$ and $|L_b \cap [b, n]|$ have the same parity and $|L_a \cap [a, b-1]|$ and $|L_b \cap [a, b-1]|$ have the same parity, $|L_a \cap [1, a-1]|$ and $|L_b \cap [1, a-1]|$ have the same parity. Since $a - 1 \notin L_a$, we have that $P = L_a \cap [1, a-1]$. On the other hand $|Q| \leq |L_b \cap [1, a-1]| \leq |Q| + 1$. If $|P| = |Q|$, then for parity reasons $Q = L_b \cap [1, a-1]$ and thus $a - 1 \notin L_b$. So $(I_a \cap [1, a-1], I_b \cap [1, a-1])$ are a sorted pair, as we have

$$r_1 \leq s_1 \leq \cdots \leq r_i \leq s_i.$$

Similarly, if $|P| = |Q| + 1$, then $a - 1 \in L_b$ and $(I_a \cap [1, a-1], I_b \cap [1, a-1])$ again are a sorted pair, since we have

$$r_1 \leq s_1 \leq \cdots \leq r_i \leq s_i \leq r_{i+1} \leq a - 1.$$

Since $(I_a \cap [1, a-1], I_b \cap [1, a-1])$ is a sorted pair ending in an element of I_b and $(I_a \cap [a, n], I_b \cap [a, n])$ is a sorted pair, it follows that (I_a, I_b) is a sorted pair. □

Using Theorem 10.10, we can conclude that positroid tiles are unions of amplituhedron w-chambers, just as tree positroid polytopes are unions of hypersimplex w-simplices. More precisely, we have Corollary 10.17, which we sharpen further in Proposition 11.1.

Corollary 10.17 (Positroid tiles are unions of w-chambers). *Let Z_π be a positroid tile for $\mathcal{A}_{n,k,2}(Z)$. Then*

$$Z_\pi = \bigcup_{\substack{\hat{\Delta}_w(Z): \\ \hat{\Delta}^\circ_w(Z) \cap Z^\circ_\pi \neq \emptyset}} \hat{\Delta}_w(Z).$$

11. T-DUALITY AND POSITROID TILINGS

In this section we show one of our main results: we prove that a collection $\{\Gamma_\pi\}$ of positroid polytopes is a positroid tiling of $\Delta_{k+1,n}$ if and only if for all $Z \in \mathrm{Mat}^{>0}_{n,k+2}$, the collection of T-dual Grasstopes $\{Z_{\hat{\pi}}\}$ is a positroid tiling of $\mathcal{A}_{n,k,2}(Z)$. Along the way we show that realizable amplituhedron chambers are exactly counted by Eulerian numbers. We also explore the phenomena that w-chambers can be empty, and define the $\mathcal{G}$-amplituhedron – a Z-independent analogue of the amplituhedron in $\mathrm{Gr}_{2,n}$. Finally, we introduce the *total amplituhedron* $\mathcal{G}^{(2)}_n \subset \mathrm{Gr}_{2,n}$ which is the amplituhedron-analogue of the hypercube, and discuss positroid tilings based on descents/sign-flips.

11.1. **Positroid tilings of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}$.** Recall that w-simplices in $\Delta_{k+1,n}$ are indexed by $D_{k+1,n}$. One main tool is the following.

Proposition 11.1. *Fix $k < n$ and $Z \in \mathrm{Mat}^{>0}_{n,k+2}$. Suppose $w \in D_{k+1,n}$ and that $\hat{\Delta}_w(Z) \neq \emptyset$. For any tree positroid polytope Γ_π, $\Delta_w \subset \Gamma_\pi$ if and only if $\hat{\Delta}_w(Z) \subset Z_{\hat{\pi}}$.*

Proof. Fix a bicolored triangulation $\mathcal{T}$ so that $G(\mathcal{T})$ is a plabic tree with trip permutation π and $\hat{G}(\mathcal{T})$ has trip permutation $\hat{\pi}$. From Theorem 9.2, $Z^\circ_{\hat{\pi}}$ consists of $Y \in \mathrm{Gr}_{k,k+2}$ such that for all arcs $a \to b$ of $\mathcal{T}$

$$\begin{cases} \mathrm{sgn}\langle Yab\rangle = (-1)^{\mathrm{area}(a\to b)} & \text{if } a < b, \\ \mathrm{sgn}\langle Ya\hat{b}\rangle = (-1)^{\mathrm{area}(a\to b)} & \text{if } a > b \end{cases}$$

and Γ_π consists of the points $x \in \mathbb{R}^n$ satisfying

$$\mathrm{area}(a \to b) \leq x_{[a,b-1]} \leq \mathrm{area}(a \to b) + 1$$

for all arcs $a \to b$ of $\mathcal{T}$. (In fact, to cut out $Z^\circ_{\hat{\pi}}$, it suffices to consider arcs with $a < b$.)

Suppose $\Delta_w \subset \Gamma_\pi$. Then the vertices $e_{I_1}, \dots, e_{I_n}$ of Δ_w satisfy the defining inequalities of Γ_π. In particular, for each arc $a \to b$ of $\mathcal{T}$,

$$\mathrm{area}(a \to b) \leq |I_a \cap [a, b-1]| \leq \mathrm{area}(a \to b) + 1.$$

By Remark 10.6, there is another vertex e_{I_r} of Δ_w satisfying $I_r = I_a \setminus \{a\} \cup \{a-1\}$. This vertex also satisfies the defining inequalities of Γ_π. Moreover, $|I_r \cap [a, b-1]|$ is 1 smaller than $|I_a \cap [a, b-1]|$, so we must have

$$|I_a \cap [a, b-1]| = \mathrm{area}(a \to b) + 1.$$

Consider $Y \in \hat{\Delta}^\circ_w(Z)$. By definition, for $a < b$, $\mathrm{sgn}\langle Yab\rangle = (-1)^{|I_a \cap [a,b-1]|-1}$. By the above computation, $\mathrm{sgn}\langle Yab\rangle = (-1)^{\mathrm{area}(a\to b)}$ for every arc $a \to b$, so we have shown $\hat{\Delta}^\circ_w(Z) \subset Z^\circ_{\hat{\pi}}$. Taking closures gives the desired containment.

Now, suppose $\hat{\Delta}_w(Z) \subset Z_{\hat{\pi}}$. This means that for all arcs $a \to b$ of $\mathcal{T}$, $\mathrm{area}(a \to b)+1$ is the same parity as $|I_a \cap [a, b-1]|$. We will show that for all q,

$$\mathrm{area}(a \to b) \leq |I_q \cap [a, b-1]| \leq \mathrm{area}(a \to b) + 1.$$

From the alcove description of w-simplices in [LP07, Section 2.3], there is some d so that Δ_w lies between the hyperplanes $\{x_{[a,b-1]} = d-1\}$ and $\{x_{[a,b-1]} = d\}$. As noted above, there is a vertex e_{I_r} of Δ_w satisfying $I_r = I_a \setminus \{a\} \cup \{a-1\}$. Since $|I_r \cap [a, b-1]| = |I_a \cap [a, b-1]| - 1$, we conclude that d is $|I_a \cap [a, b-1]|$. Thus, it suffices to show that

$$\mathrm{area}(a \to b) + 1 = |I_a \cap [a, b-1]|. \tag{11.2}$$

This is proved in Lemma 11.3. □

Lemma 11.3. *Let $\mathcal{T}$ be a bicolored triangulation of type (k, n) and let $\Delta_w \subset \Delta_{k+1,n}$ be a w-simplex with vertices $e_{I_1}, \dots, e_{I_n}$. Suppose for all arcs $a \to b$ of $\mathcal{T}$,*

$$\mathrm{area}(a \to b) + 1 \equiv |I_a \cap [a, b-1]| \pmod 2.$$

Then $\mathrm{area}(a \to b) + 1 = |I_a \cap [a, b-1]|$ *for all arcs $a \to b$ of $\mathcal{T}$.*

Proof. We use induction on n. The base cases are $n = 3$ and $k = 0, 1$, which are clear.

Without loss of generality, we may assume that $\mathcal{T}$ contains the arc $1 \to (n-1)$. Indeed, $\mathcal{T}$ contains some arc $(r+1) \to (r-1)$. We can rotate $\mathcal{T}$ by r to obtain a new triangulation with an arc $1 \to (n-1)$. We can also apply the corresponding cyclic shift $e_i \mapsto e_{i-r}$ to Δ_w to obtain a new simplex Δ_u. The vertex e_{I_p} of Δ_w is mapped to vertex $e_{J_{p-r}}$ of Δ_u, where $J_{p-r} = \{i - r : i \in I_p\}$. If the proposition is true for the new triangulation and Δ_u, it is easy to see (by shifting back) that it is true for $\mathcal{T}$ and Δ_w.

Let $\mathcal{T}'$ be the bicolored triangulation of type $(k', n-1)$ obtained by chopping the triangle with vertices $1, n-1, n$ off of $\mathcal{T}$. Note that $k' = k$ if this triangle is white, and $k' = k - 1$ otherwise. Let $v \in S_{n-1}$ be the permutation obtained from w by deleting $w_n = n$ and moving $n - 1$ to the end.

Case I. Suppose the triangle deleted from $\mathcal{T}$ is white, so $k' = k$. Then $\text{area}_{\mathcal{T}}(1 \to (n-1))$ is equal to $\text{area}_{\mathcal{T}}(1 \to n)$, so the assumption on parities means that $I_1 \cap [1, n-1]$ has the same size as $I_1 \cap [1, n-2]$. That is, $n-1 \notin I_1$, which means that $n-1$ appears to the right of $n-2$ in w. Deleting w_n and moving $n-1$ to the end results in a permutation with the same number of cyclic descents as w, meaning that $\Delta_v \subset \Delta_{k',n-1}$.

The vertices of Δ_v are $e_{J_1}, \dots, e_{J_{n-1}}$, where

$$J_a = \begin{cases} I_a & \text{if } n \notin I_a, \\ I_a \setminus \{n\} \cup \{n-1\} & \text{if } n \in I_a. \end{cases}$$

For the moment, we will denote cyclic intervals in $[n-1]$ by $[a, b]'$.

Let $a \to b$ be an arc of $\mathcal{T}'$. Because $b \neq n$, $[a, b-1]$ either contains both $n-1$ and n, or neither. So $J_a \cap [a, b-1]'$ and $I_a \cap [a, b-1]$ have the same cardinality. Also, $\text{area}_{\mathcal{T}'}(a \to b)$ is equal to $\text{area}_{\mathcal{T}}(a \to b)$, so $\mathcal{T}'$ and $\hat{\Delta}_v(Z)$ satisfy the assumptions of the proposition. By induction, we can conclude that $|J_a \cap [a, b-1]'| = \text{area}_{\mathcal{T}'}(a \to b) + 1$. In light of the equalities in this paragraph, this means that for all arcs $a \to b$ of $\mathcal{T}$ where a, b are not n, we have $|I_a \cap [a, b-1]| = \text{area}_{\mathcal{T}}(a \to b) + 1$. It remains to check that a similar equality for the arcs $1 \to n$, $(n-1) \to n$ and their reverses, which are trivial.

Case II. Suppose the triangle deleted from $\mathcal{T}$ is black, so $k' = k - 1$. Then $\text{area}_{\mathcal{T}}(1 \to (n-1))$ is equal to $\text{area}_{\mathcal{T}}(1 \to n) - 1$. The assumption on parities implies that $I_1 \cap [1, n-1]$ and $I_1 \cap [1, n-2]$ are different sizes, so $n-1 \in I_1$. This means that $n-1$ appears to the left of $n-2$ in w, and v has one fewer left descent than w. So $\Delta_v \subset \Delta_{k',n-1}$ as desired.

The vertices of Δ_v are $e_{J_1}, \dots, e_{J_{n-1}}$, where

$$J_a = \begin{cases} I_a \setminus \{n\} & \text{if } n \in I_a, \\ I_a \setminus \{n-1\} & \text{if } n-1 \in I_a, n \notin I_a. \end{cases}$$

Let $a \to b$ be an arc of $\mathcal{T}'$. Again, the cyclic interval $[a, b-1]$ either contains both $n-1$ and n, or contains neither. If $[a, b-1]$ contains neither, then clearly $|J_a \cap [a, b-1]'| = |I_a \cap [a, b-1]|$; in this case, $\text{area}_{\mathcal{T}'}(a \to b) = \text{area}_{\mathcal{T}}(a \to b)$ as well. If $[a, b-1]$ contains both, then $|J_a \cap [a, b-1]'| = |I_a \cap [a, b-1]| - 1$ and $\text{area}_{\mathcal{T}'}(a \to b) = \text{area}_{\mathcal{T}}(a \to b) - 1$. So again, $\mathcal{T}'$ and Δ_v satisfy the assumptions of the proposition. As in Case I, we can conclude that for all arcs $a \to b$ of $\mathcal{T}$ where a, b are not n, we have $|I_a \cap [a, b-1]| = \text{area}_{\mathcal{T}}(a \to b) + 1$. The equalities for the arcs $1 \to n$, $(n-1) \to n$, and their reverses are clear. □

Remark 11.4. Proposition 11.1 motivates the intuition that the w-simplex $\Delta_w \subset \Delta_{k+1,n}$ and the w-chamber $\hat{\Delta}_w(Z) \subset \mathcal{A}_{n,k,2}(Z)$ are 'T-dual' to each other. In Proposition 11.36

we will show that any w-simplex is the intersection of n distinguished positroid polytopes $\{\Gamma_\pi\}$, and the corresponding w-chamber is the intersection of the n T-dual Grasstopes $\{Z_{\hat{\pi}}\}$.

To prove the correspondence between positroid tilings, we also need the following crucial result, whose proof we delay to the following subsection.

Theorem 11.5 (w-Chambers are realizable)**.** *For each $w \in D_{k+1,n}$, there exists some $Z \in \mathrm{Mat}_{n,k+2}^{>0}$ such that the amplituhedron w-chamber $\hat{\Delta}_w(Z)$ in $\mathcal{A}_{n,k,2}(Z)$ is nonempty.*

We can now show the main result of this section.

Theorem 11.6 (Tilings of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}$ are T-dual)**.** *The collection $\mathcal{C} = \{\Gamma_\pi\}$ is a positroid tiling of $\Delta_{k+1,n}$ if and only if for all $Z \in \mathrm{Mat}_{n,k+2}^{>0}$, the collection of T-dual Grasstopes $\hat{\mathcal{C}} = \{Z_{\hat{\pi}}\}$ is a positroid tiling of $\mathcal{A}_{n,k,2}(Z)$.*

Proof. ($\Longrightarrow$): Suppose $\mathcal{C}$ is a positroid tiling of $\Delta_{k+1,n}$ and choose $Z \in \mathrm{Mat}_{n,k+2}^{>0}$. We already know that $Z_{\hat{\pi}}$ is a positroid tile from Corollary 9.1.

We first show that the Grasstopes in $\hat{\mathcal{C}}$ are dense in the amplituhedron. Consider a nonempty amplituhedron w-chamber $\hat{\Delta}_w(Z)$. Since $\mathcal{C}$ is a positroid tiling, there exists a tree positroid polytope $\Gamma_\pi \in \mathcal{C}$ which contains Δ_w. By Proposition 11.1, $\hat{\Delta}_w^\circ(Z) \subset Z_{\hat{\pi}}^\circ$, where the latter is by definition in $\hat{\mathcal{C}}$. So we have

$$\bigcup_{w} \hat{\Delta}_w^\circ(Z) \subseteq \bigcup_{\hat{\mathcal{C}}} Z_{\hat{\pi}}^\circ \subseteq \mathcal{A}_{n,k,2}(Z).$$

By Theorem 10.10, the closure of the left-most set is equal to the right, so the closure of the middle set is $\mathcal{A}_{n,k,2}(Z)$, as desired.

Now, suppose for the sake of contradiction that two distinct $Z_{\hat{\pi}}^\circ, Z_{\hat{\mu}}^\circ \in \hat{\mathcal{C}}$ are not disjoint. They are open, so their intersection is open, and thus their intersection contains a point in $\hat{\Delta}_w^\circ(Z)$ for some w. Lemma 10.9 implies that in fact the entire w-simplex $\hat{\Delta}_w^\circ(Z)$ is contained in their intersection. But then by Proposition 11.1, Δ_w is contained in $\Gamma_\pi \cap \Gamma_\mu$, a contradiction.

($\Longleftarrow$): Suppose that for all $Z \in \mathrm{Mat}_{n,k+2}^{>0}$, $\hat{\mathcal{C}}$ is a positroid tiling of $\mathcal{A}_{n,k,2}(Z)$. By Theorem 11.5, for all $w \in D_{k+1,n}$, we can choose Z so that $\hat{\Delta}_w(Z)$ is nonempty. In particular, $\hat{\Delta}_w^\circ(Z)$ must intersect one of the positroid tiles $Z_{\hat{\pi}}^\circ$ and thus by Lemma 10.9, $\hat{\Delta}_w(Z) \subset Z_{\hat{\pi}}$. Because $\hat{\mathcal{C}}$ is a positroid tiling, $\hat{\Delta}_w(Z)$ is not contained in any other positroid tile in $\hat{\mathcal{C}}$. Using Proposition 11.1, we see that every w-simplex is contained in precisely one positroid polytope in $\mathcal{C}$, and thus $\mathcal{C}$ is a positroid tiling of $\Delta_{k+1,n}$. □

In [KWZ20], they conjectured there are $\binom{n-2}{k}$ Grasstopes in a positroid tiling of $\mathcal{A}_{n,k,2}$. As noted in [ŁPW20], this is also the number of positroid polytopes in a regular positroid tiling of $\Delta_{k+1,n}$ [SW21], which are those arising from the *tropical positive Grassmannian* $\mathrm{Trop}^+ \mathrm{Gr}_{k+1,n}$ [ŁPW20]. A positroid tiling of $\mathcal{A}_{n,k,2}$ is *regular* if it is T-dual to a regular positroid tiling of $\Delta_{k+1,n}$. By Theorem 11.6, we have:

Corollary 11.7. *There are $\binom{n-2}{k}$ Grasstopes in any regular positroid tiling of $\mathcal{A}_{n,k,2}(Z)$.*

Remark 11.8. [ŁPW20] showed that all *BCFW* tilings of $\mathcal{A}_{n,k,2}(Z)$ contain $\binom{n-2}{k}$ Grasstopes; there are BCFW tilings which are not regular and regular tilings which are not BCFW.

11.2. **The $\mathcal{G}$-amplituhedron, the hypercube and the total amplituhedron.** In this subsection, we embed $\mathcal{A}_{n,k,2}(Z)$ into a full-dimensional subset of $\mathrm{Gr}_{2,n}$—the '$\mathcal{G}$-amplituhedron' $\mathcal{G}_{n,k,2}$—which does not depend on Z. We use sign chambers in the $\mathcal{G}$-amplituhedron to prove that all w-chambers of $\mathcal{A}_{n,k,2}(Z)$ are realizable (Theorem 11.5). We also draw another parallel between the hypersimplex and the amplituhedron. In Remark 7.2 we saw that the union of the (projected) hypersimplices $\tilde{\Delta}_{k+1,n}$ is the hypercube $\boxtimes_{n-1}$. Analogously, we take the union of $\mathcal{G}$-amplituhedra varying over all k to obtain the *total amplituhedron* $\mathcal{G}_n^{(2)}$, which is the amplituhedron-analogue of $\boxtimes_{n-1}$.

Definition 11.9 is intended to be a Z-independent version of the amplituhedron, inspired by Corollary 5.3.

Definition 11.9 (The $\mathcal{G}$-amplituhedron)**.** Fix $k < n$ and let

$$\mathcal{G}^{\circ}_{n,k,2} := \{z \in \mathrm{Gr}_{2,n} \mid p_{i,i+1}(z) > 0 \text{ for } 1 \le i \le n-1, \text{ and } p_{n,\hat{1}}(z) > 0, \\ \text{and } \mathrm{var}((p_{12}(z), p_{13}(z), \dots p_{1n}(z)) = k\}.$$

The closure $\mathcal{G}_{n,k,2} := \overline{\mathcal{G}^{\circ}_{n,k,2}}$ in $\mathrm{Gr}_{2,n}$ is the *$\mathcal{G}$-amplituhedron*.

Remark 11.10. Following the sign-flip descriptions from [AHTT18, KW19], one can generalize most of the definitions in this section for any m. We leave this to future work.

Comparing with Corollary 5.3 we have:

Proposition 11.11. *Fix $k < n$, and $W \in \mathrm{Gr}^{>0}_{k+2,n}$. Then*

$$\mathcal{G}^{\circ}_{n,k,2}(W) = \{z \in \mathcal{G}^{\circ}_{n,k,2} \mid z \subset W\} = \mathcal{G}^{\circ}_{n,k,2} \cap \mathrm{Gr}_2(W) \quad \textit{and} \quad \mathcal{B}_{n,k,2}(W) = \overline{\mathcal{G}^{\circ}_{n,k,2} \cap \mathrm{Gr}_2(W)}.$$

Remark 11.12. Note that $\mathcal{G}_{n,k,2}$ is full-dimensional in $\mathrm{Gr}_{2,n}$, i.e. it has dimension $2(n-2)$, whereas $\mathcal{G}^{\circ}_{n,k,2}(W)$ and $\mathcal{B}_{n,k,2}(W)$ are full-dimensional in $\mathrm{Gr}_2(W)$, i.e. have dimension $2k$.

Motivated by the decomposition of $\mathcal{A}_{n,k,2}(Z)$ into w-chambers, we analogously define w-chambers for $\mathcal{G}_{n,k,2}$.

Definition 11.13. Let $w \in D_{k+1,n}$ and let $I_a := \mathrm{cDes}_{\mathrm{L}}(w^{(a-1)})$. Then the *open $\mathcal{G}$-amplituhedron w-chamber* $\hat{\Delta}^{\circ}_w(\mathcal{G})$ consists of $z \in \mathcal{G}_{n,k,2}$ with all nonzero Plücker coordinates such that for $a = 1, \dots, n$,

$$\mathrm{Flip}(p_{a\hat{1}}(z), p_{a\hat{2}}(z), \dots, p_{a\widehat{a-1}}(z), p_{aa}(z), p_{aa+1}(z), \dots, p_{an}(z)) = I_a \setminus \{a\}.$$

Equivalently, $\Delta^{\circ}_w(\mathcal{G})$ consists of $z \in \mathrm{Gr}_{2,n}$ such that

(11.14)

$$\mathrm{sgn}\, p_{aj}(z) = (-1)^{|I_a \cap [a,j-1]|-1} \text{ for } j > a \quad \text{and} \quad \mathrm{sgn}\, p_{a\hat{j}}(z) = (-1)^{|I_a \cap [a,j-1]|-1} \text{ for } j < a.$$

The *closed $\mathcal{G}$-amplituhedron w-chamber* is the closure $\hat{\Delta}_w(\mathcal{G}) := \overline{\hat{\Delta}^{\circ}_w(\mathcal{G})}$. Abusing notation, we will often omit 'closed' when referring to closed $\mathcal{G}$-amplituhedron w-chambers.

The situation for $\mathcal{G}$-amplituhedron w-chambers is quite straightforward. We will see that the second part of (11.14) follows from the first part, so each $\hat{\Delta}^{\circ}_w(\mathcal{G})$ is an oriented matroid stratum, whose underlying matroid is the rank 2 uniform matroid on $[n]$.

Proposition 11.15. *Let $w \in D_{k+1,n}$. Then $\hat{\Delta}^{\circ}_w(\mathcal{G})$ is nonempty and is contractible.*

Proof. Consider n vectors $v_1, v_2, \dots, v_n$ in $\mathbb{R}^2$ so that the matrix

$$\begin{bmatrix} v_1 & v_{w_1+1} & v_{w_2+1} & \cdots & v_{w_{n-1}+1} \end{bmatrix}$$

has all maximal minors positive. In particular, drawing the vectors in the plane and going counterclockwise, we see $v_1, v_{w_1+1}, v_{w_2+1}, \dots, v_{w_{n-1}+1}$ in that order.

Now, set $z_1 := v_1$ and $z_b := (-1)^{|I_1 \cap [1,b-1]|-1} v_b$ for $b \geq 2$. We claim that

$$z = \begin{bmatrix} z_1 & z_2 & z_3 & \dots & z_n \end{bmatrix}$$

represents a point in $\hat{\Delta}^\circ_w(\mathcal{G})$.

Clearly $p_{1b}(z)$ has the correct sign. Consider $1 \neq a < j$. We will assume $\det[v_a v_j] > 0$; the other case is similar. Note that $p_{aj}(z)$ has sign $(-1)^{|I_1 \cap [a,j-1]|}$; we would like to show that this is equal to $(-1)^{|I_a \cap [a,j-1]|-1}$. Because $\det[v_a v_j] > 0$, $a-1$ occurs before $j-1$ in w, written in one-line notation. Recall from Remark 10.6 that $I_{w_i+1} = I_{w_{i-1}+1} \setminus \{w_i\} \cup \{w_i+1\}$. That is, I_a can be obtained from I_1 by removing w_1 and adding w_1+1, then removing w_2 and adding w_2+1, and so on until one removes $w_q = a-1$ and adds a. Note that for $c = w_1, \dots, w_{q-1}$, the numbers c and $c+1$ are either both in $[a, j-1]$ or both not in $[a, j-1]$, so $|I_1 \cap [a, j-1]| = |I_{c+1} \cap [a, j-1]|$. Removing $a-1$ from $I_{w_{q-1}+1}$ and adding a increases the size of the intersection with $[a, j-1]$ by one, so $|I_1 \cap [a, j-1]| = |I_a \cap [a, j-1]| - 1$. This shows $p_{aj}(z)$ has the correct sign for $a < j$; a similar argument shows that for $a > j$, $p_{aj}(z)$ has the desired sign so long as $p_{ja}(z)$ does.

So $\hat{\Delta}^\circ_w(\mathcal{G})$ is an oriented matroid stratum for a rank 2 oriented matroid. By [BLVS+99, Corollary 8.2.3], all rank 2 oriented matroid strata are contractible. □

Example 11.16. Let $w = (2, 6, 1, 4, 5, 3, 7) \in D_{k+1,n}$ with $k = 3$ and $n = 7$. We have $I_1 = \{1, 2, 4, 6\}$. Following the proof of Proposition 11.15, we can choose

$$\begin{aligned}(v_1, v_{w_1+1}, v_{w_2+1}, v_{w_3+1}, v_{w_4+1}, v_{w_5+1}, v_{w_6+1}) &= (v_1, v_3, v_7, v_2, v_5, v_6, v_4) \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}.\end{aligned}$$

We then get

$$z = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 4 & -2 & -7 & 5 & 6 & -3 \end{pmatrix}.$$

One can check that z lies in $\hat{\Delta}^\circ_w(\mathcal{G})$. Also note that both row vectors $z^{(1)}$ and $z^{(2)}$ of z have $\mathrm{var}(z^{(1)}) = \mathrm{var}(z^{(2)}) = k$ by construction.

Remark 11.17. The w-chambers of the $\mathcal{G}$-amplituhedron do *not* depend on Z. Roughly speaking, the amplituhedron w-chambers are linear slices of $\mathcal{G}$-amplituhedron w-chambers. More precisely, for $Z \in \mathrm{Mat}^{>0}_{n,k+2}$ with column span $W \in \mathrm{Gr}^{>0}_{k+2,n}$, we have

$$f_Z(\hat{\Delta}^\circ_w(\mathcal{G}) \cap \mathrm{Gr}_2(W)) = \hat{\Delta}^\circ_w(Z),$$

where f_Z is the homeomorphism from Proposition 3.3.

Our next goal is to use Proposition 11.15 and the connection with the $\mathcal{B}$-amplituhedron from Proposition 11.11 to deduce Theorem 11.5 on realizability of w-chambers. We start by proving Lemma 11.18.

Lemma 11.18. *Given a* $2 \times n$ *matrix* z *as constructed in the proof of Proposition* 11.15, *we can construct a* $(k+2)\times n$ *matrix* A' *representing a point* $W \in \mathrm{Gr}_{k+2,n}^{\geq 0}$ *which contains* $\mathrm{rowspan}(z)$ *as a subspace.*

Proof. Let $z^{(1)} = (z_1^{(1)}, \dots, z_n^{(1)})$ and $z^{(2)} = (z_1^{(2)}, \dots, z_n^{(2)})$ denote the rows of z. By construction, $\mathrm{var}(z^{(1)}) = \mathrm{var}(z^{(2)}) = k$ and moreover we can partition $[n]$ into disjoint consecutive intervals $H_1 \sqcup \dots \sqcup H_{k+1}$ such that the entries of $z^{(1)}$ and $z^{(2)}$ in positions H_i are positive if i is odd and negative if i is even.

By [Kar17, Lemma 4.1], since $\mathrm{var}(z^{(2)}) = k$, we can construct a $(k+1)\times n$ matrix A with maximal minors nonnegative whose row sum is $z^{(2)}$. More explicitly, we define the ith row of A to be the vector $(a_{i1}, \dots, a_{i2})$ such that $a_{ij} = z_j^{(2)}$ for $j \in H_i$ and $a_{ij} = 0$ for $j \notin H_i$. Therefore the nonvanishing Plücker coordinates of A are precisely the $p_B(A)$ such that $B = \{b_1 < b_2 < \dots < b_{k+1}\}$ with $b_i \in H_i$.

Let A' be the matrix obtained from A by adding $z^{(1)}$ as a new top (0th) row. We will label the rows of A' from 0 to $k+1$. The nonvanishing Plücker coordinates of A' are precisely the $p_{B'}(A')$ where $B' = \{b_1 < b_2 < \dots < b_{k+1}\} \cup \{b'_j\}$ with $b_i \in H_i$ and both b_j, b'_j lie in H_j.

Now we can compute the Plücker coordinates of A' in terms of Plücker coordinates of z and minors of A. Let $B' = \{b_1 < b_2 < \dots < b_{k+1}\} \cup \{b'_j\}$ as above. Then we have

$$\begin{aligned} p_{B'}(A') &= (-1)^{j-1}\Delta_{0j,b_jb'_j}(A') \cdot \Delta_{[k+1]\setminus j, B'\setminus\{b_j,b'_j\}}(A') \\ &= (-1)^{j-1} p_{b_jb'_j}(z) \prod_{i\neq j} z_{b_i}^{(2)}, \end{aligned}$$

where $\Delta_{R,C}(A')$ denotes the minor of A' on rows R and columns C. Now it follows from the construction of z that since both b_j, b'_j lie in H_j, we have $p_{b_jb'_j}(z) > 0$. Additionally, we have that the sign of $\prod_{i\neq j} z_{b_i}^{(2)}$ is $(-1)^{j+1}$. Therefore $p_{B'}(A')$ is positive, as desired. □

Example 11.19. We illustrate the proof of Lemma 11.18 using our running example from Example 11.16. We have

$$z = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 4 & -2 & -7 & 5 & 6 & -3 \end{pmatrix},$$

so

$$A = \begin{pmatrix} 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix} \text{ and } A' = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}.$$

Both matrices have maximal minors nonnegative. If $B' = \{2,3,5,6,7\}$ then $2 \in H_1, 3 \in H_2, 5, 6 \in H_3, 7 \in H_4$ and we have

$$p_{B'}(A') = \Delta_{03,56}(A')\Delta_{124,237}(A') = p_{56}(z) \cdot (4 \cdot (-2) \cdot (-3)).$$

Proof of Theorem 11.5. By Proposition 3.3, we know that $\mathcal{B}_{n,k,2}(W)$ is homeomorphic to $\mathcal{A}_{n,k,2}(Z)$, where $W \in \mathrm{Gr}_{k+2,n}^{\geq 0}$ is the column span of Z. Moreover the Plücker coordinates of the former agree with the twistor coordinates of the latter. Proposition 11.15 gives an explicit construction of a $2 \times n$ matrix z representing a point in $\hat{\Delta}_w^\circ(\mathcal{G})$, and by

Proposition 11.11 we have $\mathcal{B}_{n,k,2}(W) = \overline{\mathcal{G}^{\circ}_{n,k,2} \cap \mathrm{Gr}_2(W)}$, so to prove the theorem, we just need to realize z as a two-dimensional subspace contained in some $(k+2)$-plane $W \in \mathrm{Gr}^{>0}_{k+2,n}$.

By Lemma 11.18, we can realize z as a two-dimensional subspace contained in a $(k+2)$-plane $W \in \mathrm{Gr}^{\geq 0}_{k+2,n}$. (Here $W = \mathrm{rowspan}(A')$.) We want to now slightly deform A' to make it totally positive.

We claim that $A' \in \mathrm{Gr}^{\geq 0}_{k+2,n}$ is the limit of a sequence of points $\{\tilde{A}_t\} \in \mathrm{Gr}^{>0}_{k+2,n}$ where $\mathrm{rowspan}(\tilde{A}_t)$ contains a 2-plane $z(t)$ which lies in the same sign-chamber as z. To see this, we use the fact that $\mathrm{Gr}^{\geq 0}_{k+2,n} = \overline{\mathrm{Gr}^{>0}_{k+2,n}}$ (see Remark 2.3). We can therefore write A' as the limit of a sequence of matrices of the form $A' + (\epsilon_{ij}(t)) \in \mathrm{Gr}^{>0}_{k+2,n}$, where $(\epsilon_{ij}(t))$ is a $(k+2) \times n$ matrix, and each $\epsilon_{ij}(t)$ is a function of t with small absolute value and $\epsilon_{ij}(t) \to 0$ as $t \to 0$.

We denote the rows of $A' + (\epsilon_{ij}(t))$ by $r_i(t)$ for $0 \leq i \leq k+1$. Let $z^{(1)}(t) := r_0(t)$, let $z^{(2)}(t) := r_1(t) + r_2(t) + \cdots + r_{k+1}(t)$, and let $z(t)$ be the matrix with rows $z^{(1)}(t)$ and $z^{(2)}(t)$.

Then when $t = 0$, we have $z = z(t)$. Moreover for small t, the Plücker coordinates of $z(t)$ have the same signs as the Plücker coordinates of z, so $z(t)$ lies in the same w-chamber $\hat{\Delta}^{\circ}_w(\mathcal{G})$ as z. But now by construction, $\mathrm{rowspan}(z(t))$ lies in the positive $(k+2)$-plane $W = \mathrm{rowspan}(A' + (\epsilon_{ij}(t)))$. This completes the proof of the theorem. □

Recall the definition of realizable amplituhedron chamber from Definition 3.10.

Corollary 11.20 (Amplituhedron chambers and Eulerian numbers). *The realizable amplituhedron chambers $\mathcal{A}^{\sigma}_{n,k,2}$ are exactly the w-chambers $\hat{\Delta}^{\circ}_w$ where $w \in D_{k+1,n}$.*

Proof. Theorem 11.5 shows that each w-chamber is realizable. Theorem 10.10 shows that no other sign chambers are realizable. □

We now turn to the $\mathcal{G}$-amplituhedron. The proof of Theorem 10.10 implies the following.

Theorem 11.21. *Fix $k < n$, then*

$$\mathcal{G}_{n,k,2} = \bigcup_{w \in D_{k+1,n}} \hat{\Delta}_w(\mathcal{G}).$$

Using the sign characterization of a positroid tile $Z_{\mathcal{J}}$ of $\mathcal{A}_{n,k,2}(Z)$ (Theorem 4.28), one can define a *positroid tile* $\mathcal{G}_{\mathcal{J}}$ in $\mathcal{G}_{n,k,2}$ as (the closure of) the region in $\mathrm{Gr}_{2,n}$ whose Plücker coordinates satisfy the same sign conditions as the twistor coordinates of $Z_{\mathcal{J}}$. Analogously to Corollary 10.17, $\mathcal{G}_{\mathcal{J}}$ is a union of $\mathcal{G}$-amplituhedron w-chambers. Moreover, $Z_{\mathcal{J}}$ is a linear slice of $\mathcal{G}_{\mathcal{J}}$ (analogously to Remark 11.17). We say a *positroid tiling* of $\mathcal{G}_{n,k,2}$ is a collection of positroid tiles which cover $\mathcal{G}_{n,k,2}$ and have disjoint interiors. Since all $\mathcal{G}$-amplituhedron w-chambers are nonempty, the analogue of Theorem 11.6 holds for the $\mathcal{G}$-amplituhedron (without any dependence on Z): T-duality gives a bijection between positroid tilings of $\Delta_{k+1,n}$ and positroid tilings of $\mathcal{G}_{n,k,2}$.

Definition 11.22 (Total amplituhedron). The *total amplituhedron* $\mathcal{G}^{(2)}_n$ is

$$\mathcal{G}^{(2)}_n := \bigcup_{k=0}^{n-2} \mathcal{G}_{n,k,2}.$$

Note that $\mathcal{G}_n^{(2)}$ has top dimension $2(n-2)$ in $\mathrm{Gr}_{2,n}$, and it does *not* depend on Z.

Recall that the hypercube $\square_{n-1} \subset \mathbb{R}^{n-1}$ can be decomposed into $(n-1)!$ w-simplices in a way which is compatible with its slicing into (projected) hypersimplices $\tilde{\Delta}_{1,n}$, $\tilde{\Delta}_{2,n}$, $\dots$, $\tilde{\Delta}_{n-1,n}$. Each $\tilde{\Delta}_{k+1,n}$ is a union of exactly $E_{k,n-1}$ simplices, where $E_{k,n-1}$ is the Eulerian number.

Analogously, by Theorem 11.21, the total amplituhedron $\mathcal{G}_n^{(2)} \subset \mathrm{Gr}_{2,n}$ can be decomposed into $(n-1)!$ w-chambers in a way which is compatible with its decomposition into the $\mathcal{G}$-amplituhedra $\mathcal{G}_{n,0,2}$, $\mathcal{G}_{n,1,2}, \dots, \mathcal{G}_{n,n-2,2}$. Each $\mathcal{G}_{n,k,2}$ is a union of exactly $E_{k,n-1}$ w-chambers. This is the '$m = 2$' equivalent of encoding all helicity sectors at once for tree-level scattering amplitudes of $\mathcal{N} = 4$ SYM for $m = 4$. A related space was discussed in the context of the $\mathcal{B}$-amplituhedron [KW19, Section 3.4].

11.3. **Empty w-chambers and tilings of $\mathcal{A}_{n,k,2}$.** In this section we provide algorithms to find *all* positroid tilings of the hypersimplex and the amplituhedron using w-simplices and w-chambers. As mentioned in Remark 10.8, $\hat{\Delta}_w(Z)$ may be empty for some choices of $Z \in \mathrm{Mat}_{n,k+2}^{>0}$. We take a closer look at this phenomenon and give some examples.

Remark 11.23. It is *a priori* possible for an amplituhedron $\mathcal{A}_{n,k,2}(Z)$ to have a positroid tiling $\hat{\mathcal{C}}$ which is not T-dual to a hypersimplex positroid tiling. However, Theorem 11.6 tells us that the collection of Grasstopes $\hat{\mathcal{C}}$ will fail to be a tiling for some other amplituhedron $\mathcal{A}_{n,k,2}(Z')$. We have not found any instances of such "sporadic" tilings.

Proposition 11.24 (Algorithm for positroid tilings of $\Delta_{k+1,n}$). *In order to find all positroid tilings of $\Delta_{k+1,n}$ proceed as follows. Call two positroid tiles Γ_{π_1} and Γ_{π_2}* compatible *if they do not contain any common w-simplex.*

Step 1. *Define a graph $\mathcal{G}$ whose vertices are positroid tiles of $\Delta_{k+1,n}$ and edges connect compatible positroid tiles.*
Step 2. *Compute the set $Cl(\mathcal{G})$ of all maximal cliques of $\mathcal{G}$;*
Step 3. *For each clique $\mathcal{C} \in Cl(\mathcal{G})$, compute the list $\mathcal{L}_{\mathcal{C}}$ of all w-simplices contained in any positroid tile $\Gamma_\pi \in C$;*
Step 4. *If $\mathcal{L}_{\mathcal{C}}$ consists of all w-simplices of $\Delta_{k+1,n}$, then $\mathcal{C}$ is a positroid tiling of $\Delta_{k+1,n}$. Otherwise it is not.*

Proposition 11.25 (Algorithm for positroid tilings of $\mathcal{A}_{n,k,2}$). *In order to find all positroid tilings of $\mathcal{A}_{n,k,2}(Z)$ proceed as follows. Let $\mathcal{E}_Z$ be the list of all w-simplices Δ_w in $\Delta_{k+1,n}$ such that $\hat{\Delta}_w(Z) = \emptyset$. Call two positroid tiles $Z_{\hat{\pi}_1}, Z_{\hat{\pi}_2}$* compatible *if and only if $\Gamma_{\pi_1} \cap \Gamma_{\pi_2}$ is empty or is the union of w-simplices which are in $\mathcal{E}_Z$.*

Step 1. *Make a graph $\hat{\mathcal{G}}$ whose vertices are positroid tiles of $\mathcal{A}_{n,k,2}(Z)$ and edges connect compatible positroid tiles;*
Step 2. *Compute the set $Cl(\hat{\mathcal{G}})$ of all maximal cliques of $\hat{\mathcal{G}}$;*
Step 3. *For each clique $\hat{\mathcal{C}} \in Cl(\hat{\mathcal{G}})$, consider the collection $\mathcal{C}$ of T-dual positroid tiles in $\Delta_{k+1,n}$. Compute the list $\mathcal{L}_{\mathcal{C}}$ of all w-simplices in $\Delta_{k+1,n}$ contained in any positroid tile $\Gamma_\pi \in \mathcal{C}$;*
Step 4. *If the (possibly empty) complement of $\mathcal{L}_{\mathcal{C}}$ is contained in $\mathcal{E}_Z$, then $\hat{\mathcal{C}}$ is a positroid tiling of $\mathcal{A}_{n,k,2}(Z)$. Otherwise it is not.*

Remark 11.26. If we would like to find a positroid tiling $\hat{\mathcal{C}}$ of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$ which is *not* a positroid tiling of $\Delta_{k+1,n}$, then after Step 3 we need check that either: (i) the complement of $\mathcal{L}_{\mathcal{C}}$ is *nonempty* and contained in $\mathcal{E}_Z$; or (ii) $\mathcal{L}_{\mathcal{C}}$ is the set of all w-simplices of $\Delta_{k+1,n}$ and there is a pair of positroid tiles $\Gamma_{\pi_1}, \Gamma_{\pi_2}$ in $\mathcal{C}$ which both contain a w-simplex in $\mathcal{E}_Z$.

Below, we report some results on empty w-chambers in the cases $k = 1, 2$.

$k = 1$ *case.* The amplituhedron $\mathcal{A}_{n,1,2}(Z)$ is just an n-gon $\mathbf{P}_n(Z)$ in $\mathbb{P}^2$ with vertices $Z_1, \dots, Z_n$ going clockwise. Let $i \to j$ be a side or a diagonal of $\mathbf{P}_n(Z)$, with $i < j$. The twistor coordinate $\langle Yij \rangle$ is positive, negative or zero if Y lies to the right, left, or on the diagonal $i \to j$ respectively. Then the nonempty w-chambers $\hat{\Delta}_w(Z)$ are the connected components of the complement of all diagonals of $\mathbf{P}_n(Z)$ (see Figure 10). If no three diagonals of $\mathbf{P}_n(Z)$ intersect at a point in the interior, it is well known the number of connected components is given by:

$$N_n = \sum_{r=2}^{4} \binom{n-1}{r} = \binom{n}{4} + \binom{n-1}{2}.$$

The number of empty w-chambers in this case is shown in Table 2.

If three diagonals of $\mathbf{P}_n(Z)$ intersect at a point in its interior, then the number of empty w-chambers is larger (as the number of regions realized is smaller).

TABLE 2. Empty w-chambers vs. Eulerian numbers for $k = 1$

n	3	4	5	6	7	8	9
N_n	1	4	11	25	50	91	154
$E_{1,n-1}$	1	4	11	26	57	120	247
# Empty $\hat{\Delta}_w$	0	0	0	1	7	29	93

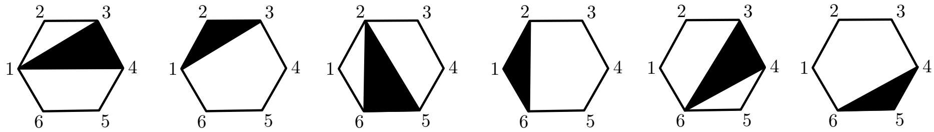
Example 11.27. Consider $\mathcal{A}_{6,1,2}(Z)$, which is a hexagon. Let us consider the permutations $w^{(+)} = 145236$ and $w^{(-)} = 341256$. Points in $\hat{\Delta}_{w^{(+)}}(Z)$ and $\hat{\Delta}_{w^{(-)}}(Z)$ have all twistor coordinates with the same sign, except for $\{\langle Y14\rangle, \langle Y25\rangle, \langle Y36\rangle\}$, whose signs are $\{+ - +\}$ and $\{- + -\}$, respectively. Let Z^* be the intersection of the diagonals $(1, 4)$ and $(2, 5)$. Then $\hat{\Delta}_{w^{(+)}}(Z)$ (respectively, $\hat{\Delta}_{w^{(-)}}(Z)$) is nonempty if and only if Z^* is to the right (respectively, left) of the diagonal $3 \to 6$. This happens when

$$\langle Z_1, Z_2, Z_5\rangle\langle Z_4, Z_3, Z_6\rangle - \langle Z_1, Z_3, Z_6\rangle\langle Z_4, Z_2, Z_5\rangle \tag{11.28}$$

is positive (respectively, negative), see Figure 10. So for any choice of Z, either $\hat{\Delta}_{w^{(+)}}(Z) = \emptyset$ or $\hat{\Delta}_{w^{(-)}}(Z) = \emptyset$, and both are empty if (11.28) vanishes.

If a collection of Grasstopes covers $\mathcal{A}_{n,k,2}(Z)$, the T-dual positroid polytopes may not cover $\Delta_{k+1,n}$.

Example 11.29. Let Z_0 be the point in $Gr^{>0}_{3,6}$ invariant under cyclic symmetry, so $\mathcal{A}_{6,1,2}(Z_0)$ is a regular hexagon in $\mathbb{P}^2$. Consider the positroid tiles $Z_{\hat{\pi}_1}, \dots, Z_{\hat{\pi}_6}$ below.



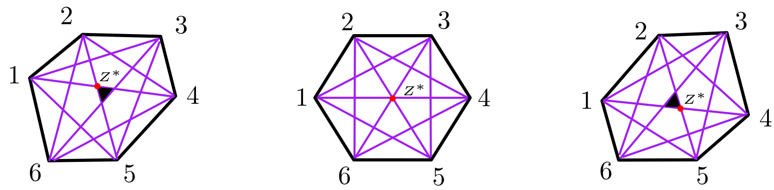


FIGURE 10. From left to right: For $\langle Z^*, Z_3, Z_6\rangle > 0$, $\hat{\Delta}_{w(+)}$ is nonempty (in black) but $\hat{\Delta}_{w(-)}$ is empty; if $\langle Z^*, Z_3, Z_6\rangle = 0$ then $\hat{\Delta}_{w(+)}$ and $\hat{\Delta}_{w(-)}$ are both empty; for $\langle Z^*, Z_3, Z_6\rangle < 0$ then $\hat{\Delta}_{w(+)}$ is empty but $\hat{\Delta}_{w(-)}$ is not (shown in black)

Clearly, they *do* cover $\mathcal{A}_{6,1,2}(Z_0)$ (and overlap). However, $\Delta_{w(+)}$ is not contained in $\Gamma_{\pi_1} \cup \cdots \cup \Gamma_{\pi_6} \subset \Delta_{2,6}$. Therefore the T-dual positroid tiles do *not* cover $\Delta_{2,6}$.

Despite the presence of empty w-chambers, for *any* Z in $\mathrm{Mat}_{k+2,n}^{>0}$, positroid tilings of $\Delta_{2,n}$ and $\mathcal{A}_{n,1,2}(Z)$ are still in bijection:

Proposition 11.30. *A collection of tree positroid polytopes $\{\Gamma_\pi\}$ is a positroid tiling of $\Delta_{2,n}$ if and only if $\{Z_{\hat{\pi}}\}$ is a positroid tiling of $\mathcal{A}_{n,1,2}(Z)$. All such tilings are regular.*

Proof. The forward direction comes from Theorem 11.6. The other direction comes from the fact that $\mathcal{A}_{n,1,2}(Z)$ is just an n-gon. Its positroid tilings are in bijection with the regular positroid tilings of $\Delta_{2,n}$ described in [ŁPW20, Proposition 10.7] (of 'Catalan' type). □

$k = 2$ case. We used Mathematica and the package 'positroid' [Bou12].

For $n = 6$, there are choices of Z such that all w-chambers of $\mathcal{A}_{6,2,2}(Z)$ are nonempty.

For $n = 7$ and some choices of Z, there are empty w-chambers $\hat{\Delta}_w$ for which Δ_w is the intersection of just 2 positroid tiles of $\Delta_{3,7}$. This implies that in general the compatibility graph $\mathcal{G}$ of positroid tiles of $\Delta_{k+1,n}$ differs from the one $\hat{\mathcal{G}}$ of positroid tiles of $\mathcal{A}_{n,k,2}(Z)$ (cf. Proposition 11.25). For example, if $w = 1645237$, then $\Delta_w = \Gamma_{\pi_1} \cap \Gamma_{\pi_2}$, with $\pi_1 = 2371645$ and $\pi_2 = 6745231$. The positroid polytopes $\{\Gamma_{\pi_1}, \Gamma_{\pi_2}\}$ are *not* compatible in $\Delta_{3,7}$, but there are choices of $Z = Z^*$ for which the T-dual Grasstopes $\{Z_{\pi_1}, Z_{\pi_2}\}$ are compatible in $\mathcal{A}_{7,2,2}(Z^*)$, as $Z_{\pi_1} \cap Z_{\pi_2} = \hat{\Delta}_w = \emptyset$. Nevertheless, the 3073 positroid tilings of $\mathcal{A}_{7,2,2}(Z^*)$ are still in bijection with the 3073 positroid tilings of $\Delta_{3,7}$.

For $n = 8$, we checked only a few choices of Z, but found that there are more than 100 w-chambers which can be empty depending on Z. As in the $n = 7$ case, the compatibility graph of $\Delta_{3,8}$ differs from that of $\mathcal{A}_{8,2,2}(Z)$. Nevertheless, for all such choices of Z, the 6443460 positroid tilings of $\mathcal{A}_{8,2,2}(Z)$ are in bijection with the positroid tilings of $\Delta_{3,8}$.

11.4. **Descent and sign-flip tilings.** Recall that permutations and their cyclic descents were used to define both the w-simplices in $\Delta_{k+1,n}$ and the w-chambers in $\mathcal{A}_{n,k,2}(Z)$. In the same spirit, by refining the set of permutations based on the *positions* of the descents, we will obtain a distinguished positroid tiling of $\Delta_{k+1,n}$ and a distinguished positroid tiling of $\mathcal{A}_{n,k,2}(Z)$. These tilings are T-dual to each other.

Recall that

$$\Delta_{k+1,n} = \bigcup_{w \in D_{k+1,n}} \Delta_w.$$

Since 1 is always a cyclic descent of $w \in D_{k+1,n}$, we have that $D_{k+1,n}$ is the set of permutations $w \in S_n$ with k left descents and $w(n) = n$. The Eulerian numbers have a very natural refinement by descent set $\mathrm{Des_L}(w)$. If $w \in S_n$ has $w(n) = n$, then neither 1 nor n is a left descent of w, so we have

$$E_{k,n-1} = \sum_{I \in \binom{[2,n-1]}{k}} \#\{w \in S_n : w(n) = n, \mathrm{Des_L}(w) = I\}.$$

This inspires the following decomposition of $\Delta_{k+1,n}$. For $I \in \binom{[2,n-1]}{k}$, let

$$\Gamma_I := \bigcup_{\substack{w \in D_{k+1,n} \\ \mathrm{Des_L}(w) = I}} \Delta_w.$$

Clearly, the collection of Γ_I cover the hypersimplex and their interiors are pairwise disjoint. There are also $\binom{n-2}{k}$ of them, which is exactly the number of full-dimensional positroid polytopes in a regular positroid tiling of $\Delta_{k+1,n}$ [SW21]. We will show that each Γ_I is in fact a positroid polytope, and that $\{\Gamma_I\}$ is a (regular) positroid tiling of $\Delta_{k+1,n}$. We will refer to it as the *descent tiling*.

On the other hand, given the sign-flip characterization of the amplituhedron from Theorem 5.1, it is natural to subdivide $\mathcal{A}_{n,k,2}(Z)$ into regions based on where the sequence $(\langle Y1a\rangle)_{a=1}^n$ has sign flips. That is, for each $I \in \binom{[2,n-1]}{k}$, we define[15]

$$Z_I^\circ := \{Y \in \mathcal{A}_{n,k,2}(Z) \mid \mathrm{Flip}(\langle Y11\rangle, \langle Y12\rangle, \langle Y13\rangle, \dots, \langle Y1n\rangle) = I\}$$

and define Z_I to be the closure of Z_I°.

[AHTT18, Section 7] conjectured that $\{Z_I\}$ is a positroid tiling of $\mathcal{A}_{n,k,2}(Z)$. The authors referred to $\{Z_I\}$ as a *sign-flip* (*or kermit*)[16] *tiling*. In this section we prove this conjecture. Moreover we show that sign-flip tilings of $\mathcal{A}_{n,k,2}(Z)$ and descent tilings of the hypersimplex $\Delta_{k+1,n}$ are T-dual to each other and also regular.

Definition 11.31 (Bicolored triangulations of kermit type)**.** Let $I = \{i_1, \dots, i_k\} \in \binom{[2,n-1]}{k}$ and let $\mathcal{T}_I$ be the bicolored triangulation whose black triangles have vertices $\{1, i_\ell, i_\ell + 1\}$ for $\ell = 1, \dots k$. We say $\mathcal{T}_I$ is *kermit type* and denote the plabic graph $\hat{G}(\mathcal{T}_I)$ by K_I. We also denote the plabic graph $G(\mathcal{T}_I)$ by C_I, and call it a *caterpillar* tree.

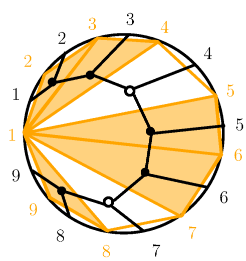


FIGURE 11. In orange, the bicolored triangulation $\mathcal{T}_I$ of kermit-type for $I = \{2, 3, 5, 6, 8\}$. In black, the dual caterpillar tree C_I.

[15]Because of our conventions regarding sign flips, Z_I would be empty if $1, n \in I$.

[16]For $k = 2$, $\mathcal{A}_{n,2,2}(Z)$ provides the integrand for the 1-loop n-point scattering amplitude in $\mathcal{N} = 4$ SYM. The name 'kermit' comes from the resemblance of the pictorial expansion of such amplitude (e.g. see [AHBC+11, pg. 18]) with the Muppet character 'Kermit the Frog'.

Proposition 11.32 (Descent and sign-flips tilings are T-dual)**.** *Let I run over $\binom{[2,n-1]}{k}$. The collections $\{\Gamma_I\}$ and $\{Z_I\}$ are T-dual regular positroid tilings of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$. Furthermore, $\Gamma_I = \Gamma_{C_I}$ and $Z_I = Z_{K_I}$ where C_I and K_I are as in Definition* 11.31.

Proof. By the sign description of $Z^{\circ}_{K_I}$ in Theorem 4.28, it is straightforward that $Z_I = Z_{K_I}$. Moreover, using Definition 10.7 and Corollary 10.17, we have

$$Z_{K_I} = \bigcup_{w:\mathrm{Des_L}(w)=I} \hat{\Delta}_w(Z). \tag{11.33}$$

Using Proposition 9.6, it is not hard to check that the positroid polytope Γ_{C_I} satisfies

$$\Gamma_{C_I} = \bigcup_{w:\mathrm{Des_L}(w)=I} \Delta_w. \tag{11.34}$$

But this is exactly Γ_I.

Finally, it is easy to check that $\{\Gamma_{C_I}\}_{I\in\binom{[2,n-1]}{k}}$ is a positroid tiling of $\Delta_{k+1,n}$ of the sort appearing in [ŁPW20, Proposition 10.7] ('Catalan type'), hence is a regular positroid tiling. It follows that $\{Z_{K_I}\}_{I\in\binom{[2,n-1]}{k}}$ is the T-dual regular positroid tiling. □

Remark 11.35. Sign-flip tilings of $\mathcal{A}_{n,k,2}(Z)$ and descent tilings of $\Delta_{k+1,n}$ are of BFCW type (in particular, of 'Catalan type', see [ŁPW20, Proposition 10.7]).

We end this section by describing each w-simplex (resp. w-chamber) as an intersection of cyclically shifted caterpillar positroid polytopes (resp. kermit Grasstopes). For $I \subset [n]$ and $a \in [n]$, let $I^{(a)}$ denote the cyclic shift of I such that $1 \mapsto a$. Similarly, for G a plabic graph, let $G^{(a)}$ denote the cyclic shift of G such that $1 \mapsto a$.

Proposition 11.36. *Let $\Delta_w, \hat{\Delta}_w(Z)$ be a w-simplex and a w-chamber in $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ respectively. Let $I_1,\dots,I_n$ give the vertices of Δ_w, and let $J_a := (I_a \setminus \{a\})^{(2-a)}$. Then:*

$$\Delta_w = \bigcap_{a\in[n]} \Gamma_{C^{(a)}_{J_a}} \quad \textit{and} \quad \hat{\Delta}_w(Z) = \bigcap_{a\in[n]} Z_{K^{(a)}_{J_a}}.$$

Proof. First, $C^{(a)}_{J_a}$ is dual to a kermit-type bicolored triangulation whose black triangles all use vertex a.

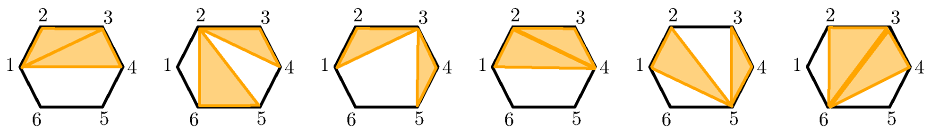
To see the statement about Δ_w, note that another way to phrase (11.34) is that Γ_{C_I} is the union of all w-simplices with 1st vertex given by $I \cup \{1\}$. Using the cyclic shift on the hypersimplex, it is not hard to see that $\Gamma_{C^{(a)}_{J_a}}$ is the union of all w-simplices with ath vertex given by I_a. So taking the intersection gives exactly the w-simplex with vertices $e_{I_1},\dots,e_{I_a}$.

The statement about $\hat{\Delta}_w(Z)$ follows from a similar argument, using (11.33) and the cyclic shift on $\mathrm{Gr}_{k,n}$. □

Example 11.37. Let us consider $w = 324156$ from Example 10.4. We have:

$I_1=\{1,2,3\}$, $I_2=\{2,3,5\}$, $I_3=\{1,3,4\}$, $I_4=\{1,2,4\}$, $I_5=\{1,3,5\}$, $I_6=\{2,3,6\}$;
$J_1=\{2,3\}$, $J_2=\{2,4\}$, $J_3=\{2,5\}$, $J_4=\{4,5\}$, $J_5=\{3,5\}$, $J_6=\{3,4\}$.

Then Δ_w is the intersection of $\Gamma_{C^{(1)}_{J_1}}, \dots, \Gamma_{C^{(6)}_{J_6}}$ and $\hat{\Delta}_w$ is the intersection of $Z_{K^{(1)}_{J_1}}, \dots, Z_{K^{(6)}_{J_6}}$. The cyclically rotated kermit-type bicolored triangulations $\mathcal{T}^{(1)}_{J_1},\dots,\mathcal{T}^{(6)}_{J_6}$ are displayed below.



Notice that $\mathcal{T}_{J_1}^{(1)}$ is equivalent to $\mathcal{T}_{J_4}^{(4)}$.

12. Schröder numbers: Separable permutations and positroid tiles

Recall from Corollary 9.1 that positroid tiles for both $\mathcal{A}_{n,k,2}(Z)$ and $\Delta_{k+1,n}$ are in bijection with bicolored subdivisions of type (k, n) and tree positroids in $\mathrm{Gr}_{k+1,n}^{\geq 0}$. [ŁPSV19] provided experimental evidence that the number $R_{k,n-2}$ of positroid tiles for $\mathcal{A}_{n,k,2}$ is given by [S$^+$, A175124], a refinement of the *large Schröder numbers* (see Table 3). In this section we prove this statement by giving a bijection between tree positroids in $\mathrm{Gr}_{k+1,n}^{\geq 0}$ and *separable* permutations on $[n-1]$ with k descents (enumerated by $R_{k,n-2}$).

Definition 12.1. A permutation $w = w_1 \dots w_n$ (in one-line notation) is *separable* if it is 3142- and 2413-avoiding, i.e. there are not four indices $i_1 < i_2 < i_3 < i_4$ such that $w_{i_3} < w_{i_1} < w_{i_4} < w_{i_2}$ or $w_{i_2} < w_{i_4} < w_{i_1} < w_{i_3}$.

Table 3. Large Schröder numbers R_{n-2} and their refinement $R_{k,n-2}$ which count the number of bicolored subdivisions of type (k, n)

n \ k	0	1	2	3	4	5	R_{n-2}
2	1						1
3	1	1					2
4	1	4	1				6
5	1	10	10	1			22
6	1	20	48	20	1		90
7	1	35	161	161	35	1	394

Definition 12.2. Let π and ν be permutations on $[k]$ and $[l]$, respectively. The *direct sum* $\pi \oplus \nu$ and the *skew sum* $\pi \ominus \nu$ of π and ν are permutations on $[k + l]$ defined by:

$$(\pi \oplus \nu)_i = \begin{cases} \pi_i, & i \in [1, k] \\ \nu_{i-k} + k, & i \in [k+1, k+l] \end{cases}, \quad (\pi \ominus \nu)_i = \begin{cases} \pi_i + l, & i \in [1, k] \\ \nu_{i-k}, & i \in [k+1, k+l] \end{cases}.$$

For example, $123 \oplus 21 = 12354$ and $123 \ominus 21 = 34521$.

Proposition 12.3 ([Kit11]). *A permutation is separable if and only if w can be built from the permutation* 1 *by repeatedly applying* $\oplus$ *and* $\ominus$.

For example, the permutation $w = 231654$ can be written as

$$((1 \oplus 1) \ominus 1) \oplus ((1 \ominus 1) \ominus 1) = (12 \ominus 1) \oplus (21 \ominus 1) = 231 \oplus 321 = 231654.$$

Proposition 12.4. *Let β be the map sending a permutation $w = w_1 \dots w_{n-1}$ in one-line notation to the permutation $\beta(w) = (w_1, \dots, w_{n-1}, n)$ in cycle notation. Then β is a bijection between separable permutations on $[n-1]$ with k descents and trip permutations of tree positroids in $\mathrm{Gr}_{k+1,n}^{\geq 0}$.*

Proof. We use strong induction on n; the base case $n = 2$ is trivial. It is enough to show that β is well-defined and surjective. Suppose that $w \in S_{n-1}$ is separable. Then either $w = u \oplus v$ or $w = u \ominus v$, for some $u \in S_{\ell-1}, v \in S_{r-1}$ separable, with $\ell - 1 + r - 1 = n - 1$. By the induction hypothesis, $\beta(u) \in S_\ell$ and $\beta(v) \in S_r$ are the trip permutations of tree plabic graphs S and T. We now "glue" together S and T in order to obtain a tree plabic graph with boundary vertices $\{1, 2, \dots, n\}$ with trip permutation $\beta(w) \in S_n$ (see Figure 12).

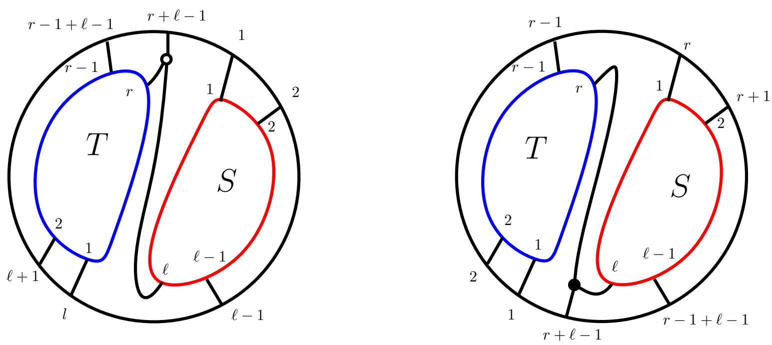


FIGURE 12. How to glue S, T together when $w = u \oplus v$ (on the left) and when $w = u \ominus v$ (on the right)

It is straightforward to check that the trip permutation of the resulting tree is $\beta(w)$. This shows that β is well-defined.

For surjectivity, consider a trivalent tree plabic graph G on $[n]$. Let v be the internal vertex adjacent to the boundary vertex n. Then deleting v gives two trees: S on $[\ell]$ and T on $[\ell + 1, n - 1]$. Let π be the trip permutation of S. Subtract ℓ from the boundary labels of T to get a tree T' on $[1, n - \ell - 1]$ and let ν be its trip permutation. Then define w to be either $\beta^{-1}(\pi) \oplus \beta^{-1}(\nu)$ or $\beta^{-1}(\pi) \ominus \beta^{-1}(\nu)$, based on whether v is white or black. By the argument used above to show well-definedness, $\beta(w)$ is the trip permutation of G. □

Remark 12.5. If S and T are tree plabic graphs, the positroids associated to $S \oplus T$ and $S \ominus T$ are the *parallel-connection* and *series-connection* of the matroids associated to S, T.

The large Schröder number R_{n-2} counts separable permutations on $[n-1]$ [Wes95] and $R_{k,n-2}$ counts separable permutations on $[n-1]$ with k descents [FLZ18, Theorem 1.1].

Corollary 12.6. *Positroid tiles of Δ_{k+1} and $\mathcal{A}_{n,k,2}(Z)$ are in bijection with separable permutations on $[n-1]$ with k descents. They are enumerated by $R_{k,n-2}$ from* [S$^+$, A175124].

Appendix A. Combinatorics of the totally nonnegative Grassmannian

In [Pos06], Postnikov defined several families of combinatorial objects which are in bijection with cells of the positive Grassmannian, including *decorated permutations*, and equivalence classes of *reduced plabic graphs*. He also used these objects to give concrete descriptions of the cells. Here we review some of this technology.

Definition A.1. A *decorated permutation* on $[n]$ is a bijection $\pi : [n] \to [n]$ whose fixed points are each colored either black (loop) or white (coloop). We denote a black fixed point i by $\pi(i) = \underline{i}$, and a white fixed point i by $\pi(i) = \overline{i}$. An *anti-exceedance* of the decorated permutation π is an element $i \in [n]$ such that either $\pi^{-1}(i) > i$ or $\pi(i) = \overline{i}$. We say that a decorated permutation on $[n]$ is of *type* (k, n) if it has k anti-exceedances.

For example, $\pi = (3, \underline{2}, 5, 1, 6, 8, \overline{7}, 4)$ has a loop in position 2, and a coloop in position 7. It has three anti-exceedances 1, 4, 7.

Decorated permutations can be equivalently thought of as affine permutations [KLS13]. An *affine permutation* on $[n]$ is a bijection $\pi : \mathbb{Z} \to \mathbb{Z}$ such that $\pi(i + n) = \pi(i) + n$ and $i \le \pi(i) \le i + n$, for all $i \in \mathbb{Z}$. It is additionally (k, n)*-bounded* if $\sum_{i=1}^{n}(\pi(i) - i) = kn$.

There is a bijection between decorated permutations of type (k, n) and (k, n)-bounded affine permutations. Given a decorated permutation π_d we can define an affine permutation π_a by the following procedure: if $\pi_d(i) > i$, then define $\pi_a(i) := \pi_d(i)$; if $\pi_d(i) < i$, then define $\pi_a(i) := \pi_d(i) + n$; if $\pi_d(i)$ is a loop then define $\pi_a(i) := i$; if $\pi_d(i)$ is a coloop then define $\pi_a(i) := i + n$. For example, under this map, the decorated permutation $\pi_d = (3, \underline{2}, 5, 1, 6, 8, \overline{7}, 4)$ in the previous example gives rise to $\pi_a = (3, 2, 5, 9, 6, 8, 15, 12)$.

Given a $k \times n$ matrix $C = (c_1, \dots, c_n)$ written as a list of its columns, we associate a decorated permutation π as follows. Given $i, j \in [n]$, let $r[i, j]$ denote the rank of $\langle c_i, c_{i+1}, \dots, c_j \rangle$, where we list the columns in cyclic order, going from c_n to c_1 if $i > j$. We set $\pi(i) := j$ to be the label of the first column j such that $c_i \in \operatorname{span}\{c_{i+1}, c_{i+2}, \dots, c_j\}$. If c_i is the all-zero vector, we decorate i as loop, and if c_i is not in the span of the other column vectors, we decorate i as coloop.

The map $C \mapsto \pi$ extends to a map on positroid cells. Moreover, Postnikov showed that the positroids for $Gr_{k,n}^{\ge 0}$ are in bijection with decorated permutations of $[n]$ with exactly k anti-exceedances (equivalently, by (k, n)-bounded affine permutations) [Pos06, Section 16]. One may read off the dimension of the cell S_π from the affine permutation π as follows. Let $\operatorname{inv}(\pi)$ be the number of pairs (i, j) such that $i \in [n], j \in \mathbb{Z}, i < j$, and $\pi(i) > \pi(j)$. Then the dimension of S_π equals $k(n - k) - \operatorname{inv}(\pi)$.

Definition A.2. A *planar bicolored graph* (or "plabic graph") is a planar graph G properly embedded into a closed disk, such that each internal vertex is colored black or white; each internal vertex is connected by a path to some boundary vertex; there are (uncolored) vertices lying on the boundary of the disk labeled $1, \dots, n$ for some positive n; and each of the boundary vertices is incident to a single edge. See Figure 13 for an example.

If the connected component of G attached to a boundary vertex i is a path ending at a black (resp., white) leaf, we call this component a black (resp., white) *lollipop*. *We will require that our plabic graphs have no internal leaves except for lollipops.*

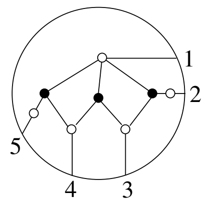


FIGURE 13. A plabic graph

There is a natural set of local transformations (moves) of plabic graphs:

(M1) *Square move* (or *urban renewal*). If a plabic graph has a square formed by four trivalent vertices whose colors alternate, then we can switch the colors of these four vertices.

(M2) *Contracting/expanding a vertex.* Two adjacent internal vertices of the same color can be merged. This operation can also be reversed.

(M3) *Middle vertex insertion/removal.* We can remove/add degree 2 vertices.

See Figure 14 for depictions of these three moves.

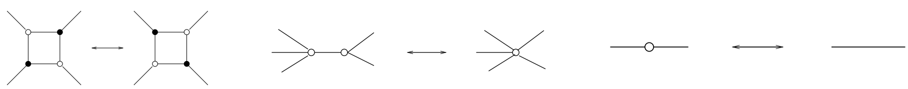


FIGURE 14. Local moves (M1), (M2), (M3) on plabic graphs

Definition A.3. Two plabic graphs are called *move-equivalent* if they can be obtained from each other by moves (M1)-(M3). The *move-equivalence class* of a given plabic graph G is the set of all plabic graphs which are move-equivalent to G. A plabic graph is called *reduced* if there is no graph in its move-equivalence in which two adjacent vertices u and v are connected by more than one edge

Note that given a plabic graph G, we can always apply moves to G to obtain a new graph G' which is bipartite.

Definition A.4. Let G be a reduced plabic graph as above with boundary vertices $1, \dots, n$. For each boundary vertex $i \in [n]$, we follow a path along the edges of G starting at i, turning (maximally) right at every internal black vertex, and (maximally) left at every internal white vertex. This path ends at some boundary vertex $\pi(i)$. By [Pos06, Section 13], the fact that G is reduced implies that each fixed point of π is attached to a lollipop; we color each fixed point by the color of its lollipop. In this way we obtain the *decorated permutation* $\pi_G = \pi$ of G. We say that G is of *type* (k, n), where k is the number of anti-exceedances of π_G.

The decorated permutation of the plabic graph G of Figure 13 is $\pi_G = (3, 4, 5, 1, 2)$, which has $k = 2$ anti-exceedances.

Definition A.5. Let G be a bipartite plabic graph. Use move (M3) to ensure that each boundary vertex is incident to a white vertex. An *almost perfect matching* M of a plabic graph G is a subset M of edges such that each internal vertex is incident to exactly one edge in M (and each boundary vertex i is incident to either one or no edges in M). We let $\partial M = \{i \mid i \text{ is incident to an edge of } M\}$.

We associate to each graph G as above a collection of subsets $\mathcal{M}(G) \subset [n]$ as follows.

Proposition A.6 ([Pos06, Proposition 11.7, Lemma 11.10]). *Let G be a plabic graph as in Definition* A.5*, and let $\mathcal{M}(G) = \{\partial M \mid M \text{ an almost perfect matching of } G\}$. Then $\mathcal{M}(G)$ is the set of bases of a positroid on $[n]$. Its rank is*

$$\#\{\text{white vertices of } G\} - \#\{\text{black vertices of } G\},$$

which is the size of ∂M for any almost perfect matching M of G.

Postnikov used plabic graphs to give parameterizations of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. These parameterizations of cells can be recast as a variant of a theorem of Kasteleyn, as was made explicit in [Spe16]. We follow the exposition there.

Theorem A.7 ([Spe16]). *Let G be a bipartite graph with boundary embedded in a disk, such that all of the boundary vertices are black. Suppose there are $N + k$ white vertices $W_1, \dots, W_{N+k}$, N internal black vertices $B_1, \dots, B_N$, and n boundary vertices $B_{N+1}, \dots, B_{N+n}$, labeled in clockwise order. Let w : $\mathrm{Edges}(G) \to \mathbb{R}_{>0}$ be any weighting function; if there is an edge between vertices i and j, we denote the weight on this edge by w_{ij}. For a perfect matching M, define $w(M) = \prod_{e \in M} w(e)$ and define ∂M to be the indices of the boundary vertices covered by an edge in M. For a subset I of $\{W_{N+1}, \dots, W_{N+n}\}$, define $\mathbb{D}(G, I, w) = \sum_{\partial M = I} w(M)$.*

Then there is a real $k \times n$ Kasteleyn matrix *L such that for each k-element subset I of ∂G, the determinant* $\det L_I$ *of the $k \times k$ submatrix of L using the columns indexed by I is* $\det L_I = \mathbb{D}(G, I, w)$*. In particular, all Plücker coordinates of L are nonnegative.*

The *positroid cell* $S_G \subset \mathrm{Gr}_{k,n}$ associated to the plabic graph G is the set of all k-planes in $\mathbb{R}^n$ spanned by matrices L as in Theorem A.7. If G is a tree, we call S_G a *tree positroid*.

Remark A.8. The Kasteleyn matrix L is constructed as follows. First construct an $(N+k)\times(N+n)$ matrix K, with rows indexed by white vertices and columns indexed by black vertices, with $K_{ij} = \pm w_e$ if there is an edge e between vertices i and j (otherwise $K_{ij} = 0$). Then, assuming G has at least one perfect matching, we can apply row operations to transform K into a matrix of block form $\begin{pmatrix} \mathrm{Id}_N & \star \\ 0 & L \end{pmatrix}$.

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