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# Only-phase Popov action: thermodynamic derivation and superconducting electrodynamics

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**Abstract**

We provide a thermodynamic derivation of the only-phase Popov action functional, which is often adopted to study the low-energy effective hydrodynamics of a generic nonrelativistic superfluid. It is shown that the crucial assumption is the use of the saddle point approximation after neglecting the quantum-pressure term. As an application, we analyze charged superfluids (superconductors) coupled to the electromagnetic field at zero temperature. Our only-phase and minimally-coupled theory predicts the decay of the electrostatic field inside a superconductor with a characteristic length much smaller than the London penetration depth of the static magnetic field. This result is confirmed also by a relativistic only-phase Popov action we obtain from the Klein–Gordon Lagrangian.

Keywords: superconductors, electrodynamics, hydrodynamics, superfluids

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## 1. Introduction

The phenomenological description of superconductive hydrodynamics has gained recent interest, due to the possibility of writing effective hydrodynamic Lagrangians describing the interaction of electromagnetic fields with charged superfluids [1–3]. In this context, the only-phase Popov action is a valuable tool to predict the behavior of the Nambu–Goldstone phase field [4, 5]. On the other hand, the problem of the screening of the electromagnetic field inside a superconductor has been deeply investigated since the first phenomenological models [6, 7], where it was shown that the magnetostatic field is exponentially screened with a characteristic length called London penetration depth $\lambda_{\mathrm{L}}$. Typically $\lambda_{\mathrm{L}}$ is in the order of hundreds of nanometers. Past theories suggest that the same cannot be said for the electrostatic field screening, that, analogously to the case of normal conductors, is decaying with a much shorter length scale, i.e. the Thomas–Fermi screening length, in the order of few angströms [8].

In the present work, on a theoretical point of view, we first review the Popov prescription for obtaining an only-phase action for a nonrelativistic zero-temperature fluid [9] in section 2. Then, in section 3 we show how it is possible to derive the same result from the familiar hydrodynamic action of a self-interacting nonrelativistic bosonic field. We observe that the result by Popov can be related to the latter action, given that the quantum pressure term has been neglected. By introducing a path integration over the number density field and an additional field, and performing a saddle-point approximation of the grand-canonical partition function, the original prescription by Popov for obtaining the only-phase action is retrieved. Working at zero temperature, we identify the additional field as the Nambu–Goldstone field, that appears in the resulting Lagrangian density in its gradient squared. In section 4 we introduce minimal coupling of the Nambu–Goldstone phase field to the electromagnetic field, and we obtain the equations of motion including one-loop corrections of the superconductive dynamics. They involve the Maxwell equations, and constitutive relations for the charge density and current density. Comparing the penetration depth of the magnetostatic field to the electrostatic field, we conclude that, within our formalism, the electrostatic field penetration depth can be put on the same footing as the magnetostatic field one, but with a penetration depth $\lambda_{\mathrm{E}}$ many orders of magnitude smaller than $\lambda_{\mathrm{L}}$. Finally, in section 5 we develop a relativistic only-phase Popov model which confirms that $\lambda_{\mathrm{E}} \ll \lambda_{\mathrm{L}}$ in realistic superconductors. Experiments measuring the penetration depth in superconductors are hindered by the necessity to use, at the same time, precise field measurements with nanometer-scale resolution, and cryogenic apparatus to keep the superconductor well below the transition temperature [10]. We remark that, until now, experiments have not been able to measure the electrostatic field penetration depth to a sufficient accuracy to discriminate between the prediction of the present work, i.e. the penetration depth indicated by $\lambda_{\mathrm{E}}$, and the Thomas–Fermi screening effect.

## 2. Popov superfluid Lagrangian

In the grand canonical framework, at zero temperature the pressure $P$ of a fluid can be written in terms of its chemical potential $\mu$, i.e.

$$P = P(\mu) \ . \tag{1}$$

This is the zero-temperature equation of the state of the fluid [11, 12]. For instance, in the case of a weakly-interacting bosonic gas it is given by $P(\mu) = \mu^2/(2g)$, where $g$ is the strength of the effective Bose–Bose contact interaction. Instead, for a two-spin-component superfluid

Fermi gas one has $P(\mu) = (2/(15\pi^2))(2m/\hbar^2)^{3/2}\mu^{5/2}$ neglecting the Fermi–Fermi interaction. The number density $n$ can be obtained from the pressure $P(\mu)$ using the thermodynamic formula [12]

$$n = \frac{\partial P}{\partial \mu}(\mu) \ . \tag{2}$$

For a fluid of identical particles of mass $m$ and chemical potential $\mu(n)$, the zero-temperature speed of sound $c_s$ is defined as [12]

$$c_s = \sqrt{\frac{n}{m}\frac{1}{\frac{\partial^2 P}{\partial \mu^2}(\mu)}} \ . \tag{3}$$

The main idea of Popov [9, 13], later adopted and extended by other authors (see, for instance, [14–16]), is that the only-phase action functional

$$\tilde{S}[\theta] = \int \mathrm{d}t \int \mathrm{d}[D]\,\mathbf{r}\,\tilde{\mathscr{L}} \tag{4}$$

of a nonrelativistic superfluid, which is characterized by the Nambu-Goldstone [4, 5] real scalar field $\theta(\mathbf{r},t)$, is obtained with the prescription

$$\mu \to \mu - \hbar\partial_t\theta - \frac{\hbar^2}{2m}|\nabla\theta|^2 \tag{5}$$

into the pressure $P(\mu)$ such that

$$\tilde{\mathscr{L}} = P\left(\mu - \hbar\partial_t\theta - \frac{\hbar^2}{2m}|\nabla\theta|^2\right) \tag{6}$$

is the real-time only-phase Lagrangian density. This approach has been also extended to the relativistic case [17–21].

Expanding (6) with respect to $\theta$ around $\mu$, taking into account equations (2) and (3), we find

$$\tilde{\mathscr{L}} = P(\mu) - n\left(\hbar\partial_t\theta + \frac{\hbar^2}{2m}|\nabla\theta|^2\right) + \frac{1}{2}\frac{n}{mc_s{}^2}\left(\hbar\partial_t\theta + \frac{\hbar^2}{2m}|\nabla\theta|^2\right)^2 + \dots \tag{7}$$

Removing the dots $(\dots)$, equation (7) becomes exactly the zero-temperature low-wavenumber effective Lagrangian density one finds at the one-loop level from the microscopic beyond-mean-field BCS-like model of attractive fermions [22] and also from the microscopic model of weakly-interacting bosons [23]. Instead, considering only the first two terms of equation (7) one recovers the familiar hydrodynamic Lagrangian density

$$\tilde{\mathscr{L}}_0 = P(\mu) - n\left(\hbar\partial_t\theta + \frac{\hbar^2}{2m}|\nabla\theta|^2\right) \tag{8}$$

of classical inviscid and irrotational fluids (see, for instance, [24]).

## 3. Deriving the only-phase Popov action

The derivation of the only-phase Popov action, equation (6), was performed by Popov [9, 13] starting from a bosonic action and separating fast and slowly varying components of the bosonic field. Here we obtain the same result by using a different procedure: the saddle-point functional integration over the density field of a peculiar density-phase action functional, given by

$$S[n,\theta] = \int_0^{+\infty} \mathrm{d}t \int_{L^D} \mathrm{d}[D]\,\mathbf{r} \left[ -\mathcal{E}_0(n) - n\hbar\partial_t\theta - n\frac{\hbar^2}{2m}|\boldsymbol{\nabla}\theta|^2 + \mu n \right] , \tag{9}$$

where $\mathcal{E}_0(n)$ is the zero-temperature internal energy of the system as a fuction of the local number density $n(\mathbf{r},t)$. For instance, in the case of weakly-interacting bosons $\mathcal{E}_0(n) = gn^2/2$ with $g$ the interaction strength, while for superfluid fermions $\mathcal{E}_0(n) = (5/3)(3\pi^2)^{2/3}n^{5/3}$ again neglecting the residual inter-particle interaction between fermions. Equation (9) is nothing else than the hydrodynamic action of a self-interacting nonrelativistic bosonic field $\psi(\mathbf{r},t)$ characterized by the action functional

$$S[\psi] = \int_0^{+\infty} \mathrm{d}t \int_{L^D} \mathrm{d}[D]\,\mathbf{r} \left[ \mathrm{i}\frac{\hbar}{2}\left(\psi^*\partial_t\psi - \psi\,\partial_t\psi^*\right) - \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi|^2 - \mathcal{E}_0\left(|\psi|^2\right) \right] , \tag{10}$$

under the familiar Madelung decomposition

$$\psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\,\mathrm{e}^{\mathrm{i}\left(\theta(\mathbf{r},t) - \frac{\mu}{\hbar}t\right)} \tag{11}$$

but then neglecting the quantum pressure term $\hbar^2(\nabla\sqrt{n})^2/(2m)$. This assumption is reliable in the spatial regions where the condition $|\hbar^2(\nabla\sqrt{n})^2/(2m)| \ll |\mathcal{E}_0(n)|$ is satisfied. The inequality says that the quantum pressure term can be neglected if the local number density is much larger than its gradient. Usually, the quantum pressure term is relevant, in the presence of a confinement potential, only near the surface or, in the present case, only at very small wavelengths.

### 3.1. Grand Canonical partition function, free energy and grand potential

It is well known that the Grand Canonical partition function $\mathcal{Z}$, that is a function of the chemical potential $\mu$, is related to the Helmholtz free energy $F$, that is a function of the total number $N$ of particle, by the thermodynamic formula

$$\mathcal{Z} = \sum_{N=0}^{\infty} \mathrm{e}^{-\beta[F(N)-\mu N]} , \tag{12}$$

where $\beta = 1/(k_\mathrm{B}T)$ with $k_\mathrm{B}$ the Boltzmann constant and $T$ the absolute temperature. Remember that the Helmholtz free energy $F$ is defined as $F = E - TS$ with $E$ the internal energy and $S$ the entropy.

In the low-temperature regime, where $\beta$ becomes very large, one can adopt the saddle-point approximation finding

$$\mathcal{Z} \simeq \mathrm{e}^{-\beta[F(N_s)-\mu N_s]} , \tag{13}$$

where $N_s$ is the saddle-point number of particles, obtained by inverting the formula

$$\frac{\partial}{\partial N}\left[F(N) - \mu N\right] = 0 , \tag{14}$$

which extremizes the exponent of the exponential function. It is important to stress that $N_s$ is a function of the chemical potential $\mu$, i.e. $N_s = N_s(\mu)$. Thus, we can write

$$\mathcal{Z} \simeq \mathrm{e}^{-\beta\Omega(\mu)} , \tag{15}$$

where $\Omega(\mu)$ is the thermodynamic grand potential, such that

$$\Omega(\mu) = F(N_s(\mu)) - \mu N_s(\mu) . \tag{16}$$

### *3.2. Path-integral representation adding an arbitrary field*

Making explicit the procedure briefly discussed in [25], let us introduce the number density field $n(\mathbf{r},\tau)$ as a function of the position vector $\mathbf{r}$ and imaginary time $\tau$. It must satisfy the relation

$$\beta N = \frac{1}{\hbar}\int_0^{\hbar\beta} \mathrm{d}\tau \int_{L^D} \mathrm{d}\left[D\right]\mathbf{r}\, n\left(\mathbf{r},\tau\right) \ . \tag{17}$$

We also introduce the local free energy density $\mathcal{F}(n(\mathbf{r},t),\theta(\mathbf{r},t))$ which depends on the local number density $n(\mathbf{r},\tau)$ and another generic field $\theta(\mathbf{r},t)$. We impose that

$$\beta F = \frac{1}{\hbar}\int_0^{\hbar\beta} \mathrm{d}\tau \int_{L^D} \mathrm{d}\left[D\right]\mathbf{r}\, \mathcal{F}\left(n\left(\mathbf{r},\tau\right),\theta\left(\mathbf{r},\tau\right)\right) \ . \tag{18}$$

Then, taking into account the thermodynamic limit and the fact that the particle number density is not uniform, we write the relationship (see also [25])

$$\sum_{N=0}^{\infty} \rightarrow \int_0^{\infty} \mathrm{d}N \rightarrow \int \mathcal{D}\left[n\left(\mathbf{r},t\right)\right] \ , \tag{19}$$

immediately obtaining the following path-integral representation of the Grand Canonical partition function

$$\mathcal{Z}\left[\theta\right] = \int \mathcal{D}\left[n\left(\mathbf{r},\tau\right)\right] \mathrm{e}^{-\frac{1}{\hbar}\int_0^{\hbar\beta}\mathrm{d}\tau \int_{L^D}\mathrm{d}[D]\mathbf{r}\,[\mathcal{F}(n(\mathbf{r},\tau),\theta(\mathbf{r},\tau))-\mu n(\mathbf{r},\tau)]} \ . \tag{20}$$

We use also here the saddle-point approximation. In this way, we have

$$\mathcal{Z}\left[\theta\right] \simeq \mathrm{e}^{-\frac{1}{\hbar}\int_0^{\hbar\beta}\mathrm{d}\tau \int_{L^D}\mathrm{d}[D]\mathbf{r}\,[\mathcal{F}(n_s(\mathbf{r},\tau),\theta(\mathbf{r},\tau))-\mu n_s(\mathbf{r},\tau)]} \ , \tag{21}$$

where the saddle-point density field $n_s(\mathbf{r},\tau)$ is obtained by inverting the equation

$$\mu = \frac{\delta\mathcal{F}}{\delta n}\left(n_s,\theta\right) \tag{22}$$

which involves the functional derivative of the local free energy. Clearly, this saddle-point density $n_s(\mathbf{r},\tau)$ is a function of the chemical potential $\mu$, i.e. $n_s(\mathbf{r},\tau;\mu)$.

We introduce a local pressure $P(\theta(\mathbf{r},\tau);\mu)$ that is a function of the arbitrary field $\theta(\mathbf{r},t)$ and also of the chemical potential $\mu$. This local pressure, that is given by

$$P\left(\theta\left(\mathbf{r},\tau\right);\mu\right) = -\mathcal{F}\left(n_s\left(\mathbf{r},\tau;\mu\right),\theta\left(\mathbf{r},\tau\right)\right) + \mu n_s\left(\mathbf{r},\tau;\mu\right) \ , \tag{23}$$

is related to the grand potential by the formula

$$-\beta\Omega = \frac{1}{\hbar}\int_0^{\hbar\beta} \mathrm{d}\tau \int_{L^D} \mathrm{d}\left[D\right]\mathbf{r}\, P\left(\theta\left(\mathbf{r},\tau\right);\mu\right) \ . \tag{24}$$

### *3.3. Zero temperature limit*

In the zero-temperature limit, i.e. setting $\beta \rightarrow +\infty$, where the local free energy density $\mathcal{F}(n,\theta)$ becomes a local internal energy density $\mathcal{E}(n,\theta)$ because the entropic contribution $TS$ vanishes, and performing the Wick rotation

$$\tau = \mathrm{i}t \,, \tag{25}$$

from equation (20) we get

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\tilde{S}[\theta]} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[n_s(\mu),\theta]} \simeq \int \mathcal{D}\left[n\right] \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[n,\theta]} \ , \tag{26}$$

where

$$\tilde{S}[\theta] = \int_0^{+\infty} \mathrm{d}t \int_{L^D} \mathrm{d}[D]\,\mathbf{r}\, P(\theta(\mathbf{r},t);\mu) \tag{27}$$

is the action functional without the local density. Instead,

$$S[n,\theta] = \int_0^{+\infty} \mathrm{d}t \int_{L^D} \mathrm{d}[D]\,\mathbf{r}\, [-\mathcal{E}(n(\mathbf{r},t),\theta(\mathbf{r},t)) + \mu\, n(\mathbf{r},t)] \tag{28}$$

is the action functional with the local density. The last equality of equation (26) is the zero-temperature version of equations (13) and (15).

### 3.4. Nambu–Goldstone phase field

Let us suppose that the field $\theta(\mathbf{r},t)$ is the Nambu–Goldstone phase field of a superfluid [4, 5], such that

$$\mathbf{v}_s = \frac{\hbar}{m}\boldsymbol{\nabla}\theta \tag{29}$$

is the superfluid velocity of the system composed of identical bosonic (or bosonic-like) particles of mass $m$. Here $\hbar$ is the reduced Planck constant. Equation (28) is the density-phase Popov action imposing that the internal energy density $\mathcal{E}(n(\mathbf{r},t),\theta(\mathbf{r},t))$ is given by

$$\mathcal{E}(n,\theta) = \mathcal{E}_0(n) + n\hbar\partial_t\theta + n\frac{\hbar^2}{2m}|\boldsymbol{\nabla}\theta|^2\,, \tag{30}$$

where $\mathcal{E}_0(n) = \mathcal{E}(n,\theta=0)$ is the zero-temperature internal energy in the absence of the phase field $\theta(\mathbf{r},t)$.

At zero temperature, the thermodynamic formula which connects the saddle-point local density $n_s(\mathbf{r},t)$ to the chemical potential $\mu$ is

$$\mu = \frac{\partial\mathcal{E}}{\partial n}(n_s,\theta)\ . \tag{31}$$

Explicitly, we have

$$\mu = \frac{\partial\mathcal{E}_0}{\partial n}(n_s) + \hbar\partial_t\theta + \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\theta|^2 \tag{32}$$

or, equivalently

$$\mu - \hbar\partial_t\theta - \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\theta|^2 = \frac{\partial\mathcal{E}_0}{\partial n}(n_s)\ . \tag{33}$$

The inversion of equation (33), namely

$$n_s = n_s(\mu,\theta) = \left(\frac{\partial\mathcal{E}_0}{\partial n}\right)^{-1}\left(\mu - \hbar\partial_t\theta - \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\theta|^2\right)\,, \tag{34}$$

gives $n_s$ as a function of $\mu$ and $\theta$. In this way we can then write the formal expression

$$\begin{aligned} P(\mu,\theta) &= P(n_s(\mu,\theta)) = P\left(\left(\frac{\partial\mathcal{E}_0}{\partial n}\right)^{-1}\left(\mu - \hbar\partial_t\theta - \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\theta|^2\right)\right) \\ &= P\left(\mu - \hbar\partial_t\theta - \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\theta|^2\right)\,, \end{aligned} \tag{35}$$

that is a Legendre transformation and $P(\mu,\theta)$ is the local pressure which appears in equation (27).

It is important to observe that in equation (27) there is the peculiar Lagrangian density

$$\tilde{\mathscr{L}} = P(\mu, \theta) \ . \tag{36}$$

Clearly, if $\theta(\mathbf{r},t) = 0$, the Lagrangian density is nothing else than the zero-temperature pressure $P$ written in terms of its chemical potential $\mu$, i.e. $P(\mu) = P(\mu, \theta = 0)$. On the basis of equation (35), the Lagrangian (36) is given exactly by equation (6), thus $P(\mu,\theta) = P(\mu - \hbar\partial_t\theta - \hbar^2|\boldsymbol{\nabla}\theta|^2/(2m))$.

## 4. Superconducting Lagrangian

In the case of a superconductor, i.e. a charged superfluid with $q$ the electric charge of each particle of mass $m$, one can generalize the Lagrangian density (6) introducing the following coupling [15, 25, 26]

$$\partial_t\theta \to \partial_t\theta + \frac{q}{\hbar}\Phi \tag{37}$$

$$\boldsymbol{\nabla}\theta \to \boldsymbol{\nabla}\theta - \frac{q}{\hbar}\mathbf{A} \tag{38}$$

to the electromagnetic field. Here $\Phi(\mathbf{r},t)$ is the scalar potential and $\mathbf{A}(\mathbf{r},t)$ is the vector potential, such that

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \partial_t\mathbf{A} \tag{39}$$

$$\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A} \tag{40}$$

with $\mathbf{E}(\mathbf{r},t)$ the electric field and $\mathbf{B}(\mathbf{r},t)$ the magnetic field. In this way, the total Lagrangian density $\mathscr{L}_{\text{tot}}$ of the Goldstone mode coupled to the electromagnetic field is given by

$$\mathscr{L}_{\text{tot}} = \mathscr{L}_{\text{shift}} + \mathscr{L}_{\text{em}} + \mathscr{L}_{\text{bg}} \ , \tag{41}$$

where

$$\mathscr{L}_{\text{shift}} = P\left(\mu - \hbar\left(\partial_t\theta + \frac{q}{\hbar}\Phi\right) - \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\theta - \frac{q}{\hbar}\mathbf{A}|^2\right) \tag{42}$$

is the Lagrangian density of the shifted Goldstone mode,

$$\mathscr{L}_{\text{em}} = \frac{\epsilon_0}{2}|\mathbf{E}|^2 - \frac{1}{2\mu_0}|\mathbf{B}|^2 \tag{43}$$

is the Lagrangian density of the free electromagnetic field, with $\epsilon_0$ the dielectric constant in the vacuum and $\mu_0$ the paramagnetic constant in the vacuum. Remember that $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in the vacuum. We also added a term

$$\mathscr{L}_{\text{bg}} = \bar{n}_{\text{bg}}q\Phi \tag{44}$$

that takes into account the role of a uniform background of positive charges, i.e. the average number density of the ions $\bar{n}_{\text{bg}}$ times the electric charge $q$, to ensure net neutrality of the material, similarly to the Jellium model of a conductor.

The Euler–Lagrange equations of the total Lagrangian (41) with respect to the scalar potential $\Phi(\mathbf{r},t)$ and the vector potential $\mathbf{A}(\mathbf{r},t)$ are nothing else than the Maxwell equations

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{45}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{46}$$

$$\nabla \wedge \mathbf{E} = \partial_t \mathbf{B} \tag{47}$$

$$\nabla \wedge \mathbf{B} = \mu_0 \, \mathbf{j} + \epsilon_0 \mu_0 \, \partial_t \mathbf{E} \tag{48}$$

where, however, the expressions of the local charge density $\rho(\mathbf{r},t)$, including Cooper pairs and the uniform positive background, and the local current density $\mathbf{j}(\mathbf{r},t)$ are highly nontrivial

$$\rho = -\frac{\partial \left(\mathcal{L}_{\text{shift}} + \mathcal{L}_{\text{bg}}\right)}{\partial \Phi} \tag{49}$$

$$\mathbf{j} = -\frac{\partial \mathcal{L}_{\text{shift}}}{\partial \mathbf{A}} \tag{50}$$

Notice that, within the approximation of using the Lagrangian (8) instead of (6), one gets

$$\rho = qn - q\bar{n}_{\text{bg}} \tag{51}$$

$$\mathbf{j} = qn\mathbf{v}_s - \frac{q^2 n}{m}\mathbf{A} \tag{52}$$

where the second term in the current density is nothing else than the London current [6], which gives rise to the expulsion of a magnetic field from a superconductor (Meissner–Ochsenfeld effect) [27].

With an improved approximation, namely working with the expansion (7) and including next-to-leading terms, we find instead

$$\rho = q\,n - q\bar{n}_{\text{bg}} - \epsilon_0 \, \frac{q^2 n \mu_0}{m} \left(\frac{c^2}{{c_s}^2}\right) \Phi \tag{53}$$

$$\mathbf{j} = q\,n\mathbf{v}_s - \frac{q^2 n}{m}\mathbf{A} \tag{54}$$

taking into account equation (3) which gives $P''(\mu) = n/(m{c_s}^2) = \epsilon_0\mu_0(n/m)(c/c_s)^2$ with $c = 1/\sqrt{\epsilon_0\mu_0}$ the speed of light and $c_s$ the speed of sound. It is important to stress that $qn$ is the electric charge density of Cooper pairs, $-q\bar{n}_{\text{bg}}$ is the electric charge density of the uniform background, and $-\epsilon_0\mu_0 q^2 n\Phi/m$ is a sort of interaction charge density related to the coupling with the scalar potential $\Phi$. In section 4.2 we will show that this term is crucial to get the correct penetration depth of the electric field. Instead, $q\,n\mathbf{v}_s$ is the electric current density of Cooper pairs and $-q^2 n\mathbf{A}/m$ is the London current [6] related to the coupling with the vector potential $\mathbf{A}$. We remark that a result similar to the one of equation (53) can be obtained from the relativistic equation of motion for the Cooper pair field, i.e. Klein–Gordon equation, by coupling the equation to the electromagnetic field, and taking the nonrelativistic limit [28]. However, within that mean-field relativistic approach [28] the ratio $(c^2/c_s^2)$ does not appear in the last term of equation (53).

### *4.1. London penetration depth for the magnetostatic field*

In a static configuration with a zero superfluid velocity $\mathbf{v}_s$ and in the absence of the electric field, i.e. $\mathbf{E} = \mathbf{0}$, the curl of equation (48) gives

$$-\nabla^2 \mathbf{B} = \mu_0 \, \nabla \wedge \left(-\frac{q^2 n}{m}\mathbf{A}\right) , \tag{55}$$

taking into account that $\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{B}) = -\nabla^2 \mathbf{B} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{B}) = -\nabla^2 \mathbf{B}$ due to the Gauss law, equation (46). Assuming that the local density $n(\mathbf{r})$ is uniform, i.e. $n(\mathbf{r}) = \bar{n}$, by using equation (40) we get

$$\nabla^2 \mathbf{B} = \frac{q^2 \bar{n}_s \mu_0}{m} \mathbf{B} . \tag{56}$$

Choosing the magnetic field as $\mathbf{B} = B(x)\,\mathbf{u}$, with $\mathbf{u}$ a unit vector, the previous equation has the following physically relevant solutions for a superconducting slab defined in the region $x \geqslant 0$:

$$B(x) = B(0)\ \mathrm{e}^{-x/\lambda_{\mathrm{L}}} , \tag{57}$$

where

$$\lambda_{\mathrm{L}} = \sqrt{\frac{m}{q^2 \bar{n}_s \mu_0}} \tag{58}$$

is the so-called London penetration depth [6], which is around 100 nanometers [29]. Equation (57) says that inside a superconductor the magnetostatic field decays exponentially. This is a well-known Meissner–Ochsenfeld effect [27].

### 4.2. Penetration depth for the electrostatic field

It is well known that normal metals screen an external electric field $\mathbf{E}$, which can penetrate at most few angströms (Thomas-Fermi screening length) [8]. For superconducting materials, our equations (45), (46), (53) and (54) suggest that the electric field $\mathbf{E}$ exponentially decays inside a zero-temperature superconductor with a characteristic penetration depth

$$\lambda_{\mathrm{E}} = \lambda_{\mathrm{L}} \frac{c_s}{c} \tag{59}$$

which is many orders of magnitude much smaller than the London penetration depth $\lambda_{\mathrm{L}}$. Let us show how to derive this result within our theoretical framework. In a static configuration, in the absence the magnetic field, i.e. $\mathbf{B} = \mathbf{0}$, and assuming a uniform number density, the gradient of equation (45), with equations (53) and (58), gives

$$\nabla^2 \mathbf{E} = -\frac{1}{{\lambda_{\mathrm{E}}}^2} \boldsymbol{\nabla} \Phi , \tag{60}$$

taking into account that $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) = \nabla^2 \mathbf{E} - \boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{E}) = \nabla^2 \mathbf{E}$ . In addition, equation (39) with the $\partial_t \mathbf{A} = \mathbf{0}$ implies $\mathbf{E} = -\boldsymbol{\nabla} \Phi$ and consequently we find

$$\nabla^2 \mathbf{E} = \frac{1}{{\lambda_{\mathrm{E}}}^2} \mathbf{E} . \tag{61}$$

Choosing $\mathbf{E} = E(x)\,\mathbf{u}$, with $\mathbf{u}$ a unit vector, the previous equation has the following physically relevant solutions for a superconducting slab defined in the region $x \geqslant 0$:

$$E(x) = E(0)\ \mathrm{e}^{-x/\lambda_{\mathrm{E}}} . \tag{62}$$

Equation (62) says that inside a zero-temperature superconductor the electrostatic field decays exponentially. The characteristic decay length $\lambda_{\mathrm{E}}$ of the electric field is quite different with respect to the London penetration depth $\lambda_{\mathrm{L}}$ of the magnetic field. Equation (62) was predicted by the London brothers [6] with $\lambda_{\mathrm{L}}$ instead of our $\lambda_{\mathrm{E}}$ but, in the absence of experimental validation [7], subsequently, Fritz London rejected it [30].

## 5. Relativistic only-phase Popov action

On the basis of the procedure previously discussed, it is possible to obtain a rel oativistic only-phase Popov action. Following [28] we start from the relativistic Klein–Gordon complex scalar field $\varphi(\mathbf{r},t)$ with Lagrangian density

$$\mathscr{L}_{\mathrm{R}} = \frac{\hbar^2}{2mc^2}|\partial_t\varphi|^2 - \frac{\hbar^2}{2m}|\nabla\varphi|^2 - \frac{mc^2}{2}|\varphi|^2 - \mathcal{E}_0\left(|\varphi|^2\right) + \mu\frac{\mathrm{i}\hbar^2}{2mc^2}\left(\varphi^*\partial_t\varphi - \varphi\,\partial_t\varphi^*\right) \ . \quad (63)$$

where the last term takes into account the conserved quantity

$$Q = \frac{\mathrm{i}\hbar^2}{2mc^2}\int d^3\mathbf{r}\left(\varphi^*\partial_t\varphi - \varphi\,\partial_t\varphi^*\right) \quad (64)$$

that is the number of particles minus the number of anti-particles [31]. This Lagrangian can be rewritten in a Schrödinger-like form setting

$$\varphi(\mathbf{r},t) = \psi(\mathbf{r},t)\,\mathrm{e}^{-\mathrm{i}mc^2t/\hbar} \quad (65)$$

with the aim of removing the mass term $mc^2|\varphi|^2/2$. In this way we get

$$\begin{aligned}\mathscr{L}_{\mathrm{R}} = {} & \frac{\hbar^2}{2mc^2}|\partial_t\psi|^2 + \frac{\mathrm{i}\hbar}{2}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right) - \frac{\hbar^2}{2m}|\nabla\psi|^2 - \mathcal{E}_0\left(|\psi|^2\right) \\ & + \mu\left[|\psi|^2 + \frac{\mathrm{i}\hbar}{2mc^2}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right)\right] \ . \end{aligned} \quad (66)$$

The terms with $mc^2$ at the denominator make the relativistic Lagrangian different with respect to the nonrelativistic one. We now insert

$$\psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\,\mathrm{e}^{\mathrm{i}\theta(\mathbf{r},t)} \quad (67)$$

into the last Lagrangian density obtaining

$$\mathscr{L}_{\mathrm{R}} = n\frac{\hbar^2}{2mc^2}\left(\partial_t\theta\right)^2 - n\hbar\partial_t\theta - n\frac{\hbar^2}{2m}|\nabla\theta|^2 - \mathcal{E}_0(n) + \mu n \quad (68)$$

after neglecting the terms that depend on the space and time derivatives of the density $n(\mathbf{r},t)$, i.e. $\hbar^2(\nabla\sqrt{n})^2/(2m)$ and $\hbar^2(\partial_t\sqrt{n})^2/(2m)$, and also the direct coupling between $\mu$ and $\partial_t\theta$. Equation (68) is a density-phase Popov Lagrangian with a relativistic correction, i.e. the term $n\hbar^2(\partial_t\theta)^2/(2mc^2)$. Then, as a direct consequence of the Legendre transformation discussed in section 3:

$$\tilde{S}_{\mathrm{R}}[\theta] = \int_0^{+\infty}\mathrm{d}t\int_{L^D} d^D\mathbf{r}\,\tilde{\mathscr{L}}_{\mathrm{R}} \quad (69)$$

with

$$\tilde{\mathscr{L}}_{\mathrm{R}} = P\left(\mu + \frac{\hbar^2}{2mc^2}\left(\partial_t\theta\right)^2 - \hbar\partial_t\theta - \frac{\hbar^2}{2m}|\nabla\theta|^2\right) \quad (70)$$

the relativistic only-phase Popov Lagrangian density. Please, compare it with the non-relativistic one, equation (6). Expanding (70) with respect to $\theta$ around $\mu$, we find

$$\tilde{\mathscr{L}}_{\rm R} = P(\mu) - n\left(-\frac{\hbar^2}{2mc^2}(\partial_t\theta)^2 + \hbar\partial_t\theta + \frac{\hbar^2}{2m}|\nabla\theta|^2\right)$$
$$+\frac{1}{2}\frac{n}{mc_s{}^2}\left(-\frac{\hbar^2}{2mc^2}(\partial_t\theta)^2 + \hbar\partial_t\theta + \frac{\hbar^2}{2m}|\nabla\theta|^2\right)^2 + \dots$$
$$= P(\mu) - n\left(\hbar\partial_t\theta + \frac{\hbar^2}{2m}|\nabla\theta|^2\right) + n\frac{\hbar^2}{2mc^2}\left(1+\frac{c^2}{c_s^2}\right)(\partial_t\theta)^2 + \dots \tag{71}$$

### 5.1. Still on the penetration depth for the electric field

On the ground of relativistic invariance, one should expect the penetration lengths for electric and magnetic fields to be the same [28]. However, using the nonrelativistic only-phase Popov action we have found that the two penetration lengths differ by five orders of magnitude. We now show that by adopting the relativistic only-phase Popov action we still have $\lambda_{\rm E} \ll \lambda_{\rm L}$.

Inserting the electromagnetic potentials of equations (39) and (40) into equation (71) but using equations (37) and (38) we find an extension of equation (53), namely

$$\rho = qn - q\bar{n}_{\rm bg} - \epsilon_0\frac{q^2 n\mu_0}{m}\left(1+\frac{c^2}{c_s{}^2}\right)\Phi. \tag{72}$$

It is important to stress that in the relativistic approach of [28] it was found instead

$$\rho = qn - q\bar{n}_{\rm bg} - \epsilon_0\frac{q^2 n\mu_0}{m}\Phi\ . \tag{73}$$

The difference is due to the fact that the relativistic only-phase Popov Lagrangian contains a term that is missing in the mean-field treatment of the Klein–Gordon Lagrangian developed in [28]. The crucial point is that the Legendre transformation from the density-phase action to the only-phase action introduces beyond-mean-field contributions. This is indeed the crucial idea, developed in 1972 by Popov [9], but not yet fully appreciated. In our case, the beyond-mean-field contribution is directly related to the speed of sound $c_s$ of the system. As a consequence of equation (72) the penetration depth of the static electric field reads

$$\lambda_{\rm E} = \lambda_{\rm L}\frac{1}{\sqrt{1+\frac{c^2}{c_s^2}}} \simeq \lambda_{\rm L}\frac{c_s}{c} \tag{74}$$

because $c_s \ll c$ for available superconductors, as discussed in the previous section.

## 6. Conclusions

We have corroborated the main idea of Popov, by establishing that the hydrodynamic Lagrangian density of a nonrelativistic superfluid, which is characterized by the Nambu–Goldstone real scalar field $\theta(\mathbf{r},t)$, is obtained with the prescription of equation (5) into the pressure $P(\mu)$. The Lagrangian (6) is Galilei invariant and it is known as the only-phase Popov Lagrangian. Clearly, the real-time action functional of equation (4) is the only-phase Popov action we were looking for. Our derivation of this nice only-phase action is strictly based on the assumption given by equation (30), namely that the starting density-phase action is exactly equation (9), where the quantum-pressure term $\hbar^2(\nabla\sqrt{n})^2/(2m)$ has been neglected. In other words, we have demonstrated that $\tilde{S}[n]$ of equations (4) and (6) is obtained from $S[n,\theta]$ of equations (28) and (30) by performing the saddle-point approximation of the functional integration with respect to the local density $n$, as shown in equation (26).

As an application of the only-phase Popov action, we have studied a zero-temperature charged superfluid (superconductor) minimally coupled to the electromagnetic field. Notice that our specific application needs the Popov action but not the full theoretical formalism used to get the action. With the help of the only-phase Popov action we have obtained quite peculiar dependences of charged density and charged current density on the electromagnetic scalar and vector potentials. Our findings suggest (see also [1–3]) that, close to zero temperature, there is a strong screening of the electrostatic field inside a superconductor with a characteristic length much smaller than the London penetration depth. In solids $c_s \simeq 10^4$ meters/seconds and consequently we expect $\lambda_{\rm E} \simeq 10^{-5}\lambda_{\rm L}$ that is well below the Thomas–Fermi screening length of normal metals. In this paper the model for superconductivity is nonrelativistic, while the fully relativistic model is analyzed in [28], where the same penetration length is found for both electric and magnetic fields at zero temperature, as expected from the relativistic invariance of the two fields. In the last part of this paper we have compared the two theories developing a relativistic only-phase Popov action. This relativistic model contains a beyond-mean-field term, not taken into account in [28], that depends on the speed of sound $c_s$ of the system implying $\lambda_{\rm E} \simeq (c_s/c)\lambda_{\rm L}$ for $c_s/c \ll 1$, with $c$ the speed of light. From the experimental point of view, it is not an easy task to measure the penetration depth of the electric field in a superconductor: in 2016 an attempt on a Niobium sample was inconclusive [10]. The main difficulty in the measurement was in combining atomic force microscopy and cryogenic cooling. Unfortunately, the experimental error was so large that no conclusive statement can be made: more accurate experiments at ultra-low temperatures are needed.

## Data availability statement

No new data were created or analysed in this study.

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