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PHASE DYNAMICS NEAR TRANSITION ENERGY IN THE FERMILAB MAIN RING

KEN TAKAYAMA†

Fermi National Accelerator Laboratory,‡ Batavia, IL 60510, USA

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The phase dynamics of small-amplitude synchrotron oscillations in the vicinity of the transition energy is discussed with kinematical nonlinearities included. We introduce a synchrotron amplitude function analogous to the betatron amplitude function and solve analytically the time evolution of bunch shapes, where the kinematical nonlinearities result in unsymmetric bunch shapes. In addition, the above synchrotron oscillation is singular at transition crossing because of the kinematical nonlinearity. From this simple fact, we identify an inherent source of bunch diffusion. A method for estimating its size is presented. When this theory is applied to the case of the Fermilab Main Ring, the predictions are in good agreement with numerical simulations and are not inconsistent with experimental results.

§1. INTRODUCTION

The effect of nonlinear kinematic terms[1-4] is studied for energies below and above transition energy. These nonlinear kinematic terms are stronger the narrower the bunches. The momentum height of the bunch passes through a maximum at transition and the kinematic terms therefore have a maximum at transition. They can distort the particle orbits in different ways, and they may lead to longitudinal emittance blow-up.

In recent experiments[5] in the Fermilab Main Ring, bunch lengths were measured at two energies, 14 GeV and 19.7 GeV, below and above transition, at an average intensity of 2.6×10^{10} protons per bunch (total Main Ring intensity 2.8×10^{13} protons per cycle). Values for the longitudinal emittance at the two energies have been derived from these measurements. The results were 0.22 eV-sec at 14 GeV and 0.28 eV-sec at 19.7 GeV, indicating an emittance increase in the region of transition. Bunch lengths were also measured at transition (17.6 GeV), where they become very narrow (about 2.5 nsec). Furthermore, in order to clarify the reasons which lead to this longitudinal emittance blow-up at transition crossing, many extensive computer simulations have been performed independently by several people, including the present author. The simulation results, which strongly imply that the effects of the nonlinear kinematic term are large, are surprisingly consistent with the measurements.

It is the purpose of this paper to calculate the effects of nonlinear kinematic terms in the range around the transition energy and compare to results of computer simulations and real machine studies.

This paper is divided in four main parts: In the first part (§2,3), we derive difference equations for acceleration in an explicit form and transform them into a differential form, which enables us to construct a Hamiltonian formulation for longitudinal motion. Here we shall restrict ourselves to small-amplitude oscillations. In addition,

† Now at National Laboratory for High Energy Physics (KEK), Japan.
‡ Operated by Universities Research Association, Inc. under contract with the U.S. Dept. of Energy.

only the lowest-order nonlinear kinematic term will be retained in this formulation. In the second part (§4), introducing the notion of a synchrotron amplitude function, we construct the "linear classical theory" of transition in a form analogous to betatron oscillations, where the nonlinear kinematical term is neglected. In the third part (§5,6), using perturbation theory, we calculate increments of the longitudinal emittance as an effect of the nonlinear kinematic term on linear motion. We identify this effect as a source of the unsymmetric bunch shape just at transition that has been recognized in the computer simulations. In the fourth part (§7), from a general point of view with respect to time-reversability, it will be shown that only such nonlinear kinematical terms can give net effects over transition crossing. Finally, a theoretical formula for the emittance blow-up ratio will be presented.

In the present discussions, effects of longitudinal space-charge forces and timing error of the phase jump at transition[7] are not included, because the former is negligible, at least for the present situation of the Fermilab Main Ring, and exact information about the latter has not been obtained.

§2. DIFFERENCE AND DIFFERENTIAL EQUATIONS FOR ACCELERATION

The theory of longitudinal phase motion, describing the energy and phase oscillations that occur when a particle passes repeatedly through one or more accelerating cavities situated at localized points around the accelerator ring, is well known. Since the oscillations normally are at a relatively low frequency, it is often legitimate as well as convenient to analyze them theoretically with differential equations derived by spreading the accelerating field uniformly around the orbit. In reality, the energy changes experienced by a particle are better represented by difference equations and depend on the sine of the electrical phase angle ϕ at which the particle traverses the cavity. The corresponding equations of motion are therefore both nonlinear and discrete.

We consider here the case of synchrotron oscillations during acceleration. To obtain the actual transformation, we consider a short rf cavity system operating at a harmonic number h, an angular frequency $\omega_{\mathrm{rf}}(t)$ and a rf peak voltage $V(t)$. We assume that $\omega_{\mathrm{rf}}(t)$ and $V(t)$ are independently controlled during acceleration. The quantities denoted by E^n and ϕ^n are, respectively, the energy and the electrical phase angle with which a particle enters the cavity at the time of transit. Then the nonlinear transformation may be written in the form

$$E^{n+1} = E^n + eV(n) \sin \phi^n, \tag{2-1a}$$

$$\phi^{n+1} = \left\{ \frac{\omega_{\mathrm{rf}}(n+1)}{\omega_{\mathrm{rf}}(n)} \phi^n + \omega_{\mathrm{rf}}(n+1) \cdot \frac{2\pi}{\omega(E^{n+1})} \right\}, \tag{2-1b}$$

where $eV(n) \sin \phi^n$ is the energy gain at the n-th transit and the revolution period is described in the form

$$\frac{2\pi}{\omega(E^{n+1})} = \frac{C_0(1 + \alpha_p Z^{n+1})}{c[1 - (m_0c^2/E^{n+1})^2]^{1/2}} \left(Z^{n+1} \equiv \frac{E^{n+1} - E_s^{n+1}}{\beta^2(E_s^{n+1})E_s^{n+1}} \right), \tag{2-2}$$

where C_0 is the length of the closed orbit corresponding to the synchronous energy E, c is the velocity of light, m_0c^2 is the proton rest energy, Z is the momentum deviation from the synchronous momentum, and α_p is the momentum compaction factor. The betatron acceleration can be neglected here. The synchronous particle is defined by the equations

$$E_s^{n+1} = E_s^n + \Delta(n), \tag{2-3a}$$

$$\phi_s^{n+1} = \left\{\frac{\omega_{rf}(n+1)}{\omega_{rf}(n)}\phi_s^n + \omega_{rf}(n+1)\cdot\frac{2\pi}{\omega(E_s^{n+1})}\right\}, \tag{2-3b}$$

where

$$\Delta(n) = eV(n)\sin\phi_s^n, \tag{2-3c}$$

$$\omega_{rf}(n) = h\omega(E_s^n) + \frac{\omega(E_s^n)}{2\pi}\left\{\sin^{-1}\frac{\Delta(n)}{eV(n)} - \sin^{-1}\frac{\Delta(n-1)}{eV(n-1)}\right\} \tag{2-3d}$$

Note that $\Delta(n)$ is determined by the change in the external guide field $B(t)$. Now the momentum compaction factor α_p may be written in the form[3]

$$\alpha_p = \alpha^{(0)} + \alpha^{(1)}Z^{n+1} + \alpha^{(2)}(Z^{n+1})^2 + O\big((Z^{n+1})^3\big), \tag{2-4}$$

where $O\big((Z^{n+1})^3\big)$ is the Landau symbol. Expanding the right-hand side of Eq. (2-2) with respect to Z^{n+1}, we have the expression

$$\begin{aligned}\frac{2\pi}{\omega(E^{n+1})} = \frac{C_0}{c\beta(E_s^{n+1})}[1 &+ \eta^{(0)}(n+1)Z^{n+1} + \eta^{(1)}(n+1)(Z^{n+1})^2 \\ &+ \eta^{(2)}(n+1)(Z^{n+1})^3 + \cdots],\end{aligned} \tag{2-5}$$

where

$$\frac{C_0}{c\beta(E_s^{n+1})} = \frac{2\pi}{\omega(E_s^{n+1})} \simeq \frac{2\pi h}{\omega_{rf}(n+1)},$$

$$\eta^{(0)}(n+1) = \alpha^{(0)} - 1/\gamma^2(E_s^{n+1}), \qquad (\alpha^{(0)} = 1/\gamma_T^2), \tag{2-6a}$$

$$\eta^{(1)}(n+1) = \alpha^{(1)} - \frac{\eta^{(0)}(n+1)}{\gamma^2(E_s^{n+1})} + \frac{3\beta^2(E_s^{n+1})}{2\gamma^2(E_s^{n+1})} \simeq \alpha^{(1)} + \frac{3\beta^2(E_s^{n+1})}{2\gamma^2(E_s^{n+1})}, \tag{2-6b}$$

$$\begin{aligned}\eta^{(2)}(n+1) &= \alpha^{(2)} - \frac{2\beta^2(E_s^{n+1})}{\gamma^2(E_s^{n+1})} + \frac{\alpha^{(1)}}{\gamma^2(E_s^{n+1})} + \frac{\eta^{(0)}(n+1)}{\gamma^4(E_s^{n+1})} \\ &\quad + \frac{\beta^2(E_s^{n+1})}{2\gamma^2(E_s^{n+1})}\left(3\alpha^{(0)} - \frac{5}{\gamma^2(E_s^{n+1})}\right) \\ &\simeq \alpha^{(2)} - \frac{2\beta^2(E_s^{n+1})}{\gamma^2(E_s^{n+1})}.\end{aligned} \tag{2-6c}$$

We note that all particle simulations for a real acceleration mode stated in the Introduction have been performed by following the difference equations (2-1a) and (2-1b). The pair E and ϕ is recognized not to be canonical because of the time dependence of the rf frequency. We are interested in small-amplitude oscillations around the synchronous point $(\phi_s{}^n, E_s{}^n)$ as a guiding center. Setting

$$\epsilon^n = E^n - E_s{}^n, \tag{2-7a}$$

$$\chi^n = \phi^n - \phi_s{}^n, \tag{2-7b}$$

we may write the difference equations for a small amplitude oscillation as

$$\begin{aligned} \epsilon^{n+1} &= \epsilon^n + eV(n)[\sin\phi^n - \sin\phi_s{}^n] \\ &\simeq \epsilon^n + eV(n)\cos\phi_s{}^n\,\chi^n, \end{aligned} \tag{2-8a}$$

$$\begin{aligned} \chi^{n+1} = \Bigg\{ & \frac{\omega_{\mathrm{rf}}(n+1)}{\omega_{\mathrm{rf}}(n)}\chi^n + \left(2\pi h + \sin^{-1}\frac{\Delta(n+1)}{eV(n+1)} - \sin^{-1}\frac{\Delta(n)}{eV(n)}\right) \\ & \times [\overline{\eta^{(0)}}(n+1)\epsilon^{n+1} + \overline{\eta^{(1)}}(n+1)(\epsilon^{n+1})^2]\Bigg\}, \end{aligned} \tag{2-8b}$$

where

$$\overline{\eta^{(0)}}(n+1) = \frac{\eta^{(0)}(n+1)}{\beta^2(E_s{}^{n+1})E_s{}^{n+1}}, \tag{2-9a}$$

$$\overline{\eta^{(1)}}(n+1) = \frac{\eta^{(1)}(n+1)}{[\beta^2(E_s{}^{n+1})E_s{}^{n+1}]^2}. \tag{2-9b}$$

To write down the difference equations in the form of exact differential ones we may use a δ-function

$$\dot{\epsilon} = \frac{2\pi eV(t)\cos\phi_s(t)}{T_s(t)}\chi\,\delta_{2\pi}(t'), \tag{2-10a}$$

$$\dot{\chi} = \frac{1}{\omega_{\mathrm{rf}}(t)}\frac{d\omega_{\mathrm{rf}}(t)}{dt}\chi + \frac{(2\pi h + T_s(t)\dot{\phi}_s(t))}{T_s(t)}[\overline{\eta^{(0)}}(t)\epsilon + \overline{\eta^{(1)}}(t)\epsilon^2], \tag{2-10b}$$

where $T_s(t)$ is the period of the synchronous particle and one iteration of the mapping. Here $t' = \Omega(t)t + t_0'$; $\Omega(t) = 2\pi/T_s(t)$ and the δ-function of period 2π is given by the Fourier expansion

$$\delta_{2\pi}(t') = \frac{1}{2\pi}\left(1 + 2\sum_{n+1}^{\infty}\cos nt'\right). \tag{2-11}$$

After neglecting rapidly oscillating terms in Eq. (2-10a), we have

$$\dot{\epsilon} = \frac{eV(t)\cos\phi_s(t)}{T_s(t)}\chi, \tag{2-12a}$$

$$\dot{\chi} = \frac{\dot{\omega}_{\mathrm{rf}}}{\omega_{\mathrm{rf}}}\chi + \frac{(2\pi h + T_s(t)\dot{\phi}_s(t))}{T_s(t)}[\overline{\eta^{(0)}}(t)\epsilon + \overline{\eta^{(1)}}(t)\epsilon^2]. \tag{2-12b}$$

§3. HAMILTONIAN FORMALISM

Under the assumption that the damping term in Eq. (2-12b) is negligible in the short period of transition crossing, we can construct a Hamiltonian formalism for small-amplitude oscillations. Neglecting the damping term, we find (χ, ϵ) to be a canonical pair that yields the Hamiltonian

$$H(\chi, \epsilon; t) = \frac{(2\pi h + T_s(t)\dot{\phi}_s(t))}{T_s(t)}\left[\frac{1}{2}\overline{\eta^{(0)}}(t)\epsilon^2 + \frac{1}{3}\overline{\eta^{(1)}}(t)\epsilon^3\right] - \frac{eV(t)\cos\phi_s(t)}{2T_s(t)}\chi^2. \quad (3\text{-}1)$$

We shall assume that the synchronous phase angle ϕ_s jumps discontinuously at $t = 0$ in such a way that $\sin\phi_s$ is constant and

$$\mathrm{sgn}(\cos\phi_s) = -\mathrm{sgn}(t).$$

We shall measure t from this instant. Now we introduce a scale change of the independent variable t by

$$\tau(t) = \int_0^t \left[-\frac{eV(t)\cos\phi_s(t)}{T_s(t)}\right] dt. \quad (3\text{-}2)$$

Note that the new independent variable τ has the dimension of energy. With the above origin for the time t, τ is always positive at all t, namely,

$$t \to 0- \text{ (approaching transition), then } \tau \to 0+,$$
$$0+ \to t \text{ (leaving transition), then } 0+ \to \tau.$$

Such a scale change yields the new Hamiltonian

$$H'(\chi, \epsilon; \tau) = -\frac{(2\pi h + T_s(t)\dot{\phi}_s(t))}{eV(t)\cos\phi_s(t)}\left[\frac{1}{2}\overline{\eta^{(0)}}(t)\epsilon^2 + \frac{1}{3}\overline{\eta^{(1)}}(t)\epsilon^3\right] + \frac{1}{2}\chi^2. \quad (3\text{-}3)$$

For later convenience, we change the canonical variables to

$$\chi = p, \quad (3\text{-}4a)$$

$$\epsilon = -x. \quad (3\text{-}4b)$$

Thus we obtain the Hamiltonian in the simple form

$$H''(x, p; \tau) = \frac{1}{2}p^2 + \frac{1}{2}\lambda_0(\tau)x^2 - \frac{1}{3}\lambda_1(\tau)x^3, \quad (3\text{-}5)$$

where

$$\lambda_0(\tau) = -\frac{(2\pi h + T_s(t)\dot{\phi}_s(t))\overline{\eta^{(0)}}(t)}{eV(t)\cos\phi_s(t)} = -\frac{(2\pi h + T_s(t)\dot{\phi}_s(t))[\alpha^{(0)} - 1/\gamma_s^2(t)]}{\beta_s^2(t)E_s(t)eV(t)\cos\phi_s(t)}, \quad (3\text{-}6a)$$

$$\lambda_1(\tau) = -\frac{(2\pi h + T_s(t)\dot{\phi}_s(t))\overline{\eta^{(1)}}(t)}{eV(t)\cos\phi_s(t)} = -\frac{(2\pi h + T_s(t)\dot{\phi}_s(t))\left[\alpha^{(1)} + \dfrac{3\beta_s^2(t)}{2\gamma_s^2(t)}\right]}{[\beta_s^2(t)E_s(t)]^2 eV(t)\cos\phi_s(t)}. \quad (3\text{-}6b)$$

The form of Eq. (3-5) reminds us of the betatron oscillations in a transport line with a nonlinear component. In analogy to perturbed betatron oscillations, we separate the right-hand side of Eq. (3-5) into unperturbed and perturbing terms

$$K^{(0)}(x, p; \tau) = \frac{1}{2} p^2 + \frac{1}{2} \lambda_0(\tau) x^2, \tag{3-7a}$$

$$K^{(1)}(x, p; \tau) = -\frac{1}{3} \lambda_1(\tau) x^3. \tag{3-7b}$$

In the next section, we shall discuss the phase dynamics in the vicinity of transition by studying the above linear Hamiltonian $K^{(0)}$.

§4. LINEAR MOTION

We consider the linear system described by

$$K^{(0)}(x, p; \tau) = \frac{1}{2} [p^2 + \lambda_0(\tau) x^2]. \tag{4-1}$$

We make the assumption that the peak rf voltage and the synchronous phase angle are constant near transition. At the transition energy, the quantity

$$\alpha^{(0)} - 1/\gamma_s^2(t),$$

vanishes. Then in the vicinity of this energy this quantity can be approximated by the first term in a Tayler series expansion of Eq. (3-6a) for deviations of γ from γ_T. We can therefore write

$$\lambda_0(\tau) \simeq -\frac{4\pi h[\gamma_s(t) - \gamma_T]}{\beta_s{}^2(0) E_s(0) eV \cos \phi_s(0) \gamma_T{}^3}. \tag{4-2}$$

Further, the quantity $\gamma_s(t) - \gamma_T$ can be written in terms of

$$\begin{aligned} \gamma_s(t) - \gamma_T &= \dot{\gamma}_s t \\ &= \frac{eV \sin \phi_s}{m_0 c^2 T_s} \tau \\ &= -\frac{\sin \phi_s(0)}{m_0 c^2 \cos \phi_s(0)} \tau. \end{aligned} \tag{4-3}$$

Substitution of Eq. (4-3) into Eq. (4-2) yields

$$\lambda_0(\tau) \simeq \frac{4\pi h \sin \phi_s}{\beta_s{}^2 E_s{}^2 \gamma_T{}^2 eV \cos^2 \phi_s} \tau. \tag{4-4}$$

Here all quantities are evaluated at transition. For the sake of later simplicity we set

$$k^2 = \frac{4\pi h \sin \phi_s}{\beta_s{}^2 E_s{}^2 \gamma_T{}^2 eV \cos^2 \phi_s}.$$

We point out that the system represented by Eq. (4-1) has an exact dynamical invariant, the Courant-Snyder invariant[8,9]

$$I(x, p; \tau) = \frac{1}{2S(\tau)}\left\{x^2 + \left[\frac{1}{2}\dot{S}(\tau)x - S(\tau)p\right]^2\right\}, \tag{4-5}$$

where $S(\tau)$ satisfies the auxiliary differential equation

$$\frac{1}{2}S\ddot{S} - \frac{1}{4}\dot{S}^2 + \lambda_0(\tau)S^2 = 1. \tag{4-6}$$

When $\lambda_0(\tau)$ is constant, the invariant I is identical to the action variable of the system, if we take the initial condition

$$S(+\infty) = 1/\sqrt{\lambda_0(\infty)}, \qquad \dot{S}(+\infty) = 0. \tag{4-7}$$

In the following, $S(\tau)$ will be called a synchrotron amplitude function. For a time-varying function $\lambda_0(\tau)$, from Eq. (4-5), we know that an infinite sequence of phase points that have a certain constant value of I at an arbitrary time behaves as a deformable moving ellipse in the phase space $(x, p; \tau)$ after that time. The form of such a ellipse, called an "invariant curve" in the following, is uniquely determined by the auxiliary differential equation (4-6) alone. We consider the invariant curve described in terms of

$$I(x, p; \tau_1) = I_0 \tag{4-8}$$

with constant I_0. The quantity I is equal to the value of the action variable of the infinite set of phase points that comprise the invariant curve, as mentioned above.

We may characterize the ellipse by two parameters ξ, δ, which are functions of $S(\tau)$ and $\dot{S}(\tau)$

$$\xi(\tau) = \sqrt{2I_0 S(\tau)}, \tag{4-9a}$$

$$\delta(\tau) = \sqrt{2I_0 \frac{[1 + \dot{S}^2(\tau)/4]}{S(\tau)}}. \tag{4-9b}$$

They are the maximum extent of the ellipse in x and p, respectively. If we assume that all parameters change adiabatically after $\tau = \tau_1$, we can choose the approximate initial conditions

$$S(\tau_1) = 1/\sqrt{\lambda_0(\tau_1)}, \qquad \dot{S}(\tau_1) = 0. \tag{4-10}$$

At $\tau = \tau_1$, the ellipse begins to move, following the time-evolution of $S(\tau)$ which is determined by Eq. (4-6). If we know the values of $S(\tau)$, $\dot{S}(\tau)$ at $\tau = 0$, which is the transition time, we can evaluate exactly the maximum upper or lower height from the synchronous energy $\xi(0)$ and the half phase spread $\delta(0)$.

For the present $\lambda_0(\tau)$, we know the general solution of Eq. (4-6) can be written in terms of Bessel functions (see Appendix A),

$$S(\tau) = \left(\frac{\pi}{3}\right)^2 \tau\left[aN_{1/3}^2(z) + bJ_{1/3}^2(z) - 2\left(ab - \frac{9}{\pi^2}\right)^{1/2} J_{1/3}(z)N_{1/3}(z)\right] \quad (4\text{-}11)$$

where a and b are arbitrary coefficients that must be determined by the initial conditions (4-10), and

$$z = \frac{2}{3} k\tau^{3/2}.$$

After mathematical manipulation (see Appendix B), we have the coefficients

$$a = \frac{3}{2} z_1 J_{1/3}^2(z_1) + \frac{3}{2z_1 N_{1/3}^2(z_1)}\left[z_1 J_{-2/3}(z_1)N_{1/3}(z_1) + \frac{2}{\pi}\right]^2 - \frac{6}{\pi z_1 N_{1/3}^2(z_1)}\left[z_1 J_{-2/3}(z_1)N_{1/3}(z_1) + \frac{1}{\pi}\right], \quad (4\text{-}12a)$$

$$b = \frac{3}{2} z_1 N_{1/3}^2(z_1) + \frac{3}{2z_1 J_{1/3}^2(z_1)}\left[z_1 J_{-2/3}(z_1)N_{1/3}(z_1) + \frac{2}{\pi}\right]^2, \quad (4\text{-}12b)$$

where

$$z_1 = \frac{2}{3} k\tau_1^{3/2}. \quad (4\text{-}13)$$

The coefficients a and b have been uniquely determined by the initial conditions and we now know the exact time-evolution of the invariant curve. In particular, we are interested in the invariant curve just at transition; it represents a bunch envelope. From Eq. (4-11) we obtain the values of $S(\tau)$ and $\dot{S}(\tau)$ at $\tau = 0$ (see Appendix C)

$$S(0) = \frac{4}{3}\left(\frac{\pi}{3}\right)^2 \left(\frac{k}{3}\right)^{-2/3} \frac{a}{\Gamma^2(2/3)}, \quad (4\text{-}14a)$$

$$\dot{S}(0) = \frac{2\pi}{3}\left[-\frac{a}{\sqrt{3}} + \left(ab - \frac{9}{\pi^2}\right)^{1/2}\right]. \quad (4\text{-}14b)$$

Introduction of Eqs. (4-14a), (4-14b) into Eqs. (4-4a), (4-4b) leads to analytical expressions for the maximum upper or lower height from the synchronous energy and the half spread around the synchronous phase at transition, that is,

$$\xi(0) = \xi(\tau_1)\sqrt{S(0)/S(\tau_1)}, \quad (4\text{-}15a)$$

$$\delta(0) = \delta(\tau_1)\sqrt{\left[1 + \frac{\dot{S}(0)^2}{4}\right] S(\tau_1)/S(0)}. \quad (4\text{-}15b)$$

We consider the case with the initial conditions at $\tau_1 = +\infty$ where the linear τ-dependence of $\lambda_0(\tau)$ still holds to good approximation. In such a region, the Bessel and Neumann functions become sinusoidal with equal amplitude and quadrature phase relationship, namely,

$$J_{1/3}(z_1) = \sqrt{2/\pi z_1}\cos(z_1 - 5\pi/12),$$
$$N_{1/3}(z_1) = \sqrt{2/\pi z_1}\sin(z_1 - 5\pi/12).$$

Also (4-16)

$$J_{-2/3}(z_1) = \sqrt{2/\pi z_1}\cos(z_1 + \pi/12)$$
$$= -\sqrt{2/\pi z_1}\sin(z_1 - 5\pi/12).$$

Substitution of Eq. (4-16) into Eqs. (4-12a) and (4-12b) yields

$$a = b = 3/\pi. \tag{4-17}$$

From Eq. (4-17), we can obtain a universal relationship between ξ and δ

$$\xi(0)\delta(0) = \xi(\tau_1)\delta(\tau_1)\left[1 + \frac{\dot{S}(0)^2}{4}\right]^{1/2}$$
$$= \frac{2}{\sqrt{3}}\xi(\tau_1)\delta(\tau_1). \tag{4-18}$$

Equation (4-18) is equivalent to the result obtained by Hereward.[11]

Predictions of the linear theory are not discussed in detail here. Nevertheless they are in very good agreement with numerical simulations. These simulations have been performed following the exact mapping equations (2-1a) and (2-1b) where nonlinear kinematical terms are not included in order to verify the validity of the linear theory. In addition, the linear theory discussed here which provides exact time evolution of bunch shapes can re-establish the well-known story[8] associated with transition crossing.

§5. NONLINEAR MOTION

The nonlinear kinematic term has been distinguished as a perturbing term in Eq. (3-7b). It gives unignorable effects to the linear oscillation only during a very short period when a particle crosses the transition energy. In order to assess its quantitative effects, it is convenient to use the action-angle formalism.

Under the linear canonical transformation

$$Q = \rho^{-1}x,$$
$$P = -\dot{\rho}x + \rho p, \tag{5-1}$$

where $\rho(\tau) = \sqrt{S(\tau)}$ satisfies the auxiliary equation

$$\ddot{\rho} + \lambda_0(\tau)\rho = \rho^{-3}, \tag{5-2}$$

the Hamiltonian (3-5) reduces to

$$K(Q, P; \tau) = \rho^{-2}\left[\frac{1}{2}P^2 + \frac{1}{2}Q^2 - \frac{\rho^5}{3}\lambda_1(\tau)Q^3\right]. \tag{5-3}$$

If a change of independent variable

$$\theta(\tau) = \int_{\tau_2}^{\tau} \rho^{-2}(\tau)\, d\tau + \theta_0, \tag{5-4}$$

is made, the Hamiltonian (5-3) becomes

$$K'(Q, P; \theta) = \frac{1}{2}(P^2 + Q^2) - \frac{\rho^5}{3}\lambda_1(\tau)Q^3. \tag{5-5}$$

Furthermore, introduction of the action-angle variables (ψ, J)

$$\begin{aligned} Q &= \sqrt{2J}\sin\psi, \\ P &= \sqrt{2J}\cos\psi, \end{aligned} \tag{5-6}$$

yields the Hamiltonian

$$G(\psi, J; \theta) = J - \frac{\rho^5}{3}\lambda_1(\tau)(2J)^{3/2}\sin^3\psi. \tag{5-7}$$

From Eq. (5-7), we derive the canonical equations

$$\psi' = \frac{\partial G}{\partial J} = 1 - \rho^5\lambda_1(\tau)(2J)^{1/2}\sin^3\psi, \tag{5-8a}$$

$$J' = -\frac{\partial G}{\partial \psi} = \rho^5\lambda_1(\tau)(2J)^{3/2}\sin^2\psi\cos\psi. \tag{5-8b}$$

If the perturbing term in Eq. (5-8a) is much smaller than the unperturbed term, that is, small compared with unity, we obtain to first order

$$J(\theta(\tau)) \simeq J(\theta(\infty)) + [2J(\theta(\infty))]^{3/2}\int_{\theta(\infty)}^{\theta(\tau)} \rho^5(\tau)\lambda_1(\tau)\sin^2\theta(\tau)\cos\theta(\tau)\, d\theta. \tag{5-9}$$

Using relation (5-4), we find a more convenient integral expression

$$J(\tau) \simeq J(\tau_2) + [2J(\tau_2)]^{3/2}\int_{\tau_2}^{\tau} \rho^3(\tau)\lambda_1(\tau)\sin^2\theta(\tau)\cos\theta(\tau)\, d\tau, \tag{5-10}$$

where the lower limit of the integration will be determined by the following considerations.

We note that the instantaneous period of phase oscillations is the same as the synchrotron-amplitude function $S(\tau)$ because the oscillation frequency is described by

$$\nu = \dot{\psi}(S) = \frac{1}{S(\tau)}. \tag{5-11}$$

From Eq. (5-8b), the typical modulation period of the perturbing term is $S(\tau)/2$. Generally, the value of $S(\tau_1)/2$ is much smaller than τ_1. This fact means that the effects of the perturbing term are averaged out, at least, at the early stage of non-adiabatic motion ($0 \ll \tau \leqslant \tau_1$). So it is reasonable for us to take $S(0)/2$ as the typical lower limit τ_2 when the perturbation begins to retain net effects. Fortunately, in many real situations, τ_2 is sufficiently small so as to satisfy the condition

$$z(\tau_2) = \frac{2}{3} k \tau_2{}^{3/2} \ll 1. \tag{5-12}$$

The relation (5-12) enables us to describe $\rho(\tau)$ or $S(\tau)$ by elementary functions. In order to do this, it is necessary to know the Bessel function for small value of z. In Ref. 14 we observe that

$$J_{1/3}(z) = \frac{1}{\Gamma(4/3)}\left(\frac{z}{2}\right)^{1/3}, \tag{5-13a}$$

$$N_{1/3}(z) = \frac{1}{\sin \pi/3}\left[\frac{\cos \pi/3}{\Gamma(4/3)}\left(\frac{z}{2}\right)^{1/3} - \frac{1}{\Gamma(2/3)}\left(\frac{z}{2}\right)^{-1/3}\right]. \tag{5-13b}$$

Considering Eqs. (5-13a) and (5-13b), we obtain to first order with respect to τ

$$S(\tau) = \rho^2(\tau) = \left(\frac{\pi}{3}\right)^2 (-q\tau + r), \tag{5-14}$$

where

$$q = \frac{4a}{3\Gamma(4/3)\Gamma(2/3)}, \tag{5-15a}$$

$$r = \frac{4a}{3\Gamma^2(2/3)}\left(\frac{k}{3}\right)^{-2/3} + \frac{4}{\sqrt{3}}\left(ab - \frac{9}{\pi^2}\right)^{1/2} \cdot \frac{1}{\Gamma(2/3)\Gamma(4/3)}. \tag{5-15b}$$

Further substitution of the above result (5-14) into (5-4) yields

$$\begin{aligned}\theta(\tau) &= \int_{\tau_2}^{\tau} 1/S(\tau')\, d\tau' + \theta_0 \\ &= \left(\frac{3}{\pi}\right)^2\left(-\frac{1}{q}\right)\left[\log\left(\frac{\pi}{3}\right)^2(-q\tau + r) - \log\left(\frac{\pi}{3}\right)^2(-q\tau_2 + r)\right] + \theta_0. \end{aligned} \tag{5.16}$$

On the other hand, $\lambda_1(\tau)$ can be expanded with respect to $\gamma_s - \gamma_T$ and to first order

$$\lambda_1(\tau) = -\frac{2\pi h[\alpha_n + 3\beta_s{}^2/2]}{\beta_s{}^4(m_0c^2)^2 eV\cos\phi_s\gamma_T{}^4}\left[1 + \frac{4\tan\phi_s}{m_0c^2\gamma_T}\tau\right], \tag{5-17}$$

where α_n ($\equiv \alpha^{(1)}/\alpha^{(0)}$) is the nonlinear lattice parameter. Using Eqs. (5-14) and (5-17), we write the perturbing integration in Eq. (5-10) in terms of

$$\Delta J = [2J(\tau_2)]^{3/2}\lambda_1(0)S^{3/2}(0)\int_{\tau_2}^{\tau}\left(1 - \frac{q}{r}\tau\right)^{3/2}\left(1 + \frac{4\tan\phi_s}{m_0c^2\gamma_T}\tau\right)\sin^2\theta(\tau)\cos\theta(\tau)\,d\tau$$
$$\simeq [2J(\tau_2)]^{3/2}\lambda_1(0)S^{3/2}(0)\int_{\tau_2}^{\tau}\left[1 + \left(\frac{4\tan\phi_s}{m_0c^2\gamma_T} - \frac{3q}{2r}\right)\tau\right]\sin^2\theta(\tau)\cos\theta(\tau)\,d\tau. \tag{5-18}$$

Here we set

$$g_1 = \frac{4\tan\phi_s}{m_0c^2\gamma_T} - \frac{3q}{2r}, \tag{5-19}$$

$$\theta(\tau) = g_2\log\left(\frac{\tau}{g_2} + g_3\right) + \bar{\theta}_0, \tag{5-20}$$

where

$$g_2 = \left(\frac{3}{\pi}\right)^2\left(-\frac{1}{q}\right),$$

$$g_3 \equiv S(0) = \left(\frac{\pi}{3}\right)^2 r,$$

$$\bar{\theta}_0 = -g_2\log\left(\frac{\tau_2}{g_2} + g_3\right) + \theta_0.$$

Using the identity

$$\sin^2\theta(\tau)\cos\theta(\tau) = \frac{1}{4}[\cos\theta(\tau) - \cos 3\theta(\tau)],$$

with together (5-19), (5-20), we have

$$\Delta J \simeq \frac{1}{4}[2J(\tau_2)]^{3/2}\lambda_1(0)S^{3/2}(0)\int_{\tau_2}^{\tau}(1 + g_1\tau)\left\{\cos\left[g_2\log\left(\frac{\tau}{g_2} + g_3\right) + \bar{\theta}_0\right]\right.$$
$$\left. - \cos 3\left[g_2\log\left(\frac{\tau}{g_2} + g_3\right) + \bar{\theta}_0\right]\right\}d\tau. \tag{5-21}$$

We are interested in the value of ΔJ at $\tau = 0$. This value is evaluated in the following way.

The integration to be performed is

$$I = \int_{\tau_2}^{0} d\tau\,(1 + g_1\tau)\left\{\cos\left[g_2 \log\left(\frac{\tau}{g_2} + g_3\right) + \bar{\theta}_0\right] - \cos 3\left[g_2 \log\left(\frac{\tau}{g_2} + g_3\right) + \bar{\theta}_0\right]\right\}. \tag{5-22}$$

A change of the integration variable to

$$w = g_2 \log\left(\frac{\tau}{g_2} + g_3\right) + \bar{\theta}_0,$$

leads to

$$I = e^{-\bar{\theta}_0/g_2}\int_{w_1}^{w_2} e^{w/g_2}(1 - g_1 g_2 g_3 + g_1 g_2 e^{-\bar{\theta}_0/g_2} e^{w/g_2}) \times (\cos w - \cos 3w)\, dw, \tag{5-23}$$

where

$$w_1 = g_2 \log\left(\frac{\tau_2}{g_2} + g_3\right) + \bar{\theta}_0 = \theta_0, \tag{5-24a}$$

$$w_2 = g_2 \log g_3 + \bar{\theta}_0 = g_2 \log\left[g_3 \Big/ \left(\frac{\tau_2}{g_2} + g_3\right)\right] + \theta_0. \tag{5-24b}$$

The integration of Eq. (5-23) is trivial. We obtain

$$\begin{aligned} I = e^{-\bar{\theta}_0/g_2}\Big\langle (1 - g_1 g_2 g_3)&\left\{\frac{1}{1/g_2^{\,2} + 1}\left| e^{w/g_2}\left(\frac{1}{g_2}\cos w + \sin w\right)\right|_{w_2}^{w_2}\right. \\ &\left. - \frac{1}{1/g_2^{\,2} + 9}\left| e^{w/g_2}\left(\frac{1}{g_2}\cos 3w + 3\sin 3w\right)\right|_{w_2}^{w_2}\right\} \\ &+ g_1 g_2 e^{-\bar{\theta}_0/g_2}\left\{\frac{1}{4/g_2^{\,2} + 1}\left| e^{2w/g_2}\left(\frac{2}{g_2}\cos w + \sin w\right)\right|_{w_1}^{w_2}\right. \\ &\left. - \frac{1}{4/g_2^{\,2} + 9}\left| e^{2w/g_2}\left(\frac{2}{g_2}\cos 3w + 3\sin 3w\right)\right|_{w_1}^{w_2}\right\}\Big\rangle. \end{aligned} \tag{5-25}$$

Some of the intermediate steps in the calculation are explained in Appendix D. The final results are

$$\begin{aligned} I = \frac{g_2 g_3(1 - g_1 g_2 g_3)}{1 + g_2^{\,2}}&\{[-\sin\Theta + g_2\cos\Theta - g_2(1/2g_2 + 1)]\sin\theta_0 \\ &+ [\cos\Theta + g_2\sin\Theta - (1/2g_2 + 1)]\cos\theta_0\} - \frac{g_2 g_3(1 - g_1 g_2 g_3)}{1 + 9g_2^{\,2}} \\ &\times \{[-\sin 3\Theta + 3g_2\cos 3\Theta - 3g_2(1/2g_2 + 1)]\sin 3\theta_0 \end{aligned}$$

$$+ [\cos 3\Theta + 3g_2 \sin 3\Theta - (1/2g_2 + 1)] \cos 3\theta_0\}$$

$$+ \frac{g_1 g_2^2 g_3^2}{4 + g_2^2} \{[-2 \sin \Theta + g_2 \cos \Theta - g_2(1/2g_2 + 1)^2] \sin \theta_0$$

$$+ [2 \cos \Theta + g_2 \sin \Theta - 2(1/2g_2 + 1)^2] \cos \theta_0\}$$

$$- \frac{g_1 g_2^2 g_3^2}{4 + 9g_2^2} \{[-2 \sin 3\Theta + 3g_2 \cos 3\Theta - 3g_2(1/2g_2 + 1)^2] \sin 3\theta_0$$

$$+ [2 \cos 3\Theta + 3g_2 \sin 3\Theta - 2(1/2g_2 + 1)^2] \cos 3\theta_0\}, \tag{5-26}$$

where

$$\Theta = g_2 \log\left[1 \bigg/ \left(\frac{1}{2g_2} + 1\right)\right]. \tag{5-27}$$

The value of I is dependent of the initial phase θ_0. If the maximum and minimum values of I are denoted by $I_{max}(>0)$ and $I_{min}(<0)$, we can write the positive and negative changes of the action variable in terms of

$$\Delta J(0)_{\pm} = \frac{[2J(\tau_2)]^{3/2}}{4} \lambda_1(0) S^{3/2}(0) I_{\substack{max \\ min}}, \tag{5-28}$$

where the suffix symbols of the left-hand side do not always correspond to those of the right-hand side because of the negative sign of $\lambda_1(0)$. We define the emittance-increase parameter, which is independent of the initial emittance (the initial value of the action variable) of a particle, by

$$K_{\pm} = \frac{1}{2} \lambda_1(0) S^{3/2}(0) I_{\substack{max \\ min}}. \tag{5-29}$$

It is assumed that there are no net effects of the perturbing term up to $\tau = \tau_2$ and we can take the value of $J(\tau_1)$ as $J(\tau_2)$. Then

$$J(\tau_1)[1 + K_- \sqrt{2J(\tau_1)}] \leqslant J(0) \leqslant J(\tau_1)[1 + K_+ \sqrt{2J(\tau_1)}]. \tag{5-30}$$

Thus incoherent changes of the action variable give an unsymmetric ellipse in the phase space. In particular, we can obtain expressions for the maximum upper and lower height from the synchronous energy and the maximum and minimum excursion from the synchronous phase as

$$\xi(0)_{\substack{max \\ min}} = \{2J(\tau_1)S(0)[1 + K_{\pm}\sqrt{2J(\tau_1)}]\}^{1/2}$$

$$= \xi(\tau_1)\sqrt{\frac{S(0)}{S(\tau_1)}}\,[1 + K_{\pm}\sqrt{2J(\tau_1)}]^{1/2}, \tag{5-31a}$$

$$\delta(0)_{\substack{max \\ min}} = \delta(\tau_1)\sqrt{\frac{S(\tau_1)}{S(0)}}[1 + K_{\pm}\sqrt{2J(\tau_1)}]^{1/2}. \tag{5-31b}$$

where the value of $S(0)/S(\tau_1)$ has already been derived in the previous section.

§6. APPLICATIONS

The theoretical considerations are applied to an example that corresponds to the nominal acceleration mode in Fermilab Main Ring.[12] For this example, several parameters of rf acceleration are listed in Table I.

TABLE I

Acceleration Parameters of the Fermilab Main Ring

Harmonic number	$h = 1113$
RF Voltage	$eV = -2$ MV
Synchronous phase	$\phi_s \sim 235.87°$
Transition gamma	$\gamma_T = 18.8$
Transition energy	$E = 17.639$ GeV
Nonlinear lattice parameter	$\alpha^{(1)}/\alpha^{(0)} = 0.14$ (Ref. 14)

We choose τ_1 as

$$\begin{aligned}\tau_1 &= \cos\phi_s(0)[E_s(0) - E_s(\tau_1)]\\ &= 559.9428 \text{ MeV}\end{aligned} \tag{6-1}$$

where $E_s(\tau_1)$ is enough far from transition. Using the parameters in the table and the value for τ_1, we get

$$k = \left[\frac{4\pi h|\sin\phi_s|}{E_s{}^2(0)\gamma_T{}^2\cos^2\phi_s|eV|}\right]^{1/2} = 4.0891 \times 10^{-4} \text{ MeV}^{-3/2}. \tag{6-2}$$

Then

$$\begin{aligned}z_1 &= \frac{2}{3}k\tau_1{}^{3/2} = 3.6120\\ S(\tau_1) &= 1/k\tau_1{}^{1/2} = 103.3476 \text{ MeV}.\end{aligned} \tag{6-3}$$

From a table of Bessel functions,[14] we read

$$J_{1/3}(z_1) = -0.2736, \qquad N_{1/3}(z_1) = 0.3172, \qquad J_{-2/3}(z_1) = -0.309. \tag{6-4}$$

Substitution of these values for z_1, $J_{1/3}(z_1)$, $N_{1/3}(z_1)$, and $J_{-2/3}(z_1)$ into Eqs. (4-12a) and (4-12b) gives

$$\begin{aligned}a &= 0.4056 + 0.3296 + 0.1877 = 0.9229\\ b &= 0.5451 + 0.443 = 0.9881.\end{aligned} \tag{6.5}$$

From Eqs. (6-2) and (6-5) we get the value of the synchrotron amplitude function at transition

$$S(0) = \frac{4}{3}\left(\frac{\pi}{3}\right)^2\left(\frac{k}{3}\right)^{-2/3}\frac{a}{\Gamma^2(2/3)} = 277.503 \text{ MeV} \tag{6-6}$$

Here, we use

$$\Gamma(1/3) = 2.6801, \qquad \Gamma(2/3) = 1.3550, \quad \Gamma(4/3) = 0.89338^{15}.$$

The value of the coefficient $\lambda_1(0)$ is

$$|\lambda_1(0)| = \frac{2\pi h[\alpha^{(1)}/\alpha^{(0)} + 3\beta_s^2(0)/2]}{\gamma_T^4 \beta_s^4(0)(m_0 c^2)^2 eV(0) \cos\phi_s(0)} = 1.2343 \times 10^{-7}\text{MeV}^{-3}. \tag{6-7}$$

The $S(0)$ above leads to the value of the typical boundary

$$\tau_2 = S(0)/2 = 138.7515 \text{ MeV} \tag{6-8}$$

The parameters g_1, g_2 in the integration of Eq. (5-18) are

$$g_2 = -0.8279/a = -0.8971$$

$$g_1 = 3.6118 \times 10^{-4} - \frac{3}{2}\frac{\Gamma(2/3)}{\Gamma(4/3)}\left(\frac{k}{3}\right)^{2/3} = -5.66 \times 10^{-3}. \tag{6-9}$$

Then

$$\Theta = g_2 \log\left[1\Big/\left(\frac{1}{2g_2} + 1\right)\right] = -0.7312 \tag{6-10}$$

Substituting these values for g_1, g_2, g_3, Θ into Eq. (5-26), we obtain the integral I in the form

$$\begin{aligned} I &= (-39.1614)\sin\theta_0 + (-72.9502)\cos\theta_0 \\ &\quad + (71.9840)\sin 3\theta_0 + (-5.2259)\cos 3\theta_0. \end{aligned} \tag{6.11}$$

Thus

$$I_{\max} = 96, \qquad I_{\min} = -174. \tag{6-12}$$

From Eq. (6-12), we have

$$\begin{aligned} K_+ &= 1/2 \times 0.9577 \times 10^{-7} \times 4622.7619 \times 179 = 3.85 \times 10^{-2} \\ K_- &= -1/2 \times 0.9577 \times 10^{-7} \times 4722.7619 \times 96 = -2.1241 \times 10^{-2}. \end{aligned} \tag{6-13}$$

Finally, substituting these values for $S(0)$, $S(\tau_1)$ into Eq. (5-31a), we get

$$\begin{aligned} \xi(0)_{\max} &= \xi(\tau_1) \times 1.6386 \times [1.0 + 3.7871 \times 10^{-3}\xi(\tau_1)]^{1/2} \\ \xi(0)_{\min} &= \xi(\tau_1) \times 1.6386 \times [1.0 - 2.0894 \times 10^{-3}\xi(\tau_1)]^{1/2}. \end{aligned} \tag{6-14}$$

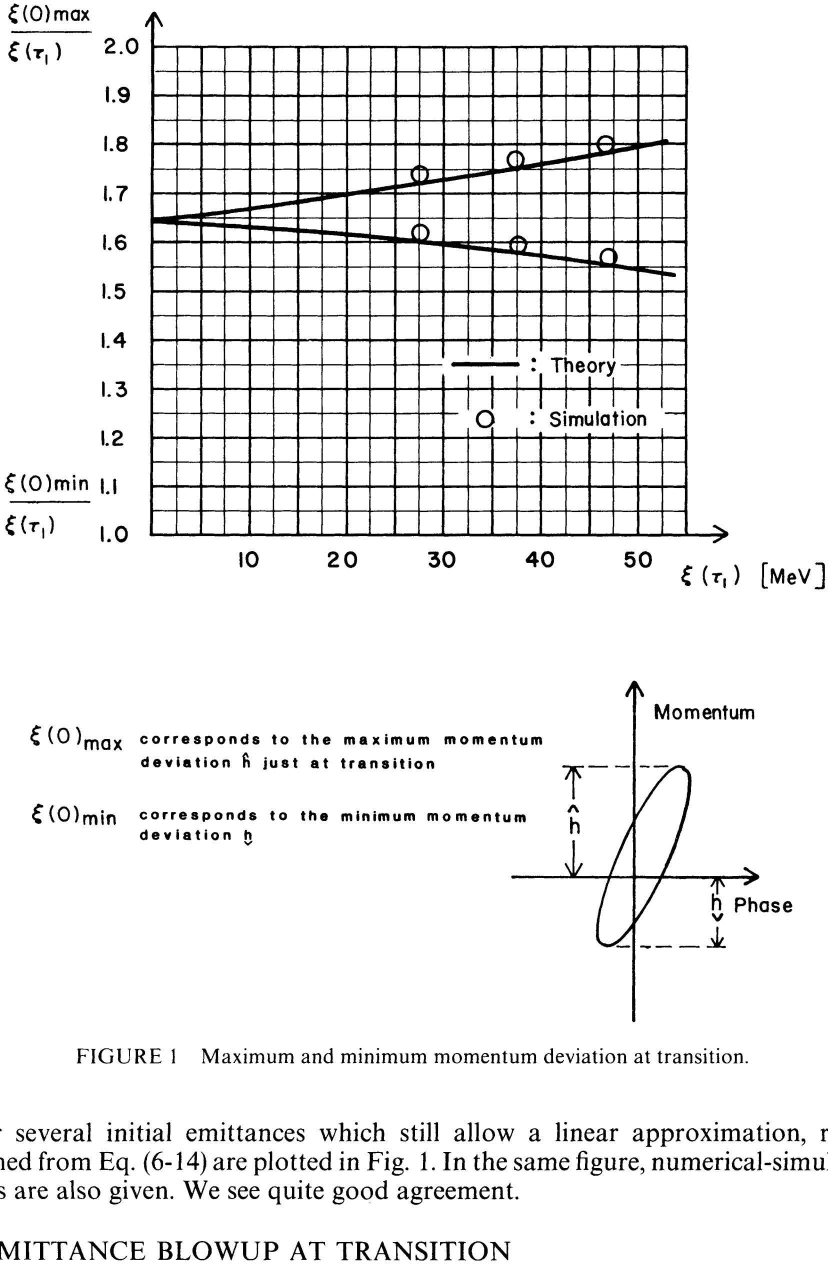

FIGURE 1 Maximum and minimum momentum deviation at transition.

For several initial emittances which still allow a linear approximation, results obtained from Eq. (6-14) are plotted in Fig. 1. In the same figure, numerical-simulation results are also given. We see quite good agreement.

§7. EMITTANCE BLOWUP AT TRANSITION

When a particle crosses the transition energy, the electric phase of rf is abruptly changed externally to $\pi - \phi_s$. Such a manipulation yields time-reversal of the phase motion for the dynamical system described by

$$H(x, p; \tau) = \frac{1}{2}[p^2 + \lambda_0(\tau)x^2], \tag{7-1}$$

because the change of sign of the cosine function in

$$\tau(t) = \int_0^t -\left[\frac{eV(t)\cos\phi_s(t)}{T_s(t)}\right] dt \tag{7-2}$$

will lead to time-reversing while the sign of $\lambda_0(\tau)$,

$$\lambda_0(\tau) = -\frac{4\pi h(\gamma_s(t) - \gamma_T)}{\beta_s^2(t)E_s(t)eV(t)\cos\phi_s(t)\gamma_T^3} \tag{7-3}$$

still remains unchanged due to sign changes of the denominator and numerator. This holds even if all higher-order terms with respect to phase are included. Consequently, emittance blowup during transition crossing cannot in principle be explained by synchrotron-oscillation theory, which restricts itself to ordinary pendulum oscillations with adiabatically changing coefficients.

On the other hand, the coefficient of the nonlinear kinematic term,

$$\lambda_1(\tau) = -\frac{2\pi h[\alpha^{(1)} + 3\beta_s^2(t)/2\gamma_s^2(t)]}{[\beta_s(t)E_s(t)]^2 eV(t)\cos\phi_s(t)}, \tag{7-4}$$

changes its sign after the phase jump. The dynamical system including such a term is therefore no longer time reversible. In other words, synchrotron oscillations accompanied with kinematic nonlinearity are singular at transition. Just after passing transition, a bunch suddenly meets an unmatched bucket. This leads to actual emittance blowup. The magnitude of the blowup is proportional to the amount of emittance distortion due to the nonlinear kinematic term. Thus we may write the final emittance blowup ratio during transition crossing as

$$R = 1 + 2\cdot K_+ \cdot \sqrt{2J(\tau_1)}, \tag{7-5}$$

where $J(\tau_1)$ is the emittance or the value of the action variable far below transition. The final blowup ratio R is plotted as a function of the initial emittance for the normal acceleration mode ($h = 1113$) in Fig. 2. Results of measurements are also given in the figure. The small overestimate seen may imply that the exact nonlinear lattice parameter is somewhat smaller than the value used in the present calculations.

§8. CONCLUSION

A linear classical theory of transition, which is equivalent to the usual one of matrix form,[11,16] has been developed by introducing the synchrotron amplitude function. As a natural extension of this linear theory, a general method which relies on perturbation techniques to assess effects of the lowest order nonlinear kinematic term is presented. When these theories are applied to the case of the Fermilab Main Ring, they agree very well with results of computer simulations and real measurements. This emphasizes the importance of the higher-order chromaticity control, which can be done by adjusting the nth and $2n$th Fourier components of the sextupole magnetic fields (n horizontal betatron tune).[3] If $\lambda_1(\tau)$ is reduced by such higher-order chromaticity control, the emittance blowup discussed here will be improved.

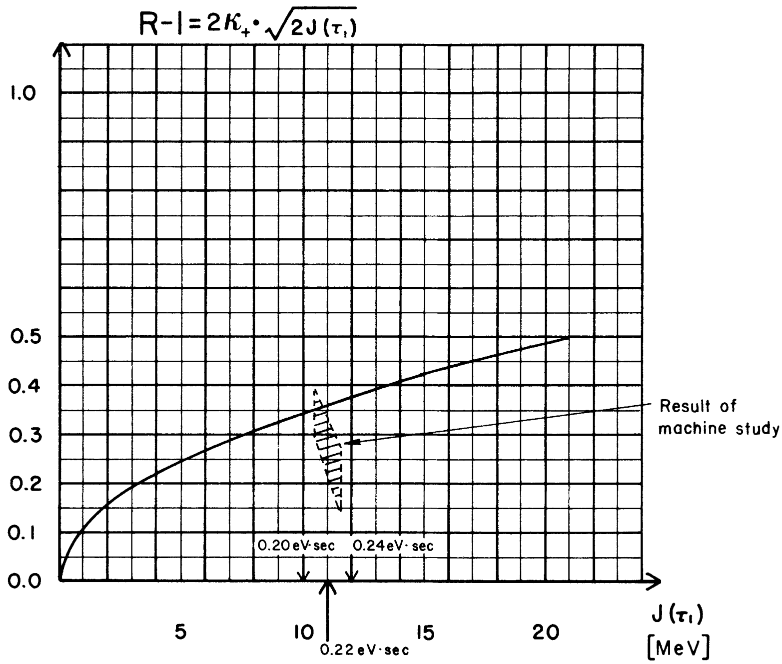

FIGURE 2 Blowup ratio R as a function of the initial emittance.

The present analytical approach can be also used to derive explicit expressions for emittance increments resulting from other nonlinear forces which become significant, in particular, in the vicinity of transition.

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APPENDIX A

General Solution of Auxiliary Equation

When $x_1(\tau)$ and $x_2(\tau)$ are linearly independent solutions of the time-dependent linear equation

$$\ddot{x} + \lambda_0(\tau)x = 0, \tag{A-1}$$

we can write a general solution of its modified nonlinear auxiliary equation (or envelope equation)

$$\ddot{\rho} + \lambda_0(\tau)\rho = \frac{1}{\rho^3}, \tag{A-2}$$

where $\rho(\tau)$ is the square-root function of $S(\tau)$, in terms of $x_1(\tau)$ and $x_2(\tau)$ as

$$\rho(\tau) = (C_1 x_1^2 + C_2 x_2^2 + C_3 x_1 x_2)^{1/2}. \tag{A-3}$$

Squaring both sides of (A-3) and differentiating with respect to the independent variable τ, we have

$$\begin{aligned} 2\dot{\rho}^2 + 2\rho\ddot{\rho} = {} & 2C_1\dot{x}_1^2 + 2C_1 x_1\ddot{x}_1 + 2C_2\dot{x}_2^2 + 2C_2 x_2\ddot{x}_2 \\ & + C_3\ddot{x}_1 x_2 + C_3 x_1\ddot{x}_2 + 2C_3\dot{x}_1\dot{x}_2. \end{aligned} \tag{A-4}$$

From (A-1) and (A-2), Eq. (A-4) reduces to an equation including first time derivatives alone. Further, using (A-3), we obtain

$$\begin{aligned} \text{LHS} = {} & \frac{(2C_1 x_1\dot{x}_1 + 2C_2 x_2\dot{x}_2 + C_3\dot{x}_1 x_2 + C_3 x_1\dot{x}_2)^2}{2(C_1 x_1^2 + C_2 x_2^2 + C_3 x_1 x_2)} + \frac{2}{C_1 x_1^2 + C_2 x_2^2 + C_3 x_1 x_2} \\ & - 2\lambda(\tau)(C_1 x_1^2 + C_2 x_2^2 + C_3 x_1 x_2), \end{aligned}$$

$$\text{RHS} = 2C_1\dot{x}_1^2 + 2C_2\dot{x}_2^2 + 2C_3\dot{x}_1\dot{x}_2 - 2\lambda(\tau)(C_1 x_1^2 + C_2 x_2^2 + C_3 x_1 x_2).$$

Equating both sides and eliminating terms, we find

$$(C_3^2 - 4C_1C_2)(\dot{x}_1 x_2 - x_1\dot{x}_2)^2 + 4 = 0. \tag{A-5}$$

From (A-5), the arbitrary constant C_1, C_2, and C_3 are not independent. Namely, C_3 is determined from C_1 and C_2 as

$$C_3 = \pm\sqrt{4C_1C_2 - \frac{4}{W^2}}, \tag{A-6}$$

where W is the Wronskian $x_1\dot{x}_2 - \dot{x}_1x_2$, which is a constant. Consequently, we can write the general solution of Eq. (4-6) as

$$S(\tau) = C_1{x_1}^2(\tau) + C_2{x_2}^2(\tau) - 2\sqrt{C_1C_2 - \frac{1}{W^2}}\, x_1(\tau)x_2(\tau). \tag{A-7}$$

For the present case

$$\lambda(\tau) = k^2\tau; \tag{A-8}$$

the independent solutions of the linear equation can be written in terms of Bessel and Neumann functions of order 1/3,

$$x_1(\tau) = \tau^{1/2}N_{1/3}\left(\frac{2}{3}k\tau^{3/2}\right), \tag{A-9a}$$

$$x_2(\tau) = \tau^{1/2}J_{1/3}\left(\frac{2}{3}k\tau^{3/2}\right). \tag{A-9b}$$

Thus the general solution becomes

$$\begin{aligned} S(\tau) = \tau\Bigg[& aN_{1/3}^2\left(\frac{2}{3}k\tau^{3/2}\right) + bJ_{1/3}^2\left(\frac{2}{3}k\tau^{3/2}\right) \\ & -2\sqrt{ab - \frac{1}{W^2}}\, N_{1/3}\left(\frac{2}{3}k\tau^{3/2}\right)J_{1/3}\left(\frac{2}{3}k\tau^{3/2}\right)\Bigg], \end{aligned} \tag{A-10}$$

with

$$W = x_1(0)\dot{x}_2(0) - \dot{x}_1(0)x_2(0) = 3/\pi.$$

This is in agreement with the result obtained by Lewis.[10]

APPENDIX B

Particular Solution of the Auxiliary Equation (4-6)

Since the initial conditions are

$$S(\tau_1) = 1/k{\tau_1}^{1/2}, \tag{B-1a}$$

$$\dot{S}(\tau_1) = 0, \tag{B-1b}$$

we have immediately two algebraic equations for a and b

$$aN_\nu^{\,2} + bJ_\nu^{\,2} - 2\left(ab - \frac{9}{\pi^2}\right)^{1/2} J_\nu N_\nu = \frac{6}{\pi^2 z_1}, \tag{B-2a}$$

$$aN_\nu N_\nu' + bJ_\nu J_\nu' - \left(ab - \frac{9}{\pi^2}\right)^{1/2} (J_\nu' N_\nu + J_\nu N_\nu') = -\frac{2}{\pi^2 z_1^{\,2}}, \tag{B-2b}$$

where ν is 1/3, the prime denotes derivatives with respect to z, and all Bessel functions are to be given their values at $z_1 = z(\tau_1)$. From (B-2a), we have

$$\left(ab - \frac{9}{\pi^2}\right)^{1/2} = \frac{aN_\nu^{\,2} + bJ_\nu^{\,2} - 6/\pi^2 z_1}{2J_\nu N_\nu}. \tag{B-3}$$

Substitution of (B-3) into (B-2b) yields

$$\frac{N_\nu(N_\nu' J_\nu - N_\nu J_\nu')}{2J_\nu} a + \frac{J_\nu(J_\nu' N_\nu - J_\nu N_\nu')}{2N_\nu} b = -\frac{3(J_\nu' N_\nu + J_\nu N_\nu')}{\pi^2 z_1 J_\nu N_\nu} - \frac{2}{\pi^2 z_1^{\,2}}. \tag{B-4}$$

Using the formula

$$J_\nu N_\nu' - J_\nu' N_\nu = 2/\pi z, \tag{B-5}$$

we have

$$a = \frac{J_\nu^{\,2}}{N_\nu^{\,2}} b - \frac{3}{\pi N_\nu^{\,2}}\left(J_\nu' N_\nu + J_\nu N_\nu' + \frac{2J_\nu N_\nu}{3z_1}\right). \tag{B-6}$$

If we substitute (B-6) into (B-3), square both sides, and compare corresponding terms, we obtain

$$b = \frac{3}{2} z_1 N_\nu^{\,2} + \frac{1}{6z_1 J_\nu^{\,2}}\left[\frac{3}{2}(J_\nu' N_\nu + J_\nu N_\nu') + J_\nu N_\nu + \frac{3}{\pi}\right]^2. \tag{B-7}$$

Further, from a recursion equation for Bessel functions

$$J_\nu' = J_{\nu-1} - \nu z^{-1} J_\nu, \tag{B-8}$$

and the relation (B-5), we find

$$\begin{aligned} J_\nu' N_\nu + J_\nu N_\nu' &= J_\nu' N_\nu + J_\nu' N_\nu + \frac{2}{\pi z_1} \\ &= 2(J_{\nu-1} - \frac{\nu}{z_1} J_\nu) N_\nu + \frac{2}{\pi z_1}. \end{aligned} \tag{B-9}$$

Introduction of (B-9) into (B-7) leads to

$$b = \frac{3}{2} z_1 N_\nu^{\,2} + \frac{3}{2z_1 J_\nu^{\,2}}\left(z_1 J_{\nu-1} N_\nu + \frac{2}{\pi}\right)^2. \tag{B-10}$$

Thus we also find

$$a = \frac{3}{2} z_1 J_\nu^2 + \frac{3}{2z_1 N_\nu^2}\left(z_1 J_{\nu-1} N_\nu + \frac{2}{\pi}\right)^2 - \frac{6(z_1 J_{\nu-1} N_\nu + 1/\pi)}{\pi z_1 N_\nu^2}. \tag{B-11}$$

APPENDIX C

Evaluation of S(0) and Ṡ(0)

We set

$$S(0) = \lim_{\tau\to 0}\left(\frac{\pi}{3}\right)^2 \tau\left[aN_{1/3}^2(z(\tau)) + bJ_{1/3}^2(z(\tau)) - 2\left(ab - \frac{9}{\pi^2}\right)^{1/2} J_{1/3}(z(\tau))N_{1/3}(z(\tau))\right] \tag{C-1}$$

where $z(\tau)$ is $\frac{2}{3}k\tau^{3/2}$. Using the formula

$$N_{1/3}(z) = \frac{1}{\sin\pi/3}\left[\cos\frac{\pi}{3}\cdot J_{1/3}(z) - J_{-1/3}(z)\right], \tag{C-2}$$

$S(0)$ becomes

$$\left(\frac{\pi}{3}\right)^2 \lim_{\tau\to 0}\tau\left\{\frac{a}{3}\left[J_{1/3}^2 - 4J_{1/3}J_{-1/3} + 4J_{-1/3}^2\right] + bJ_{1/3}^2 - \frac{2}{\sqrt{3}}\left(ab - \frac{9}{\pi^2}\right)^{1/2}(J_{1/3}^2 - 2J_{1/3}J_{-1/3})\right\}. \tag{C-3}$$

If we retain only the lowest-order term of the series expansion of the Bessel functions, we have

$$J_{1/3}(z(\tau)) = \left(\frac{k}{3}\right)^{1/3}\frac{\tau^{1/2}}{\Gamma(4/3)}, \tag{C-4a}$$

$$J_{-1/3}(z(\tau)) = \left(\frac{k}{3}\right)^{-1/3}\frac{\tau^{-1/2}}{\Gamma(2/3)}. \tag{C-4b}$$

Substituting (C-4a) and (C-4b) into (C-3) and taking its limit at $\tau = 0$, we obtain

$$S(0) = \frac{4}{3}\left(\frac{\pi}{3}\right)^2\left(\frac{k}{3}\right)^{-2/3}\frac{1}{\Gamma^2(2/3)}. \tag{C-5}$$

$\dot{S}(0)$ can also be evaluated in a similar way. We set

$$\dot{S}(0) = \lim_{\tau\to 0}\left(\frac{\pi}{3}\right)^2\left\{\left[aN_{1/3}^2 + bJ_{1/3}^2 - 2\left(ab - \frac{9}{\pi^2}\right)^{1/2} J_{1/3}N_{1/3}\right] + 2\tau\dot{z}\left[aN_{1/3}N'_{1/3} + bJ_{1/3}J'_{1/3} - \left(ab - \frac{9}{\pi^2}\right)^{1/2}(J'_{1/3}N_{1/3} + J_{1/3}N'_{1/3})\right]\right\}, \tag{C-6}$$

where dots denote derivatives with respect to τ, primes denote derivatives with respect to z, and all Bessel functions are functions of $z(\tau)$. Using formulas (B-9), (C-2), and the recursion relation (B-8), we have

$$N_{1/3}N'_{1/3} = \frac{1}{3}(J_{1/3} - 2J_{-1/3})\left(J_{-2/3} + 2J_{2/3} - \frac{J_{1/3}}{3z} + \frac{2}{3z}J_{-1/3}\right), \quad \text{(C-7a)}$$

$$J_{1/3}J'_{1/3} = J_{1/3}\left(J_{-2/3} - \frac{1}{3z}J_{1/3}\right), \quad \text{(C-7b)}$$

$$J'_{1/3}N_{1/3} + J_{1/3}N'_{1/3} = \frac{2}{\sqrt{3}}\left(J_{-2/3} - \frac{1}{3z}J_{1/3}\right)(J_{1/3} - 2J_{-1/3}) + \frac{2}{\pi z}. \quad \text{(C-7c)}$$

Retaining only the lowest-order term of the series expansion for $J_{2/3}(z(\tau))$ and $J_{-2/3}(z(\tau))$, we have

$$J_{2/3}(z(\tau)) = \left(\frac{k}{3}\right)^{2/3} \frac{\tau}{\Gamma(5/3)}, \quad \text{(C-8a)}$$

$$J_{-2/3}(z(\tau)) = \left(\frac{k}{3}\right)^{-2/3} \frac{\tau^{-1}}{\Gamma(1/3)}. \quad \text{(C-8b)}$$

From $\tau\dot{z} = k\tau^{3/2}$, the limiting values of the component terms at $\tau = 0$ in (C-6) become

$$\lim_{\tau\to 0} \tau\dot{z}J_{1/3}J_{-2/3} = \lim_{\tau\to 0} k\left(\frac{k}{3}\right)^{1/3-2/3} \frac{\tau^{3/2+1/2-1}}{\Gamma(4/3)\Gamma(1/3)} = 0,$$

$$\lim_{\tau\to 0} \tau\dot{z}J_{1/3}J_{2/3} = \lim_{\tau\to 0} k\left(\frac{k}{3}\right)^{1/3+2/3} \frac{\tau^{3/2+1/2+1}}{\Gamma(4/3)\Gamma(5/3)} = 0,$$

$$\lim_{\tau\to 0} \tau\dot{z}J_{-1/3}J_{-2/3} = \lim_{\tau\to 0} k\left(\frac{k}{3}\right)^{-1/6-2/3} \frac{\tau^{3/2-1/2-1}}{\Gamma(2/3)\Gamma(1/3)} = \frac{3}{\Gamma(2/3)\Gamma(1/3)}, \quad \text{(C-9)}$$

$$\lim_{\tau\to 0} \tau\dot{z}J_{-1/3}J_{2/3} = \lim_{\tau\to 0} k\left(\frac{k}{3}\right)^{-1/3+2/3} \frac{\tau^{3/2-1/2+1}}{\Gamma(2/3)\Gamma(5/3)} = 0,$$

$$\lim_{\tau\to 0} \tau\dot{z}J_{1/3}J_{1/3} = \lim_{\tau\to 0} \frac{3}{2}\left(\frac{k}{3}\right)^{1/3+1/3} \frac{\tau^{1/2+1/2}}{\Gamma(4/3)\Gamma(4/3)} = 0.$$

Substitution of (C-9) into (C-6) yields

$$\begin{aligned}\dot{S}(0) = \left(\frac{\pi}{3}\right)^2 \Bigg\{ \lim_{\tau\to 0} \Bigg[a\left(\frac{4}{3}J^2_{-1/3} - \frac{8\tau\dot{z}}{9z}J^2_{-1/3}\right) + a\left(-\frac{4}{3}J_{1/3}J_{-1/3} + \frac{8\tau\dot{z}}{9z}J_{1/3}J_{-1/3}\right) \\ + \left(ab - \frac{9}{\pi^2}\right)^{1/2}\left(\frac{4}{\sqrt{3}}J_{1/3}J_{-1/3} - \frac{8\tau\dot{z}}{3\sqrt{3}z}J_{1/3}J_{-1/3}\right)\Bigg] \\ + \left[-\frac{4a}{\Gamma(2/3)\Gamma(1/3)} - 2\left(ab - \frac{9}{\pi^2}\right)^{1/2}\left(-\frac{12}{\sqrt{3}\Gamma(2/3)\Gamma(1/3)} + \frac{2\tau\dot{z}}{\pi z}\right)\right]\Bigg\}.\end{aligned} \quad \text{(C-10)}$$

Thus, we have

$$\dot{S}(0) = \frac{2\pi}{3}\left[-\frac{a}{\sqrt{3}} + \left(ab - \frac{9}{\pi^2}\right)^{1/2}\right]. \tag{C-11}$$

Here we use the relation

$$\Gamma(2/3)\Gamma(1/3) = \frac{2\pi}{\sqrt{3}}.$$

APPENDIX D

Calculation of Perturbing Integration

One of the four parts in the perturbation integration,

$$I_A = e^{-\bar{\theta}_0/g_2}\left\langle\frac{1 - g_1 g_2 g_3}{1/g_2{}^2 + 1}\right| e^{w/g_2}\left(\frac{1}{g_2}\cos w + \sin w\right)\Bigg|_{w_1}^{w_2}, \tag{D-1}$$

is calculated as follows:

$$\begin{aligned} I_A &= \frac{g_2(1 - g_1 g_2 g_3)}{1 + g_2{}^2}\, e^{-\bar{\theta}_0/g_2}[e^{w_2/g_2}(\cos w_2 + g_2 \sin w_2) \\ &\quad - e^{w_1/g_2}(\cos w_1 + g_2 \sin w_1)] \\ &= \frac{g_2(1 - g_1 g_2 g_3)}{1 + g_2{}^2}\,[e^{\log g_3}(\cos w_2 + g_2 \sin w_2) \\ &\quad - e^{\log g_3((1/2g_2)+1)}(\cos w_1 + g_2 \sin w_1)] \\ &= \frac{g_2(1 - g_1 g_2 g_3)}{1 + g_2{}^2}\left\{g_3 \cos\left[g_2 \log\left(\frac{1}{((1/2g_2) + 1)} + \theta_0\right)\right.\right. \\ &\quad \left.\left. + g_2 g_3 \sin\left[g_2 \log\left(\frac{1}{((1/2g_2) + 1)} + \theta_0\right)\right] - g_3\left(\frac{1}{2g_2} + 1\right)(\cos\theta_0 + g_2 \sin\theta_0)\right\} \\ &= \frac{g_2 g_3(1 - g_1 g_2 g_3)}{1 + g_2{}^2}\left\{\left[-\sin\Theta + g_2\cos\Theta - g_2\left(\frac{1}{2g_2} + 1\right)\right]\sin\theta_0\right. \\ &\quad \left. + \left[\cos\Theta + g_2\sin\Theta - \left(\frac{1}{2g_2} + 1\right)\right]\cos\theta_0\right\}, \end{aligned} \tag{D-2}$$

where

$$\Theta = g_2 \log\frac{1}{1/2g_2 + 1}.$$

The other three parts can be calculated in a similar way.