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Bound States for the Spin-1/2 Aharonov-Bohm Problem in a Rotating Frame

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Abstract: In this paper, we study the effects of rotation in the spin-1/2 non-relativistic Aharonov-Bohm problem for bound states. We use a technique based on the self-adjoint extension method and determine an expression for the energies of the bound states. The inclusion of the spin element in the Hamiltonian guarantees the existence of bound state solutions. We perform a numerical analysis of the energies and verify that both rotation and the spin degree of freedom affect the energies of the particle. The main effect we observe in this analysis is a cutoff value manifested in the Aharonov-Bohm flux parameter that delimits the values for the positive and negative energies.

Keywords: Aharonov-Bohm effect; self-adjoint extension; rotating frame; bound states

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1. Introduction

The Aharonov-Bohm effect [1] is one of the most remarkable phenomena in the branch of quantum theory. The effect is frequently associated with studying geometrical phases in quantum mechanics, being also related to a new way of thinking about the significance of the vector potential. Thus, the idea of the Aharanov-Bohm (AB) effect itself constitutes an essential aspect regarding the foundations of quantum mechanics. The interpretation given by Aharonov and Bohm has been widely discussed, including the non-locality of the AB effect. Besides, several experiments have been performed to confront the theoretical predictions. The main result behind the AB effect is that a quantum particle can be affected by an electromagnetic potential, even if no electromagnetic fields are acting on the particle. At this point, it is worth noting that in their famous manuscript, Aharonov and Bohm proposed two kinds of experiments, consisting of a magnetic AB effect and its electric counterpart. Despite that, the magnetic version has attracted more attention. Thus, in this manuscript, we use the "AB effect" to refer to its magnetic version. The influence of the magnetic interaction in the AB effect appears through a geometrical phase in the wave function, which depends on the vector potential when an interferometry experiment is performed.

The work of Aharonov and Bohm influenced novel theoretical developments in several research fields, and various analog effects were proposed. For instance, Aharonov and Casher have predicted that it is possible to obtain a quantum phase for a neutral particle that has a magnetic moment. In that case, there is an electric field due to a charge line, but there is no force acting on the particle. That effect is known as the "Aharonov-Casher" effect. Similarly, a particle constrained to move in a region in which there is no curvature can be influenced by curvature effects coming from an inaccessible region to it, corresponding to a version of a gravitational analog [2,3]. Other versions of gravitational analogs were also investigated [4]. Another example of an analog effect consists of a hydrodynamic one, in which sound waves are scattered by a quantized vortex in a superfluid helium [5]. Besides, there is an optical analog in the context of metamaterials [6] and a photonic analog, which arises from photon–phonon interactions [7]. A non-Abelian version of the AB effect was also presented [8].

Studying the AB effect is not restricted to the case of scattering quantum states through interferometric setups. The influence of an electromagnetic flux can also affect a quantum particle in the case of bound states. One can say that the presence of magnetic flux, for instance, can affect the energy eigenvalues of a particle for bound states even if there is no electromagnetic field acting on such energy states [9]. The AB effect for bound states has been analyzed in many different scenarios. For instance, it can take place in graphene rings [10], affecting its valley degeneracy [11]. In carbon nanotubes, the AB effect also can be related to exciton structures [12].

In the context of bound states, there are also analogs of the AB effect. A molecular version of that effect occurs due to an effective vector potential, which appears in triatomic molecules of Li_3, for example [13]. A topological analog due to a defect in a solid is capable of modifying its energy spectrum [14]. Modifications on the energy spectrum of relativistic spin-0 particles also occur due to AB-like effects [15]. Likewise, is possible to obtain an Aharonov-Casher for bound states [16]. AB-like effects for bound states can also take place in the framework of Kaluza-Klein theory [17,18]. The AB effect was also considered in several situations involving the Schrödinger equation. Ref. [19], for example, deals with the AB effect in planar systems with disclinations vortices by using the Schrödinger equation and the effective mass approximation. The AB effect in the presence of a density of topological defects in the non-relativistic domain is considered in [20]. The Schrödinger equation allows studying the AB effect for non-commutative quantum mechanics [21,22]. Ref. [23] deals with the problem of electron gases in the presence of the AB effect, including the case of a non-relativistic Hamiltonian.

Despite the importance of studying all of the aspects concerning the AB effect mentioned above, there is a foundational one that we have not cited yet. It consists of examining the role of the spin degree of freedom in describing the AB effect. We can think about how to incorporate that degree of freedom and how it modifies the quantum mechanical description of the system. The study of this subject requires different mathematical tools and reveals novel meaningful physical aspects. In a sequence of pioneer works, Hagen has demonstrated that it is necessary to take into account the appearance of a Zeeman term containing a δ interaction when the spin degree is present [24–26]. This type of interaction modifies the Hamiltonian, introducing a singularity to it. Then, it is necessary to employ appropriate techniques to deal with it. We can use the method of self-adjoint extensions to treat this issue [27,28].

A possible generalization of these studies involves describing the quantum dynamics of such a system in a non-inertial frame. When a given system is spinning, its physical properties can change in comparison with the static case [29,30]. In this context, several analogies between electromagnetic interactions and non-inertial effects take place. The Einstein-de Haas effect [31], and the Barnett effect [32] are examples of these similarities. Another important aspect in this scenario is the manifestation of a similar Hall effect due to rotation [33]. Non-inertial effects also play an equivalent role of electromagnetic interactions in rotating plasmas [34,35]. Besides, there is a rotational analog of the AB effect, called the Aharanov-Carmi effect, in which a quantum phase can emerge in a spinning system in the presence of suitable electromagnetic fields [36,37]. Additionally, a rotating semiconductor can serve as a rotational diode [38]. Concerning the spin degree of freedom in rotating frames, it can generate an analog of the Zeeman effect due to its coupling with the rotation [39].

From the motivation of including the spin degree of freedom in the description of the AB effect and having in mind the similarities between electromagnetic fields and rotation, in this manuscript, we study the AB effect for the bound states for a rotating system. The manuscript is organized as follows. In Section 2, we obtain the equation of motion for a quantum particle in a rotating frame in the presence of electromagnetic interactions related to the rising of the AB effect. In Section 3, by employing the self-adjoint approach, we obtain the energy spectrum for the system. Then, we investigate that spectrum concerning the existence of bound states and the role of the spin degree of freedom. We make our conclusions in Section 4.

2. The Equation of Motion

In this section, we study the non-relativistic quantum motion of a charged particle of mass m_e in a rotating frame in the presence of the Aharonov-Bohm effect by taking into account the spin effects of the particle.

The equation that describes the non-relativistic motion of spin-1/2 particles in electromagnetic fields is the Pauli-Schrödinger equation. There are some works in the literature in which this equation is obtained for different physical models involving inertial effects. In addition, in the literature, the Pauli-Schrödinger equation is found by using the low-energy expansion and block diagonalization of the Dirac Hamiltonian developed by Foldy, Wouthuysen [40], and Tani [41], and provides several relativistic corrections. The lowest order of the expansion of the Hamiltonian to the order of $1/m_e$ contains the spin-independent energy (kinetic energy and potential energy) and the interaction energy (magnetic dipole energy, spin-rotation coupling, and the energy due to inertial effects). The expansion of the order of $1/m_e^2$ yields the spin-orbit coupling and the Darwin term. These two terms contain corrections in the electric field due to the rotating frame. For our purposes, it is sufficient to consider the Hamiltonian only up to order $1/m_e$. We can justify it by noticing that the first order already provides the terms involving both the spin degree of freedom and rotation. Then, we follow the alternative model presented in Chapter 17 of Ref. [42], which takes into account only the spin and rotation without the inclusion of relativistic correction terms. In the absence of electromagnetic fields, this model was studied in Ref. [42] in the context of Sagnac phase shift.

To include the rotating effects in the Schrödinger equation, we use the minimal coupling procedure for the rotating frame, given by [42]

$$p^\mu \to p^\mu - m_e \mathscr{A}^\mu, \quad \mathscr{A}^\mu = \left(-\frac{1}{2}(\boldsymbol{\Omega}\times\mathbf{r})^2, \boldsymbol{\Omega}\times\mathbf{r}\right), \tag{1}$$

with $\mathscr{A}^\mu$ being the gauge field for the rotating frame, $\boldsymbol{\Omega}$ the angular velocity, and $\mu = 0,1,2,3$. To include the Aharonov-Bohm effect, we must consider the particle interacting with the electromagnetic gauge field, $A^\mu = (A^0, \mathbf{A})$, which can be introduced into the Schrödinger equation by using the minimal substitution as defined above. We define the magnetic flux tube for the Aharonov-Bohm problem as

$$e\mathbf{A} = \left(0, -\frac{\phi}{\rho}, 0\right), \quad e\mathbf{B} = \left(0, 0, -\phi\frac{\delta(\rho)}{\rho}\right), \tag{2}$$

with $\nabla\cdot\mathbf{A} = 0$, $A^0 = 0$. The quantity $\phi = e\Phi/2\pi\hbar = -\Phi/\Phi_0$ is related to the Aharonov-Bohm flux, in which Φ denotes the magnetic flux and $\Phi_0 = h/e$ indicates the quantum of magnetic flux. Then, to implement the above substitutions in the Schrödinger equation, we use the substitution $p^\mu \to p^\mu - m_e\mathscr{A}^\mu - eA^\mu$. Since the Aharonov-Bohm problem has translational invariance in the z-direction, we can exclude the z degree of freedom by imposing $p_z = z = 0$ [24,27,43,44].

After adding the spin-rotation term to the Schrödinger Hamiltonian, we arrive at the Pauli-Schrödinger equation of motion

$$H\psi = \mathscr{E}\psi, \tag{3}$$

where

$$H = \frac{1}{2m_e}(\mathbf{p} - e\mathbf{A} - m_e\boldsymbol{\Omega}\times\mathbf{r})^2 - \frac{1}{2}m_e(\boldsymbol{\Omega}\times\mathbf{r})^2 - \frac{e\hbar}{2m_e}\sigma\cdot\mathbf{B} - \frac{\hbar}{2}\sigma\cdot\boldsymbol{\Omega} \tag{4}$$

is the Hamiltonian of the system and $\sigma^i = (\sigma^x, \sigma^y, \sigma^z)$ are the standard Pauli matrices. The interpretation of the terms in the Hamiltonian (4) is given as follows. The first term corresponds to the kinetic one, while the second term corresponds to the scalar potential related to the rotation. The third term describes the Zeeman interaction, while the last one is the spin-rotation coupling. At this point, it is worth making a brief comment on the Zeeman term, which is written as $-\boldsymbol{\mu} \cdot \boldsymbol{B}$, with $\boldsymbol{\mu} = -\mu\boldsymbol{\sigma}$ corresponding to the electron magnetic moment, and $\mu = e\hbar/2m_e$. The gyromagnetic factor is usually considered to be $g = 2$. However, if we consider the anomalous magnetic of the electron, then we have $g = 2(1+a)$, with $a = 0.00115965218279$ [45]. It corresponds to the deviation with respect to the usual case [1]. In this way, we can make the substitution $\mu \to \mu(1+a)$. In our numerical analysis below, we use the value of g above. This value is crucial for the appearance of the curves of bound states energies. Since this deviation is small, it also justifies considering only up to order $1/m_e$.

Returning to the problem, let us admit that $\boldsymbol{\Omega}$ is along the z-axis. In cylindrical coordinates, $\mathbf{r} = \rho\,\hat{\boldsymbol{\rho}}$, we write $\boldsymbol{\Omega} = \Omega\,\hat{\mathbf{z}}$ and $\boldsymbol{\Omega} \times \mathbf{r} = \Omega\rho\,\hat{\boldsymbol{\varphi}}$. Equation (4) takes the form of

$$H = \frac{1}{2m_e}\left(p^2 + e^2A_\varphi^2 - 2eA_\varphi p_\varphi - 2m_e\Omega\rho p_\varphi + 2em_eA_\varphi\Omega\rho\right) - \left(\frac{g}{2}\right)\frac{e\hbar}{2m_e}\sigma^z B^z - \frac{\hbar}{2}\sigma^z\Omega. \quad (5)$$

After using the field and potential configurations from Equation (2) in Equation (5), we find

$$\begin{aligned} H = & -\frac{\hbar^2}{2m_e}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial}{\partial\varphi} + i\phi\right)^2 + 2m_e\Omega\frac{1}{i\hbar}\frac{\partial}{\partial\varphi}\right] \\ & - \frac{\hbar^2}{2m_e}\left(-\frac{g\phi}{2}\sigma^z\frac{\delta(\rho)}{\rho} + \frac{m_e\Omega}{\hbar}\sigma^z + \frac{2m_e\Omega\phi}{\hbar}\right). \end{aligned} \quad (6)$$

Besides the terms involving the influence of the magnetic field and the rotation separately, we can notice that the last term in the above expression corresponds to a combined effect of rotation and magnetic field. As pointed out in Ref. [24], the δ function cannot be neglected from the model if we intend to solve the problem taking into account the spin of the particle. In fact, the absence of the Zeeman interaction term in Equation (4) leads to the spin-0 Aharonov-Bohm effect, where the solution of the problem for bound states is no longer possible. From Equation (3), we can note that ψ is an eigenfunction of σ^z, whose eigenvalues are $s = \pm 1$, satisfying $\sigma^z\psi = \pm\psi = s\psi$. Thus, assuming the eigenfunctions to be of the form

$$\psi(\rho, \varphi) = f(\rho)e^{im\varphi}, \quad (7)$$

where $m = 0, \pm 1, \pm 2, \pm 3, \ldots$ is the angular momentum quantum number, we obtain the radial equation

$$\mathscr{H} f(\rho) = \mathscr{K}^2 f(\rho), \quad (8)$$

where

$$\mathscr{H} = \mathscr{H}_0 + \frac{1}{2}gs\phi\,\frac{\delta(\rho)}{\rho} \quad (9)$$

and

$$\mathscr{H}_0 = -\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + \frac{(m+\phi)^2}{\rho^2}, \quad (10)$$

$$\mathscr{K}^2 = \frac{2m_e\mathscr{E}}{\hbar^2} + \frac{2m_e\Omega}{\hbar}\left(m + \phi + \frac{s}{2}\right). \quad (11)$$

As we can see in Equation (9), the inclusion of the spin of the particle in the approach introduces a singularity at the origin. With the presence of this singular term, the operator loses the self-adjointness property. From the theory of self-joint extensions, we know that self-adjointness is a crucial property of an operator since only self-adjoint operators always have a spectral decomposition. We shall solve Equation (8) using a technique based on the self-adjoint extension method of operators in quantum mechanics. We perform this in the next section. There are several examples of works in the literature, including what we use here, which use the self-adjoint extensions in various physical contexts addressing the problem for both scattering and bound states [46–60].

3. Self-Adjoint Extensions

In this section, we solve Equation (8) using the approach based on the self-joint extension method developed in Ref. [61]. We think there is no need to expose the mathematical details of this technique because it is already present in several articles in the literature, including the papers mentioned in the previous section. Then, following the procedure of Ref. [61], we temporarily neglect the δ-function potential in Equation (9) and replace the problem in Equation (8) by the eigenvalue equation for $\mathscr{H}_0$

$$\mathscr{H}_0 f_\xi = \mathscr{K}^2 f_\xi, \tag{12}$$

plus the self-adjoint extensions. In Equation (12), the wave function is labeled by the parameter ξ of the self-adjoint extension. This parameter is related to the behavior of f_ξ at the origin. The next step of the approach is to require that the operator $\mathscr{H}_0$ be converted into a self-adjoint operator. This is accomplished by extending its domain of definition by the deficient subspace, which is encompassed by the solutions of the eigenvalue equation

$$\mathscr{H}_0^\dagger f_\pm = \pm i \mathscr{K}_0^2 f_\pm, \tag{13}$$

where $\mathscr{K}_0^2 \in \mathbb{R}$ is introduced for dimensional reasons. Since $\mathscr{H}_0$ is a Hermitian operator, then $\mathscr{H}_0^\dagger = \mathscr{H}_0$. This implies that the only square integrable functions that are solutions of Equation (13) are the modified Bessel functions of second kind

$$f_\pm = K_{|m+\phi|}(\sqrt{\mp i}\mathscr{K}_0 \rho), \tag{14}$$

with $\mathrm{Im}\sqrt{\pm i} > 0$. It can be verified that the functions (14) are square integrable only in the range $(m+\phi) \in (-1,1)$, for which $\mathscr{H}_0$ is not self-adjoint. In this case, the dimension of the deficiency subspace is found to be $(n_+, n_-) = (1,1)$ [28,43,44,59]. In this manner, the $\mathscr{D}(\mathscr{H}_{\xi,0})$ in $L^2(\mathbb{R}^+, \rho d\rho)$ is given by the set of functions [62]

$$f_\xi(\rho) = f_m(\rho) + C\Big[K_{|m+\phi|}(\sqrt{-i}\mathscr{K}_0 \rho) + e^{i\xi} K_{|m+\phi|}\Big(\sqrt{i}\mathscr{K}_0 \rho\Big)\Big], \tag{15}$$

where $f(\rho)$, with $f(0) = \dot{f}(0) = 0$, is the regular wave function and the self-adjoining extension parameter $\xi \in [0, 2\pi)$ represents a choice for the boundary condition. The expression (15) is a consequence of the von Neumman theory [62]. In addition, we know that, for different values of ξ, we have different domains for $\mathscr{H}_0$. The adequate boundary condition will be determined by the physical system [46,53,60,63]. This important property allows us to make use of the following physical regularization procedure for the potential vector of the magnetic field [24,25,64,65]:

$$e\mathbf{A} = \begin{cases} -\dfrac{\phi}{\rho}\hat{\varphi}, & \rho > R, \\ 0, & \rho < R, \end{cases} \tag{16}$$

where R is the defect core radius [53,61], which is a very small radius smaller than the Compton wave length λ_C of the electron [50]. We can make the following substitution for the δ function in Equation (9):

$$\frac{\delta(\rho)}{\rho} \to \frac{\delta(\rho - R)}{R}. \tag{17}$$

This regularized form for the δ function allows for the physics of the problem to self-select a value for the self-adjoint extension parameter. The next stage consists of making use of Equation (17) and consider the zero-energy solutions f_0 and $f_{\xi,0}$ for $\mathscr{H}$ and $\mathscr{H}_0$, respectively. We write

$$\left[-\frac{d^2}{d\rho^2}-\frac{1}{\rho}\frac{d}{d\rho}+\frac{(m+\phi)^2}{\rho^2}+\frac{1}{2}gs\phi\,\frac{\delta(\rho-R)}{R}\right]f_0=0, \tag{18}$$

$$\left[-\frac{d^2}{d\rho^2}-\frac{1}{\rho}\frac{d}{d\rho}+\frac{(m+\phi)^2}{\rho^2}\right]f_{\xi,0}=0. \tag{19}$$

The value for the self-adjoint extension parameter ξ is determined by the following boundary condition for $f(\rho)$ at the origin:

$$\lim_{R\to 0^+} R\frac{\dot{f}_0}{f_0}\Big|_{\rho=R}=\lim_{R\to 0^+} R\frac{\dot{f}_{\xi,0}}{f_{\xi,0}}\Big|_{\rho=R}. \tag{20}$$

The left-hand side of Equation (20) can be obtained by the direct integration of (18) from 0 to R. After performing this calculation, we get the result

$$\lim_{R\to 0^+} R\frac{\dot{f}_0}{f_0}\Big|_{\rho=R}=\frac{1}{2}gs\phi. \tag{21}$$

To calculate the right-hand side of Equation (20), we seek the bound states solutions of $\mathscr{H}_0$, which allow us to obtain bound states solutions for $\mathscr{H}$. However, to find an expression for such bound state energies, we consider $\mathscr{K}$ as a pure imaginary quantity, $\mathscr{K}\to i\mathscr{K}_b$. In this way, we have the modified Bessel equation

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\left(\frac{(m+\phi)^2}{\rho^2}+\mathscr{K}_b^2\right)\right]f_\xi(\rho)=0, \tag{22}$$

whose solution is found to be

$$f_\xi(\rho)=K_{|m+\phi|}(\mathscr{K}_b\,\rho). \tag{23}$$

At this stage, we can observe that the solution (23) belongs to $\mathscr{D}(\mathscr{H}_{\xi,0})$. Thus, we can assume a solution having the form (15) for some ξ selected from the physics of the problem. To calculate the right-hand side of Equation (20) we need to consider the asymptotic representation of (23) in the limit $\rho\to 0$, which is given by

$$K_{|m+\phi|}(\mathscr{K}_b\rho)\sim\frac{\pi}{2\sin(\pi|m+\phi|)}\left[\frac{(\mathscr{K}_b\rho)^{-|m+\phi|}}{2^{-|m+\phi|}\Gamma(1-|m+\phi|)}-\frac{(\mathscr{K}_b\rho)^{|m+\phi|}}{2^{|m+\phi|}\Gamma(1+|m+\phi|)}\right]. \tag{24}$$

Using this result and Equation (21) and substituting them into Equation (20), we arrive at the following expression:

$$\lim_{R\to 0^+} R\frac{\dot{f}_{\xi,0}}{f_{\xi,0}}\Big|_{r=R}=\frac{\mathscr{A}_+}{\mathscr{A}_-}|m+\phi|=\frac{1}{2}gs\phi, \tag{25}$$

with

$$\mathscr{A}_\pm=R^{2|m+\phi|}\Gamma(1-|m+\phi|)\left(\frac{\mathscr{K}_b}{2}\right)^{|m+\phi|}\pm 2^{|m+\phi|}\Gamma(1+|m+\phi|). \tag{26}$$

Finally, substituting $\mathscr{K}$ given in Equation (11) into Equation (25) and solving for $\mathscr{E}$, we find the following expression:

$$\mathscr{E} = -\frac{2\hbar^2 \Lambda^{\frac{1}{|m+\phi|}}}{m_e R^2} - \hbar\Omega\left(m + \phi + \frac{s}{2}\right), \tag{27}$$

with

$$\Lambda = \frac{\Gamma(1+|m+\phi|)}{\Gamma(1-|m+\phi|)} \frac{\left(\frac{1}{2}gs\phi + |m+\phi|\right)}{\left(\frac{1}{2}gs\phi - |m+\phi|\right)} \geqslant 0. \tag{28}$$

For $\Omega = 0$, only the first term in Equation (27) remains and the explicit dependence on the element of spin is in the parameter Λ.

It is important to emphasize that the energy (27) depends only on the parameters involved in the model. In other words, no arbitrary parameters that come from the self-adjoint extension appear. This means that the self-adjoint extension parameter was automatically self-selected through the physical regularization procedure for the δ function in Equation (17). However, according to Refs. [28,44], it is possible to obtain an explicit form for the expression that characterizes the self-adjoint extension parameter in terms of the physical parameters of the problem. In our case, such an expression is found to be

$$\xi = -\frac{1}{R^{2\,|m+\phi|}} \left(\frac{\frac{1}{2}gs\phi + |m+\phi|}{\frac{1}{2}gs\phi - |m+\phi|} \right). \tag{29}$$

This relation shows us that parameter ξ can assume different values because of its explicit dependence on m, ϕ, s, g and the radius of the solenoid, R. Although the energy (27) is a simple expression, if we analyze it in more detail, we notice that the presence of rotation implies several unusual features. We find that the principal impact due to rotation is the appearance of positive energies. When there is no rotation, the spectrum (27) is purely negative. In the next section, we discuss in detail some profiles we can access from energy (27).

4. Numerical Analysis and Discussion of the Results

In this section, we investigate the changes in the energy of the particle when the physical parameters of the model are varied, in particular, the rotation and the Aharonov-Bohm flux. We shall show that the energy (27) can exhibit different profiles depending on the values of the parameters and the quantum number m. We made some numerical analyses to study the behavior of $\mathscr{E}$ as a function of ϕ and Ω. For $\Omega = 0$, the energy (27) is always negative, regardless of the values of the parameters involved. For $\Omega > 0$, there is a threshold value depending on the values we choose for the parameters, and the energy can be positive or negative. This effect is due to the presence of rotation. In Figure 1, we plot the energy as a function of the flux corresponding to the state with $m = 0$. In Figure 1a–c we consider three different intervals for Ω and $R = 5$ nm. We can see that the energy is zero when $\phi = 0.5$.

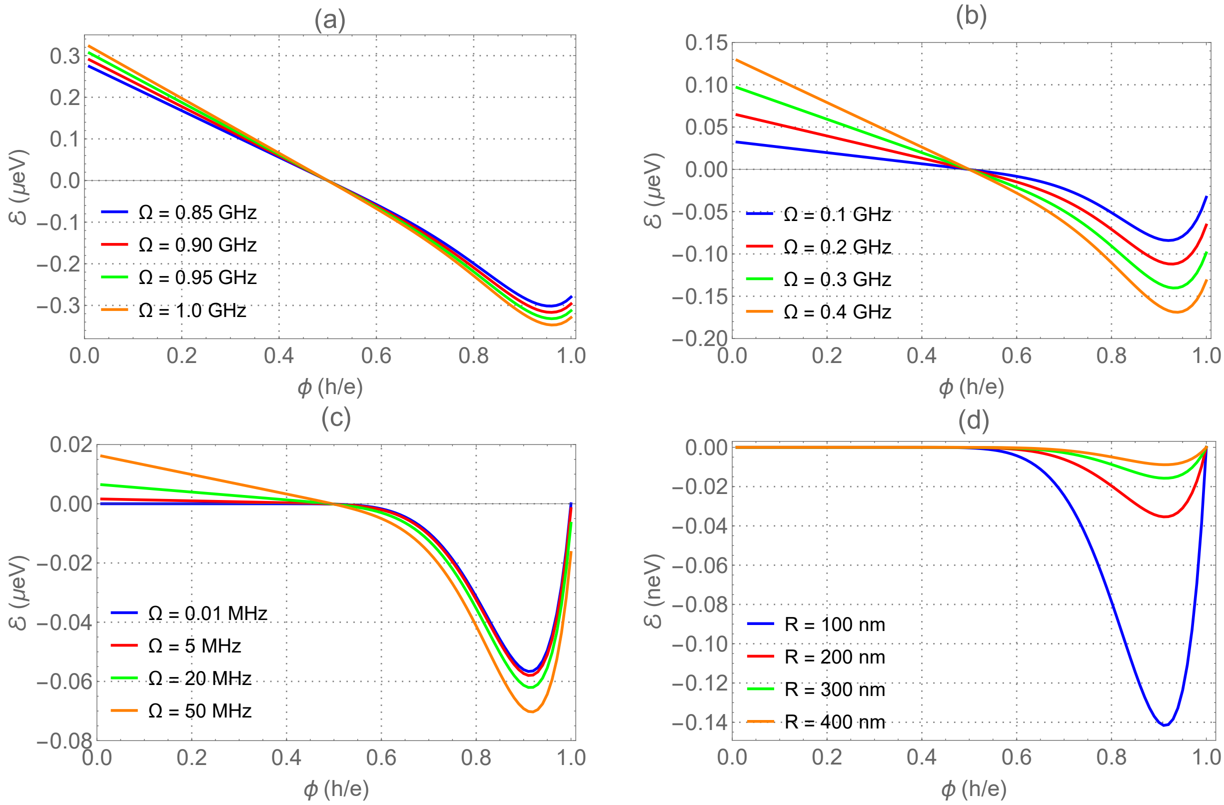

Figure 1. Energy as a function of the flux. In Figure (**a**–**c**) the profile considering three different frequency ranges for $m = 0$ and $R = 5$ nm. In Figure (**d**), we consider some values of R for $m = 0$ and $\Omega = 0$.

In the interval with Ω between 0.85 GHz and 1.0 GHz [2], we observe that the energy changes more slowly (Figure 1a). In this interval, we can also see that for zero flux, the energies are positive, with magnitudes of the order of 0.3 μeV. When we investigate the profile in the interval between 0.1 GHz and 0.4 GHz, the energy values for null flux are smaller and different in magnitude, as well as the energies for $\phi > 0.5$ (Figure 1b). For decreasing values of Ω, both positive and negative energies tend to decrease (Figure 1c). In Figure 1d we show the particular case when $\Omega = 0$ and some values of R. We can see that when R decreases, the energy tends to increase. For values of ϕ in the range between 0 and 0.5, the values of energies are very small, but not null. Table 1 shows some values for energy in this range. The second column refers to the case in which $R = 100$ nm (blue line), while the third column corresponds to $R = 400$ nm (orange line). When we investigate the behavior of the energy as a function of Ω, we find a linear profile for any value of the parameters (Figure 2). However, for fixed R and m there is a threshold value for ϕ and Ω that can provide either positive or negative energy. This can be easily identified by directly comparing the two energy terms in Equation (27). Figure 3 displays a more general profile of the energy as a function of both flux and rotation. We can see the linear behavior (exhibiting positive and negative energies) in the plane with constant ϕ and the linear-parabolic profile in the plane in which Ω is constant.

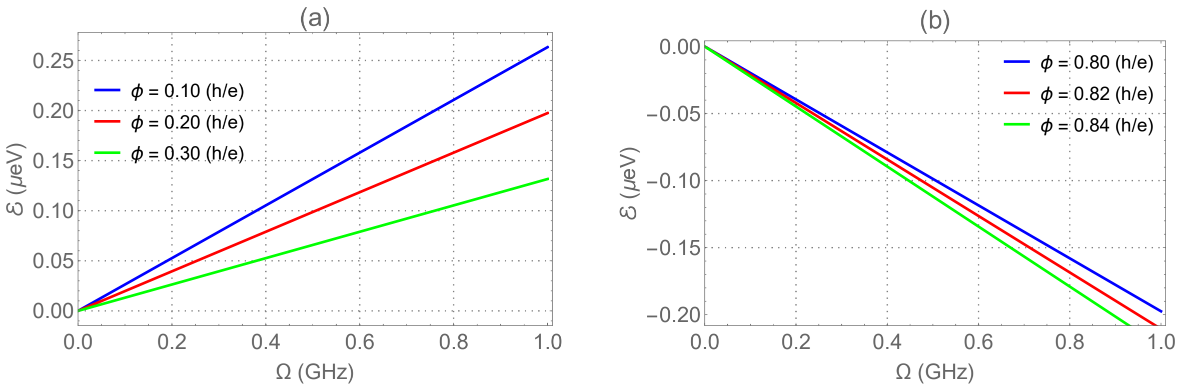

Figure 2. Energy as a function of Ω for two different ranges of ϕ with $m = 0$ and $R = 5$ nm. For values of ϕ smaller than 0.5 the energies are positive (panel (**a**)) while for larger values they are negative (panel (**b**)).

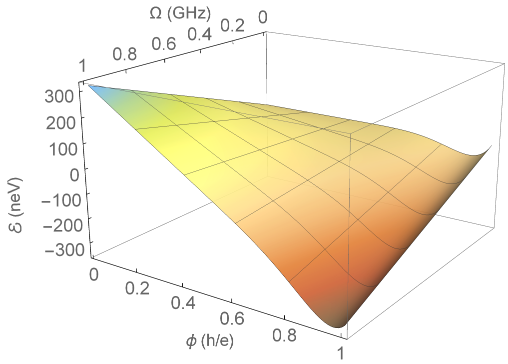

Figure 3. Energy as a function Ω and ϕ for $m = 0$ and $R = 5$ nm. For values of ϕ larger than 0.5 the energies are negative while for smaller values they are positive. This effect is due to the presence of rotation.

Table 1. Some values for the energy $\mathscr{E}$, corresponding to the case displayed in Figure 1d, where we consider $\Omega = 0$. In the second and third columns we show the energies with $R = 100$ nm and $R = 400$ nm, respectively.

ϕ (h/e)	$\mathscr{E}$ (neV)	$\mathscr{E}$ (neV)
1.0000×10^{-2}	0.0000	0.0000
2.0000×10^{-2}	-6.2385×10^{-174}	-3.8990×10^{-175}
2.9999×10^{-2}	-5.7152×10^{-115}	-3.5720×10^{-116}
4.0000×10^{-2}	-1.7292×10^{-85}	-1.0807×10^{-86}
5.0000×10^{-2}	-8.4369×10^{-68}	-5.2731×10^{-69}
5.9999×10^{-2}	-5.2267×10^{-56}	-3.2666×10^{-57}
7.0000×10^{-2}	-1.3833×10^{-47}	-8.6460×10^{-49}
8.0000×10^{-2}	-2.8693×10^{-41}	-1.7933×10^{-42}
8.9999×10^{-2}	-2.3479×10^{-36}	-1.4674×10^{-37}
0.1000	-1.9991×10^{-32}	-1.2494×10^{-33}
0.1100	-3.2825×10^{-29}	-2.0516×10^{-30}
0.1200	-1.5685×10^{-26}	-9.8034×10^{-28}
0.1300	-2.8999×10^{-24}	-1.8124×10^{-25}
0.1400	-2.5423×10^{-22}	-1.5889×10^{-23}
0.1499	-1.2269×10^{-20}	-7.6686×10^{-22}
0.1600	-3.6457×10^{-19}	-2.2785×10^{-20}
0.1700	-7.2650×10^{-18}	-4.5406×10^{-19}
0.1799	-1.0377×10^{-16}	-6.4860×10^{-18}
0.1900	-1.1199×10^{-15}	-6.9995×10^{-17}
0.2000	-9.5226×10^{-15}	-5.9516×10^{-16}

5. Conclusions

In the present manuscript, we have investigated the role of the spin degree of freedom in the context of the Aharonov-Bohm effect for bound states. In particular, we incorporate the rotating effects in the description of that problem, which have revealed novel relevant physical aspects. Because of the point interaction term present in the Hamiltonian of the model, we have employed the self-adjoint extension method to solve the radial equation of motion for bound states. We have examined how the Aharonov-Bohm effect, combined with the presence of a non-inertial frame, modifies the particle motion when the spin degree of freedom is taking into account. The combination of the Aharonov-Bohm effect and rotation produces an asymmetric behavior regarding the spin states. More specifically, the spin states labeled by $s = \pm 1$ have different contributions to the energy spectrum of the problem. In particular, we have shown that only for the spin state with $s = -1$, there are bound state solutions for the Aharonov-Bohm effect when the particle is in a rotating frame. Usually, we expect that both the Zeeman term and the spin-rotation coupling would be related to the spin-breaking degeneracy solely. However, the self-adjoint extension method employed here shows us that the combination of the Aharonov-Bohm effect and rotation, considering the spin degree of freedom, reveals novel effects on the energy spectrum of the particle. Hence, the eigenvalues of the spin operator σ_z correspond to situations dramatically different concerning the behavior of the system. It shows us the relevance of the spin degree of freedom in the description of the Aharonov-Bohm effect. Thus, we believe that this manuscript can contribute to the understanding of how the spin degree of freedom affects the quantum mechanical description of a system in the presence of the Aharonov-Bohm effect.

As a final word, we think that the self-adjoint extension method can be helpful for examining condensed matter systems in the presence of the Aharonov-Bohm effect and the spin degree of freedom. In this case, we could employ the spin degree of freedom to manipulate the energy spectrum of a given system, since the method can provide either bound state solutions or scattering solutions depending on which spin states $s = \pm 1$ we consider. We will continue studying these scenarios in future studies.

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Notes

1. Value for the deviation in the year 2008.
2. We chose that magnitude based on the literature involving rotation of nanosystems. See, for instance Refs. [66,67].

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