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Multifractality of Inelastic Events

Igor M.Dremin
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois, U.S.A.
Permanent address:
P.N.Lebedev Physical Institute, Moscow, U.S.S.R.

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Abstract

The multifractal approach to multiparticle processes at high energies is advocated. It provides the quantitative measure of inhomogeneity of particles' positions in the phase-space volume. Its relationship to intermittency and large fluctuations is briefly discussed.

1 Introduction and main ideas

With the advent of SSC and nuclei colliders we are entering the new era of tremendous multiplicities with high statistics. The adequate Monte-Carlo simulation becomes sophisticated and,even more important, the data handling and presentation is not so clear and transparent. One should look for simple quantitative measures to compare theory and experiment.

At present, any classification of multiple production processes stems from prejudices inspired by numerous theoretical models. Even being correct, they often suffer from controversies and insufficiently strictly defined borderlines between different physical phenomena. In some way, it seems to be an inevitable consequence of such a classification. Thus one speaks about pionization, fragmentation, inelastic diffraction, jets, spikes etc. Nevertheless, with all its shortcomings, this classification is very useful.

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I propose to look upon inelastic processes from a somewhat different point of view[1,2]. Namely, they can be considered as purely geometrical objects and, correspondingly, classified as such ones. This mathematical approach provides some quantitative criteria. Being applied to the same physical processes, both classifications should add to each other and, I hope, to our understanding of the dynamical origin of different phenomena.

To decipher these statements, let us treat an inelastic event as a set of points (in the three-dimensional momentum space) each of which corresponds to the end-point of the momentum vector of a particle produced. One can ask whether the sets for different events can be described by any measure and classified according to it. Actually, the similar problems appeared when treating purely mathematical objects like the Cantor set or solving some non-linear physical problems with strange attractors. The generalized fractal dimensions were proposed to be used for the description of such objects. Fractality provided some new insights into their nature (for the review, see[3]).

Inspired by the analogy, I propose to apply the fractal dimension technique to multiparticle production processes. As explained above, their geometry is defined by the dynamics involved. The most far-reaching program includes the classification of *individual* events according to their dimension what helps to reveal the underlying dynamical mechanisms. Our hope for a success is supported by the fact that all nowadays-popular models of soft multiparticle production use the cascading dynamics which,as well known, can be characterized as a self-similar branching process with definite fractal dimensions.

Independently of our theoretical prejudices, any classification is important by itself. In our case, we hope to learn about the substructure of inelastic events, their fluctuations, related to intermittency and, possibly, to the spatial inhomogeneity of the regions of energy concentration during the collision. The measure singularities are defined by the dynamics and they are important for understanding of the non-linearity of effective equations governing the process.

With these goals in mind, we shall proceed through the mathematical interlude to physical definitions and concrete measures of inelastic processes (with simplest applications of them) to be proposed for experimentalists dealing with multiple production. At this stage there are no final recipes and the whole procedure is still developing. The hope for its success is based on positive experience gained in many applications to other branches of physics.

2 The mathematical interlude

The fractal dimension can be considered as a generalization of a notion of the commonly used topological dimension to non-integer numbers. If the latest one is defined as the number of independent directions which characterize the object, it is hard to imagine the non-integer dimension. However, one can use another definition due to Kolmogorov or Hausdorff. According to it, the fractal dimension D_F is defined as a value which provides the finite limit

$$0 < lim_{\epsilon \to 0} N(\epsilon)\epsilon^{D_F} < \infty \tag{1}$$

to the product of the minimal number of covering the object hypercubes $N(\epsilon)$ with a size $l = \epsilon$ (Kolmogorov definition) or $l \leq \epsilon$ (Hausdorff definition) and of a factor ϵ^{D_F} when $\epsilon \to 0$.

It becomes more apparent for physicists if one considers the relation between the size of an object l and its mass M as a scaling law:

$$M \sim l^{D_F}. \tag{2}$$

For usual objects it coincides with the topological dimension (for a line $D_F = 1$, for a square $D_F = 2$ etc)

Now, let us consider the so-called Koch curve (Fig.1). It is built up according to the following algorithm. One splits the straight line interval into three equal pieces, builds up the equilateral triangle on the middle piece and omits this piece. The whole procedure is repeated on four intervals left and so on. The resulting self-similar curve is a fractal with the dimension $D_F = ln4/ln3$ which is easily calculated according to eq.(2).

If one just omits the middle pieces without building up the triangles, one gets the so-called Cantor set (Fig.2) of the infinite number of points with the dimension $D_F = ln2/ln3$.

The probability $p_i(l)$ to be in one of the hypercubes $N(l)$ is proportional to l^{D_F} at small l. Therefore the sum of their moments for a fractal is given by

$$\sum_i p_i^q(l) \sim l^{qD_F}(D_F = const) \tag{3}$$

The multifractals generalize the notion of fractals since for them

$$\sum_i p_i^q(l) \sim l^{\phi(q)} \tag{4}$$

where

$$\phi(q) = q d_{q+1}. \tag{5}$$

d_{q+1} are called the Renyi dimensions[4] and depend on q (generally, they are decreasing functions of q for multifractals[3]).

The rough analogy of the transition from fractals to multifractals is the replacement of a homogeneous stick by an inhomogeneous one when one has to change the mass definition by introducing its measure—the density distribution $\rho(x)$:

$$M = \int_0^l \rho(x)dx. \tag{6}$$

The deficiency of this analogy is the absence of the self-similarity.

What concerns the multifractals, the probability to be in the hypercube $N_i(l)$ is defined now by

$$p_i(l) = \int_{N_i(l)} d\omega(x) \tag{7}$$

where $d\omega(x)$ is the natural measure. Let us group all the boxes with the singularity α ($p_i(l) \sim l^\alpha, l \to 0$) into a subset $S(\alpha)$. α is called a local mass dimension. The number of boxes $dN_\alpha(l)$ needed to cover $S(\alpha)$ is

$$dN_\alpha(l) = d\rho(\alpha) l^{-f(\alpha)} \tag{8}$$

where $f(\alpha)$ is a fractal dimension of the set $S(\alpha)$ and it is related to Renyi dimension. One gets:

$$\sum_{i=1}^{N_i(l)} p_i^q(l) \sim \int d\rho(\alpha) l^{\alpha q - f(\alpha)} \tag{9}$$

wherefrom one gets by the saddle-point method:

$$d_q = \frac{1}{q-1} \min_\alpha(\alpha q - f(\alpha)) = \frac{1}{q-1}(\bar{\alpha} q - f(\bar{\alpha})) \tag{10}$$

with $\bar{\alpha}$ defined from the equation:

$$\frac{df}{d\alpha}|_{\alpha=\bar{\alpha}} = q(\bar{\alpha}). \tag{11}$$

The notion of the Renyi dimension generalizes the fractal dimension:

$$d_0 = D_F = -\phi(-1), \tag{12}$$

the information dimension:

$$d_1 = D_1 = \phi'(0), \tag{13}$$

the correlation dimension:

$$d_2 = \nu = \phi(1) \tag{14}$$

with $D_F > D_1 > \nu$ (i.e. $dd_q/dq < 0$).

What can be revealed by the dimension analysis?

- The number of the degrees of freedom of a system n_f is given by the integer part of the fractal dimension

$$n_f = [D_F] + 1 \tag{15}$$

- The measure singularities $f(\alpha)$ are determined from the formula (10).
- The fractal dimensions are related to the non-linearities of the underlying dynamical equations. However, this relation is indirect and usually is established by computer calculations.
- It can provide some knowledge about the regions of the most active energy dissipation, intermittency and the fractal space-time structure of the interaction region (for example, in the quark-gluon plasma).
- The properties of the cascade models are closely related to the Renyi dimensions of the cascades.
- The last but not the least to us is an attempt to use dimensions for classification of inelastic processes which is described below.

3 The multifractal classification of inelastic processes

The multifractal approach is widely used[3] in many branches of physics— turbulence, spin glasses, strange attractors, cloud formation, percolation, polymers etc.

The idea to apply it to multiple production [1,2] was inspired by the similarity with strange attractors. Let us consider any inelastic process with many hadrons produced as a set of points within the three-dimensional phase-space volume. Each particle corresponds to the end-point of its momentum vector. One can ask whether these sets can be classified in a similar way to classification of strange attractor' points or like a Cantor set i.e. as geometrical objects with the geometry given by the dynamics.

To simplify the initial stage, I propose to consider the projection of the sets on the rapidity y axis. Then the probabilities $p_i(l)$ can be defined in the following way:

$$p_i(l) = \frac{1}{n-1}\sum_{j\neq i}\theta(l - |y_i - y_j|) \tag{16}$$

where n is the multiplicity of an event, $y_{i,j}$ are the rapidities of particles i, j (normalized to the maximum available rapidity interval). For an n-particle event, the moments are wrtitten as

$$C_q(l) = \frac{1}{n}\sum_{i=1}^{n} p_i^q(l) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n-1}\sum_{j\neq i}\theta(l - |y_i - y_j|)\right)^q \tag{17}$$

and at small l should behave as

$$C_q(l) \sim l^{\phi(q)} = l^{qd_{q+1}}. \tag{18}$$

Ideally, the definite Renyi dimension should be ascribed to each event and all events can be classified according to the values of the dimensions.

In practice, there arise some complications. First of all, the mathematical procedure is valid at $n \to \infty$ while we have to deal with events at finite multiplicities n which are not yet large enough. The only remedy is to use as high multiplicities as possible.

Second, as for any physical system one can not consider the mathematical limit $l \rightarrow 0$ but should stop at the values of l which are physically meaningful. (For example, for usual objects one is not allowed to use the size close to the molecular sizes.) On the other side, the lengths close to the maximal value also have nothing to do with the fractal dimensions. Therefore, the choice of the "meaningful range" of l to determine the Renyi dimensions depends on our intuition and experience.

As an example how to overcome the difficulties, let us consider two inelastic events of pp-interaction at 400GeV/c with 20 charged particles produced. One of them is rather homogeneous on the (pseudo)rapidity axis while another one is a representative of the so-called "ring" or "spike" events where many particles are grouped nearby. The moments $C_1(l)$ drastically differ for them (see Fig.3) with a rather smooth behavior for the homogeneous event (number 1 in Fig.3) and more irregular shape for the ring event (number 2). Both of them do not obey the simple scaling law (17) in the whole region of l but satisfy it in the restricted region of l from .2 to .4. If one chooses a "meaningful interval" near the point $l_m = .3$ one gets completely different values of the Renyi dimensions for the two events ranging from almost 1 for the homogeneous event to about .1 in second case. They provide the quantitative measures for our definition of the homogeneous (spread along the line) and spike (concentrated near the point) events. Whether the q-dependence of the Renyi dimensions (Fig.4) is a signal of multifractality or an artifact of the finite multiplicity should be studied separately but first possibility seems preferable. The height of the plateau and its starting point (at $l \sim .2$ in Fig.3) show the strength and the width of the spike.

Thus we see how the proposed method is used for a separation of homogeneous and spike events. One needs larger statistics of high-multiplicity events to classify them according to the proposal. Some preliminary results[5] are shown in Fig.5 for π^+p-interactions at 250GeV/c. The distributions of the whole set of events with 10 charged particles and of the spike events over the correlation dimension d_2 differ strongly from each other as expected. They are rather stable within the meaningful interval of l.

Let me note that alternative (but not so simple, in my opinion) methods can be used. In particular, those are the studies of the behavior of multiplets of points on the rapidity axis[1,3] or the multiplicity dependence of moments of intervals between neighbors[6,7,8]. In the latest case, one defines the moments

$$P_\gamma(n) = \frac{1}{n}\sum_{i=1}^{n}\delta_{i,min}^{\gamma} \tag{19}$$

of the minimal distances in rapidities

$$\delta_{i,min} = min(y_i - y_{i-1}; y_{i+1} - y_i). \tag{20}$$

According to [6] they should behave as

$$P_\gamma(n) \sim n^{-\gamma/D(\gamma)} \tag{21}$$

with

$$D(\gamma = (1-q)d_q) = d_q. \tag{22}$$

Therefore if one chooses events with the same values of d_q for different multiplicities n_1 and n_2 and calculates $P_\gamma(n)$ at the values $\gamma = (1-q)d_q$ one gets the relation without any free parameter:

$$ln(\frac{P_\gamma(n_1)}{P_\gamma(n_2)})/ln(\frac{n_2}{n_1}) = 1 - q. \tag{23}$$

It would be a powerful relation if it did not suffer from some diseases common to the whole method. It is valid for pure multifractals obeying the law (17) in the whole region of rapidities while the real events show up such a behavior in rather narrow "meaningful" interval (see Fig.3). One can not apply the method at $\gamma < 0$ if two particles have equal rapidities, but it can be easily generalized to the distances in the three-dimensional phase space. Anyway, some generalization of the simplest procedure (18)-(22) is needed.

I would like to stress once again that the proposed method is aimed at classification of individual events to use afterwards the subsets with different dimensions separately while other methods exploit an inclusive approach.

4 General discussion

It has been shown above that the Renyi dimensions can be used to separate the spike events from the whole set of data. Combined with other methods of spike identification, they can provide deeper understanding of the events substructure.

Evidently,there should exist some correlation between the Renyi dimension and intermittency[9]. One is tempted to assume that intermittency becomes larger at smaller Renyi dimensions. It would be appealing to verify it experimentally.

In macroscopic problems, large fluctuations are commonly related to inhomogeneities within the space volume and to non-linearities of the underlying dynamical equations. In our case there seems to be no straightforward way to such a conclusion. However, some arguments in favor of it are provided by very complicated motion of exchanged partons[10] and by the polymer-like model of the vacuum due to instantons[11].

Another interesting problem related to the non-linearity of the dynamical equations is whether the basic dynamics is stochastic or regular. The fluctuations should appear randomly for stochastic processes and could prefer definite positions in regular dynamics (as happens, for example, for Cherenkov gluons[12]). The preliminary results of our analysis of spike positions in *pp*-interactions at 205GeV/c and 360GeV/c[5] show some irregularities over the smooth background i.e. possible coexistence of both mechanisms (see Fig.6).

To conclude, there is some hope that quantitative characteristics of inelastic processes provided by the multifractal analysis can shed some light on the dynamics of hadron interactions and I would appeal to experimentalists to be innovative in correlation measures and to pay more attention to fluctuations in multiparticle production.

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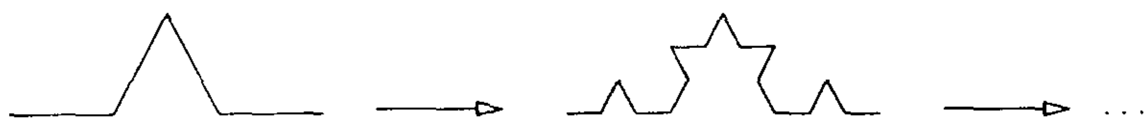

Fig.1 The Koch curve is built up step by step replacing the middle part of the line by two sides of the equilateral triangle formed on the omitted part

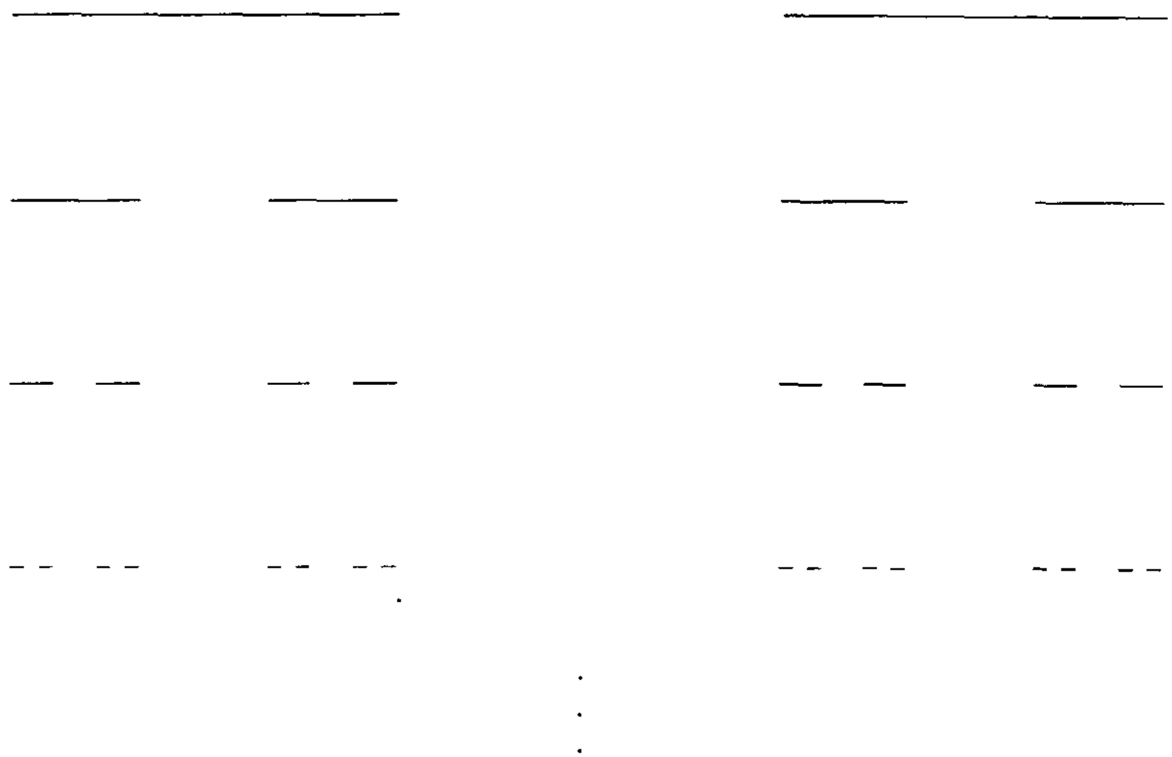

Fig.2 The Cantor set is formed omitting all the middle parts of the lines

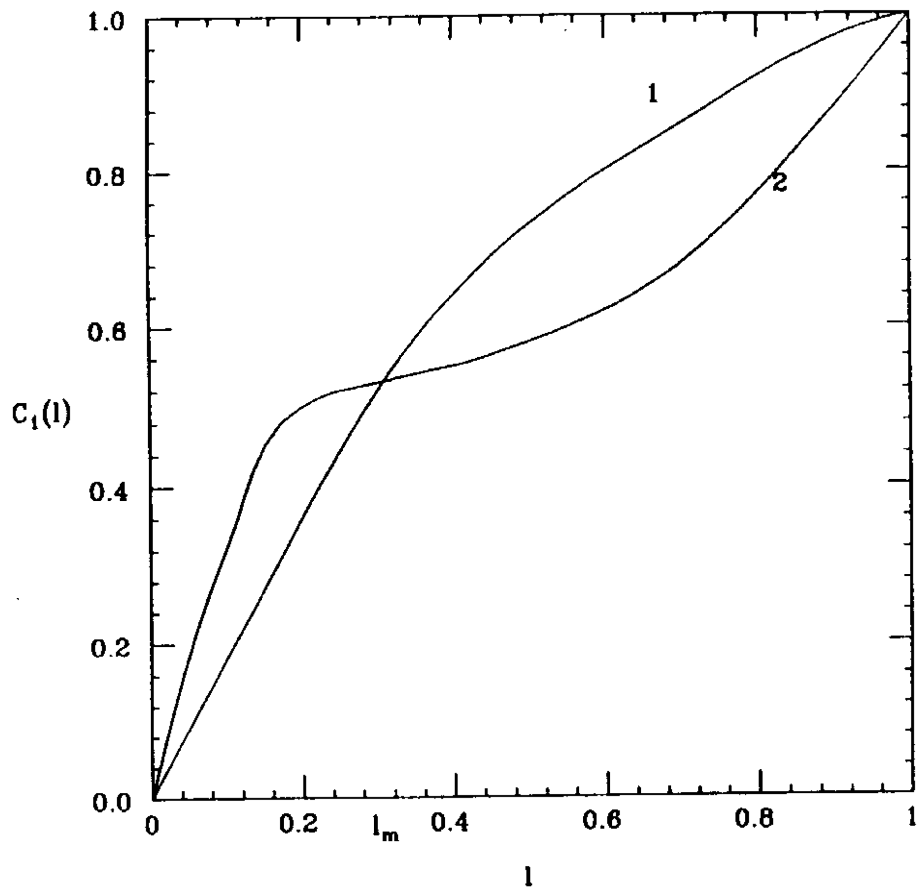

Fig.3 The moments $C_1(l)$ for two inelastic pp−events at 400GeV/c with 20 charged particles produced show different behavior for the smooth distribution of particles on the rapidity axis (1) and for spike event (2)

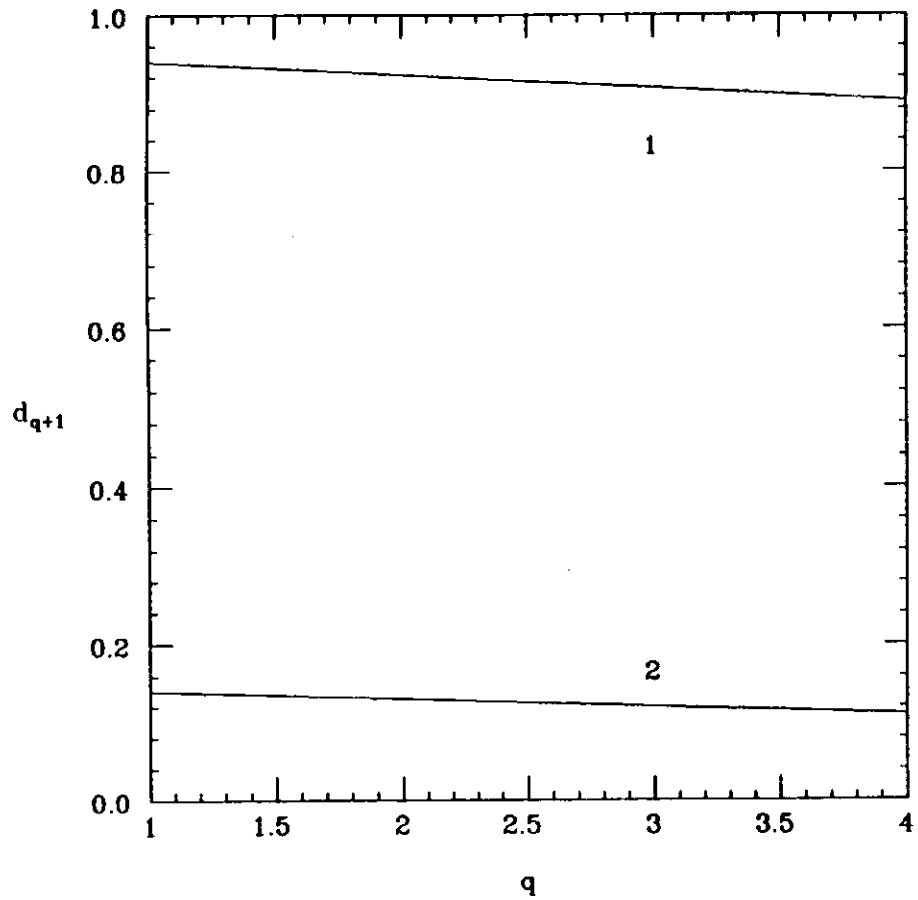

Fig.4 The Renyi dimensions for two events differ in values and their q-dependence favors multifractal interpretation

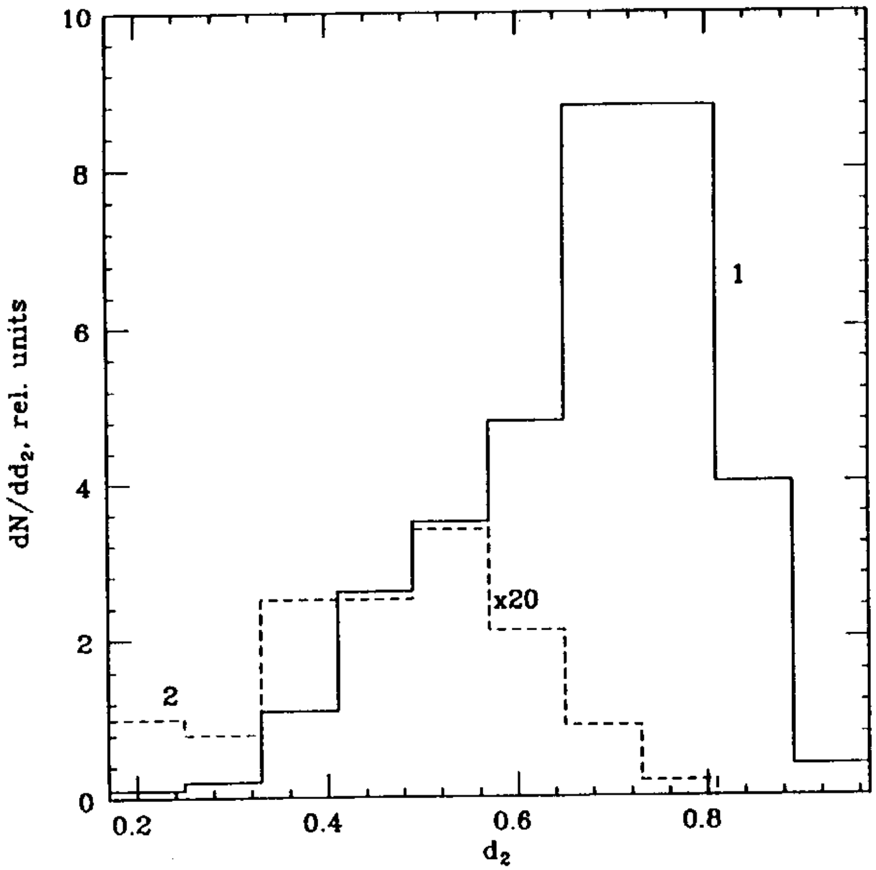

Fig.5 The distribution of 6263 π^+p–events with ten particles produced at 250GeV/c over the Renyi dimension shows rather large average dimension (solid line) and smaller values for spike events (dashes)

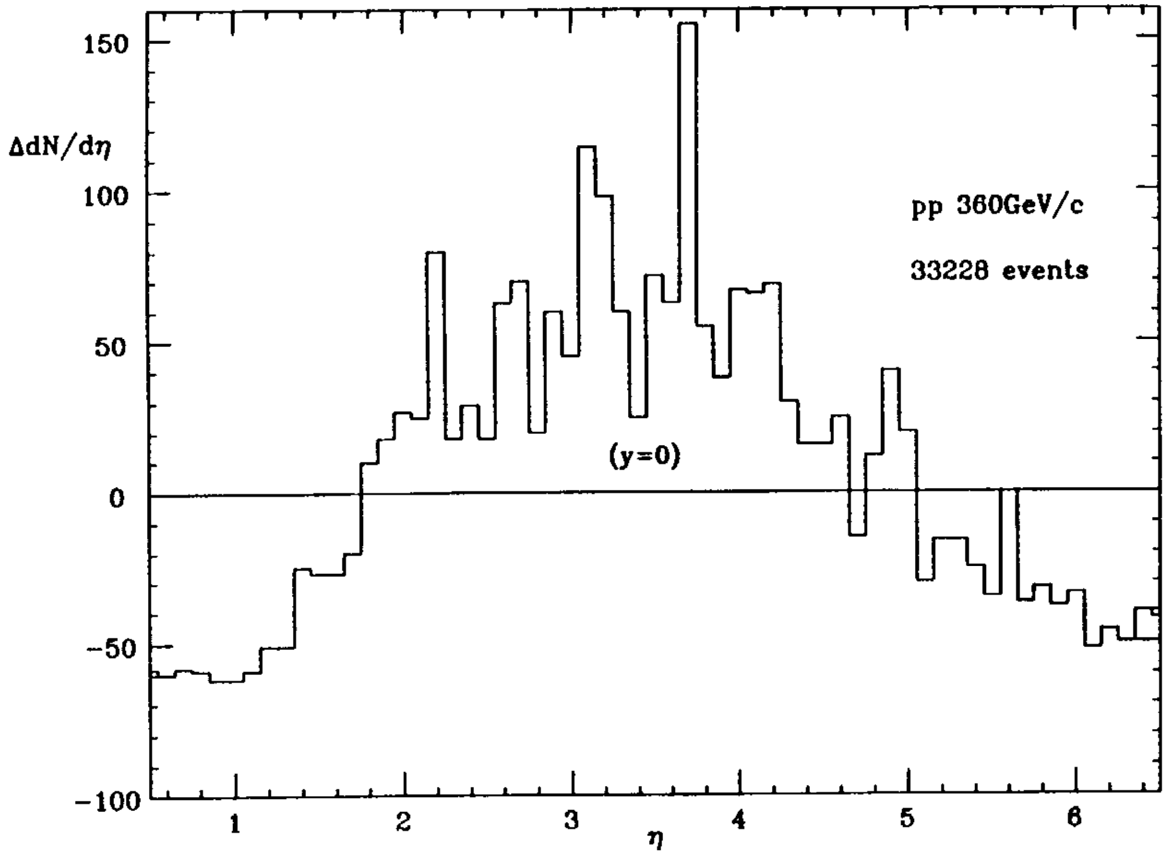

Fig.6 The difference between particle distributions for spike events and for the whole set of data (normalized to the same value) shows that dense groups of particles prefer the central rapidity region and favor some irregularities at special values of the rapidity