Thèse (Dissertation)

"Topology and mass generation mechanisms in abelian gauge field theories"

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Abstract

Among a number of fundamental issues, the origin of inertial mass remains one of the major open problems in particle physics. Furthermore, topological effects related to non perturbative field configurations are poorly understood in those gauge theories of direct relevance to our physical universe. Motivated by such issues, this Thesis provides a deeper understanding for the appearance of topological effects in abelian gauge field theories, also in relation to the existence of a mass gap for the gauge interactions. These effects are not accounted for when proceeding through gauge fixings as is customary in the literature. The original Topological-Physical factorisation put forth in this work enables to properly identify in topologically massive gauge theories (TMGT) a topological sector which appears under formal limits within the Lagrangian formulation. Our factorisation then allows for a straightforward quantisation of TMGT, accounting for all the topological features inherent to such dynamics. Moreover dual actions are constructed while preserving the gauge symmetry also in the pr[...]

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Topology and mass generation mechanisms in abelian gauge field theories

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“ Sa rencontre ressuscitait tout le passé ; non pas qu’elle l’eût oublié : mais la puissante joie de vivre de la jeunesse, dont la chair expulse si vite les corps étrangers, adoucit le contour des mauvais souvenirs, en obscurcit les cruelles couleurs, et finit par leur donner l’irréalité d’une histoire lue dans un livre. ”

M. Pagnol, *“Manon des Sources”*, (1963).

Ces dernières années d’aucuns ont analysé un curieux spécimen, une espèce de scientifique, sportif à ses heures et artiste refoulé. Sa principale occupation ces dernières années a été l’élaboration d’une thèse. Bien étrange occupation que voilà, qui ne s’est cependant pas concrétisée entièrement seul…

Merci à Jan Govaerts de m’avoir offert une chance de grandir en tant que chercheur. Son aide a été précieuse pour déjouer les rouages administratifs tant au FRIA qu’à l’UCL. Merci de m’avoir lancé sur un projet que j’ai adoré, pour ses commentaires précieux lors de la relecture de nos articles et de cette thèse. Enfin la collaboration finale pour la construction des défauts topologiques a été des plus agréable et efficace.

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Le gentil et charmant bipède Bruno Bertrand, espèce rare à protéger, peut être classifié en plusieurs groupes.

Le club des physiciens de CP3, des cyclotrolls bien élevés et pas si méchants que ça (Simon, Xavier, Séverine, Philippe, Céline,. . .).
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Le club Teletubbies de l'Université de Mons (Georges, Michel, Nicolas).
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Le club des artistes dramatiques pour tous ses beaux moments de vie partagés (Laëti, Gaël, Caro, Barbaro, Charles, Lucie, Francesco, Mara, et la troupe de Jean-Yves).
L'ASBL Equithéa pour le ressourcement loin du superficiel et aseptisé Louvain-la-Neuve.

Cet omnivore de la vie est plutôt sociable, avec une certaine aspiration à partager avec ses semblables.

Le chemin d'une vie serpente des rencontres, des hasards, des carrefours, des routes droites et des chemins de traverse. A ces quelques personnes qui ont été importantes... A ces quelques personnes qui sont clef... A ces quelques personnes dont le parfum est tenace ou dont l'effluve est éphémère. Docteur Hupez, Eric M., Philippe Carlier, Maïté S., Mr Bailly, Olivier M., Sabrina B., Eléonore D., Alice, Jean-Robert H., Mara P., Serafina, Lodia P., Laure-Anne M., Charlène J.

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“Transformer le plomb en or
Tu as tant de belles choses à vivre encore
Tu verras au bout du tunnel se dessiner un arc-en-ciel
Comme la brume voilant l’aurore
Il y a tant de belles choses que tu ignores
La foi qui abat les montagnes, la source blanche dans ton âme
Penses-y quand tu t’endors : l’amour est plus fort que la mort.”

F. Hardy, “*Tant de belles choses*”, (2004).

A ma petite Sicilienne, à sa fille, à son père, à ses petits-enfants. À toutes ces personnes à qui vous avez tendu la main. Je voudrais vous dire que la chaîne de la douleur et de la violence est brisée, que vous allez construire quelque chose de beau sur des racines solides. Ces phrases me font terriblement penser à vous.

“La blessure de la misère, c’est terrible. J’ai revêtu le manteau de la misère trop longtemps. Et quand on arrive à le retirer, ou il vous en reste encore quelques lambeaux, ou vous avez arraché un peu de vous-même. C’est une blessure qui ne guérit pas. On reste écorché vif”

R. Devos

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“Je vous souhaite des rêves à n’en plus finir et l’envie furieuse d’en réaliser quelques-uns.”

J. Brel

J’espère que vous m’accompagnerez dans le nouveau chapitre de ma vie que je vais entamer, après avoir tourné cette dernière page.

CONTENTS

Preamble

A harmonic world

From the day Human beings were standing up, they have never ceased seeking their origins. This ultimate search motivates Human beings to reach towards an understanding of the mysteries of Nature and this quest takes on many forms. Harmony of sounds gave birth to music, that of colors to painting, that of body to theatre and dance, that of language to eloquence. In the antique world these realms of human creativity were then all considered on a par with the birth of scientific method. Pythagore de Samos for instance tried to describe the orbits of celestial bodies in terms of musical harmony. What distinguishes physics as being an exact science is the powerful mathematical formulation of its models and the test of these models through the experimental method. With the emergence of modern science the search of musical harmony as a fundamental paradigm to describe physical phenomena became a chimera. Nevertheless periodic phenomena like pendulum motion or acoustic waves are described by a very common quadratic frequency term, or quadratic “mass” term. Hence a vocabulary which is very reminiscent of harmony ; words like “harmonic oscillator” and frequency are to remain in mathematics and physics even though the physical concepts they evoke turn out to be very different. In particular within the context of particle physics, the harmonic oscillator, through Fourier modes decomposition, is a basic building block towards the identification of the physical spectrum of quantum field theories and their interactions.

The concept of field was first introduced by Michael Faraday in [1, 2] in a very intuitive way in order to explain the electric and magnetic couplings to point charges and electric currents: "The point (...) was, whether it was not possible that the vibrations which in a certain theory are assumed to account for radiation and radiant phenomena may not occur in the lines of force which connect particles, and consequently masses of matter together ; a notion which, as far as it is admitted, will dispense with the ether (...)". Faraday reasoned by deduction in terms of mental pictures with a constant appeal to the experimental research he pursued himself. Maxwell managed to justify Faraday's intuition mathematically speaking by introducing the notion of electromagnetic field of which the dynamics is described by what is known in our contemporary view as the Maxwell equations [3].

As of today models of particle physics are constructed from quantum gauge field theories. This implies that to each spacetime event there corresponds a quantum harmonic oscillator. These harmonic oscillators are all coupled to one another in a manner consistent with the required Lorentz covariance. A particle then is defined as a relativistic propagating energy-momentum *eigen*-quantum of that field. The introduction of gauge symmetry in (quantum) field theories enables to explain (perturbatively) the local interactions of particles. Indeed the realisation of symmetries requires the existence of gauge bosons, which mediate the interactions from one spacetime event to another. However global effects related to topological considerations on the one hand, and non perturbative field configurations on the other hand, are poorly understood in those gauge theories of direct relevance to our physical universe.

The context of our research work

Topological aspects are playing a central role in many physical phenomena and theoretical models : topological defects and phase transitions, Aharonov-Bohm effect, fractional statistics, models with compactified extra space dimensions, topological (quantum) field theories, etc. Indeed these physical effects may not be described locally and refer thus to global aspects. These global aspects are for instance the homology of the underlying manifold on which the theory is defined, the homotopy of the compact submanifold defining holonomies or global boundary conditions, and likewise in the spaces of field configurations. Hence global variables (or modes of zero momentum) are "topological invariants" associated to cohomology groups while physical configurations are sometimes classified according to a "topological number" like the winding number, related to homotopy groups. However although (quantum) gauge field theories are well understood locally, difficulties may arise as soon as global aspects, related to low energy configurations, are taken into account. A way of eluding this issue is by taking the lead from some theories of condensed physics which are in some cases non relativistic limits of (gauge) field theories at very low energy, where the sector of global variables resides. Often our research used such techniques as a laboratory test for dealing with topological aspects of gauge field theories. However, in the case of interacting theories, the covariant extension of condensed matter theories to field theories is non trivial since then dynamical modes become mixed with global modes.

A better control of their global sector could enable one to gain more insight into some of the current open issues related to gauge field theories. A typical example of such issues is confinement and chiral symmetry breaking in quantum chromodynamics describing the strong interactions among quarks and gluons. Within this context, this quantum gauge field theory is well described at high energy through usual perturbation schemes in the coupling constant. However phenomenological evidence concurs with the expectation that in such a theory, quarks and gluons are confined into massive colourless bound states. Unfortunately there does not exist a definitive understanding of the actual dynamics responsible for this feature. The problem that arises in Yang-Mills theories at low energy is that the coupling constant becomes large and perturbative methods no longer apply. Hence the observed physical field configurations of colourless hadrons are by essence non perturbative since they reside in the strong coupling sector of the theory. Even in the case of pure Yang-Mills theory, a rigorous mathematical proof that glueball states of bound gluons develop a mass gap remains yet to be established. Many techniques have been put forward and developed in order address this type of issues. A non exhaustive list includes for instance

- *Lattice gauge theory*
  Lattice simulations have shown that Yang-Mills models exhibit confinement. However these techniques do not offer a genuine understanding of the underlying dynamics (see [4] for a recent review).

- *Effective field theories*
  Let us mention for example the dual superconducting model of chromoelectric strings introduced by 't Hooft. This technique consists in introducing a projection onto the Cartan subalgebra of the gauge group, this sector being expected to be the main culprit for confinement at low energy. This kind of projection techniques has known developments of interest through the Cho-Faddeev-Niemi decomposition (see [5, 6] and references therein). Notice the profound connection of abelian projection with other methods like field correlators or compact QED (see [7] for a review). Let us also mention the recent introduction of the spin-charge decomposition in [8, 9].

- *Duality properties of gauge field theories and string theory*
  A recent avenue of investigations through Seiberg-Witten duality and holographic QCD, for example.

Whatever their category, all these techniques seem to show that topological aspects play a crucial part in the understanding of the mechanism of confinement. Since confinement arises at low energy where global modes become dominant, a complete understanding of that mechanism may require first to tame all the global topological effects involved in the ground state and next to be able to bridge the gap between local and global aspects as soon as dynamical effects are taken into account.

For example, physical states of the pure Yang-Mills theory in the confining phase are glueballs described by non perturbative field configurations. The original fields involved in the definition of any gauge theory do not generate physical configurations since these fields are not gauge invariant degrees of freedom. As a matter of fact, several approaches to isolate genuine physical degrees of freedom are available. The gauge fixing procedure effectively removes redundant gauge variant degrees of freedom. However such gauge fixing usually suffers Gribov problems, except in some exceptional cases. Another approach consists in constructing a factorised dual formulation. Indeed, following a convenient redefinition of the fields gauge variant degrees of freedom are decoupled from the physical ones. Although locally trivial, the sector of gauge variant variables may be sensitive to global effects. Therefore, important topological content may be lost when a gauge fixing procedure is applied, especially within the context of order in condensed matter physics or confining states in QCD. The purpose of our research is to show how these non trivial topological effects may arise and have direct consequences on the physical spectrum.

For the sake of simplicity we have chosen first to illustrate these notions within the simplest case of theories generating a mass gap where topological effects are of prime importance: the so-called topologically massive gauge theories. These abelian gauge theories describe the dynamics of a p-form and $d-p$ form in a spacetime of dimension $d+1$ coupled through a topological "BF"term. This term generates a mass for the physical degrees of freedom without breaking the abelian gauge invariances.

This work

This Thesis describes the original research I have pursued these last four years, some of which having already been partially published, see [10, 11]. Briefly speaking, this Thesis actually consists in two original issues addressed in careful detail:

- *How to properly define the formal limits in the coupling constants of gauge fields theories through which topological sectors appear ?*
- *What is the influence of this topological sector on the physical spectrum of the theory ?*

Furthermore in Science, every answer implies new questions. In particular we formulate in this Thesis two novel unexpected results of our research

- *The most famous mass generation mechanisms preserving the abelian gauge symmetry are related through an intricate network of dualities, modulo the presence of topological terms generating possible topological effects.*
- *Our analysis could open a new avenue towards the construction of generalised topological defects in any dimension.*

Our concluding remarks will summarise how the answers to the first two questions led to our new propositions.

CHAPTER 1

Aspects of abelian gauge field theories

The present introductory Chapter is dedicated to a general overview of abelian gauge field theories which focusses onto some issues which are addressed in this Thesis. The simplest example of all such theories, the Maxwell theory, is first extended to gauge fields of any tensorial rank. Another simple extension may also be obtained by transmuting the coupling constant into a real dynamical scalar field. The Euler-Lagrange equations of motion then describe electromagnetic fields propagating in a medium. Finally the most famous examples of local mass generation mechanisms preserving the abelian gauge symmetry are recalled.

Any gauge field theory possesses spurious degrees of freedom arising from the construction principle of gauge invariance. In abelian gauge field theories, the pure gauge degrees of freedom are the longitudinal ones which may be isolated through Hodge decomposition. The non explicitly covariant Hamiltonian formulation – which requires a complete analysis of constraints in the case of gauge theories – enables to select those among the physical degrees of freedom which are actually propagating. The first order Lagrangian formulation provides the link between the Hamiltonian and Lagrangian formulations and offers the great advantage of directly involving physical degrees of freedom. The study of the formal limits of the first order actions of abelian gauge theories is the first step of the programme developed in this Thesis of which one of the main purposes is the identification of topological sectors in gauge field theories.

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1.1 Maxwell theory

1.1.1 Usual Maxwell theory for the photon field

The familiar Maxwell Lagrangian density describing the dynamics of the photon field is a gauge theory defined in terms of the fundamental gauge connection $A_\mu(x)$,

$$\mathcal{L}_{\mathrm{M}}^{d+1} = -\frac{1}{4\,e^2} F_{\mu\nu}\, F^{\mu\nu} - A_\mu\, J^\mu\,, \tag{1.1}$$

where the Greek index $\mu = 0, 1 \ldots d$ refers to the Minkowski spacetime of dimension $(d+1)$ endowed with metric $\eta_{\mu\nu} \equiv \mathrm{diag}(1, -1, \ldots, -1)$. Indeed, for the sake of simplicity, the choice of the underlying spacetime manifold is restricted to a flat connected manifold but possibly not homotopically trivial. This Lagrangian density is expressed in terms of the field strength curvature

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,,$$

where ∂_μ is the derivative with respect to the vector coordinate x^μ while the matter current $J^\mu(x)$ is conserved : $\partial_\mu J^\mu = 0$. This latter continuity condition which directly results from the Maxwell equations,

$$\eta^{\nu\sigma}\, \eta^{\mu\rho}\, \partial_\rho F_{\mu\nu} = e^2\, J^\sigma\,,$$

is in fact related the invariance of the theory under the abelian $U(1)$ gauge transformations. If we consider $J^\mu(x)$ as the current associated to dynamical matter fields which couple to the gauge field, this conservation law is an expression of Noether's theorem.

As a matter of fact, abelian gauge transformations are defined as smooth maps

$$U : \mathcal{M} \to U(1) : (t, \vec{x}) \to U(t, \vec{x}) = e^{\mathrm{i}\alpha(t,\vec{x})}\,,$$

where the parameter $\alpha(x)$ of the transformation depends on spacetime coordinates. The transformation $U(t, \vec{x})$ acts on the connection $A_\mu(x)$ and its associated field strength curvature $F_{\mu\nu}(x)$ in the usual way:

$$A'_\mu = U\, A_\mu\, U^{-1} + \mathrm{i}\, U\, \partial_\mu U^{-1}\,, \qquad F'_{\mu\nu} = U F_{\mu\nu} U^{-1}\,. \tag{1.2}$$

In the case of a non simply connected space(time) manifold, single-valuedness of the transformations $U(t, \vec{x})$ implies that $\alpha(t, \vec{x})$ may be decomposed into a periodic and a non periodic part[1]. Hence, this general notation covers all the possible gauge transformations including the ones which are not connected to the identity transformation and thus are associated to winding numbers around the non trivial homology cycles of the

[1] In the case of $\mathcal{M} = \mathbb{R} \times T^d$, where T^d is the torus of dimension d, see the discussion in [12].

space(time) manifold, also referred to as "large gauge transformations". This group of transformations is also called the "modular group" since it is defined as the quotient of the group of gauge transformations by the subgroup of transformations homotopic to the identity. As far as these latter "small gauge transformations" are concerned, any transformation of this type may be reached through successive infinitesimal transformations of the form

$$A'_\mu = A_\mu + \partial_\mu \alpha \,.$$

Hence the Maxwell Lagrangian density is manifestly invariant under small gauge transformations since $\partial_\mu J^\mu = 0$ and $F'_{\mu\nu} = 0$.

Given these gauge transformations, a physical variable is a variable defined without gauge ambiguity, namely a gauge invariant variable. Hence the space of physical field configurations of any gauge theory is defined as the quotient of the space of field configurations by the full gauge group, namely the space of gauge orbits relating fields configurations equivalent up to gauge transformations. There exist thus in any gauge field theory physical degrees of freedom and spurious gauge degrees of freedom redundant in the description of the system. For what concerns the Maxwell theory, such a splitting applies to the gauge field $A_\mu(x)$ through the Hodge decomposition into longitudinal and transverse parts,

$$A_\mu = A^{\rm L}_\mu + A^{\rm T}_\mu = \partial_\mu \theta + \eta^{\rho\sigma}\, \partial_\rho \xi_{\mu\sigma} \,. \tag{1.3}$$

When a homotopically non trivial manifold is considered[2], a harmonic term must be added in order to account for global degrees of freedom associated to the period (winding number) of the field along non homologically trivial circles on the spacetime manifold. The longitudinal part of $A_\mu(x)$ possesses a single degree of freedom while the remaining degrees of freedom reside in the transverse part. The longitudinal part carries all the gauge variant character of the gauge field $A_\mu(x)$ since it transforms under the abelian $U(1)$ gauge transformation according to

$$\theta' = \theta + \alpha \,.$$

This spurious degree of freedom is referred to as being "pure gauge". The remaining gauge invariant transverse degrees of freedom are the genuine physical ones. This decomposition already shows that the longitudinal part of 2-form fields $B_{\mu\nu}(x)$ consists of d degrees of freedom. The conservation law for the current implies that $J^\mu(x)$ is a transverse vector field, hence the gauge invariant content of this field.

Finally, when the time coordinate is privileged, propagating degrees of freedom reduce to the $(d-1)$ polarisation states of the photon field described in terms of the electric and magnetic fields. These propagating physical fields are described in section 1.3 within the context of the non explicitly covariant Hamiltonian formulation.

[2]It has been also assumed that the periodicity properties of $A_\mu(x)$ on the spacetime manifold are trivial.

1.1.2 *p*-form Maxwell theory in any dimension

Notations

The differential form formalism is introduced in order to keep to compact notations while introducing a straightforward generalisation of the previous concepts to fields of any tensorial rank defined on any connected spacetime manifold $\mathcal{M}$ of dimension (d+1). Let $\omega(x)$ belong to $\Omega^p(\mathcal{M})$, namely, the set of real-valued p-forms on $\mathcal{M}$

$$\omega_p = \frac{1}{p!}\, \eta_{\mu_1\cdots\mu_p}\, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} \,,$$

where Greek indices, $\mu, \nu = 0, 1, \ldots, d$, denote the coordinate indices of the spacetime manifold $\mathcal{M}$. The Hodge operator $*$ maps p-forms to ($d-p+1$)-forms provided that $\mathcal{M}$ is endowed with a Lorentzian metric structure (of mostly negative signature) and is defined as

$$*\omega_p = \frac{\sqrt{h}}{p!\,(d-p+1)!}\epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{d-p+1}}\, h^{\mu_1\rho_1} \ldots h^{\mu_p\rho_p}\, \omega_{\rho_1\cdots\rho_p}\, \mathrm{d}x^{\nu_1}\wedge\ldots\wedge\mathrm{d}x^{\nu_{d-p+1}} \,,$$

where h is the absolute value of determinant of the metric $h_{\mu\nu}$ while $\epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{d-p+1}}$ is the totally antisymmetric Levi-Civita symbol such that $\epsilon_{0\cdots d} = 1$. This pseudo-tensor obeys the following contraction rule:

$$\epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{d-p+1}}\, \epsilon^{\mu_1\cdots\mu_p\rho_1\cdots\rho_{d-p+1}} = p!\, \delta^{\rho_1}_{[\nu_1} \ldots \delta^{\rho_{d-p+1}}_{\nu_{d-p+1}]} \,,$$

and is related to the totally antisymmetric tensor ε through $\varepsilon_{\mu_1\cdots\mu_d+1} = \sqrt{h}\,\epsilon_{\mu_1\cdots\mu_{d+1}}$.

The real-valued inner product on $\Omega^p(\mathcal{M}) \times \Omega^p(\mathcal{M})$ is defined as

$$\begin{aligned}(\omega_p, \eta_p) &= \int_{\mathcal{M}} \omega_p \wedge *\eta_p \\ &= \frac{1}{p!}\int_{\mathcal{M}} \sqrt{h}\, \omega_{\mu_1\cdots\mu_p}\, \eta_{\nu_1\cdots\nu_p}\, h^{\mu_1\nu_1} \ldots h^{\mu_p\nu_p}\, \mathrm{d}^{d+1}x \,.\end{aligned} \tag{1.4}$$

from the definition of the wedge product $\wedge$ and Hodge operator. The exterior derivative operator maps a p-form ω_p to a (p+1)-form $\mathrm{d}\omega_p$.

$$\mathrm{d} : \Omega^p(\mathcal{M}) \to \Omega^{p+1}(\mathcal{M}) : \omega_p \to \frac{1}{p!}\, \partial_{\mu_1}\omega_{\mu_2\cdots\mu_{p+1}}\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_{p+1}} \,.$$

From this definition, the following operators are constructed

$$\begin{aligned}\mathrm{d}^\dagger &: \Omega^p(\mathcal{M}) \to \Omega^{p-1}(\mathcal{M}) : \omega_p \to -\sigma^{p\,(d-p)} * \mathrm{d} * \omega_p \,, \\ \triangle &: \Omega^p(\mathcal{M}) \to \Omega^p(\mathcal{M}) : \omega_p \to (\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d})\omega_p \,,\end{aligned}$$

where we have defined $\sigma = (-1)$, and correspond to the coderivative and Laplacian operators, respectively.

The generalised formulation

The p-form Maxwell theory is obtained through the substitution of the electromagnetic field $A_\mu(x)$ in the usual formulation of the Maxwell theory by a p-form field of spacetime components $A_{\mu_1\cdots\mu_p}(x)$, a connection for a source of elementary extended objects of dimension $(p-1)$, see [13, 14]. Hence, given a real valued p-form field A in $\Omega^p(\mathcal{M})$, this generalised Maxwell action is of the form

$$S_{\text{Max}}[A] = \frac{\sigma^p}{2\,e^2}\,(F,F) - (A,J)\,. \tag{1.5}$$

Given a choice of units such that $c=1$, the physical dimensions of $A_{\mu_1\ldots\mu_p}(x)$ are L^{-p}, whereas the coupling constant e^2 is of physical dimension $E^{-1}\,L^{d-4}$. According to our conventions, the components of the field strength tensor $F=\mathrm{d}A$ read

$$F_{\mu_1\cdots\mu_{p+1}} = \frac{1}{p!}\,\partial_{[\mu_1}\,A_{\mu_2\cdots\mu_{p+1}]}\,, \tag{1.6}$$

where square brackets on indices denote total antisymmetrisation.

Unfortunately, it is no longer possible to express abelian gauge transformations for p-form gauge fields like in (1.2) in terms of univalued scalar phase factors. In that case such a generalisation is possible if each spacetime event is associated to a coordinatised manifold $x^\mu(\tilde\sigma)$, with $\tilde\sigma\equiv(\sigma^1,\ldots,\sigma^{p-1})$, embedded in the spacetime manifold. Hence the 1-form connection $A^c[x(\tilde\sigma)]$ is defined on the principal bundle of extended objects of dimension $(p-1)$ spanned by $x^\mu(\tilde\sigma)$. The local character of $A^c[x(\tilde\sigma)]$ requires that this connection is related to the p-form field $A(x)$ in (1.5) through

$$A^c[x(\tilde\sigma)]\,\mathrm{d}\tau = \frac{1}{p!}\int_{x(\tilde\sigma)} A_{\mu_1\cdots\mu_p}\;\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_p}\,,$$

where the integral covers the manifold $x^\mu(\tilde\sigma)$. Hence $A^c[x(\tilde\sigma)]$ transforms like any (abelian) connection[3] according to

$$(A^c)' = \mathrm{i}\,g\,\mathrm{d}g^{-1}\,, \qquad \text{with } g=\exp\left(-\mathrm{i}\int_{x(\tilde\sigma)}\alpha\right)\,, \tag{1.7}$$

where d is the appropriate exterior derivative defined on the principal bundle whereas α is a $(p-1)$-form. It may be then proved that abelian gauge transformations act on the p-form field $A(x)$ according to

$$A' = A + \tilde\alpha\,. \tag{1.8}$$

[3]It is worth to note however that this generalisation is compatible with spacetime locality only if the gauge group is $U(1)$, see [15].

Often in the literature, the p-form $\tilde{\alpha}(x)$ is required to be exact, namely $\tilde{\alpha}=\mathrm{d}\alpha$. However, if we relax now this condition and consider $\tilde{\alpha}(x)$ as being a closed form ($\mathrm{d}\tilde{\alpha}=0$), the cohomology structure of the spacetime manifold offers an elegant generalisation of the concepts of small and large gauge transformations for p-form gauge fields.

In the case of a homologically trivial manifold, any closed form is also exact. In the case of a homologically non trivial manifold, according to the Hodge theorem any closed form $\tilde{\alpha}(x)$ may uniquely be decomposed (for a given metric structure) into the sum of an exact and a harmonic form[4], see [16] for a review. The exact part of $\tilde{\alpha}$ defines small gauge transformations whereas the modular group is the quotient of the full gauge group by the subgroup of small gauge transformations, namely essentially the set of large gauge transformations. These transformations cannot be built from a succession of infinitesimal transformations. They correspond to the cohomologically[5] non trivial, namely the harmonic components of $\tilde{\alpha}(x)$.

The spacetime components of the p-form current $J(x)$ may be equally written in terms of a pseudo-tensor or a tensor with covariant Lorentz indices. These two notations are related through

$$J_{\mu_1\ldots\mu_p} = \frac{\sigma^p}{\sqrt{h}}\, h_{\mu_1\nu_1}\ldots h_{\mu_p\nu_p}\, J^{\nu_1\cdots\nu_p}\,.$$

The invariance of the action (1.5) under the gauge transformation defined in (1.8) requires the conservation of this current

$$*\mathrm{d}*J = 0 \quad \Leftrightarrow \partial_{\mu_1} J^{\mu_1\cdots\mu_p} = 0\,.$$

This condition also follows from the equation of motion

$$*\mathrm{d}*F = \sigma^p\, e^2\, J\,, \tag{1.9}$$

which is the generalisation to gauge fields of any tensorial rank of the Maxwell equations. Here we consider the current in its generic form and do not associate it neither to the history of any extended charged object nor to dynamical matter fields of extended objects. In such specific cases, the equations of motion should be conveniently completed. Nevertheless, the Maxwell equation (1.9) is always recovered if we assume a minimal coupling to the p-form gauge field.

Finally, every p-form field defined on the spacetime manifold of dimension $(d+1)$ accounts for C_p^{d+1} degrees of freedom at any given space point, with

$$C_p^{d+1} = \frac{(d+1)!}{p!\,(d-p+1)!}\,.$$

[4] A p-form ω_p is said harmonic if $\triangle\omega_p = 0$.

[5] The p^{th} de Rham cohomology group, denoted $H^p(\Sigma,\mathbb{R})$, characterises the topology of the manifold and is associated to global variables of non zero periods around homology cycles.

Moreover by virtue of the Hodge theorem [16], any p-form field $A(x)$ may uniquely be decomposed into the sum of an exact, a co-exact and a harmonic form with respect to the inner product specified in (1.4),

$$A = A^e + A^c + A_h \,. \tag{1.10}$$

Hence such a decomposition amounts to a split of the fields into a longitudinal part, a transverse part and a "global" part. Starting from the 1-form gauge field, an iterative process enables to count for each p-form field, $p=1,\dots d$, the number of "pure gauge" longitudinal degrees of freedom

$$N^p_{\mathrm{L}} = \sum_{i=0}^{p-1} \sigma^{p-i+1} \, \frac{(d+1)!}{i!\,(d-i+1)!} = C^d_{p-1} \,,$$

and likewise the number of physical transverse degrees of freedom $N^p_{\mathrm{T}} = C^d_p$. We will consider later the global degrees of freedom sensitive to large gauge transformations on homotopically non trivial manifolds.

1.2 Gauge invariant mass generation mechanisms

Often observable massive vector fields arise in diverse phenomena, for example in particle and condensed matter physics. However it is also well-known that the presence of a "direct" mass term like the Proca term spoils gauge invariance of field theories, and consequently, their renormalisability in a quantum context. The compromise found between these two antinomic requirements is the spontaneous breaking of the local symmetry where the action is invariant under a given gauge symmetry but this symmetry is hidden to the observer since not manifest in the physical spectrum. The secret of this mechanism relies on the presence of complex scalar field(s) of which the condensation in the non vanishing vacuum expectation value, renders the gauge fields massive. The Brout-Englert-Higgs (BEH) mechanism was first developed within the context of condensed matter physics when a physical system undergoes a phase transition. Indeed, phase transitions are often associated with spontaneous symmetry breaking, for example in the BCS theory of superconductivity where condensation of Cooper pairs arises below the critical temperature.

Furthermore, the BEH mechanism provides masses for the weakly interacting gauge vector bosons in the Standard Model, while remaining consistent with the renormalisability and unitarity constraints. However, the predicted Higgs boson has yet to be discovered. Within this context, the quest for alternative mass generation mechanisms is quite fascinating. The two other mass generation mechanisms which compete with the BEH mechanism in the sense that they are local and preserve the gauge symmetry are the Stueckelberg and the topological mass generation mechanisms. However these theories do not offer presently a serious alternative to the electroweak model since they do not admit a satisfying non abelian generalisation (in the sense of the Occam's razor and renormalisability) while their coupling to fermions remains mysterious.

1.2.1 Brout-Englert-Higgs mechanism

The basic concepts of the celebrated BEH mechanism are presently reviewed within the context of the Maxwell-Higgs model. A more advanced analysis may be found in [17] or any other book dedicated to gauge field theories. The Maxwell-Higgs model is a model of scalar electrodynamics of which the Lagrangian density reads

$$\mathcal{L}_{\mathrm{AH}} = -\frac{1}{4\,e^2} F_{\mu\nu} F^{\mu\nu} + |\mathrm{D}_\mu \phi|^2 - V\left(2\,|\phi|^2\right)\,. \tag{1.11}$$

The scalar field couples to the gauge connection through the covariant derivative,

$$\mathrm{D}_\mu \phi = \partial_\mu \phi - \mathrm{i}\, A_\mu\, \phi\,,$$

which by construction makes the Maxwell-Higgs model invariant under the abelian gauge symmetry

$$A'_\mu = A_\mu + \partial_\mu \alpha\,, \qquad \phi' = \mathrm{e}^{\mathrm{i}\alpha}\,\phi\,, \qquad \mathrm{D}'_\mu \phi' = \mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{D}_\mu \phi\,. \tag{1.12}$$

This way of coupling a $U(1)$ complex scalar field to a gauge field is called the "minimal coupling". The then associated current reads

$$J_\mu = \mathrm{i}\,\left(\phi^*\,\mathrm{D}_\mu \phi - \phi\,(\mathrm{D}_\mu \phi)^*\right)\,, \tag{1.13}$$

and is conserved by virtue of Noether's theorem.

The usual self-interacting quartic potential for the scalar field in the Maxwell-Higgs model reads

$$V\left(2\,|\phi|^2\right) = \tilde{\mu}^2\,|\phi|^2 + \lambda\,|\phi|^4\,,$$

where $\tilde{\mu}^2 < 0$ and $\lambda > 0$ in order to recover the familiar "Mexican hat-shaped" potential. Therefore the complex scalar field $\phi(x)$ possesses a non vanishing vacuum expectation value (vev) given by

$$\langle\phi\rangle = \frac{v}{\sqrt{2}}\,, \qquad v = \sqrt{\frac{-\tilde{\mu}^2}{\lambda}}\,. \tag{1.14}$$

Any complex scalar field may be factorised into a physical part and a pure gauge part which correspond, respectively, to the modulus and the phase in the polar parametrisation. Let us then parametrise $\phi(x)$ in terms of two real scalar fields, a physical field

$\varrho(x)$ and a gauge field $\theta(x)$ which transforms as $\theta' = \theta + \alpha$ under the abelian gauge transformation introduced in (1.12). The perturbation of the physical degree of freedom carried by $\varrho(x)$ will be not considered in the vicinity of the metastable vacuum at $\phi(x) = 0$ but around any particular configuration minimising the energy,

$$\phi(x) = \frac{1}{\sqrt{2}}\,\varrho(x)\,e^{\mathrm{i}\theta(x)} = \frac{1}{\sqrt{2}}\,(\tilde{\varrho}(x) + v)\,e^{\mathrm{i}\theta(x)}\,, \tag{1.15}$$

hence the introduction of the physical Higgs field $\tilde{\varrho}(x)$ with $\langle\tilde{\varrho}\rangle = 0$ whereas $\langle\varrho\rangle = v$.

According to this representation, the Lagrangian density (1.11) takes now the form

$$\mathcal{L} = -\frac{1}{4\,e^2}F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{2}\left|\partial_\mu\tilde{\varrho} - \mathrm{i}\,(\tilde{\varrho} + v)\,\left(A_\mu - \partial_\mu\theta\right)\right|^2 - V\left((\tilde{\varrho} + v)^2\right)\,, \tag{1.16}$$

while the gauge invariant current reads

$$J_\mu = \varrho^2\,\left(\partial_\mu\theta - A_\mu\right)\,. \tag{1.17}$$

At this stage, what corresponds to the massless Goldstone boson $\theta(x)$ in the spontaneous breaking of global symmetries may be gauged away, as suggested by the form of the conserved current (1.17). Hence under this "unitary gauge" choice, the BEH mechanism of spontaneous breaking of a local symmetry implies that the phase of the complex scalar field is absorbed as a single longitudinal degree of freedom for the gauge field $A_\mu(x)$, making the latter field massive. A glance at the above Lagrangian density reveals that the gauge field then acquires a mass $m_A = e^2\,v^2$ while the scalar Higgs field $\tilde{\varrho}(x)$ is a physical mode of mass $m_\varrho = -\mu^2$.

The BEH mechanism, presently introduced in a textbook manner, will be dealt with in a slightly different way in Section 1.4 in terms of the physical propagating electromagnetic fields. The difficulties, which inevitably arise when such techniques of isolating physical variables are used, will be emphasised as a first introduction to the kind of issues which will be tackled in this Thesis.

1.2.2 Stueckelberg mechanism

In the Maxwell-Higgs model, the London limit is the limit in which the mass of the Higgs field becomes infinite while the mass of the gauge boson remains finite. As a matter of fact, within this limit the dynamics of the Higgs field is in some sense frozen to its vacuum expectation value. Hence, in terms of the parameters of the usual quadratic "Mexican hat-shaped" potential, the London limit reads

$$\lambda \to \infty\,, \qquad \mu^2 \to \infty\,, \qquad \text{such that } v = \sqrt{\frac{-\tilde{\mu}^2}{\lambda}} \text{ be finite}\,.$$

Applying now this limit to the Lagrangian density (1.16),

$$\mathcal{L}_{\text{1-Stu}} = -\frac{1}{4\,e^2} F_{\mu\nu}\, F^{\mu\nu} + \frac{v^2}{2}\,(A_\mu - \partial_\mu\theta)^2 \ ,$$

the action resulting from the decoupling of the massive Higgs field is nothing other than the Stueckelberg action for a 1-form gauge field (see [18] and references therein). Within this limit, the gauge invariant current $J_\mu(x)$ in (1.17) which may be then associated to the Stueckelberg mass term is interpreted as the transverse part of the gauge field $A_\mu(x)$ whereas $\tilde{\theta}_\mu = \partial_\mu\theta(x)$ is associated to its longitudinal part.

In contradistinction to the Maxwell-Higgs model, the Stueckelberg action may be readily extended to any p-form field $A(x)$ in $\Omega^p(\mathcal{M})$

$$S_{\text{p-Stu}}[A,\theta] = \frac{\sigma^p}{2\,e^2}\ (F,F) - \sigma^p\,\frac{v^2}{2}\left(A - \tilde{\theta}\right)^2 \ .$$

In the above action, $\tilde{\theta}(x)$ is a closed p-form of which the transformation under the abelian gauge symmetry

$$\tilde{\theta}' = \tilde{\theta} + \alpha$$

compensates for that of the p-form gauge field $A(x)$ defined in (1.8) is order to preserve the gauge invariance of the mass term. In the present Chapter, we do not allow $\tilde{\theta}(x)$ and $A(x)$ to possess a harmonic component, hence $\tilde{\theta} = \mathrm{d}\theta$. Thus the p-form $\tilde{\alpha} = \mathrm{d}\alpha$ only parametrises small gauge transformations. Notice that a more general action for the Stueckelberg mechanism will be introduced in Chapter 4 in terms of a real arbitrary parameter ξ. Very interesting aspects like Stueckelberg extensions of the Standard Model through a mass term for the photon or for the Z' boson (see for example [18, 19]) are outside the scope of this Thesis.

1.2.3 Topological mass generation

Topological field theories

Topological field theories (TFT, see [20] for a review) have played an important role in a wide range of fields in mathematics and physics ever since they were first constructed by A. S. Schwarz [21, 22] and E. Witten [23]. These theories actually possess so large a gauge freedom that their physical, namely their gauge invariant observables solely depend on the topology (more precisely, the diffeomorphism equivalence class) of the underlying manifold. Another related feature of TFT is the absence of propagating physical degrees of freedom. Upon quantisation, these specific properties survive, possibly modulo some global aspects related to quantum anomalies. As a consequence, topological quantum field theories (TQFT) often have a finite dimensional Hilbert space and are quite generally solvable, even though their formulation requires an infinite number of degrees of freedom. There exists a famous classification scheme for TQFT, according to whether they are of the Schwarz or Witten type [20].

As a class of great interest, TFT of the Schwarz type have a classical action which is explicitly independent of any metric structure on the underlying manifold and does not reduce to a surface term. The present Subsection focuses on all such theories defined by a sequence of abelian BF theories for manifolds $\mathcal{M}$ of any dimension $(d+1)$ [21, 22, 24, 25]. Given a p-form field $A(x)$ in $\Omega^p(\mathcal{M})$ and a $(d-p)$-form field $B(x)$ in $\Omega^{d-p}(\mathcal{M})$, the general TFT action of interest is of the form

$$S_{B\wedge F}[A,B] = \kappa \int_{\mathcal{M}} (1-\xi)\, F\wedge B - \sigma^p\, \xi\, A\wedge H, \tag{1.18}$$

the constant κ being some real normalisation parameter of which the properties are specified throughout the discussion hereafter. This action is invariant under two independent classes of finite abelian gauge transformations acting separately in either the A- or B-sector,

$$A' = A + \alpha, \qquad B' = B + \beta, \tag{1.19}$$

where α and β are closed p- and $(d-p)$-forms on $\mathcal{M}$, respectively. The derived quantities $F = \mathrm{d}A$ and $H = \mathrm{d}B$ are the gauge invariant field strengths associated to $A(x)$ and $B(x)$. The arbitrary real variable ξ we have introduced in order to parametrise any possible surface term is physically irrelevant for an appropriate choice of boundary conditions on $\mathcal{M}$. Given the definition of the wedge product, the integrand in (1.18) is a $(d+1)$-form, the integration of which over $\mathcal{M}$ does not require a metric.

Hence the Lagrangian density for BF theories reads, for example in 3+1 dimensions,

$$\mathcal{L}^4_{BF} = \xi\, \frac{\kappa}{6}\epsilon^{\mu\nu\rho\sigma}\, A_\mu\, H_{\nu\rho\sigma} + (1-\xi)\, \frac{\kappa}{4}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}\, B_{\rho\sigma}\,, \tag{1.20}$$

in terms of two fields $A_\mu(x)$ and $B_{\mu\nu}(x)$ on which gauge transformations (1.19) act according to

$$A'_\mu = A_\mu + \partial_\mu \Lambda^{(\alpha)}, \qquad B'_{\mu\nu} = B_{\mu\nu} + \partial_{[\mu}\, \tilde{\Lambda}^{(\beta)}_{\nu]}\,.$$

Let us recall that this notation does not cover possible large gauge transformations.

In the particular situation when the number of space dimensions d is even and such that $d = 2p$ with p itself being odd, in addition to the BF theories defined by (1.18) there exist TFT of the Schwarz type involving only the single p-form field $A(x)$ with the following action[6],

$$S_{A\wedge F}[A] = \kappa \int_{\mathcal{M}} A\wedge F. \tag{1.21}$$

[6] If p is even with $d = 2p$, this action reduces to a surface term.

These theories are said to be of the AF type throughout this Thesis. They include the well-known (abelian) Chern-Simons theory in 2+1 dimensions [21, 22, 26] of which the Lagrangian density reads

$$\mathcal{L}_{\text{CS}} = \frac{1}{2}\kappa\, \epsilon^{\mu\nu\rho}\, \partial_\mu A_\nu\, A_\rho\,, \tag{1.22}$$

in terms of the spacetime components $A_\mu(x)$ of the 1-form field $A(x)$.

This sequence of TFT of the Schwarz type formulated in any dimension, and related to one another through dimensional reduction [27], possesses some fascinating properties. First, the space of gauge inequivalent classical solutions is isomorphic to $H^p(\mathcal{M}) \times H^{d-p}(\mathcal{M})$, $H^p(\mathcal{M})$ being the p^{th} cohomology group of the manifold $\mathcal{M}$. Second, the types of topological terms contributing to these actions define generalisations to arbitrary dimensions of ordinary two-dimensional anyons. Namely, non local holonomy effects give rise to exotic statistics for the extended objects which may be coupled to the higher order tensor fields [28, 29, 30]. Third, these types of quantum field theories display profound connections between mathematics and physics for what concerns topological properties related, say, to the motion group, the Ray-Singer torsion and link theory. These connections appear within the canonical quantisation[7] of these systems [31, 32].

Action for topologically massive gauge theories

Analogies between theoretical particle physics and condensed matter may again be fruitful. For example, Chern-Simons terms account for fermionic collective phenomena, such as in the quantum Hall effect. Generally speaking the statistical transmutation of extended objects in higher dimensions may rely on topological couplings of the BF type. From a relativistic point of view, (quantum) field theories in any spacetime dimension are a general framework of potential relevance to high energy physics as well as mathematical investigations for their own sake. Within this context such topological BF terms define couplings between two independent tensor fields whose dynamics is characterised by the action

$$\begin{aligned} S_{\text{TMGT}}[A,B] &= \int_{\mathcal{M}} \frac{1}{2\,e^2}\,\sigma^p\, F\wedge *F + \frac{1}{2\,g^2}\,\sigma^{d-p}\, H\wedge *H \\ &\quad +\kappa \int_{\mathcal{M}} (1-\xi)\, F\wedge B - \sigma^p\,\xi\, A\wedge H\,, \end{aligned} \tag{1.23}$$

provided the spacetime manifold $\mathcal{M}$ is endowed now with a Lorentzian metric structure allowing for the introduction of the Hodge $*$ operator. The parameters e and g are

[7] When $\mathcal{M} = \mathbb{R}\times\Sigma$, the physical Hilbert space is the set of square integrable functions on $H^p(\Sigma)$ [24].

arbitrary real constants corresponding to coupling constants when matter fields coupled to A and B are introduced. Without loss of generality for the present analysis, these parameters are assumed to be strictly positive. In 3+1 dimensions, one recovers the famous Cremmer-Scherk action [33, 34, 35] and in 2+1 dimensions, the doubled Chern-Simons theory [36]. Given the total action (1.23) written out in component form, the Cremmer-Scherk Lagrangian density thus reads

$$\begin{aligned} \mathcal{L}^4_{\text{TMG}} &= -\frac{1}{4\,e^2}F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{12\,g^2}H_{\mu\nu\rho}\,H^{\mu\nu\rho} \\ &+ \kappa\,\epsilon^{\mu\nu\rho\sigma}\left(\frac{\xi}{6}A_\mu\,H_{\nu\rho\sigma} + \frac{1-\xi}{4}F_{\mu\nu}\,B_{\rho\sigma}\right) . \end{aligned} \tag{1.24}$$

It is well known that the topological terms generate a mass gap,

$$M_{BF} = \hbar\,\mu = \hbar\,\kappa\,e\,g\,,$$

for the dynamical tensor fields without breaking gauge invariance. Unfortunately, the topological mass generation mechanism of the BF-type is not generalisable to a renormalisable non abelian gauge theory, unless further fields are introduced (see [37] and references therein).

In the particular circumstance that $d = 2p$ with p odd, a topological term of the AF type (1.21) generates also a mass gap,

$$M_{AF} = \hbar\,\mu = \hbar\,\kappa\,e^2\,,$$

even though the action involves a single p-form field A,

$$S_{\text{TMGT}}[A] = \int_{\mathcal{M}} \frac{-1}{2e^2}\,F\wedge *F + \frac{\kappa}{2}\,A\wedge F. \tag{1.25}$$

In 2+1 dimensions, this action defines the famous Maxwell-Chern-Simons theory [38, 39, 40] of which the Lagrangian density is of the form

$$\mathcal{L}_{\text{MCS}} = -\frac{1}{4\,e^2}F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{4}\kappa\,\epsilon^{\mu\nu\rho}\,F_{\mu\nu}\,A_\rho\,, \tag{1.26}$$

when expressed in terms of the spacetime components of the 1-form field. As a matter of fact, nothing forbids to add this topological term to the Maxwell theory since this term preserves invariance under abelian gauge transformations[8] and does not require the introduction of extra degrees of freedom. The Chern-Simons term breaks the invariance under time reversal $(x^0,\vec{x}\,) \to (-x^0,\vec{x}\,)$ and parity in 2+1 dimensions, $(x^0,x^1,x^2) \to (x^0,-x^1,x^2)$ as is the case for the topological θ-term in Maxwell or Yang-Mills theories in 3+1 dimensions. But contrary to the θ-term, the Chern-Simons term is not a pure derivative and generates a “topological” mass for the gauge field.

[8]At least for what concerns small gauge transformations. Considering the invariance under large gauge transformations implies further restriction for κ as will be discussed hereafter.

1.3 Hamiltonian formulation and propagating variables

We have expressed so far gauge field theories within the Lagrangian formulation which is explicitly Lorentz covariant. Let us single out now the time coordinate in order to highlight the dynamical evolution of the system throughout time. The dynamical system may be thus expressed in terms of the non explicitly Lorentz covariant Hamiltonian operator which generates time translations and is therefore associated to the (conserved) energy. The Hamiltonian formulation thus enables to identify among physical variables those which are actual propagating dynamical variables. In order to define this formulation, the spacetime manifold $\mathcal{M}$ is now taken to have the topology of $\mathcal{M}=\mathbb{R}\times\Sigma$, where Σ is a compact orientable d-dimensional Riemannian space manifold without boundary. Adopting then synchronous coordinates on $\mathcal{M}$, the spacetime metric takes the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \tilde{h}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \tag{1.27}$$

where $\tilde{h}_{ij}(\vec{x}\,)$ is the positive definite Riemannian metric on Σ. Here Latin indices, $i=1,\ldots,d$, label the space directions in Σ while 0 denotes the time component in $\mathbb{R}$.

The Hamiltonian formulation is defined in terms of a canonical Hamiltonian H_0 given by the Legendre transformation of its associated Lagrangian and dependent on phase space variables. This phase space is endowed with a geometric symplectic structure. The built-in gauge invariance of the Lagrangian of any gauge field theory implies that the configuration space within the Lagrangian formulation is not in one-to-one correspondence with the phase space within the Hamiltonian one[9], hence the existence of constraints. Therefore gauge field theories require a careful treatment which is known as the "analysis of constraints" in order to identify their Hamiltonian formulation [41, 42]. This algorithm will be discussed presently in a particular case.

1.3.1 Maxwell-Chern-Simons theory by way of example

The Maxwell-Chern-Simons theory (1.26) introduced in the previous Section as a particular case of topological mass generation mechanism is chosen to illustrate by way of example the algorithm of analysis of constraints in gauge field theories. This natural extension of the Maxwell theory is not chosen in a haphazard way. The great advantage in considering such an extension is that the pure Maxwell theory may be recovered in each step of the Hamiltonian analysis of constraints by simply setting $\kappa=0$. Moreover, the Hamiltonian formulation of the MCS theory will be useful in Chapter 2 in order to introduce to our factorisation of degrees of freedom.

[9] Or equivalently, the Hessian matrix of the Lagrangian describing singular systems has zero modes.

Phase space and symplectic structure

The Lagrangian formulation of the MCS theory consists in the Lagrangian density (1.26) defined in terms of the configuration space variables $A_\mu(t,\vec{x}\,)$ and their generalised velocities $\partial_0 A_\mu(t,\vec{x}\,)$. These latter variables will be henceforth denoted as $\dot{A}_\mu(t,\vec{x}\,)$, where a dot stands for differentiation with respect to the time coordinate. The phase space of the associated Hamiltonian formulation is spanned by the field variables $A_\mu(t,\vec{x}\,)$ and their conjugate momenta $P^\mu(t,\vec{x}\,)$ and is endowed with a symplectic structure which is related to the following canonical Poisson bracket algebra

$$\begin{aligned}\left\{A_i(t,\vec{x}\,)\,,\,P^j(t,\vec{y})\right\} &= \delta_i^j\,\delta^2(\vec{x}-\vec{y}\,)\,,\\ \left\{A_0(t,\vec{x}\,)\,,\,P^0(t,\vec{y})\right\} &= \delta^2(\vec{x}-\vec{y}\,)\,.\end{aligned} \tag{1.28}$$

These Poisson brackets are to be evaluated at equal time. According to the definition of the Legendre transformation, the conjugate momenta are defined as functions of configuration space variables through the following relations

$$P^i(t,\vec{x}\,) = \frac{\delta L_{\text{MCS}}\left[A,\dot{A}\right]}{\delta\dot{A}_i(t,\vec{x}\,)} = \frac{1}{e^2}\,\delta^{ij}\,F_{0j}(t,\vec{x}\,) + \frac{\kappa}{2}\,\epsilon^{ij}\,A_j(t,\vec{x}\,)\,, \tag{1.29}$$

$$P^0(t,\vec{x}\,) = \frac{\delta L_{\text{MCS}}\left[A,\dot{A}\right]}{\delta\dot{A}_0(t,\vec{x}\,)} = 0\,. \tag{1.30}$$

In the case of the Maxwell-Chern-Simons theory it appears clearly through the definition of the time component of the conjugate momentum (1.30) that not all conjugate momenta $P^0(t,\vec{x}\,)$ and $P^i(t,\vec{x}\,)$ are locally independent as functions of the configuration space variables $\dot{A}_0(t,\vec{x}\,)$ and $\dot{A}_i(t,\vec{x}\,)$, implying the existence of a primary constraint $P^0 \approx 0$ like in the pure Maxwell theory.

Hamiltonian and constraints

Having defined the phase space along with its symplectic structure, the dynamics of the system is generated through the Poisson brackets from the primary Hamiltonian. This primary Hamiltonian consists in the extension of the canonical Hamiltonian, resulting from the Legendre transformation, where the primary constraints have been included through their associated Lagrange multipliers. However the primary constraints are required to be preserved under time evolution and thus their total time derivative must weakly vanish. In the case of the MCS theory this condition generates a secondary constraint,

$$\dot{P}^0 = \left\{P^0, H\right\} \approx 0 \quad \Rightarrow \quad \varphi = \partial_i P^i + \frac{\kappa}{2}\,\epsilon^{ij}\,\partial_i A_j \approx 0\,, \tag{1.31}$$

which is nothing other than the specific Gauss Law associated to the MCS theory. The time consistency condition of this last constraint is trivially satisfied, halting here the constraints algorithm.

The primary and secondary constraints for the MCS theory define a complete set of independent first-class constraints since their Poisson brackets are weakly vanishing and since they are linearly independent. From their definition, it may be easily shown that first-class constraints generate small gauge transformations. In the case of the MCS theory, given any linear combination of the first class-constraint, for example,

$$\zeta_\alpha(t) = \int_\Sigma \mathrm{d}^2x\, \alpha(t,\vec{x})\, \varphi(t,\vec{x})\,,$$

the infinitesimal action of such a combination $\zeta_\alpha(t)$ on the phase space degrees of freedom $A_i(t,\vec{x}\,)$ and $P^i(t,\vec{x}\,)$,

$$\begin{aligned} \delta_\alpha A_i &= -\left\{A_i(t,\vec{x}\,), \zeta_\alpha(t)\right\} = \partial_i\alpha\,, \\ \delta_\alpha P^i &= -\left\{P^i(t,\vec{x}\,), \zeta_\alpha(t)\right\} = \frac{\kappa}{2}\,\epsilon^{ij}\,\partial_j\alpha\,, \end{aligned} \tag{1.32}$$

corresponds to those gauge transformations in (1.2) which are connected to the identity transformation.

This discussion has showed that the Legendre transformation, which relates the Lagrangian formulation of gauge theories to their Hamiltonian formulation, is well defined modulo a complete analysis of constraints. For further details, see [41, 42, 43]. For what concerns the MCS theory, its total Hamiltonian density reads

$$\begin{aligned} \mathcal{H} &= \frac{e^2}{2}\left(P^i - \frac{\kappa}{2}\,\epsilon^{ij}\,A_j\right)^2 + \frac{1}{4\,e^2}\,(F_{ij})^2 \\ &- uP^0 + (u' - A_0)\,\partial_i\left(P^i + \frac{\kappa}{2}\,\epsilon^{ij}\,A_j\right) + \partial_i(A_0\,P^i)\,, \end{aligned} \tag{1.33}$$

where the last term is a total divergence. The Lagrange multipliers u and u' are arbitrary functions of time and phase space variables. The Hamiltonian density for the pure Maxwell theory is obtained by setting $\kappa = 0$ without any other redefinition.

1.3.2 Propagating variables

Fundamental Hamiltonian formulation of MCS theory

The non explicitly covariant first order action in field theories is defined as an action of which the associated Lagrangian density is linear in time derivative of the fields.

Any gauge field theory described in this Thesis may be formulated as a non explicitly covariant first order action, which is obtained by defining the Lagrangian density as the Legendre transformation of the total Hamiltonian density, thus including all the first class constraints. In the case of the MCS theory this action $S[A, P; u, u']$ reads

$$\begin{aligned} S &= \int \mathrm{d}t\, \mathrm{d}^2x \left\{ \dot{A}_0\, P^0 + \partial_0 A_i\, P^i - \frac{e^2}{2} \left(P^i - \frac{\kappa}{2}\, \epsilon^{ij}\, A_j \right)^2 - \frac{1}{4\, e^2}\, (F_{ij})^2 \right. \\ &\quad \left. + u\, P^0 - (u' - A_0)\, \partial_i \left(P^i + \frac{\kappa}{2}\, \epsilon^{ij}\, A_j \right) - \partial_i (A_0\, P^i) \right\} \end{aligned} \tag{1.34}$$

The constraint $\varphi(t, \vec{x}\,) \approx 0$ generates the physically relevant small gauge transformations, see (1.31). However, we did not comment about the second constraint $P^0 \approx 0$. Actually, this constraint generates symmetries related to the presence of Lagrange multipliers treated as dynamical degrees of freedom and therefore is without real physical significance. The presence of such a constraint is a sign that some physical variables within the Lagrangian formulation are non propagating, namely decouple from the system within the Hamiltonian formulation where time is privileged. Then an elegant reduction process of these physically irrelevant degrees of freedom has been introduced in [41] in terms of the removal of "layers" in order to move toward the innermost core of the "nested" structure of Hamiltonian formulations.

In the first order formulation of the MCS theory defined in (1.34), the Hamiltonian nested structure appears clearly. Indeed, this expression does not yet refer to a fundamental Hamiltonian described with the minimum of physically relevant degrees of freedom. Actually, the phase space variable $A_0(t, \vec{x}\,)$ and its conjugate momentum $P^0(t, \vec{x}\,)$ may be removed from the first order action through the simple redefinition $\tilde{u}' = A_0 - u'$ and by setting $u = \dot{A}_0$. Provided that we keep in mind that $A_0(t)$ is understood as a Lagrange multiplier, we may define $\tilde{u}' = A_0$. This extra layer being removed, the first order action is of the form

$$S = \int \mathrm{d}t\, \mathrm{d}^2x \left\{ \partial_0 A_i\, P^i - \mathcal{H}_{\text{MCS}} \right\} . \tag{1.35}$$

The fundamental Hamiltonian density for the MCS theory reads up to a surface term

$$\mathcal{H}_{\text{MCS}} = \frac{e^2}{2} \left(P^i - \frac{\kappa}{2}\, \epsilon^{ij}\, A_j \right)^2 + \frac{1}{4\, e^2}\, (F_{ij})^2 + A_0\, \partial_i \left(P^i + \frac{\kappa}{2}\, \epsilon^{ij}\, A_j \right) + \text{ST} .$$

Here $A_0(t)$ is no longer a degree of freedom of the system but rather plays the role of a Lagrange multiplier enforcing Gauss' law.

The great advantage offered by the fundamental first order formulation is the explicit introduction of the conjugate momenta within the Lagrangian formulation. These conjugate momenta are related to the variables of the (extended) configuration space through second-class constraints, without affecting the gauge symmetries, thus the first-class constraints, of the theory. Within the first order formulation of the MCS theory (1.35), for instance, the gauge invariant combination

$$\frac{1}{e^2} E^i_{\rm el} = P^i - \frac{\kappa}{2}\,\epsilon^{ij}\,A_j\,, \tag{1.36}$$

turns out to be an auxiliary field reducible through Gaussian integration. Within the Hamiltonian formulation, any “Gaussian variable” is readily reducible like a second-class constraint, which is solved as a strict equality, provided Dirac brackets are introduced for the remaining degrees of freedom. The second-class constraint reads

$$P^i - \frac{\kappa}{2}\,\epsilon^{ij}\,A_j = \frac{1}{e^2} F_{0i}\,,$$

and the physical variables $E^i_{\rm el}(t,\vec{x}\,)$ are thus readily identified to be the electric field. Furthermore, the original relation between the configuration space and phase space variables, see (1.29), is recovered. In other words, there exists a one-to-one correspondence between non explicitly covariant first order actions and Hamiltonian structures, provided that the Lagrange multipliers associated to the first-class constraints are not considered as dynamical degrees of freedom.

Propagating variables in theories of the Maxwell-type

The non explicitly covariant first order action for the pure Maxwell theory is simply obtained by setting $\kappa = 0$ without any other redefinition,

$$S = \int {\rm d}t\,{\rm d}^2x \left\{ \frac{1}{e^2}\dot{A}_i\,E^i_{\rm el} - \mathcal{H}_{\rm M} \right\}\,. \tag{1.37}$$

The fundamental Hamiltonian density reads in terms of the electric vector field

$$\mathcal{H}_{\rm M} = \frac{1}{2\,e^2}\,(E^i_{\rm el})^2 + \frac{1}{4\,e^2}\,\left(F_{ij}\right)^2 - \frac{A_0}{e^2}\,\partial_i E^i_{\rm el} + {\rm ST}\,,$$

with the formulation of the canonical Poisson Brackets and gauge invariance remaining unaltered. Notice that the number of spacetime dimensions does not appear explicitly in the above expression which remains the same whatever this number. In 3+1 dimensions, from the three physical degrees of freedom within the Lagrangian formulation, there remain only two actual propagating degrees of freedom. In the Maxwell theory they correspond to the polarisation states of the massless photon field.

The Maxwell-Chern-Simons and pure Maxwell theories share a common formulation of their canonical Hamiltonian in terms of the electric field. However the presence of the Chern-Simons topological term modifies Gauss' Law as well as the symplectic structure of the theory since the electric field components no longer commutes. Hence this CS term generates a mass gap of topological origin.

1.4 Physical formulation of gauge field theories

1.4.1 First order actions and Maxwell theories

Maxwell theory in 3+1 dimensions

A manifest realisation of the gauge invariance principle implies that the original fields used to define any gauge theory do not generate physical configurations since these fields are not gauge invariant degrees of freedom. In fact a decomposition into pure gauge longitudinal and physical transverse degrees of freedom is feasible within the Lagrangian formulation, see (1.3), but manifest spatial locality is lost when we proceed to such a change of variables,

$$A^{\mathrm{T}}_\mu = \left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)\eta^{\nu\rho}\,A_\rho\,, \qquad A^{\mathrm{L}}_\mu = \frac{\partial_\mu\partial_\nu}{\partial^2}\eta^{\nu\rho}\,A_\rho\,.$$

The other way to isolate physical degrees of freedom is to fix the gauge, for instance, the covariant Lorentz gauge $\partial^\mu A_\mu = 0$. The purpose of this gauge fixing procedure in Maxwell theory is to remove the longitudinal part carrying the symmetry of the gauge connection. Within the Hamiltonian formulation, it is possible to decompose the classical phase space into a non propagating temporal part, a pure gauge longitudinal part and a physical transverse part. However, this decomposition suffers the same problem of locality as its covariant counterpart. Hence the introduction of the temporal and Coulomb gauges, respectively $A_0 = 0$ and $\partial^i A_i = 0$, in order to isolate the two transverse propagating degrees of freedom of the photon. However such gauge fixings usually suffer Gribov problems. Furthermore as will be highlighted in this Thesis, a careless gauge fixing procedure often implies that topological content is lost, as shown by the counter-examples to the Fradkin-Vilkovisky theorem [44].

However, without resorting to any gauge fixing procedure, physical variables are already manifest within the first order Lagrangian formulation. The explicitly covariant first order Lagrangian formulation for the Maxwell theory in 3+1 dimensions simply results from the Lorentz covariant extension of the first order action defined in (1.37),

$$\mathcal{L}^{\mathrm{f.o.}}_{\mathrm{Max}} = -\frac{e^2}{4}\,E_{\mu\nu}\,E^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\,\partial_\mu A_\nu\,E_{\rho\sigma} - A_\mu\,J^\mu\,, \tag{1.38}$$

where a conserved source current $J^\mu(x)$ of matter fields has been added. By covariant extension is meant the extension of the electric field to a tensor with spacetime covariant indices $E_{\mu\nu}(x)$. This expression may be also obtained from the original Lagrangian formulation (1.1) after the introduction of the Gaussian auxiliary field $E_{\mu\nu}(x)$.

Two Euler-Lagrange equations may be extracted from the above first order formulation. The first relates the gauge invariant field $E_{\mu\nu}(x)$ to the current

$$\epsilon^{\mu\nu\rho\sigma}\,\partial_\nu E_{\rho\sigma} = 2\,J^\mu\,, \tag{1.39}$$

while the second corresponds to the equation relating the physical variable $E_{\mu\nu}(x)$ to the original gauge field $A_\mu(x)$ through Gaussian integration

$$E_{\alpha\beta} = \frac{1}{e^2}\,\eta_{\alpha\mu}\,\eta_{\beta\nu}\,\epsilon^{\mu\nu\rho\sigma}\,\partial_\rho A_\sigma\,. \tag{1.40}$$

Within the Hamiltonian formulation this latter equation may be divided into two second-class constraints when space and time indices are split. The first represents the relation between the variables $A_i(x)$ and their conjugate momenta

$$E_{ij} = \frac{1}{e^2}\,\delta_{ik}\,\delta_{jl}\,\epsilon^{klm}\,F_{om} \quad\Rightarrow\quad E^i_{\text{el}} = \frac{e^2}{2}\epsilon^{ijk}\,E_{ij}$$

which are, as already seen, related to the electric vector field. The second second-class constraint relates the field strength tensor $F_{ij}(x)$ to the thus physical vector field $B^i_{\text{mg}}(x)$

$$E_{0i} = -\frac{1}{2\,e^2}\,\delta_{ij}\,\epsilon^{jkl}\,F_{kl} = \frac{1}{e^2}\,\delta_{ij}\,B^j_{\text{mg}}$$

which is known to be the magnetic field. The well-known Maxwell equations,

$$\begin{aligned} \vec{\nabla}\cdot\vec{E}_{\text{el}} = e^2\,J^0\,, &\qquad \vec{\nabla}\times\vec{E}_{\text{el}} = -\partial_t\vec{B}_{\text{mg}}\,, \\ \vec{\nabla}\cdot\vec{B}_{\text{mg}} = 0\,, &\qquad \vec{\nabla}\times\vec{B}_{\text{mg}} - \partial_t\vec{E}_{\text{el}} = e^2\,\vec{J}\,, \end{aligned}$$

are recovered if the above correspondences are applied to the equations of motion (1.39) and (1.40).

Usually any abelian theory of the Maxwell-type is expressed in terms of the gauge connection $A_\mu(x)$. This formulation highlights the elegant connection between the dynamical connection on an abelian fiber bundle from the mathematical point of view and the physical interpretation in terms of the mediation of the electromagnetic force. However, we will prefer within the context of this Thesis to consider the first order Lagrangian formulation which is fundamental from a dynamical perspective since it involves directly the physical propagating variables, in the present case the electromagnetic fields. Unfortunately, even within the first order formulation of the pure Maxwell theory, it is impossible to isolate the physical propagating field $E_{\mu\nu}$ from the gauge field $A_\mu(x)$ through a local and linear change of variables. The role of this latter field is to generate dynamics for the former through the equations of motion as well as to introduce minimal local couplings to matter fields. At the level of the Maxwell equations, it further implies Faraday's law of induction and the absence of magnetic monopoles.

Maxwell theory for p-form gauge fields

The present discussion of the familiar Maxwell theory in 3+1 dimensions readily generalises to p-form Maxwell theories defined in any dimension. Adopting synchronous coordinates on $\mathcal{M}=\mathbb{R}\times\Sigma$, see (1.27), the number of propagating degrees of freedom of a massless p-form field $A(x)$ equals the number of transverse degrees of freedom of a form defined on $\Omega^p(\Sigma)$, that is C_p^{d-1}. These propagating degrees of freedom are explicitly involved within the covariant Lagrangian first order formulation of (1.5),

$$S^{\text{f.o.}}_{\text{Max}} = \sigma^{p+1}\,\frac{e^2}{2}\,(E,E) + \int_{\mathcal{M}} F\wedge E - (A,J)\,, \tag{1.41}$$

obtained after the introduction of the Gaussian auxiliary $(d-p)$-form field $E(x)$. The equations of motion derived from this first order formulation read

$$E = \frac{\sigma^p}{e^2}\,*_{(d+1)}\,\mathrm{d}A\,,\qquad *\mathrm{d}E = J\,, \tag{1.42}$$

and may be identified as the Maxwell equations expressed in terms of the generalised electric and magnetic fields when space and time Lorentz indices are distinguished.

Usually the p-form electric field $E_{\text{el}}(x)$ is expressed in terms of its pseudo-tensorial contravariant space components,

$$E_{\text{el}} = \frac{1}{p!}\,\frac{1}{\sqrt{h}}\,E_{\text{el}}^{i_1\cdots i_p}\,\tilde{h}_{i_1 j_1}\ldots\tilde{h}_{i_p j_p}\,\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_p}\,,$$

and likewise for the $(d–p–1)$-form magnetic field of space components $B_{\text{mg}}^{i_1\cdots i_{d-p-1}}(x)$. Then, knowing the relation between the electromagnetic vector fields and the 1-form gauge field which generates these physical fields, the generalisation to any p-form gauge field is straightforward:

$$\begin{aligned} E_{\text{el}}^{i_1\cdots i_p} &= \sqrt{h}\,\tilde{h}^{i_1 j_1}\ldots\tilde{h}^{i_p j_p}\,F_{0 i_1\cdots i_p}\,,\\ B_{\text{mg}}^{i_1\cdots i_{d-p-1}} &= \sigma^d\,\frac{\sigma^{p\,(d-p)}}{(p+1)!}\,\epsilon^{i_1\cdots i_{d-p-1} j_1\cdots j_{p+1}}\,F_{j_1\cdots j_{p+1}}\,. \end{aligned}$$

This definition enables one to formulate the canonical Hamiltonian of the Maxwell theory for p-form fields as the sum of the squares of the electric and magnetic fields, where the electric field is proportional to the conjugate momentum of the phase space variable $A_{i_1\cdots i_p}(x)$.

Finally, the physical $(d-p)$-form field $E(t,\vec{x}\,)$ may be separated into its temporal component $\mathrm{d}t\wedge E_0(t,\vec{x}\,)$ with $E_0(t,\vec{x}\,)$ being a $(d–p–1)$-form on Σ, and its remaining components $\tilde{E}(t,\vec{x}\,)$ restricted to $\Omega^{d-p}(\Sigma)$,

$$\begin{aligned} E_0(t,\vec{x}\,) &= \frac{1}{(d-p-1)!}\,E_{0i_1\cdots i_{d-p-1}}(t,\vec{x}\,)\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_{d-p-1}}\,,\\ \tilde{E}(t,\vec{x}\,) &= \frac{1}{p!}\,E_{i_1\cdots i_{d-p}}(t,\vec{x}\,)\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_{d-p}}\,. \end{aligned} \tag{1.43}$$

Hence within the Hamiltonian formulation, the first equation of (1.42) may be interpreted as two second-class constraints which relate $E_0(t,\vec{x}\,)$ to the magnetic field,

$$B_{\text{mg}} = e^2\,E_0\,,$$

on the one hand, while the variable $\tilde{E}(t,\vec{x}\,)$ and the electric field must be Hodge dual,

$$E_{\text{el}} = e^2\,\sigma^{p\,(d-p)}\,*_{(d-p)}\,\tilde{E}\,, \tag{1.44}$$

on the other hand. Furthermore this latter relation establishes the link between the configuration space variables of the Lagrangian first order formulation and the conjugate momenta within the Hamiltonian formulation, to which the electric field is equal up to a multiplicative constant.

1.4.2 Dielectric Maxwell theories

A natural extension of the Maxwell theory in 3+1 dimensions, of which the Lagrangian density is defined in (1.1), may be obtained by transmuting the coupling constant $1/e^2$ into a real-valued function of spacetime $\varepsilon(x)$. The resulting Lagrangian density then reads

$$\mathcal{L}_{\text{DMax}} = -\frac{1}{4}\,\varepsilon(x)\,F_{\mu\nu}\,F^{\mu\nu} - A_\mu\,J^\mu\,, \tag{1.45}$$

and is often called by misuse of language[10] the "dielectric Maxwell theory". Whatever its origin, we assume that the conserved current $J^\mu(x)$ should not explicitly depend on the function $\varepsilon(x)$. This function is considered as a new (possibly dynamical) scalar field and describes the properties of the medium in which the electromagnetic field propagates. However the present formulation is not the most general. Indeed the function $\varepsilon(x)$ should be replaced by a symmetric matrix $\varepsilon_{\mu\nu}(x)$ contracting the field strength tensor in the kinetic term of (1.45). This symmetric tensor accounts for the anisotropy of the medium of which the response differs whether submitted to electric or magnetic fields. For the sake of simplicity, we will consider here a scalar field $\varepsilon(x)$, implying that the response of the isotropic medium to the propagating electric

[10] This function equally characterises the polarisability or the magnetisation of the medium.

field is of the same magnitude as the magnetisation of this medium resulting from the magnetic field. This rude hypothesis is justified in Chapter 5 where soliton solutions describing an electric field embedded in a dielectric medium are constructed within an *ansatz* setting the magnetic field to zero.

The first order Lagrangian formulation of the "dielectric" Maxwell theory in 3+1 dimensions,

$$\mathcal{L}^{\text{f.o.}}_{\text{Max}} = -\frac{1}{4}\,\frac{1}{\varepsilon(x)}\,E_{\mu\nu}\,E^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\,\partial_\mu A_\nu\,E_{\rho\sigma} - A_\mu\,J^\mu\,,$$

is constructed in such a way that the scalar field $\varepsilon(x)$ only couples to the physical electromagnetic field $E_{\mu\nu}(x)$. Hence this model shares in common with the pure Maxwell theory the equation (1.39) relating the electromagnetic field to the current, but differs from it through the physical meaning of this electromagnetic field

$$E_{\alpha\beta} = \frac{1}{2}\,\varepsilon(x)\,\eta_{\alpha\mu}\,\eta_{\beta\nu}\,\epsilon^{\mu\nu\rho\sigma}\,F_{\rho\sigma}\,.$$

Under the above assumption that the current $J^\mu(x)$ is independent of $\varepsilon(x)$, as befits such Maxwell theories in a medium, this equation relates a field constructed from the Faraday tensor to a free current four-vector. Considering now the time component of this equation, the relation between the electric displacement $\vec{D}_{\text{el}}(x)$ and the free charge density $\rho_{\text{f}}(x)$ embedded in a dielectric medium is recovered

$$\vec{\nabla}\cdot\vec{D}_{\text{el}} = \rho_{\text{f}}\,, \quad \text{where we define } D^i_{\text{el}} = \frac{1}{2}\epsilon^{ijk}E_{jk} \text{ and } \rho_f = J^0\,,$$

whereas if the space components are taken into account, a link between the magnetising vector field $\vec{H}_{\text{mg}}(x)$ and a free current density $\vec{J}(x)$ is obtained

$$\vec{\nabla}\times\vec{H}_{\text{mg}} - \partial_t\vec{D}_{\text{el}} = \vec{J}\,, \quad \text{where we define } H^i_{\text{mg}} = \delta^{ij}\,E_{0j}\,.$$

The formulation of the Maxwell equations in terms of the free charge and current is thus recovered.

Let us now establish how to recover the Maxwell equations in terms of the total charge and current. The measured electric field $\vec{E}_{\text{el}}(x)$ is generated both by the net free charge density $\rho_{\text{f}}(x)$ and by the structural (or bounded) charge density $\rho_{\text{s}}(x)$ associated to the response of the dielectric medium. Likewise the total magnetic vector field $\vec{B}_{\text{mg}}(x)$ is generated by the total current density $\vec{J}_{\text{T}}(x)$. This total current includes the contributions of the usual conduction current, referred to as the free current density $\vec{J}(x)$, the bounded current associated to the magnetisation of the medium, and the current related to bounded charges. Given the familiar relations between the electric displacement and the electric vector field

$$\vec{D}_{\text{el}} = \varepsilon\,\vec{E}_{\text{el}}\,,$$

and defining the relation between the magnetisation and magnetic vector fields as

$$\vec{H}_{\rm mg} = \varepsilon\, \vec{B}_{\rm mg}\,,$$

the Maxwell equations read

$$\begin{aligned} \vec{\nabla}\cdot\vec{E}_{\rm el} &= \frac{\rho_{\rm T}}{\varepsilon_0}\,, \qquad \vec{\nabla}\times\vec{E}_{\rm el} = -\partial_t\vec{B}_{\rm mg}\,, \\ \vec{\nabla}\cdot\vec{B}_{\rm mg} &= 0\,, \qquad \vec{\nabla}\times\vec{B}_{\rm mg} - \partial_t\vec{E}_{\rm el} = \vec{J}_{\rm T}\,. \end{aligned}$$

The scalar field $\varepsilon(x)$ is assumed to possess a vacuum expectation value denoted ε_0 which is associated to the permittivity and permeability in the vacuum. The expressions for the bounded charge and current densities obviously follow from the above definitions and equations.

Finally the present analysis of the Maxwell theory in a medium is readily generalisable to tensor gauge fields of any rank wathever the number of spacetime dimensions. For example, the Maxwell theory for a 2-form gauge field propagating in a medium described by a function $\varepsilon(x)$ in 3+1 dimensions reads

$$\mathcal{L}_{\text{2-Max}} = \frac{1}{12}\,\varepsilon(x)\,H_{\mu\nu\rho}\,H^{\mu\nu\rho} + \frac{1}{2}\,B_{\mu\nu}\,K^{\mu\nu}\,, \tag{1.46}$$

where the conserved current $K^{\mu\nu}$ does not explicitly depend on the function $\varepsilon(x)$.

1.4.3 Physical formulation of the Maxwell-Higgs model

In this Subsection a rather unfamiliar perspective on the Maxwell-Higgs model in terms of its physical first order formulation will be introduced and some troubles related to gauge fixing procedures will be brought to the fore. Let us first start with the first order Lagrangian formulation of the Maxwell-Higgs model,

$$\mathcal{L}^{\rm f.o.}_{\rm AH} = -\frac{e^2}{4}\,E_{\mu\nu}\,E^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\,\partial_\mu A_\nu\,E_{\rho\sigma} + \frac{1}{2}\,|\partial_\mu\varrho - {\rm i}\,\varrho\,(A_\mu - \partial_\mu\theta)|^2 - V(\varrho^2)\,, \tag{1.47}$$

originally built from the Lagrangian density (1.16) resulting from the polar parametrisation of the complex scalar field, see (1.15). Within the context of abelian gauge field theories, there exist plenty of ways to define actions in terms of gauge invariant quantities. The most common of course remains the gauge fixing procedure. Indeed, for gauge field theories like abelian Higgs or Stueckelberg theories, a tempting possibility is to express the Lagrangian density in terms of the gauge invariant current $J^\mu(x)$. The conservation law for this current, $\partial^\mu J_\mu = 0$, in such theories acts like a

Lorentz gauge fixing procedure but in this case this "on shell" constraint results from Noether's theorem. Therefore the change of variables (1.15) and

$$A_\mu - \partial_\mu \theta = \frac{1}{\varrho^2} J_\mu = \eta \, G_\mu \, ,$$

introduced in [45, 46] for example[11], seems at first sight to be equivalent to the fixing of the unitary gauge. However the identification of the argument of the complex scalar field as longitudinal part of the gauge field arises only at the level of the equations of motion. Within the unitary gauge, this identification is first required independently of whether the equations of motion are satisfied or not.

Hence under such a reparametrisation, the first order Lagrangian formulation of the Maxwell-Higgs model now reads

$$\begin{aligned} \mathcal{L}^4_{\mathrm{MHP}} &= -\frac{e^2}{4} E_{\mu\nu} \, E^{\mu\nu} + \frac{\eta}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\mu G_\nu \, E_{\rho\sigma} + \frac{\eta^2}{2} \, \varrho^2 \, G_\mu \, G^\mu + \mathcal{L}_\varrho \\ &+ \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \Sigma_{\mu\nu} \, E_{\rho\sigma} \ , \end{aligned} \tag{1.48}$$

where $\Sigma_{\mu\nu}$ is related to the scalar field $\theta(x)$ through

$$\Sigma_{\mu\nu} = \partial_{[\mu} \, \partial_{\nu]} \theta \, .$$

The Lagrangian density $\mathcal{L}_\varrho(\varrho, \partial_\mu \varrho)$,

$$\mathcal{L}_\varrho = \frac{1}{2} \, \partial_\mu \varrho \, \partial^\mu \varrho - \frac{1}{2} \, \tilde{\mu}^2 \, \varrho^2 + \frac{1}{4} \, \lambda \, \varrho^4 \, , \tag{1.49}$$

describes the dynamics for the self-interacting scalar field $\varrho(x)$. We define the first line of (1.48) as the physical formulation of the Maxwell-Higgs model which only involves the electromagnetic field $E_{\mu\nu}(x)$ and the current generating this field.

However contrary to what happens under the choice of the unitary gauge, it is crucial to keep the term in the second line of (1.48), since otherwise, important topological content would be lost. Indeed the univalued phase in the polar decomposition of the complex scalar field $\phi(x)$ does not forbid the field $\theta(x)$ to possess a non periodic part at infinity, associated to a non zero winding number. In that case, topological considerations then require that $\phi(x)$ vanishes on some spacetime submanifold, where its phase is undefined. Therefore the unitary gauge fixing procedure becomes pathological while the introduced parametrisation of the field is problematic since boundary conditions must be specified already at the level of the Lagrangian formulation. Indeed in the physical formulation (1.48), $\Sigma_{\mu\nu}(x)$ is interpreted as the current of a vortex string of which the worldsheet coincides with the subset in spacetime of zeros of $\varrho(x)$. Models for effective magnetic strings based on this observation were built in [47, 48, 49], see also [46]. In Chapter 5 we will shed new light on this issue.

[11] We have introduced the normalisation parameter η of physical dimension $E\,L$ for later convenience.

1.5 Topological sector of gauge field theories

1.5.1 Formal limit of the Maxwell-Chern-Simons theory

In contradistinction to the usual formulation of the Maxwell theory we proceeded to a rescaling of the gauge field by the coupling constant e, so that this coupling constant now multiplies the kinetic term for the gauge field, see (1.1). This rescaling, already known within the context of the scale anomaly in QCD, enables to make the covariant derivative independent of the coupling constant whenever dynamical matter fields, whether of fermionic (QED) or bosonic character (scalar QED) are introduced. This rescaling further implies that the coupling constant multiplies the squared term in the physical electromagnetic field within the first order Lagrangian formulation of gauge field theories like the Maxwell theory, see (1.38) and (1.41), the Maxwell-Higgs model (1.47) and the Maxwell-Chern-Simons theory,

$$\mathcal{L}^{2+1}_{\mathrm{master}} = \frac{e^2}{2}\, E_\mu\, E^\mu + \frac{1}{4}\epsilon^{\mu\nu\rho}\, F_{\mu\nu}\, \left(2\, E_\rho + \kappa\, A_\rho\right) \,, \tag{1.50}$$

through an inverse coupling compared to that of the original formulation. The study of the asymptotic formal limit in the parameter e within the first order formulation of abelian gauge field theories may be considered as a first step towards a more ambitious project whose basic ideas were suggested in a heuristic way in [50]. In that paper such limits have been studied within the context of theories of the Yang-Mills type for non abelian semi-simple Lie groups in 1+1, 2+1 and 3+1 dimensions.

Usually the problem of confinement is tackled by isolating a non perturbative sector of physical field configurations. However such a procedure requires a careful treatment of topological effects which may be lost when one proceeds to gauge fixing procedures, as highlighted in Section 1.4. Furthermore current works based on lattice simulations, effective and dual field theories approaches show that the dynamics responsible for confinement may possibly be dominated by topological effects, generated by a topological sector of configuration space. In [50] such topological sector candidates appear in an heuristic way through formal limits in the gauge coupling constant in non abelian gauge field theories. Notice that abelian gauge field theories already exhibit such characteristics. For example the naive limit of an infinite coupling constant $e \to \infty$ in the usual Lagrangian the MCS theory (1.26) leads to a pure topological field theory (TFT) of the AF-type, see (1.21), while a TFT of the BF type is recovered in the limit $e \to 0$ within the first order Lagrangian formulation (1.50).

In this Thesis, we have managed to isolate properly this topological sector and to show in which sense it generates topological effects in abelian topological mass generation mechanisms. In fact working with abelian gauge theories is not unrealistic since lattice simulations, studies at lower dimensions and effective models seem to show that the physical sector at low energy resides in the Cartan subalgebra of the non abelian gauge group. Furthermore, topologically massive gauge theories generate a mass gap of topological origin. Therefore the systematic consideration of topological effects for this type of theories, often neglected in the literature, could provide crucial insights towards the study of confinement in non abelian gauge field theories.

1.5.2 Our solution: a dual projection method

The main difficulty which arises in such a programme is to give meaning to the formal limits introduced in [50]. The analysis of the limit $e \to 0$ did not reach significant results. Within the pure Maxwell theory it has been shown in [12] that the limit $e \to \infty$ is ill-defined in the physical spectrum of the theory. Our approach to deal with this type of issue is totally different and only applies to theories generating a mass gap, that is in this case TMGT. Considering in Chapters 2 and 3 a dual projection method and a generalisation of the Lowest Landau level projection, we will properly define the formal limit mentioned above, at the classical as well as at the quantum level.

The notion of duality (for a review, see [51]) has played a vital role in a large range of fields in theoretical physics as distinct as statistical mechanics, confinement in Yang-Mills theories or the quest for a fundamental quantum unification of all particles and interactions. Duality relies on the existence of two equivalent descriptions of a model using different fields and turns out to be especially interesting when an explicit relation between the fields is established since it typically exchanges strong and weak coupling regimes. It may then allow for perturbative calculations in the variables of the dual theory. In this Thesis another interesting feature which may arise from dualisation processes is introduced. Indeed, upon a convenient redefinition of fields, gauge variant degrees of freedom are decoupled from the physical ones and thus the dual formulation is factorised. There exists quite a number of examples of gauge theories where this kind of technique has been developed and which is sometimes referred to as “dual projections” [52, 53]. The difficulty then relates to the fairly rare existence of such a reparametrisation being local, linear and conserving the number of degrees of freedom. That is not the case for the Maxwell theory, for example, as recalled in Section 1.4. However, we will show for the first time that the dual factorisation is possible for TMGT whether of the BF or the AF type in any dimension.

Then more realistic theories are obtained by extending our analysis to non abelian gauge groups and by coupling to matter fields. In such cases, the action resulting from our reparametrisation turns out to be split into two sectors partially decoupled. This may offer perspectives in the development of new approximation schemes from the non perturbative topological sector. Finally, Chapters 4 and 5 of this Thesis will be dedicated to other unexpected consequences of our factorisation, when topologically massive gauge theories interact with real scalar fields through dielectric couplings. In fact, we will properly establish in Chapter 4 that all the local gauge invariant mass generation mechanisms introduced in Section 1.2, including the celebrated BEH mechanism, are related through an intricate network of dualities, taking into account in Chapter 5 the presence of topological terms generating topological effects.

CHAPTER 2

Factorisation of topologically massive gauge theories

The original fields involved in the definition of any gauge theory do not generate physical configurations since these fields are not gauge invariant. Hence two approaches to isolate genuine physical degrees of freedom are available. The first involves some gauge fixing procedure which usually suffer Gribov problems. The second consists in constructing a factorised dual formulation. Indeed, following a convenient redefinition of the fields, gauge variant degrees of freedom are decoupled from the physical ones. The main difficulty arising for this program is the fairly rare existence of such reparametrisations which at the same time are local and conserve the number of degrees of freedom. Moreover field redefinitions within the covariant Lagrangian formulation are not necessarily associated to equivalent canonical transformations within the corresponding Hamiltonian formulation while preserving at each step gauge invariance. However, topologically massive gauge theories do not encounter such restrictions. Through a local and linear field redefinition within the first order Lagrangian formulation, or the associated canonical transformation within the Hamiltonian one, the dual action possesses the same gauge symmetry structure as the original theory and is decoupled into a propagating sector of massive physical variables and a sector with gauge variant variables defining a topological field theory. These results hence provide a complete understanding of a novel general structure for topologically massive gauge theories, referred to as "Topological-Physical" (TP) factorisation, which involves both the Lagrangian and Hamiltonian formulations.

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2.1 Hamiltonian formalism and TP Factorisation

This Chapter discusses a new salient property of the abelian TMGT of the BF type (1.23) or AF type (1.25) valid whatever the number of space dimensions d and the tensorial rank of the fields: the Topological-Physical (TP) factorisation of their degrees of freedom. This new result, presented in one of our paper [10], was first achieved within the Hamiltonian formulation through a canonical transformation of classical phase space leading to two independent and decoupled sectors[1]. The first of these sectors, namely the "physical" one, consists of gauge invariant variables which are canonically conjugate and describes massive propagating physical degrees of freedom. The second sector, namely the "topological" one, consists of canonically conjugate gauge variant variables which are decoupled from the total Hamiltonian and, hence, are non dynamical. This sector is equivalent to a pure TFT of the BF or AF type. This factorisation enables the identification of a mass generating mechanism for any p-form or, by Hodge dualisation, any $(d-p)$-form. These two possible "pictures" are constructed without introducing any gauge fixing condition or second-class constraint whatsoever as has heretofore always been the case in the literature.

2.1.1 The case of the Maxwell-Chern-Simons theory

The famous Maxwell-Chern-Simons (MCS) theory defined in (1.26) represents a typical example of topologically massive gauge theories of the AF type which is defined in 2+1 dimensions. For pedagogical reasons, our factorisation will be first introduced within the Hamiltonian formulation of the MCS theory in the plane, which is the simplest case of the topological mass generation mechanism where our factorisation applies. Next the general case of topologically massive gauge theories of the BF type defined on spacetime manifolds of any dimension will be addressed.

The total first-class Hamiltonian density (1.33) for the Maxwell-Chern-Simons theory has already been obtained in Section 1.3 through an exhaustive analysis of constraints. However the kinematic variables which span the phase space (1.28) of the MCS theory are not physical since the gauge field $A_i(t,\vec{x}\,)$ and its conjugate momentum $P^i(t,\vec{x}\,)$ are not invariant under the $U(1)$ abelian gauge transformation. However instead of proceeding to a gauge fixation procedure or solving the first-class constraint, the physical Hamiltonian for the MCS theory may be obtained through a novel local field redefinition which preserves canonical commutation relations.

[1] In other words, the Poisson brackets of variables belonging to the two distinct sectors vanish identically.

Let us first consider the new field variable

$$2\,\mathcal{A}_i = A_i - \frac{2}{\kappa}\,\epsilon_{ij}\,P^j\,. \tag{2.1}$$

This choice is made in such a way that Gauss' law in (1.33) be expressed in terms of this variable only :

$$\kappa\,\mathcal{A}_0\,\epsilon^{ij}\,\partial_i\mathcal{A}_j \approx 0\,. \tag{2.2}$$

The field components of the new variables $\mathcal{A}_i(t,\vec{x}\,)$ form a pair of canonically conjugate variables of which the Poisson bracket reads

$$\{\mathcal{A}_1(t,\vec{x}\,)\,,\,\mathcal{A}_2(t,\vec{y}\,)\} = \frac{1}{\kappa}\,\delta^2(\vec{x}-\vec{y}\,)\,, \tag{2.3}$$

The abelian gauge transformation defined in its local form in (1.32) acts on these variables according to

$$\mathcal{A}'_i = \mathcal{A}_i + \partial_i\alpha\,. \tag{2.4}$$

A second new field variable $E^i(t,\vec{x}\,)$ may be defined as

$$E^i = P^i - \frac{\kappa}{2}\,\epsilon^{ij}\,A_j\,, \tag{2.5}$$

and turns out to be complementary to the first one introduced above. When considered in combination with the equations of motion, this variable is proportional to the electric field $E^i_{\rm el}(t,\vec{x}\,)$, see (1.36), of which the Poisson bracket of its field components no longer vanishes,

$$\left\{E^1(t,\vec{x}\,)\,,\,E^2(t,\vec{y}\,)\right\} = -\kappa\,\delta^2(\vec{x}-\vec{y}\,)\,, \tag{2.6}$$

in contradistinction to the pure (massless) Maxwell theory.

This canonical transformation defines a new parametrisation of phase space which is then decoupled into two independent and orthogonal sectors. By orthogonal is meant the fact that their Poisson brackets are mutually vanishing:

$$\left\{\mathcal{A}_i(t,\vec{x}\,)\,,\,E^j(t,\vec{y}\,)\right\} = 0\,,$$

and by independent the fact that these two sectors do not speak to one another at the level of the Hamiltonian density, provided that the Lagrange multipliers in the associated first order formulation are redefined in an appropriate way, namely

$$u = \dot{A}_0\,, \qquad (u' - A_0) = -\mathcal{A}_0 - \frac{1}{e^2\,\kappa^2}\,\partial_i\left(E^i - \frac{\kappa}{2}\,\epsilon^{ij}\,\mathcal{A}_j\right)\,. \tag{2.7}$$

where a dot stands for differentiation with respect to the time coordinate, $t\in\mathbb{R}$.

Hence the fundamental Hamiltonian density of the Hamiltonian nested structures for this gauge invariant dynamics [41] reads within its factorised formulation

$$\mathcal{H} = \frac{e^2}{2}\,\left(E^i\right)^2 + \frac{1}{2}\frac{1}{e^2\,\kappa^2}\left(\partial_i E^i\right)^2 - \kappa\,\mathcal{A}_0\,\epsilon^{ij}\,\partial_i\mathcal{A}_j + \mathrm{ST}\,. \tag{2.8}$$

Obviously, $\mathcal{A}_0$ is a Lagrange multiplier enforcing the first-class constraint which generates the small gauge transformations (2.4).

When restricted to the physical subspace for which this constraint is satisfied, the above gauge invariant Hamiltonian reduces to a functional depending only on the dynamical physical sector which solely consists of the pseudo-vector field $E^i(t,\vec{x}\,)$. This sector, which is invariant under parity, describes the dynamics of a single propagating massive "field degree of freedom" in 2+1 dimensions.

The second sector which consists of the variable $\mathcal{A}_i(t,\vec{x}\,)$ whose components form a pair of canonically conjugate variables actually shares the same symplectic structure, (2.3), Gauss law constraint, (2.2) and gauge transformation, (2.4), as the phase space description of the pure Chern-Simons theory. Hence this "Chern-Simons sector" accounts for the parity violating Chern-Simons theory embedded into the Maxwell-Chern-Simons theory. The Chern-Simons theory a being topological field theory, its Hamiltonian density vanishes identically. In the same way, as we will see in Chapter 3, this "CS sector" may imply, depending on the topology of the underlying space manifold, a degeneracy in the energy spectrum, hence its close relation with the Landau problem in quantum mechanics.

2.1.2 The case of TMGT of *BF* type in any dimension

Hamiltonian formulation

Because of the built-in gauge invariances of these systems, the analysis of the constraints [41, 42] of topologically massive gauge theories is required in order to identify their Hamiltonian formulation. Given the total action (1.23) written out in component form the associated Lagrangian density reads,

$$\begin{aligned}
\mathcal{L}^{d+1}_{\mathrm{TMGT}} &= \frac{\sqrt{h}}{2\,e^2}\,\frac{\sigma^p}{(p+1)!}\,F_{\mu_1\cdots\mu_{p+1}}\,F^{\mu_1\cdots\mu_{p+1}} \\
&+ \frac{\sqrt{h}}{2\,g^2}\,\frac{\sigma^{d-p}}{(d-p+1)!}\,H_{\nu_1\cdots\nu_{d-p+1}}\,H^{\nu_1\cdots\nu_{d-p+1}} \\
&+ \kappa\,\frac{(1-\xi)}{(1+p)!\,(d-p)!}\,\epsilon^{\mu_1\cdots\mu_{p+1}\nu_1\cdots\nu_{d-p}}\,F_{\mu_1\cdots\mu_{p+1}}\,B_{\nu_1\cdots\nu_{d-p}} \\
&- \kappa\,\frac{\xi\,\sigma^p}{p!\,(d-p+1)!}\,\epsilon^{\mu_1\cdots\mu_p\nu_1\cdots\nu_{d-p+1}}\,A_{\mu_1\cdots\mu_p}\,H_{\nu_1\cdots\nu_{d-p+1}}\,,
\end{aligned} \tag{2.9}$$

where our conventions introduced in Section 1.1 apply again. The above expression, with the single parameter κ multiplying each of the topological $B\wedge F$ and $A\wedge H$ terms while ξ parametrises a possible surface term, does not entail any loss of generality. Had two independent parameters κ and λ multiplying each of the topological terms been introduced, only their sum, $(\kappa+\lambda)$, would have been physically relevant, the other combination corresponding in fact to a pure surface term.

Adopting synchronous coordinates on $\mathcal{M} = \mathbb{R}\times\Sigma$, see (1.27), the configuration space variable $A(t,\vec{x}\,)$ may be separated into its temporal component $\mathrm{d}t\wedge A_0(t,\vec{x}\,)$ with $A_0(t,\vec{x}\,)$ belonging to $\Omega^{p-1}(\Sigma)$, and its components $\tilde{A}(t,\vec{x}\,)$ restricted to $\Omega^p(\Sigma)$,

$$\begin{aligned} A_0(t,\vec{x}\,) &= \frac{1}{(p-1)!}\,A_{0i_1\cdots i_{p-1}}(t,\vec{x}\,)\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_{p-1}}\,, \\ \tilde{A}(t,\vec{x}\,) &= \frac{1}{p!}\,A_{i_1\cdots i_p}(t,\vec{x}\,)\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}\,. \end{aligned} \tag{2.10}$$

A similar decomposition applies to the $(d-p)$-form $B(t,\vec{x}\,)$. The actual phase space variables are then the spatial components $\tilde{A}$ and $\tilde{B}$ along with their conjugate momenta $\tilde{P}$ and $\tilde{Q}$ defined to be the following differential forms on Σ,

$$\begin{aligned} \tilde{P} &= \frac{1}{p!}\frac{1}{\sqrt{h}}\,\tilde{h}_{i_1j_1}\ldots\tilde{h}_{i_pj_p}\,P^{i_1\cdots i_p}\,\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_p}\,, \\ \tilde{Q} &= \frac{1}{(d-p)!}\frac{1}{\sqrt{h}}\,\tilde{h}_{i_1j_1}\ldots\tilde{h}_{i_{d-p}j_{d-p}}\,Q^{i_1\cdots i_{d-p}}\,\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_{d-p}}\,, \end{aligned} \tag{2.11}$$

of which the pseudo-tensorial space components are $P^{i_1\cdots i_p}$ and $Q^{i_1\cdots i_{d-p}}$. Expressed in terms of the configuration space variables, these latter quantities are given as

$$\begin{aligned} P^{i_1\cdots i_p} &= \frac{\sqrt{h}}{e^2}\,F_{0j_1\cdots j_p}\,\tilde{h}^{i_1j_1}\ldots\tilde{h}^{i_pj_p} + \kappa\,\frac{(1-\xi)}{(d-p)!}\,\epsilon^{i_1\cdots i_pj_1\cdots j_{d-p}}\,B_{j_1\cdots j_{d-p}}\,, \\ Q^{i_1\cdots i_{d-p}} &= \frac{\sqrt{h}}{g^2}\,H_{0j_1\cdots j_{d-p}}\,\tilde{h}^{i_1j_1}\ldots\tilde{h}^{i_{d-p}j_{d-p}} \\ &\quad -\ \kappa\,\frac{\xi}{p!}\,\sigma^{p(d-p)}\,\epsilon^{i_1\cdots i_{d-p}j_1\cdots j_p}A_{j_1\cdots j_p}\,, \end{aligned} \tag{2.12}$$

while the canonical brackets,

$$\begin{aligned} \left\{A_{i_1\cdots i_p}(t,\vec{x}\,),P^{j_1\cdots j_p}(t,\vec{y}\,)\right\} &= \delta^{j_1}_{[i_1}\ldots\delta^{j_p}_{i_p]}\,\delta^{(d)}(\vec{x}-\vec{y}\,)\,, \\ \left\{B_{i_1\cdots i_{d-p}}(t,\vec{x}\,),Q^{j_1\cdots j_{d-p}}(t,\vec{y}\,)\right\} &= \delta^{j_1}_{[i_1}\ldots\delta^{j_{d-p}}_{i_{d-p}]}\,\delta^{(d)}(\vec{x}-\vec{y}\,)\,, \end{aligned} \tag{2.13}$$

characterise the symplectic structure of Poisson brackets. *A priori*, phase space also includes the canonically conjugate variables A_0 and P^0, and B_0 and Q^0.

The Legendre transform of the Lagrangian (2.9) leads to the total gauge invariant Hamiltonian,

$$\begin{aligned} H &= \frac{e^2}{2}\left(*\tilde{P}-\kappa\,(1-\xi)\,\tilde{B}\right)^2+\frac{1}{2\,e^2}\left(\mathrm{d}\tilde{A}\right)^2+\left(u,P^0\right) \\ &+ \frac{g^2}{2}\left(*\tilde{Q}+\kappa\,\xi\,\sigma^{p(d-p)}\tilde{A}\right)^2+\frac{1}{2\,g^2}\left(\mathrm{d}\tilde{B}\right)^2+\left(v,Q^0\right)+(\text{surface term}) \\ &+ \int_\Sigma \sigma^p\,(u'+A_0)\wedge\mathrm{d}\left(*\tilde{P}+\kappa\,\xi\,\tilde{B}\right) \\ &+ \int_\Sigma \sigma^{d-p}(v'+B_0)\wedge\mathrm{d}\left(*\tilde{Q}-\kappa\,(1-\xi)\,\sigma^{p(d-p)}\,\tilde{A}\right). \end{aligned} \tag{2.14}$$

In this expression as well as throughout hereafter, the Hodge $*$ operation is now considered only on the space manifold Σ endowed with the Riemannian metric $\tilde{h}_{ij}$. In (2.14) the inner product on $\Omega^k(\Sigma)\times\Omega^k(\Sigma)$ is constructed as

$$(\omega_k)^2=(\omega_k,\omega_k) \quad \text{with} \quad (\omega_k,\eta_k)=\int_\Sigma \omega_k\wedge *\eta_k. \tag{2.15}$$

The quantities u' and v' are Lagrange multipliers for the two first-class constraints associated to the two abelian gauge symmetries while u and v are those for the first-class constraints $P^0=0$ and $Q^0=0$ arising because the fields A_0 and B_0 are auxiliary degrees of freedom of which the time derivatives do not contribute to the action. Upon reduction to the basic layer of the Hamiltonian nested structure [41], P^0 and Q^0 decouple from the system whereas A_0 and B_0 play the role of Lagrange multipliers enforcing the two Gauss laws. These constraints generate those gauge transformations in (1.19) which are continuously connected to the identity transformation, namely the small gauge symmetries, one generated by the fields $\tilde{P}$ and $\tilde{B}$ and the other by $\tilde{A}$ and $\tilde{Q}$, respectively. Note that given Hodge duality, the phase space variables are associated to isomorphic spaces, $\Omega^p(\Sigma)\equiv\Omega^{d-p}(\Sigma)$. Hence at any given spacetime point, phase space has dimension $4C_d^p$.

Topological-Physical (TP) factorisation

The above results are well-known. However the fields used to construct the theory do not necessarily create physical states since these are not gauge invariant variables. Therefore, let us now introduce the new Physical-Topological factorisation of the classical theory, by also requiring that these field redefinitions are canonical and preserve canonical commutation relations. First consider the quantities

$$\mathcal{A}=-\frac{1}{\kappa}\,\sigma^{p(d-p)}\,*\tilde{Q}+(1-\xi)\,\tilde{A}, \qquad \mathcal{B}=\frac{1}{\kappa}*\tilde{P}+\xi\,\tilde{B}, \tag{2.16}$$

defined on the dual sets $\Omega^p(\Sigma)$ and $\Omega^{d-p}(\Sigma)$. This choice is made in such a way that the two Gauss laws are expressed in term of these variables only, as is the case for a topological BF theory,

$$\kappa\,\sigma^{p(d-p)}\,\mathrm{d}\mathcal{A}=0,\qquad \sigma^p\,\kappa\,\mathrm{d}\mathcal{B}=0. \tag{2.17}$$

As a matter of fact, these variables are canonically conjugate,

$$\left\{\mathcal{A}_{i_1\cdots i_p}(t,\vec{x}\,),\mathcal{B}_{j_1\cdots j_{d-p}}(t,\vec{y}\,)\right\}=\frac{1}{\kappa}\,\epsilon_{i_1\cdots i_p j_1\cdots j_{d-p}}\,\delta^{(d)}(\vec{x}-\vec{y}\,). \tag{2.18}$$

The two finite gauge transformations in (1.19) act on these new variables according to the relations,

$$\mathcal{A}'=\mathcal{A}+\alpha,\qquad \mathcal{B}'=\mathcal{B}+\beta. \tag{2.19}$$

At a given spacetime point, these canonically conjugate variables carry $2C_d^p$ degrees of freedom. The remaining $2C_d^p$ degrees of freedom are associated to the following pair of gauge invariant variables,

$$G=\tilde{Q}+\kappa\,\xi\,*\tilde{A},\qquad E=\tilde{P}-\kappa\,(1-\xi)\,\sigma^{p(d-p)}\,*\tilde{B}. \tag{2.20}$$

Their pseudo-tensor Lorentz components are defined as in (2.11) while they possess the following non vanishing canonical Poisson brackets,

$$\left\{E^{i_1\cdots i_p}(t,\vec{x}\,),G^{j_1\cdots j_{d-p}}(t,\vec{y}\,)\right\}=-\kappa\,\epsilon^{i_1\cdots i_p j_1\cdots j_{d-p}}\,\delta^{(d)}(\vec{x}-\vec{y}\,). \tag{2.21}$$

When considered in combination with the equations of motion, these variables are proportional to the non commutative electric fields associated, respectively, to the field strength tensors of A and B. Consequently, we have achieved a coherent reparametrisation of phase space which, in fact, factorises the system into two orthogonal sectors, namely sectors of which mutual Poisson brackets vanish identically,

$$\begin{aligned}
&\left\{\mathcal{A}_{i_1\cdots i_p}(t,\vec{x}\,),E^{j_1\cdots j_p}(t,\vec{y}\,)\right\}=0, &&\left\{\mathcal{A}_{i_1\cdots i_p}(t,\vec{x}\,),G^{j_1\cdots j_{d-p}}(t,\vec{y}\,)\right\}=0,\\
&\left\{\mathcal{B}_{i_1\cdots i_{d-p}}(t,\vec{x}\,),E^{j_1\cdots j_p}(t,\vec{y}\,)\right\}=0, &&\left\{\mathcal{B}_{i_1\cdots i_{d-p}}(t,\vec{x}\,),G^{j_1\cdots j_{d-p}}(t,\vec{y}\,)\right\}=0.
\end{aligned}$$

Finally in order to obtain the factorised formulation of the fundamental Hamiltonian, the Lagrange multipliers in (2.14) may be redefined in a convenient way as

$$\begin{aligned}
u&=\dot{A}_0, &\qquad \mathcal{A}_0&=A_0+u'+\frac{\sigma^{(p-1)(d-p)}}{2\,g^2\,\kappa^2}*\mathrm{d}\left(\kappa\,\mathcal{B}-2*E\right)\,, \\
v&=\dot{B}_0, &\qquad \mathcal{B}_0&=B_0+v'+\frac{\sigma^p}{2\,e^2\,\kappa^2}*\mathrm{d}\left(\kappa\,\mathcal{A}+(1)^{p(d-p)}\,2\,*G\right)\,.
\end{aligned} \tag{2.22}$$

Consequently the basic total first-class Hamiltonian of the system reads,

$$\begin{aligned} H[E,G,\mathcal{A},\mathcal{B}] &= \frac{e^2}{2}\,(E)^2 + \frac{1}{2\,\kappa^2\,g^2}\,\left(\mathrm{d}^\dagger E\right)^2 + \frac{g^2}{2}\,(G)^2 + \frac{1}{2\,e^2\,\kappa^2}\,\left(\mathrm{d}^\dagger G\right)^2 \\ &+ \kappa\int_\Sigma \sigma^p\,\mathcal{A}_0\wedge\mathrm{d}\mathcal{B} - \sigma^{(p+1)(d-p)}\,\mathcal{B}_0\wedge\mathrm{d}\mathcal{A}\,. \end{aligned} \tag{2.23}$$

Obviously, $\mathcal{A}_0$ and $\mathcal{B}_0$ are Lagrange multipliers enforcing the first-class constraints which generate the small gauge transformations in (2.19),

$$\mathcal{G}^{(1)} = \mathrm{d}\mathcal{A}\,, \qquad \mathcal{G}^{(2)} = \mathrm{d}\mathcal{B}. \tag{2.24}$$

When restricted to the physical subspace for which these constraints are satisfied, the above gauge invariant Hamiltonian reduces to a functional depending only on the dynamical physical sector, given by the expression in the first line of (2.23).

The topological sector

This reparametrisation of phase space have indeed achieved the announced factorisation. A first sector is comprised of the variables constructed in (2.16), which decouple from the physical Hamiltonian and are therefore non propagating degrees of freedom. Furthermore, the canonically conjugate variables $\mathcal{A}$ and $\mathcal{B}$ actually share the same Poisson brackets, Gauss law constraints and gauge transformations as the phase space description of a pure BF topological field theory constructed only from the topological terms in the action (2.9). Hence this "topological field theory (TFT) sector" accounts for the BF theory embedded into the topologically massive gauge theory.

The physical sector

Physical and non physical degrees of freedom are mixed in the original phase space. Our redefinition of fields deals with the original degrees of freedom in such a way that within the Hamiltonian formalism, non propagating (and gauge variant) degrees of freedom are decoupled from the dynamical sector. This latter sector describes only physical degrees of freedom, namely the gauge invariant canonically conjugate electric fields (up to a multiplicative constant), which diagonalise the physical Hamiltonian (2.23) in such a way that they acquire a mass through a mixing of the original variables (2.20). However the Poisson bracket structure remains unaffected since these field redefinitions define merely a canonical transformation. Hence within the physical sector, the Hamiltonian formulation of a Proca theory is recovered which offers then two equivalent interpretations or "pictures".

On account of Hodge duality between $\Omega^p(\Sigma)$ and $\Omega^{d-p}(\Sigma)$, one readily identifies in the dynamical sector the Hamiltonian of a massive p-form field of mass $m=\hbar\mu$,

$$H[C,E,\mathcal{A},\mathcal{B}] = \frac{\mu^2}{2}\,(C)^2 + \frac{1}{2}\,(\mathrm{d}C)^2 + \frac{1}{2}\,(E)^2 + \frac{1}{2\,\mu^2}\,(\mathrm{d}E)^2 + H_{\text{TFT}}[\mathcal{A},\mathcal{B}].$$

In comparison with (2.23) the following identifications have been applied,

$$\mu = |\kappa\, e\, g|\,, \qquad E \to \frac{E}{e}\,, \qquad *G = e\,\kappa\,\sigma^{p(d-p)}\,C,$$

where C is a p-form field of which the Lorentz components are covariant in the manner of (2.10). Physical phase space is then endowed with the elementary Poisson brackets

$$\left\{C_{i_1\cdots i_p}(t,\vec{x}\,), E^{j_1\cdots j_p}(t,\vec{y}\,)\right\} = \delta^{j_1}_{[i_1}\,\ldots\delta^{j_p}_{i_p]}\,\delta^{(d)}(\vec{x}-\vec{y}\,)\,.$$

Alternatively one may also obtain the Hamiltonian of a massive $(d-p)$-form field of mass $m=\hbar\mu$,

$$H[C,G,\mathcal{A},\mathcal{B}] = \frac{\mu^2}{2}\,(C)^2 + \frac{1}{2}\,(\mathrm{d}C)^2 + \frac{1}{2}\,(G)^2 + \frac{1}{2\,\mu^2}\,(\mathrm{d}G)^2 + H_{\text{TFT}}[\mathcal{A},\mathcal{B}],$$

in which, in comparison with (2.23), the following identifications have been applied,

$$G \to \frac{G}{g}\,, \qquad *E = -g\,\kappa\,C.$$

In this case, C is a $(d-p)$-form field with covariant Lorentz components and

$$\left\{C_{i_1\cdots i_{d-p}}(t,\vec{x}\,), G^{j_1\cdots j_{d-p}}(t,\vec{y}\,)\right\} = \delta^{j_1}_{[i_1}\,\ldots\delta^{j_{d-p}}_{i_{d-p}]}\,\delta^{(d)}(\vec{x}-\vec{y}\,)\,,$$

are the elementary Poisson brackets for these physical phase space variables.

To conclude this discussion of the factorised Hamiltonian formulation of these TMGT, let us emphasize once more that no gauge fixing procedure whatsoever was applied, in contradistinction to all discussions available until now in the literature, leading to an identification of the physical content of these theories. Through the present approach, the TFT content of TMGT is made manifest in a transparent and simple manner, with in addition a decoupling of the actual physical and dynamical sector of the system from its purely topological one, the latter carrying only topological information characteristic of the topology of the underlying spacetime manifold.

2.2 Covariant extension

This Section presents results of one of our papers [11] where it has been shown that in 2+1 dimensions the canonical transformation introduced in the previous Section within the Hamiltonian formulation is complementary to a dual projection for the Lagrangian first order formulation of the Maxwell-Chern-Simons theory [52, 53]. In the same way, the covariant extension of the factorisation identified within their Hamiltonian formulation leads to a dual projection for topologically massive gauge theories in any dimension and for all tensorial ranks, which has not been considered previously. The covariant extension proceeds then by extending the relations (2.16) and (2.20) between the phase space fields to a covariant reparametrisation of their associated spacetime components within the Lagrangian first order formulation, introduced in Section 1.4 (keeping in mind the relation (2.12) between the conjugate momenta and the time derivative of the original gauge fields).

Contrary to what is sometimes tacitly taken for granted in the literature, the correspondence between a change of variables within the Lagrangian first order formulation and its associated canonical transformation within the Hamiltonian formulation is far from being obvious. Actually, the covariant extension of canonical transformations is trivial in the infrared limit, namely when only the global sector of zero momentum modes is retained. In that case, any covariant factorisation or soldering technique is associated to a corresponding canonical transformation within the Hamiltonian formulation. However this feature does not necessarily survive for field theories. As an example, the soldering that fuses self-dual and anti-self-dual Lagrangians into the Maxwell-Chern-Simons-Proca theory cannot be associated to a canonical transformation within the Hamiltonian formulation [54], although it is the case in the infrared limit [55, 56]. However gauge field theories like TMGT do not encounter such restrictions.

2.2.1 The case of the Maxwell-Chern-Simons theory

The reduction of a "master" Lagrangian [57] accounts for the common origin of both the Maxwell-Chern-Simons (MCS) and "self-dual" Lagrangians [58]. The master Lagrangian is the first order form of the MCS Lagrangian after the introduction of gauge invariant auxiliary fields f_μ, readily reducible through Gaussian integration,

$$\mathcal{L}^{2+1}_{\rm master} = \frac{1}{2}e^2\, f_\mu\, f^\mu + \frac{1}{4}\epsilon^{\mu\nu\rho}\, F_{\mu\nu}\, \left(2\, f_\rho + \kappa\, A_\rho\right) . \tag{2.25}$$

However, to a certain extent the reduction of the master Lagrangian as introduced in [57] is analogous to a procedure of gauge fixing. Indeed the reduction of gauge variant variables within the Lagrangian formulation is analogous to the resolution of the associated first-class "Gauss" constraint within the Hamiltonian formulation (1.31).

In contradistinction to the master Lagrangian method [57], the dual factorised theory is constructed through a local and linear field redefinition, hence of which the path integral Jacobian is field independent, leading to a redefinition of the master action $S_{\rm master}[A,f] \to S_{\rm SD}[E,\mathcal{A}]$, namely

$$E_\mu\left(A_\mu, f_\mu\right) = f_\mu, \qquad \mathcal{A}_\mu\left(A_\mu, f_\mu\right) = \frac{1}{\kappa} f_\mu + A_\mu. \tag{2.26}$$

This transformation resulting from the Lorentz covariant extension of the phase space canonical transformation (2.1) and (2.5) is equivalent to that used in [52, 53] and so the dual projection technique is recovered. Note that this field redefinition is well defined provided only the topological mass parameter κ is non vanishing, $\kappa \neq 0$. Upon reduction through Gaussian integration, the gauge invariant variables E_μ are found to correspond to the electric and magnetic field components,

$$E_i \equiv \frac{1}{e^2}\,\epsilon_{ij}\,E^j_{\rm el}, \qquad E_0 \equiv \frac{1}{e^2}\,B_{\rm mg}.$$

Consequently, a coherent reparametrisation of configuration space is achieved. In fact, it factorises the action into two decoupled contributions,

$$\mathcal{L}^{2+1}_{\rm fact} = \mathcal{L}_{\rm SD}\left(E_\mu, \partial_\mu E_\nu\right) + \mathcal{L}_{\rm CS}\left(\mathcal{A}_\mu, \partial_\mu \mathcal{A}_\nu\right) + \text{ST}\,.$$

In deriving this expression a total surface term “ST” mixing the two types of field variables has been ignored, since it does not contribute for any appropriate choice of boundary conditions. It may, however, play a role when the quantum field theory is defined on a manifold with boundaries.

The physical self-dual part $\mathcal{L}_{\rm SD}$ consists of Proca and topological mass terms,

$$\mathcal{L}_{\rm SD} = \frac{1}{2} e^2\,E_\mu\,E^\mu - \frac{1}{2\kappa}\epsilon^{\mu\nu\rho}\,\partial_\mu E_\nu\,E_\rho.$$

This part describes a single propagating spin one free excitation of mass $m = \hbar\,\kappa\,e^2$ and violates parity. The Legendre transformation along with the reduction of second-class constraints inherent to such singular systems give back the physical part of the factorised Hamiltonian density (2.8). This is also the case for the topological sector, showing the closed structure formed by our TP factorisation between the Hamiltonian and Lagrangian formulations.

The second part $\mathcal{L}_{\rm CS}$ consists of gauge variant variables defining a purely topological Chern-Simons theory,

$$\mathcal{L}_{\rm CS} = \frac{1}{2}\kappa\,\epsilon^{\mu\nu\rho}\,\partial_\mu \mathcal{A}_\nu\,\mathcal{A}_\rho. \tag{2.27}$$

This last part, already expected within the path integral quantisation approach [59], is absent from the dual Lagrangian when the master action method [57] is used in which case all the topological content inherited from the original Chern-Simons term is lost. In particular, non trivial topological features become manifest in the presence of external sources, or when the space manifold Σ has non trivial topology. It is also noteworthy to mention that in the infrared limit dual projection techniques bring to the fore the existence of the Z_2 quantum anomaly of topological origin [56, 60].

As far as the local part is concerned, the fact that the pure Chern-Simons theory describes gauge fields of flat connection implies, in combination with (2.26), that

$$\kappa\,\epsilon^{\mu\nu\rho}\,\partial_\mu\mathcal{A}_\nu = \kappa\,\epsilon^{\mu\nu\rho}\,\partial_\mu A_\nu + \epsilon^{\mu\nu\rho}\,\partial_\mu f_\nu \approx 0.$$

One recovers of course the condition for the reduction of the master action in [57], but in the present approach this condition is required as a weak constraint preserving the gauge symmetry content between the dual formulations. The present factorisation may be generalised to TMGT of the AF type in any dimension, see (1.25).

2.2.2 The case of TMGT of *BF* type in any dimension

The equivalence between gauge non-invariant first order mass generating theories for any p-form and topologically massive gauge theories (TMGT) of the AF or BF type has so far been shown in diverse dimensions through the Hamiltonian embedding due to Batalin, Fradkin and Tyutin (BFT), either partial [61, 62] or complete [63, 64], through the covariant gauge embedding method [65, 66] within the Lagrangian formulation, through the master action [67], etc. All methods developed so far share a common characteristic, namely that in fact the dual action does not possess the same gauge symmetry content as the original formulation. Hence at the quantum level the equivalence between the two dual formulations applies only for pure theories defined on space manifolds of trivial topology.

The TP factorisation approach readily applies to topological mass generation in any dimension and for all tensorial ranks, whatever the topology of the space manifold. In order to construct the dual factorised action of TMGT of the BF type, the original action (1.23) must be written in its first order form after the introduction of gauge invariant auxiliary $(d-p)$- and p-form fields $\mathfrak{f}$ and $\mathfrak{h}$, respectively,

$$\begin{aligned} S_{\rm master} &= \frac{e^2}{2}\,(\mathfrak{f})^2 + \frac{g^2}{2}\,(\mathfrak{h})^2 + \int_{\mathcal{M}} F\wedge\mathfrak{f} + H\wedge\mathfrak{h} \\ &+ \kappa\int_{\mathcal{M}} (1-\xi)\,F\wedge B - \sigma^p\,\xi\,A\wedge H\,. \end{aligned} \tag{2.28}$$

The convenient notation $(\omega_k)^2 = -\sigma^k\,(\omega_k,\omega_k)$ is defined from the inner product (1.4). A simple local and linear transformation in the master action (2.28), of field independent path integral Jacobian and inducing the redefinition $S_{\rm master}[A,B,\mathfrak{f},\mathfrak{h}] \rightarrow S_{\rm fact}[E,G,\mathcal{A},\mathcal{B}]$, namely,

$$\begin{aligned} E &= \mathfrak{f}, \qquad & \mathcal{A} &= A - \frac{1}{\kappa}\sigma^{p(d-p)}\,\mathfrak{h}, \\ G &= \mathfrak{h}, \qquad & \mathcal{B} &= B + \frac{1}{\kappa}\mathfrak{f}, \end{aligned} \tag{2.29}$$

enables the factorisation of the theory into two decoupled sectors,

$$S_{\rm fact}[E,G,\mathcal{A},\mathcal{B}] = S_{\rm dyn}[E,G] + S_{BF}[\mathcal{A},\mathcal{B}] + \int_{\mathcal{M}} {\rm ST}\,. \tag{2.30}$$

Once again this transformation is well defined provided the topological coupling κ does not vanish. This transformation is nothing other than the Lorentz covariant extension, in combination with the expressions for conjugate momenta, of the canonical transformation (2.16) and (2.20) in the phase space of the original TMGT within their Hamiltonian formulation (2.14). The total divergences, referred to as "ST" and mixing the variables $\mathcal{A}$ and $\mathcal{B}$ with E and G, respectively, are again parametrised by ξ.

The first contribution $S_{\rm dyn}[E,G]$ consisting of dynamical physical variables reads

$$S_{\rm dyn} = \frac{e^2}{2}\,(E)^2 + \frac{g^2}{2}\,(G)^2 + \frac{1}{\kappa}\int_{\mathcal{M}} \sigma^{d-p}\,\xi\,E\wedge {\rm d}G - (1-\xi)\,{\rm d}E\wedge G\,. \tag{2.31}$$

The gauge independent "self-dual" action generalised to any dimension of [66, 67] is recovered. The constant ξ we have introduced in (1.23) appears explicitly in the factorised Lagrangian density (2.30), in contradistinction to its associated Hamiltonian density (2.23). This means that the factorised Langrangian density leads to the same factorised Hamiltonian formulation through Lengendre transformation and analysis of constraints, independently of the parameter ξ. However, the interpretation in terms of one particular picture depends on the value of ξ. Then the first order formulation of the Proca action for a p- or a $(d-p)$-form field is readily identified. Indeed, by setting $\xi=1$ and integrating out the then Gaussian auxiliary $(d-p)$-form field E, one derives the action of a p-form field G of mass $m=\hbar\mu$,

$$S_{\rm dyn}[G] = \frac{g^2}{2}\,(G)^2 + \frac{g^2}{2}\,\frac{\sigma^d}{\mu^2}\,({\rm d}G)^2\ ,$$

with $\mu=\kappa\,e\,g$. Alternatively one may also obtain the action of a $(d-p)$-form field E of mass $m=\hbar\mu$, by fixing $\xi=0$ and eliminating the Gaussian p-form field G,

$$S_{\rm dyn}[E] = \frac{e^2}{2}\,(E)^2 + \frac{e^2}{2}\,\frac{\sigma^d}{\mu^2}\,({\rm d}E)^2\ .$$

Finally some works [68] have fixed the value of our constant ξ to $\frac{1}{2}$ for the action of TMGT of the BF type (1.23) in 2+1 dimensions, modulo surface terms. One of the convincing arguments is that upon setting $\xi = \frac{1}{2}$, this action may be obtained through the dimensional reduction of $U_e(1) \times U_g(1)$ Maxwell theory in 3+1 dimensions with a gauge field and a pseudo-gauge field[2].

The second contribution $S_{BF}[\mathcal{A}, \mathcal{B}]$ to the dual factorised action (2.30) involves gauge variant variables transforming as follows under the abelian gauge symmetries (1.19),

$$\mathcal{A}' = \mathcal{A} + \alpha, \qquad \mathcal{B}' = \mathcal{B} + \beta, \tag{2.32}$$

and defines in fact once again a pure topological field theory of the BF type,

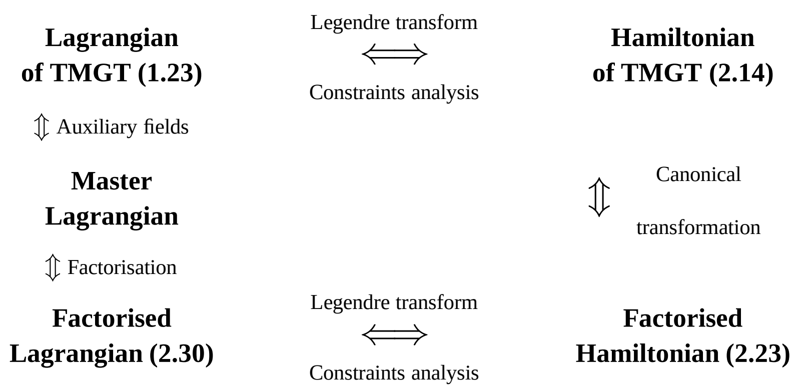
$$S_{BF} = \kappa \int_{\mathcal{M}} (1-\xi)\, \mathcal{F} \wedge \mathcal{B} - \sigma^p \, \xi \, \mathcal{A} \wedge \mathcal{H},$$

where $\mathcal{F} = \mathrm{d}\mathcal{A}$ and $\mathcal{H} = \mathrm{d}\mathcal{B}$. This decoupled TFT sector thus insures that the gauge structure of the original theory is preserved through dual factorisation. Moreover, as in the MCS case, the presence of this topological term, so far hardly evoked in the literature for very particular types of TMGT [70], has very intriguing consequences. Indeed non trivial topological effects arise from this topological term when TMGT are defined on topologically non trivial manifolds or are coupled to matter fields.

2.3 Conclusion and schematic overview

2.3.1 TP factorisation : a general and completed structure

This covariant generalisation emphasizes the universal character of the Topological-Physical factorisation, whatever the formulation of the theory, hence leading to the following general and completed structure.



[2]Notice however the relative sign between the kinetic terms for the photon and the pseudo-photon [69].

At first sight the introduction of the first order form of the action (1.23) and thus the extension of the configuration space by auxiliary Gaussian fields seems artificial. As a matter of fact, to express directly the fields of the original Lagrangian formulation of TMGT as explicit functions of those of its dual formulation (2.30) turns out to be impossible because the two formulations do not possess the same numbers of degrees of freedom. Although the two formulations describe the same physics, there are extra auxiliary degrees of freedom in the dual formulation. Therefore, a convenient Lagrangian must be chosen among those leading to the same constrained Hamiltonian [41]. The convenient formulation is the first order one (2.28) for which the comparison with the dual formulation is readily achieved from the local and linear transformation (2.29). This transformation simply redistributes the degrees of freedom, conserving the number of auxiliary fields and maintaining the gauge symmetry structure of the theory. In section 2.1 where the Topological-Physical factorisation was achieved within the Hamiltonian formulation, all second-class constraints are being reduced using Dirac brackets. Therefore, the two phase spaces possess already the same number of degrees of freedom at any given spacetime point and dualisation is directly achieved. The first order Lagrangian formulation of TMGT makes manifest the relation between the covariant field redefinitions and the associated canonical transformations within the Hamiltonian formulation.

2.3.2 Conclusion

The main new result of this Chapter is the identification of the Physical-Topological (PT) factorisation of abelian topologically massive gauge theories (TMGT) in any dimension, into a manifestly gauge invariant and dynamical sector and a gauge variant purely topological sector of the BF or AF type. Our novel approach considers the most general action for abelian TMGT in any dimension and for any p-form fields, including the two possible types of topological terms related through an integration by parts. The possibility of the factorisation is intimately related to the fact that TMGT generate a mass gap. Indeed within the Hamiltonian formulation this mass gap involves the non trivial dynamical global (or "zero-mode") sector (which carries the structure of harmonic oscillators). It turns out that the same change of variables factorises also the local sector. It was then possible in Section 2.1 to factorise phase space through a canonical transformation, see (2.16) and (2.20), which is obviously local, using the mass gap parameter μ.

In Section 2.2 this change of variables has been extended in a manifestly Lorentz covariant way by considering the first order form of the original Lagrangian of TMGT. In comparison to other methods developed so far in the literature, the technique consisting in constructing the dual action for TMGT by a local and linear redefinition of the fields is, firstly, much more direct and, secondly, preserves the entire gauge symmetry content of the original action, while at each step maintaining manifest Lorentz covariance. In this sense, this type of dual projection method enables to isolate the physical content of the theory in a gauge invariant way, the entire gauge variant contributions residing only in the second sector of the action which reduces to a pure topological field theory. The relevance of our conclusions for general TMGT is confirmed by some partial results already achieved for particular types of TMGT within the path integral framework [59, 70].

The appearance of the topological sector, which insures that the gauge symmetry content is maintained, has dramatic consequences. First, this topological term could be of prime importance for theories where the p-form fields are connections coupled to extended objects carrying the associated relevant charges. Second, as described in Chapter 3 within the context of canonical quantisation, this term controls the degeneracy of the physical spectrum of the original TMGT through topological invariants of the space manifold when it is of non trivial topology. Our TP factorisation will then lead to a generalisation of the lowest Landau level projection of the Landau problem, which makes sense even at the classical level.

CHAPTER 3

A new perspective on the lowest Landau level projection

The Landau model and its quantum Hall limit is known to be an idealisation for a physical realisation of the simplest example of non commutative geometry. This actual physical model decribes charged electrons moving in the plane in the presence of an external uniform magnetic field transverse to that plane. The limit of strong magnetic field is the famous Landau projection onto the fundamental quantum state which implies the non commutativity of the space coordinates. The Maxwell-Chern-Simons theory may be seen as the gauge field theoretic realisation of this mechanism since the Landau problem is recovered in the long wavelentgh limit of this theory [55, 56]. Furthermore the Landau projection onto the ground state of the MCS theory boils down to a projection onto a pure topological Chern-Simons theory.

The clue to our new TP factorisation introduced in Chapter 2 relies on the identification of a topological field theory embedded in the full topologically massive gauge theory, not manifest within the original Hamiltonian formulation. Actually, the TFT sector with its reduced phase space appears already at the classical level, independently of any projection onto the ground state. Furthermore the TP factorisation allows for a straightforward quantisation of these systems and the identification of their spectrum of physical states, accounting also for all the topological features inherent to such dynamics. This Chapter not only offers a novel picture of the lowest Landau level projection through our new TP factorisation already valid at the classical level, but also generalises it to TMGT of the BF *type in any dimension.*

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* *

3.1 Lowest Landau level projection at the classical level

3.1.1 Hodge decomposition : local and global sectors

The space manifold Σ having been assumed to be orientable and compact, let us now consider the consequences of its cohomology group structure, especially in the case when the latter could be non trivial. Throughout the discussion it is implicitly assumed that the p- and $(d-p)$-form fields A and B are globally defined differentiable forms in $\Omega^p(\mathcal{M})$ and $\Omega^{d-p}(\mathcal{M})$. When parametrising the theory in terms of the PT factorised variables, the latter assumption of a topological character concerns only the TFT sector. The variables of the dynamical sector are already globally defined whatever the topological properties of the original variables. By virtue of the Hodge theorem [16], the phase space variables of the TFT sector, thus globally defined on Σ, may uniquely be decomposed for each time slice into the sum of an exact, a co-exact and a harmonic form, see (1.10), with respect to the inner product specified in (2.15). A likewise decomposition applies to the dynamical sector.

The local sector

Such a decomposition amounts to a split of the fields into a longitudinal part (subscript L), a transverse part (subscript T) and a "global" part, using the derivative operator d and the coderivative operator $\mathrm{d}^\dagger$ introduced in Section 1.1. The transverse and longitudinal parts are associated to idempotent orthogonal projection operators,

$$\begin{aligned} \Pi^T_{(p)} &= \frac{1}{\triangle^\perp_{(p)}}\mathrm{d}^\dagger_{(p+1)}\mathrm{d}_{(p)}, & \Pi^L_{(p)} &= \frac{1}{\triangle^\perp_{(p)}}\mathrm{d}_{(p-1)}\mathrm{d}^\dagger_{(p)}, \\ \Pi^T_{(p)} &: \Omega^p(\Sigma) \to (Z^\dagger_\perp)^p(\Sigma), & \Pi^L_{(p)} &: \Omega^p(\Sigma) \to Z^p_\perp(\Sigma), \end{aligned} \tag{3.1}$$

where $\triangle^\perp_{(p)}$ is the Laplacian operator acting on the space $\Omega^p_\perp(\Sigma)$ of p-forms from which the kernel $\mathrm{k}er\,\triangle_{(p)}$ of the Laplacian $\triangle_{(p)}$ has been subtracted, while $(Z^\dagger_\perp)^p$ (resp. $Z^p_\perp$) is the space of co-closed (resp. closed) p-forms non cohomologous to zero. One therefore has the following properties,

$$(-1)^{p(d-p)}\,\Pi^T_{(p)} = *\Pi^L_{(d-p)}*, \qquad \Pi^T_{(p)} + \Pi^L_{(p)} = \mathrm{Id}^\perp_{(p)},$$

where $*$ is the Hodge operator on Σ and $\mathrm{Id}^\perp_{(p)}$ the identity operator on $\Omega^p_\perp(\Sigma)$.

In order that the longitudinal and transverse components possess the same physical dimensions as the original fields, the Hodge decomposition of fields may be expressed in terms of a convenient normalisation,

$$\sqrt{\triangle^\perp}\mathcal{A} = \mathrm{d}A_L + \mathrm{d}^\dagger A_T, \qquad \sqrt{\triangle^\perp}\mathcal{B} = \mathrm{d}B_L + \mathrm{d}^\dagger B_T. \tag{3.2}$$

Let us then define a new set of variables in the TFT sector, using the projection operators (3.1),

$$\begin{aligned} \varphi = \Pi^T_{(p-1)}\, A_L, \qquad & *Q_\vartheta = \Pi^L_{(p+1)} A_T, \\ \vartheta = \Pi^T_{(d-p-1)} B_L, \qquad & *P_\varphi = \Pi^L_{(d-p+1)} B_T\,, \end{aligned} \tag{3.3}$$

where the components of $*P_\varphi$ and $*Q_\vartheta$ are pseudo-tensors defined in a manner analogous to the conjugate momenta in (2.11). In terms of these new variables the non vanishing Poisson brackets are

$$\begin{aligned} \left\{\varphi_{i_1\cdots i_{p-1}}(t,\vec{x}\,), P_\varphi^{j_1\cdots j_{p-1}}(t,\vec{y}\,)\right\} &= \frac{1}{\kappa}\left(\Pi^T\right)^{j_1\cdots j_{p-1}}_{i_1\cdots i_{p-1}}\,\delta^{(d)}(\vec{x}-\vec{y}\,), \\ \left\{\vartheta_{i_1\cdots i_{d-p-1}}(t,\vec{x}\,), Q_\vartheta^{j_1\cdots j_{d-p-1}}(t,\vec{y}\,)\right\} &= -\frac{1}{\kappa}\left(\Pi^T\right)^{j_1\cdots j_{d-p-1}}_{i_1\cdots i_{d-p-1}}\,\delta^{(d)}(\vec{x}-\vec{y}\,). \end{aligned}$$

In conclusion, in the TFT sector, rather than working in terms of the phase space variables $\mathcal{A}$ and $\mathcal{B}$ one may parametrise these degrees of freedom in terms of the "longitudinal" fields φ and ϑ as well as their conjugate momenta, namely the "transverse" fields P_φ and Q_ϑ, to which the harmonic components A_h and B_h must still be adjoined. The same procedure may be applied to the variables of the dynamical sector. The Hamiltonian (2.23) then decomposes into a transverse, a longitudinal and a harmonic contribution from these latter variables only.

The global sector

A natural consequence of the Hodge decomposition is the isomorphism between the p^{th} de Rham cohomology group, $H^p(\Sigma,\mathbb{R})$, and the space of harmonic p-forms, $\ker\triangle_{(p)}$. This means that each equivalence class of $H^p(\Sigma,\mathbb{R})$ has an unique harmonic p-form representative identified through the inner product (2.15). It is possible to choose a basis for $\ker\triangle_{(p)}$ in such a way that the harmonic component of any p-form is expressed in a topological invariant way. This may be achieved by defining a topological invariant isomorphism between the components of an equivalence class of the p^{th} (singular) homology group $H_p(\Sigma,\mathbb{R})$ and the components of a form in $\ker\triangle_{(p)}$ (the p-homology group is the set of equivalence classes of p-cycles differing by a p-boundary). Thus, instead of constructing the basis from the Hodge decomposition inner product (2.15), one uses the bilinear, non degenerate and topological invariant inner product Λ defined by

$$\Lambda: \quad H_p(\Sigma)\times H^p(\Sigma)\to\mathbb{R}: \quad \Lambda\left([\Gamma]\,,[\omega]\right) = \int_\Gamma \omega, \tag{3.4}$$

making explicit the Poincaré duality between homology and cohomology groups [16]. Given the Hodge theorem, this inner product naturally induces a topological invariant inner product between the equivalent classes of $H_p(\Sigma)$ and the elements of $\ker \triangle_{(p)}$. Therefore, if one introduces generators of the free abelian part of the p^{th} singular homology group of rank N_p, $\left\{\Sigma^{\gamma}_{(p)}\right\}_{\gamma=1}^{N_p}$, a convenient dual basis $\{X^{\gamma}\}$ of $\ker \triangle_{(p)}$ may be chosen such that

$$\Lambda\left(\left[\Sigma^{\alpha}_{(p)}\right], X^{\beta}\right) = \delta^{\alpha\beta}.$$

Using the duality (3.4), the harmonic component A_h of the p-form variable $\mathcal{A}$ is thus decomposed according to

$$A_h = \sum_{\gamma=1}^{N_p} \Lambda\left(\left[\Sigma^{\gamma}_{(p)}\right], A_h\right) \, X^{\gamma} \, .$$

These components of A_h in the basis $\{X^{\gamma}\}$ are topological invariants because they express the periods of $\mathcal{A}$ over the cycle generators of $H_p(\Sigma)$. This is thus nothing other than the classical Wilson loop argument over these generators,

$$a^{\gamma} = \oint_{\Sigma^{\gamma}_{(p)}} A_h \, . \tag{3.5}$$

In other words, the variables $a^{\gamma}(t)$ specify the complete set of remaining "global" degrees of freedom in the TFT sector for the field $\mathcal{A}$,

$$A_h(t, \vec{x}\,) = \sum_{\gamma=1}^{N_p} a^{\gamma}(t)\, X^{\gamma}(\vec{x}\,).$$

In a likewise manner, the harmonic component of the $(d-p)$-form variable $\mathcal{B}$ may be decomposed according to

$$B_h = \sum_{\gamma=1}^{N_p} \Lambda\left(\left[\Sigma^{\gamma}_{(d-p)}\right], B_h\right) \, Y^{\gamma},$$

where $\{Y^{\gamma}\}$ is the dual basis of the cycle generators in $H_{d-p}(\Sigma)$, $\left\{\Sigma^{\gamma}_{(d-p)}\right\}_{\gamma=1}^{N_p}$. Hence, the components of harmonic $(d-p)$-forms are expressed as

$$b^{\gamma} = \oint_{\Sigma^{\gamma}_{(d-p)}} B_h, \tag{3.6}$$

leading to a similar decomposition of the global degrees of freedom for the field $\mathcal{B}$,

$$B_h(t,\vec{x}\,) = \sum_{\gamma=1}^{N_p} b^\gamma(t)\, Y^\gamma(\vec{x}\,).$$

The Poisson brackets between the above global variables are topological invariants,

$$\left\{a^\gamma, b^{\gamma'}\right\} = \frac{1}{\kappa} I^{\gamma\gamma'}, \tag{3.7}$$

namely the signed intersection matrix of which each entry is the sum of the signed intersections of the generators of $H_p(\Sigma)$ and $H_{d-p}(\Sigma)$,

$$I^{\gamma\gamma'} = I\left[\Sigma^\gamma_{(p)}, \Sigma^{\gamma'}_{(d-p)}\right]. \tag{3.8}$$

Within our approach, we recover the results of [71, 31, 32] in 2+1, 3+1 and d+1 dimensions, respectively, for pure topological field theories of the AF and BF types.

3.1.2 Large and small gauge transformations

Only the TFT sector is not gauge invariant. Its phase space variables transform exactly like in a pure BF theory, see (2.19). Let us recall that in (2.19), α and β are, respectively, closed p- and $(d-p)$-forms on Σ and their respective exact parts define small gauge transformations, generated by the two Gauss law first-class constraints (2.17). Given the Hodge decompositions in the TFT sector (3.2) and (3.3), these constraints, which require that the phase space variables $\mathcal{A}$ and $\mathcal{B}$ of the TFT sector be closed forms, reduce to

$$\mathcal{G}^{(1)} = \sqrt{\triangle^\perp} * Q_\vartheta, \qquad \mathcal{G}^{(2)} = \sqrt{\triangle^\perp} * P_\varphi. \tag{3.9}$$

Small gauge transformations act only on the exact part of the TFT sector fields by translating them, namely in terms of the longitudinal $(p$–$1)$- and $(d$–p–$1)$-form fields defined in (3.3),

$$\varphi' = \varphi + \alpha_L, \qquad \vartheta' = \vartheta + \beta_L,$$

where α_L and β_L are, respectively, the longitudinal $(p-1)$- and $(d-p-1)$-forms defining the exact components of the gauge transformation forms α and β through a construction similar to that in (3.3). The harmonic components of α and β define the associated large gauge transformations.

Small gauge transformations

The physical classical phase space in the TFT sector is the set of all field configurations $\mathcal{A}$ and $\mathcal{B}$ obeying the first-class constraints setting to zero their transverse degrees of freedom, see (3.9), and identified modulo the action of all gauge transformations, whether small or large. Since under small transformations the longitudinal modes φ and ϑ are gauge equivalent to the trivial configuration of vanishing longitudinal fields, like in any pure BF TFT the physical phase space of the TFT sector, so far for what concerns small gauge symmetries, is thus finite dimensional and isomorphic to the ensemble of harmonic forms defined modulo exact forms,

$$\mathcal{P} = H^p(\Sigma, \mathbb{R}) \oplus H^{d-p}(\Sigma, \mathbb{R}), \tag{3.10}$$

where $H^p(\Sigma, \mathbb{R})$ is the p^{th} de Rham cohomology group. Let us recall that according to Poincaré duality, $H^p(\Sigma)$ is isomorphic to $H^{d-p}(\Sigma)$. Hence, whether one considers functionals of harmonic p-forms or $(d-p)$-forms is of no consequence. The finite dimension of this group is given by the corresponding Betti number N_p (for example in the case of the torus, $\Sigma = T_d$, $N_p = C_d^p$). The physical phase space of the TFT sector is thus spanned by the global degrees of freedom $a^\gamma(t)$ and $b^\gamma(t)$, which are indeed obviously invariant under all small gauge transformations. However, this phase space is subjected to further restrictions still, stemming from large gauge transformations.

Large gauge transformations

In a manner similar to the above characterisation of the physical phase space in the TFT sector, the modular group is the quotient of the full gauge group by the subgroup of small gauge transformations generated by the first-class constraints, namely essentially the set of large gauge transformations. Hence large gauge transformations correspond to the cohomologically non trivial, namely the harmonic components of α and β. However the strict invariance of the global phase space variables a^γ and b^γ under large gauge transformations will be not required. Rather, having in mind abelian gauge symmetries for p-forms fields defined in terms of the univalued exponential (1.7), the global physical observables to be required to remain invariant under large gauge transformations are the holonomy or Wilson loop operators of the TFT sector around compact orientable submanifolds Σ_p and Σ_{d-p} in Σ. The only non trivial Wilson loops are those around homotopically non trivial cycles, namely elements $[\Gamma_p]$ of $H_p(\Sigma, \mathbb{Z})$ which may be decomposed in the basis $\left\{\Sigma^\gamma_{(p)}\right\}_{\gamma=1}^{N_p}$. Consequently, given the basis of $\ker \triangle_{(p)}$ constructed from (3.4) one has the following set of global

Wilson loop observables

$$W[\Gamma_{(p)}] = \exp\left(\mathrm{i}\sum_{\gamma=1}^{N_p}\sigma^\gamma \oint_{\Sigma^\gamma_{(p)}} \mathcal{A}\right) = \exp\left(\mathrm{i}\sum_{\gamma=1}^{N_p}\sigma^\gamma\, a^\gamma\right),$$
$$W[\Gamma_{(d-p)}] = \exp\left(\mathrm{i}\sum_{\gamma=1}^{N_p}\tilde{\sigma}^\gamma \oint_{\Sigma^\gamma_{(d-p)}} \mathcal{B}\right) = \exp\left(\mathrm{i}\sum_{\gamma=1}^{N_p}\tilde{\sigma}^\gamma\, b^\gamma\right),$$

where σ^γ, $\tilde{\sigma}^\gamma$ are arbitrary integers. Large gauge transformations associated to closed forms α and β act on the global variables a^γ and b^γ according to

$$a'^\gamma = a^\gamma + \alpha^\gamma, \qquad b'^\gamma = b^\gamma + \beta^\gamma, \tag{3.11}$$

where α^γ and β^γ are given by

$$\alpha^\gamma = \oint_{\Sigma^\gamma_{(p)}} \alpha, \qquad \beta^\gamma = \oint_{\Sigma^\gamma_{(d-p)}} \beta.$$

Although the Wilson loops are constructed on the free abelian p^{th} homology group $H_p(\Sigma,\mathbb{Z})$, the cohomology group including the large gauge transformation parameters is dual to the singular homology group $H_p(\Sigma,\mathbb{R})$. Hence, the only allowed large gauge transformations correspond to components of the harmonic content of the forms α and β which are discrete and quantised,

$$\alpha^\gamma = \oint_{\Sigma^\gamma_{(p)}} \alpha = 2\pi\,\ell^\gamma_{(p)}, \qquad \beta^\gamma = \oint_{\Sigma^\gamma_{(d-p)}} \beta = 2\pi\,\ell^\gamma_{(d-p)}. \tag{3.12}$$

Here $\ell^\gamma_{(p)}$ and $\ell^\gamma_{(d-p)}$ are integers which characterise the winding numbers of the large gauge transformations, namely the periods of these transformations around the homology cycle generators. The requirement of gauge invariance of all Wilson loops hence constrains the parameters of large gauge transformations to belong to the dual of the free abelian homology group. As a consequence, finally the physical classical phase space in the TFT sector is the quotient of the de Rham cohomology group $H^p(\Sigma,\mathbb{R})\oplus H^{d-p}(\Sigma,\mathbb{R})$ by the additive lattice group defined by the transformations,

$$a'^\gamma = a^\gamma + 2\pi\,\ell^\gamma_{(p)}, \qquad b'^\gamma = b^\gamma + 2\pi\,\ell^\gamma_{(d-p)},$$

namely a finite dimensional compact space having the topology of a torus of dimension $2N_p$.

3.1.3 Classical TP Factorisation and Landau projection

To venture so deeply into topological aspects is essential within the context of our TP factorisation of topologically massive gauge theories. Indeed we have extended the analysis usually developed for pure topological field theories to dynamical topological mass generation theories. As a result the topological sector consists in a finite dimensional phase space (3.10) of global variables as is the case for any pure AF or BF topological field theory. This phase space endowed with a symplectic structure is further restricted into a compact torus by the action of large gauge transformations. The isolation of the TFT sector is very reminiscent of a Landau projection onto the quantum ground state of topologically massive gauge theories (TMGT), already known for the Maxwell-Chern-Simons theory (1.33) which is the gauge field theoretic extension of the Landau problem [55, 56]. But now the TFT sector is made manifest already at the classical level through a simple reparametrisation of phase space.

Likewise the dynamical sector may also be split into local and global sectors following the same process, which is a novel approach within this context[1]. Hence, in contradistinction to what is sometimes advocated in the literature, the dynamical sector contains also global variables belonging to a symplectic vector space isomorphic to (3.10) and which are therefore topological invariants. Of course this global dynamical subsector may not be referred to as being topological since any dynamical sector of the Hamiltonian necessarily requires the introduction of a metric structure. Therefore, not only our analysis enables to distinguish the topological and dynamical sectors but also to split these sectors into subsectors of local and global variables, leading to the structure in Fig.3.1 valid for all types of TMGT in any dimension.

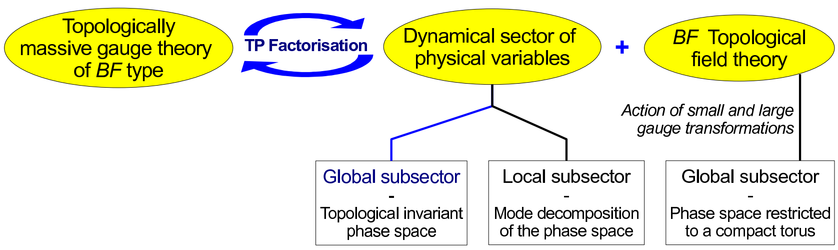

Figure 3.1: *Classical factorisation of TMGT within the Hamiltonian formulation, including the local and global subsectors.*

[1] Actually, what is usually called "global sector" is the sector of modes of zero momentum resulting from a metric dependent spectral decomposition of the phase space variables, for a given space manifold Σ. Therefore, these "0-modes" are not topological invariant. However in our present approach in terms of the topological invariant inner product (3.4), a global sector of topological invariant phase space variables is isolated without specifying any metric structure.

3.2 Hilbert space factorisation in canonical quantisation

The BRST formalism offers a powerful and elegant quantisation procedure for TMGT but requires the introduction of ghosts. In some respects, this formalism has also been used for the definition and characterisation of topological quantum field theories [20]. In a related manner, the path integral quantisation of these theories also brings to the fore the characterisation of topological invariants through concepts of quantum field theory. For example, the two-point correlation function of BF (and AF) theories provides a quantum field theoretic realisation of the linking number of two surfaces of dimensions p and $(d-p)$ embedded in $\mathcal{M}$ and its path integral representation through the Ray-Singer analytic torsion of the underlying manifold. Notwithstanding these achievements, this Chapter will not rely on such methods which necessarily require some gauge fixing procedure. Rather, ordinary Dirac canonical quantisation methods will be implemented to unravel the physical content of TMGT. First, this quantisation procedure is best adapted to a condensed matter interpretation. It also enables to deal with large gauge transformations on homologically non trivial manifolds. Second, the new TP factorisation identified within the Hamiltonian formulation independently of any gauge fixing procedure makes canonical quantisation especially attractive.

3.2.1 Dynamical and topological sectors

Canonical quantisation readily proceeds from the correspondence principle. According to this principle, classical Poisson brackets are mapped onto equal time quantum commutation relations for the classical variables which are promoted to linear self-adjoint operators acting on the Hilbert space of quantum states in the Schrödinger picture at the reference time $t = t_0$,

$$\begin{aligned}
\left[\hat{\mathcal{A}}_{i_1\cdots i_p}(t_0,\vec{x}\,),\hat{\mathcal{B}}_{j_1\cdots j_{d-p}}(t_0,\vec{y}\,)\right] &= \frac{\mathrm{i}\,\hbar}{\kappa}\,\epsilon_{i_1\cdots i_p j_1\cdots j_{d-p}}\,\delta^{(d)}(\vec{x}-\vec{y}\,),\\
\left[\hat{E}^{i_1\cdots i_p}(t_0,\vec{x}\,),\hat{G}^{j_1\cdots j_{d-p}}(t_0,\vec{y}\,)\right] &= -\frac{\mathrm{i}\,\hbar}{\kappa}\,\epsilon^{i_1\cdots i_p j_1\cdots j_{d-p}}\,\delta^{(d)}(\vec{x}-\vec{y}\,).
\end{aligned}$$

A possible representation of the associated Hilbert space is in terms of functionals $\Psi[\mathcal{A},E]$ with their canonical hermitian inner product defined in terms of the field degrees of freedom $\mathcal{A}(\vec{x}\,)$ and $E(\vec{x}\,)$.

It should be clear that the PT factorisation identified at the classical level extends to the quantum system. The full Hilbert space of the system factorises into the tensor product of two separate and independent Hilbert spaces, each of which is the representation space of the operator algebra of either the gauge invariant dynamical sector or the TFT sector. As a consequence of the complete decoupling of these two sectors, one of which contributes to the physical Hamiltonian only, the other to the first-class constraint operators only, a basis of the space of quantum states may be constructed in terms of a likewise factorisation of wave functionals. Symbolically one has

$$\Psi[\mathcal{A}, E] = \Phi[E]\,\Psi[\mathcal{A}].$$

The component $\Phi[E]$ associated to the dynamical sector is manifestly gauge invariant and is the only one which contributes to the energy spectrum. The physical Hilbert subspace associated to the TFT sector, $\mathcal{H}_\Psi$, consists of those wave functionals $\Psi^P[\mathcal{A}]$ which are invariant under small gauge transformations, namely which belong to the kernel of the first-class constraint operators generating these transformations. Furthermore the physical wave functional carries a projective representation of the group of large gauge transformations (LGT) [32] or may be required to be invariant under these transformations from a more restrictive perspective.

When the space manifold Σ is topologically trivial, for instance in the case of the hyperplane, quantisation of TMGT does not offer much interest *per se* besides the free dynamics of the dynamical sector, since the TFT sector then possesses a single gauge invariant quantum state. However in the presence of external sources, or when the space manifold Σ does have non trivial topology, new and interesting features arise. In the latter situation, to be addressed hereafter, the finite though multi-dimensional gauge invariant content of the TFT sector, $\Psi^P[\mathcal{A}]$, does not contribute to the energy spectrum. As demonstrated later, the resulting degeneracy of the energy eigenstates of the complete system depends on topological invariants of Σ. Since the physical wave functional $\Psi^P[\mathcal{A}]$ in the topological sector coincides with that of a pure topological quantum field theory, one recovers the results of R. J. Szabo [32] who solved within the Schrödinger picture the pure topological BF theory in any dimension.

3.2.2 Local and global subsectors

At the classical level, phase space has been separated into two decoupled sectors: the TFT and the dynamical sectors. According to the Hodge decomposition theorem (1.10), each of the corresponding fields may in turn be decomposed into three further subsectors in terms of their longitudinal, transverse and global components. The Gauss law constraints in conjunction with invariance under small gauge transformations reduce the TFT sector to its global variables only, characterised by the vector space $\mathcal{P}$ of the de Rham cohomology group in (3.10), which is to be restricted further into a compact torus by the lattice action of the appropriate discrete large gauge transformations. Likewise in the dynamical sector, the global degrees of freedom of phase space are also purely topological and are again isomorphic to the $2N_p$-dimensional symplectic vector space $\mathcal{P}$ in (3.10). In each case, these spaces are spanned by the global variables defined as in (3.5) and (3.6), namely (a^γ, b^γ) and (E^γ, G^γ), respectively. It is intuitive to introduce for these even dimensional vector spaces a complex structure parametrised by a $N_p \times N_p$ complex symmetric matrix, $\tau = \Re(\tau) + \mathrm{i}\,\rho$, such that $(-\tau)$ takes its values in the Siegel upper half-space[2]. Such a complex structure introduced over the phase space of global degrees of freedom enables the definition of a holomorphic phase space polarisation, hence quantisation of these sectors. The same decomposition in terms of longitudinal, transverse and global degrees of freedom applies at the quantum level. Through the correspondence principle, these three subsectors of quantum operators obey the Heisenberg algebra, whether for the TFT or the dynamical sector.

3.3 The topological sector

3.3.1 Hilbert space and holomorphic polarisation

For what concerns the local operators, one has

$$\begin{aligned} \left[\hat{\varphi}_{i_1\cdots i_{p-1}}(t_0,\vec{x}), \hat{P}_\varphi^{j_1\cdots j_{p-1}}(t_0,\vec{y})\right] &= \frac{\mathrm{i}\,\hbar}{\kappa}\,\left(\Pi^T\right)^{j_1\cdots j_{p-1}}_{i_1\cdots i_{p-1}}\,\delta(\vec{x}-\vec{y}), \quad (3.13)\\ \left[\hat{\vartheta}_{i_1\cdots i_{d-p-1}}(t_0,\vec{x}), \hat{Q}_\vartheta^{j_1\cdots j_{d-p-1}}(t_0,\vec{y})\right] &= -\frac{\mathrm{i}\,\hbar}{\kappa}\,\left(\Pi^T\right)^{j_1\cdots j_{d-p-1}}_{i_1\cdots i_{d-p-1}}\,\delta(\vec{x}-\vec{y}), \end{aligned}$$

while for the global operators,

$$\left[\hat{a}^\gamma(t_0), \hat{b}^{\gamma'}(t_0)\right] = \mathrm{i}\,\frac{\hbar}{\kappa}\,I^{\gamma\gamma'}.$$

Introducing now the holomorphic combinations of the latter operators[3],

$$\begin{aligned} \hat{c}_\gamma &= \sqrt{\frac{\kappa}{2\,\hbar}}\,\sum_{\delta=1}^{N_p}\left(I_{\gamma\delta}\,\hat{a}^\delta + \tau_{\gamma\delta}\,\hat{b}^\delta\right)\,, \quad (3.14)\\ \hat{c}^\dagger_\gamma &= \sqrt{\frac{\kappa}{2\,\hbar}}\,\sum_{\delta=1}^{N_p}\left(I_{\gamma\delta}\,\hat{a}^\delta + \overline{\tau}_{\gamma\delta}\,\hat{b}^\delta\right)\,, \end{aligned}$$

where $I_{\gamma\delta}$ is the inverse of the intersection matrix,

$$\sum_{\delta=1}^{N_p} I_{\gamma\delta}\,I^{\delta\gamma'} = \delta_\gamma^{\gamma'},$$

[2] Namely the set of $N_p \times N_p$ complex symmetric matrix whose imaginary part is positive definite.

[3] It is implicitly assumed here that the parameter κ is strictly positive. If κ is negative, the roles of the operators $\hat{a}^\gamma$ and $\hat{b}^\gamma$ are simply exchanged in the discussion hereafter.

one finds the Fock type algebra

$$\left[\hat{c}_\gamma, \hat{c}^\dagger_{\gamma'}\right] = \Im(\tau)_{\gamma\gamma'} = \rho_{\gamma\gamma'}, \tag{3.15}$$

all other possible commutators vanishing identically. Note that this result implies that the inner product in this sector of Hilbert space is to be defined in terms of the imaginary part $(\rho^{-1})^{\gamma\gamma'}$, in a manner totally independent from the Riemannian metric structure of the compact space submanifold Σ. *A priori*, physical observables in pure topological quantum field theories ought nevertheless to be independent from any extraneous *ad hoc* structure introduced through the quantisation process such as the present complex structure.

Gauss law constraints and large gauge transformations are to be considered in the wave functional representation of Hilbert space. The latter is spanned by the direct product of basis vectors for the representation spaces of the algebras (3.13) and (3.15). These consist of functionals $\Psi[\varphi, \vartheta, c]$ of the infinite dimensional space of field configurations in the TFT sector. Accordingly, the inner product of such states is defined by

$$\langle \Psi_1 \, | \Psi_2 \rangle = \int [\mathcal{D}\varphi] \; [\mathcal{D}\vartheta] \; \left[\prod_\gamma \mathrm{d}c_\gamma\right] \; (\det\rho)^{-1/2} \; \Psi_1^*[\varphi, \vartheta, c] \, \Psi_2[\varphi, \vartheta, c],$$

which requires the specification of a functional integration measure. This measure is taken to be the gaussian measure for fluctuations in the corresponding fields, which is induced by the Riemannian metric on Σ for fluctuations in φ and ϑ,

$$\begin{aligned} \delta\varphi^2 &= \int_\Sigma \mathrm{d}\vec{x} \, h^{i_1 k_1}(\vec{x}\,) \ldots h^{i_{p-1} k_{p-1}}(\vec{x}\,) \, \delta\varphi_{i_1 \cdots i_{p-1}}(\vec{x}\,) \, \delta\varphi_{k_1 \cdots k_{p-1}}(\vec{x}\,) \,, \\ \delta\vartheta^2 &= \int_\Sigma \mathrm{d}\vec{x} \, h^{j_1 l_1}(\vec{x}\,) \ldots h^{j_{d-p-1} l_{d-p-1}}(\vec{x}\,) \, \delta\vartheta_{j_1 \cdots j_{d-p-1}}(\vec{x}\,) \, \delta\vartheta_{l_1 \cdots l_{d-p-1}}(\vec{x}\,) \,, \end{aligned}$$

or else by the complex structure τ in the global sector,

$$\delta c^2 = \sum_{\gamma,\gamma'=1}^{N_p} (\rho^{-1})^{\gamma\gamma'} \delta c_\gamma \, \delta c_{\gamma'} \,. \tag{3.16}$$

In contradistinction to an ordinary pure topological quantum field theory, such a space metric is readily available within the context of TMGT, being necessary for the specification of the dynamical fields. Independently from the complex structure introduced in the global sector, independence of the physical Hilbert space measure in the (φ, ϑ) sector on the metric on Σ will be established hereafter. Consequently the canonical commutation relations (3.13) and (3.15) in the TFT sector are represented by the following functional operators acting on the Hilbert space wave functionals. On the one hand those refering to the local operators $\hat{\varphi}$ and $\hat{\vartheta}$,

$$\hat{\varphi}_{i_1\cdots i_{p-1}}(\vec{x}\,) \equiv \varphi_{i_1\cdots i_{p-1}}(\vec{x}\,) \quad , \quad \hat{\vartheta}_{i_1\cdots i_{d-p-1}}(\vec{x}\,) \equiv \vartheta_{i_1\cdots i_{d-p-1}}(\vec{x}\,)\,,$$

and their conjugate momenta,

$$\begin{aligned}
\hat{P}_{\varphi}^{i_1\cdots i_{p-1}}(\vec{x}\,) &\equiv -\frac{\mathrm{i}\,\hbar}{\kappa}\left(\Pi^T\right)^{i_1\cdots i_{p-1}}_{j_1\cdots j_{p-1}}\,\frac{\delta}{\delta\varphi_{j_1\cdots j_{p-1}}(\vec{x}\,)}\,,\\
\hat{Q}_{\vartheta}^{i_1\cdots i_{d-p-1}}(\vec{x}\,) &\equiv \frac{\mathrm{i}\,\hbar}{\kappa}\left(\Pi^T\right)^{i_1\cdots i_{d-p-1}}_{j_1\cdots j_{d-p-1}}\,\frac{\delta}{\delta\vartheta_{j_1\cdots j_{d-p-1}}(\vec{x}\,)}\,,
\end{aligned}$$

and on the other hand those constructed from the global sector (3.15),

$$\hat{c}_\gamma \equiv c_\gamma \quad , \quad \hat{c}^\dagger_\gamma \equiv -\sum_{\gamma'=1}^{N_p}\rho_{\gamma\gamma'}\,\frac{\partial}{\partial c_{\gamma'}}\,. \tag{3.17}$$

3.3.2 Gauss law constraints

The physical Hilbert space is invariant under all gauge transformations. A first restriction arises by requiring the physical quantum states to be invariant under small gauge transformations generated by the first-class constraints. This set is the kernel of the Gauss law constraint operators (2.24) which remain defined as in the classical theory since no operator ordering ambiguity is encountered,

$$\begin{aligned}
\hat{\mathcal{G}}^{(1)}\left|\Psi^P\right\rangle = 0 \quad &\Rightarrow \quad \frac{\delta}{\delta\vartheta_{i_1\cdots i_{d-p-1}}(\vec{x}\,)}\Psi^P[\varphi,\vartheta,c] = 0,\\
\hat{\mathcal{G}}^{(2)}\left|\Psi^P\right\rangle = 0 \quad &\Rightarrow \quad \frac{\delta}{\delta\varphi_{i_1\cdots i_{p-1}}(\vec{x}\,)}\Psi^P[\varphi,\vartheta,c] = 0.
\end{aligned}$$

Hence physical quantum states necessarily consist of wave functionals which are totally independent of the longitudinal variables (φ,ϑ). When restricted to such states and properly renormalised, the inner product integration measure is constructed from the definition (3.16) of the gaussian metric on the space of fluctuations in the global coordinates,

$$\langle\Psi_1\,|\Psi_2\rangle = \int\prod_\gamma \mathrm{d}c_\gamma\;(\det\rho)^{-1/2}\;\Psi_1^*(c)\,\Psi_2(c).$$

This measure on the physical Hilbert space is thus indeed independent of the Riemannian metric on Σ, and involves only the *ad hoc* complex structure τ introduced towards the quantisation of the global TFT sector.

3.3.3 LGT and global variables

Construction of the operator generating LGT

The structure of the physical Hilbert space $\mathcal{H}_\Psi$ dramatically depends on the way one deals with LGT. Given the holomorphic parametrisation (3.14), under the lattice action of LGT of periods $(\ell^\gamma_{(p)}, \ell^\gamma_{(d-p)}) \equiv (\underline{\ell}_{(p)}, \underline{\ell}_{(d-p)})$ as defined in (3.12) the new global operators should transform as,

$$\begin{aligned} c'_\gamma &= c_\gamma + \sqrt{\frac{2\,\pi^2\,\kappa}{\hbar}} \sum_{\gamma'=1}^{N_p} \left(I_{\gamma\gamma'}\,\ell^{\gamma'}_{(p)} + \tau_{\gamma\gamma'}\,\ell^{\gamma'}_{(d-p)} \right), \\ {c'_\gamma}^{\dagger} &= c^\dagger_\gamma + \sqrt{\frac{2\,\pi^2\,\kappa}{\hbar}} \sum_{\gamma'=1}^{N_p} \left(I_{\gamma\gamma'}\,\ell^{\gamma'}_{(p)} + \overline{\tau}_{\gamma\gamma'}\,\ell^{\gamma'}_{(d-p)} \right). \end{aligned} \tag{3.18}$$

Using the following Baker-Campbell-Hausdorff (BCH) formula for any two operators $\hat{A}$ and $\hat{B}$ commuting with their own commutator,

$$e^{\hat{A}}\,\hat{B}\,e^{-\hat{A}} = \hat{B} + \left[\hat{A},\hat{B}\right], \qquad e^{\hat{A}+\hat{B}} = e^{-\frac{1}{2}[\hat{A},\hat{B}]}\,e^{\hat{A}}\,e^{\hat{B}}, \tag{3.19}$$

it may be seen that the quantum operator $\hat{U}_{\rm LGT}\left(\underline{\ell}_{(p)}, \underline{\ell}_{(d-p)}\right)$ generating the large gauge transformations of periods $\underline{\ell}_{(p)}$ and $\underline{\ell}_{(d-p)}$ is

$$\begin{aligned} \hat{U}_{\rm LGT} &= C_{\underline{\ell}}\left(\underline{\ell}_{(p)}, \underline{\ell}_{(d-p)}\right) \prod_{\gamma,\gamma',\epsilon}^{N_p} \exp\left\{ 2\,\pi\,\sqrt{\frac{\kappa}{2\,\hbar}}\,(\rho^{-1})^{\gamma\gamma'} \right. \\ &\quad \times \left. \left[\left(I_{\gamma\epsilon}\,\ell^\epsilon_{(p)} + \overline{\tau}_{\gamma\epsilon}\,\ell^\epsilon_{(d-p)} \right) \hat{c}_{\gamma'} - \left(I_{\gamma\epsilon}\,\ell^\epsilon_{(p)} + \tau_{\gamma\epsilon}\,\ell^\epsilon_{(d-p)} \right) \hat{c}^\dagger_{\gamma'} \right] \right\}. \end{aligned} \tag{3.20}$$

The 1-cocycle $C_{\underline{\ell}}\left(\underline{\ell}_{(p)}, \underline{\ell}_{(d-p)}\right)$ will be determined presently. This operator (3.20) defines the action of LGT on the Hilbert space in the global TFT sector,

$$\begin{aligned} \hat{U}_{\rm LGT}\,\Psi\left(c_\gamma\right) &= \prod_{\gamma,\gamma',\delta,\epsilon}^{N_p} {\rm e}^{\pi^2\,\frac{\kappa}{\hbar}\left[I_{\gamma\delta}\,\ell^\delta_{(p)} + \overline{\tau}_{\gamma\delta}\,\ell^\delta_{(d-p)}\right](\rho^{-1})^{\gamma\gamma'}\left[I_{\gamma'\epsilon}\,\ell^\epsilon_{(p)} + \tau_{\gamma'\epsilon}\,\ell^\epsilon_{(d-p)}\right]} \\ &\quad \times \prod_{\gamma,\gamma',\delta}^{N_p} {\rm e}^{\pi\sqrt{\frac{2\kappa}{\hbar}}\left[I_{\gamma\delta}\,\ell^\delta_{(p)} + \overline{\tau}_{\gamma\delta}\,\ell^\delta_{(d-p)}\right](\rho^{-1})^{\gamma\gamma'}\,c_{\gamma'}} \\ &\quad \times C_{\underline{\ell}}\;\Psi\left(c_\gamma + \pi\sqrt{\frac{2\kappa}{\hbar}} \sum_{\gamma'=1}^{N_p} \left[I_{\gamma\gamma'}\,\ell^{\gamma'}_{(p)} + \tau_{\gamma\gamma'}\,\ell^{\gamma'}_{(d-p)} \right] \right), \end{aligned} \tag{3.21}$$

where the BCH formula (3.19) has been used. However, a U(1)×U(1) 2-cocycle $\omega_2(k;\ell)$ appears in the composition law of this quantum representation,

$$\hat{U}\left(\underline{k}_{(p)}+\underline{\ell}_{(p)},\underline{k}_{(d-p)}+\underline{\ell}_{(d-p)}\right)=e^{-2\mathrm{i}\pi\omega_2(k;\ell)}\,\frac{C_{\underline{k+\ell}}}{C_{\underline{k}}\,C_{\underline{\ell}}}\,\hat{U}\left(\underline{k}_{(p)},\underline{k}_{(d-p)}\right)\hat{U}\left(\underline{\ell}_{(p)},\underline{\ell}_{(d-p)}\right),$$

$$\omega_2\left(\underline{k}_{(p)},\underline{k}_{(d-p)};\underline{\ell}_{(p)},\underline{\ell}_{(d-p)}\right)=-\frac{\pi\,\kappa}{\hbar}\sum_{\gamma,\gamma'=1}^{N_p}I_{\gamma\gamma'}\left[\ell^{\gamma}_{(d-p)}\,k^{\gamma'}_{(p)}-k^{\gamma}_{(d-p)}\,\ell^{\gamma'}_{(p)}\right].$$

The 1-cocycle $C\left(\underline{\ell}_{(p)},\underline{\ell}_{(d-p)}\right)$ appearing in (3.20) may be determined by requiring that the abelian group composition law for LGT is recovered. This implies that $\omega_2(k;\ell)$ is a coboundary,

$$\begin{aligned}\omega_2(k;\ell) &= \mathcal{C}_1\left(\underline{k}_{(p)}+\underline{\ell}_{(p)},\underline{k}_{(d-p)}+\underline{\ell}_{(d-p)}\right)\\ &\quad -\mathcal{C}_1\left(\underline{k}_{(p)},\underline{k}_{(d-p)}\right)-\mathcal{C}_1\left(\underline{\ell}_{(p)},\underline{\ell}_{(d-p)}\right)\qquad (\mathrm{mod}\ \mathbb{Z}),\\ C_{\underline{\ell}} &\equiv C\left(\underline{\ell}_{(p)},\underline{\ell}_{(d-p)}\right)=\mathrm{e}^{2\,\mathrm{i}\,\pi\,\mathcal{C}_1\left(\underline{\ell}_{(p)},\underline{\ell}_{(d-p)}\right)}.\end{aligned}$$

A careful analysis, analogous to the one in [72], finds that the unique solution to this coboundary condition is

$$\kappa=\frac{\hbar}{2\pi}\,\mathcal{I}\,k,\qquad C\left(\underline{\ell}_{(p)},\underline{\ell}_{(d-p)}\right)=\prod_{\gamma,\gamma'=1}^{N_p}\mathrm{e}^{\mathrm{i}\,\pi\,k\,\mathcal{I}\,\ell^{\gamma}_{(d-p)}\,I_{\gamma\gamma'}\,\ell^{\gamma'}_{(p)}},\tag{3.22}$$

where $k\in\mathbb{Z}$ and[4] $\mathcal{I}=\det\left(I^{\gamma\gamma'}\right)\in\mathbb{N}$. It is noteworthy to recall that although $I_{\gamma\gamma'}$ is a rational valued matrix, $\mathcal{I}\,I_{\gamma\gamma'}$ is integer valued. Note also the quantisation condition arising for the coefficient κ multiplying the topological terms in the original action of TMGT.

Characterisation of the physical Hilbert space

If k is rational, namely if $k=k_1/k_2$ with k_1,k_2 strictly positive natural numbers, invariance of physical states under LGT cannot be achieved. However in this case the LGT group has a finite dimensional projective representation which may be constructed by finding a normal subgroup generated by the LGT operators. As demonstrated in [32], the TFT part of the physical wave functions carries a projective repre-

[4] Recall that κ, hence k is assumed to be strictly positive in the present discussion, while the situation for a negative κ or k is obtained through the exchange of the sectors a^γ and b^γ.

sentation of the group of LGT while the above discussion establishes that the dimension of the physical Hilbert space[5] is

$$\dim\left(\mathcal{H}_{\Psi}^{k_1,k_2}\right) = \prod_{\delta=1}^{N_p} k_1\, k_2\, \mathcal{I}\, \mathrm{Min}(I_{\delta\delta'})\,.$$

Any state of a given irreducible representation gives the same matrix element for a physical observable. The characterisation of Hilbert space changes qualitatively for integer or rational values of k, but the theory remains well-defined.

If we take k to be an integer, see (3.22), wave functions of the physical Hilbert space may be classified in terms of irreducible representations of the group of LGT (3.21),

$$\begin{aligned}
&\Psi\left(\eta_1; c_\gamma + \sqrt{\pi\,\mathcal{I}\,k}\,\sum_{\gamma'}^{N_p}\left[I_{\gamma\gamma'}\,k_{(p)}^{\gamma'} + \tau_{\gamma\gamma'}\,k_{(d-p)}^{\gamma'}\right]\right) &(3.23)\\
&= \prod_{\substack{\gamma,\gamma'\\ \delta,\epsilon}}^{N_p} \exp\left\{\frac{-\pi\mathcal{I}k}{2}\left[I_{\gamma\delta}k_{(p)}^{\delta} + \overline{\tau}_{\gamma\delta}k_{(d-p)}^{\delta}\right](\rho^{-1})^{\gamma\gamma'}\left[I_{\gamma'\epsilon}k_{(p)}^{\epsilon} + \tau_{\gamma'\epsilon}k_{(d-p)}^{\epsilon}\right]\right\}\\
&\times \prod_{\gamma,\gamma',\delta}^{N_p} \exp\left\{-\sqrt{\pi\mathcal{I}\,k}\left[I_{\gamma\delta}\,k_{(p)}^{\delta} + \overline{\tau}_{\gamma\delta}\,k_{(d-p)}^{\delta}\right](\rho^{-1})^{\gamma\gamma'}c_{\gamma'}\right\}\\
&\times \prod_{\gamma,\gamma'}^{N_p} \exp\left\{2\,i\,\pi\,\eta_1(\underline{k}_{(p)},\underline{k}_{(d-p)}) - i\,\pi\,k\,\mathcal{I}\,k_{(d-p)}^{\gamma}\,I_{\gamma\gamma'}\,k_{(p)}^{\gamma'}\right\}\,\Psi\left(\eta_1; c_\gamma\right)\,,
\end{aligned}$$

where the 1-cocycle $\eta_1\left(\underline{k}_{(p)},\underline{k}_{(d-p)}\right)$ characterises the irreducible representation. Since for an abelian group each of its irreducible representations is one-dimensional, physical states corresponding to a given irreducible representation are singlet under LGT.

As is well-known, functions obeying such a double periodicity condition are nothing other than the generalised Riemann theta functions defined in any dimension on the complex N_p-torus [32], with the compact reduced phase space resulting from the requirement of invariance under LGT,

$$\begin{aligned}
\Psi^{r_\delta}\begin{pmatrix} a_\delta \\ b_\delta \end{pmatrix}(c_\gamma) \;=\; &\prod_{\gamma,\gamma'=1}^{N_p}\left(e^{-\frac{1}{2}\,c_\gamma\,(\rho^{-1})^{\gamma\gamma'}\,c_{\gamma'}}\right) &(3.24)\\
&\Theta\left[\begin{array}{c}\sum_{\delta'=1}^{N_p}\frac{I^{\delta'\delta}}{\mathcal{I}\,k}\left(a_{\delta'} + r_{\delta'}\right)\\ b_\delta\end{array}\right]\left(\sqrt{\frac{\mathcal{I}\,k}{\pi}}\,c_\gamma\right| - \mathcal{I}\,k\,\tau\right)\,,
\end{aligned}$$

[5] Notice that we obtain a slightly different result from that of Ref. [32].

where

$$r_\delta \in [0, k\,\mathcal{I}\,\mathrm{Min}(I_{\delta\delta'}) - 1] \subset \mathbb{N}\,.$$

Each physical subspace, characterised by the 1-cocycle

$$\eta_1^{(ab)}(\underline{k}_{(p)}, \underline{k}_{(d-p)}) = a_\gamma\, k^\gamma_{(p)} + b_\gamma\, k^\gamma_{(d-p)}\,, \quad \text{where} \quad a_\gamma, b_\gamma \in [0, 1[\subset \mathbb{R}\,,$$

is invariant under a particular irreducible representation of LGT. The TFT component of each physical Hilbert space is of dimension

$$\dim\left(\mathcal{H}^k_\Psi\right) = \prod_{\delta=1}^{N_p} k\,\mathcal{I}\,\mathrm{Min}(I_{\delta\delta'})\,.$$

In general, the choice of physical Hilbert space which is invariant under all LGT is the representation space with $\eta_1(\underline{k}_{(p)}, \underline{k}_{(d-p)}) \in \mathbb{Z}$, namely corresponding to $a_\gamma, b_\gamma = 0$.

3.4 The dynamical sector : Hamiltonian diagonalisation

Based on Hodge's theorem, (1.10) and (3.2) define the decomposition of the dynamical sector into three decoupled subsectors of canonically conjugate variables: the global harmonic sector and the local (E_L, P_E) and (G_L, Q_G) sectors. In turn the classical Hamiltonian (2.23) decomposes into three separate contributions, one for each subsector. When quantising the system in each subsector, the total quantum Hamiltonian follows from the classical one without any operator ordering ambiguity,

$$\hat{H}[\hat{E}, \hat{G}] = \hat{H}_h[\hat{E}_h, \hat{G}_h] + \hat{H}_1[\hat{E}_L, \hat{P}_E] + \hat{H}_2[\hat{G}_L, \hat{Q}_E].$$

The physical spectrum is thus identified by diagonalising each of these contributions separately.

3.4.1 Global degrees of freedom

The choice of normalisation used previously in the harmonic sector relies on the Poincaré duality between the basis elements $[X^\gamma]$ and $[Y^\gamma]$ of the relevant cohomology groups and their associated homology generators $\Sigma^\gamma_{(p)}$ and $\Sigma^\gamma_{(d-p)}$, respectively, see (3.4). This choice is of a purely topological character. However in the dynamical sector, there is a remaining freedom as far as the normalisation of the choice of the harmonic representative of the cohomology group is concerned, depending on the metric structure, and thus fixing the basis elements X^γ of $\ker \triangle_{(p)}$ and Y^γ of $\ker \triangle_{(d-p)}$.

This choice involves the inner product in (2.15) on which the Hodge decomposition relies. Hence one sets

$$\int_{\Sigma} X_\gamma \wedge *X_{\gamma'} = \frac{e}{g}\,\Omega_{\gamma\gamma'}, \qquad \int_{\Sigma} Y_\gamma \wedge *Y_{\gamma'} = \frac{g}{e}\,\tilde{\Omega}_{\gamma\gamma'}, \tag{3.25}$$

where $\Omega_{\gamma\gamma'}$ and $\tilde{\Omega}_{\gamma\gamma'}$ are $N_p \times N_p$ real symmetric matrices. Given this normalisation, the global part of the metric dependent quantum Hamiltonian operator constructed from (2.23) is expressed as

$$\hat{H}_h[\hat{E}^\gamma, \hat{G}^{\gamma'}] = \frac{1}{2} e\, g \sum_{\gamma,\gamma'=1}^{N_p} \left[\hat{E}^\gamma\, \hat{E}^{\gamma'}\, \Omega_{\gamma\gamma'} + \hat{G}^\gamma \hat{G}^{\gamma'}\, \tilde{\Omega}_{\gamma\gamma'}\right], \tag{3.26}$$

while the non vanishing commutation relations between the global phase space operators read

$$\left[\hat{E}^\gamma, \hat{G}^{\gamma'}\right] = -\mathrm{i}\, \hbar\, \kappa\, I^{\gamma\gamma'}. \tag{3.27}$$

As in the TFT sector, see (3.14), the following holomorphic polarisation of the global dynamical sector is used,

$$\begin{aligned} d_\gamma &= \frac{1}{\sqrt{2\hbar\,\kappa}} \sum_{\alpha=1}^{N_p} \left(I_{\gamma\alpha}\, \hat{E}^\alpha - \upsilon_{\gamma\alpha}\, \hat{G}^\alpha\right)\,, \\ d_\gamma^\dagger &= \frac{1}{\sqrt{2\hbar\,\kappa}} \sum_{\alpha=1}^{N_p} \left(I_{\gamma\alpha}\, \hat{E}^\alpha - \overline{\upsilon}_{\gamma\alpha}\, \hat{G}^\alpha\right)\,, \end{aligned}$$

where $\upsilon = \Re(\upsilon) + \mathrm{i}\, \sigma$ is the $N_p \times N_p$ complex symmetric matrix characterising the complex structure introduced in the global dynamical phase space sector, of which the imaginary part determines the non vanishing commutation relations of the Fock like algebra

$$\left[d_\gamma, d_{\gamma'}^\dagger\right] = \sigma_{\gamma\gamma'}. \tag{3.28}$$

In order to readily diagonalise the Hamiltonian in the global sector which is of the harmonic oscillator form, it is convenient to make the following choice for the complex structure matrix υ as well as for the normalisation quantities specified in (3.25),

$$\Re(\upsilon) = 0, \qquad \sigma_{\gamma\gamma'} = \tilde{\Omega}_{\gamma\gamma'} = \delta_{\gamma\gamma'}, \qquad \Omega_{\gamma\gamma'} = \sum_{\alpha,\beta=1}^{N_p} I_{\alpha\gamma}\, I_{\beta\gamma'}\, \delta^{\alpha\beta}, \tag{3.29}$$

where $\delta^{\gamma\gamma'}$ is the $N_p \times N_p$ Kronecker symbol. With these choices, the contribution of the global variables to the Hamiltonian is indeed diagonal,

$$H_g = \frac{1}{2}\,\hbar\,\mu\,N_p + \hbar\,\mu \sum_{\gamma,\gamma'=1}^{N_p} d^\dagger_\gamma\, d_{\gamma'}\,\delta^{\gamma\gamma'}, \qquad \mu = e\,g\,\kappa.$$

One recognizes the Hamiltonian of a collection of N_p independent harmonic oscillators of angular frequency[6] μ, which turns out to be the mass gap of the quantum field theory. The operators d_γ and $d^\dagger_\gamma$ are, respectively, annihilation and creation operators obeying the Fock algebra (3.28) now with $\sigma_{\gamma\gamma'} = \delta_{\gamma\gamma'}$. The energy spectrum in the global dynamical sector of the system is readily identified. The normalised fundamental state is the kernel of all annihilation operators,

$$d_\alpha\,|0\rangle = 0, \qquad \varepsilon^h_{(0)} = \frac{1}{2}N_p\,\hbar\,\mu, \qquad \langle 0|0\rangle = 1,$$

where $\varepsilon^h_{(0)}$ is the vacuum energy. Excited states, $|n_\gamma\rangle$, are obtained through the action of the N_p creation operators $d^\dagger_\gamma$ on the fundamental state. This leads to the energy eigenvalue for any of these states,

$$|n_\gamma\rangle = \prod_{\gamma=1}^{N_p} \frac{1}{\sqrt{n_\gamma!}}\left(d^\dagger_\gamma\right)^{n_\gamma}|0\rangle, \qquad \varepsilon_{(n_\gamma)} = \varepsilon^h_{(0)} + \hbar\,\mu \sum_{\gamma=1}^{N_p} n_\gamma, \tag{3.30}$$

$\{n_\gamma\}_{\gamma=1}^{N_p}$ being the eigenvalues of each of the number operators $d^\dagger_\gamma d_\gamma$, hence positive integers.

3.4.2 Local degrees of freedom on the torus

The canonical treatment of the global degrees of freedom in both the TFT and dynamical sectors does not require the explicit specification of the space manifold Σ with its topology and Riemannian metric, yet allowing the general discussion of the previous Sections. However, in order to identify the full spectrum of dynamical physical states, the space manifold Σ including its geometry has now to be completely specified. The explicit choice to be made for the purpose of the present discussion is that of the d-dimensional Euclidean torus, $\Sigma = T_d$, enabling Fourier mode analysis of the then infinite discrete, thus countable set of degrees of freedom, and diagonalisation of the harmonic oscillator structure of the Hamiltonian.

[6] Recall that under the assumptions of the analysis, this combination of parameters is indeed positive.

Notations

This particular choice of the d-torus is motivated by the fact that this manifold is the simplest flat yet non simply connected manifold. This topological space may be defined in an equivalent way as the product

$$T_d = \underbrace{S^1 \otimes \cdots \otimes S^1}_{d},$$

or as a quotient space $T^d = R^2/\sim$. The first definition highlights the structure of the first homology (homotopy group) of T_d while the second definition relies on any choice of lattice vectors specifying the geometry of the d-torus.

The lattice vectors are to be denoted $\vec{e}_a$, $a=1,\ldots,d$. This basis of which the associated metric is denoted g_{ab} generates the lattice

$$\Lambda \equiv \left\{\vec{\ell} = \ell^i \vec{e}_i | \ell^i = \ell^a\, e^i_a, \ell^a \in \mathbb{Z}\right\} ,$$

where an orthonormal basis of vectors $\vec{e}_i$ is introduced as a linear combination of vectors of the original basis $\vec{e}_a$, and conversely

$$\vec{e}_a = e^i_a\, \vec{e}_i \,, \qquad g_{ab} = e^i_a\, \delta_{ij}\, e^j_b \,.$$

Then the elementary cell of volume $V = \sqrt{\det(g)}$ is defined as

$$C(\Lambda) \equiv \left\{\vec{x} = x^i \vec{e}_i | x^i = x^a\, e^i_a, x^a \in [0,1]\right\} .$$

The dual bases $\underline{e}^i$ and $\underline{e}^a$ of these lattice vectors $\vec{e}_i$ and $\vec{e}_a$ are respectively defined as

$$\underline{e}^a(\vec{e}_b) = \delta^a_b \,, \qquad \underline{e}^i(\vec{e}_j) = \delta^i_j \,.$$

Then the relation between the dual bases is expressed in terms of the coefficients $\tilde{e}^a_i$,

$$\underline{e}^a = \tilde{e}^a_i\, \underline{e}^i \,, \qquad \tilde{e}^a_i\, e^i_b = \delta^a_b \,.$$

Given these definitions, the dual lattice is generated by 1-form basis vectors such that

$$\tilde{\Lambda} = \left\{\underline{k} = k_i \underline{e}^i | k_i = k_a \tilde{e}^a_i, k_a \in [0,1]\right\} ,$$

where $\underline{k}$ are discrete vectors of which the components are measured in units of L^{-1}. Their norm is expressed as $\omega(\underline{k}) = \sqrt{k_i\, k_j\, \delta^{ij}} = \sqrt{k_a\, k_a\, g^{ab}}$.

Therefore the equivalence relation $\sim$ defining the d-Torus reads

$$\vec{x} \sim \vec{y} \Leftrightarrow \vec{x} = \vec{y} + \vec{\ell}, \quad \vec{\ell} = \ell^i \vec{e}_i \in \Lambda \,,$$

where the representative of each equivalence class is chosen to belong to $C(\Lambda)$.

Fourier mode expansion

Accordingly, the variables E and G of the dynamical sector are periodic around the torus p- and $(d-p)$-cycles, respectively. Their Fourier mode expansions read

$$E_{\perp}^{i_1\ldots i_p}(\vec{x}\,) = \delta^{i_1 j_1}\ldots\delta^{i_p j_p}\sum_{\underline{k}\neq\underline{0}}\sum_{\substack{\alpha_1=1\\ \cdots\\ \alpha_{p-1}=1}}^{d-1}\varepsilon_{j_1\cdots j_p}^{\alpha_1\cdots\alpha_p}(\underline{k})\,E^{\alpha_1\cdots\alpha_p}(\underline{k})\,\mathrm{e}^{2\,\mathrm{i}\,\pi\,\underline{k}(\vec{x}\,)}\,,$$

where $E^{\alpha_1\cdots\alpha_p}(\underline{k})$ is a complex valued antisymmetric tensor. Note that the zero modes of the fields are not included in these expressions, as emphasized by the subscript $\perp$. In fact, these zero modes are the global degrees of freedom which have already been dealt with in the previous Section. The real valued tensors $\varepsilon_{i_1\cdots i_p}^{\alpha_1\cdots\alpha_p}(\underline{k})$ define a basis of orthonormalised polarisation tensors for each $\underline{k}\neq\underline{0}$. In our conventions, these tensors are constructed from a orthonormalised basis of polarisation vectors $\varepsilon_i^{\alpha}(\underline{k})$ for a vector field such that

$$\varepsilon_i^{\alpha}(\underline{k})\,\varepsilon_j^{\beta}(\underline{k})\,\delta^{ij} = \delta^{\alpha\beta}, \tag{3.31}$$

where $\delta^{\alpha\beta}$ is the Kronecker symbol in polarisation space. This basis is chosen in such a way that, for each $\underline{k}\neq\underline{0}$, the dual lattice vector $\underline{\varepsilon}^{d}(\underline{k})$ is longitudinal whereas the vectors $\underline{\varepsilon}^{\alpha}(\underline{k})$ are transverse for $\alpha = 1,\cdots,d-1$. Finally, it is convenient to choose for the longitudinal vector

$$\underline{\varepsilon}^{d}(\underline{k}) = \frac{\underline{k}}{\omega(\underline{k})}, \qquad \underline{k}\neq\underline{0}.$$

Given the recursion relation induced by the Hodge decomposition theorem, the general polarisation tensor of any p-tensor field may be expressed as

$$\varepsilon_{i_1\cdots i_p}^{\alpha_1\cdots\alpha_p}(\underline{k}) = \frac{1}{p!}\,\varepsilon_{[i_1}^{\alpha_1}(\underline{k})\ldots\varepsilon_{i_p]}^{\alpha_p}(\underline{k}),$$

which may likewise be decomposed into transverse and longitudinal components,

$$\begin{aligned}\text{Longitudinal} \quad &: \quad \left\{\varepsilon_{i_1\cdots\, i_{p-1}\, i_p}^{\alpha_1\cdots\alpha_{p-1}\, d}(\underline{k})\right\}_{\alpha_1,\ldots,\alpha_{p-1}=1}^{d-1}\;;\\ \text{Transverse} \quad &: \quad \left\{\varepsilon_{i_1\cdots i_p}^{\alpha_1\cdots\alpha_p}(\underline{k})\right\}_{\alpha_1,\ldots,\alpha_p=1}^{d-1}\;.\end{aligned} \tag{3.32}$$

Given any mode, the C_d^p degrees of freedom of a phase space field then separate into C_{d-1}^{p-1} longitudinal and C_{d-1}^{p} transverse degrees of freedom. These notations having been specified, and using the decompositions defined in (3.2), the relevant quantum operators $\hat{E}_\perp^{i_1\cdots i_p}(\vec{x}\,)$ and $\hat{G}_\perp^{i_p\cdots i_{d-p}}(\vec{x}\,)$ are Fourier expanded as

$$\begin{aligned}
\hat{E}_\perp^{i_1\cdots i_p} &= \sum_{\underline{k}\neq\underline{0}} \Bigg\{ \delta^{i_1 j_1}\ldots\delta^{i_p j_p}\, p \sum_{\substack{\alpha_1=1\\ \cdots\\ \alpha_{p-1}=1}}^{d-1} \varepsilon^{\alpha_1\cdots\alpha_{p-1}\,d}_{j_1\cdots\, j_{p-1}\,j_p}(\underline{k})\,\hat{E}_L^{\alpha_1\cdots\alpha_{p-1}}(\underline{k}) \\
&+ \kappa\,\frac{\epsilon^{i_1\cdots i_p\, j_1\cdots j_{d-p}}}{(d-p-1)!} \sum_{\substack{\alpha_1=1\\ \cdots\\ \alpha_{d-p-1}}}^{d-1} \varepsilon^{\alpha_1\cdots\alpha_{d-p-1}\,d}_{j_1\cdots\, j_{d-p}}(\underline{k})\,\hat{Q}_G^{\alpha_1\cdots\alpha_{d-p-1}}(\underline{k}) \Bigg\}\, \mathrm{e}^{2\,\mathrm{i}\,\pi\,\underline{k}(\vec{x}\,)}, \\
\hat{G}_\perp^{i_p\cdots i_{d-p}} &= \sum_{\underline{k}\neq\underline{0}} \Bigg\{ \delta^{i_1 j_1}\ldots\delta^{i_{d-p} j_{d-p}}(d-p) \sum_{\substack{\alpha_1=1\\ \cdots\\ \alpha_{d-p-1}}}^{d-1} \varepsilon^{\alpha_1\cdots\alpha_{d-p-1}\,d}_{j_1\cdots\, j_{d-p}}(\underline{k})\,\hat{G}_L^{\alpha_1\cdots\alpha_{d-p-1}}(\underline{k}) \\
&+ \frac{\kappa}{(p-1)!}\,\epsilon^{j_1\cdots j_p\, i_1\cdots i_{d-p}} \sum_{\substack{\alpha_1=1\\ \cdots\\ \alpha_{p-1}}}^{d-1} \varepsilon^{\alpha_1\cdots\alpha_{p-1}\,d}_{j_1\cdots\, j_{p-1}\,j_p}(\underline{k})\,\hat{P}_E^{\alpha_1\cdots\alpha_{p-1}}(\underline{k}) \Bigg\}\, \mathrm{e}^{2\,\mathrm{i}\,\pi\,\underline{k}(\vec{x}\,)}\,.
\end{aligned}$$

The self-adjoint property of the operator $\hat{E}^{i_1\cdots i_p}(\vec{x}\,)$ translates into the following relations between the associated mode operators and their adjoint,

$$\sum_{\substack{\alpha_1=1\\ \cdots\\ \alpha_{p-1}=1}}^{d-1} \varepsilon^{\alpha_1\cdots\alpha_{p-1}\,d}_{j_1\cdots\, j_{p-1}\,j_p}(\underline{k})\,\hat{E}_L^{\alpha_1\cdots\alpha_{p-1}}(\underline{k}) = \sum_{\substack{\alpha_1=1\\ \cdots\\ \alpha_{p-1}=1}}^{d-1} \varepsilon^{\alpha_1\cdots\alpha_{p-1}\,d}_{j_1\cdots\, j_{p-1}\,j_p}(-\underline{k})\,\hat{E}_L^{\dagger\,\alpha_1\cdots\alpha_{p-1}}(-\underline{k}),$$

$$\sum_{\substack{\alpha_1=1\\ \cdots\\ \alpha_{d-p-1}}}^{d-1} \varepsilon^{\alpha_1..\alpha_{d-p-1}\,d}_{i_1\cdots\, i_{d-p}}(\underline{k})\,\hat{Q}_G^{\alpha_1..\alpha_{d-p-1}}(\underline{k}) = \sum_{\substack{\alpha_1=1\\ \cdots\\ \alpha_{d-p-1}}}^{d-1} \varepsilon^{\alpha_1..\alpha_{d-p-1}\,d}_{i_1\cdots\, i_{d-p}}(-\underline{k})\,\hat{Q}_G^{\dagger\,\alpha_1..\alpha_{d-p-1}}(-\underline{k}).$$

Similar relations apply for the modes of the self-adjoint operator $\hat{G}^{i_p\cdots i_{d-p}}(\vec{x}\,)$.

Consequently, this decomposition of the non zero modes of the field operators in the dynamical sector leads to two decoupled subsectors, each of which is comprised of a countable set of mode operators with $\underline{k}\neq\underline{0}$. In the first subsector one has the operators $\hat{E}_L(\underline{k})$ and $\hat{P}_E(\underline{k})$ with the following non vanishing commutation relations,

$$\left[\hat{E}_L^{\dagger\,\alpha_1\cdots\alpha_{p-1}}(\underline{k}),\hat{P}_E^{\beta_1\cdots\beta_{p-1}}(\underline{k}')\right] = \mathrm{i}\,\frac{\hbar}{V}\,\delta^{\alpha_1[\beta_1}\ldots\delta^{\alpha_{p-1}\,\beta_{p-1}]}\delta_{\underline{k}\underline{k}'}\,, \qquad (3.33)$$

while in the second subsector the operators $\hat{G}_L(\underline{k})$ and $\hat{Q}_G(\underline{k})$ possess the commutator algebra,

$$\left[\hat{G}_L^{\dagger\ \alpha_1\cdots\alpha_{d-p-1}}(\underline{k}), \hat{Q}_G^{\beta_1\cdots\beta_{d-p-1}}(\underline{k}')\right] = \mathrm{i}\,\frac{\hbar}{V}\,\delta^{\alpha_1[\beta_1}\ldots\delta^{\alpha_{d-p-1}\,\beta_{d-p-1}]}\delta_{\underline{k}\underline{k}'}\,. \quad (3.34)$$

Diagonalisation of the Hamiltonian

This Fourier mode decomposition reduces the problem of diagonalising the Hamiltonian to a simple exercise in decoupled quantum oscillators, with

$$\begin{aligned} \hat{H}_1[\hat{E}_L,\hat{P}_E] &= \frac{V}{2}\,\frac{\kappa^2\,g^2}{(p-1)!}\sum_{\underline{k}\neq\underline{0}}\left(\hat{P}_E^{\alpha_1\cdots\alpha_{p-1}}(\underline{k})\right)^2 \quad (3.35)\\ &+ \frac{V}{2}\,\frac{1}{(p-1)!}\,\frac{1}{\kappa^2\,g^2}\sum_{\underline{k}\neq\underline{0}}\tilde{\omega}^2(\underline{k})\,\left(\hat{E}_L^{\alpha_1\cdots\alpha_{p-1}}(\underline{k})\right)^2\,,\\ \hat{H}_2[\hat{G}_L,\hat{Q}_E] &= \frac{V}{2}\,\frac{\kappa^2\,e^2}{(d-p-1)!}\sum_{\underline{k}\neq\underline{0}}\left(\hat{Q}_G^{\alpha_1\cdots\alpha_{d-p-1}}(\underline{k})\right)^2 \quad (3.36)\\ &+ \frac{V}{2}\,\frac{1}{(d-p-1)!}\,\frac{1}{\kappa^2\,e^2}\sum_{\underline{k}\neq\underline{0}}\tilde{\omega}^2(\underline{k})\,\left(\hat{G}_L^{\alpha_1\cdots\alpha_{d-p-1}}(\underline{k})\right)^2\,. \end{aligned}$$

In these expressions the following notation is being used,

$$\left(\hat{E}_L^{\alpha_1\cdots\alpha_{p-1}}(\underline{k})\right)^2 = \sum_{\substack{\alpha_1,\ldots,\alpha_{p-1}=1\\ \beta_1,\ldots,\beta_{p-1}=1}}^{d-1}\hat{E}_L^{\alpha_1\cdots\alpha_{p-1}}(\underline{k})\,\hat{E}_L^{\dagger\ \beta_1\cdots\beta_{p-1}}(\underline{k})\,\delta^{\alpha_1\beta_1}\ldots\delta^{\alpha_{p-1}\beta_{p-1}}.$$

The operators (3.35) and (3.36) are nothing other than the Hamiltonians of a collection of C_{d-1}^{p-1} and C_{d-1}^{p} independent harmonic oscillators, respectively, all of angular frequency

$$\tilde{\omega}(\underline{k}) = \sqrt{4\,\pi^2\,\omega^2(\underline{k})+\mu^2}, \qquad \mu = e\,g\,\kappa.$$

The physical spectrum may easily be constructed by introducing annihilation and creation operators associated to the algebras (3.33) and (3.34). The annihilation operators are defined by

$$\mathfrak{a}^{\alpha_1\cdots\alpha_{p-1}}(\underline{k}) = \frac{1}{\kappa\,g}\sqrt{\frac{V\,\tilde{\omega}(\underline{k})}{2\,\hbar}}\,\left(\hat{E}_L^{\alpha_1\cdots\alpha_{p-1}}(\underline{k}) + \mathrm{i}\,\frac{g^2\,\kappa^2}{\tilde{\omega}(\underline{k})}\,\hat{P}_E^{\alpha_1\cdots\alpha_{p-1}}(\underline{k})\right)\,,$$

$$\mathfrak{b}^{\alpha_1\cdots\alpha_{d-p-1}}(\underline{k}) = \frac{1}{\kappa\,e}\sqrt{\frac{V\,\tilde{\omega}(\underline{k})}{2\,\hbar}}\,\left(\hat{G}_L^{\alpha_1\cdots\alpha_{d-p-1}}(\underline{k}) + \mathrm{i}\,\frac{\kappa^2\,e^2}{\tilde{\omega}(\underline{k})}\,\hat{Q}_G^{\alpha_1\cdots\alpha_{d-p-1}}(\underline{k})\right)\,,$$

whereas the creation operators $\mathfrak{a}^{\dagger\ \alpha_1\cdots\alpha_{p-1}}(\underline{k})$ and $\mathfrak{b}^{\dagger\ \alpha_1\cdots\alpha_{d-p-1}}(\underline{k})$ are merely the adjoint operators of $\mathfrak{a}^{\alpha_1\cdots\alpha_{p-1}}(\underline{k})$ and $\mathfrak{b}^{\alpha_1\cdots\alpha_{d-p-1}}(\underline{k})$, respectively. One then establishes the Fock algebras,

$$\begin{aligned}\left[\mathfrak{a}^{\alpha_1\cdots\alpha_{p-1}}(\underline{k}),\mathfrak{a}^{\dagger\ \beta_1\cdots\beta_{p-1}}(\underline{k}')\right] &= \delta^{\alpha_1\,[\beta_1}\ldots\delta^{\alpha_{p-1}\ \beta_{p-1}]}\,\delta_{\underline{k}\underline{k}'}, \\ \left[\mathfrak{b}^{\alpha_1\cdots\alpha_{d-p-1}}(\underline{k}),\mathfrak{b}^{\dagger\ \beta_1\cdots\beta_{d-p-1}}(\underline{k}')\right] &= \delta^{\alpha_1\,[\beta_1}\ldots\delta^{\alpha_{d-p-1}\ \beta_{d-p-1}]}\,\delta_{\underline{k}\underline{k}'}\,,\end{aligned} \tag{3.37}$$

whereas (3.35) and (3.36) then reduce to the simple expressions,

$$\hat{H}_1[\mathfrak{a},\mathfrak{a}^\dagger] = \hbar\sum_{\underline{k}\neq\underline{0}}\tilde{\omega}(\underline{k})\left(\frac{1}{2}\,C^{p-1}_{d-1} + \sum_{\alpha_1<\cdots<\alpha_{p-1}}^{d-1}\mathfrak{a}^{\dagger\ \alpha_1\cdots\alpha_{p-1}}(\underline{k})\,\mathfrak{a}^{\alpha_1\cdots\alpha_{p-1}}(\underline{k})\right)\,,$$

$$\hat{H}_2[\mathfrak{b},\mathfrak{b}^\dagger] = \hbar\sum_{\underline{k}\neq\underline{0}}\tilde{\omega}(\underline{k})\left(\frac{1}{2}C^{p}_{d-1} + \sum_{\alpha_1<\cdots<\alpha_{d-p-1}}^{d-1}\mathfrak{b}^{\dagger\ \alpha_1\cdots\alpha_{d-p-1}}(\underline{k})\,\mathfrak{b}^{\alpha_1\cdots\alpha_{d-p-1}}(\underline{k})\right).$$

The Fock space representation is based on the normalised Fock vacuum $|0\rangle$, $\langle 0|0\rangle = 1$, which is the kernel of all annihilation operators

$$\mathfrak{a}^{\alpha_1\cdots\alpha_{p-1}}(\underline{k})\,|0\rangle = 0, \qquad \mathfrak{b}^{\alpha_1\cdots\alpha_{d-p-1}}(\underline{k})\,|0\rangle = 0\,,$$

where the divergent total vacuum energy $\varepsilon^{1+2}_{(0)}$ associated to this fondamental quantum state reads

$$\varepsilon^{1+2}_{(0)} = \frac{1}{2}\hbar\,C^p_d\sum_{\underline{k}\neq\underline{0}}\tilde{\omega}(\underline{k})\,.$$

Excited states are obtained through the action onto the Fock vacuum of all $C^p_d = C^{p-1}_{d-1} + C^p_{d-1}$ creation operators, see (3.37). This leads to states $|n_\gamma(\underline{k})\rangle$ with energy eigenvalues

$$\varepsilon_{(n_\gamma(\underline{k}))} = \varepsilon^{1+2}_{(0)} + \hbar\sum_{\underline{k}\neq\underline{0}}\sum_{\gamma=1}^{C^p_d} n_\gamma(\underline{k})\,\tilde{\omega}(\underline{k}), \tag{3.38}$$

where $\{n_\gamma(\underline{k})\}_{\gamma=1}^{C^d_p}$ are positive integers corresponding to number operator eigenvalues. A shorthand notation is used in (3.38) with the index γ labelling the C^p_d possible combinations of a set of p distinct integers in the range $[1,d]$, $\{\alpha_1,\ldots,\alpha_i,\ldots,\alpha_p\}^d_{\alpha_i=1}$, which will be referred to as Γ^p_d.

3.5 Spectrum and projection onto the TFT sector

3.5.1 Physical spectrum on the torus

Combining all the results of the previous Sections for what concerns the diagonalisation of the physical TMGT Hamiltonian on the spatial d-torus $\Sigma = T_d$, the complete energy spectrum of states is given as

$$\varepsilon_{(n_\gamma(\underline{k}))} = \varepsilon_{(0)} + \hbar \sum_{\underline{k}\in\mathbb{Z}^d} \sum_{\gamma} n_\gamma(\underline{k})\, \tilde{\omega}(\underline{k})\,, \tag{3.39}$$

which is the sum of the contributions (3.30) and (3.38), see Fig.3.2. Note that on the d-torus, the p^{th} Betti number, N_p, equals C_d^p. The components of the vector $\underline{k}$ of the dual lattice may take any integer values since it is implicit in (3.39) that $\{n_\gamma(\underline{0})\}_{\gamma=1}^{C_d^p} = \{n_\gamma\}_{\gamma=1}^{N_p}$. However, the index γ has a different meaning whether $\underline{k} \neq \underline{0}$ or $\underline{k} = \underline{0}$. In the first case it refers to a value in the set Γ_d^p and denotes one of the possible C_d^p polarisations, while in the second case it is a (co)homology index, $\gamma = 1, \cdots, C_d^p$. The total vacuum energy $\varepsilon_{(0)}$ in (3.39) is divergent,

$$\varepsilon_{(0)} = \frac{1}{2}\hbar\, C_d^p \sum_{\underline{k}\in\mathbb{Z}^d} \tilde{\omega}(\underline{k})\,,$$

and must be subtracted from the energy spectrum.

The positive integer valued functions $n_\gamma(\underline{k})$ count, for each $\underline{k} \neq \underline{0}$, the number of massive quanta of a p- or $(d-p)$-tensor field of momentum $2\pi\hbar\,\underline{k}$, of polarisation (3.32), namely

$$\begin{aligned} \text{Transverse} \quad &: \quad \varepsilon^{\gamma}_{i_1\cdots i_p}(\underline{k}), \quad \gamma \in \Gamma^p_{d-1}\,; \\ \text{Longitudinal} \quad &: \quad \varepsilon^{\gamma\, d}_{i_1\cdots i_p}(\underline{k}), \quad \gamma \in \Gamma^{p-1}_{d-1}\,, \end{aligned} \tag{3.40}$$

and of rest mass[7]

$$M = \hbar\,\mu = \hbar\,\kappa\, e\, g. \tag{3.41}$$

There are also the contributions of the global quanta of the p- and $(d-p)$-tensor fields, where $\{n_\gamma(\underline{0})\}_{\gamma=1}^{C_d^p}$ count the numbers of excitations along the homology cycle generators $\Sigma^\gamma_{(p)}$ and $\Sigma^\gamma_{(d-p)}$. In the particular case when $p = 1$, the integers $\{n_\gamma(\underline{k})\}_{\gamma=1}^{d}$ count, for each $\underline{k} \neq \underline{0}$, the number of massive photons of momentum $2\pi\hbar\,\underline{k}$, of rest mass M and of polarisation

$$\text{Transverse}: \ \{\varepsilon_i^\gamma(\underline{k})\}_{\gamma=1}^{d-1}\,, \qquad \text{Longitudinal}: \varepsilon_i^d(\underline{k})\,.$$

[7] A quantity indeed positive under the assumptions made.

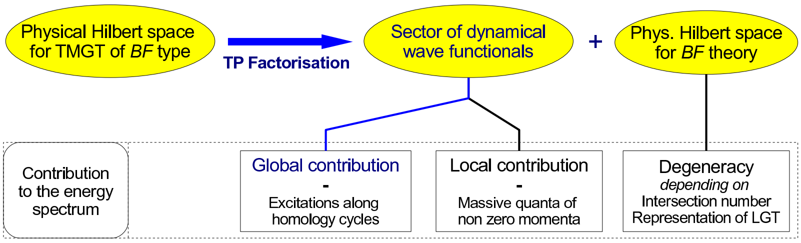

Figure 3.2: *Factorisation of Hilbert space for TMGT and the contribution of each sector to the energy spectrum. The physical Hilbert space of the TFT sector is defined modulo the action of SGT and according to the way we deal with LGT.*

Depending on how one deals with large gauge transformations in the TFT sector, each energy state is either infinitely degenerate for a real valued k, see (3.22), or $(\prod_{\delta=1}^{N_p} k_1\, k_2\, \mathcal{I}\, \mathrm{Min}(I_{\delta\delta'}))$ times degenerate if k is a rational number of the form $k = k_1/k_2$. If k is an integer, each energy state is $(\prod_{\delta=1}^{N_p} k\, \mathcal{I}\, \mathrm{Min}(I_{\delta\delta'}))$ times degenerate and the mass gap is then quantised,

$$M = \frac{\hbar^2}{2\pi}\, \mathcal{I}\, k\, e\, g\,.$$

In the Maxwell-Chern-Simons (MCS) case in $2+1$ dimensions (1.26), we recover in the global sector a quantum mechanical system corresponding to the Landau problem of condensed matter physics on the 2-torus.

3.5.2 Projection onto the topological field theory sector

As introduced in Section 1.5, the naive limits of infinite coupling constants, $e \to \infty$ and $g \to \infty$, in the classical Lagrangian of topologically massive gauge theories, see (1.26) and (1.23), must lead to a pure topological field theory (TFT) of the AF or BF type. However, as pointed out by several authors (see for example [71, 56]), a paradox seems to arise at the quantum level (as well as within the classical Hamiltonian formulation) when the pure Chern-Simons (CS) theory is viewed as the limit $e \to \infty$ of the Maxwell-Chern-Simons (MCS) theory. The Hilbert space of the CS theory is constructed from the algebra of the non commuting configuration space operators which are in fact canonically conjugate phase space operators. As far as the MCS theory is concerned, its Hilbert space is constructed from the Heisenberg algebras of twice as many phase space operators. This problem is generic whenever a pure quantum TFT (TQFT) is considered as the limit of its associated TMGT because two distinct Hilbert spaces are being compared. Actually, due to the second-class constraints appearing in the Hamiltonian analysis of a TFT which is already in Hamiltonian form, non vanishing commutation relations apply to the configuration space operators. Furthermore, the Gauss law constraints of pure TFT are not the limit of the Gauss law constraints of TMGT. The former operators tend to restrict too drastically the physical Hilbert space in comparison to the limit of the TMGT physical Hilbert space.

This problem of an ill-defined limit is usually handled by projecting from the Hilbert space of the TMGT onto its degenerate ground state. This projection acts in a manner similar to second-class constraints which then lead to a reduced phase space and non vanishing configuration space commutation relations determined from the associated Dirac brackets. The global sector of the MCS theory is analogous to the classical Landau problem of a charged point particle of mass m moving in a two dimensional surface in the presence of an uniform external magnetic field B perpendicular to that surface. The mass gap (3.41) then corresponds to the cyclotron frequency ω_c [55, 56, 73]. The spectrum of the quantised model is organised into Landau levels (with a degeneracy dependent on the homology structure of the underlying manifold), of which the energy separation ω_c is proportional to the ratio B/m. The limit $B \to \infty$ or $m \to 0$ effectively projects onto the lowest Landau level (LLL) in which one obtains a non commuting algebra for the space coordinates. By analogy, projection onto the ground state reduces the phase space of the MCS theory (1.33) to the canonically conjugate configuration space operators of a pure CS theory. In the global sector, the projection from a TMGT, (2.14), onto a pure TQFT offers in some sense a generalisation of the LLL projection in any dimension. The mass gap (3.41) of the TMGT becoming infinite for coupling constants running to infinity, all excited states decouple from the physical spectrum, leaving over only the degenerate ground states. Projection onto these ground states restricts the Hilbert space to that of a TQFT.

Interestingly, the PT factorisation established in Chapter 2 enables the usual projection from TMGT to TQFT to be defined in a natural way. Already in the classical Hamiltonian formulation phase space is separated into two decoupled sectors, the first being dynamical and manifestly gauge invariant, and the second being equivalent to a pure TFT with identical Gauss law constraints and commutation relations. The present approach does not require any gauge fixing procedure whatsoever. Actually the non commuting sector of a CS theory or, more generally, the reduced phase space of a TQFT appears no longer after the projection onto the ground state at the quantum level (or after the introduction of Dirac brackets) but is manifest already at the classical Hamiltonian level. By letting e or g grow infinite, the mass gap (3.41) becomes infinite, hence dynamical massive excitations decouple whereas the TFT sector, which is independent of the coupling constants, remains unaffected. In this limit, the system looses any dynamics, the latter being intimately related to the Riemannian metric structure of the spacetime manifold, while all that is then left is a wave function depending on global variables only, namely the quantum states of a TQFT.

3.6 Discussion

Duality relations are especially interesting when an explicit relation between fields of the two descriptions is established since it typically exchanges strong and weak coupling regimes. In chapter 2 and 3 another feature which may arise from some dualisation processes has been addressed, namely the isolation of a dynamical sector of physical variables. Applied for the first time in our papers [10, 11] to topologically massive gauge theories (TMGT) in any dimension and irrespective of whether the Lagrangian or Hamiltonian formulation is being used, this duality transformation is referred to as TP factorisation and sheds new light onto the concept of Lowest Landau Level projection.

As a matter of fact, it was already established in [74] that it is possible to identify among the phase space variables of a TMGT combinations corresponding to those of a TFT. Indeed in a particular case of an underlying manifold with boundary, edge states may be understood in terms of a TFT, already at the classical level. Nevertheless, this paper did not realise the powerful gauge fixing free factorisation leading to a dual theory decoupled into two sectors as described in Chapters 2 and 3. Incidentally, it should be of interest to analyse how this new approach may shed new light onto this paper, in a manner akin to that in which it properly defines the projection onto a topological field theory through the limits $e, g \to \infty$. In the present approach, the TFT sector is actually made manifest already at the classical level (see Fig.3.1), independently of any projection onto the quantum ground state, or onto physical edge states in the case of a manifold with boundary. This TFT sector accounts for the degeneracy of the physical spectrum depending only on topological invariants. The energy spectrum includes two types of contributions, as illustrated in Fig.3.2. The first one originates from a sector of global variables where a metric structure is introduced but not explicitly specified in order to diagonalise the Hamiltonian. The second one originates from a sector of dynamical variables where the spacetime manifold endowed with its metric structure must be specified in order to diagonalise the Hamiltonian through a spectral decomposition.

The formalism of TMGT defined by the actions (1.25) or (1.23) offers a possible description of some phenomena such as effective superconductivity [75, 74], Josephson arrays [76] for compact gauge groups, confinement by flux tube [77, 78], etc. In this case, the TP factorisation stands for a generalisation of the Landau projection which enables to isolate the essential topological content. Then the denomination "topological sector" has to be understood not only in terms of topological couplings, but in a more general context as being the sector where all non trivial topological effects arise.

CHAPTER 4

Dual formulation of abelian Higgs models

The TP factorisation technique introduced in Chapter 2 consists in constructing a dual formulation of TMGT which is factorised into a dynamical sector of massive physical variables and a gauge dependent sector defining a topological field theory. This technique enables one to establish that the Maxwell-Higgs model in the symmetry breaking phase shares a common physical sector with a particular form of TMGT of the BF-type coupled to a real scalar "Higgs" field in a specific way. This duality relation may be extended to the usual formulation of the abelian Higgs action in terms of a complex scalar field. Nevertheless, it then turns out to be impossible to maintain at the same time the gauge content of gauge theories and locality. Our dual factorisation techniques also apply to TMGT of the AF type and novel duality relations then arise. The general case of TMGT in any dimension is next addressed for specific couplings to scalar fields constructed in such a way that our TP factorisation be not broken.

Furthermore, following a similar approach, this Chapter introduces a generalisation of the already known duality between TMGT of the BF-type and Stueckelberg theories, based on the "London limit" of the dualities mentioned above. Hence a structure of dual equivalences between mass generation mechanisms, namely the Higgs, the Stueckelberg and the topological mass generation mechanisms, is easily obtained, provided that the scalar Higgs field does not vanish. In fact any zero of the Higgs field on some subset of space which is of zero measure is associated to the existence of a topological defect localised on this subset, as will be discussed in Chapters 5 and 6.

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4.1 The dual Maxwell-Higgs model in 3+1 dimensions

4.1.1 The dielectric Cremmer-Scherk action: two pictures

The dielectric Cremmer-Scherk Lagrangian density

As is to be discussed presently, a dual formulation of the Maxwell-Higgs model in $3{+}1$ dimensions in the symmetry breaking phase may be obtained from the topologically massive Cremmer-Scherk action (1.24) by transmuting the scale factor g into a real dynamical scalar field $\varrho(x)$. The resulting Lagrangian density then reads

$$\begin{aligned}\mathcal{L}^4_{\mathrm{TMG}\varrho} &= -\frac{1}{4\,e^2}F_{\mu\nu}\,F^{\mu\nu} + \frac{1}{12\,\eta^2}\frac{1}{\varrho^2}H_{\mu\nu\rho}\,H^{\mu\nu\rho} + \mathcal{L}_\varrho \\ &+ \kappa\,\epsilon^{\mu\nu\rho\sigma}\left(\frac{\xi}{6}A_\mu\,H_{\nu\rho\sigma} + \frac{1-\xi}{4}F_{\mu\nu}\,B_{\rho\sigma}\right)\,. \end{aligned} \tag{4.1}$$

The field $\varrho(x)$ thus interacts with the tensor field $B_{\mu\nu}(x)$ through a dielectric coupling. Such types of Lagrangian densities in 3+1 dimensions will therefore be called "dielectric Cremmer-Scherk theories". In the present Chapter it will be assumed that the field $\varrho(x)$ does not vanish anywhere on the spacetime manifold. This assumption will be relaxed later on but at this stage it ensures that the action remains finite for a field strength $H_{\mu\nu\rho}(x)$ which is well defined everywhere on the spacetime manifold. As is established in Chapters 5 and 6, zeros of the scalar field are intimately related to the existence of topological defects. As usual, $\mathcal{L}_\varrho$ is the Lagrangian density for a massive real scalar field:

$$\mathcal{L}_\varrho\left(\varrho, \partial_\mu\varrho\right) = \frac{1}{2}\partial_\mu\varrho\,\partial^\mu\varrho - V\left(\varrho^2\right)\,, \tag{4.2}$$

where there is no need to specify the self-interaction potential $V\left(\varrho^2\right)$ at this stage.

The arbitrary real variable $\xi \in [0,1]$ we have introduced is physically irrelevant for an appropriate choice of boundary conditions since it parametrises any possible surface term. This implies for instance that the equations of motion are independent of ξ. Nevertheless this does not mean that the introduction of this parameter is useless. We have already proved in Section 2.2.2 without having recourse to any gauge fixing choice whatsoever that ξ parametrises a classification of the generalised Proca theories. These theories account for the physical sector of pure topologically massive gauge theories as made manifest through our TP factorisation techniques. Another reason for introducing the parameter ξ is to offer an elegant interpretation of the dielectric Cremmer-Scherk theory (4.1) in terms of the p-form field $A(x)$ or the $(d{-}p)$-form field $B(x)$ whether $\xi = 1$ or $\xi = 0$, respectively. In fact the two possible interpretations already apply within the original non factorised formulation of the theory.

The "B-field picture" : $\xi = 0$

Let us first consider the Lagrangian density (4.1) with $\xi = 0$, and isolate the contributions involving the gauge field $B_{\mu\nu}(x)$,

$$\mathcal{L}^{\xi=0}_{\mathrm{TMG}\varrho} = -\frac{1}{4\,e^2} F_{\mu\nu}\, F^{\mu\nu} + \mathcal{L}_\varrho + \mathcal{L}^{B}_{\mathrm{TMG}\varrho}\,.$$

We then have:

$$\mathcal{L}^{B}_{\mathrm{TMG}\varrho} = \frac{1}{12\,\eta^2}\frac{1}{\varrho^2} H_{\mu\nu\rho}\, H^{\mu\nu\rho} + \frac{1}{4}\,\kappa\,\epsilon^{\mu\nu\rho\sigma}\, F_{\mu\nu}\, B_{\rho\sigma}\,.$$

In comparison with the "dielectric" Maxwell theories introduced in Section 1.1, the part where the field $B_{\mu\nu}(x)$ is involved describes the dynamics of a 2-form gauge field embedded in a medium described by a suitable dielectric function $\varepsilon(x)$, see (1.46). Hence the scalar field $\varrho(x)$ plays the role of this dielectric function:

$$\varepsilon = \frac{1}{\eta^2\,\varrho^2}\,,$$

where we recall that it is assumed in the present Chapter that $\varrho(x)$ does not vanish anywhere on the spacetime manifold.

If the associated 2-form conserved current $K^{\mu\nu}(x)$ is written[1] as

$$K^{\mu\nu} = \kappa\, e^2\, \eta^{\mu\alpha}\, \eta^{\nu\beta}\, E_{\alpha\beta}\,, \tag{4.3}$$

the source for the electromagnetic field associated to $B_{\mu\nu}(x)$ reads

$$E_{\alpha\beta} = \frac{1}{e^2}\,\eta_{\alpha\mu}\,\eta_{\beta\nu}\,\epsilon^{\mu\nu\rho\sigma}\,\partial_\rho A_\sigma\,, \qquad \eta^{\mu\alpha}\partial_\mu E_{\alpha\beta} = 0\,. \tag{4.4}$$

The fact that this current involves the dynamical gauge field $A_\mu(x)$ does not imply any particular consequence in the present discussion. This just characterises the underlying dielectric model described by the dielectric Cremmer-Scherk theory (4.1). However it is of prime importance that the current $E_{\mu\nu}(x)$ does not explicitly depend on the scalar field $\varrho(x)$ as befits any dielectric theory of the form (1.46).

The choice $\xi = 0$ is naturally related to a realisation of the dielectric Cremmer-Scherk theory as a dielectric theory for the 2-form field $B_{\mu\nu}(x)$ or, more generally, to an interpretation of the underlying physical system in terms of the electromagnetic field constructed from $B_{\mu\nu}(x)$. For this reason we will henceforth refer to this case as the "B-field picture".

[1]This choice for the normalisation of the current involving κ and the scale factor e^2 is made for later convenience.

The "A-field picture" : $\xi = 1$

Let us next consider the Lagrangian density (4.1) with $\xi = 1$, and isolate the contributions involving the gauge field $A_\mu(x)$,

$$\mathcal{L}^{\xi=1}_{\mathrm{TMG}\varrho} = \frac{1}{12\,\eta^2}\frac{1}{\varrho^2} H_{\mu\nu\rho}\, H^{\mu\nu\rho} + \mathcal{L}_\varrho + \mathcal{L}^{A}_{\mathrm{TMG}\varrho}\,. \tag{4.5}$$

Now we obtain:

$$\mathcal{L}^{A}_{\mathrm{TMG}\varrho} = -\frac{1}{4\,e^2} F_{\mu\nu}\, F^{\mu\nu} + \frac{1}{6}\,\kappa\,\epsilon^{\mu\nu\rho\sigma}\, A_\mu\, H_{\nu\rho\sigma}\,.$$

In comparison with the Maxwell theories introduced in Section 1.1, the part where the field $A_\mu(x)$ is involved describes the dynamics of a 1-form gauge field, where the conserved current $J^\mu(x)$, written for later convenience as

$$J^\mu = \kappa\,\eta^2\,\varrho^2\,\eta^{\mu\alpha}\,G_\alpha, \tag{4.6}$$

reads, according to (1.1) and (4.5),

$$\eta^2\,\varrho^2\,G_\alpha = \frac{1}{2}\,\eta_{\alpha\mu}\,\epsilon^{\mu\nu\rho\sigma}\,\partial_\nu\,B_{\rho\sigma}\,, \qquad \eta^{\mu\alpha}\partial_\mu(\eta^2\,\varrho^2\,G_\alpha) = 0\,, \tag{4.7}$$

and thus generates the electromagnetic field associated to $A_\mu(x)$.

As is to be discussed presently, the Lagrangian density (4.5) offers a dual formulation of the Maxwell-Higgs model in $3+1$ dimensions. Hence within this context, the scalar field $\varrho(x)$ plays the role of the Higgs field. In fact, the Lagrangian density (4.5) was already obtained previously by K. Lee [79] through a path integral formulation but has so far never been analysed in detail except in the London limit or within the context of effective vortex-string theories. Our approach highlights the necessary careful treatment of the gauge content of such theories by means of a local and linear reparametrisation of the first-order Lagrangian formulation. Indeed we are going to prove that the Lagrangian density (4.5) is not dual to the $U(1)$ abelian Higgs model but rather more specifically to the first order formulation of the latter in terms of physical variables (1.48). Again the equivalence is established modulo a topological BF term. The other great advantage of this new approach is to display clearly the possibility of constructing new topological defect solutions for this dual formulation and to give some clues towards that goal. The case $\xi = 1$ naturally relates to an interpretation of the underlying physical system in terms of the electromagnetic field constructed from a 1-form gauge field. For this reason we will henceforth refer to this case as the "A-field picture".

4.1.2 Factorisation and duality within the Lagrangian formulation

Physical and topological sectors

This duality identification first proceeds from the definition of the first order Lagrangian formulation associated to (4.1), hence the introduction of the Gaussian auxiliary fields $E_{\mu\nu}(x)$ and $G_\mu(x)$:

$$\begin{aligned}\mathcal{L}^{\rm m}_{{\rm TMG}\varrho} &= -\frac{e^2}{4}\,E_{\mu\nu}\,E^{\mu\nu} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\,\partial_\mu A_\nu\,E_{\rho\sigma} \\ &+ \frac{\eta^2}{2}\,\varrho^2\,G_\mu\,G^\mu + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\,\partial_\mu B_{\nu\rho}\,G_\sigma \\ &+ \kappa\,\epsilon^{\mu\nu\rho\sigma}\left(\frac{\xi}{6}A_\mu\,H_{\nu\rho\sigma} + \frac{1-\xi}{4}F_{\mu\nu}\,B_{\rho\sigma}\right) + \mathcal{L}_\varrho\,. \end{aligned} \tag{4.8}$$

When considered in combination with the equations of motion defining the Gaussian integration, these physical variables $E_{\mu\nu}(x)$ and $G_\mu(x)$ correspond to the currents introduced in (4.4) and (4.7) within the B-field and A-field pictures, respectively.

The Topological-Physical factorisation is not really affected by the presence of the dielectric coupling. Indeed under the same parametrisation as the one introduced for the free TMGT in Section 2.2, that is

$$A_\mu = \mathcal{A}_\mu - \frac{1}{\kappa}G_\mu\,, \qquad B_{\mu\nu} = \mathcal{B}_{\mu\nu} + \frac{1}{\kappa}E_{\mu\nu}\,, \tag{4.9}$$

the dual action is again factorised into a dynamical sector of physical variables and a topological sector of gauge variant variables, modulo a total surface term, “ST”,

$$S_{\rm fac}\,[E,G,\mathcal{A},\mathcal{B},\varrho] = S_{\rm MHP}\,[E,G,\varrho] + S^4_{BF}\,[\mathcal{A},\mathcal{B}] + \int_{\mathcal{M}} {\rm ST}\,. \tag{4.10}$$

Similarly to the free case, the surface term is ignored while the topological sector,

$$\mathcal{L}^4_{BF} = \xi\,\frac{\kappa}{6}\epsilon^{\mu\nu\rho\sigma}\,\mathcal{A}_\mu\,\mathcal{H}_{\nu\rho\sigma} + (1-\xi)\,\frac{\kappa}{4}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\,\mathcal{B}_{\rho\sigma}\,,$$

is as usual a pure BF topological field theory.

In fact only the physical sector significantly changes since it now inherits the dynamical scalar field $\varrho(x)$ to which the physical field $G_\mu(x)$ couples:

$$\begin{aligned}\mathcal{L}_{\rm MHP} &= -\frac{e^2}{4}E_{\mu\nu}\,E^{\mu\nu} + \frac{\eta^2}{2}\,\varrho^2\,G_\mu\,G^\mu + \mathcal{L}_\varrho \\ &+ \frac{1}{2\,\kappa}\epsilon^{\mu\nu\rho\sigma}\left(\xi\,\partial_\mu G_\nu\,E_{\rho\sigma} - (1-\xi)\,\partial_\mu E_{\nu\rho}\,G_\sigma\right)\,. \end{aligned} \tag{4.11}$$

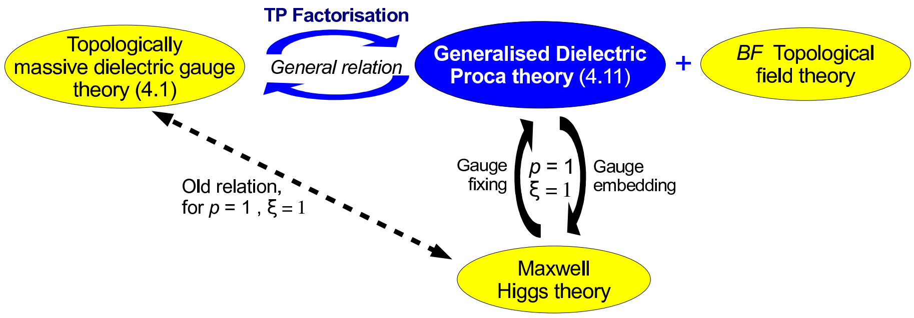

Figure 4.1: *Duality relation between the physical sector of TMDGT (4.1) and the generalised first order formulation of dielectric Proca theories obtained through TP factorisation techniques. The dual Maxwell-Higgs model is recovered through a gauge embedding procedure in a very specific case. In blue: our contribution.*

This new Lagrangian density may be considered as the most general formulation for dielectric Proca theories, for which the first order formulations known so far in the literature are recovered upon setting $\xi=0$ or $\xi=1$. In particular for the latter value of ξ, namely within the A-field picture, we have already encountered such a Lagrangian density in Section 1.2. It is nothing other than the first order Lagrangian formulation of the Maxwell-Higgs model expressed in terms of physical variables, see (1.48), with $\eta = 1/\kappa$. In this sense, $\varrho(x)$ plays the role of the Higgs field provided that $\mathcal{L}_\varrho$ defined in (4.2) be the Lagrangian density for the decoupled part of the Higgs field (1.49), namely the kinetic term of the scalar field together with the symmetry breaking quartic potential. This potential may be thus chosen to be of the form

$$V\left(\varrho^2\right) = \frac{\tilde{\mu}^2}{2}\,\varrho^2 + \frac{\lambda}{4}\,\varrho^4\,, \tag{4.12}$$

where $\tilde{\mu}^2 < 0$ and $\lambda > 0$, in order to recover the usual "Mexican hat-shaped" potential. Therefore the Higgs field possesses a non vanishing vacuum expectation value,

$$\langle\varrho\rangle = v = \sqrt{\frac{-\tilde{\mu}^2}{\lambda}} \neq 0\,. \tag{4.13}$$

In contradistinction to discussions available until now in the literature, see for example [79], we have precisely established to which Lagrangian the dielectric Cremmer-Scherk theory (4.1) is dual, that is the first order physical formulation of the Maxwell-Higgs model along with a topological BF term, as illustrated in Fig.4.1. Moreover this duality is constructed from a linear and local redefinition of the first order formulation of the dynamics.

Restoration of the $U(1)$ broken symmetry

At this stage, the usual formulation of the Maxwell-Higgs Lagrangian (1.11) is thus recovered through the inverse procedure to that which leads to the physical formulation, discussed in Section 1.4. This procedure requires a non trivial extension of the gauge content. The gauge invariant variable $G_\mu(x)$ is associated to the Hodge dual of the field strength of $B_{\mu\nu}(x)$ through Gaussian integration, see (4.7), in the first order formulation (4.8). After the transformation (4.9), $G_\mu(x)$ obeys the following equation of motion associated to the physical sector of the factorised Lagrangian (4.11),

$$\eta^2\,\varrho^2\,G_\alpha = -\eta_{\alpha\mu}\,\frac{1}{2\,\kappa}\,\epsilon^{\mu\nu\rho\sigma}\,\partial_\nu\,E_{\rho\sigma}\,.$$

A conservation law naturally arises from this latter equation: $\partial^\mu\left(\varrho^2\,G_\mu\right)=0$ (on shell). In this sense the variable $J_\mu \ltimes \varrho^2\,G_\mu$ is interpreted as a current to which a gauge connection couples within the A-field picture, see (4.6). This current may be that of the complex scalar field of which the Higgs field $\varrho(x)$ is the radial part in the polar parametrisation, following the example of (1.17) which results from (1.13).

However, in order to proceed to a consistent reparametrisation of the field configuration space, the connection may not be the original $A_\mu(x)$ connection as often advocated in the literature (see for example [35, 80]). First, this latter choice would lead to an inconsistent counting of the degrees of freedom as far as the transformations (4.9) are concerned, although it remains consistent with the equations of motion[2] (at least as long as couplings to fermions are not considered). Second, an extension of the gauge content of the system is required in order to restore the $U(1)$ symmetry under which the complex scalar field transforms. This extension is similar in spirit to gauge embedding or other related dualisation [81] procedures. Thus a new gauge connection $\tilde{A}_\mu(x)$ is introduced, which allows to redefine the gauge invariant variable $G_\mu(x)$ as

$$\kappa\,\eta^2\,G_\mu = -\frac{1}{\varrho^2}J_\mu = \kappa^2\,\eta^2\,\left(\tilde{A}_\mu - \partial_\mu\theta\right)\,, \tag{4.14}$$

where the transformation of the variable $\theta(x)$ under the new $U(1)$ gauge symmetry compensates for that of the connection $\tilde{A}_\mu(x)$ in order to preserve the gauge invariance of $J_\mu(x)$,

$$\tilde{A}'_\mu = \tilde{A}_\mu + \partial_\mu\tilde{\alpha}, \qquad \theta' = \theta + \tilde{\alpha}\,. \tag{4.15}$$

Here the local notation for the gauge transformation is used because global variables have no influence in this specific case on account of the restrictive assumptions made in this Chapter (topologically trivial manifold and nowhere vanishing scalar field $\varrho(x)$).

[2] The equations of motion relate the transverse part of the gauge field $A_\mu(x)$ to the physical field $G_\mu(x)$.

Given the redefinition of the physical field $G_\mu(x)$ in (4.14) the Lagrangian density of the Maxwell-Higgs model is recovered from the physical sector (4.11) upon setting $\xi = 1$,

$$\mathcal{L}^4_{\mathrm{AH}\varrho\theta} = -\frac{1}{4\,e^2}\,\tilde{F}_{\mu\nu}\,\tilde{F}^{\mu\nu} + \frac{1}{2}\left|\partial_\mu\varrho - \mathrm{i}\,\kappa\,\eta\,\varrho\,\left(\tilde{A}_\mu - \partial_\mu\theta\right)\right|^2 - V(\varrho^2)\,.$$

Therefore, the total gauge embedded action dual to (4.1) through TP-factorisation is decoupled into the $U(1)$ Maxwell-Higgs action and a pure topological BF action,

$$S\left[\tilde{A},\mathcal{A},\mathcal{B},\varrho,\theta\right] = S_{\mathrm{AH}}\left[\tilde{A},\varrho,\theta\right] + S_{\mathrm{TFT}}\left[\mathcal{A},\mathcal{B}\right] + \int_{\mathcal{M}} \mathrm{ST}\,.$$

This action turns out to be invariant under three independent classes of finite abelian gauge transformations acting separately in either the $\mathcal{A}$-, the $\mathcal{B}$-, the ($\tilde{A}$,θ)-sector. This latter restored transformation allows to define a complex scalar field of which the polar parametrisation follows in terms of the two real scalar fields $\varrho(x)$ and $\theta(x)$:

$$\phi(x) = \frac{1}{\sqrt{2}}\,\varrho(x)\,\mathrm{e}^{\mathrm{i}\kappa\eta\,\theta(x)}\,. \tag{4.16}$$

Thus the usual formulation of the Maxwell-Higgs Lagrangian density before symmetry breaking is recovered from S_{AH}:

$$\mathcal{L}_{\mathrm{AH}} = -\frac{1}{4\,e^2}\,\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \left|\tilde{\mathrm{D}}_\mu\phi\right|^2 - V\left(2\,|\phi|^2\right)\,. \tag{4.17}$$

In this Lagrangian density, the covariant derivative,

$$\tilde{\mathrm{D}}_\mu\phi = \partial_\mu\phi - \mathrm{i}\,\kappa\,\eta\,\tilde{A}_\mu\,\phi\,, \tag{4.18}$$

is defined from the connection $\tilde{A}$ of which the field strength tensor reads

$$\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu\,,$$

as usual.

This conclude the discussion about the local dual formulation of the Maxwell-Higgs model in 3+1 dimensions. It will be shown in Section 4.2 that the structure of dualities in Fig.4.1, hence the dual formulation of the Maxwell-Higgs model is extensible to any dimension, notwithstanding the non renormalisable character of the scalar field quadratic potential,

$$V\left[2\,|\phi|^2\right] = \tilde{\mu}^2\,|\phi|^2 + \lambda\,|\phi|^4\,, \tag{4.19}$$

in more than four spacetime dimensions.

4.1.3 Duality within the Hamiltonian formulation

The Hamiltonian analysis of constraints applied to the Lagrangian formulation of the dielectric Cremmer-Scherk (DCS) theory in (4.1) shows that the simple transmutation of the scale factor g into the dynamical scalar field $\varrho(x)$ readily leads to the Hamiltonian formulation of this DCS theory from the uncoupled Hamiltonian (2.14),

$$\begin{aligned}
\mathcal{H}^4_{\text{TMG}\varrho} &= \frac{e^2}{2}\left(P^i - (1-\xi)\,\frac{\kappa}{2}\,\epsilon^{ijk}\,B_{jk}\right)^2 + \frac{1}{4\,e^2}\,(F_{ij})^2 \\
&+ \frac{\eta^2}{4}\,\varrho^2\,\left(Q^{ij} + \xi\,\kappa\,\epsilon^{ijk}\,A_k\right)^2 + \frac{1}{12\,\eta^2}\,\frac{1}{\varrho^2}\,(H_{ijk})^2 + \mathcal{H}_\varrho \\
&- (u' + A_0)\,\partial_i\left(P^i + \xi\,\frac{\kappa}{2}\,\epsilon^{ijk}\,B_{jk}\right) + u\,P^0 \\
&+ (v'_i + B_{0i})\,\partial_j\left(Q^{ij} - (1-\xi)\,\kappa\,\epsilon^{ijk}\,A_k\right) + v_{0i}\,Q^{0i} \\
&+ \partial_i\left(A_0\,P^i + B_{0j}\,Q^{ij}\right)\,,
\end{aligned} \tag{4.20}$$

where the last term is a pure surface term characteristic of the analysis of constraints for such gauge theories. Again the symplectic structure of Poisson brackets is specified by the canonical brackets between phase space variables defined in (2.13), adapted to this specific number of dimensions. This Hamiltonian density shares the same gauge structure and thus the same first-class constraints as that of the pure Cremmer-Scherk theory. The only difference resides in the presence of the dynamical dielectric scalar field $\varrho(x)$ along with its conjugate momentum $\pi(x)$ such that

$$\{\varrho(t,\vec{x}\,)\,,\,\pi(t,\vec{y}\,)\} = \delta^2(\vec{x} - \vec{y}\,)\,. \tag{4.21}$$

To this new set of phase space variables is associated the Hamiltonian density $\mathcal{H}_\varrho$,

$$\mathcal{H}_\varrho = \frac{1}{2}\,\pi^2 + \frac{1}{2}\,(\partial_i\varrho)^2 + V\left(\varrho^2\right)\,, \tag{4.22}$$

which is decoupled from the gauge fields and their conjugate momenta.

In the same way as within the Lagrangian formulation, the presence of the dielectric coupling does not affect our factorisation techniques. Therefore, under the same canonical transformations, (2.16) and (2.20), and redefinition of Lagrange multipliers, (2.22), as in the uncoupled case, the factorised fundamental Hamiltonian reads

$$\begin{aligned}
\mathcal{H}^{\text{fac}}_{\text{TMG}\varrho} &= \frac{e^2}{2}\left(E^i\right)^2 + \frac{1}{2\,\eta^2\kappa^2}\left(\frac{\partial_i E^i}{\varrho}\right)^2 + \frac{\eta^2}{4}\left(\varrho\,G^{ij}\right)^2 + \frac{1}{2\,e^2\kappa^2}\left(\partial_j G^{ij}\right)^2 \\
&+ \mathcal{H}_\varrho + \frac{\kappa}{2}\,\mathcal{A}_0\,\epsilon^{ijk}\,\partial_i\mathcal{B}_{jk} - \kappa\,\mathcal{B}_{0i}\,\epsilon^{ijk}\,\partial_j\mathcal{A}_k + \text{ST}\,.
\end{aligned} \tag{4.23}$$

This factorised density is explicitly independent of ξ. This means that on account of Hodge duality, one readily identifies in the physical sector the formulation for either the A- or the B-field picture. In particular, when considered in combination with the equations of motions (2.12) and our TP canonical transformations (2.20), the physical variable $G^{ij}(x)$ reads

$$G^{ij} = Q^{ij} + \kappa\,\epsilon^{ijk}\,A_k\,,$$

and therefore, on the ground of the analysis of Section 1.4, may be interpreted as the electric displacement tensor $D^B_{\rm el}(x)$ associated to the original field $B_{\mu\nu}(x)$,

$$\left(D^B_{\rm el}\right)^{ij} = \frac{1}{\eta^2\,\varrho^2}\,\delta^{ik}\,\delta^{jl}\,H_{0kl} = G^{ij}\,, \quad \text{within the } B\text{-field picture.} \tag{4.24}$$

Likewise the variable $E^i(x)$ is proportional to the electric vector field $\vec{E}^A_{\rm el}(x)$,

$$\left(E^A_{\rm el}\right)^i = \delta^{ij}\,F_{0j} = e^2\,E^i\,, \quad \text{within the } A\text{-field picture,} \tag{4.25}$$

associated to the field strength tensor of the original variable $A_i(x)$.

The equations for Gaussian integration, see as a reminder (1.40) and (1.44), make manifest the Hodge duality between the space components of the physical fields within the first order Lagrangian formulation (4.8), E_{ij} and G_i, and the phase space variables of the physical sector within the Hamiltonian formulation (4.23), E^i and G^{ij},

$$E^i = \frac{1}{2}\,\epsilon^{ijk}\,E_{jk}\,, \qquad G^{ij} = \epsilon^{ijk}\,G_k\,. \tag{4.26}$$

In terms of these latter variables, the dynamical part of the factorised Hamiltonian formulation (4.23) reads within the B-field picture

$$\begin{aligned} \mathcal{H}^B_{\rm dyn} &= \frac{\eta^2}{4}\left[\varrho\left(D^B_{\rm el}\right)^{ij}\right]^2 + \frac{1}{2\,e^2\kappa^2}\left[\partial_j\left(D^B_{\rm el}\right)^{ij}\right]^2 \\ &+ \frac{e^2}{4}\,\left(E_{ij}\right)^2 + \frac{1}{12\,\eta^2\,\kappa^2}\left(\frac{1}{\varrho}\,\partial_{[i}\,E_{jk]}\right)^2 + \mathcal{H}_\varrho\,, \end{aligned} \tag{4.27}$$

with the following non-vanishing Poisson brackets between the phase space variables

$$\left\{\left(D^B_{\rm el}\right)^{ij}(t,\vec{x}\,), E_{kl}(t,\vec{y}\,)\right\} = \kappa\,\delta^k_{[i}\,\delta^l_{j]}\,\delta^{(3)}(\vec{x}-\vec{y}\,)\,.$$

Similarly, the dynamical part of (4.23) reads within the A-field picture

$$\begin{aligned} \mathcal{H}^A_{\rm dyn} &= \frac{1}{2\,e^2}\,\left(\vec{E}^A_{\rm el}\right)^2 + \frac{1}{2\,\eta^2\,\kappa^2}\left(\frac{\vec{\nabla}\vec{E}^A_{\rm el}}{\varrho}\right)^2 \\ &+ \frac{\eta^2}{2}\,\left(\varrho\,G_i\right)^2 + \frac{1}{4\,e^2\kappa^2}\,\left(\partial_{[i}\,G_{j]}\right)^2 + \mathcal{H}_\varrho\,, \end{aligned}$$

with the following non-vanishing Poisson brackets between the phase space variables

$$\left\{\left(E^A_{\rm el}\right)^i(t,\vec{x}\,), G_j(t,\vec{y}\,)\right\} = \kappa\,e^2\,\delta^i_j\,\delta^{(3)}(\vec{x}-\vec{y}\,)\,.$$

The present analysis is readily generalised to any number of spacetime dimensions.

4.2 Abelian Higgs models in any dimension

4.2.1 The dual Chern-Simons-Higgs theory in 2+1 dimensions

Throughout this Chapter we are constructing dual equivalences from topologically massive gauge theories of the BF-type with specific dielectric couplings. A query which naturally comes to mind is to wonder what happens in the case of topological mass generation theories of the AF-type, (1.25), with a dielectric coupling between the scalar field and the single p-form gauge field $A(x)$. Surprisingly the construction of this kind of dual equivalence has never been considered until now. In this Subsection, we address duality relations specifically in 2+1 dimensions, but the equivalences may be extended whatever the even number of space dimensions, provided that this number be such that $d = 2p$ with p odd.

Let us first consider the Maxwell-Chern-Simons theory (1.26) of which the gauge field $A_\mu(x)$ couples dielectrically to a dynamical scalar field $\varrho(x)$,

$$\mathcal{L}_{\mathrm{MCS}\varrho} = -\frac{1}{4\,\eta^2}\frac{1}{\varrho^2}F_{\mu\nu}\,F^{\mu\nu} + \frac{\kappa}{4}\epsilon^{\mu\nu\rho}\,F_{\mu\nu}\,A_\rho + \mathcal{L}_\varrho\,. \tag{4.28}$$

As a reminder, in this expression $\mathcal{L}_\varrho\,(\varrho, \partial_\mu\varrho)$ is the non coupled Lagrangian density for a self-interacting dynamical scalar field (4.2). We are going to establish by means of a field transformation that the above Lagrangian density shares a common first order physical formulation with the Chern-Simons-Higgs (CSH) theory. This latter theory is a particular type of an abelian Higgs model in 2+1 dimensions where the gauge field is governed by a Chern-Simons Lagrangian instead of the usual Maxwell Lagrangian (see [73] and references therein).

Let us start as usual from the first order formulation of (4.28),

$$\mathcal{L}^{\mathrm{master}}_{\mathrm{MCS}\varrho} = \frac{1}{2}\eta^2\,\varrho^2\,E_\mu\,E^\mu + \frac{1}{2}\epsilon^{\mu\nu\rho}\,F_{\mu\nu}\,E_\rho + \frac{\kappa}{4}\epsilon^{\mu\nu\rho}\,F_{\mu\nu}\,A_\rho + \mathcal{L}_\varrho\,,$$

obtained after the introduction of the Gaussian auxiliary field $E_\mu(x)$. Under the same local and linear transformation of the fields as in the pure Maxwell-Chern-Simons case, see (2.26), the above first order Lagrangian density is factorised, modulo a surface term ST, into a sector $\mathcal{L}_{\mathrm{dyn}}$ of physical variables and a topological sector $\mathcal{L}_{\mathrm{CS}}$ consisting of a pure Chern-Simons theory (2.27) where the entire original gauge symmetry content resides:

$$\mathcal{L}^{\mathrm{fac}}_{\mathrm{MCS}\varrho} = \mathcal{L}_{\mathrm{dyn}}\,(E_\mu, \partial_\mu E_\nu, \varrho, \partial_\mu\varrho) + \mathcal{L}_{\mathrm{CS}}\,(\mathcal{A}_\mu, \partial_\mu\mathcal{A}_\nu) + \mathrm{ST}\,.$$

This physical sector having been made manifest through our TP factorisation procedure turns out to be also the first order physical formulation of the Chern-Simons-

Higgs theory which reads

$$\mathcal{L}_{\rm dyn} = \frac{1}{2}\eta^2\, \varrho^2\, E_\mu\, E^\mu + \frac{1}{2\,\kappa}\epsilon^{\mu\nu\rho}\, \partial_\mu E_\nu\, E_\rho + \mathcal{L}_\varrho\,.$$

Hence the announced dual equivalence is thus established.

In comparison to duality relations established for the Maxwell-Higgs model, a further step may be taken in order to recover the usual formulation of the Chern-Simons-Higgs Lagrangian. This implies the restoration of the broken $U(1)$ symmetry in the physical sector, with its cortege of subtleties related to the gauge embedding procedures. Indeed, the physical variable $G_\mu(x)$ may be written again as the $U(1)$ current of a complex scalar field, see (4.14) and (4.15). Hence the physical sector turns into a $U(1)$ gauge (embedded) theory and reads

$$\mathcal{L}_{{\rm CSH}\varrho\theta} = \frac{1}{2}\,\kappa^2\,\eta^2\,\varrho^2\,\left(\tilde{A}_\mu - \partial_\mu\theta\right)^2 + \frac{\kappa}{2}\,\epsilon^{\mu\nu\rho}\,\partial_\mu\tilde{A}_\nu\,\tilde{A}_\rho + \mathcal{L}_\varrho\,.$$

Once the gauge symmetry is restored, the $U(1)$ gauge variant complex scalar field $\phi(x)$ is readily constructed in its polar parametrisation in terms of the Higgs field $\varrho(x)$ and the gauge variant field $\theta(x)$, see (4.16). Then the Chern-Simons-Higgs Lagrangian before symmetry breaking is recovered:

$$\mathcal{L}_{\rm CSH} = \frac{\kappa}{2}\,\epsilon^{\mu\nu\rho}\,\partial_\mu\tilde{A}_\nu\,\tilde{A}_\rho + \left|\tilde{\rm D}_\mu\phi\right|^2 - V\left(2\,|\phi|^2\right)\,,$$

where $\tilde{\rm D}^\mu$ denotes the usual covariant derivative (4.18) with the connection $\tilde{A}_\mu(x)$.

As usual in abelian Higgs models a Bogomol'nyi self-dual structure exists for soliton solutions, but within this model this type of structure arises from a specific sixth order potential for the scalar field:

$$V\left(2|\phi|^2\right) = \lambda\,|\phi|^2\,\left(|\phi|^2 - v^2\right)^2\,.$$

This potential possesses two degenerate vacua at $|\phi| = v$ and $|\phi| = 0$ but only the former breaks the $U(1)$ gauge symmetry. The main interest of this covariant model resides in its unusual soliton solutions of two distinguished types emerging from this sixth-order potential at the self-dual point and carrying both magnetic flux and electric charge. The first type, referred to as "topological soliton solution" [82, 83] interpolates from the origin, where the Higgs scalar field vanishes, to the circle at infinity where the Higgs field takes its non zero vacuum expectation value $|\phi| = v$. The second type, the so-called "non topological soliton solution" [84], is characterised by a Higgs field keeping its zero expectation value on the circle at infinity. However the study of these soliton solutions in the dual formulation and their possible generalisations is beyond the scope of this work and thus will not be addressed here.

4.2.2 The most general network of dualities in any dimension

The unbroken TP factorisation

The dual formulation of the Maxwell-Higgs model in 3+1 dimensions (4.1) pertains to a very specific form of topological mass generation for the 1-form gauge field $A(x)$ with a dielectric coupling between a real scalar field and the complementary $(d-1)$-form field $B(x)$. Otherwise a general formulation of the action of topologically massive dielectric gauge theories with couplings to dynamical real scalar fields, $S_{\text{TMG}\varrho\varpi}[A, B, \varpi, \varrho]$, may be written as

$$\begin{aligned} S^{d+1}_{\text{TMG}\varrho\varpi} &= \int_{\mathcal{M}} \frac{\sigma^p}{2}\frac{1}{e^2(\varpi)}\,F\wedge *F + \frac{\sigma^{d-p}}{2}\frac{1}{g^2(\varrho)}\,H\wedge *H \\ &+ S_{BF}[A,B] + \mathcal{S}_{\varpi}\left[\varpi\right] + \mathcal{S}_{\varrho}\left[\varrho\right]\ , \end{aligned} \tag{4.29}$$

where the scalar fields $\varpi(x)$ and $\varrho(x)$ couple through the dielectric functions $e\left(\varpi(x)\right)$ and $g\left(\varrho(x)\right)$ to the kinetic terms of the p-form $A(x)$ and the $(d-p)$-form $B(x)$, respectively. The usual topological coupling $S_{BF}[A, B]$ appears in the action (4.29),

$$S_{BF}[A,B] = \kappa\int_{\mathcal{M}} (1-\xi)\,F\wedge B - \sigma^p\,\xi\,A\wedge H\ . \tag{4.30}$$

In fact, this action is the most general construction of topologically massive gauge fields coupled to real scalar fields of which the decoupled part is minimal

$$\mathcal{S}_{\varrho}\left[\varrho\right] = \int_{\mathcal{M}} \frac{1}{2}\,\mathrm{d}\varrho\wedge *\mathrm{d}\varrho - *V\left(\varrho^2\right)\ ,$$

and which at the same time preserves our factorisation into decoupled physical and topological sectors. There is no need at this stage to specify the potentials for the two scalar fields, $\tilde{V}\left(\varpi^2\right)$ and $V\left(\varrho^2\right)$.

Despite the presence of dielectric scalar fields in the topological mass generation model (4.29), the dual factorised formulation is obtained in the same way as within the non coupled case, see Section 2.2, and thus first proceeds through the extension of the field content of the theory. Indeed the covariant first order formulation of the action (4.29) reads

$$\begin{aligned} S^{\text{master}}_{\text{TMG}\varrho\varpi} &= \frac{1}{2}\left(e(\varpi)\,E\right)^2 + \frac{1}{2}\left(g(\varrho)\,G\right)^2 + \int_{\mathcal{M}} F\wedge E + H\wedge G \\ &+ S_{BF}[A,B] + \mathcal{S}_{\varpi}\left[\varpi\right] + \mathcal{S}_{\varrho}\left[\varrho\right]\ , \end{aligned}$$

after the introduction of the auxiliary $(d-p)$- and p-form fields $E(x)$ and $G(x)$. Then through a reparametrisation similar to that of the non coupled case,

$$A = \mathcal{A} + \frac{1}{\kappa}\sigma^{p(d-p)}\,G\quad,\quad B = \mathcal{B} - \frac{1}{\kappa}E\,,$$

the action is factorised into a pure topological field theory of the BF-type, $S_{BF}[\mathcal{A},\mathcal{B}]$, and a dynamical sector S_{dyn} of the form

$$\begin{aligned} S_{\text{dyn}} &= \frac{1}{2}\left(e(\varpi)\,E\right)^2 + \frac{1}{2}\left(g(\varrho)\,G\right)^2 + \mathcal{S}_{\varpi}\left[\varpi\right] + \mathcal{S}_{\varrho}\left[\varrho\right] \\ &+ \frac{1}{\kappa}\int_{\mathcal{M}} \sigma^{d-p}\,\xi\,E\wedge \mathrm{d}G - (1-\xi)\,\mathrm{d}E\wedge G + \int_{\mathcal{M}} \mathrm{ST}\,, \end{aligned} \tag{4.31}$$

modulo a surface term, irrelevant for an appropriate choice of boundary conditions on $\mathcal{M}$. Hence a similar structure to that of Fig.4.1 in 3+1 dimension is obtained whatever the number of spacetime dimensions from the generalised action in (4.29).

The TP factorisation within the Hamiltonian formulation of topologically massive dielectric gauge theories may be cast in the same mould as its Lagrangian counterpart since the canonical transformation leading to the factorised fundamental Hamiltonian remains the same as in the non coupled case. Therefore all the discussion pursued in the particular case of the dielectric Cremmer-Scherk theory in Section 4.1 is readily extended to the general case in any dimension, from the Lagrangian density (4.29). Hence within the A-field picture, the scalar field $\varpi(x)$ acts like of a dielectric field while the scalar field $\varrho(x)$ plays the role of a "Higgs" field associated to generalised London equations for a p-form. Likewise, within the B-field picture, the scalar field $\varrho(x)$ acts like a dielectric field while the scalar field $\varpi(x)$ plays the role of a "Higgs" field associated to generalised London equations for a ($d-p$)-form.

Recovering the Maxwell-Higgs models

Finally, the range of dual formulations for Maxwell-Higgs models in any dimension reduces actually to very particular cases of topologically massive dielectric gauge theories subjected to the following restrictions:

- One of the two gauge fields is an 1-form field. Let us choose A to be this field.
- The dielectric function $e\left(\varrho(x)\right)$ reduces to a constant e.
- The dielectric function $g\left(\varrho(x)\right)$ has the simple form : $g(\varrho) = \eta\,\varrho$.

The following action results from these assumptions,

$$S_{\text{MH}} = \int_{\mathcal{M}} \frac{\sigma^p}{2e^2} F\wedge *F + \frac{\sigma^{d-p}}{2\,\eta^2}\,\frac{1}{\varrho^2} H\wedge *H + S_{B\wedge F}[A,B] + \mathcal{S}_{\varrho}\left[\varrho\right]\ , \tag{4.32}$$

from which the Maxwell-Higgs model (4.17) is recovered through our TP factorisation techniques and some gauge embedding procedure in the physical sector.

The $U(1)$ symmetry breaking in this range of Maxwell-Higgs models in any dimension is dictated by the complex scalar field $\phi(x)$ in (4.16) possessing a non vanishing expectation value in the vacuum. Otherwise, if $p \neq 1$, it is not possible to reconstruct a complex scalar field from the polar parametrisation (4.16), since the current $J(x)$ introduced in (4.14) is then no longer a 1-form.

4.2.3 The London limit and the dual Stueckelberg models

The "London limit" for TMDGT

In the Maxwell-Higgs model, the London limit is the limit in which the mass of the Higgs field becomes infinite while the mass of the gauge field remains finite, as is seen in Section 1.2. Likewise an equivalent limit may be readily identified for the topologically massive dielectric gauge theories (TMDGT) defined in (4.29), within the context of the duality relations displayed in Fig.4.1. However, as there is no need to specify explicitly the shape of the self-interacting scalar potentials, by "London limit" we shall refer to any asymptotic limit or relation between the diverse coupling constants which leads to the "freezing" of the scalar fields to their vacuum expectation values. In this sense, the action for pure topologically massive gauge theories (1.23) is recovered through the London limit,

$$e\,(\zeta\,\varpi)\;\rightarrow\;e\,(\zeta\,\langle\varpi\rangle)\equiv e\,,\qquad g\,(\eta\,\varrho)\;\rightarrow\;g\,(\eta\,\langle\varrho\rangle)\equiv g\,,$$

of the action (4.29) for TMDGT.

Equivalence between TMGT and Stueckelberg theories revisited

We have already seen in Section 1.2 that the Stueckelberg theory for a 1-form gauge field is recovered from the London limit of the Maxwell-Higgs model. This latter model shares a common physical sector with the Cremmer-Scherk theory as made manifest through our TP factorisation. In the same way one may expect that generalised Stueckelberg theories be obtained from the London limit of the physical sector of TMDGT, modulo a suitable gauge embedding procedure. In fact, several dualisation techniques have been used until now in order to establish the dual equivalence between topologically massive gauge theories and Stueckelberg theories. As far as gauge embedding procedures are concerned, this duality relation is considered as an intermediate step in order to establish the duality relation between theories of the Proca-type and TMGT. This type of methods, developed whether within the Hamiltonian [64] or the Lagrangian [66] formulation, are characterised by an intricate maze of successive gauge fixing and unfixing procedures. There also exist other techniques which establish such duality relations through a master Lagrangian approach, see [81]. In this latter case however the gauge fixing procedures are hidden behind the successive integrations of the equations of motion and thus no special care has been taken with regards to the treatment of the gauge symmetry content of these theories.

In contradistinction with the above-mentioned procedures of dualisation, our method offers several advantages and consists of two steps. First we have already established in Chapter 2, through our TP factorisation, the duality relation between pure TMGT of the BF-type (1.23) and a generalised first order formulation of Proca theories[3],

$$S_{\rm dyn} = \frac{e^2}{2}\,(E)^2 + \frac{g^2}{2}\,(G)^2 + \frac{1}{\kappa}\int_{\mathcal{M}} \sigma^{d-p}\,\xi\,E\wedge {\rm d}G - (1-\xi)\,{\rm d}E\wedge G\,, \quad (4.33)$$

parametrised by the constant ξ we have introduced, see Section 2.2. The two theories are dual modulo a topological BF term in which all the gauge content resides. This procedure is free of gauge fixing and consistent for what concerns the counting of the numbers of degrees of freedom.

Second it is only at this stage that gauge embedding procedures apply in order to make manifest the duality between the Proca theories obtained in the physical sector and the Stueckelberg theories. Hence according to our procedure, the Proca theories are now an intermediate step in establishing the duality relation. We have briefly introduced this type of generic procedure consisting in the extension of the gauge content in Section 4.1. Indeed, knowing that ${\rm d}G=0$ and ${\rm d}E=0$ and following the example of (4.14), these two gauge invariant p and $(d-p)$-form fields may be written as

$$G = \sigma^{p(d-p)}\,\kappa\left(\tilde{A}-\theta\right), \quad (4.34)$$

$$E = -\kappa\left(\tilde{B}-\chi\right), \quad (4.35)$$

where $\theta(x)$ is a closed p-form while $\chi(x)$ is a closed $(d-p)$-form. Under the assumptions we have made which do not allow for the presence of topological effects, $\theta(x)$ and $\chi(x)$ are also exact[4]. In this case, the gauge embedding procedure is thus well-defined. The transformations of the fields $\theta(x)$ and $\chi(x)$ under the gauge symmetries,

$$\theta' = \theta + \tilde{\alpha}\,, \qquad \chi' = \chi + \tilde{\beta}\,,$$

compensates for that of the two independent classes of finite abelian gauge transformations acting separately in either the $\tilde{A}$- or $\tilde{B}$-sector,

$$\tilde{A}' = \tilde{A} + \tilde{\alpha}\,, \quad (4.36)$$

$$\tilde{B}' = \tilde{B} + \tilde{\beta}\,, \quad (4.37)$$

[3] Namely the non gauge invariant "self-dual" action of [65, 67] generalised to any dimension.

[4] Indeed, we have assumed a topologically trivial spacetime manifold and non vanishing scalar fields. These assumptions do not allow $\theta(x)$ and $\chi(x)$ to have a global component, see [48, 79] in 3+1 dimensions.

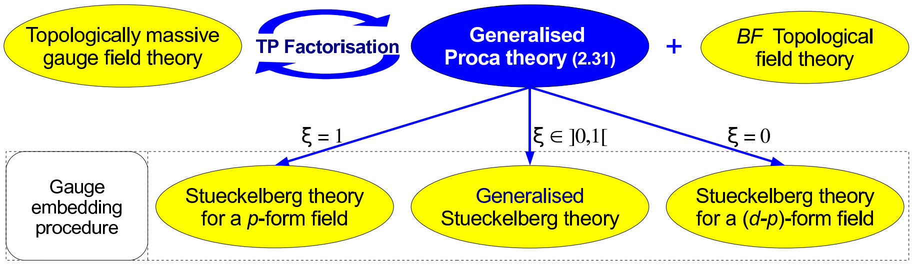

Figure 4.2: *Duality relation between TMGT (1.23) and a generalised formulation of the Stueckelberg theories (4.40) obtained through TP factorisation and gauge embedding techniques. The usual formulation of the Stueckelberg mechanism for a p-form field (4.38) and a $(d-p)$-form field (4.39) is recovered through Gaussian integration for specific values of ξ. In blue: our contribution.*

in order to preserve the gauge invariance of $G(x)$ and $E(x)$. We have used the same notation as in (1.19), where $\tilde{\alpha}(x)$ and $\tilde{\beta}(x)$ are two exact p- and $(d-p)$-forms, respectively. In this sense the physical variable $G(x)$ may be considered as the gauge invariant transverse part of the gauge field variable $\tilde{A}(x)$ while $\theta(x)$ is associated to its longitudinal part. A likewise identification applies for the $(d-p)$-form field $B(x)$.

The parametrised dual formulation of Stueckelberg theories

It is worthwile to note that the dual gauge embedded Stueckelberg theory constructed from the physical sector of the factorised TMGT depends dramatically on the value of our parameter ξ. Let us first consider the two extreme values $\xi = 0$ and $\xi = 1$, starting from the factorised Lagrangian density (4.33). Indeed, by setting $\xi = 1$, integrating out the then Gaussian auxiliary $(d-p)$-form field $E(x)$ and extending the gauge content of the theory at the level of the p-form field $G(x)$ through the transformation (4.34), one derives within the A-field picture the Stueckelberg action of a p-form field $\tilde{A}(x)$,

$$S_{\rm dyn} = \frac{\sigma^p}{2\,e^2} \int_{\mathcal{M}} \tilde{F} \wedge *\tilde{F} + \frac{\mu^2}{2\,e^2} \left(\tilde{A} - \theta\right)^2 , \qquad (4.38)$$

where $\tilde{F} = \mathrm{d}\tilde{A}$ and $\mu = \kappa\, e\, g$. Alternatively one may also obtain within the B-field picture the Stueckelberg action of a $(d-p)$-form field by fixing $\xi = 0$, eliminating the Gaussian p-form field $G(x)$ and applying the gauge embedding transformation (4.35),

$$S_{\rm dyn} = \frac{\sigma^{d-p}}{2\,g^2} \int_{\mathcal{M}} \tilde{H} \wedge *\tilde{H} + \frac{\mu^2}{2\,g^2} \left(\tilde{B} - \chi\right)^2 , \qquad (4.39)$$

where $\tilde{H} = \mathrm{d}\tilde{B}$. These two actions above are invariant under one abelian gauge symmetry only, of which the associated transformation reads as in (4.36) or (4.37) whether one considers the transformation acting on the p-form field $A(x)$ in (4.38) or on the $(d-p)$-form field $B(x)$ in (4.39), respectively.

Finally a generalised formulation of the Stueckelberg theory may be obtained starting from the first order formulation of the Proca theory and applying the gauge embedding procedure to the two gauge invariant fields $G(x)$ and $E(x)$, see (4.35) and (4.34),

$$\begin{aligned} S_{\mathrm{dyn}} &= \frac{e^2}{2}\kappa^2\left(\tilde{B}-\chi\right)^2 + \frac{g^2}{2}\kappa^2\left(\tilde{A}-\theta\right)^2 \\ &+ \kappa\int_{\mathcal{M}} \sigma^{d-p}\,(1-\xi)\,\tilde{A}\wedge \mathrm{d}\tilde{B} - \xi\,\mathrm{d}\tilde{A}\wedge\tilde{B}\,. \end{aligned} \tag{4.40}$$

Our parameter ξ, which was originally introduced in order to parametrise the possible surface terms, is thus not irrelevant since it determines to which (gauge embedded) Stueckelberg theory the topologically massive gauge theories are dual[5], as illustrated in Fig.4.2. Contrary to (4.38) and (4.39), this generalised formulation of the abelian Stueckelberg theory possesses two independent classes of finite abelian gauge transformations acting separately in either the $\tilde{A}$- or the $\tilde{B}$-sector, see (4.36) and (4.37).

4.3 Conclusion and schematic overview

In 3+1 dimensions, a real scalar field freely propagates like a 2-tensor field since the exterior derivative of a 0-form is Hodge dual to that of a 2-form. In a more general context, this equivalence holds in $(d+1)$ spacetime dimensions between a real scalar field and a $(d-1)$-tensor field. This explains why the $(d-1)$-tensor field in topological mass generation mechanisms is often considered as playing the same role as the argument of the phase of the complex scalar field in the Maxwell-Higgs model. Indeed these fields generate a longitudinal part to the 1-form gauge field and so make it massive. However, again in $3+1$ dimensions the exterior derivative of the 2-form field may not be directly replaced by a real scalar field. In fact this assumption made until now in the literature causes some troubles because the total number of degrees of freedom, including the pure gauge ones, of a 2-form field does not match with those of the scalar field. In the same way some gauge symmetry content is lost since gauge transformations acting on the 2-form field do not correspond to those acting on the argument of the complex scalar field. Actually these pure gauge degrees of freedom are of prime importance as soon as topological effects appear. Furthermore this procedure

[5] The analysis as developed so far in the literature considered the duality relations between TMGT and the different types of Stueckelberg theories as distinct cases [80, 81].

is non generalisable to any topologically massive (dielectric) gauge theory for p-form fields, with $p > 1$.

Our approach is completely different. We consider the $(d-p)$-form field as a current for the p-form field and conversely, the p-form field as current for the $(d-p)$-form field. According to our terminology, the first case corresponds to the A-field picture while the second case is associated to the B-field picture. In this sense, this procedure is close in spirit to gauge embedding procedures. We have already obtained in Chapter 2 the dual equivalence between topologically massive gauge theories (TMGT) and a generalised first order formulation of Proca theories. The extension of the gauge symmetry content enabled us to obtain in this Chapter a novel intricate network of dualities between the most common (local) mechanisms generating a mass gap in abelian gauge field theories, as illustrated in Fig.4.3. Namely, a dual equivalence between the Maxwell-Higgs model, (4.32), and topologically massive dielectric gauge theories (TMDGT), (4.29), in the particular case $p = 1$. This duality relation extends in the London limit between the pure TMGT (1.23) and a generalised formulation of Stueckelberg theories (4.40), where extreme care has been taken with the gauge symmetry content. Indeed we have also established that all these equivalences are true modulo a topological BF term and, as usual, a specific type of gauge embedding procedure. Although they share the same local formulation in terms of physical fields, the gauge symmetry content of these mass generation mechanisms differs dramatically. It implies that these theories share a common dynamics but are globally distinct as soon as non trivial topological effects appear.

We think that the correct procedure is to avoid to establish duality relations between gauge theories at any price and to give up in this case the gauge embedding procedures. We may perfectly admit that two theories are locally dual in their physical sector of dynamical variables but are not dual if we consider their entire gauge symmetry content. In this sense, TMDGT offer the great advantage that the topological sector of gauge variant variables is isolated from the physical sector through our TP factorisation, independently of any gauge fixing procedure. Such a decoupled topological sector does not arise in the Maxwell-Higgs model which possesses one type only of abelian gauge invariance. Furthermore, this topological term was until now swamped by a mass of successive procedures of gauge embeddings and/or gauge fixings characteristic of the dualisation techniques previously introduced in the literature. This topological BF term, which seems to be a simple spectator term when constructing duality relations between mass generation mechanisms, turns out to play a pivotal role as soon as topological effects are taken into account, as will be discussed in Chapters 5 and 6.

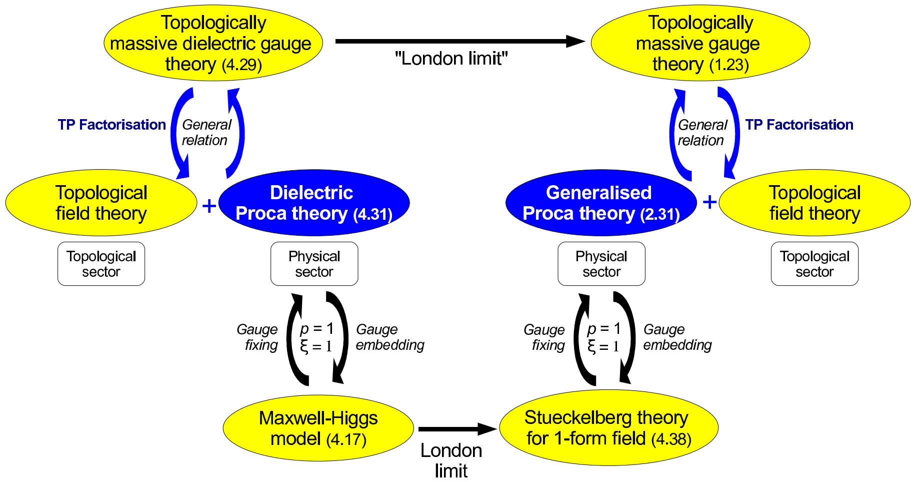

Figure 4.3: *Network of (local) duality relations between mass generation mechanisms in abelian gauge field theories. Gaussian integrations of auxiliary fields are not mentioned. In blue: our contributions.*

The duality network established in this Chapter holds for pure abelian gauge field theories. The understanding of the fate of these dualities when couplings to fermions are introduced is certainly of interest but is beyond the scope of this Thesis. Another case of study may be the dual formulation of the Maxwell-Higgs model on topologically non trivial manifolds or manifolds with boundaries, as is the case in superconductivity. Again the topological BF term will generate non trivial topological effects in these cases. Let us conclude this Chapter by noticing that there exists an alternative non local formulation of the Maxwell-Higgs model, see [48, 49], where a topological BF term is involved in the construction of the Lagrangian. Although this formulation is less usual in the literature it offers the advantage to have the same gauge symmetry structure as the dielectric Cremmer-Scherk (DCS) theory. In fact we have proved elsewhere [85] that this non local formulation is totally equivalent to the local DCS theory, even in the presence of topological defect solutions.

CHAPTER 5

Magnetic vortices and (di)electric monopoles in the plane

The network of dualities in Fig.4.3 has a local character and only holds for a scalar field $\varrho(x)$ nowhere vanishing on the spacetime manifold. However if we relax this assumption, global aspects ought to be taken into account. Hence to any submanifold of zero measure of zeros in $\varrho(x)$ is there associated a topological defect. In particular for TMDGT in $2{+}1$ dimensions introduced in the present Chapter, we have managed to construct novel magnetic vortex-type topological defects analogous to the Nielsen-Olesen vortices arising in the Maxwell-Higgs model. Likewise, these topological defects may be seen within a physically equivalent picture as electric monopoles distributed within a dielectric medium. This type of topological defects in gauge field theories, never considered until now, makes the search for soliton solutions in topological mass generation models of prime interest.

Our TP factorisation enables one to isolate the physical part, in this case common with that of the Maxwell-Higgs model, from the topological part, including global boundary conditions specific to the topological defect solution while conserving the gauge content of the original theory. In fact the topological part was up to now always missing in the dualisation processes of the Maxwell-Higgs model due to gauge fixings characteristic of such procedures. Perhaps this is the reason why our new topological defect solutions have so far never been considered. Starting from scratch, the procedure leading to the construction of topological defects will be described in detail while at the same time highlighting the analogies with the Maxwell-Higgs model.

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5.1 A topologically massive dielectric theory in the plane

We have established in Chapter 4 that the dielectric Cremmer-Scherk theory (4.1) shares, within the A-field picture, a dynamical sector of physical variables common with that of the Maxwell-Higgs theory. In the same way, one may naturally expect such a local duality relation in 2+1 dimensions between the Maxwell-Higgs model and a topologically massive dielectric gauge theory specific to that number of spacetime dimensions of the form

$$\begin{aligned}\mathcal{L}^3_{\mathrm{TMG}\varrho} &= -\frac{1}{4\,e^2}F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{4\,\eta^2}\frac{1}{\varrho^2}H_{\mu\nu}\,H^{\mu\nu} + \mathcal{L}_\varrho \\ &+ \frac{\kappa}{2}\,\epsilon^{\mu\nu\rho}\left(\xi\,A_\mu\,H_{\nu\rho} + (1-\xi)\,F_{\mu\nu}\,B_\rho\right)\,. \end{aligned} \tag{5.1}$$

As usual, $\mathcal{L}_\varrho(\varrho, \partial_\mu\varrho)$ is the decoupled Lagrangian density for the self-interacting scalar field $\varrho(x)$ defined in (4.2). Unfortunately, a factorisation process analogous to that which leads to the double Maxwell-Chern-Simons theory [36, 75] in the free case is no longer feasible due to this specific coupling to the scalar field. However, under the assumptions[1] made in Chapter 4, it is still possible to isolate the dynamical sector from the topological one since our TP factorisation technique is applicable, as for any topologically massive dielectric gauge theory of the general form (4.29).

In this Section, we relax one of these restrictions in allowing the scalar field $\varrho(x)$ in the Lagrangian density (5.1) to possess zeros on some spacetime events and analyse the implications of such an assumption. A brief glance at this Lagrangian density already suggests that something has to happen when the scalar field $\varrho(x)$ vanishes on a subset of the spacetime manifold, namely that the kinetic term for the gauge field $B_\mu(x)$ should then cancel accordingly in order to keep the energy functional finite. As a matter of fact we prove in Section 5.3 that corresponding to any submanifold of zeros in $\varrho(x)$ there is associated a topological defect, of which the characteristic topological invariant is intimately related to the orders of these zeros. A natural query which comes to mind is the validity of our TP factorisation when such topologically non trivial soliton solutions arise. The answer may be obtained with the help of the equations of motion which offer the hints towards the convenient parametrisation which takes into account the possible existence of such topological effects.

5.1.1 Equations of motion and physical variables

The action (5.1) may be written in its covariant first order form after the introduction of the auxiliary fields $G_\mu(x)$ and $E_\mu(x)$ so that physical variables are already manifest

[1] Indeed, we have assumed a topologically trivial spacetime manifold and non vanishing scalar fields.

in the Euler-Lagrange equations of motion :

$$\begin{aligned}\mathcal{L}^m_{\mathrm{TMG}\varrho} &= \frac{\eta^2}{2}\varrho^2\, G_\mu\, G^\mu + \frac{1}{2}\epsilon^{\mu\nu\rho}\, H_{\mu\nu}\, G_\rho + \frac{e^2}{2} E_\mu\, E^\mu + \frac{1}{2}\epsilon^{\mu\nu\rho}\, F_{\mu\nu}\, E_\rho \\ &+ \frac{\kappa}{2}\,\epsilon^{\mu\nu\rho}\,(\xi\, A_\mu\, H_{\nu\rho} + (1-\xi)\, F_{\mu\nu}\, B_\rho) + \mathcal{L}_\varrho\,. \end{aligned} \tag{5.2}$$

This is in general the first step in establishing duality relations through our TP factorisation technique within the Lagrangian formulation. The equations obtained from the variation of $G_\mu(x)$ and $E_\mu(x)$ relate these physical variables to their gauge variant counterparts, $B_\mu(x)$ and $A_\mu(x)$ respectively, through the Gaussian integration:

$$\begin{aligned} 2\,\eta^2\,\varrho^2\,\eta^{\rho\mu}\,G_\mu &= -\epsilon^{\rho\mu\nu}\,H_{\mu\nu}\,, \\ 2\,e^2\,\eta^{\rho\mu}\,E_\mu &= -\epsilon^{\rho\mu\nu}\,F_{\mu\nu}\,. \end{aligned} \tag{5.3}$$

It is on this ground that the field strentgh tensor $F_{\mu\nu}(x)$ has been interpreted in Section 4.1 as a current for the dynamical gauge field $B_\mu(x)$ within the B-field picture and conversely within the A-field picture, see (4.3) and (4.6) in 3+1 dimensions. The equations resulting from the variation of $B_\mu(x)$ and $A_\mu(x)$ are pure divergences:

$$\begin{aligned} \epsilon^{\rho\mu\nu}\,\partial_\mu\,(G_\nu + \kappa\,A_\nu) &= 0\,, \\ \epsilon^{\rho\mu\nu}\,\partial_\mu\,(E_\nu + \kappa\,B_\nu) &= 0\,, \end{aligned} \tag{5.4}$$

and correspond to the Gauss laws associated to each of these gauge fields. Finally, the equation describing the dynamics of the scalar field $\varrho(x)$ reads

$$\Box\varrho = \eta^2\,\varrho\,G_\mu\,G^\mu - \tilde{\mu}^2\,\varrho - \lambda\,\varrho^3\,, \tag{5.5}$$

where $\Box$ denotes the spacetime d'Alembertian, $\Box \equiv \eta^{\mu\nu}\,\partial_\mu\partial_\nu$.

5.1.2 The dynamical sector and local variables

These equations of motion may be expressed in terms of physical fields alone from (5.3) and (5.4), independently of the gauge fields $A_\mu(x)$ and $B_\mu(x)$,

$$\begin{aligned} \kappa\,\eta^2\,\varrho^2\,\eta^{\rho\mu}\,G_\mu &= \epsilon^{\rho\mu\nu}\,\partial_\mu E_\nu\,, \\ \kappa\,e^2\,\eta^{\rho\mu}\,E_\mu &= \epsilon^{\rho\mu\nu}\,\partial_\mu G_\nu\,. \end{aligned} \tag{5.6}$$

As a matter of fact, one of the possible Lagrangian densities which directly gives in return the equations of motion (5.5) and (5.6) expressed in terms of the physical variables $E_\mu(x)$, $G_\mu(x)$ and $\varrho(x)$ is precisely the dynamical part,

$$\begin{aligned}\mathcal{L}^3_{\rm dyn} &= \frac{e^2}{2}\,E_\mu\,E^\mu + \frac{\eta^2}{2}\,\varrho^2\,G_\mu\,G^\mu + \mathcal{L}_\varrho \\ &- \frac{1}{\kappa}\epsilon^{\mu\nu\rho}\left(\xi\,\partial_\mu G_\nu\,E_\rho - (1-\xi)\,\partial_\mu E_\nu\,G_\rho\right)\,,\end{aligned} \tag{5.7}$$

of the dual Lagrangian density resulting from our TP factorisation. This correspondence established locally remains globally true. It will be shown hereafter that the dynamical sector is in fact not affected by the possible existence of a subset of space-time where the scalar field $\varrho(x)$ vanishes, or in other words, by the presence of the non trivial topological content associated to topological defect solutions.

The set of first order equations (5.6) may be locally reorganised into a set of partially decoupled second order equations related by the scalar field $\varrho(x)$ only:

$$\begin{aligned}\eta^{\nu\rho}\,\partial_\nu\left(\partial_\mu E_\rho - \partial_\rho E_\mu\right) &= \eta^{\nu\rho}\,\partial_\nu \ln\frac{\varrho^2}{v^2}\,\left(\partial_\mu E_\rho - \partial_\rho E_\mu\right) + 2\,\mu^2\frac{\varrho^2}{v^2}E_\mu \\ \eta^{\nu\rho}\,\partial_\nu\left(\partial_\mu G_\rho - \partial_\rho G_\mu\right) &= 2\,\mu^2\,\frac{\varrho^2}{v^2}G_\mu\,,\end{aligned} \tag{5.8}$$

to which the equation (5.5) must be added in order to have a complete set of local equations of motion. Let us recall that $v = \langle\varrho\rangle$ refers to the vacuum expectation value of the scalar field which will be assumed to be non vanishing[2]. Hence from the different available coupling constants and parameters we have also defined

$$\mu = \kappa\,e\,\eta\,v\,, \tag{5.9}$$

which is associated to the mass gap generated in this type of TMDGT according to the above equations. In fact, the second of the equations (5.8) is associated within the A-field picture to the Lagrangian density resulting from (5.7) where $\xi = 1$, after the integration of the then Gaussian auxiliary field $E_\mu(x)$. Alternatively one may also obtain the first of the equations (5.8) from (5.7) by setting $\xi = 0$, namely working within the B-field picture, and eliminating the Gaussian field $G_\mu(x)$. Notice that the Gaussian integration of local variables does not alter the topological (global) content.

At least locally, the topological part of the TP factorised Lagrangian density is decoupled from the physical part and corresponds to a pure topological field theory of the BF type. Indeed, it is worthy to recall here that the correspondence between the transverse part of $A_\mu(x)$ and the physical variable $G_\mu(x)$, and likewise for $B_\mu(x)$ and $E_\mu(x)$, is required as a first-class constraint resulting from the local equations of motion in the topological sector. However it is natural to wonder whether our factorisation survives in the presence of non trivial topological effects.

[2]This condition has to be required in order to construct topological defect solutions.

5.1.3 The topological sector and global variables

The set of equations (5.5) and (5.6) is only locally valid and so does not encode important topological information. As we shall see later, any topological defect associated to a given worldline Γ on which the scalar field $\varrho(x)$ vanishes in the spacetime of dimension three, defines a connection on the manifold $\mathbb{R}^3 \setminus \Gamma$ which has the same homotopy as $S^1 \times \mathbb{R}$. On this homotopically non trivial manifold, closed forms like the curvature $F = \mathrm{d}A$ are not necessarily exact. Hence, Gauss' laws (5.4) may also be integrated on a two dimensional surface $\mathcal{S}$ in the spacetime manifold with $\mathcal{C}$ as a boundary. But rather than requiring strict equality of the harmonic components of $F_{\mu\nu}$ and $\partial_{[\mu}\, G_{\,\nu]}$, a weaker relation between complex phases resulting from the exponentiation of (5.4) will be introduced. Using Stokes' theorem, this reads

$$\exp\left(\mathrm{i}\,\Omega\,\oint_{\mathcal{C}}(\kappa\, A_\mu + G_\mu)\right) = 1\,. \tag{5.10}$$

A priori, the new constant Ω of which the physical dimensions must be $E^{-1}\,L^{-1}$ may be equal to any linear combination of the available constants η and $1/\kappa$. Within the context of the Maxwell-Higgs model, obtained after the restoration of the broken $U(1)$ symmetry, (5.10) is very reminiscent of compact abelian gauge symmetries defined in terms of uni-valued pure imaginary exponential phase factors. Therefore, given the polar formulation of the complex $U(1)$ scalar field in (4.16) and the definition of the associated current, one should fix $\Omega = \eta$. However without any reference to a symmetry breaking mechanism as should be the case for the construction of topological defect solutions of TMDGT, the value of Ω will be fixed later through the study of the behaviour of the fields close to zeros of $\varrho(x)$.

At this stage, a new physical variable may be introduced, namely the flux associated to the gauge field $A_\mu(x)$ through the set of all compact surfaces in spacetime

$$\Phi[\mathcal{C}] = \int_{\mathcal{S}} F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \qquad \mathcal{C} = \partial\mathcal{S}\,.$$

Thus the above relation (5.10) reduces to an equality between holonomies while the physical interpretation of the flux $\Phi[\mathcal{C}]$ depends on the contour. For example, in the static case for a given inertial frame, this Lorentz invariant quantity measures either the total magnetic flux through a purely spacelike contour or the electrostatic potential difference between two spacetime events of which space coordinates are fixed. By virtue of Stokes' theorem, the flux reads

$$\Phi\,[\mathcal{C}] = \oint_{\mathcal{C}} A_\mu\,\mathrm{d}u^\mu\,.$$

Consequently, (5.10) takes the form

$$\Omega\,\kappa\,\Phi\,[\mathcal{C}] = 2\,\pi\,L\,[\mathcal{C}] - \Omega\,\oint_{\mathcal{C}} G_\mu\,\mathrm{d}u^\mu\,, \tag{5.11}$$

where $L\,[\mathcal{C}]$ is an integer-valued functional the dependence of which on the contour $\mathcal{C}$ has yet to be specified.

The fact that the global equation (5.11) is not expressed in terms of the gauge invariant field $G_\mu(x)$ alone is to be related to the fact that the holonomy of this physical variable does not provide a coherent description of the global degrees of freedom of the system. Without wishing to launch into the analysis which will be developed throughout the next Sections, let us briefly comment this fact. In (5.11) the holonomy of $A_\mu(x)$ around the set of non contractible loops has also been introduced, namely the flux $\Phi[\mathcal{C}]$ through the surface $S = \partial\mathcal{C}$ including zeros of the scalar field. Along with the holonomy of $G_\mu(x)$, this flux is also considered as a physical variable because it is defined on the space of field configurations in an unique way. However neither the former nor the latter are coherent global physical variables on $\mathbb{R}^3 \setminus \Gamma$ since they are not topological invariant. In fact, the actual global physical variable is their difference for every contour $\mathcal{C}$, namely the integer-valued functional $L\,[\mathcal{C}]$ introduced in (5.11) and which will be referred to henceforth as "vorticity". Hence the knowledge of $L\,[\mathcal{C}]$ for all possible closed contours in spacetime encodes all the information about the topological structure for a given field configuration, namely the position and the degeneracy of each topological defect core, and its dynamics throughout the spacetime of dimension three.

Finally, likewise definitions apply to the other vector gauge field $B_\mu(x)$ and the associated flux and vorticity are referred to as $\tilde{\Phi}\,[\mathcal{C}]$ and $\tilde{L}\,[\mathcal{C}]$. However it will be seen that the physical variable $E_\mu(x)$ is not affected by the existence of events in spacetime where the scalar field $\varrho(x)$ vanishes. Moreover the most general formulation of TMDGT introduced in (4.29) considers the dielectric coupling between the kinetic term of the vector gauge field $A_\mu(x)$ and a scalar field $\varpi(x)$. In the present situation, since this function $\varpi(x)$ keeps its non vanishing vacuum expectation value throughout spacetime, the vorticity $\tilde{L}[\mathcal{C}]$ is identically zero whatever the contour $\mathcal{C}$ in spacetime.

5.1.4 The Topological-Physical factorisation

As seen in Section 1.2 a local formulation of the Maxwell-Higgs model in terms of physical variables is pathological as soon as non trivial topological effects associated to topological defect solutions are present. Indeed in this case, global boundary conditions accounting for the singular phase of the complex scalar field must be explicitly specified in the Lagrangian density (see for example [79, 45, 46]). As far as TMDGT are concerned the Lagrangian or Hamiltonian density may again be split into a dynamical part of physical variables and a topological part where the gauge symmetry content resides through a local and linear reparametrisation, without specifying any global boundary condition. However our TP factorisation is in this case no longer complete since the two sectors are only partially decoupled.

As already stated before, the TMDGT introduced in (5.1) and defined in 2+1 dimensions is recovered from the general formulation (4.29) by freezing the scalar field ϖ to its vacuum expectation value. Hence no topological content is lost when introducing the decoupled gauge variant variable $\mathcal{B}_\mu$ given by

$$\mathcal{B}_\mu = \frac{1}{\kappa}\, E_\mu + B_\mu\,,$$

since it will be shown hereafter that the fluxes of $B_\mu(x)$ and of $E_\mu(x)$ are identically zero in that case. Then, the equations of motion set the transverse part of $\mathcal{B}_\mu(x)$ to zero while its longitudinal part is pure gauge. Moreover the field $\mathcal{B}_\mu(x)$ does not include any singular part giving rise to a non trivial flux.

Indeed within the factorised formulation of TMDGT introduced in Chapter 4, the decoupled term describing a pure topological field theory implies that on topologically trivial manifolds, the gauge fields have neither local nor global parts when solving the classical equations of motion. However this kind of topological term is sensitive to topological effects which are associated, for example, to topological defect solutions. In fact the physical variable $G_\mu(x)$ inherits a singular global component from the local behaviour of the scalar field $\varrho(x)$ close to its zeros, making the local and global degrees of freedom to mix and become interdependent. Therefore, the sum $\kappa\, A_\mu + G_\mu$ may not be associated to a new variable which decouples totally from the system in the topological sector. Hence the factorised action does not consist in two independent sectors,

$$\mathcal{S}^3_{\rm fac}[E,G,A,\mathcal{B},\varrho] = S_{\rm dyn}[E,G,\varrho] + S_{\rm top}[A,G,\mathcal{B}] + \int_{\mathcal{M}} {\rm ST}\,,$$

but is again split into a dynamical part and a topological part, modulo a total divergence. The factorised Lagrangian density,

$$\begin{aligned}\mathcal{L}^3_{\rm fac} \;&=\; \frac{e^2}{2}\, E_\mu\, E^\mu + \frac{\eta^2}{2}\, \varrho^2\, G_\mu\, G^\mu + \mathcal{L}_\varrho\,(\varrho, \partial_\mu \varrho) \\ &-\; \frac{1}{\kappa} \epsilon^{\mu\nu\rho}\,(\xi\, \partial_\mu G_\nu\, E_\rho - (1-\xi)\, \partial_\mu E_\nu\, G_\rho) + {\rm ST} \\ &+\; (1-\xi)\, \varepsilon^{\mu\nu\rho}\, \partial_\mu\, (\kappa\, A_\nu + G_\nu)\; \mathcal{B}_\rho + \xi\, \varepsilon^{\mu\nu\rho}\; (\kappa\, A_\mu + G_\mu)\; \partial_\nu \mathcal{B}_\rho\,,\end{aligned} \tag{5.12}$$

then gives in return the same partially decoupled (and totally decoupled locally) equations of motion in terms of physical variables as those we obtained in the beginning of this Chapter: (5.5), (5.6) and (5.11). The dynamical part consisting in the two first lines of (5.12) is not affected by the presence of topological content associated, for example, to topological defect solutions. However such topological effects dramatically modify the formulation of the topological sector. Hence the role of the topological terms in the third line is to implement the global equation (5.11). The global part of the fields of singular origin then comes to the fore through the study of vorticity.

5.2 Two equivalent types of topological defects

5.2.1 *Ansatz* for static field configurations

We are presently interested in the emergence of classical soliton solutions from equations of motion (5.5), (5.6) and (5.11). Henceforth we shall thus restrict our analysis to time independent field configurations making the action stationary. These static field configurations obey two sets of partially decoupled first order equations for a given spacelike sheet. On the one hand the equations relating $G_0(x)$ and $E_i(x)$:

$$\begin{aligned}\epsilon^{ij}\,\partial_i E_j &= \kappa\,\eta^2\,\varrho^2\,G_0\,,\\ \epsilon^{ij}\,\partial_j G_0 &= -\kappa\,e^2\,\delta^{ij}\,E_j\,,\end{aligned}$$

on the other hand the equations relating $E_0(x)$ and $G_i(x)$:

$$\begin{aligned}\epsilon^{ij}\,\partial_j E_0 &= -\kappa\,\eta^2\,\varrho^2\,\delta^{ij}\,G_j\,,\\ \epsilon^{ij}\,\partial_i G_j &= \kappa\,e^2\,E_0\,.\end{aligned}$$

These two sets of equations are coupled through the dynamics of the scalar field $\varrho(x)$:

$$\partial_i\partial^i\varrho = -\eta^2\,(G_0)^2\,\varrho + \eta^2\,(G_i)^2\,\varrho + \tilde{\mu}^2\varrho + \lambda\,\varrho^3\,.$$

Finally, the global equation (5.11) renders complete the set of equations of motion. According to the context, these equations may describe two equivalent types of soliton solutions, whether interpreted within the A- or the B-field picture.

In order to make the set of equations simpler, it is useful at this stage to introduce a further *ansatz* in addition to time independence of the fields. This means that the ensemble of solutions is restricted to a subset possessing given symmetries and properties. Let us recall that in our case, the *ansätze* are not associated to some gauge fixing since topological defect solutions are constructed within the physical sector. Rather the possible *ansätze* are related to the presence of electric or magnetic charges or flux associated to topological defects. A first naive assumption would be to set $G_i = 0$. However, a brief look at the behaviour of the field $G_0(x)$ close to the vortex location readily leads to the conclusion that this physical variable is either equal to zero or a pure imaginary number. Obviously this latter case makes no sense. The *ansatz* which will be introduced here is $G_0 = 0$. The local equations of motion under this extra *ansatz* then reduce to

$$\begin{aligned} \epsilon^{ij}\,\partial_j E_0 &= -\kappa\,\eta^2\,\varrho^2\,\delta^{ij}\,G_j \\ \epsilon^{ij}\,\partial_i G_j &= \kappa\,e^2\,E_0 \\ \partial_i\partial^i\varrho &= \eta^2\,(G_i)^2\,\varrho + \tilde{\mu}^2\varrho + \lambda\,\varrho^3\,. \end{aligned} \tag{5.13}$$

Notice that the case where no extra *ansatz* is introduced, namely $G_i \neq 0$ and $G_0 \neq 0$, will not be addressed here. If solutions exist, they should correspond to a kind of dyon solution within the B-field picture with the magnitude of the magnetic field bounded by that of the electric field.

Finally the local equations of motion must be completed by the global one which reads under the *ansatz* just introduced:

$$\Omega\,\kappa\,\Phi\,[\mathcal{C}] = 2\,\pi\,L\,[\mathcal{C}] - \Omega\,\oint_{\mathcal{C}} G_i\,\mathrm{d}l^i\,, \tag{5.14}$$

where the ensemble of possible contours is restricted to spacelike ones. This last global equation accounts for the non trivial topological content characteristic of the soliton solutions. Before constructing these topological defect solutions and exploring insights that this novel avenue offers, we will presently give the physical meaning of each of the A- and B-field pictures.

5.2.2 *A*-field picture: Covariant description of superconductivity

Within the A-field picture, namely for $\xi = 1$, the partially factorised Lagrangian density (5.12) is of the form

$$\begin{aligned} \mathcal{L}^3_{\mathrm{fac}} &= \frac{e^2}{2} E_\mu\,E^\mu + \frac{\eta^2}{2}\,\varrho^2\,G_\mu\,G^\mu - \frac{1}{\kappa}\epsilon^{\mu\nu\rho}\partial_\mu G_\nu\,E_\rho + \mathcal{L}_\varrho\,(\varrho, \partial_\mu\varrho) \\ &+ \frac{1}{\kappa}\,\varepsilon^{\mu\nu\rho}\,(\kappa\,A_\mu + G_\mu)\,\partial_\nu\mathcal{B}_\rho + \mathrm{ST}\,. \end{aligned} \tag{5.15}$$

As already stated in Section 4.1 the dynamical part of the factorised formulation which consists in the first line of (5.15) is very reminiscent of the first order physical formulation of the Maxwell-Higgs model. The extra topological term in the second line of (5.15) points out that the scalar field is liable to have zeros on the spacetime manifold. This term is only sensitive to global effects associated to topological defect solutions.

The local dynamical sector

By analogy with the case of the dielectric Cremmer-Scherk theory analysed in Section 4.1, the dynamical sector of the Hamiltonian density associated to (5.15) reads

$$\begin{aligned}\mathcal{H}_{\rm dyn} &= \frac{e^2}{2}\,\big(E^i\big)^2 + \frac{1}{2\,\kappa^2\eta^2}\,\left(\frac{\partial_i E^i}{\varrho}\right)^2 \\ &+ \frac{\eta^2}{2}\,\big(\varrho\,G^i\big)^2 + \frac{1}{2\,\kappa^2 e^2}\,\big(\partial_i G^i\big)^2 + \mathcal{H}_\varrho\,,\end{aligned} \tag{5.16}$$

where as a reminder $H_\varrho(\varrho, \partial_i\varrho, \pi, \partial_i\pi)$ defined in (4.22) denotes the uncoupled part of the real scalar field $\varrho(x)$ along with its associated conjugate momentum $\pi(x)$, see (4.21). Of course the usual non vanishing Poisson brackets between the other physical variables of the phase space are recovered and take the form

$$\big\{E^i(t,\vec{x}\,), G^j(t,\vec{y}\,)\big\} = -\kappa\,\epsilon^{ij}\,\delta^2(\vec{x}-\vec{y}\,)\,.$$

This Hamiltonian density will later on become useful through the analysis of the (finite) classical energy of topological defects. An analysis similar to that of Section 4.1 in 3+1 dimensions implies two different interpretations for this Hamiltonian formulation either in terms of the A-field or the B-field picture.

The physical variable $E_0(x)$ within the A-field picture may be interpreted as the magnetic scalar field $B^A_{\rm mg}(x)$, modulo a multiplicative constant, while $E_i(x)$ is related to the electric vector field $\vec{E}^A_{\rm el}(x)$:

$$B^A_{\rm mg} = e^2\,E_0 \quad , \quad \big(E^A_{\rm el}\big)^i = e^2\,\epsilon^{ij}\,E_j\,. \tag{5.17}$$

The *ansatz* $G = 0$ previously introduced sets this electric field to zero and thus the classical solutions within the A-field picture are purely magnetic. The other physical phase space variable $G^i(x)$ is Hodge dual its counterparts $G_i(x)$, introduced within the Lagrangian first order formulation,

$$G^i = \epsilon^{ij}\,G_j\,.$$

In fact $G_i(x)$ may associated to the current $J^i(x)$ in the Maxwell-Higgs model introduced in Chapter 4 (see (4.6) and (4.14), in 3+1 dimensions). Hence we have :

$$J^i = \kappa\,\eta^2\,\varrho^2\,\delta^{ij}\,G_j\,, \tag{5.18}$$

and the first of the equations (5.13) which now takes the form

$$\vec{\nabla} B^A_{\rm mg} = \frac{\kappa\,\eta}{e^2}\,*\vec{J}\,, \tag{5.19}$$

implies that this current generates the magnetic field $B^A_{\rm mg}(x)$.

Hence, applying now the *ansatz* for static field configurations, one recovers within the dynamical sector the energy functional $\varepsilon^A_{\rm dyn}(x)$,

$$\varepsilon^A_{\rm dyn} = \frac{\eta^2}{2}\,(\varrho\,G_i)^2 + \frac{1}{4\,e^2\kappa^2}\left(\partial_{[i}\,G_{j]}\right)^2 + \mathcal{H}_\varrho\,, \tag{5.20}$$

which corresponds to the classical energy in superconductivity in the local physical sector. Likewise, applying these correspondences to the set of equations (5.13), the usual Maxwell equation for the magnetic scalar field in the plane, (5.19), is recovered along with the equations specific to Landau-Ginzburg superconductivity:

$$\begin{aligned} B^A_{\rm mg} &= -\frac{1}{\kappa\,\eta}\,\vec{\nabla}\times\frac{\vec{J}}{\rho^2}\,, \\ \nabla^2\varrho &= \frac{|\vec{J}|^2}{\varrho^3} + \lambda\,\varrho\,(\varrho^2 - v^2)\,. \end{aligned} \tag{5.21}$$

The first of these equations is the famous local second London equation describing the Meissner effect in superconductors, see for example [86, 45].

The topological sector

The dynamical part of the Hamiltonian is not affected by presence of topological content associated to topological defect solutions. Such topological effects are actually associated with the topological sector which consists in the second line of (5.15) or equivalently reads within the Hamiltonian formulation

$$\mathcal{H}^A_{\rm top} = \mathcal{B}_0\,\epsilon^{ij}\,\partial_i\left(A_j - \frac{1}{\kappa}\,G_j\right)\,. \tag{5.22}$$

The topological sector generates the global equation

$$\Omega\,\kappa\,\Phi\,[\mathcal{C}] = 2\,\pi\,L\,[\mathcal{C}] - \Omega\,\oint_{\mathcal{C}} G_i\,{\rm d}l^i\,,$$

which may also be identified as the global London equation of the effective theory of superconductivity, see [45]. The relation (5.4) between the magnetic field and its associated gauge field $A_i(x)$ does not appear in the factorised formulation (5.15). Thus referring to the global variable $\Phi[\mathcal{C}]$ as the flux is at first a misuse of language which turns out afterwards to be coherent with the global London equation and the second equation of (5.13). A similar thought process applies within the B-field picture for the free electric charge.

Dual description of covariant superconductivity

The specific TMDGT in 2+1 dimensions, of which the Lagrangian density reads

$$\mathcal{L}^3_{\mathrm{TMG}\varrho} = -\frac{1}{4\,e^2} F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{4\,\eta^2}\frac{1}{\varrho^2} H_{\mu\nu}\,H^{\mu\nu} + \frac{\kappa}{2}\,\epsilon^{\mu\nu\rho}\,A_\mu\,H_{\nu\rho} + \mathcal{L}_\varrho\,, \quad (5.23)$$

and the Maxwell-Higgs model share a common first order physical formulation and thus a common set of local equations of motion. Moreover they both possess a global London equation characteristic of vortex solutions of quantised magnetic flux $\Phi\,[\mathcal{C}]$. Hence, the Lagrangian density (5.23) admits dual magnetic vortices within the A-field picture but the topological origin of the classical quantisation of the flux turns out to be rather different from that of the Maxwell-Higgs model. In the same way, the TMDGT defined in (5.23) offers a dual covariant formulation of the Landau-Ginzburg theory of superconductivity in the superconducting phase. The great difference is that within this dual formulation the formation of vortices is not related to any symmetry breaking mechanism. This dual description along with the notion of topological order has been described earlier in the London limit only and in the compact case [74, 76, 75]. It is established for the first time in this Chapter that the equations of motion associated to the Lagrangian density (5.1) admits topological defect solutions of the vortex-type within the A-field picture.

5.2.3 *B*-field picture: (di)electric "monopoles"

Let us turn now to the new B-field picture of which the associated (factorised) Lagrangian density,

$$\begin{aligned}\mathcal{L}^B_{\mathrm{fac}} &= \frac{e^2}{2}\,E_\mu\,E^\mu + \frac{\eta^2}{2}\,\varrho^2\,G_\mu\,G^\mu + \frac{1}{\kappa}\epsilon^{\mu\nu\rho}\,\partial_\mu E_\nu\,G_\rho \mathcal{L}_\varrho \\ &+ \varepsilon^{\mu\nu\rho}\,\partial_\mu\,(\kappa\,A_\nu + G_\nu)\,\mathcal{B}_\rho + \mathrm{ST}\,,\end{aligned} \quad (5.24)$$

describes the dynamics of electromagnetic fields propagating in a medium of which the absolute permittivity $\varepsilon(x)$ is given by the real scalar field $\varrho(x)$:

$$\varepsilon(x) = \frac{1}{\eta^2\,\varrho^2(x)}\,,$$

as already stated in Chapter 4. The dielectric vacuum constant ε_0, obviously related to $\langle\varrho\rangle$, and the relative permittivity $\varepsilon_{\mathrm{R}}(x)$,

$$\varepsilon_0 = \frac{1}{\eta^2\,\langle\varrho\rangle^2} = \frac{1}{\eta^2\,v^2} \quad , \quad \varepsilon_{\mathrm{R}}(x) = \frac{\varepsilon(x)}{\varepsilon_0}\,,$$

are then defined from this absolute permittivity.

At this stage, an analysis similar to that of Section 4.1 in 3+1 dimensions implies that the physical variable $G_i(x)$ may be associated with the electric field $\vec{E}^B_{\rm el}(x)$,

$$\left(E^B_{\rm el}\right)^i = \frac{1}{\varepsilon(x)}\,\epsilon^{ij}\,G_j\,,$$

while the *ansatz* $G_0\!=\!0$ sets the magnetic field to zero and thus the classical solutions within the B-field picture are purely electric. The measured electric field $\vec{E}^B_{\rm el}(x)$ is generated both by the net free charge density[3] $\rho_{\rm f}(x)$ and by the structural charge density $\rho_{\rm s}(x)$ associated to the response of the dielectric medium. By virtue of this very definition, the usual Maxwell static equations for the electric field in a dielectric medium are recovered from (5.13) and read

$$\vec{\nabla}\times\vec{E}^B_{\rm el} = 0 \quad , \quad \vec{\nabla}\cdot\vec{E}^B_{\rm el} = \frac{1}{\varepsilon_0}(\rho_f+\rho_s) = \frac{1}{\varepsilon_0}\rho_T\,.$$

According to the equation of motions (5.13), the total charge density $\rho_{\rm T}$ reads

$$\rho_T = \vec{\nabla}\varepsilon_R\times\vec{G} + \varepsilon_R\,\kappa\,e^2\,E_0\,.$$

If we assume that the scalar field $\varrho(x)$ is everywhere frozen to its vacuum expectation value, it may already be observed in this equation that the other physical variable $E_0(x)$ has a very natural interpretation within the B-field picture in terms of the free charge density in the vacuum

$$\rho_{\rm f} = \kappa\,e^2\,E_0\,.$$

The integration of this latter relation on any surface $\mathcal{S}$ in the plane bounded by $\mathcal{C}$,

$$Q_{\rm f}[\mathcal{C}] = \int_{\mathcal{S}}\kappa\,e^2\,E_0{\rm d}x^2 = \kappa\,\Phi[\mathcal{C}]\,,$$

provides a link between the magnetic flux of the vortex solution through the surface $\mathcal{S}$ within the A-field picture and the free electric charge enclosed by $\mathcal{C}$ of the (di)electric defect solution within the B-field picture.

Having in mind the discussion about the interpretation of the Hamiltonian density (4.27) within the B-field picture in Section 4.1, it is also useful to introduce the electric displacement $\vec{D}^B_{\rm el}(x)$ which is Hodge dual to the physical variable $G_i(x)$

$$\left(D^B_{\rm el}\right)^i = \epsilon^{ij}\,G_j\,. \tag{5.25}$$

[3]Do not confuse the scalar field $\varrho(x)$ with the free, structural and total charge densities, denoted by $\rho_{\rm f}(x)$, $\rho_{\rm s}(x)$ and $\rho_{\rm T}(x)$, respectively.

Hence, applying now the *ansatz* for static field configurations, one recovers within the dynamical sector the energy functional $\varepsilon^B_{\text{dyn}}(x)$,

$$\varepsilon^B_{\text{dyn}} = \frac{\eta^2}{2} \left(\varrho\, \vec{D}^B_{\text{el}}\right)^2 + \frac{1}{2\, e^2 \kappa^2} \left(\vec{\nabla} \vec{D}^B_{\text{el}}\right)^2 + \mathcal{H}_\varrho\,, \tag{5.26}$$

in terms of the electric displacement $\vec{D}^B_{\text{el}}(x)$ which is associated to the free charge $\rho_{\text{f}}(x)$ density through the dielectric Maxwell equation:

$$\vec{\nabla} \cdot \vec{D}^B_{\text{el}} = \rho_{\text{f}}\,.$$

The well-known relation : $\vec{D}^B_{\text{el}}(x) = \varepsilon(x)\, \vec{E}^B_{\text{el}}(x)$ is then recovered. Finally, apart from the usual static Maxwell equations, two other local relations may be written in terms of the electric displacement and the free charge density,

$$\mu\, \vec{D}^B_{\text{el}} = \varepsilon_R\, \vec{\nabla} \rho_{\text{f}} \quad , \quad \nabla^2 \varrho = \eta^2\, |\vec{D}^B_{\text{el}}|^2\, \varrho + \lambda\, \varrho\, (\varrho^2 - v^2)\,,$$

which are specific to our model describing electric defects.

The topological sector consists in the second line of (5.15) or equivalently reads within the Hamiltonian formulation

$$\mathcal{H}^B_{\text{top}} = -\frac{1}{\kappa}\, \mathcal{B}_0 \left(\vec{\nabla} \vec{D}^B_{\text{el}}(x) - \kappa\, \epsilon^{ij}\, \partial_i A_j\right).$$

This sector generates the global equation

$$\Omega\, Q_{\text{f}}[\mathcal{C}] = 2\pi\, L[\mathcal{C}] - \Omega \int_{\mathcal{C}} \epsilon_{ij} \, \left(D^B_{\text{el}}\right)^i \, \mathrm{d}x^j\,, \tag{5.27}$$

taking into account the relevant topological aspects of field configurations. These aspects are related to new topological defect solutions describing the electric field associated to a charge $Q_{\text{f}}[\mathcal{C}]$, of which the classical quantisation is of topological origin, embedded in a dielectric medium. This kind of topological defects will henceforth be referred to as “dielectric monopoles”.

5.3 Construction of topological defect solutions

The different steps leading to the construction of topological defect solutions in the model of topological mass generation defined in (5.1) will be addressed in this Section. Within the A-field picture, our vortex solutions are dual to those which have been put forward by Nielsen and Olesen [47] in analogy with the Ginzburg-Landau theory of superconductivity of which the Maxwell-Higgs model is the covariant extension. What is also new and even more intriguing is the existence of an equivalent formulation of these topological defects in terms of electric monopoles in a dielectric medium within the B-field picture. Our dual vortex solutions within the A-field picture are characterised by a different origin of the magnetic flux quantisation in comparison to the Maxwell-Higgs model. The identification of topological content of the field configurations turns out to be original accordingly. This quantised flux is related to the quantised free charge of the (di)electric monopole defects within the B-field picture.

The presence of the dielectric coupling in the Lagrangian density (5.1) gives credence to the intuition that some non trivial effects arise each time the real scalar field $\varrho(x)$ vanishes on the space plane. These local effects are already unraveled through a brief analysis of the behaviour of the physical fields close to zeros of the Higgs field. Moreover, an analysis of the asymptotic behaviour of the physical fields also shows how these local effects have direct influence on the topological structure of the theory.

5.3.1 Physical fields close to a single topological defect

We first allow the scalar field $\varrho(x)$ to vanish at some discrete points in the plane. These points are defined to be the positions of the (possibly degenerate) topological defects. For the sake of simplicity, we will concentrate our attention on the particular case of topological defects whose positions all coincide with the origin. Physical field configurations corresponding to such topological defects possess a rotational symmetry on the plane. Consequently, convenient coordinates used to express the physical variables are the polar ones (r, φ),

$$x = r \cos\varphi \quad , \quad y = r \sin\varphi \, , \tag{5.28}$$

where $r = \sqrt{x^2 + y^2}$ is the radius while φ is the polar angle, $\varphi \in [0, 2\pi[$. On account of the polar symmetry *ansatz*, all the physical variables are required to depend only on the radial coordinate: $G(r)$, $E(r)$, $\varrho(r)$. Then the scalar field $\varrho(r)$ may be written as a series expansion in powers of r of the form:

$$\varrho(r) = \sum_{n=1}^{\infty} \chi_{(n)} \, r^n \, ,$$

on the assumption of real analyticity.

Equations of motion under the polar symmetry *ansatz*

Let us now establish how the set of equations of motion (5.13) and (5.14) transforms under the polar symmetry *ansatz*. The covariant space components of any 1-form

$A = A_i(\vec{x})\,\mathrm{d}x^i$ restricted on $\Omega^1(\mathbb{R}^2)$ transform under polar coordinates as

$$\begin{aligned} r\,A_x &= r\,\cos\varphi\,A_r - \sin\varphi\,A_\varphi\,, \\ r\,A_y &= r\,\sin\varphi\,A_r + \cos\varphi\,A_\varphi\,, \end{aligned}$$

while the inverse relation reads

$$\begin{aligned} A_\varphi &= r\,\cos\varphi\,A_y - r\,\sin\varphi\,A_x\,, \\ A_r &= \cos\varphi\,A_x + \sin\varphi\,A_y\,. \end{aligned}$$

Then the first two local equations of (5.13) relate $E_0(r)$ and $G_\varphi(r)$:

$$\kappa\,\eta^2\,\varrho^2(r)\,G_\varphi = r\,\partial_r E_0\,, \tag{5.29}$$

$$\partial_r G_\varphi = r\,\kappa\,e^2\,E_0\,, \tag{5.30}$$

while the radial component $G_r(r)$ vanishes identically. The third local equation of (5.13) now writes

$$\frac{1}{r}\partial_r\left(r\,\partial_r\varrho(r)\right) = \frac{\eta^2}{r^2}\,\varrho(r)\,G_\varphi^2 + \tilde{\mu}^2\,\varrho(r) + \lambda\,\varrho^3(r)\,, \tag{5.31}$$

and determines the radial profile of $\varrho(r)$.

Finally, these local equations must be completed through the global equation,

$$\frac{2\,\pi}{\Omega}\,L\,[\mathcal{C}] = \kappa\,\Phi\,[\mathcal{C}] + \int_0^{2\pi} G_\varphi\,\mathrm{d}\varphi\,, \tag{5.32}$$

where $\mathcal{C}$ is any possible choice of circular contour centered around the origin. The assumption that the functional $\Phi[\mathcal{C}]$ vanishes identically for any contour shrunk to a point, even though this point may be the location of a topological defect,

$$\lim_{\mathcal{S}\to 0}\oint_{\partial\mathcal{S}} A_\varphi\,\mathrm{d}\varphi = \lim_{\mathcal{S}\to 0}\Phi\,[\partial\mathcal{S}] = 0\,, \tag{5.33}$$

is also considered as a boundary condition. This extra condition bears a relation to the global equation (5.32), making a link between local and global aspects. It ensures that the magnetic field within the A-field picture (5.14) or the free charge density within the B-field picture (5.27) is not singular at the location of topological defects.

Behaviour of the fields close to the origin

The analysis of the behaviour of the fields close to the origin leads to the following conclusions.

- A finite number of the first consecutive expansion coefficients of the scalar field $\varrho(r)$ may all be set to zero. The index of the first non vanishing coefficient $\chi_{(N)}$ is called the order of the zero of $\varrho(r)$, denoted by N, $N \geq 1$.

- The physical variables possess a fixed parity required in order to have a consistent power series expansion close to the origin. Among them, $G_\varphi(r)$ and $E_0(r)$ are even while the parity of $\varrho(r)$ depends on the order N, namely even (resp. odd) if N is even (resp. odd). The power series expansion for $\varrho(r)$,

$$\varrho(r) = r^N \sum_{n=0}^{\infty} \chi_{(2n)}\, r^{2n} \,, \tag{5.34}$$

  is written again so as to reproduce the correct behavior of the field at the origin and to take into account the parity properties.

- The power series expansion of the physical variable $G_\varphi(r)$ includes a coefficient of zero order which depends on the order N.

$$\eta\, G_\varphi = \pm N \pm \frac{2}{N}\left((N+1)\frac{\chi_{(2)}}{\chi_{(0)}} - \frac{\tilde{\mu}^2}{4}\right) r^2 + \mathcal{O}(r^{2N+2}), \quad \text{for } r \to 0\,. \tag{5.35}$$

  The physical variable $G_i(r)$ is thus ill-defined at the origin where the topological defect is localised. As far as the B-field picture is concerned, only the radial component of the measured electric vector field $\vec{E}^B_{\rm el}(r)$ survives under the polar symmetry *ansatz* and behaves as

$$\left(E^B_{\rm el}\right)^r = \eta^2\, \frac{\varrho^2}{r} G_\varphi \to \pm\eta\, N\, \chi^2_{(0)}\, r^{2N-1} + \mathcal{O}(r^{2N+1}) \,, \qquad \text{for } r \to 0\,,$$

  while the associated electric displacement $\vec{D}^B_{\rm el}(r)$ is singular at the origin,

$$\left(D^B_{\rm el}\right)^r \to \pm\frac{1}{\eta}\,\frac{N}{r} + \mathcal{O}(r) \,, \qquad \text{for } r \to 0\,, \tag{5.36}$$

  where the B-electric monopole is located.

- The physical variable E_0 is regular close to the origin and may also be expanded in even powers of r, with the zero order coefficient of the form

$$\kappa\, e^2\, \eta\, E_0 = \pm\frac{1}{N}\left(4\,(N+1)\,\frac{\chi_{(2)}}{\chi_{(0)}} - \tilde{\mu}^2\right) + \mathcal{O}(r^2)\,, \quad \text{for } r \to 0\,. \tag{5.37}$$

  The magnetic field associated to the vortices within the A-field picture or the free charge density associated to the monopole defects within the B-field picture are thus well defined at the origin.

5.3.2 Vorticity

A careful analysis of the definition of vorticity in the topological sector of the Lagrangian density (5.1) enables us to get a precise idea of the origin of its topological invariant character. In what follows, we will work mainly within the A-field picture in order to highlight the analogies and differences with the already known vortex solutions in the Maxwell-Higgs model. Within this picture, the existence of our new type of topological defects is not related to any symmetry breaking whatsoever, but rather is in direct relation with the fact that the scalar field vanishes at some points in the plane. Of course all of the following discussion remains valid within the B-field picture for the quantisation of the free electric charge.

The case of a single topological defect

It will again be assumed in what follows that the scalar field $\varrho(r)$ possesses a single zero of order N in the plane, placed at the origin. The global equation,

$$\int_S \mathrm{d}(\kappa\, A + G) = \frac{2\pi}{\Omega}\, L[\mathcal{C}]\,, \tag{5.38}$$

implies that the differential $\mathrm{d}(\kappa\, A + G)$ is a closed form which is not exact. This relation remains true in a small neighbourhood around the origin and implies the following relation between the order N and the vorticity $L[\mathcal{C}]$

$$\lim_{\mathcal{S}\to(0,0)} L\,[\mathcal{C}] = \frac{\Omega}{\eta}\, N\,, \qquad \mathcal{C} = \partial\mathcal{S}\,, \tag{5.39}$$

according to the boundary condition (5.33) and the behaviour of $G_\varphi(r)$ close to the origin (5.35). This relation established for an infinitely small neighbourhood around the origin may be extended to any contour $\mathcal{C}$ enclosing the origin, by virtue of the topological invariance of $L\,[\mathcal{C}]$ to be discussed presently.

In fact, the zero of order N of the scalar field endows the physical variable $G_i(x)$ with a singular part at the origin, while the magnetic flux is well defined. Therefore, the "non exact" part of $\mathrm{d}(\kappa\, A{+}G)$ has a singular character in the plane since

$$\epsilon^{ij}\,\partial_i(\kappa\, A_i + G_i) = 2\pi\,\frac{N}{\eta}\,\delta^2(\vec{x})\,.$$

Then, by virtue of Stokes' theorem,

$$\int_S \mathrm{d}(\kappa\, A + G) = \oint_{\mathcal{C}=\partial S} (\kappa\, A + G)\,,$$

whenever $\mathrm{d}(\kappa A{+}G)$ is singular at the origin, this is equivalent to the statement that the holonomy of $(\kappa A{+}G)$ is defined on the manifold $\mathbb{R}^2 \setminus \{(0,0)\}$, henceforth denoted

$\mathbb{R}^2_0$. This topologically non trivial manifold has the same homotopy as the circle S^1. Consequently, any contour in the plane surrounding the origin is a 1-cycle defined modulo a 1-boundary, thus belonging to the first homology group $H_1(\mathbb{R}^2_0, \mathbb{Z})$.

The global equation (5.38) selects in fact the global harmonic parts of $A(x)$ and $G(x)$ which are thus equal modulo a global form, the vorticity $L[\mathcal{C}]$. Hence, although the holonomies of $(\kappa\, A_i)$ and G_i are not actual global variables, the global equation implies that their sum must necessarily be topological invariant. The functional $L[\mathcal{C}]$ is thus constant for the set of homotopically equivalent contours, namely the (non contractible) loops surrounding the origin. Indeed, choosing such two contours $\mathcal{C}$ and $\tilde{\mathcal{C}}$ and evaluating their dual cohomology group with respects to eq. (5.38):

$$\oint_{\mathcal{C}} (\kappa\, A + G) - \oint_{\tilde{\mathcal{C}}} (\kappa\, A + G) = \frac{2\,\pi}{\eta}\,(L\,[\mathcal{C}] - L[\tilde{\mathcal{C}})\,,$$

a difference between vorticities is obtained. Clearly the difference between the two contours belonging to the same equivalence class of $H_1(\tilde{\mathbb{R}}_0, \mathcal{Z})$ is a pure boundary and thus $L[\mathcal{C}]-L[\tilde{\mathcal{C}}]=0$. Following Poincaré duality, equivalent arguments lead to the invariance of the magnetic flux $\Phi[\mathcal{C}]$ under small gauge transformations $A' = A + \alpha$ for any contour $\mathcal{C}$, see [45],

$$\Phi'[\mathcal{C}] - \Phi[\mathcal{C}] = \oint_{\tilde{\mathcal{C}}} \alpha = 0\,,$$

since under these transformations α is an exact form.

The topological invariance of vorticity implies that the global equation reads

$$\oint_{\mathcal{C}} (\kappa\, A + G) = \frac{2\,\pi}{\Omega}\, L\,[\mathcal{C}]\ , \qquad \eta\, L[\mathcal{C}] = \Omega\, N\,, \tag{5.40}$$

for all contours $\mathcal{C}$ in $\mathbb{R}^2_0$ enclosing the origin. The procedure just described is very reminiscent of that which leads to the explanation of the Arahonov-Bohm effect. Actually we will see later that $A_i(x)$ and $G_i(x)$ interchange their global part when interpolating from contours shrunk to zero to contours at the infinity, since $G_i(x)$ tends to zero for large radius. Thus the (classical) Arahonov-Bohm effect is asymptotically recovered. However, in contradistinction to what happens in the Arahonov-Bohm effect, the quantisation of the (asymptotic) magnetic flux already arises at the classical level and is related to the order N, according to (5.40). One may further enforce $\Omega = \eta$ in order that $L\,[\mathcal{C}]$ be integer-valued. However we introduce hereafter a new interpretation of the vorticity, specific to TMDGT, which implies the integer-valued character of $L[\mathcal{C}]$ in a very elegant way.

The global equation (5.40) states that on any curve in $H_1(\mathbb{R}^2_0, \mathbb{Z})$, the harmonic parts of $(\kappa\, A)$ and G belong to the same gauge orbit under the modular group of transformations of non zero winding number on this contour. Requiring the weaker equality of the Wilson loop as in Section 3.1, see (3.12), the vorticity takes integer values only and is related to the winding number around the non contractible loops. Hence $L[\mathcal{C}] = N$ is a topological invariant belonging to the lattice group $H^1(\mathbb{R}_0, \mathbb{Z})$. However, acting on the harmonic part of singular origin, these "large gauge transformations" are themselves singular and field configurations associated to topological defect solutions break the invariance under these LGT. Two topological defect field configurations differing by their vorticity are topologically distinct[4]. In fact, the only allowed gauge transformations are regular and single-valued throughout spacetime, namely the "small" gauge transformation of zero winding number for a given contour $\mathcal{C}$. Rather the "singular" large gauge transformations classify different fields configurations according to the integer-valued vorticity functional for any loop $\mathcal{C}$.

Multi-topological defects solutions

By virtue of the topological invariance of the vorticity, the relation (5.40) is true for all contours $\mathcal{C}$ such that the surface delimited by this contour does not encounter an other zero of the scalar field. It is obviously always true within the simple case of a single topological defect at the origin. We will now generalise this case in allowing the scalar field $\varrho(x)$ to vanish on a discrete set of points $z_k = (x_k, y_k)$. The requirement of a finite energy functional, of which the physical part is of the form (5.20) within the A-field picture, generates the following condition at the topological defect location:

$$\varrho^2\,(G_i)^2 < \infty\,, \qquad \text{for } z \to z_k\,.$$

Consequently the condition of finite energy field configurations does not forbid the physical variable G_i to possess a $1/|z - z_k|$ singularity at the set of points $\{z_k\}$. Actually that is indeed the case as shown previously (5.35) through the analysis in polar coordinates of the behaviour of the fields close to the origin, where the single topological defect had been placed.

This conclusion may readily be extended to field configurations with topological defects located at the set of points z_k in the plane. The topologically non trivial manifold on which the global London equation (5.38) is defined then reduces to $\mathbb{R}^2 \setminus \{z_k\}$. Consequently $L[\mathcal{C}]$ is non zero when the contour shrinks to a point z_k where the singular harmonic part of the global London equation (5.38) is only carried by the physical variable $G_i(x)$, following the boundary condition (5.33). The value of $L[\mathcal{C}]$ is then equal to the order at this point, which will be referred to as N_k. By virtue of the topological invariance of the vorticity, $L[\mathcal{C}]$ does not change for two different contours which can be deformed to each other. However, when the surface $\mathcal{S}$ bounded by $\mathcal{C}$

[4] This explains why the dynamical and topological sectors of (5.1) are no longer completely decoupled through our TP factorisation.

encounters another point where the field $\varrho(x)$ possesses another zero of order N', and thus another point where $\mathrm{d}G$ is singular, the vorticity increases by the value N'. The functional $L[\mathcal{C}]$ depending on the contour $\mathcal{C}$ is subsequently piece-wise constant and equals the net multiplicity of zeros inside this contour

$$L[\mathcal{C}] = \sum_{k=1}^{K[\mathcal{C}]} N_k \, .$$

In this equation, $K[\mathcal{C}]$ denotes the number of points z_k inside the contour $\mathcal{C}$. In other words the vorticity $L[\mathcal{C}]$ is the sum of the contributions of a collection $K[\mathcal{C}]$ topological defects surrounded by $\mathcal{C}$ along with their respective degeneracy N_k. Two topological defect configurations which cannot be deformed into each other are topologically distinct. Singular large gauge transformations classify these topologically distinct vortex solutions according to the associated lattice group,

$$H^1(\mathbb{R}^2 \setminus \{z_k\} \, , \mathbb{Z}) = \mathbb{Z} \, ,$$

to which the vorticity belongs.

For a single topological defect solution localised at the origin, field configurations of different vorticities are topologically distinct. Once the solution consists in more than one topological defect, their relative positions and respective degeneracies ought to be also taken into account. All this information is encoded in the vorticity functional $L[\mathcal{C}]$. In fact, it may be proved that the knowledge of vorticity $L[\mathcal{C}]$ for any contour $\mathcal{C}$ characterises the field configuration associated to a given topological defect in an unique way. Further, the piece-wise constant character of this functional $L[\mathcal{C}]$ implies that the space of multi-topological defect solutions is $2L$ dimensional, with

$$L = \lim_{S \to \mathbb{R}^2} L[\partial S] \, ,$$

for space manifolds without boundary.

Finally, the last important point of this Section is the general character of our analysis of the topological content specific to TMDGT. Often we have not specified purposely the number of space dimensions, or even the rank of the physical variables. Actually our approach of the vorticity is generic for any topological mass generation model with a specific dielectric coupling to a scalar field whatever the number of spacetime dimensions and the tensorial rank of the p-form fields, see (4.29). Hence although our approach of vorticity may seem heavier than that of the Maxwell-Higgs model, its range of application might possibly be far more general.

5.3.3 Asymptotic behaviour of the physical fields

As has been shown in the previous Subsections, the message carried by the global equation (5.32) generated from the topological sector is that there exist topological defect solutions for which physical observables constructed from physical variables possess non trivial topological content. In contradistinction to the case of pure TMGT, discussed in Chapters 2 and 3, the global "0-modes" and local "k-modes" do not factorise in our interacting model. Hence topologically massive dielectric gauge theories may no longer be completely decoupled into dynamical and topological sectors. Even in the London limit, despite the fact that the non interacting case (pure TMGT) is recovered in the physical sector, a non trivial topological content subsists again.

The asymptotic London limit

One way to reach the London limit is to consider the asymptotic behaviour of the fields at infinity, requiring at a same time the (local) energy functional, introduced in (5.20) within the A-field picture, to remain finite. Under the commonly made assumption that the scalar field $\varrho(r)$ reaches its vacuum expectation value rapidly when $r \to \infty$, the asymptotic values of the physical variables have to obey:

$$\lim_{r\to\infty} \left(\frac{\varrho}{r}\right)^2 (G_\varphi)^2 = 0\,, \qquad \lim_{r\to\infty} \varrho^2\,(E_0)^2 = 0\,,$$

according to (5.20) and (5.30). The following boundary conditions are thus obtained:

$$\rho \to v\,, \qquad G_\varphi = k + \mathcal{O}(1/r)\,, \qquad E_0 = \mathcal{O}(1/r)\,, \qquad \text{for } r \to \infty\,, \tag{5.41}$$

and are compatible with (5.29) provided that $k = 0$. In fact it may be easily seen that the scalar field $\varrho(r)$ approaches its asymptotic value as

$$\varrho(r) \to v + \mathcal{O}\left(e^{-\sqrt{2}\,|\tilde{\mu}|\,r}\right)\,, \qquad \text{for } r \to \infty\,, \tag{5.42}$$

see for example [47, 87]. The characteristic decay rate $(\sqrt{2}\,|\tilde{\mu}|)^{-1}$ is the inverse of the mass of the field $\varrho(r)$ in the theoretical particle physics interpretation or the coherence length[5] in the condensed matter physics one.

As $\varrho \to v$ rapidly enough at large radius, local equations (5.29) and (5.30) read under this London limit

$$\begin{aligned} \partial_r\,(r\,\partial_r E_0) &= r\,\mu^2\,E_0\,, \\ r\,\partial_r\left(\frac{1}{r}\partial_r G_\varphi\right) &= \mu^2\,G_\varphi\,, \qquad \text{for } r \to \infty\,, \end{aligned}$$

[5] Modulo a normalisation factor $\sqrt{2}$, see (4.16).

where μ is defined in (5.9), and are then exactly solvable. Equation (5.31) is redundant with the asymptotic boundary conditions (5.41) since $\lambda\, v^2 = -\tilde{\mu}^2$ for the quadratic potential (4.12). In fact, a free theory is recovered at large distances and corresponds to the physical part[6] of TMGT, see (2.31), to which a topological sector must be added in order to reproduce the global equation (5.32). The above couple of equations reads as follows in a more appealing decoupled form upon setting $G_\varphi = r\,\tilde{G}_\varphi$:

$$\begin{aligned} \partial_r^2 E_0 + \frac{1}{r}\partial_r E_0 &= \mu^2\, E_0\,, \\ \partial_r^2 \tilde{G}_\varphi + \frac{1}{r}\partial_r \tilde{G}_\varphi &= \left(\mu^2 + \frac{1}{r^2}\right)\tilde{G}_\varphi\,, \qquad \text{for } r \to \infty\,. \end{aligned} \tag{5.43}$$

The well known solutions to these second order equations are the modified Bessel functions[7] $I_n(x)$ and $K_n(x)$, where in the present situation $n \in \mathbb{Z}$. These types of Bessel functions are real-valued and do not manifest a periodic behaviour, as expected for the physical fields considered. If the physical variables $E_0(r)$ and $G_\varphi(r)$ are assumed to vanish asymptotically (5.41), then

$$\begin{aligned} G_\varphi^\infty(r) &= r\,\tilde{G}_\varphi^\infty(r) = c_{el}\, r\, K_1(\mu\, r)\,, \\ \text{and} \quad E_0^\infty(r) &= c_{mg}\, K_0(\mu\, r)\,, \end{aligned}$$

are the only acceptable local solutions to the asymptotic equations of motion (5.43).

Considering this asymptotic behaviour, the surface term "ST" which arises when factorising the Lagrangian (or the Hamiltonian) density through our TP factorisation technique, see (5.12), vanishes at the boundary of the plane

$$\int_{S^2} \varepsilon^{ij}\,\partial_i \left((A_j + \frac{1}{\kappa}G_i)\, E_0 \right) \mathrm{d}^2x = \oint_\infty \left(A_\varphi + \frac{1}{\kappa}G_\varphi \right) E_0 = 0\,,$$

and so does not contribute to the action. Finally, before leaving the analysis of the asymptotic behaviour for the physical variables, it is of prime importance to consider global boundary conditions not yet taken into account since the set of equations (5.43) is only locally defined. These global boundary conditions fix the remaining integration constants c_{mg} and c_{el}.

Physical fields within the A-field picture

The A-field picture where topological defects of vortex type arise is first considered. As already stated, the asymptotic boundary conditions (5.41) do not allow the closed

[6] Or equivalently the associated Proca theory or gauge fixed Stueckelberg theory, see Fig.4.3.

[7] Also known as Bessel functions of pure imaginary argument.

form $\mathrm{d}G$ to possess a global (or "non exact") part of singular character at large radius. This is to be related to the fact that the conserved current (5.18) vanishes at infinity while the magnetic flux through the plane reads

$$\Phi_\infty = \lim_{S\to\mathbb{R}^2} \int_S (B^A_{\text{mg}})\, \mathrm{d}x^2 = 2\pi\, \frac{L}{\eta\,\kappa}\,, \qquad \lim_{S\to\mathbb{R}^2} L[\partial S] = L\,.$$

By virtue of topological invariance of vorticity, the global equation (5.32) implies that the magnetic field inherits asymptotically the singular part of $\mathrm{d}G$ at the origin.

Within the A-field picture, the limit $\varrho\to v$ corresponds to the asymptotic London limit of fluxon strings [47] since vortex cores are considered as being infinitely thin at large distance. Let us consider now the magnetic field $B^\infty_{\text{mg}}(r)$ in this London limit which is obviously related to $E^\infty_0(r)$, see (5.17). Hence the integration of this latter physical field over the plane,

$$L = c_{\text{mg}}\,\eta\,\kappa\,e^2 \int_0^\infty r\,K_0(\mu\,r) = c_{\text{mg}}\,\frac{\eta\,\kappa\,e^2}{\mu^2} = \frac{c_{\text{mg}}}{\eta\,\kappa\,v^2}\,,$$

fixes the value of the constant c_{mg} to

$$c_{\text{mg}} = \eta\,\kappa\,v^2\,L\,.$$

This integration constant can only take discrete values depending on the vorticity L.

Going back to vortices of finite core, the magnetic field $B^A_{\text{mg}}(r)$ shares with its counterpart $B^\infty_{\text{mg}}(r)$ in the London limit the same asymptotic behaviour which thus reads

$$(B^A_{\text{mg}}) = \frac{L}{\eta\,\kappa}\sqrt{\frac{\pi\,\mu^3}{2\,r}}\;\mathrm{e}^{-\mu\,r}\left[1+\mathcal{O}\left(\frac{1}{r}\right)\right]\,, \qquad \text{for } r\to\infty\,. \tag{5.44}$$

The behaviour of the magnetic field for vortices in the Maxwell-Higgs formulation is recovered, where $(1/\mu)$ is the penetration depth.

Physical fields within the B-field picture

As far as the B-field picture is concerned, under the approximation of an infinitely thin dielectric monopole core at large distance, the electric displacement $\vec{D}^B_{\text{el}}(r)$ defined in (5.25) reads

$$\left(D^B_{\text{el}}\right)^r \xrightarrow{\varrho=v} \left(D^B_\infty\right)^r = \frac{1}{r}G^\infty_\varphi = c_{\text{el}}\,K_1(\mu\,r)\,. \tag{5.45}$$

This approximation at large radius is analogous to the London limit within the A-field picture since it is again assumed that the field $\varrho(r)$ reaches its vacuum expectation value very rapidly, see (5.42).

Considering now this London limit $\varrho = v$ throughout the plane, the electric displacement behaves at the origin as

$$\left(D_{\infty}^{B}\right)^{r} \rightarrow \frac{c_{\text{el}}}{\mu}\,\frac{1}{r} \qquad \text{for } r \rightarrow 0\,.$$

This limit in combination with the behaviour of the physical variable $G_{\varphi}(r)$ close to the origin (5.35) implies that the constant c_{el} can only take discrete values

$$c_{\text{el}} = \frac{\mu}{\eta}\,L\,.$$

The relation (5.45) implies that the electric displacement $\vec{D}_{\infty}^{B}(r)$ falls exponentially to zero at large radius. Therefore, the total free electric charge Q_f carried by the monopole at infinity, given by the global equation (5.27), reads

$$Q_f^{\infty} = \frac{L}{\eta}\,,$$

fixing again the value c_{mg}. Hence within the B-field picture, vorticity measures the free electric charge in units of one quantum of charge of the dielectric monopole.

As already stated within the A-field picture, the London limit $\varrho = v$ provides a very good asymptotic approximation to our model. In this limit, the "measured" electric field $\vec{E}_{\infty}^{B}(r)$ propagates in the vacuum and is therefore proportional to the electric displacement, $\vec{E}_{\infty}^{B} = \eta^2 v^2 \vec{D}_{\infty}^{B}$. Therefore, within the B-field picture, the "measured" electric field $\vec{E}_{\text{el}}^{B}(r)$ which takes into account the response of the dielectric medium to the free electric charge Q_f behaves at large distance as:

$$(E_{\text{el}}^{B})^{r} = L\,\eta\,v^2\,\sqrt{\frac{\pi\,\mu}{2\,r}}\,\mathrm{e}^{-\mu\,r}\,\left[1 + \mathcal{O}\left(\frac{1}{r}\right)\right]\,, \qquad \text{for } r \rightarrow \infty\,, \tag{5.46}$$

and decreases exponentially at infinity with $(1/\mu)$ as characteristic decay length.

5.3.4 Numerical solutions: two types of topological defects

Properties of the solutions

In this Subsection the solutions to the set of local equations (5.29), (5.30) and (5.31), completed through the global boundary conditions (5.32) and (5.33), will be presented. These solutions were obtained through numerical simulation in collaboration with Térence Delsate and, as a reminder, account for a single topological defect localised at the origin and possibly degenerate. Introducing the radial coordinate u

normalised to the length scale μ, that is $u = \mu\, r$, and redefining the physical fields as variables without physical dimension:

$$G = \eta\, G_\varphi\,, \qquad \tilde{\varrho} = \frac{\varrho}{v}\,, \qquad \mu\, E = \frac{e}{v}\, E_0\,, \qquad \tilde{\lambda} = \frac{|\tilde{\mu}|}{\mu}\,,$$

the ensuing system of equations reads:

$$\begin{aligned} \partial_u^2 G - \frac{1}{u}\partial_u G - \tilde{\varrho}^2\, G &= 0\,, \\ \partial_u^2 \tilde{\varrho} + \frac{1}{u}\,\partial_u \tilde{\varrho} - \frac{1}{u^2}\, G^2\, \tilde{\varrho} + \tilde{\lambda}\, \tilde{\varrho}\,(1 - \tilde{\varrho}^2) &= 0\,, \\ \partial_u G - u\, E &= 0\,. \end{aligned} \tag{5.47}$$

These equations depend on two parameters $|\tilde{\mu}|$ and μ of which only the ratio $\tilde{\lambda}$ is physically relevant. This set of coupled differential equations is subjected to the following asymptotic boundary conditions

$$\lim_{u\to\infty} \tilde{\varrho}(u) = 1\,, \qquad \lim_{u\to\infty} G(u) = 0\,, \qquad \lim_{u\to\infty} E(u) = 0\,, \tag{5.48}$$

according to our discussion of the asymptotic London limit while the physical variables take the following values at the origin

$$\tilde{\varrho}(0) = 0\,, \qquad G(0) = L\,, \qquad L \in \mathbb{Z}\,, \tag{5.49}$$

in agreement with the behaviour of the physical fields close to the origin.

The numerical resolution of these coupled differential equations is far from being trivial. Indeed, on the one hand, these equations are non linear and on the other hand a part of the above boundary conditions is delocalised on the boundary of the plane at infinity. Hence, the first coefficients of the expansion in power of u at the origin must be adjusted in order to meet the required boundary conditions at infinity. The FORTRAN subroutine "COLSYS" has been used in order to tackle this kind of difficulties arising in the numerical integration of the set of equations (5.47) for the ensemble of boundary conditions (5.48) and (5.49). As seen hereafter, these solutions are localised (particle-like) topological defects in the sense that the physical fields approach their asymptotic vacuum values exponentially. This is confirmed through our study of the asymptotic behaviour of the physical variables $B^A_{\rm mg}(r)$ and $E^B_{\rm el}(r)$, see (5.44) and (5.46) respectively, which have a characteristic decay length equal to one while the decay length of $\tilde{\varrho}(u)$ is $(2\,\tilde{\lambda})^{-\frac{1}{2}}$, see (5.42).

Considering the global boundary conditions (5.32), the asymptotic vorticity L has different meaning whether one considers the A-field or the B-field picture. Actually, L is the total quantised magnetic flux (in unit of flux) through the plane of the vortex field configuration. Within the B-field picture, vorticity around each monopole location is associated to the quantised free electric charge carried by this monopole. As far as the free energy is concerned, vorticity is related to the topological sector, see (5.22), generating the global equation (5.32) and thus not decoupled from the physical sector. This implies that topological effects are no longer associated to the simple degeneracy of the energy spectrum[8] but really affect it. Hence two topological defects with different topological invariant, that is different vorticity, have different energy.

B-field picture: (di)electric monopoles

As already stated, the set of coupled equations (5.47) along with the global equation (5.32) describes new topological defect solutions within the B-field picture. These (di)electric monopoles account for a free charge $Q_{\mathrm{f}}[\mathcal{C}]$, of which the classical quantisation is of topological origin, embedded in a dielectric medium. Usually the electric displacement $\vec{D}^B_{\mathrm{el}}(r)$ is used to describe free electric fields in a dielectric medium. However, the behaviour of this physical variable close to the origin is pathological, see (5.36). The electric displacement is associated to the free charge density $\rho_{\mathrm{f}}(r)$ through the Maxwell equation (5.27). However, strictly speaking, $\vec{D}^B_{\mathrm{el}}(r)$ is not generated by $\rho_{\mathrm{f}}(r)$ since $\vec{\nabla}\times\vec{D}^B_{\mathrm{el}} \neq 0$. Indeed although the free charge density is finite at the defect location, see (5.37), the electric displacement is singular. This fact is already made manifest through the global equation (5.27). The radial free charge distribution $\rho_{\mathrm{f}}(u)$, expressed in terms of the dimensionless variable $E(u)$,

$$\rho_{\mathrm{f}} = \frac{\mu^2}{\eta}\, E\,,$$

is plotted in Fig.5.1, Fig.5.2 and Fig. 5.3.

As usual for this type of topological defect solutions, the value of the scalar field $\tilde{\varrho}(u)$, presented in Fig.5.1, is bounded by its vacuum expectation value: $\tilde{\varrho}(u) < 1$, in order that the physical fields reach their vacuum value at infinity. Within the context of superconductivity, the value of this scalar field interpolates between the completely disordered state ($\tilde{\varrho}(u) = 0$) at the vortex location and the completely ordered state ($\tilde{\varrho}(u) = 1$) at infinity, where the Meissner effect is maximal. Likewise within the B-field picture, the scalar field $\tilde{\varrho}(u)$ effectively acts like a dynamical dielectric medium, of relative permittivity $\varepsilon_R = 1/\tilde{\varrho}$, which counterbalances the free electric field in the vacuum since $\tilde{\varrho}(r) < 1$. The contribution of the dielectric medium is maximal and even singular at the electric defect location. According to (5.34), this singularity in $1/r^2$ compensates for that in $1/r$ of the electric displacement, see (5.36), with the

[8] As was the case for pure TMGT defined on non simply connected manifolds, see Chapter 3.

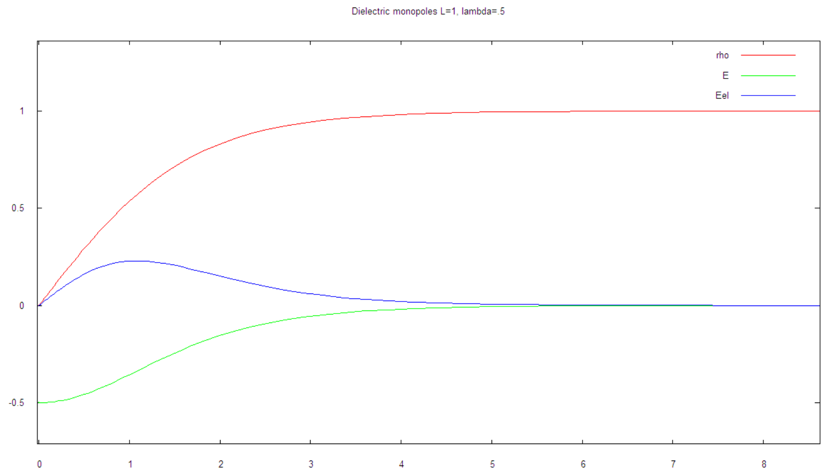

Figure 5.1: *Dielectric monopole of one quantum of free electric charge at infinity within the B-field picture at the Bogomol'nyi point $\tilde{\lambda}=1/2$. This figure represents the rescaled electric field, $E_{el}=(1/\mu^2\,\eta\,v^2)(E^B_{el})^u$, and free charge density, $E=(\eta/\mu^2)\,\rho_{\rm f}$, along with the inverse relative permittivity $\tilde{\varrho}=(1/\varepsilon_{\rm R})$ as functions of u.*

result that the total "measured" electric field $\vec{E}^B_{\rm el}(r)$ given by $\varepsilon(x)\,\vec{E}^B_{\rm el}(x)$=$\vec{D}^B_{\rm el}(x)$ be zero at the origin. In the non zero coherence length case, the measurable electric field is screened by the permittivity and hence protected from a bare singularity. On the contrary at large radius, the total electric field propagates in the vacuum as the influence of the dielectric medium becomes insignificant. This total electric field is related to the variable without dimension $G(u)$,

$$\frac{1}{\mu^2\,\eta\,v^2}(E^B_{\rm el})^u=\frac{\tilde{\varrho}^2}{u}\,G\,.$$

Fig.5.1, Fig.5.2 and Fig.5.3 present the results of numerical investigations for this radial field which is well defined all over the plane.

A-field picture: Dual magnetic vortices

Let us now concentrate our attention on the dual formulation within the A-field picture of the Nielsen-Olesen vortex solutions of quantised magnetic flux L (in unit of flux). The magnetic field $B^A_{\rm mg}(u)$ associated to a vortex solution localised at the origin is

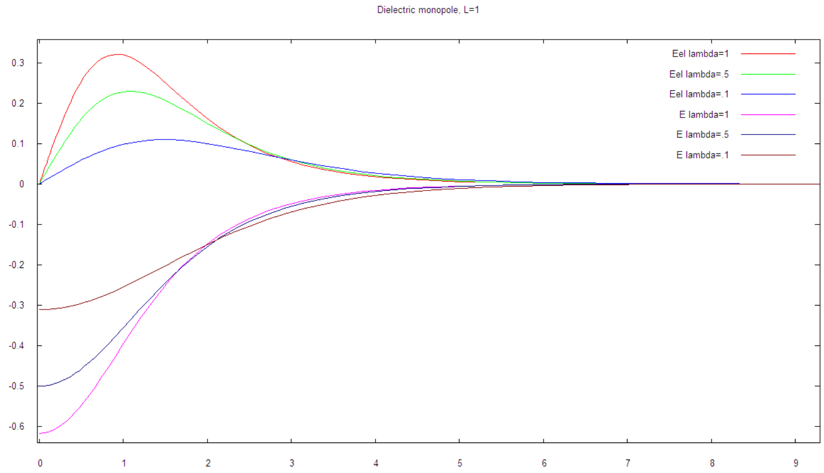

Figure 5.2: *Dielectric monopole solutions of one quantum of free electric charge at infinity within the B-field picture for $\tilde{\lambda} = 1, 1/2, 0.1$. This figure represents the rescaled electric field, $E_{el} = (1/\mu^2\,\eta\,v^2)(E^B_{el})^u$ and the rescaled free charge density, $E = (\eta/\mu^2)\,\rho_{\mathrm{f}}$, as functions of the normalised radius u*

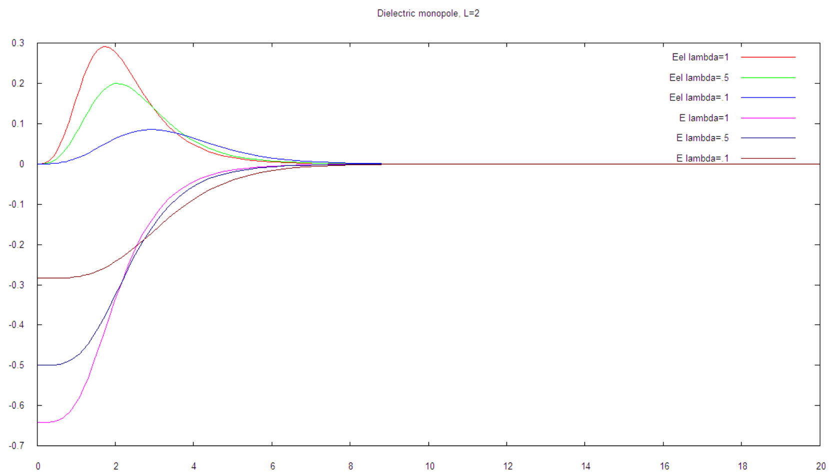

Figure 5.3: *Dielectric monopole solutions of two quantum of free electric charge at infinity within the B-field picture for $\tilde{\lambda} = 1, 1/2, 0.1$. This figure represents the rescaled electric field, $E_{el} = (1/\mu^2\,\eta\,v^2)(E^B_{el})^u$ and the rescaled free charge density, $E = (\eta/\mu^2)\,\rho_{\mathrm{f}}$, as functions of the normalised radius u.*

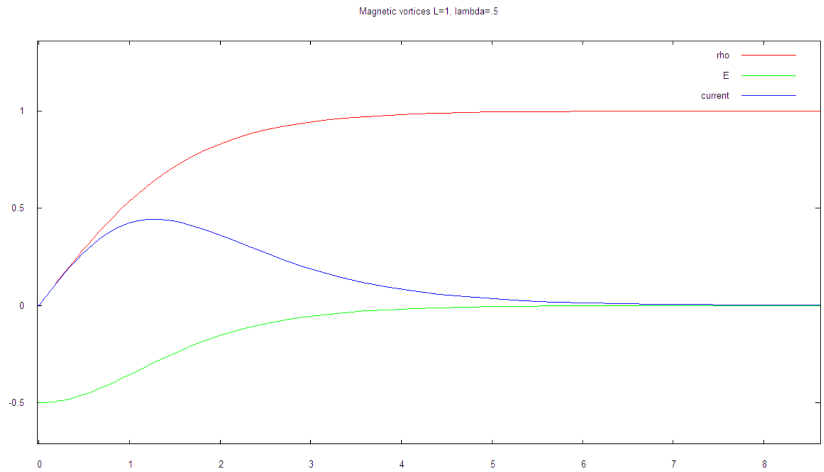

Figure 5.4: *Dual vortex solution of one quantum of magnetic flux at infinity within the A-field picture at the Bogomol'nyi "self-dual" point $\tilde{\lambda} = 1/2$. This figure represents the rescaled magnetic field, $E = (\mu^2/\kappa\,\eta)\,B^A_{mg}$, the rescaled current density $(e/\mu\,v)\,J_\varphi$ and the renormalised Higgs field $\tilde{\varrho}$ as functions of u.*

related to the dimensionless variable $E(u)$ according to

$$\frac{\mu^2}{\kappa\,\eta}\,B^A_{\rm mg} = E\,.$$

This transverse magnetic field, represented in Fig.5.4, is generated through the usual Maxwell equation,

$$e^2\,\vec{\nabla} B^A_{\rm mg} = \kappa\,\eta\,*\,\vec{J},$$

by the azimuthal component $J_\varphi(u)$ of the current density which reads in term of the dimensionless variable $G(u)$

$$\frac{e}{\mu\,v}\,J_\varphi = \tilde{\varrho}^2\,G\,.$$

Fig.5.4 presents the results of numerical investigations for this azimuthal field well defined all over the plane and for the normalised Higgs field $\tilde{\varrho}(u)$. Here we do not describe in detail the vortex solutions we obtained since they are already well-known in the literature through the study of the effective model of type II superconductivity (see [87] and references therein). We prefer to dwell on analogies and differences between vortices in the Maxwell-Higgs model and their present dual formulation.

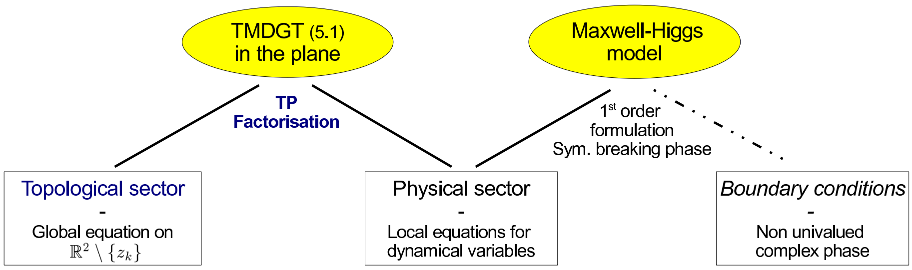

Figure 5.5: *Duality relation between vortex solutions in our dielectric model of topological mass generation within the A-field picture introduced in (5.1) and vortex solutions of the Maxwell-Higgs model in $2+1$ dimensions. The two vortex solutions share a common local sector of physical variables but the origin of the classical quantisation of the magnetic flux, of topological character, is different.*

As already stated our TMDGT in 2+1 dimensions (5.23) within the A-field picture and the Maxwell-Higgs model share a common physical sector of dynamical variables but they differ through their topological content. Hence these theories admit both vortices as static solutions but the origin of the classical quantisation of the magnetic flux is different, see Fig.5.5. The vorticity defined in our dielectric model of topological mass generation and the vorticity associated to boundary conditions on the complex $U(1)$ scalar field in the Maxwell-Higgs model, see Section 1.4, are related through

$$\begin{array}{ccccc} & & \textit{Maxwell-Higgs} & & \textit{TMGT} \\ 2\pi\, L[\mathcal{C}] & = & \int_0^{2\pi} \partial_\varphi \theta \, \mathrm{d}\varphi & = & \eta\, \kappa\, \Phi\, [\mathcal{C}] + \eta \oint_{\mathcal{C}} G \\ & & \text{Non univalued} & & \text{Global part} \\ & & \text{complex phase} & & \text{on } \mathbb{R}^2 \setminus \{z_k\} \end{array}$$

Topological defect configurations which cannot be deformed into each other are topologically distinct. Hence singular large gauge transformations classify our dual vortex solutions according to the first cohomology group, $H^1(\mathbb{R}\setminus\{z_k\}, \mathbb{Z})$. Vorticity, associated to the number and degeneracy of topological defects, is related to the parameters of this lattice group and may only take integer values accordingly. Recalling that for non pathological manifolds, the first homology group $H^1(\mathcal{M}, \mathbb{Z})$ is isomorphic to the first homotopy group, $\pi_1(\mathcal{M})$, modulo the commutator subgroup, enforces the comparison with the boundary conditions in the Maxwell-Higgs model. However our dual approach remains noticeably different. In contradistinction with the usual vortices, the construction of our dual vortices proceeds first from the analysis of the behaviour of the physical variables close to the position of vortices. Then the consequences of this local behaviour on the topological structure of the solutions are addressed.

5.4 Conclusion

Let us conclude this Chapter by some comments about the extreme care which has to be taken in dealing with global effects. Two "pure" theories sharing a common formulation in terms of physical variables through some dualisation procedures but differing by their gauge symmetry content are equivalent if they are defined on topologically trivial manifolds. As far as topologically massive dielectric gauge theories (TMDGT) are concerned a further restriction applies: the scalar field does not vanish anywhere on the spacetime manifold since then global effects, associated to topological defect solutions, arise. Otherwise, the process which consists in establishing the dual equivalence between the topologically massive action (5.23) within the A-field picture and the Maxwell-Higgs model does not apply unless global boundary conditions are defined (see for example [45]). Up to now the occurrence of these global boundary conditions was considered in the literature only within the study of some effective limit in the Nielsen-Olesen vortex solutions which leads to the formulation of Nambu strings [47, 79]. In the context of our work, this problem is a direct consequence of the formulation of the Maxwell-Higgs model as a first order theory described by local physical variables, see Section 1.4.

The gauge embedding procedure relying on the Fradkin-Vilkovisky theorem enables to establish duality relations between the most common mass generation mechanisms which possess different gauge symmetry structures, see Fig.4.3. However such dual equivalences no longer hold as soon as topological defect solutions arise. Indeed, considering for instance the Maxwell-Higgs model and the TMDGT within the A-field picture defined in 2+1 dimensions, these theories are locally equivalent but the topological origin of the quantisation of the magnetic flux is totally different, see Fig.5.5. As far as the Maxwell-Higgs model is concerned, the classical quantisation of the flux comes from non trivial boundary conditions for the $U(1)$ complex scalar field which is required to reach its vev at infinity. It is therefore intimately related to the symmetry breaking potential. Turning now to the case of the TMDGT defined in (5.1), we have proved in this Chapter that the topological content of topological defect solutions resides in the topological sector which maintains two types of gauge invariances. Therefore it makes no sense to relate the formation of these topological defects to any symmetry breaking mechanism. Usually in the literature, the topological sector was until now swamped by a mesh of successive procedures of gauge embeddings and/or gauge fixings characteristic of the dualisation methods. This is probably the reason why our topological defect solutions have not been discovered until now. In fact, the formation of these topological defects perfectly illustrates the restricted validity of the gauge embedding procedures relying on the Fradkin-Vilkovisky theorem (see [44] and references therein).

Finally, the strong interdependence of the two equivalent pictures implies that all insights gained through the study of magnetic vortices within the A-field picture might be applied to the new dielectric monopoles within the B-field picture. Among an exhaustive list, let us just mention the careful analytical analysis developed in order to prove the existence and smoothness of finite energy vortex solutions (see [87] and references therein), the quantum corrections around the classical non perturbative vortex configurations, vortex dynamics and interactions [88], etc. In some sense this thought process already applies in this Chapter. Conversely a complete and systematic analysis of our novel dielectric monopoles picture might shed new light on the properties of the magnetic vortex solutions. Finally our work offers a new insight in the analysis of solitons in abelian gauge field theories since our procedure could allow to construct topological defect solutions in any dimension never considered before. We will hardly explore this avenue in Chapter 6 by addressing in $3+1$ dimensions a novel generalisation of vortices to 2-form gauge fields and their associated dielectric monopole solutions for 1-form gauge fields.

CHAPTER 6
New horizons

Vortex solutions in abelian gauge theories have been restricted until now by the stringent condition $p = 1$, namely they have been associated with 1-form gauge fields. However the topological sector of TMDGT retains the same formulation whatever the number of spacetime dimensions or the tensorial rank of the gauge fields. Therefore our work offers new insights into the analysis of brane-type topological defects in abelian gauge field theories. We will hardly explore this new avenue in this Chapter by addressing for the first time vortex-type topological defects generalised to 2-form gauge fields in $3{+}1$ dimensions, within the B-field picture. In other respects, the (di)electric "monopoles" obtained in Chapter 5 do not offer much interest per se besides the novel character of their construction. Indeed to the best of our knowledge, there do not exist applications where electric fields polarise a dielectric medium in the plane. However such (di)electric monopoles for 1-form gauge fields admit a generalisation in $3{+}1$ dimensions, namely a more realistic case, within the A-field picture.

Finally this Chapter also mentions a number of possible avenues for research leading beyond this Thesis in cosmology, particle physics and condensed matter physics.

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6.1 A zoo of topological defects ?

6.1.1 Dynamical and topological sectors

The construction of magnetic vortices within the A-field picture, and their equivalent dielectric monopoles within the B-field picture, which was described in Chapter 5 for a specific number of spacetime dimensions, is generalisable to any dimension for a large range of TMDGT. Indeed the action for topologically massive dielectric gauge theories in any dimension introduced in (4.29),

$$\begin{aligned} S_{\text{TMG}\varrho\varpi} &= \int_{\mathcal{M}} \frac{\sigma^p}{2} \frac{1}{e^2(\varpi)} F \wedge *F + \frac{\sigma^{d-p}}{2} \frac{1}{g^2(\varrho)} H \wedge *H \\ &+ S_{BF}[A,B] + \mathcal{S}_{\varpi}[\varpi] + \mathcal{S}_{\varrho}[\varrho] \ , \end{aligned} \tag{6.1}$$

should possibly admit generalised topological defects provided that at least one of the two scalar fields, $\varrho(x)$ or $\varpi(x)$, is allowed to vanish on a spacelike submanifold. The worldsheet associated to the topological defect is defined as the trajectory of the center of the core of these defects. It may also be defined as the submanifold where a dielectric scalar field vanishes, by analogy with the (dual) Maxwell-Higgs model where vortices are related to zeros of the Higgs field. Then a large range of extended topological defect solutions may emerge from (6.1) with complementary physical interpretations whether one works within the A- or the B-field picture.

As usual, our TP factorisation enables to isolate the physical and the topological sectors of these novel topological defects of which the range of solutions should be restricted by the stringent condition of finite energy. In contradistinction with the topologically trivial case addressed in Section 4.2, the dual factorised action should be only partially decoupled if one allows for possible topological defect solutions and reads

$$S_{\text{par}}[A,B,E,G,\varrho,\varpi] = S_{\text{dyn}}[E,G,\varrho,\varpi] + S_{\text{top}}[A,B,E,G] \,,$$

up to a physically irrelevant surface term for boundary conditions associated to topological defect configurations.

The physical sector $S_{\text{dyn}}[E,G]$ does not involve any topological content associated to topological defect solutions and therefore may be written again as in (4.31),

$$\begin{aligned} S_{\text{dyn}} &= \frac{\zeta^2}{2} e^2(\varpi)\ (E)^2 + \frac{\eta^2}{2} g^2(\varrho)\ (G)^2 + \mathcal{S}_{\varpi}[\varpi] + \mathcal{S}_{\varrho}[\varrho] \\ &+ \frac{1}{\kappa} \int_{\mathcal{M}} \sigma^{d-p}\, \xi\, E \wedge \mathrm{d}G - (1-\xi)\, \mathrm{d}E \wedge G \,. \end{aligned}$$

According to the usual properties of topological defects, the space of physically inequivalent field configurations is the moduli space, that is the space of classical soliton solutions modulo gauge transformations (connected to identity). However in our approach, the topological defects are constructed from a sector of already gauge invariant variables. So any soliton solution is physical per se and the space of topologically inequivalent solutions is defined in an one-to-one manner.

The second sector $S_{\text{top}}[A, B, E, G]$ of the form

$$\begin{aligned} S_{\text{top}} &= \frac{1}{\kappa} \int_{\mathcal{M}} (1-\xi)\, \mathrm{d}(\kappa\, A - \sigma^{p(d-p)}\, G) \wedge (\kappa\, B + E) \\ &- \sigma^p\, \xi\, (\kappa\, A - \sigma^{p(d-p)}\, G) \wedge \mathrm{d}(\kappa\, B + E)\,, \end{aligned}$$

is identified as being topological not only because the fields are topologically coupled but also because this sector generates the generalised “London” equations in any dimension. By analogy with the Nielsen-Olesen vortex, solutions in 2+1 dimensions the local form of these equations is trivial while their global form embraces all the topological content which characterizes a given topological defect field configuration. By virtue of Stokes’ theorem, these global equations are of the form

$$\Omega \oint_{\mathcal{C}_p} \kappa\, A - \sigma^{p(d-p)}\, G = 2^p \pi\, L[\mathcal{C}_p]\,, \quad \tilde{\Omega} \oint_{\mathcal{C}_{d-p}} \kappa\, B + E = 2^{d-p} \pi\, \tilde{L}[\mathcal{C}_{d-p}]\,,$$

where $\mathcal{C}_p$ and $\mathcal{C}_{d-p}$ are boundaries of spacelike submanifolds of dimension p+1 and d-p+1 respectively, while $\tilde{L}[\mathcal{C}_p]$ and $\tilde{L}[\mathcal{C}_{d-p}]$ are generalised vorticities. The fact that the homotopy groups $\Pi_p(\mathbb{R}^{d+1} \setminus \Gamma)$ and $\Pi_{d-p}(\mathbb{R}^{d+1} \setminus \tilde{\Gamma})$ must be non trivial constrains the dimension of the worldsheets Γ and $\tilde{\Gamma}$ associated to each topological defect solution of respective vorticity L and $\tilde{L}$.

6.1.2 Some comments about our terminology

Let us specify the terminology we use for abelian monopole and vortex solutions, in order to avoid any confusion. A monopole will refer in our convention to a point topological defect in space, hence a worldline in spacetime, carrying electric or magnetic charge. The generated electric vector field (resp. magnetic tensor field of rank $(d-2)$) is radial in space and is associated to an 1-form (resp. $(d-2)$-form) gauge field. More generally, an electric pole associated to p-form gauge field is a topological defect of dimension $(p-1)$ in the space manifold, whatever the number of spacetime dimensions. We will use the terminology of electric “monopole of rank p” as usual, or p-brane topological defect of the monopole type. For the same p-form gauge field in a spacetime of dimension $d+1$, the magnetic pole is also an extended defect of which the associated worldsheet is of dimension $(d-p-1)$.

A vortex will be defined as a topological defect of which the magnetic (resp. electric) field is generated by a vector current defined on a spacelike two dimensional surface in the spacetime manifold of dimension $d+1$. This topological defect thus extents along a $(d-2)$-dimensional surface, and so a worldsheet of dimension $d-1$. The generated magnetic (electric) field is transverse to the plane along which the planar current lies and is associated to a 1-form ($(d-2)$-form) gauge field. More generally, a magnetic vortex associated to p-form gauge field is a topological defect of dimension $(d-p-1)$ in the space manifold of dimension d. We will use the terminology of p-vortex, or $(d-p-1)$-brane topological defect of vortex type. For the same p-form gauge field in a spacetime of dimensions $d+1$, the electric vortex is also an extended defect of which the worldsheet is dual to that of the magnetic vortex. Finally, an instanton will be defined as a defect of which the worldhistory reduces to a point.

In the specific case of TMDGT of which the action (6.1) involves a 1-form field $A(x)$ and its associated ($d-1$)-form field $B(x)$, the following table is obtained:

	Extension in 2 dim. space	**Extension in 3 dim. space**	**Extension in d dim. space**
Electric monopole	point	point	point
Electric vortex		line	line
Magnetic vortex	point	line	($d-2$)-hypersurface
Magnetic monopole	Instanton	point	($d-3$)-hyperline

In particular, the electric monopoles are generated by point electric charges while the magnetic vortices are generated by a current of point electric charges whatever the number of spacetime dimensions.

6.1.3 Perspectives

One might expect the emergence of a zoo of brane-type topological defects from the ready extension to any dimension and to forms of any rank of topologically massive dielectric gauge theories, with a convenient form for the self-interacting scalar potentials. If they exist, these brane-type defects have a different origin than those arising in compact QED and their range should be limited by the stringent condition of a finite energy density. The coupling to gravity would also be an interesting avenue to explore in order to construct gravitational brane-type defects in theories with extra dimensions. Then the formation of these brane-type topological defects involves fields which naturally arise in processes of dimensional reductions (and effective string theories), namely a real scalar field (dilaton, radion,etc.) and real-valued tensor fields.

6.2 Monopoles and vortices in 3+1 dimensions

6.2.1 The situation in 3+1 dimensions

Let us consider the most general dielectric extension of the Cremmer-Scherk theory,

$$\begin{aligned}\mathcal{L}^4_{\mathrm{TMG}\varrho\varpi} &= -\frac{1}{4}\frac{1}{e^2(\varpi)}\,F_{\mu\nu}\,F^{\mu\nu}+\frac{1}{12}\frac{1}{g^2(\varrho)}\,H_{\mu\nu\rho}\,H^{\mu\nu\rho}+\mathcal{L}_\varpi+\mathcal{L}_\varrho\\ &+ \kappa\,\epsilon^{\mu\nu\rho\sigma}\left(\frac{\xi}{6}A_\mu\,H_{\nu\rho\sigma}+\frac{1-\xi}{4}F_{\mu\nu}\,B_{\rho\sigma}\right)\,,\end{aligned}$$

where the general action (6.1) is restricted to the specific case $p=1$ in 3+1 dimensions. As usual, $\mathcal{L}_\varrho(\varrho,\partial_\mu\varrho)$ and $\mathcal{L}_\varpi(\varpi,\partial_\mu\varpi)$ are the decoupled Lagrangian densities for the dynamical scalar fields $\varrho(x)$ and $\varpi(x)$ of respective potentials $V(\varrho^2)$ and $\tilde{V}(\varpi^2)$ and arbitrary dielectric functions $g^2(\varrho)$ and $e^2(\varpi)$. Our TP factorisation obviously applies and the resulting partially decoupled Lagrangian density reads

$$\begin{aligned}\mathcal{L}^4_{\mathrm{fac}} &= -\frac{1}{4}\,e^2(\varpi)\,E_{\mu\nu}\,E^{\mu\nu}+\frac{1}{2}\,g^2(\varrho)\,G_\mu\,G^\mu+\mathcal{L}_\varpi+\mathcal{L}_\varrho \\ &+ \frac{1}{2\,\kappa}\epsilon^{\mu\nu\rho\sigma}\left(\xi\,\partial_\mu G_\nu\,E_{\rho\sigma}-(1-\xi)\,\partial_\mu E_{\nu\rho}\,G_\sigma\right)\\ &+ \xi\,\frac{\kappa}{2}\epsilon^{\mu\nu\rho\sigma}\left(A_\mu-\frac{1}{\kappa}G_\mu\right)\partial_\nu\left(B_{\rho\sigma}+\frac{1}{\kappa}E_{\rho\sigma}\right)\\ &+ (1-\xi)\,\frac{\kappa}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A_\nu-\frac{1}{\kappa}G_\nu\right)\left(B_{\rho\sigma}+\frac{1}{\kappa}E_{\rho\sigma}\right)+\mathrm{ST}\,,\end{aligned} \tag{6.2}$$

taking into account the possible non trivial topological effects associated to zeros of the scalar fields.

The local sector

The first two lines of the factorised Lagrangian density (6.2) account for the local dynamical sector expressed in terms of the physical propagating variables. The local Euler-Lagrange equations resulting from this sector read

$$\begin{aligned}\kappa\,g^2(\varrho)\,\eta^{\mu\alpha}\,G_\alpha &= -\frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu E_{\rho\sigma}\\ \kappa\,e^2(\varpi)\,\eta^{\mu\alpha}\,\eta^{\nu\beta}\,E_{\alpha\beta} &= \varepsilon^{\mu\nu\rho\sigma}\,\partial_\rho G_\sigma\end{aligned} \tag{6.3}$$

This local sector is not affected by the possible existence of a subset of spacetime where the scalar fields $\varrho(x)$ and/or $\varpi(x)$ vanish. The dynamics of the scalar field

$\varpi(x)$ is described through the equation

$$\Box\varpi = -\frac{1}{2}\, e(\varpi)\, e'(\varpi)\, E_{\mu\nu}\, E^{\mu\nu} - \tilde{V}'(\varpi^2)\,, \tag{6.4}$$

and likewise for the dynamics of the other scalar field $\varrho(x)$. Notice that we have also defined in the above equation

$$e'(\varpi) = \frac{\mathrm{d}e(\varpi)}{\mathrm{d}\varpi}\,, \qquad \tilde{V}'(\varpi^2) = \frac{\mathrm{d}V(\varpi^2)}{\mathrm{d}\varpi}\,.$$

A similar convention is used to express the derivative with respect of $\varrho(x)$.

The Global sector

As shown in the previous Section, the topological sector generates two global equations of motion, one relating the global part of $A_\mu(x)$ and $G_\mu(x)$

$$\Omega \oint_{\mathcal{C}_1} (\kappa\, A_\mu - G_\mu)\; \mathrm{d}u^\mu = 2\pi\, L[\mathcal{C}_1] \tag{6.5}$$

in terms of the vorticity $L[\mathcal{C}_1]$, and the other relating the global part of $B_{\mu\nu}(x)$ and $E_{\mu\nu}(x)$

$$\frac{\tilde{\Omega}}{2} \oint_{\mathcal{C}_2} (\kappa\, B_{\mu\nu} + E_{\mu\nu})\; \mathrm{d}u^\mu \wedge \mathrm{d}u^\nu = 4\pi\, \tilde{L}[\mathcal{C}_2] \tag{6.6}$$

in terms of the vorticity $\tilde{L}[\mathcal{C}_2]$. As far as these two global equations are concerned, let us assume now that the curve integrals are defined along spacelike contours $\mathcal{C}_q$ of dimension $q = 1, 2$ and that we focus on single topological defects at the origin. Then the vorticity is a non trivial topological invariant if the spacetime manifold on which the associated physical field of rank q is regular is of the same homotopy as $S^q{\times}\mathbb{R}^{4-q-1}$. This condition of topological origin constrains the dimensional extension of topological defects of vorticity $L[\mathcal{C}_1]$ or $\tilde{L}[\mathcal{C}_2]$, thus the dimension of the worldsheet on which the scalar field $\varrho(x)$ or $\varpi(x)$ respectively vanishes. Hence topological defect solutions obeying the first global equation (6.5) must be string-like since

$$\Pi_1\left(\mathbb{R}^4 \setminus \Gamma_2\right) \equiv \Pi_1\left(S^1 \times \mathbb{R}^2\right) = \mathbb{Z}\,,$$

where $\mathbb{R}^4 \setminus \Gamma_2$ is the submanifold on which $G_i(x)$ is regular and Γ_2 is the associated worldsheet of dimension two. Likewise topological defect solutions obeying the second global equation (6.6) must be point-like since

$$\Pi_2\left(\mathbb{R}^4 \setminus \tilde{\Gamma}_1\right) \equiv \Pi_2\left(S^2 \times \mathbb{R}\right) = \mathbb{Z}\,,$$

where $\mathbb{R}^4 \setminus \Gamma_1$ is the submanifold on which $E_{ij}(x)$ is regular and $\tilde{\Gamma}_1$ is the associated worldline.

Topological defects in 3+1 dimensions

As is well known, vortex (or monopole) field configurations sufficiently distant in comparison to the size of their core may be thought as a set of interacting Newtonian particles. However, the question of the multi-soliton solutions, their interaction and dynamics will not be addressed at all. The purpose of the present Chapter will be reduced to a preliminary classification of our novel solutions in 3+1 dimensions. Within the context of this Thesis only dielectric monopoles and magnetic vortices will be considered, as shown in the following table where the equivalence relation between the two possible pictures is also indicated:

	A**-field picture**	$\Longleftrightarrow$	B**-field picture**
$\varrho(x) = \langle\varrho\rangle\,,\,\forall x$	Electric monopole	**point**	Magnetic vortex
$\varpi(x) = \langle\varpi\rangle\,,\,\forall x$	Magnetic vortex	**line**	Electric monopole

We will assume that one of the two scalar fields is frozen to its vacuum expectation value. Of course other types of topological defect solutions might exist without this assumption but will not be addressed. For the first time, a possible generalisation of vortex solutions for 2-form fields $B_{\mu\nu}(x)$ will be considered under the spherical symmetry *ansatz*. These topological defects may be interpreted in an equivalent way as electric monopoles embedded in a dielectric medium, of which the polarisability function is related to the scalar field $\varpi(x)$. Under the cylindrical symmetry *ansatz*, the usual vortex string solutions are also recovered, in a generalisation of those that were constructed in the plane in Chapter 5.

6.2.2 Topological defects within spherical symmetry

Let us now consider in particular the point-like topological defect solutions satisfying the global equation (6.6). For the sake of simplicity we will assume that the scalar field $\varrho(x)$ is frozen to its vacuum expectation value. This allows us to denote

$$g(\varrho) \equiv g\left(\langle\varrho\rangle\right) = g\,. \qquad (6.7)$$

Consequently the topological sector which generates the other global equation (6.5) is trivial. Only one type of topological defects may be constructed from the Lagrangian density (6.2) which now reduces to

$$\begin{aligned}
\mathcal{L}^4_{\rm fac} \;=\; & -\frac{1}{4}\,e^2(\varpi)\,E_{\mu\nu}\,E^{\mu\nu} + \frac{g^2}{2}\,G_\mu\,G^\mu + \mathcal{L}_\varpi \qquad (6.8)\\
& + \frac{1}{2\,\kappa}\epsilon^{\mu\nu\rho\sigma}\left(\xi\,\partial_\mu G_\nu\,E_{\rho\sigma} - (1-\xi)\,\partial_\mu E_{\nu\rho}\,G_\sigma\right)\\
& + \frac{\xi}{2}\epsilon^{\mu\nu\rho\sigma}\,\mathcal{A}_\mu\,\partial_\nu\left(\kappa\,B_{\rho\sigma} + E_{\rho\sigma}\right) + \frac{1-\xi}{4}\,\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\,\left(\kappa\,B_{\rho\sigma} + E_{\rho\sigma}\right)\,.
\end{aligned}$$

Static field configurations

We are presently interested in the existence of classical soliton solutions to the equations of motion (6.3), (6.4) and (6.6) under the further condition (6.7). These static field configurations are thus solution to two sets of partially decoupled first order equations for a given spacelike sheet. The first set relates $G_0(x)$ and $E_{ij}(x)$

$$\begin{aligned} \kappa\, e^2(\varpi)\, \delta^{ik}\, \delta^{jl}\, E_{ij} &= -\epsilon^{klm}\, \partial_m G_0 \\ \kappa\, g^2\, G_0 &= -\frac{1}{2}\, \epsilon^{ijk}\, \partial_i E_{jk}\,, \end{aligned}$$

while the second set relates $E_{0i}(x)$ and $G_i(x)$

$$\begin{aligned} \kappa\, g^2\, G_i\, \delta^{ij} &= \epsilon^{jkl}\, \partial_k E_{0l} \\ \kappa\, e^2(\varpi)\, E_{0i}\, \delta^{ij} &= -\epsilon^{jkl}\, \partial_k G_l\,. \end{aligned}$$

These two pairs of equations are coupled through the scalar field $\varpi(x)$ of which the second order static equation reads

$$\nabla^2 \varpi = e(\varpi)\, e'(\varpi)\, \left(\frac{1}{2}\, (E_{ij})^2 - (E_{0i})^2 \right) + \tilde{V}'(\varpi^2)\,. \tag{6.9}$$

Finally the global equation (6.6) accounts for the non trivial topological content characteristic of the present soliton solutions.

Spherical symmetry *ansatz*

For the sake of simplicity our analysis is restricted to topological defects all of whose positions coincide with the origin. Hence convenient coordinates to describe our new point-like topological defect solutions are the spherical ones since the associated physical field configurations possess the spherical symmetry. According to our conventions, the spherical coordinates are defined as

$$\begin{aligned} x &= r\, \cos\varphi\, \sin\vartheta\,, \\ y &= r\, \sin\varphi\, \sin\vartheta\,, \\ z &= r\, \cos\vartheta\,, \end{aligned}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the radius, φ is the azimuthal angle, $\varphi \in [0, 2\pi[$, while ϑ is the polar angle, $\vartheta \in [0, \pi[$. The covariant space components of any 1-form

$A = A_i(\vec{x})\,\mathrm{d}x^i$ restricted on $\Omega^1(\mathbb{R}^3)$ transform under spherical coordinates as

$$
\begin{aligned}
r\,A_x &= r\,\cos\varphi\,\sin\vartheta\,A_r - \frac{\sin\varphi}{\sin\vartheta}\,A_\varphi + \cos\varphi\,\cos\vartheta\,A_\vartheta\,,\\
r\,A_y &= r\,\sin\varphi\,\sin\vartheta\,A_r + \frac{\cos\varphi}{\sin\vartheta}\,A_\varphi + \sin\varphi\,\cos\vartheta\,A_\vartheta\,,\\
r\,A_z &= r\,\cos\vartheta\,A_r - \sin\vartheta\,A_\vartheta\,,
\end{aligned}
$$

while any 2-form $B = \frac{1}{2}B_{ij}(\vec{x})\,\mathrm{d}x^i\wedge\mathrm{d}x^j$ restricted on $\Omega^2(\mathbb{R}^3)$ transform as

$$
\begin{aligned}
r^2\,B_{yz} &= -r\,\cos\varphi\,\cot\vartheta\,B_{r\varphi} - \cos\varphi\,B_{\varphi\vartheta} - r\,\sin\varphi\,B_{r\vartheta}\,,\\
r^2\,B_{zx} &= -r\,\sin\varphi\,\cot\vartheta\,B_{r\varphi} - \sin\varphi\,B_{\varphi\vartheta} + r\,\cos\varphi\,B_{r\vartheta}\,,\\
r^2\,B_{xy} &= r\,B_{r\varphi} - \cot\vartheta\,B_{\varphi\vartheta}\,.
\end{aligned}
$$

Finally, by analogy with the case in 2+1 dimensions developed in Chapter 5, the topological defect solutions possess two possible physical interpretations whether within the A- or the B-field picture. Each of the two pictures requires specific *ansätze* which turn out to be equivalent on account of the equations of motion.

A-field picture: Dielectric monopoles

On the grounds of an analysis similar to that of Section 4.1, the Lagrangian density (6.8) within the A-field picture ($\xi = 1$) is very reminiscent of a dielectric Maxwell theory for the gauge field $A_\mu(x)$, see Section 1.4. At this stage, it turns out to be useful to introduce a further *ansatz* which cancels the magnetic field

$$E_{0i} = \delta_{ij}\,(H^A_{\mathrm{mg}})^j = 0\,,$$

in order to restrict to purely electric topological defect solutions. These point-like solutions describe a free electric charge density $\rho_{\mathrm{f}}(x)$ embedded in a dielectric medium of dielectric function $\varepsilon(x) = e^{-2}(\varpi)$ to which is associated the dielectric displacement $\vec{D}^A_{\mathrm{el}}(x)$

$$(D^A_{\mathrm{el}})^i = \frac{1}{2}\epsilon^{ijk}E_{jk}\,. \tag{6.10}$$

On account of the spherical symmetry *ansatz*, the electric displacement is required to be radial and to only depend on the radial coordinate. Hence we require a kind of hedgehog *ansatz* for this vector field

$$(D^A_{\mathrm{el}})^i = D\,\frac{x^i}{r}\,,\quad \text{where } (D^A_{\mathrm{el}})^r(r) = D(r)\,.$$

Given the relation (6.10), this implies that the physical variable $E_{ij}(x)$ is of the form

$$E_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j = \frac{1}{2}\,\sin\vartheta\,\tilde{E}\,\partial_{[i}\,\varphi\,\partial_{j]}\vartheta\,\mathrm{d}x^i \wedge \mathrm{d}x^j = \frac{1}{2}\,E_{\varphi\vartheta}\,\mathrm{d}\varphi \wedge \mathrm{d}\vartheta\,.$$

where the variable $\tilde{E}(r)$ only depends on the radius,

$$E_{\varphi\vartheta}(r,\vartheta) = -\sin\vartheta\,\tilde{E}(r)\,. \tag{6.11}$$

Combining all the contributions of the above *ansätze*, the static equations of motion reduce to

$$\begin{aligned} r^2\,\partial_r G_0 &= \kappa\,e^2(\varpi)\,\tilde{E}\,, \\ \partial_r\tilde{E} &= \kappa\,g^2\,r^2\,G_0\,. \end{aligned} \tag{6.12}$$

Therefore the time component of $G_\mu(x)$ must depend only on the radius, while its space components G_r, G_φ and G_ϑ vanish identically. Similarly to the case in 2+1 dimensions, the variable $G_0(r)$ may be associated to the free charge density

$$\varrho_{\mathrm{f}} = \kappa\,g^2\,G_0\,.$$

Finally, the local equation (6.9) now writes

$$\frac{1}{r^2}\,\partial_r\left(r^2\,\partial_r\varpi\right) = e(\varpi)\,e'(\varpi)\,\frac{1}{r^4}\,\tilde{E}^2 + \tilde{V}'(\varpi^2)\,, \tag{6.13}$$

and determines the radial profile of $\varpi(r)$.

These local equations must be completed through the global equation (6.6) which now becomes

$$\kappa\,\tilde{\Omega}\,\tilde{\Phi}[\mathcal{C}_2] - \tilde{\Omega}\iint\tilde{E}\,\sin\vartheta\,\mathrm{d}\varphi\,\mathrm{d}\vartheta = 4\pi\tilde{L}[\mathcal{C}_2]\,, \tag{6.14}$$

for a spherical contour $\mathcal{C}_2$ centered around the origin. In the above equation we have defined the flux,

$$\tilde{\Phi}[\mathcal{C}_2] = \oint_{\mathcal{C}_2} B\,, \tag{6.15}$$

which is related to the free electric charge enclosed by $\mathcal{C}_2$,

$$Q_{\mathrm{f}}[\mathcal{C}_2] = \kappa\,\tilde{\Phi}[\mathcal{C}_2]\,,$$

of the dielectric monopole field configuration within the A-field picture.

B-field picture: Magnetic 2-vortices

The physical sector of the Lagrangian density (6.8) within the B-field picture ($\xi = 0$) may be considered as the generalisation of the physical formulation of the Maxwell-Higgs model, see (1.48), to 2-form gauge fields in 3+1 dimensions. The further *ansatz* introduced restricts topological defect solutions within the B-field picture to purely magnetic ones, with the magnetic field depending only on the radius on account of the spherical symmetry *ansatz*,

$$\begin{aligned} \text{Zero electric field} \quad &: \quad (E^B_{\rm el})^{ij} = e^2\, \epsilon^{ijk}\, G_k = 0\,, \\ r\text{-dependent magnetic field} \quad &: \quad B^B_{\rm mg}(r) = e^2\, G_0(r)\,. \end{aligned}$$

According to the equations of motion, this *ansatz* implies that all the components of $E_{\mu\nu}(x)$ in spherical coordinates vanish identically, except for $E_{\varphi\vartheta}(r,\vartheta)$ which must be of the form (6.11).

The point-like topological defect solutions which obey the equations (6.12), (6.13) and (6.14) may be considered as a generalisation to 2-form gauge fields of the Nielsen-Olesen magnetic vortices. In particular (6.14) is seen to correspond to a generalisation of the global London equation for the effective theory of type II superconductivity. The magnetic field of a 2-form gauge field in 3+1 dimensions is a scalar in the same way that the magnetic field for a 1-form gauge field is a scalar in 2+1 dimensions. Therefore, in comparison with the case in 2+1 dimensions, the functional $\Phi[\mathcal{C}_2]$ defined in (6.15) may be interpreted as the transverse flux of the magnetic field through the hypersurface of dimension three embedded in a space of dimension four.

6.2.3 Towards magnetic 2-vortex and dielectric monopoles

On the assumption that the scalar field $\varpi(r)$ be an analytic function which vanishes at the origin of $\mathbb{R}^3$, $\varpi(r)$ may be written as a series expansion in powers of r of the form

$$\varpi(r) = r^N \sum_{n=1}^{\infty} \chi_{(n)}\, r^n\,.$$

Let us recall that the positive integer N is called the order of the zero of $\varpi(r)$. The assumption that the functional $\tilde{\Phi}[\mathcal{C}_2]$ vanishes identically for any contour shrunk to a point, even though this point may be the origin,

$$\lim_{\mathcal{C}_2\to 0} \frac{1}{\kappa} \oint_{\mathcal{C}_2} B_{\varphi\vartheta}\, \mathrm{d}\varphi\, \mathrm{d}\vartheta = \lim_{\mathcal{C}_2\to 0} \Phi\,[\mathcal{C}_2] = 0\,,$$

is considered as a further global boundary condition. It ensures that the magnetic field within the B-field picture or the free charge density within the A-field picture are not singular at the location of topological defects.

On the grounds of the Hamiltonian density[1] (4.23), one may readily infer the following necessary condition for a finite energy functional

$$e^2(\varpi)\,(E_{ij})^2 < \infty\,, \qquad \text{for } x \to 0\,, \tag{6.16}$$

which given the spherical symmetry *ansatz* reduces to

$$e^2(\varpi)\,\frac{1}{r^4}\,\tilde{E}^2 < \infty\,, \qquad \text{for } r \to 0\,.$$

This condition constrains the first term of the expansion of $\tilde{E}(r)$ in powers of r close to the origin, according to the form of the dielectric function $e(\varpi)$. However the variable $\tilde{E}(r)$ is also constrained by a condition of finiteness of the flux:

$$\lim_{r\to 0}\tilde{\Omega}\iint \tilde{E}\,\sin\vartheta\,\mathrm{d}\varphi\,\mathrm{d}\vartheta = -4\pi\tilde{L}\,.$$

These two conditions put together imply that the power of $\varpi(r)$ in the dielectric function $e(\varpi)$ may depend on the order N and must be at least of order four if $N=1$. The analysis of the behaviour of the fields close to the origin corroborates this conclusion.

Let us presently restrict our general discussion to the case $N = 1$ and define the dielectric function $e(\varpi)$ to be of the form

$$e(\varpi) = \zeta\,\varpi^2\,, \tag{6.17}$$

where ζ is a constant parameter of physical dimension $E^{-\frac{3}{2}}\,L^{\frac{1}{2}}$. As far as the potential $\tilde{V}(\varpi)$ is concerned one may of course consider the usual "Mexican hat-shaped" quadratic potential defined in (4.12). However the form of the dielectric function has direct influence on the formulation on the potential. Indeed, it may be proved that a BPS limit exists for the energy functional with a dielectric function of the form (6.17) for a sixth order potential. We hope to obtain in a close future numerical solutions for these new topological defect solutions which already manifest regular behaviours asymptotically and close to the origin. In the case $N=1$ the existence of topological defect solutions of one quantum of magnetic flux within the B-field picture or one quantum of free electric charge within the A-field picture is expected.

[1] In this Hamiltonian density introduced in Chapter 4, the scalar field $\varrho(x)$ is dynamical while $\varpi(x)$ is frozen to its vacuum expectation value.

6.2.4 Perspectives

The abelian Higgs models offer a common formalism for diverse phenomena where topological defects arise in different areas like condensed matter physics, particle physics, cosmology, etc. In the same way the dual formulation of topological defects might be added to the exhaustive list of synergies between very distinct fields in physics through a common mathematical formulation. The Cremmer-Scherk theory defined in (1.24) offers a possible description of effective superconductivity [75, 74], Josephson arrays [76] for compact gauge groups, confinement by flux tube [77, 78], etc. The same theory was also used as a dual formulation of cosmic strings [79]. We have properly established that the formulation of all these theories as topological mass generation mechanisms is not a happy coincidence. Although seemingly somewhat unrelated at first sight, these theories share in common an original description as Landau-Ginzburg models or their Lorentz covariant Maxwell-Higgs extensions, admitting vortex solutions of quantised magnetic flux. Likewise, within the A-field picture, the dielectric Cremmer-Scherk (DCS) theory defined in (4.1) admits topological defects dual to these vortices. The formation of vortices within this formulation is not associated to phase transitions nor to any local symmetry breaking. Within the London limit the pure Cremmer-Scherk theory generates fluxon vortex strings with a non trivial topological sector inherited from the associated dielectric theory.

As far as cosmic strings are concerned, the great advantage of the formulation in terms of (DCS) theories is the presence of a real scalar field and an antisymmetric 2-tensor field instead of the less natural complex scalar field of the Landau-Ginzburg model. For what concerns the possible applications in condensed matter physics, nothing forbids to describe the formation of magnetic vortices in type II superconductors in terms of our effective DCS theory. The interesting point here is the different origin of the topological content in comparison with the compact Cremmer-Scherk (CCS) theory introduced in [75, 76]. Within the DCS formulation, we have established that vorticity is related to the global London equation generated by a topological sector of the action. We recover the usual picture of the order in terms of condensation of Cooper pairs. In the second formulation, the quantisation of the magnetic flux results from the compact character of the gauge group and the order is referred to as being "topological". An analysis of the equivalence between the London limit of the DCS theory and the compact CCS might offer new insight in condensed matter physics.

Within this context, the TP factorisation stands for a generalisation, already at the classical level, of the Landau projection which enables to isolate the essential topological content of these models. The denomination "topological sector" has to be understood not only in terms of topological couplings, but in a more general context as being the sector where all non trivial topological effects arise. This topological sector then corresponds to the topological "ground" state, leading to a deeper understanding of these models and opening an avenue towards new perturbative developments around classical non perturbative configurations.

Finally let us briefly mention the perspective offered by the second possible formulation (6.8) of the DCS theory and its associated topological defect solutions. One might expect applications of our dielectric monopole solutions of quantised free charge in solid state physics. Within this context, the Lagrangian density (6.8) admits particle-like topological defects describing a free electric charge screened by surrounding positive charges. Then the resulting measured electric field decays exponentially. Likewise within the B-field picture, the possible construction of point-like topological defect solutions, generalising to 2-form gauge fields the Nielsen-Olesen vortices, might be relevant in any model involving antisymmetric tensor fields.

6.3 Mass gap in non abelian gauge field theories

6.3.1 Confinement in QCD

For the sake of simplicity we have chosen first to illustrate our systematic search of topological effects in gauge field theories within topologically massive (dielectric) gauge theories. This choice was quite natural since these theories represent a simple case of theories generating a mass gap where topological effects are of prime importance. Having acquired a deeper understanding of the origin of topological effects, it would be interesting to consider more realistic theories towards a description of confinement in Yang-Mills theories. Two extensions of our work then naturally come to mind : the coupling to matter fields like fermions and the extension to non abelian gauge theories. Further, through the network of dualities we have identified, our work also bears some relations with effective theories of confinement.

New perturbation schemes

The standard perturbation techniques, so useful at high energy where QCD exhibits the “property” of asymptotic freedom, are expressed in terms of unphysical and gauge variant colored objects like quarks and gluons. However the observed physical field configurations at low energy are colourless objects. Furthermore, these latter field configurations are by essence non perturbative since they reside in the strong coupling sector of the theory. This requires to develop new perturbation schemes but now from the real physical quantities which have to be identified.

Let us go back to our simple model of abelian topologically massive gauge theories (TMGT). Generically the introduction of matter fields in TMGT spoils the decoupling of the physical and topological sectors through our TP factorisation approach. Nevertheless the dual theory remains partially factorised in the case, for example, of minimal coupling with fields of fermionic character since the sector of physical propagating variables communicates with the gauge variant sector only through the dynamical matter fields, leading possibly to interesting approximation schemes. Such approximation procedures would consist in two steps.

- First to solve the topological sector coupled to matter fields in the ground state, taking into account the existence of topological effects like fractional statistics associated to non local holonomy effects.
- Second "turning on" the dynamics of the interactions, namely considering perturbatively the coupling of the matter fields to the physical sector.

Hence, such a procedure relies from the outset on a sector which is already gauge invariant, thus physical, and not on the original gauge field degrees of freedom.

Promising results have been already achieved when TMGT are coupled to matter fields. Indeed quite a number of new effective theories may be constructed depending on the way the matter fields are coupled either to the original fields or to the topological sector in the dual formulation. We again obtain a network of dualities which is more exhaustive than that analysed in the literature through gauge embedding procedures. Moreover in contradistinction with previous papers, all relevant topological terms generating topological effects are now taken into account. The construction of effective theories describing condensed matter effects makes these studies attractive. This analysis may also bring further insights into bosonisation processes in more than 2+1 dimensions.

Pure Yang Mills theory

If the TP factorisation techniques are to turn out to be applicable to large classes of gauge theories generating a mass gap, they may offer perspectives in the development of new approximation schemes for non perturbative dynamics. In particular, it would be of great interest to understand whether similar considerations could apply to pure Yang-Mills theories in order to isolate the low energy physical configurations of gluon bound states which reside in the zero-mode sector. As far as Yang-Mills theory is concerned, the global and local variables are interdependent and this non trivial mixing perhaps carries the secret of the dynamics of confinement. Hence, one might hope to isolate a topological sector of global variables which would correspond to the ground state of the theory, even though the self interaction of the gluon field spoils the decoupling of the physical and topological sectors through our TP factorisation approach.

Effective theories

In contradistinction to abelian theories of the Maxwell-type, the Lagrangian first order formulation of Yang-Mills theories does not involve actual propagating physical variables. Indeed the associated chromo-electromagnetic field belongs to the adjoint representation of the non abelian gauge group and is therefore not a relevant physical variable to describe the propagation of gluons. Furthermore, given Derrick's theorem, solitons are ruled out in Yang-Mills theory in 3+1 dimensions. However, a number of techniques like lattice simulations seem to show that the low energy confining states are controlled by topological defects of the monopole and vortex type.

A possible solution to this dilemma is to develop effective theories of QCD at low energy where it is assumed that the abelian sector dominates. For example an effective dual superconductivity model, originally obtained through dual projection method, may be reached through the Cho-Faddeev-Niemi parametrisation, see [5, 6] and references therein. In this model, a chromoelectric vortex string of linear potential appears between the quarks and antiquarks while magnetic monopoles condensates. On account of the preliminary results of this Chapter, a query which then naturally comes to mind is whether the dielectric Cremmer-Scherk (DCS) theory which admits vortex solutions may be of any relevance for a dual superconducting model. Does there exist a decomposition alternative to that of Cho-Faddeev-Niemi or to the recent spin-charge decomposition [8, 9] which reveals the presence of a DCS theory in the low energy effective theory of QCD ? Moreover, the generalisation of the DCS theory should admit a larger range of topological defects than the Maxwell-Higgs model. Mixing then topological defects of the vortex and the monopole types might lead in a long term goal to an alternative effective description of confinement.

6.3.2 Alternative mechanisms for mass generation

It is also of interest to extend this approach to Yang-Mills-Chern-Simons theories [38, 39, 40, 73] or to the nonabelian generalisation of the Cremmer-Scherk theory. However in 3+1 dimensions there exist no direct non abelian generalisation to a local, power counting renormalisable action while preserving the same field content and the same number of local symmetries as the abelian Cremmer-Scherk theory (1.24), see [37] and references therein. Indeed such extensions require the introduction of extra fields or to allow for non renormalisable couplings. Nevertheless it would be of great interest to understand how to apply our TP factorisation in this context in order to achieve a relevant generalisation to non abelian gauge symmetries of the Cremmer-Scherk theory, and to solve some current problems related to quantisation.

Concluding remarks

Among a number of fundamental concepts, the understanding of the origin of mass remains one of the major open problems in particle physics and cosmology. As a matter of fact difficulties arise both at the theoretical as well as at the phenomenological level. Thus the dynamics responsible for the mass gap in Yang-Mills theories is still under investigation while the predicted Higgs boson has yet to be discovered. While not providing definitive answers, this Thesis has shed new light on these issues through the Topological-Physical (TP) factorisation. Actually our original project was to properly identify in the simple case of abelian topologically massive gauge theories (TMGT), used as laboratories for mass gap generation, a topological sector which appears under naive formal limits within the Lagrangian first order formulation of such models.

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* *

Our solution: the TP factorisation

The basic principle of the Brout-Englert-Higgs mechanism is that the existence of massive fluctuations of vector fields in a variety of physical phenomena (such as in particle and condensed matter physics) follows from the breaking of a gauge symmetry. This breaking is triggered by the vacuum condensation of some scalar fields. This approach is successfully put to use within the Standard Model in spite of the lack of any direct experimental evidence for the existence of the Higgs boson until now. In Chapters 2 and 3 it was shown that for what concerns TMGT, the factorised physical massive degrees of freedom are simply decoupled through the rearrangement of field variables, without relying on any symmetry breaking nor any gauge fixing procedure. This technique, which we refer to as the Topological-Physical factorisation, enables us to answer the first question in the preamble:

How to properly define the formal limits in the coupling constants of gauge fields theories through which topological sectors appear ?

The TP factorisation technique, which consists in a local canonical transformation within the Hamiltonian formulation, was also extended in Chapter 2 in a manifestly Lorentz covariant way by considering the first order Lagrangian formulation. The possibility of factorisation is intimately related to the fact that TMGT generate a mass gap. The technique allows the construction of dual actions while preserving the gauge symmetry content. The entire gauge variant contributions resides only within the second sector of the action which reduces to a pure topological field theory. Let us then address the second issue raised in the preamble:

What is the influence of this topological sector on the physical spectrum ?

The appearance of this topological sector has very intriguing consequences when TMGT are defined on topologically non trivial manifolds or are coupled to matter fields, whether of a fermionic or a bosonic character, since non trivial topological effects then arise. As shown in Chapter 3 our classical factorisation readily allows for a straightforward quantisation of TMGT and the identification of their spectrum of physical states, accounting also for all the topological features inherent to such dynamics. In that case the topological “TFT” sector accounts for the degeneracy of the physical spectrum depending only on topological invariants. The resulting factorisation of quantum states makes more natural the projection onto a topological field theory, generalising the concept of projection onto the lowest Landau level within the Landau problem. In the present approach, the TFT sector is actually made manifest already at the classical level. We hardly touched upon the issue of the relevance of our factorisation techniques when TMGT are coupled to matter fields. Let us only point out within this context that the coupling to the higher order tensor fields in the topological sector should give rise to exotic statistics for extended objects [28, 29, 30].

The unexpected consequences

Generically the introduction of matter fields spoils the decoupling of the physical and topological sectors through our factorisarion approach. Nevertheless this decoupling survives at least for the local degrees of freedom when scalar dielectric couplings with the gauge fields are introduced. Within this context,

the most famous mass generation mechanisms preserving the abelian gauge symmetry are related through an intricate network of dualities, modulo the presence of topological terms generating possible topological effects.

In particular in the dual formulation of the Maxwell-Higgs model, the topological sector carries the topological content, related to vorticity, of the dual Nielsen-Olesen vortices, as established in Chapter 5. Within a physically equivalent picture, these topological defect solutions may also be interpreted as electric monopoles embedded in a dielectric medium. Furthermore, the straightforward generalisation of topologically massive dielectric gauge theories to fields of any tensorial rank whatever the number of spacetime dimensions suggests that

our analysis could open a new avenue towards the construction of generalised topological defects in any dimension.

The construction of such brane-type topological defects is currently under investigation. Promising results have already been achieved in Chapter 6 where the situation in 3+1 dimensions is partly addressed. This Chapter also mentions a number of possible avenues for research leading beyond this Thesis.

This work has provided a deeper understanding for the appearance and the role of topological effects in gauge field theories, although the discussion remained restricted to abelian gauge groups. This Thesis has also established the crucial importance of a systematic and careful consideration of topological effects which are often ignored in the literature through gauge (un)fixings, for example, which do not pay due attention to these issues. The long term goal is to gain a deeper understanding of the relevance of topological sectors to the nonperturbative dynamics of gauge field theories, ultimately such as Yang-Mills theories coupled fermionic matter fields.

BIBLIOGRAPHY

[1] M. Faraday, *Experimental Researches in Electricity*, vol. 1,2. 1844,1847.

[2] M. Faraday, *Experimental Researches in Electricity*, vol. 3. 1855.

[3] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, vol. 2 of *Clarendon Press, Oxford*. Macmillan and Co., 1873.

[4] F. Bruckmann, "Exploring the QCD vacuum - (some) recent developments in confinement and topology," in *Proceedings of the XXV International Symposium on Lattice Field Theory, July 30 - August 4, 2007, Regensburg, Germany*. arXiv:0710.2788 [hep-lat].

[5] L. D. Faddeev and A. J. Niemi, "Partially dual variables in SU(2) Yang-Mills theory," *Phys. Rev. Lett.* **82** (1999) 1624–1627, hep-th/9807069.

[6] L. D. Faddeev and A. J. Niemi, "Partial duality in SU(N) Yang-Mills theory," *Phys. Lett.* **B449** (1999) 214–218, hep-th/9812090.

[7] D. Antonov, "String representation and nonperturbative properties of gauge theories," *Surveys High Energ. Phys.* **14** (2000) 265–355, hep-th/9909209.

[8] L. D. Faddeev and A. J. Niemi, "Electric-magnetic duality in infrared SU(2) Yang-Mills theory," *Phys. Lett.* **B525** (2002) 195–200, hep-th/0101078.

[9] L. D. Faddeev and A. J. Niemi, "Spin-charge separation, conformal covariance and the SU(2) Yang-Mills theory," *Nucl. Phys.* **B776** (2007) 38–65, hep-th/0608111.

[10] B. Bertrand and J. Govaerts, "Gauge invariant factorisation and canonical quantisation of topologically massive gauge theories in any dimension," *J. Phys.* **A40** (2007) 9609–9634, `arXiv:0704.1512 [hep-th]`.

[11] B. Bertrand and J. Govaerts, "Topologically massive gauge theories and their dual factorised gauge invariant formulation," *J. Phys.* **A40** (2007) F979–F986, `arXiv:0705.3452 [hep-th]`.

[12] F. Payen, "Théories de Yang-Mills pures et théories des champs quantiques topologiques," Master's thesis, Université Catholique de Louvain, Belgium, September 2004. Unpublished.

[13] P. G. O. Freund and R. I. Nepomechie, "Unified geometry of antisymmetric tensor gauge fields and gravity," *Nucl. Phys.* **B199** (1982) 482.

[14] M. Henneaux and C. Teitelboim, "p-form electrodynamics," *Found. Phys.* **16** (1986) 593–617.

[15] C. Teitelboim, "Gauge invariance for extended objects," *Phys. Lett.* **B167** (1986) 63.

[16] M. Nakahara, *Geometry, Topology and Physics*. Graduate student series in physics. Adam Hilger, 1990. 505 p.

[17] J. Govaerts, "L'Interaction Electrofaible: une Fenêtre sur la Physique Au-delà du Modèle Standard," in *Proceedings of the "Ecole Internationale Joliot-Curie de Physique Nucléaire", Noyaux en Collisions, Maubuisson (France)*, 1995.

[18] H. Ruegg and M. Ruiz-Altaba, "The Stueckelberg field," *Int. J. Mod. Phys.* **A19** (2004) 3265–3348, `hep-th/0304245`.

[19] B. Kors and P. Nath, "Aspects of the Stueckelberg extension," *JHEP* **07** (2005) 069, `hep-ph/0503208`.

[20] D. Birmingham, M. Blau, M. Rakowski, and G. Thompson, "Topological field theory," *Phys. Rept.* **209** (1991) 129–340.

[21] A. S. Schwarz, "The partition function of degenerate quadratic functional and Ray-Singer invariants," *Lett. Math. Phys.* **2** (1978) 247–252.

[22] A. S. Schwarz, "The partition function of a degenerate functional," *Commun. Math. Phys.* **67** (1979) 1–16.

[23] E. Witten, "Supersymmetry and Morse theory," *J. Diff. Geom.* **17** (1982) 661–692.

[24] G. T. Horowitz, "Exactly soluble diffeomorphism invariant theories," *Commun. Math. Phys.* **125** (1989) 417.

[25] M. Blau and G. Thompson, "Topological gauge theories of antisymmetric tensor fields," *Ann. Phys.* **205** (1991) 130–172.

[26] E. Witten, "Quantum field theory and the Jones polynomial," *Commun. Math. Phys.* **121** (1989) 351.

[27] J. Barcelos-Neto and E. C. Marino, "A new sequence for topological terms at any spacetime dimensions," *Europhys. Lett.* **57** (2002) 473–479, `hep-th/0107109`.

[28] X. Fustero, R. Gambini, and A. Trias, "Polyakov's Fermi-Bose transmutation in (3+1)-dimensions," *Phys. Rev. Lett.* **62** (1989) 1964.

[29] J. A. Harvey and J. Liu, "Strings and statistics," *Phys. Lett.* **B240** (1990) 369–374.

[30] J. C. Baez, D. K. Wise, and A. S. Crans, "Exotic statistics for loops in 4d BF theory," `gr-qc/0603085`.

[31] M. Bergeron, G. W. Semenoff, and R. J. Szabo, "Canonical BF type topological field theory and fractional statistics of strings," *Nucl. Phys.* **B437** (1995) 695–722, `hep-th/9407020`.

[32] R. J. Szabo, "Topological field theory and quantum holonomy representations of motion groups," *Annals Phys.* **280** (2000) 163–208, `hep-th/9908051`.

[33] E. Cremmer and J. Scherk, "Spontaneous dynamical breaking of gauge symmetry in dual models," *Nucl. Phys.* **B72** (1974) 117–124.

[34] C. R. Hagen, "Action principle quantization of the antisymmetric tensor field," *Phys. Rev.* **D19** (1979) 2367.

[35] T. J. Allen, M. J. Bowick, and A. Lahiri, "Topological mass generation in (3+1)-dimensions," *Mod. Phys. Lett.* **A6** (1991) 559–572.

[36] R. Jackiw and S.-Y. Pi, "Seeking an even-parity mass term for 3-D gauge theory," *Phys. Lett.* **B403** (1997) 297–303, `hep-th/9703226`.

[37] M. Henneaux *et. al.*, "A no-go theorem for the nonabelian topological mass mechanism in four dimensions," *Phys. Lett.* **B410** (1997) 195–202, `hep-th/9707129`.

[38] R. Jackiw and S. Templeton, "How superrenormalizable interactions cure their infrared divergences," *Phys. Rev.* **D23** (1981) 2291.

[39] J. F. Schonfeld, “A mass term for three-dimensional gauge fields,” *Nucl. Phys.* **B185** (1981) 157.

[40] S. Deser, R. Jackiw, and S. Templeton, “Topologically massive gauge theories,” *Ann. Phys.* **140** (1982) 372–411. Erratum-ibid. 185, 406, (1988) and 281, 409, (2000).

[41] J. Govaerts, *Hamiltonian Quantisation and Constrained Dynamics*, vol. B4 of *Leuven notes in mathematical and theoretical physics*. Leuven University Press, 1991. 371 p.

[42] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*. Princeton University Press, 1992. 520 p.

[43] B. Bertrand, “Quantification de la théorie de Maxwell-Chern-Simons sur le tore,” Master’s thesis, Université Catholique de Louvain, Belgium, September 2004. Unpublished.

[44] J. Govaerts, “A note on the Fradkin-Vilkovisky theorem,” *Int. J. Mod. Phys.* **A4** (1989) 4487.

[45] J. Govaerts, “Half-integer winding number solutions to the Ginzburg-Landau-Higgs equations,” *J. Phys.* **A34** (2001) 8955–8993, `hep-th/0007112`.

[46] M. N. Chernodub, L. Faddeev, and A. J. Niemi, “Non-abelian Supercurrents and Electroweak Theory,” `arXiv:0804.1544 [hep-th]`.

[47] H. B. Nielsen and P. Olesen, “Vortex-line models for dual strings,” *Nucl. Phys.* **B61** (1973) 45–61.

[48] J.-L. Gervais and B. Sakita, “Quantized relativistic string as a strong coupling limit of the Higgs model,” *Nucl. Phys.* **B91** (1975) 301.

[49] P. Orland, “Extrinsic curvature dependence of Nielsen-Olesen strings,” *Nucl. Phys.* **B428** (1994) 221–232, `hep-th/9404140`.

[50] J. Govaerts, “Topological quantum field theory and pure Yang-Mills dynamics,” in *Proceedings of the Third International Workshop on Contemporary Problems in Mathematical Physics (2003), Cotonou (Republic of Benin)*, (Singapore), pp. 273–293, World Scientific, 2004. `hep-th/0408023`.

[51] S. E. Hjelmeland and U. Lindstrom, “Duality for the non-specialist,” `hep-th/9705122`.

[52] C. Wotzasek, “On the dimensional dependence of the electromagnetic duality groups,” *Phys. Rev.* **D58** (1998) 125026, `hep-th/9809136`.

[53] M. S. Guimaraes, L. Grigorio, and C. Wotzasek, "The dual of the Carroll-Field-Jackiw model," `hep-th/0609215`.

[54] R. Banerjee and S. Ghosh, "Canonical transformations and soldering," *Phys. Lett.* **B482** (2000) 302–308, `hep-th/0003092`.

[55] Y. I. Kogan and A. Y. Morozov, "Structure of (2 + 1) Photodynamics," *Sov. Phys. JETP* **61** (1985) 1–8.

[56] G. V. Dunne, R. Jackiw, and C. A. Trugenberger, "'Topological' (Chern-Simons) quantum mechanics," *Phys. Rev.* **D41** (1990) 661.

[57] S. Deser and R. Jackiw, ""Self-duality" of topologically massive gauge theories," *Phys. Lett.* **B139** (1984) 371.

[58] P. K. Townsend, K. Pilch, and P. van Nieuwenhuizen, "Self-duality in odd dimensions," *Phys. Lett.* **136B** (1984) 38.

[59] P. J. Arias and A. Restuccia, "Topological sectors of spin 1 theories in (2+1)-dimensions," *Phys. Lett.* **B347** (1995) 241–246, `hep-th/9410134`.

[60] J. L. Noronha and C. Wotzasek, "Testing quantum duality using cold Rydberg atoms," *Phys. Lett.* **B602** (2004) 144–148, `quant-ph/0308076`.

[61] R. Banerjee and H. J. Rothe, "Batalin-Fradkin-Tyutin embedding of a selfdual model and the Maxwell-Chern-Simons theory," *Nucl. Phys.* **B447** (1995) 183–192, `hep-th/9504066`.

[62] E. Harikumar and M. Sivakumar, "Topologically massive Abelian gauge theory from BFT Hamiltonian embedding of a first-order theory," *Nucl. Phys.* **B565** (2000) 385–396, `hep-th/9904183`.

[63] R. Banerjee, H. J. Rothe, and K. D. Rothe, "Hamiltonian embedding of self-dual model and equivalence with Maxwell-Chern-Simons theory," *Phys. Rev.* **D55** (1997) 6339–6343, `hep-th/9611077`.

[64] S.-K. Kim, Y.-W. Kim, and Y.-J. Park, "Complete BFT embedding of massive theory with one- and two- form gauge fields," *Mod. Phys. Lett.* **A18** (2003) 2287, `hep-th/0301096`.

[65] M. A. Anacleto, A. Ilha, J. R. S. Nascimento, R. F. Ribeiro, and C. Wotzasek, "Dual equivalence between self-dual and Maxwell-Chern-Simons models coupled to dynamical U(1) charged matter," *Phys. Lett.* **B504** (2001) 268–274, `hep-th/0104152`.

[66] R. Menezes, J. R. S. Nascimento, R. F. Ribeiro, and C. Wotzasek, "On the dual equivalence of the self-dual and topologically massive p-form models," *Phys. Lett.* **B564** (2003) 261–268, `hep-th/0305112`.

[67] M. Botta Cantcheff, "Topologically massive gauge theories from first order theories in arbitrary dimensions," *Phys. Lett.* **B533** (2002) 126–130, `hep-th/0110264`.

[68] P. Castelo Ferreira, "Ue(1)xUg(1) actions in (2+1) dimensions: Full vectorial electric and magnetic fields," *Europhys. Lett.* **79** (2007) 20004, `hep-th/0703193`.

[69] P. Castelo Ferreira, "Explicit actions for electromagnetism with two gauge fields from P and T violation by electromagnetic duality rotations," *J. Math. Phys.* **47** (2006) 072902, `hep-th/0510063`.

[70] P. J. Arias and L. Leal, "Topological sectors and gauge invariance in massive vector-tensor theories in D >= 4," *Phys. Lett.* **B404** (1997) 49–56, `hep-th/9606120`.

[71] A. P. Polychronakos, "Abelian Chern-Simons theories in (2+1)-dimensions," *Ann. Phys.* **203** (1990) 231–254.

[72] J. Govaerts and B. Deschepper, "The physical projector and topological quantum field theories: U(1) Chern-Simons theory in 2+1 dimensions," *J. Phys.* **A33** (2000) 1031–1053, `hep-th/9909221`.

[73] G. V. Dunne, "Aspects of Chern-Simons theory," `hep-th/9902115`.

[74] A. P. Balachandran and P. Teotonio-Sobrinho, "The edge states of the BF system and the London equations," *Int. J. Mod. Phys.* **A8** (1993) 723–752, `hep-th/9205116`.

[75] M. C. Diamantini, P. Sodano, and C. A. Trugenberger, "Superconductors with topological order," *Eur. Phys. J.* **B53** (2006) 19, `hep-th/0511192`.

[76] M. C. Diamantini, P. Sodano, and C. A. Trugenberger, "Gauge theories of Josephson junction arrays," *Nucl. Phys.* **B474** (1996) 641–677, `hep-th/9511168`.

[77] K. Seo and A. Sugamoto, "Phase Transition due to vortex condensation and the problem of quark confinement," *Phys. Rev.* **D24** (1981) 1630.

[78] C. Chatterjee and A. Lahiri, "Monopole confinement by flux tube," *Europhys. Lett.* **76** (2006) 1068–1073, `hep-ph/0605107`.

[79] K.-M. Lee, "The dual formulation of cosmic strings and vortices," *Phys. Rev.* **D48** (1993) 2493–2498, `hep-th/9301102`.

[80] E. Harikumar and M. Sivakumar, "On the equivalence between topologically and non-topologically massive abelian gauge theories," *Mod. Phys. Lett.* **A15** (2000) 121–132, `hep-th/9911244`.

[81] A. Smailagic and E. Spallucci, "Duality of massive gauge invariant theories in arbitrary space-time dimension," *Phys. Rev.* **D61** (2000) 067701, `hep-th/9911089`.

[82] R. Jackiw and E. J. Weinberg, "Selfdual Chern-Simons vortices," *Phys. Rev. Lett.* **64** (1990) 2234.

[83] J. Hong, Y. Kim, and P. Y. Pac, "Multivortex solutions of the Abelian Chern-Simons-Higgs theory," *Phys. Rev. Lett.* **64** (1990) 2230.

[84] R. Jackiw, K.-M. Lee, and E. J. Weinberg, "Selfdual Chern-Simons solitons," *Phys. Rev.* **D42** (1990) 3488–3499.

[85] B. Bertrand, "The dual formulations of abelian Higgs models and their generalisations : A zoo of topological defect solutions." Internal note, January, 2008.

[86] C. G. Kuper, *An Introduction to the Theory of Superconductivity*. Monographs on the Physics and Chemistry of Materials. Clarendon Press, Oxford, 1968.

[87] A. Jaffe and C. Taubes, *Vortices and Monopoles : Structure of Static Gauge Theories*, vol. 2 of *Progress in physics*. Birkh 1980.

[88] L. Jacobs and C. Rebbi, "Interaction energy of superconducting vortices," *Phys. Rev.* **B19** (1979) 4486–4494.