Neutral fermion pair production by Sauter-like magnetic step

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We review our recent results on the creation from the vacuum of neutral fermions with anomalous magnetic moments by a Sauter-like magnetic field. We construct in- and out solutions of the Dirac-Pauli equation with this field and calculate with their help pertinent quantities characterizing the vacuum instability, such as differential mean numbers and flux density of created pairs and and vacuum-to-vacuum transition amplitudes. Special attention is paid to situations where the external field lies in two particular configurations, varying either "gradually" or "sharply" along the inhomogeneity direction. We also estimate critical magnetic field intensities, near which the phenomenon could be observed.

Keywords: Dirac-Pauli equation; quantum electrodynamics; pair production; neutral fermions with anomalous magnetic moment

1. Introduction

The violation of the vacuum stability stimulated by external electromagnetic fields is commonly associated with the possibility of such backgrounds producing work on virtual pairs of particles and antiparticles. The most well-known examples are electric-like fields, as they produce work on charged particles and are able to tear apart electron/positron pairs from the vacuum if the field amplitudes approach the so-called Schwinger critical value $E_{\rm c} = m^2c^3/e\hbar \approx 1.3 \times 10^{16}\,{\rm V/cm}$.[1] The phenomenon has been a subject of intense investigation since the seminal works of Klein,[2] Sauter,[3,4] Heisenberg and Euler,[5] and Schwinger.[1] An extensive discussion about the origin of the effect, theoretical foundations, and experimental aspects can be found in some reviews and monographs; see e.g.[6–15] and references therein.

Following the above interpretation, one may ask oneself about the possibility that inhomogeneous macroscopic magnetic fields which produce a work on particles with a magnetic moment, may create pairs from the vacuum. The answer to this question is affirmative, provided the particles are neutral and have an anomalous magnetic moment. Bearing in mind, first of all, very strong magnetic fields observed in astrophysics, we can assume that this type of field is practically time-independent and steplike, that is, their gradient has a well-defined sign. At present, there exist two types of particles enjoying the properties mentioned above: the neutron and the neutrino. According to experimental data, neutrons have a magnetic moment $\mu_N \approx -1.04187563(25) \times 10^{-3}\mu_{\rm B}$,[16] where $\mu_{\rm B}$ is the Bohr magneton. As for neutrinos, there is not a general consensus because of the different types of neutrinos, mechanism under which neutrinos acquires magnetic moment, specific models, etc. Presently, experimental constraints range from $\mu_{\nu_\tau} < 3.9 \times 10^{-7}\mu_{\rm B}$ (for the tau neutrino)[17] until $\mu_{\nu_e} < 2.9 \times 10^{-11}\mu_{\rm B}$ (for the electron neutrino).[18] Moreover, stringent constraints obtained from astrophysical observations[19–24] indicate that $\mu_\nu < (2.6 - 4.5) \times 10^{-12}\mu_{\rm B}$ while lower upper bounds, predicted by effective theories above the electroweak scale, suggest that $\mu_\nu < 10^{-14}\mu_{\rm B}$.[25] It is important to point out that for some theories beyond the Standard Model (SM),[26] it was reported that the magnetic moment for the neutrinos lie within the range $\left(10^{-12} - 10^{-14}\right)\mu_{\rm B}$. For a more extensive discussion concerning experimental aspects and theoretical predictions for neutrinos' electromagnetic properties, see e.g. the reviews[27–31] and references therein.

In this work, we review our recent results on the creation of neutral fermions with anomalous magnetic moments from the vacuum by Sauter-like magnetic field.[32] We follow the general formulation developed to describe the effect nonperturbatively, which is based on the canonical quantization of fermion fields with time-independent inhomogeneous external fields.[33–35] The system under consideration is placed in the four-dimensional Minkowski spacetime, parameterized by coordinates $X = (X^\mu\,,\ \mu = 0, i) = (t, \mathbf{r})$, $X^0 = t$, $X^i = \mathbf{r} = (x, y, z)$, $i = 1, 2, 3$, with the metric tensor $\eta_{\mu\nu} = {\rm diag}\,(+1, -1, -1, -1)$. We also employ natural units, in which $\hbar = 1 = c$.

2. Solutions of Dirac-Pauli equation with Sauter-like magnetic step

The Dirac-Pauli (DP) equation[36] for a neutral spin 1/2 particle with the anomalous magnetic moment μ, the mass m interacting with external electromagnetic backgrounds $A^{\mu} = \left(A^0, \mathbf{A}\right)$ has the form:

$$\left(i\gamma^{\mu}\partial_{\mu} - m - \frac{1}{2}\mu\sigma^{\mu\nu}F_{\mu\nu}\right)\psi\left(X\right) = 0\,,$$
$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]_{-}\,, \quad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\,. \tag{1}$$

Here $\psi\left(X\right)$ is a bispinor, $\gamma^{\mu} = \left(\gamma^0, \boldsymbol{\gamma}\right)$ are Dirac matrices, and μ is the algebraic value of the anomalous magnetic moment (e.g., $\mu = -\left|\mu_N\right|$ for a neutron). In what follows we consider external electromagnetic fields of a specific type, corresponding to a time-independent magnetic field oriented along the positive direction of the z-axis, inhomogeneous along the y-direction, $\mathbf{B}\left(\mathbf{r}\right) = \left(0, 0, B_z\left(y\right)\right)$, and homogeneous at remote distances, $B_z\left(\pm\infty\right) = \text{const}$. Moreover, it is assumed that its gradient is always positive $\partial_y B\left(y\right) \geq 0$, $\forall\, y \in \left(-\infty, +\infty\right)$, meaning that $B_z\left(+\infty\right) > B_z\left(-\infty\right)$ and that the field is genuinely a step (or steplike, in short). To study neutral fermion pair production by steplike magnetic fields, we consider that the magnetic field inhomogeneity is given by the analytic function

$$B_z\left(y\right) = \varrho B' \tanh\left(y/\varrho\right)\,, \quad B' > 0\,, \quad \varrho > 0\,, \tag{2}$$

which meets the conditions discussed above and allows solving the DP equation exactly. The amplitude B' and the length scale ϱ describe, respectively, the "slope" of the field with respect to the y-axis and how "rectilinear" it is in the neighborhood of the z-axis. Thus, the larger B' and ϱ, the more "steep" and the more "rectilinear" the pattern of (2) near the z-axis. Because the field (2) resembles a step-like "potential" for charged particles and its gradient has a Sauter profile,[4] we call the field (2) Sauter-like magnetic step. For illustrative purposes, we present the field (2) and its gradient for some values of ϱ and B' in Fig. 1.

In the Schrödinger form, the DP equation (1) reads[32,33,37]

$$i\partial_t\psi\left(X\right) = \hat{H}\psi\left(X\right)\,, \quad \hat{H} = \gamma^0\left(\gamma^3\hat{p}_z + \Sigma_z\hat{\Pi}_z\right)\,. \tag{3}$$

Here $\Sigma_z = i\gamma^1\gamma^2$ and

$$\hat{\Pi}_z = \hat{\pi}_z - \mathbb{I}\mu B_z\left(y\right)\,, \quad \hat{\pi}_z = \Sigma_z\left(\boldsymbol{\gamma}\hat{\mathbf{p}}_{\perp} + m\right)\,, \tag{4}$$

is an integral-of-motion spin operator, $\left[\hat{\Pi}_z, \hat{H}\right]_{-} = 0$. The subscript "$\perp$" labels quantities perpendicular to the field, e.g. $\hat{\mathbf{p}}_{\perp} = \left(\hat{p}_x, \hat{p}_y\right)$ and $\mathbb{I}$ denotes the 4×4 identity matrix. Since the operators $\hat{p}_0$, $\hat{p}_x$, and $\hat{p}_z$ are compatible with the Hamiltonian (and also with $\hat{\Pi}_z$), the DP spinor admits the general form $\psi_n\left(X\right) = \exp\left(-ip_0t + ip_xx + ip_zz\right)\psi_n\left(y\right)$, where $\psi_n\left(y\right)$ depends exclusively on y

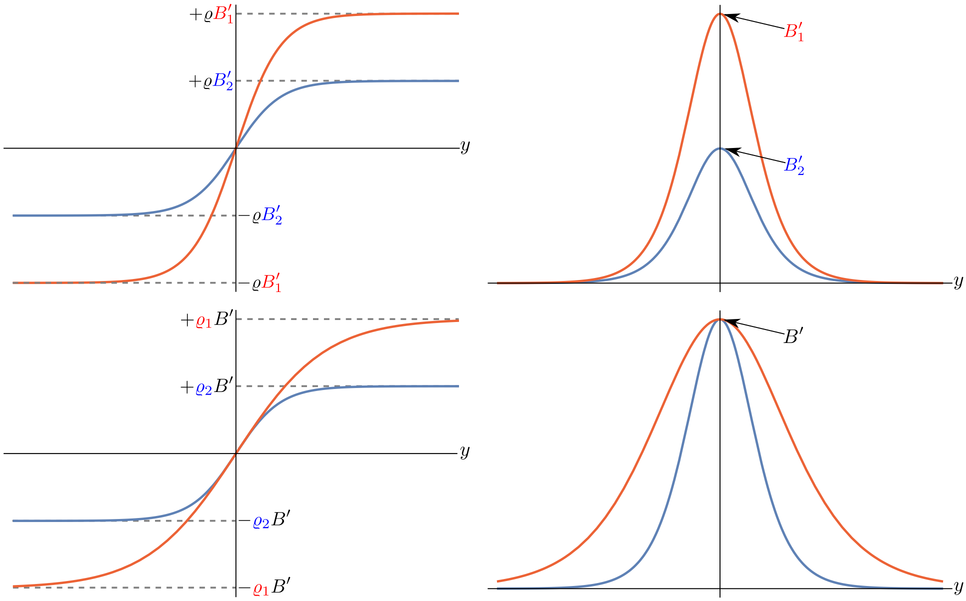

Fig. 1. Sauter-like magnetic steps (2) (pictures on the left) and its gradient (pictures on the right). In the upper panel, $B_1' = 2 \times B_2'$ while in the lower panel, $\varrho_1 = 2 \times \varrho_2$.

and is solution of the eigenvalue equation:

$$\begin{aligned} &\hat{\Pi}_z \psi_n(X) = e^{-ip_0 t + ip_x x + ip_z z} \Pi_z \psi_n(y)\,, \quad \Pi_z \psi_n(y) = s\omega \psi_n(y)\,, \quad s = \pm 1\,, \\ &\Pi_z = \hat{\pi}_z - \mathbb{I}\mu B_z(y)\,, \quad \hat{\pi}_z = \Sigma_z\left(\gamma^1 p_x + \gamma^2 \hat{p}_y + m\right)\,, \quad p_0^2 = \omega^2 + p_z^2\,. \end{aligned} \tag{5}$$

Due to the structure of the external field (2), there is an additional integral-of-motion spin operator

$$\hat{R} = \hat{H}\hat{\Pi}_z^{-1}\left[\mathbb{I} + \left(\hat{p}_z \hat{\Pi}_z^{-1}\right)^2\right]^{-1/2}\,, \tag{6}$$

which is compatible with the Hamiltonian $\left[\hat{R}, \hat{H}\right]_- = 0$ and with all previous operators, $\left[\hat{R}, \hat{\Pi}_z\right]_- = \left[\hat{R}, \hat{p}_0\right]_- = \left[\hat{R}, \hat{p}_x\right]_- = \left[\hat{R}, \hat{p}_z\right]_- = 0$. In particular, the operator (6) implies that $\psi_n(y)$ is a solution of the eigenvalue equation

$$\begin{aligned} &\hat{R}\psi_n(X) = e^{-ip_0 t + ip_x x + ip_z z} R\psi_n(y)\,, \quad R\psi_n(y) = s\psi_n(y)\,, \\ &R = \Upsilon\gamma^0\left(\Sigma_z + \frac{sp_z}{\omega}\gamma^3\right)\,, \quad \Upsilon = \frac{1}{\sqrt{1 + p_z^2/\omega^2}}\,. \end{aligned} \tag{7}$$

As a result, the complete set of commuting operators is $\hat{p}_x$, $\hat{p}_z$, $\hat{\Pi}_z$, $\hat{R}$ and the corresponding quantum numbers are $n = (p_x, p_z, \omega, s)$.

The set of equations (5) and (7) are simultaneously satisfied choosing $\psi_n(y)$ in the form

$$\psi_n(y) = (\mathbb{I} + sR)\left[\hat{\pi}_z + \mathbb{I}(\mu B_z(y) + s\omega)\right]\varphi_{n,\chi}(y)\,\upsilon_\kappa^{(\chi)}\,, \tag{8}$$

where $\upsilon_\kappa^{(\chi)}$ belongs to a set of four constant spinors, satisfying the eigenvalue equations

$$i\gamma^1\upsilon_\kappa^{(\chi)} = \chi\upsilon_\kappa^{(\chi)}\,,\quad \gamma^0\gamma^2\upsilon_\kappa^{(\chi)} = \kappa\upsilon_\kappa^{(\chi)}\,,\quad \chi = \pm 1\,,\quad \kappa = \pm 1\,, \tag{9}$$

and the orthonormality conditions $\upsilon_{\kappa'}^{(\chi')\dagger}\upsilon_\kappa^{(\chi)} = \delta_{\chi'\chi}\delta_{\kappa'\kappa}$. As for the scalar functions $\varphi_{n,\chi}(y)$, they are solutions of the second-order ordinary differential equation:

$$\left\{-\frac{d^2}{dy^2} - \left[s\omega + \mu B_z(y)\right]^2 + \pi_x^2 + i\mu\chi B_z'(y)\right\}\varphi_{n,\chi}(y) = 0\,,\quad \pi_x^2 = m^2 + p_x^2\,. \tag{10}$$

The potential energy of a neutral fermion interacting with the field is $U_s(y) = sU(y)$, where $U(y) = -\mu B_z(y)$. To facilitate subsequent discussions, it is convenient to select a fixed sign for particle magnetic moment. From now on, we choose a fermion with a negative magnetic moment as the main particle, $\mu = -|\mu|$. Because the field (2) increases monotonically with y, the maximum potential energy that may be experienced by the fermion is determined by the magnitude of the "step" $\mathbb{U}$

$$\mathbb{U} \equiv U_{\mathrm{R}} - U_{\mathrm{L}} = 2\varrho\,|\mu|\,B' > 0\,, \tag{11}$$

which is the difference between the asymptotic values $U_{\mathrm{R}} = U(+\infty) = |\mu|\,\varrho B'$, $U_{\mathrm{L}} = U(-\infty) = -|\mu|\,\varrho B'$ and is positive, by definition[a]. At remote distances–where the field can be considered homogeneous and no longer accelerates particles–the term proportional to χ in Eq. (10) is absent. Therefore, solutions of Eq. (10) have well-defined "left" ${}_\zeta\varphi_{n,\chi}(y)$ and "right" ${}^\zeta\varphi_{n,\chi}(y)$ asymptotic forms:

$$\begin{aligned}
{}_\zeta\varphi_{n,\chi}(y) &= {}_\zeta\mathcal{N}\exp\left(i\zeta\left|p^{\mathrm{L}}\right|y\right)\,,\quad \zeta = \mathrm{sgn}\left(p^{\mathrm{L}}\right)\,,\quad y \to -\infty\,,\\
{}^\zeta\varphi_{n,\chi}(y) &= {}^\zeta\mathcal{N}\exp\left(i\zeta\left|p^{\mathrm{R}}\right|y\right)\,,\quad \zeta = \mathrm{sgn}\left(p^{\mathrm{R}}\right)\,,\quad y \to +\infty\,.
\end{aligned} \tag{12}$$

Here, ${}_\zeta\mathcal{N}$, ${}^\zeta\mathcal{N}$ are normalization constants, $\left|p^{\mathrm{L/R}}\right|$ are y-components of fermions momenta at remote regions

$$\left|p^{\mathrm{L/R}}\right| = \sqrt{\left[s\pi_s(\mathrm{L/R})\right]^2 - \pi_x^2}\,,\quad \pi_s(\mathrm{L/R}) = \omega - sU_{\mathrm{L/R}}\,, \tag{13}$$

and $\pi_s(\mathrm{L/R})$ are their transverse kinetic energies at remote areas. Correspondingly, asymptotically-"left" ${}_\zeta\psi_n(X) = \exp(-ip_0t + ip_xx + ip_zz)\;{}_\zeta\psi_n(y)$ and asymptotically-"right" ${}^\zeta\psi_n(X) = \exp(-ip_0t + ip_xx + ip_zz)\;{}^\zeta\psi_n(y)$ sets of DP spinors are solutions of the eigenvalue equations

$$\begin{aligned}
&\hat{p}_y\;{}_\zeta\psi_n(X) = \zeta\left|p^{\mathrm{L}}\right|\;{}_\zeta\psi_n(X)\,,\quad \hat{h}_\perp^{\mathrm{kin}}\;{}_\zeta\psi_n(X) = s\pi_s(\mathrm{L})\;{}_\zeta\psi_n(X)\,,\quad y \to -\infty\,,\\
&\hat{p}_y\;{}^\zeta\psi_n(X) = \zeta\left|p^{\mathrm{R}}\right|\;{}^\zeta\psi_n(X)\,,\quad \hat{h}_\perp^{\mathrm{kin}}\;{}^\zeta\psi_n(X) = s\pi_s(\mathrm{R})\;{}^\zeta\psi_n(X)\,,\quad y \to +\infty\,,
\end{aligned} \tag{14}$$

[a]The labels "L" and "R" mean "asymptotic left region $y \to -\infty$" and "asymptotic right region $y \to +\infty$", respectively.

where $\hat{h}_{\perp}^{\rm kin} = \hat{\Pi}_z - \mathbb{I}\,|\mu|\,B_z(y)$ is the one-particle transverse kinetic energy operator.

Substituting the field (2) into Eq. (10) and performing a simultaneous change of variables

$$\varphi_{n,\chi}(y) = \xi^{\rho}(1-\xi)^{\sigma} f(\xi)\ ,\quad \xi(y) = \frac{1}{2}\left[1+\tanh(y/\varrho)\right]\ , \tag{15}$$

we may convert Eq. (10) to the form of the differential equation for the Gauss Hypergeometric Function[38]

$$\xi(1-\xi) f'' + \left[c-(a+b+1)\xi\right] f' - abf = 0\,, \tag{16}$$

provided the parameters ρ, σ, a, b, and c are:

$$\begin{aligned}
a &= \frac{1}{2}(1-\chi) - \frac{i\varrho}{2}\left(\mathbb{U} + \left|p^{\rm L}\right| - \left|p^{\rm R}\right|\right)\ ,\\
b &= \frac{1}{2}(1+\chi) + \frac{i\varrho}{2}\left(\mathbb{U} + \left|p^{\rm R}\right| - \left|p^{\rm L}\right|\right)\ ,\\
c &= 1 - i\varrho\left|p^{\rm L}\right|\ ,\quad \rho = -\frac{i}{2}\varrho\left|p^{\rm L}\right|\ ,\quad \sigma = \frac{i}{2}\varrho\left|p^{\rm R}\right|\ .
\end{aligned} \tag{17}$$

Among the 24 Hypergeometric functions satisfying Eq. (16),[38] we select those that tend to unity when $y \to \mp\infty$. Solutions meeting this property are proportional to Hypergeometric functions of type $F(a',b';c';\xi)$ and $F(a'',b'';c'';1-\xi)$. For example, a possible set of exact solutions to Eq. (10) behaving asymptotically like Eqs. (12) is

$$\begin{aligned}
{}_{\zeta}\varphi_{n,\chi}(y) &= {}_{\zeta}\mathcal{N}\exp\left(i\zeta\left|p^{\rm L}\right|y\right)\left[1+\exp(2y/\varrho)\right]^{-i\varrho\left(\zeta\left|p^{\rm L}\right|+\left|p^{\rm R}\right|\right)/2}\ {}_{\zeta}u(\xi)\ ,\\
{}^{\zeta}\varphi_{n,\chi}(y) &= {}^{\zeta}\mathcal{N}\exp\left(i\zeta\left|p^{\rm R}\right|y\right)\left[1+\exp(-2y/\varrho)\right]^{i\varrho\left(\left|p^{\rm L}\right|+\zeta\left|p^{\rm R}\right|\right)/2}\ {}^{\zeta}u(\xi)\ ,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
&{}_{-}u(\xi) = F(a,b;c;\xi)\ ,\quad {}_{+}u(\xi) = F(a+1-c,b+1-c;2-c;\xi)\ ,\\
&{}^{-}u(\xi) = F(a,b;a+b+1-c;1-\xi)\ ,\ {}^{+}u(\xi) = F(c-a,c-b;c+1-a-b;1-\xi)\,.
\end{aligned} \tag{19}$$

With the aid of these solutions, we may finally introduce the sets of DP spinors

$$\begin{aligned}
{}_{\zeta}\psi_n(X) &= e^{-i(p_0t-p_xx-p_zz)}\left(\mathbb{I}+sR\right)\left\{\hat{\pi}_z + \mathbb{I}\left[s\omega - |\mu|\,B_z(y)\right]\right\}\ {}_{\zeta}\varphi_{n,\chi}(y)\,v_{\kappa}^{(\chi)}\ ,\\
{}^{\zeta}\psi_n(X) &= e^{-i(p_0t-p_xx-p_zz)}\left(\mathbb{I}+sR\right)\left\{\hat{\pi}_z + \mathbb{I}\left[s\omega - |\mu|\,B_z(y)\right]\right\}\ {}^{\zeta}\varphi_{n,\chi}(y)\,v_{\kappa}^{(\chi)}\ ,
\end{aligned} \tag{20}$$

provided the quantum numbers n obey the conditions

$$\left[s\pi_s({\rm L/R})\right]^2 > \pi_x^2\,. \tag{21}$$

These inequalities ensure the nontriviality of DP spinors with real asymptotic momenta $p^{\rm L}$ and $p^{\rm R}$ in remote areas, fulfilling Eqs. (14).

To calculate the normalization constants ${}_{\zeta}\mathcal{N}$, ${}^{\zeta}\mathcal{N}$, we use the inner product on the timelike surface $y = \text{const.}$,

$$\left(\psi,\psi'\right)_y = \int dtdxdz\psi^{\dagger}(X)\,\gamma^0\gamma^2\psi'(X)\ , \tag{22}$$

after imposing specific normalization conditions[b]. We assume that all processes take place within a macroscopically large space-time box, of volume TV_y, $V_y = L_x L_z$, and impose periodic boundary conditions upon DP spinors in the variables t, x, z at the boundaries. Thus, the integrals in (22) are calculated from $(-T/2, -L_x/2, -L_z/2)$ to $(+T/2, +L_x/2, +L_z/2)$ and the limits $(T, L_x, L_z) \to \infty$ are taken at the end of calculations. Under these conditions, the inner product is y-independent and we may impose the following normalization conditions:

$$\left({}_{\zeta'}\psi_{n'},\ {}_{\zeta}\psi_n \right)_y = \zeta\eta_{\mathrm{L}}\delta_{n'n}\delta_{\zeta'\zeta}\,, \quad \left({}^{\zeta'}\psi_{n'},\ {}^{\zeta}\psi_n \right)_y = \zeta\eta_{\mathrm{R}}\delta_{n'n}\delta_{\zeta'\zeta}\,, \tag{23}$$

where $\eta_{\mathrm{L/R}} = \mathrm{sgn}\left[\pi_s\left(\mathrm{L/R}\right)\right]$. It should be noted that the time independence of the magnetic field under consideration is an idealization. Physically, it is meaningful to believe that the field inhomogeneity was switched on sufficiently fast before instant t_{in}. By this time, it had time to spread to the whole area under consideration and then acted as a constant field during a large time T. It is supposed that one can ignore effects of its switching on and off. This is a kind of regularization, which could, under certain conditions, be replaced by periodic boundary conditions in t, see Refs.[34,35] for details. Evaluating the inner product (22) for each DP spinor (20) and imposing the normalization conditions (23), we obtain

$$\left|\,{}_{\zeta}\mathcal{N}\right| = \frac{\left[TV_y\Upsilon\left(1 - s\kappa\chi\Upsilon\right)\right]^{-1/2}}{2\sqrt{\left|p^{\mathrm{L}}\right|\left|\pi_s\left(\mathrm{L}\right) - s\chi\zeta\left|p^{\mathrm{L}}\right|\right|}}\,, \quad \left|\,{}^{\zeta}\mathcal{N}\right| = \frac{\left[TV_y\Upsilon\left(1 - s\kappa\chi\Upsilon\right)\right]^{-1/2}}{2\sqrt{\left|p^{\mathrm{R}}\right|\left|\pi_s\left(\mathrm{R}\right) - s\chi\zeta\left|p^{\mathrm{R}}\right|\right|}}\,. \tag{24}$$

Considering that the "left" and "right" sets of DP spinors (20) are orthonormal and complete (with respect to the inner product (22)), we may decompose one set into another with the help of some g-coefficients

$$\begin{aligned} \eta_{\mathrm{L}}\ {}^{\zeta}\psi_n\left(X\right) &= g\left({}_{+}|^{\zeta}\right)\ {}_{+}\psi_n\left(X\right) - g\left({}_{-}|^{\zeta}\right)\ {}_{-}\psi_n\left(X\right)\,, \\ \eta_{\mathrm{R}}\ {}_{\zeta}\psi_n\left(X\right) &= g\left({}^{+}|_{\zeta}\right)\ {}^{+}\psi_n\left(X\right) - g\left({}^{-}|_{\zeta}\right)\ {}^{-}\psi_n\left(X\right)\,, \end{aligned} \tag{25}$$

which, by definition, are inner products between different sets of DP spinors

$$\left({}_{\zeta}\psi_n,\ {}^{\zeta'}\psi_{n'} \right)_y = \delta_{nn'}g\left({}_{\zeta}|^{\zeta'}\right) = \delta_{nn'}g\left({}^{\zeta'}|_{\zeta}\right)^*\,, \tag{26}$$

and play an important role in the quantization of DP spinors with steplike magnetic fields. These coefficients link different sets of creation and annihilation operators and contains all the necessary information about vacuum instability, as shall be seen below. Substituting the identities (25) into normalization conditions (23) supply us with two important identities

$$\sum_{\zeta''=\pm}\zeta'' g\left({}^{\zeta'}|_{\zeta''}\right)g\left({}_{\zeta''}|^{\zeta}\right) = \zeta\eta_{\mathrm{L}}\eta_{\mathrm{R}}\delta_{\zeta'\zeta} = \sum_{\zeta''=\pm}\zeta'' g\left({}_{\zeta'}|^{\zeta''}\right)g\left({}^{\zeta''}|_{\zeta}\right)\,, \tag{27}$$

from which we may derive a number of identities, for example $\left|g\left({}_{+}|^{-}\right)\right|^2 = \left|g\left({}_{-}|^{+}\right)\right|^2$, $\left|g\left({}_{+}|^{+}\right)\right|^2 = \left|g\left({}_{-}|^{-}\right)\right|^2$, and $\left|g\left({}_{+}|^{+}\right)\right|^2 - \left|g\left({}_{+}|^{-}\right)\right|^2 = \eta_{\mathrm{L}}\eta_{\mathrm{R}}$.

[b]Note that for $\psi' = \psi$, the inner product (22) divided by T coincides with the definition of the current density accross the y-const. hyperplane.

3. Pair production

Besides providing conditions for the existence of solutions (20), the inequalities (21) imposes certain limitations on the quantum numbers. For critical Sauter-like magnetic steps, whose step magnitudes (11) meet the condition

$$\mathbb{U} > \mathbb{U}_{\mathrm{c}} = 2m\,, \tag{28}$$

the whole manifold of quantum numbers divides into five sub-ranges, $\Omega_k\,,\ k = 1, ..., 5$. Neutral fermion pair production takes place only in a well-defined bounded set of quantum numbers[32–34]

$$\Omega_3 = \{n : U_{\mathrm{L}} + \pi_x \leq s\omega \leq U_{\mathrm{R}} - \pi_x\,,\quad \pi_{xz} \leq \mathbb{U}/2\}\,,\quad \pi_{xz} = \sqrt{\pi_x^2 + p_z^2}\,, \tag{29}$$

which is conventionally called *Klein zone*, Ω_3. In this subrange, $s\pi_s\,(\mathrm{L}) \geq \pi_x$ and $s\pi_s\,(\mathrm{R}) \leq -\pi_x$, which means that $s\eta_{\mathrm{L}} = +1$ and $s\eta_{\mathrm{R}} = -1$. As a result, there exist two linearly-independent "left" ${}_\zeta\psi_{n_3}\,(X)$ and "right" ${}^\zeta\psi_{n_3}\,(X)$ sets of DP spinors with quantum numbers within the Klein zone $n_3 = n \in \Omega_3$.

To quantize the DP field operators using sets of solutions in this subrange, we need to classify them as particle or antiparticle states and as incoming waves (waves traveling toward the "step") or outgoing waves (waves traveling outward the "step") in remote areas. The correct classification demands a careful study of the inner product on y- and t-constant hyperplanes because important quantities to the scattering problem are expressed as surface integrals on such hyperplanes. After a detailed study of these quantities, which was presented in Refs.[34,35] for charged particles and in[33] for neutral fermions, "in"-solutions (incoming waves) and "out"-solutions (outgoing waves) are

$$\text{in-solutions:}\ \ {}_-\psi_{n_3}\,(X)\,,\ {}^-\psi_{n_3}\,(X)\,,\quad \text{out-solutions:}\ \ {}_+\psi_{n_3}\,(X)\,,\ {}^+\psi_{n_3}\,(X)\,. \tag{30}$$

The above sets of solutions are complete and orthogonal with respect to the inner product on t-constant hyperplane

$$(\psi_n, \psi'_{n'}) = \int_{V_y} dxdz \int_{-K^{(\mathrm{L})}}^{K^{(\mathrm{R})}} dy \psi_n^\dagger\,(X)\,\psi'_{n'}\,(X)\,, \tag{31}$$

in which the lower/upper cutoffs $K^{(\mathrm{L/R})}$ are macroscopic but finite parameters of the volume regularization that are situated far beyond the region of a large gradient $\partial_y B_z\,(y)$; see Ref.[34,35] for details. In particular, the inner product (31) between "in" and "out" sets of DP spinors have the form

$$\left({}_\zeta\psi_n,\ {}_\zeta\psi_{n'}\right) = \left({}^\zeta\psi_n,\ {}^\zeta\psi_{n'}\right) = \mathcal{M}_n\delta_{nn'}\,,\quad \left({}_\zeta\psi_n,\ {}^\zeta\psi_{n'}\right) = 0\,,\quad n, n' \in \Omega_3\,. \tag{32}$$

where $\mathcal{M}_n = 2\left|g\left({}_+|^-\right)\right|^2 t^{(\mathrm{L/R})}/T$, $t^{(\mathrm{L/R})} = K^{(\mathrm{L/R})}\left|\pi_s\,(\mathrm{L/R})\,/p^{\mathrm{L/R}}\right|$. Because there are two linearly independent sets of spinors (30), the quantization is performed using two distinct "in" and "out" sets of annihilation & creation operators

$$\begin{aligned}
&\text{in-set: } {}_-b_{n_3}\,(\mathrm{in})\,,\ {}_-b_{n_3}^\dagger\,(\mathrm{in})\,,\ {}^-a_{n_3}\,(\mathrm{in})\,,\ {}^-a_{n_3}^\dagger\,(\mathrm{in})\,,\\
&\text{out-set: } {}_+b_{n_3}\,(\mathrm{out})\,,\ {}_+b_{n_3}^\dagger\,(\mathrm{out})\,,\ {}^+a_{n_3}\,(\mathrm{out})\,,\ {}^+a_{n_3}^\dagger\,(\mathrm{out})\,,
\end{aligned} \tag{33}$$

which, in turn, obey the following anticommutation relations

$$\left[\ {}^{-}a_{n_3'}\left(\mathrm{in}\right),\ {}^{-}a_{n_3}^{\dagger}\left(\mathrm{in}\right)\right]_+ = \left[\ {}_{-}b_{n_3'}\left(\mathrm{in}\right),\ {}_{-}b_{n_3}^{\dagger}\left(\mathrm{in}\right)\right]_+ = \delta_{n_3' n_3}\,,$$
$$\left[\ {}^{+}a_{n_3'}\left(\mathrm{out}\right),\ {}^{+}a_{n_3}^{\dagger}\left(\mathrm{out}\right)\right]_+ = \left[\ {}_{+}b_{n_3'}\left(\mathrm{out}\right),\ {}_{+}b_{n_3}^{\dagger}\left(\mathrm{out}\right)\right]_+ = \delta_{n_3' n_3}\,, \tag{34}$$

and whose annihilation operators (33) annihilate the corresponding vacuum states

$${}_{-}b_{n_3}\left(\mathrm{in}\right)|0,\mathrm{in}\rangle = {}^{-}a_{n_3}\left(\mathrm{in}\right)|0,\mathrm{in}\rangle = 0,\ {}_{+}b_{n_3}\left(\mathrm{out}\right)|0,\mathrm{out}\rangle = {}^{+}a_{n_3}\left(\mathrm{out}\right)|0,\mathrm{out}\rangle = 0. \tag{35}$$

Finally, the quantized DP field operator in the Klein zone reads

$$\hat{\Psi}\left(X\right) = \sum_{n\in\Omega_3}\mathcal{M}_n^{-1/2}\left[\ {}^{-}a_n\left(\mathrm{in}\right)\ {}^{-}\psi_n\left(X\right) + \ {}_{-}b_n^{\dagger}\left(\mathrm{in}\right)\ {}_{-}\psi_n\left(X\right)\right]\,,$$
$$= \sum_{n\in\Omega_3}\mathcal{M}_n^{-1/2}\left[\ {}^{+}a_n\left(\mathrm{out}\right)\ {}^{+}\psi_n\left(X\right) + \ {}_{+}b_n^{\dagger}\left(\mathrm{out}\right)\ {}_{+}\psi_n\left(X\right)\right]\,. \tag{36}$$

Using orthogonality relations between DP spinors (23), (32) and the relations given by Eqs. (25), one may establish a linear relation between the “in”-set of creation/annihilation operators in terms of the “out”-set and *vice-versa*. For example, two (out of four) canonical transformations have the following form

$${}_{-}b_n^{\dagger}\left(\mathrm{in}\right) = -g\left({}^{+}|{}_{-}\right)^{-1}\ {}^{+}a_n\left(\mathrm{out}\right) + g\left({}_{+}|{}^{-}\right)^{-1} g\left({}_{-}|{}^{-}\right)\ {}_{+}b_n^{\dagger}\left(\mathrm{out}\right)\,,$$
$${}^{+}a_n\left(\mathrm{out}\right) = -g\left({}_{-}|{}^{+}\right)^{-1}\ {}_{-}b_n^{\dagger}\left(\mathrm{in}\right) + g\left({}^{-}|{}_{+}\right)^{-1} g\left({}^{+}|{}_{+}\right)\ {}^{-}a_n\left(\mathrm{in}\right)\,. \tag{37}$$

With the aid of the canonical transformations (37), we may finally define vacuum instability quantities, such as the differential mean numbers of “out” particles created from the “in” vacuum,

$$N_n^{\mathrm{cr}} = \left\langle 0,\mathrm{in}\right|\ {}^{+}a_n^{\dagger}\left(\mathrm{out}\right)\ {}^{+}a_n\left(\mathrm{out}\right)\left|\mathrm{in},0\right\rangle = \left|g\left({}_{-}|{}^{+}\right)\right|^{-2}\,,\quad n\in\Omega_3\,, \tag{38}$$

and the flux density of particles created with a given s,

$$n_s^{\mathrm{cr}} = \frac{1}{V_y T}\sum_{n\in\Omega_3} N_n^{\mathrm{cr}} = \frac{1}{\left(2\pi\right)^3}\int dp_z \int dp_x \int dp_0 N_n^{\mathrm{cr}}\,. \tag{39}$$

The total flux density of particles created with both spin polarizations is $n^{\mathrm{cr}} = n_{+1}^{\mathrm{cr}} + n_{-1}^{\mathrm{cr}}$ and the vacuum-vacuum transition probability reads:

$$P_v = \left|\langle 0,\mathrm{out}|0,\mathrm{in}\rangle\right|^2 = \exp\left[\sum_{s=\pm1}\sum_{n\in\Omega_3}\ln\left(1-N_n^{\mathrm{cr}}\right)\right]\,. \tag{40}$$

It should be noted that if the total number of created particles $N^{\mathrm{cr}} = V_y T n^{\mathrm{cr}}$ is small, one may neglect higher-order terms in Eq. (40) to conclude that $P_v \approx 1 - N^{\mathrm{cr}}$. With the aid of this relation, we may link the total number of neutral fermions created from the vacuum with the imaginary part of an effective action S_{eff} provided it satisfies the Schwinger relation $P_v = \exp\left(-2\mathrm{Im}S_{\mathrm{eff}}\right)$, and it is small, so that $P_v \approx 1 - 2\mathrm{Im}S_{\mathrm{eff}}$. Therefore,

$$\mathrm{Im}S_{\mathrm{eff}} \approx V_y T n^{\mathrm{cr}}/2\,. \tag{41}$$

From these equations, we observe that all the information about pair creation by the external field is enclosed in $g\left({}_{-}|^{+}\right)$. To obtain this coefficient, we may use an appropriate Kummer relation[38] that connects three Gauss Hypergeometric functions appearing in one of the relations given by Eq. (25). After obtaining this coefficient and calculating its absolute square $\left|g\left({}_{-}|^{+}\right)\right|^{-2}$, we finally obtain the differential mean numbers of pairs created from the vacuum:

$$N_n^{\mathrm{cr}} = \frac{\sinh\left(\pi\varrho\left|p^{\mathrm{R}}\right|\right)\sinh\left(\pi\varrho\left|p^{\mathrm{L}}\right|\right)}{\sinh\left[\pi\varrho\left(\mathbb{U}+|p^{\mathrm{L}}|-|p^{\mathrm{R}}|\right)/2\right]\sinh\left[\pi\varrho\left(\mathbb{U}+|p^{\mathrm{R}}|-|p^{\mathrm{L}}|\right)/2\right]}\,. \tag{42}$$

Note that N_n^{cr} are positive-definite because the difference $\left|\left|p^{\mathrm{L}}\right|-\left|p^{\mathrm{R}}\right|\right|$ bounded in this subrange; $0 \le \left|\left|p^{\mathrm{L}}\right|-\left|p^{\mathrm{R}}\right|\right| \le \sqrt{\mathbb{U}\left(\mathbb{U}-2\pi_x\right)}$. The above expression gives the exact distribution of neutral fermions created from the vacuum by the field (2). When summed over the quantum numbers, it provides exact expressions for the flux density of the created particles (39) and the vacuum-vacuum transition probability (40). Lastly, it is noteworthy to discuss some peculiarities associated with the choice of the quantum number s and its impact on the quantization (36). As pointed out in Sec. 2, there are two species of neutral fermions, one with $s = +1$ and another with $s = -1$. In the latter case, the classification differs from the one given by Eq (30), namely ${}_{+}\psi_{n_3}(X)$, ${}^{+}\psi_{n_3}(X)$ are "in"-solutions while ${}_{-}\psi_{n_3}(X)$, ${}^{-}\psi_{n_3}(X)$ are "out"-solutions. Although this classification changes the quantization (36), it does not change the mean numbers (38). This means that the flux density of particles created with $s = -1$ equals the one with $s = +1$, $n_{+1}^{\mathrm{cr}} = n_{-1}^{\mathrm{cr}}$. Therefore, summations over s in Eqs. (39), (40) just produce an extra factor of 2 in final expressions and that is why it is enough selecting s fixed to perform specific calculations; hereafter, we select $s = +1$ for convenience. In what follows, we analyze vacuum instability quantities when the field lies in two special configurations, varying either "gradually" or "sharply" along the inhomogeneity direction.

3.1. *"Gradually"-varying field configuration*

This field configuration corresponds to the case where the amplitude B' is sufficiently large and the field inhomogeneity stretches over a relatively wide region of the space, such that the condition

$$\sqrt{\varrho\mathbb{U}/2} \gg \max\left(1, \frac{m}{\sqrt{|\mu|\,B'}}\right)\,, \tag{43}$$

is satisfied. Accordingly, the arguments of the hyperbolic functions in (42) are large, meaning that the mean numbers of pairs created acquires the following approximate form,

$$N_n^{\mathrm{cr}} \approx e^{-\pi\tau}\,,\quad \tau = \varrho\left(\mathbb{U}-\left|p^{\mathrm{R}}\right|-\left|p^{\mathrm{L}}\right|\right)\,. \tag{44}$$

The above distribution is exponentially small for large values of ω and p_x. Its most significant contribution comes from a finite range of values of quantum numbers such that the conditions $\min\left(\pi_{+1}^2(\mathrm{L}), \pi_{+1}^2(\mathrm{R})\right) \gg \pi_x^2$ remains valid. In this case, τ admits the following approximation

$$\tau = \frac{(\mathbb{U}/2)^2}{(\mathbb{U}/2)^2 - \omega^2}\lambda + O\left(\pi_x^4/\left|\pi_{+1}(\mathrm{R})\right|^3\right) + O\left(\pi_x^4/\left|\pi_{+1}(\mathrm{L})\right|^3\right). \tag{45}$$

Now, we can estimate the flux density of pairs created n^{cr} for a magnetic step evolving gradually along the y-direction according to (43). To this end, it is convenient to transform the original integral over p_0 into an integral over ω through the relation between p_0, ω, and p_z discussed before, $p_0^2 = \omega^2 + p_z^2$. Performing such a change of variables, the flux density of the particles created by the external field in the configuration (43) has the form

$$n^{\mathrm{cr}} \approx \frac{4}{(2\pi)^3}\int_0^{p_z^{\max}} dp_z \int_{-p_x^{\max}}^{p_x^{\max}} dp_x \int_0^{\omega_{\max}^2} d\omega^2 \frac{e^{-\pi\tau}}{\sqrt{\omega^2 + p_z^2}},$$
$$p_z^{\max} = \sqrt{(\mathbb{U}/2)^2 - m^2}, \quad p_x^{\max} = \sqrt{(\mathbb{U}/2)^2 - m^2 - p_z^2}, \quad \omega_{\max} = \mathbb{U}/2 - \pi_x. \tag{46}$$

The multiplicative factor 4 comes from the summation over s and from the fact that the integrand is symmetric in p_z. To obtain an analytical expression to N^{cr}, we formally extend the integration limits of the last two integrals to infinity. This procedure amounts to incorporating exponentially small contributions to n^{cr} since the differential mean numbers are exponentially small at large p_x and ω. In this case, we may technically interchange the order of the last two integrals in (46) and use the approximation given by Eq. (45) to discover that the flux density of the created particles is approximately given by

$$n^{\mathrm{cr}} \approx \frac{m}{(2\pi)^3}\mathbb{U}^2\sqrt{b'}e^{-\pi b'}I_{b'}, \quad I_{b'} = \int_0^\infty \frac{du}{(u+1)^{5/2}} \ln\left(\frac{\sqrt{1+u} + \sqrt{1+2u}}{\sqrt{u}}\right) e^{-\pi b' u}, \tag{47}$$

where $b' = m^2/|\mu| B'$ and $\mathbb{U} = 2\varrho|\mu| B'$. At last, one may use the identity $\ln\left(1 - N_n^{\mathrm{cr}}\right) = -\sum_{l=1}^{\infty}\left(N_n^{\mathrm{cr}}\right)^l/l$ and perform integrations similar to the ones discussed before to discover that the vacuum-vacuum transition probability admits the final form

$$P_v = \exp\left(-\beta V_y T n^{\mathrm{cr}}\right), \quad \beta = \sum_{l=0}^{\infty} \frac{\epsilon_{l+1}}{(l+1)^{3/2}} \exp\left(-l\pi b'\right), \quad \epsilon_l = \frac{I_{b'l}}{I_{b'}}, \tag{48}$$

with n^{cr} given by Eq. (47).

It is noteworthy mentioning that relation (40)–which is well-known for strong-field QED with external electromagnetic fields–holds for the case under consideration as well. However, a direct similarity of total quantities for both cases is absent. We see that the flux density of created neutral fermion pairs and the quantity $\ln P_v^{-1}$ are quadratic in the magnitude of the step. This is a consequence of the fact that the number of states with all possible ω and p_z excited by the magnetic-field inhomogeneity is quadratic in the increment of the kinetic momentum. This is also the reason why the flux density of created pairs and $\ln P_v^{-1}$ per unit of the length are not uniform.

3.2. *"Sharply"-varying field configuration*

A second configuration of interest is when the field (2) "sharply" steeps near the origin. Such a configuration is specified by the conditions:

$$1 \gg \sqrt{\varrho \mathbb{U}/2} \gtrsim \frac{m}{\sqrt{|\mu|\, B'}} \,. \tag{49}$$

The first inequality indicates that the gradient $\partial_y B_z(y)$ sharply peaks about the origin, while the second implicates that the Klein zone is relatively small. This configuration is particularly important due to a close analogy to charged pair production by the Klein step, see Ref.[39] for the review. For electric fields whose spatial inhomogeneity meets conditions equivalent to (49), it was demonstrated that the imaginary part of the QED effective action features properties similar to those of continuous phase transitions.[40,41] Recently,[42] we have demonstrated for the inverse-square electric field that this peculiarity also follows from the behavior of total quantities when the Klein zone is relatively small. Because of the condition (49), not only the parameter $\varrho\mathbb{U}/2$ is small but all parameters involving the quantum numbers p_x, p_z, and ω are small as well on account of the inequalities (29). As a result, the arguments of the hyperbolic functions in (42) are small, which means that we may expand the hyperbolic functions in ascending powers and truncate the corresponding series to first-order to demonstrate that the mean numbers admit the approximate form:

$$N_n^{\mathrm{cr}} \approx \frac{4\left|p^{\mathrm{R}}\right|\left|p^{\mathrm{L}}\right|}{\mathbb{U}^2 - \left(\left|p^{\mathrm{L}}\right| - \left|p^{\mathrm{R}}\right|\right)^2} \,. \tag{50}$$

To implement the conditions (49), we conveniently introduce the Keldysh parameter $\gamma = 2m/\mathbb{U}$ and observe that it obeys the condition $1-\gamma^2 \ll 1$ on account of (49). Next, we perform the change of variables

$$\frac{\omega}{m} = \frac{1}{2}\left(1-\gamma^2\right)(1-v) \,, \quad \frac{p_x^2}{m^2} = \left(1-\gamma^2\right) r \,, \tag{51}$$

and expand the asymptotic momenta $\left|p^{\mathrm{L/R}}\right|$ in ascending powers of $1-\gamma^2$ to learn that $\left|p^{\mathrm{R}}\right|/m = \left(1-\gamma^2\right)^{1/2}\sqrt{v-r} + O\left(\left(1-\gamma^2\right)^{3/2}\right)$, $\left|p^{\mathrm{L}}\right|/m = \left(1-\gamma^2\right)^{1/2}\sqrt{2-v-r} + O\left(\left(1-\gamma^2\right)^{3/2}\right)$. Substituting these approximations into (50) we obtain

$$N_n^{\mathrm{cr}} = \left(1-\gamma^2\right)\sqrt{(1-r)^2 - (1-v)^2} + O\left(\left(1-\gamma^2\right)^2\right) \,. \tag{52}$$

We now wish to estimate the total number of pairs created from the vacuum by a sharply varying external field. In this case, it is convenient to first integrate over p_z, which is allowed as long as we swap the integration limits indicated in (46), i.e.

$p_z^{\max} = \sqrt{(\mathbb{U}/2)^2 - m^2 - p_x^2}$ and $p_x^{\max} = \sqrt{(\mathbb{U}/2)^2 - m^2}$. Calculating the integral and performing the change of variables proposed in (51), we expand the result in power series of $1-\gamma^2$ to find

$$\int_0^{p_z^{\max}} \frac{dp_z}{\sqrt{\omega^2 + p_z^2}} = -\frac{1}{2}\ln\left(1-\gamma^2\right) + 2\ln 2 + \ln\sqrt{1-r} - \ln\sqrt{(1-v)^2} + O\left(1-\gamma^2\right). \tag{53}$$

The most significant contribution to total quantities in this regime comes from the logarithm $\ln\left(1-\gamma^2\right)$, as $1-\gamma^2 \ll 1$. Neglecting higher-order terms in $1-\gamma^2$, the flux density of the particles created is approximately given by

$$n^{\rm cr} \approx \frac{\left(1-\gamma^2\right)^{7/2}\left|\ln\left(1-\gamma^2\right)\right| m^3}{(2\pi)^3} \int_0^{r_{\max}} \frac{dr}{\sqrt{r}} \int_{v_{\min}}^{v_{\max}} dv\,(1-v)\sqrt{(1-r)^2 - (1-v)^2}\,, \tag{54}$$

where $v_{\min} \approx r$ and $v_{\max} \approx r_{\max} \approx 1$. After straightforward integrations, the flux density of the particles created from the vacuum by a sharply varying Sauter-like magnetic step takes the approximate form

$$n^{\rm cr} \approx \frac{4}{105\pi^3} m^3 \left(1-\gamma^2\right)^{7/2}\left|\ln\left(1-\gamma^2\right)\right| . \tag{55}$$

Due to the smallness of the coefficient $\left(1-\gamma^2\right)^{7/2}$, the total number of neutral fermions pairs created from the vacuum is also small $N^{\rm cr} = V_y T n^{\rm cr}$, which means that the vacuum-vacuum transition probability is approximately given by $P_v \approx 1 - N^{\rm cr}$. We may use this result to link the flux density of pairs created (55) with the imaginary part of the effective action, given by the approximation (41).

4. Concluding remarks

Here we review our recent results on the creation of neutral fermion pairs with anomalous magnetic moments from the vacuum by Sauter-like magnetic field.[32] We show that the problem is technically analogous to the problem of charged-particle creation by an electric step, for which the nonperturbative formulation of strong-field QED exists.[34,35] To employ this formulation, we first find exact solutions of the DP equation with Sauter-like magnetic field with well-defined spin polarization and calculated all quantities characterizing the effect, in particular when the field lies in two specific configurations. When the field varies "gradually" along the inhomogeneity direction, we found that the flux density of created neutral fermion pairs is quadratic in the magnitude of the step $\mathbb{U}$. This feature is particularly different from the case of charged pair production by electric steps, in which the the flux density features a linear dependence on the magnitude of the electric step. The quadratic dependence for neutral fermions derives from the non-cartesian geometry of the parameter space formed by the quantum numbers, and it is inherent to the dynamics of neutral fermions with anomalous magnetic moments in inhomogeneous magnetic fields. This also explains why the flux density of created pairs per unit of the length are not uniform. In particular, it means that the Schwinger method of the effective action works for the case under consideration only after a suitable parameterization. The second feature worth discussing is the behavior of total quantities when the field "sharply" varies. It is exactly the form of the Klein effect.[39] If we compare the flux density of neutral fermion pairs created with the total number of electron-positron pairs created from the vacuum by inhomogeneous electric fields (given, for example, by Eq. (88) with $d = 4$ in[42]), we observe two major differences: the first is the presence of a logarithmic coefficient $\left|\ln\left(1-\gamma^2\right)\right|$, that can be traced back to the integration over p_z (53) and therefore does not depend on the external field. To our knowledge, this term has no precedents in QED (although a logarithmic coefficient of this type may appear in scalar QED). The second, and more important, is the value of the scaling (or critical) exponent seen in (55). In contrast to QED in $3+1$ dimensions, in which $N^{\mathrm{cr}} \sim \left(1-\gamma^2\right)^3$,[40–42] the total number of neutral fermions pairs created from the vacuum features a larger exponent, 7/2. Aside from minor numerical differences, this means that the total number (55) has an extra term $\sqrt{1-\gamma^2}\left|\ln\left(1-\gamma^2\right)\right|$, which is always less than unity in the range of values to γ within the interval $0 \leq \gamma < 1$. Formally, this indicates that backreaction effects caused by neutral fermions produced by sharply-evolving inhomogeneous magnetic fields may be significantly smaller compared to QED under equivalent conditions.

The mechanism here described raises the question about the critical magnetic field intensity, near which the phenomenon could be observed. It is possible to estimate such a value based on fermion's mass and its magnetic moment. Since $\max B_z(y) = B_z(+\infty) = \varrho B' \equiv B_{\max}$, the nontriviality of the Klein zone (29) yields the following condition

$$\mathbb{U} = 2\left|\mu\right|\varrho B' > 2m \Rightarrow B_{\max} > B_{\mathrm{cr}}\,, \quad B_{\mathrm{cr}} \equiv \frac{m}{|\mu|} \approx 1.73\times 10^{8} \times \left(\frac{m}{1\,\mathrm{eV}}\right)\left(\frac{\mu_{\mathrm{B}}}{|\mu|}\right)\mathrm{G}\,, \tag{56}$$

where $\mu_{\mathrm{B}} = e/2m_e \approx 5.8\times 10^{-9}\,\mathrm{eV/G}$ is the Bohr magneton.[16] For neutrons, whose mass and magnetic moment are $m_N \approx 939.6\times 10^{6}\,\mathrm{eV}$, $\mu_N \approx -1.042\times 10^{-3}\mu_{\mathrm{B}}$, the critical magnetic field (56) is $B_{\mathrm{cr}} \approx 1.56\times 10^{20}\,\mathrm{G}$. More optimistic values can be estimated for neutrinos because of their light masses and small magnetic moments. For example, considering recent constraints for neutrinos effective magnetic moment $\mu_\nu \approx 2.9\times 10^{-11}\mu_{\mathrm{B}}$[18] and mass $m_\nu \approx 10^{-1}\,\mathrm{eV}$,[30] we find $B_{\mathrm{cr}} \approx 5.97\times 10^{17}\,\mathrm{G}$. Evidently, this value changes considering different values to neutrinos' magnetic moment and mass. Taking, for instance, the experimental estimate to the tau-neutrino magnetic moment $\mu_\tau \approx 3.9\times 10^{-7}\mu_{\mathrm{B}}$[17] and assuming its mass $m_{\nu_\tau} \approx 10^{-1}\,\mathrm{eV}$ we obtain a value to B_{cr} near QED critical field $B_{\mathrm{QED}} = m^2/e \approx 4.4\times 10^{13}\,\mathrm{G}$, namely $B_{\mathrm{cr}} \approx 4.44\times 10^{13}\,\mathrm{G}$. On the other hand, assuming the lower bound found in Ref.[25] $\mu_\nu \approx 10^{-14}\mu_{\mathrm{B}}$ and the same mass $m_\nu \approx 10^{-1}\,\mathrm{eV}$ we obtain a value to B_{cr} orders of magnitude larger than B_{QED}, $B_{\mathrm{cr}} \approx 1.73\times 10^{21}\,\mathrm{G}$. The critical magnetic field surprisingly increases if one considers the magnetic moment predicted by the SM,

$\mu_\nu \approx 3.2 \times 10^{-19} \mu_{\rm B} \times (m_\nu/1\,{\rm eV})$.[27,30] Substituting this value into (56) and considering $m_\nu \approx 1\,{\rm eV}$ we find $B_{\rm cr} \approx 5.41 \times 10^{26}\,{\rm G}$. Based on these estimates, we believe that neutral fermion pair production may occur in astrophysical enviroments, in particular during a supernova explosion or in the vicinity of magnetars, whose typical order of magnetic field intensities range from $10^{16} - 10^{18}\,{\rm G}$ (or up to 10^{20} G), as reported in Refs.[43–46].

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