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A mesoscopic quantum gravity effect

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Abstract We explore the symmetry reduced form of a non-perturbative solution to the constraints of quantum gravity corresponding to quantum de Sitter space. The system has a remarkably precise analogy with the non-relativistic formulation of a particle falling in a constant gravitational field that we exploit in our analysis. We find that the solution reduces to de Sitter space in the semi-classical limit, but the uniquely quantum features of the solution have peculiar property. Namely, the unambiguous quantum structures are neither of Planck scale nor of cosmological scale. Instead, we find a periodicity in the volume of the universe whose period, using the observed value of the cosmological constant, is on the order of the volume of the proton.

Keywords Quantum gravity, Quantum cosmology, de Sitter space, Kodama state

1 Three roads to quantum gravity phenomenology

In contrast to popular lore, quantum effects are manifest at all length scales. The quantum regime is not restricted to the microscopic, but it is generically associated with extremes, e.g. high energy, low temperature, high density, etc. This association is in part historical: had a given effect been discovered prior to the development of quantum mechanics, it would likely be called classical. This is evidenced by the occasional unambiguous quantum process that is categorized as a classical phenomenon, such as the theory of electrical conductivity whose underlying mechanism is purely quantum mechanical yet the associated Ohm's law is as classical as they come. It would be a happy occurrence if such were the case in quantum gravity—that a familiar, seemingly classical phenomenon necessarily

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had its roots in quantum gravity. In this paper we address a slightly more modest proposal with this goal in mind. In particular we draw attention to the emergence of intermediate length scale structures from canonical quantum gravity via a particular solution to the non-perturbative quantum constraints.

To categorize our proposal, it will be useful to distinguish three broad classes of quantum gravity phenomenology: the microscopic, the macroscopic, and the mesoscopic. Most discussions of quantum gravity phenomena focus on the Planck scale set by $\ell_{Pl} = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35} m$. Due to the incredibly small size of the Planck scale, direct observation of such quantum gravity effects is not likely any time soon so realistic phenomenology must appeal to other length scales. Somewhat paradoxically, recurrent themes suggest that quantum gravity effects might be manifest at macroscopic, cosmological scales as in the dualities, brane world scenarios, and possible large extra dimensions of String Theory, or the quantum gravity inspired explanations for the smallness of the cosmological constant [1]. This paper, however, concerns the possibility of an intermediate mesoscopic scale emerging from quantum gravity. Although we will stop short of a full analysis of possible mesoscopic physics with quantum gravitational roots, we will see very clearly structures of mesoscopic scale emerging out of quantum de Sitter space. We will exploit a remarkably precise analogy between a non-relativistic particle in free-fall and the Kodama state, which is a candidate solution to the constraints of non-perturbative quantum gravity corresponding to de Sitter space. The latter will inherit many of the interesting quantum features of the former.

2 Particle in free-fall

Let us start with a brief review of the quantum mechanical description of a particle in free-fall in a constant gravitational field underscoring the aspects of the solution that will carry over to quantum de Sitter space (for an excellent pedagogical discussion including experimental consequences, see [2]). In light of the weakness of the gravitational force (even at the surface of the earth when acting on subatomic particles) it may come as a surprise to many that a particle in free-fall could exhibit observable quantum effects at all. In reality, the unique quantum features of the distribution are considerably larger than one might generically expect.

The Schrodinger equation for a particle of mass m in free-fall is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dz^2} + mgz\psi = E\psi. \quad (1)$$

This is a familiar differential equation whose unique bounded solution is given by the Airy function

$$\psi(z) = \mathcal{N} Ai\left(\left(\frac{2}{\lambda_c^2 \lambda_g}\right)^{1/3} (z - z_0)\right) \quad (2)$$

where \mathcal{N} is a normalization constant, $z_0 = E/mg$ is the peak of the classical trajectory, $\lambda_c = \frac{\hbar}{mc}$ is the Compton wavelength, and $\lambda_g = \frac{c^2}{g}$ is a length scale set by the macroscopic gravitational field. In the momentum representation, the

Fig. 1 The undulatory quantum probability density is shown superimposed on the classical probability density. We clearly see a close agreement between the two that is a consequence of the WKB nature of the state. The “fringes” of the quantum state are evident in the oscillatory behavior of the wavefunction

wavefunction is pure phase, and it is both an exact solution and a zeroeth order WKB state. In this sense, the quantum state is as classical as they come, and one should expect a close agreement between the classical and quantum probability distributions. The classical probability density at a point in phase space is proportional to the amount of time spent in a small neighborhood of the point, which we can write, $\rho_{\text{class}}(z) \sim \sum_{\text{crossings}} \frac{1}{|\dot{z}(z)|} = \frac{1}{\sqrt{\frac{2}{3}(z_0 - z)}}$, summing over the number of times the particle enters the small region. The asymptotic expansion of the Airy function reveals the characteristic exponential decay of the quantum probability distribution outside the classically forbidden region, and the $1/\sqrt{z_0 - z}$ behavior within the classically allowed region away from the turning point. The classical and quantum probability distributions are shown superimposed in Fig. 1 where the classical–quantum correspondence is immediately evident.

The uniquely quantum structures of the wavefunction come from a peculiar feature of the quantum wavefunction: whereas the classical trajectory and the classical structure of the quantum wavefunction is independent of the mass (a consequence of the classical equivalence principle), the oscillatory behavior of the wave function does depend on the mass. These quantum structures are scaled by length parameter $(\lambda_c^2 \lambda_g)^{1/3}$. Although λ_c is typically very small for subatomic particles, λ_g is very large and the combination results in quantum structures that are large enough to be observable. The phase gives rise to “fringes” resulting from the oscillatory behavior of the quantum probability distribution and the exponentially damped tail in the classically forbidden region. For a neutron near the surface of the Earth, this yields a width for the first fringe of approximately $(\lambda_c^2 \lambda_g)^{1/3} \approx 10^{-5} m$, which is close to the resolving power of the naked eye! For smaller mass particles the fringe width can be much larger. For example, taking the mass limit on the electron neutrino $m_{\nu_e} \lesssim 2 \text{ eV}$, the largest fringe size is on the order of one meter or more. The peculiar balance between a very small and a very large quantity that gives rise to the intermediate quantum scale will carry over to our cosmological model.

3 Quantum de sitter space and the Kodama state

The Kodama state and its various generalizations have been argued to be non-perturbative solutions to the quantum operator analogues of the equations defining de Sitter space [3; 4; 5; 6]. Just as with the particle in free-fall, the state can be viewed as an exact state and a zeroeth order WKB state. The simplest route to the construction is to begin with the Einstein–Cartan action with a cosmological constant, and for generality we will add a non-minimal parity violating term often

referred to as the Holst modification or Immirzi term:¹

$$S = \frac{1}{k} \int_M \star e e \left(R - \frac{\lambda}{6} e e \right) - \frac{1}{\beta} e e R \quad (3)$$

de Sitter space is defined by the condition $R = \frac{\lambda}{3} e e$ (the Immirzi parameter has no effect at the classical level). Inserting this into the action and setting the three-torsion to zero (a second class constraint that emerges in the detailed constraint algebra) we arrive at the Kodama state in the connection representation

$$\Psi_{R_\Gamma}[A] = \mathcal{N} e^{iS_0} = \mathcal{P} \exp \left[\frac{3i}{4k\lambda\beta^3} \int_{\Sigma} Y[A] + 2(1+\beta^2) A \wedge R_\Gamma \right] \quad (4)$$

where the implied trace is now in the adjoint representation of $SU(2)$ we have absorbed all terms that depend only on the classical configuration E into an overall phase factor. The author has argued that the above defines an auxiliary Hilbert space of states labelled by a particular configuration of the three-curvature $R_\Gamma[E]$, and the unique diffeomorphism and gauge invariant state corresponding to $R_\Gamma = 0$ is the quantum version of de Sitter space in the flat \mathbb{R}^3 slicing.

We are primarily interested in the symmetry reduced version of the state. For simplicity we consider the Kodama state in the limit that $\beta \rightarrow \infty$. As we will see the relevant quantum structures we will find are well above the Planck scale, and the Immirzi parameter typically affects physics at the Planck scale, thus we are justified in taking this limit. Furthermore, preliminary investigations to be reported in a follow up paper suggest these structures are unaffected by the introduction of the Immirzi parameter.

Beginning with the Friedman–Robertson–Walker ansatz for the metric,²

$$ds^2 = -N^2 d\tau^2 + a^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right), \quad (5)$$

the action reduces to:

$$\begin{aligned} S &= \frac{3L^3}{8\pi G} \int_{\mathbb{R}} dt \left(a^2 \ddot{a} + a \dot{a}^2 + \kappa a - \frac{\lambda}{3} a^3 \right) \\ &= \frac{3}{8\pi G} \int_{\mathbb{R}} dt \left(\mu \dot{a} + N\sqrt{\mu} (k^2 + n - \frac{\lambda}{3} \mu) \right) \end{aligned} \quad (6)$$

where, L^3 is a fiducial volume of the cell over which the action is evaluated, and we have defined $\mu \equiv L^2 a^2$, $k \equiv L \dot{a}$, and $n \equiv L^2 \kappa$. The last variable, n , is positive, zero,

¹ In the first line we have used the index free Clifford notation, which is generally easier to work with for simple calculation. In this notation, the spin connection is valued in the Clifford bi-vector algebra, $\omega = \frac{1}{4} \gamma_I \gamma_J \omega^{IJ}$, the tetrad is valued in the vector elements of the Clifford algebra, $e = \frac{1}{2} \gamma_I e^I$, and the dual is $\star = -i\gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$. Explicit wedge products have been dropped and the trace over the Clifford matrices is assumed in the action.

² In our conventions, the coordinates carry dimensions of length so the scale factor $a(\tau)$ is unitless.

or negative corresponding to a closed spherical, open flat, or open hyperbolic three geometry respectively. From the form of the action, we identify the fundamental Poisson bracket, $\{k, \mu\} = \frac{8\pi G}{3}$, which carries over to the operator commutator

$$[\hat{k}, \hat{\mu}] = i \frac{8\pi G}{3}. \quad (7)$$

Classically, there is only one solution to the Hamiltonian constraint (assuming the three-metric is non-degenerate), and it is the defining condition of de Sitter space in symmetry reduced variables:

$$n + k^2 = \frac{\lambda}{3} \mu. \quad (8)$$

As previously, we insert this solution back into the action to arrive at the WKB solution in the k -representation:

$$\Psi[k] = \mathcal{P} e^{iS_0[k]} = \mathcal{P} \exp \left[\frac{9i}{8\pi G \lambda} \left(\frac{1}{3} k^3 + kn \right) \right]. \quad (9)$$

A similar form for the Kodama state was also obtained in the context of symmetry reduced Plebanski theory in [7]. It can easily be verified that the above state is simply the symmetry reduced form of (4) in the limit that $\beta \rightarrow \infty$. We note the integer n in the symmetry reduced state plays the role of the curvature parameter R_Γ .

The wavefunction is easier to interpret in the μ -representation. The Fourier transform of (9) is the bounded solution to the Airy differential equation yielding:

$$\Psi(\mu) = \mathcal{N} Ai \left(- \left(\frac{3}{8\pi \ell_{pl}^2 r_0} \right)^{2/3} (\mu - nr_0^2) \right) \quad (10)$$

where $r_0 = \sqrt{\frac{3}{\lambda}}$ is the de Sitter radius. The semi-classical analysis of this state follows closely with that of a particle in free-fall.

4 Semi-classical analysis of the state

To carry out the semi-classical analysis of the wavefunction we first need to identify a time variable. In fact, we have already implicitly chosen one. To see this, recall that the symmetry reduced Kodama state is in the kernel of the quantum operator version of the de Sitter condition (8). However, the Hamiltonian constraint is $C_H(N) = N\sqrt{\mu}(k^2 + n - \frac{\lambda}{3}\mu)$. If the Kodama state is to be in the kernel of the Hamiltonian constraint, it is natural to choose a dynamical lapse, $N = \frac{\alpha}{\sqrt{\mu}}$, where α is a constant so that the Hamiltonian operator is now precisely the operator corresponding to (8)³. Classically this corresponds to a choice of a non-standard time

³ As always, subject to a particular (very natural) choice of operator ordering.

variable where the
de Sitter solution now takes the form

$$\begin{aligned} k(\tau) &= \tau/r_0 \\ \mu(\tau) &= \tau^2 + nr_0^2. \end{aligned} \tag{11}$$

Thus, this choice for the lapse has effectively stretched the time variable such that the de Sitter trajectory is parabolic as opposed to hyperbolic.

Now, given this trajectory, consider the classical probability distribution in the μ -representation. The classical probability density is $\rho_{\text{class}}(\mu) \sim \sum_{\text{crossings}} \frac{1}{|\mu'(\mu)|} = \frac{1}{|\sqrt{\mu - nr_0^2}|}$ which holds for $\mu \geq nr_0^2$ and $\rho_{\text{class}} = 0$ for $\mu < nr_0^2$. The probability density blows up as we approach nr_0^2 from above because the effective “velocity” goes to zero at this point. This is the analogue of the classical turning point of the particle in free-fall, corresponding to the throat of de Sitter space in the $n = +1$ model where the universe reverses its contraction and begins to expand. Just as with the particle in free-fall, we have a very close match between the classical and quantum probability density, $\rho_{\text{quantum}} = |\Psi(\mu)|^2$. The asymptotic expansion of the Airy function yields:⁴

⁴ A similar asymptotic expansion for a candidate de Sitter solution involving a cosine rather than a sine was found using different methods long ago by Hawking [8].

$$\begin{aligned}\rho_{\text{quant}}(\mu) &\stackrel{\mu \gg nr_0^2}{\sim} \frac{\sin^2\left(\frac{1}{4\pi G\hbar r_0}(\mu - nr_0)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{(\mu - nr_0^2)}} \\ \rho_{\text{quant}}(-\mu) &\stackrel{\mu \gg -nr_0^2}{\sim} \frac{\exp\left(-\frac{1}{2\pi G\hbar r_0}(\mu - nr_0)^{3/2}\right)}{\sqrt{(\mu - nr_0^2)}}.\end{aligned}\quad (12)$$

Again we see the characteristic exponential decay of the wavefunction outside the classically forbidden region, and the purely quantum oscillatory feature of the quantum probability density superimposed on the classical distribution inside the classically allowed region.

4.1 A mesoscopic length scale

As with the particle in free-fall, the uniquely quantum structures depend on the balance of a very small and a very large length scale. Just as the kinematics of the particle in free-fall is classically independent of the mass, the de Sitter solution is classically independent of Newton's constant. However, the Kodama state does depend on $G = \ell_{pl}^2$, just as the quantum free-fall state does depend on m . Consider the asymptotic expansion, (12), of the quantum probability distribution in the classically allowed region. We clearly see an oscillatory structure superimposed on the classical probability distribution. For large μ , the distribution is oscillatory with respect to $\mu^{3/2}$. Recalling that the physical volume (in the closed model) $2\pi^2\mu^{3/2}$, the quantum probability distribution has a periodicity in the volume given by:

$$\Delta V = 8\pi^4 \ell_{pl}^2 r_0. \quad (13)$$

Thus, the scale of the uniquely quantum features of the wavefunction are neither of Planck scale nor cosmological scale, but they reside in an intermediate, mesoscopic scale. Using the observed value of the cosmological constant today, $\lambda \approx 10^{-120}/\ell_{pl}^2 \approx 10^{-50} m^{-2}$, the periodicity in the volume is approximately:

$$\Delta V \approx 10^{-42} m^3 \quad (14)$$

or converting this to a length scale:

$$(\Delta V)^{1/3} \approx 10^{-14} m \approx 10 \times d_{\text{proton}} \quad (15)$$

where d_{proton} is the diameter of the proton. Thus, just as with the case of the free-fall quantum state, the balance of microscopic and macroscopic length scales conspire to produce a mesoscopic quantum scale. Since the scale of the pure quantum structures of the Kodama state is on the length scale of the strong interaction, the possibility remains that a signature of these structures may be seen in, for example the relative abundance of matter in the early universe.

Fig. 2 The smooth paraobolic curve is the classical trajectory and the oscillating curve is the effective trajectory of the volume, V , plotted as a function of τ on the vertical axis for the values $nr_0^2 = 1$ and $(\frac{3}{8\pi\ell_{pl}^2 r_0})^{2/3} = 3$. We clearly see a strong agreement between the classical and quantum trajectories. The quantum trajectory appears to evolve via a series of quasi-discrete jumps with value $\Delta V = 8\pi^4\ell_{pl}^2 r_0$

To gain further insight into the nature of the quantum oscillations of the wavefunction, we construct an effective spacetime by identifying the quantum probability distribution as an effective probability in the WKB analysis. That is, we identify:

$$\rho_{\text{effective}} = |\psi|^2 \stackrel{\text{effective}}{\approx} \frac{2}{|\mu'(\mu)|}. \quad (16)$$

This is to be viewed as an effective equations for deducing μ as a function of τ . Rearranging terms we have $\int d\tau = \pm \int d\mu \rho_{\text{effective}}(\mu)$ so that :

$$\begin{aligned} \tau &= \pm \mathcal{N}^2 \int d\mu (Ai(-\alpha(\mu - nr_0^2)))^2 \\ &= \pm \mathcal{N}^2 \left((\mu - nr_0^2)Ai[-\alpha(\mu - nr_0^2)]^2 + \frac{Ai'[-\alpha(\mu - nr_0^2)]^2}{\alpha} \right) \equiv f(\mu) \end{aligned} \quad (17)$$

where $\alpha = (\frac{3}{8\pi\ell_{pl}^2 r_0})^{2/3}$ The function $f(\mu)$ is invertible, giving an effective trajectory for μ as a function of time:

$$\mu_{\text{effective}}(\tau) = f^{-1}(|\tau|). \quad (18)$$

This trajectory is plotted in Fig. 2 with the classical trajectory superimposed.

The WKB analysis is not valid near the classical turning point so the peculiar behavior of the effective scale factor at $\tau = 0$ can be discarded. The volume appears to evolve by a series of quasi-discrete jumps that cycle average to reproduce the classical trajectory. It should be stressed that the quantum trajectory plotted in Fig. 2 is only an effective trajectory—a proper treatment would require embedding the symmetry reduced de Sitter space in a full Hilbert space, identifying an internal time variable and plotting the expectation value of the volume in an appropriate state as a function of the internal time variable. This analysis is forthcoming in a follow-up paper.

5 Concluding remarks

We have clearly seen a mesoscopic scale emerging from the non-perturbative description of quantum de Sitter space. Furthermore, the essential features followed from a WKB analysis, and regardless of the details of the quantization procedure one uses the WKB approximation should be valid in an appropriate regime. Thus, regardless of the details of the quantum theory at the Planck scale, since the quantum structures of interest are much larger we expect that they will remain. The

numerical coincidence that these fluctuations are on the order of the scale set by the strong interaction opens up the exciting possibility that this type of quantum gravity effect might have consequences for the dynamics of the matter content of the universe. It remains to be seen whether this scaling has observational consequences.

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