String Effective Actions, Dualities, and Generating Solutions

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String Effective Actions, Dualities, and Generating Solutions

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Chapter 1

Introduction

The first part of this thesis deals with the construction of the gauge theory living on a single D-brane and supergravity theories that can arise as low-energy effective descriptions of string theory and M-theory. The latter are thought to be consistent theories of quantum gravity, which unify the four different forces and thus reconcile quantum field theory (QFT) and general relativity (GR). The second part of the thesis is concerned with deriving brane solutions of (super)gravity theories, which have turned out to play an essential role in strengthening our belief in dualities in the non-perturbative limit. To fully appreciate the emergence and the merit of string and M-theory; and the discovery of dualities, we will first sketch the historical development of particle and high-energy physics.

1.1 Historical Remarks and Motivation

The larger part of 20th century theoretical physics has been dominated by two major achievements which both brought about a radical change in physics: quantum mechanics and general relativity.

In the nineteen twenties and thirties quantum mechanics was formulated as the theory that describes the behavior of particles at (sub)-atomic scales, and is therefore the theory to be used if one is dealing with elementary particles. Based on experiments it was noticed that all particles in nature have a fundamental property called *spin*, the value of which divides them into two categories: bosons and fermions. The fermionic sector contains all matter and consists of three generations, each comprising two quarks and two leptons (an electron and a neutrino). The lightest of these three generations makes up for nearly all known matter.

Between 1905 and 1916 Einstein proposed his theory of relativity. He states that the laws of physics should be the same for all observers and must therefore be formulated in an observer-independent way (covariantly). The theory of relativity consists of two parts: the theory of special relativity, which radically changed our notions of space and time and showed how these concepts are intricately connected, and the theory of general relativity (GR) which describes spacetime itself as a dynamical entity, the metric field. In GR gravity manifests itself through the curvature of spacetime, which in turn is caused by the presence of mass and energy.

A combination of special relativity and quantum mechanics finally led to the standard model (SM) around 1970, which quite successfully describes the interaction between the elementary particles. The standard model is a particular quantum field theory (QFT) of infinitely many possible ones. Here the concept of gauge symmetry plays an important role. By making symmetry transformations local, i.e. introducing coordinate dependent transformation parameters, spin one gauge bosons are introduced that mediate the force between particles. Actually the matter particles mentioned above interact by exchanging bosons: the electromagnetic, weak and strong force are described by the exchange of photons, W/Z intermediate vector bosons and gluons, respectively. The group of SM is $SU(3) \times SU(2) \times U(1)$. The experimental confirmation of the SM is excellent up to 10^2 GeV, however, some problems remain. Firstly the Higgs sector which is responsible for giving masses to the other fundamental particles, has eluded discovery so far[1]. Secondly, there are compelling theoretical arguments to consider possible extensions: first of all the SM contains nineteen finetuned parameters[2] that can not be predicted, and hence it is not a fundamental theory. Furthermore, it is difficult to explain the smallness of the Higgs mass (with $m_H \leq 1\,\mathrm{TeV}/c^2$), which goes under the name of the hierarchy problem. Also, the occurrence of three generations of matter particles has not been understood yet. It moreover turns out that the three running coupling constants that are associated with the SM gauge group become approximately equal at the enormously high energy of 10^{15} GeV. This suggests that at this energy the three forces become unified in a single 'grand unified theory' (GUT) based on a simple gauge group. Note that the SM does not contain the fourth fundamental force, gravity, since the strengths of the other three forces are much stronger than gravity.

Let us get back to GR. The experimental and theoretical successes of GR are as impressive as those of SM. For example, GR accounts for the bending of light by massive objects like our sun. GR also predicts the occurrence of spacetime singularities inside black holes[3]. It also plays a pivotal role in contemporary cosmology, where it explains for instance the observed cosmological redshift of the light of distant galaxies as a consequence of the expansion of the universe. So far GR is used as a classical field theory. An attempt to describe gravity by using similar quantization techniques

[1] This is one of the primary goals of the new LHC accelerator of CERN, which raises the experimental scale up to $\sim 10^4$ GeV.

[2] For example the parameters which correspond to masses of elementary particles.

[3] Black holes are objects that are so massive that they are hidden behind an event horizon, a surface from which even light can not escape (at least classically).

as used for the SM failed. The theory suffers from infinities, which, contrary to SM ('t Hooft and Veltman [1]), cannot be controlled. This can be seen from the fact that the gravitational coupling constant $\kappa = 8\pi G/c^4$ is not dimensionless, and it is therefore unsuitable to be used for performing perturbative expansions. The scale at which quantum gravity becomes important is the Planck scale given by

$$l_{\text{Planck}} = \sqrt{\frac{8\pi G h}{c^3}} \approx 4.1 \times 10^{-35}\,\text{m}, \qquad M_{\text{Planck}} = \left(\frac{hc^5}{8\pi G}\right)^{1/2} \sim 10^{18}\text{GeV}, \quad (1.1.1)$$

with h Planck's constant. As one can see the Planck scale is very close to the GUT scale (10^{15} GeV). This observation shows the need for a theory of "quantum gravity" that can handle all four fundamental forces simultaneously.

As a first attempt, physicists were thinking about a theoretical improvement of SM by introducing a different type of symmetry, called supersymmetry. This is a symmetry between bosons and fermions that predicts that for every boson in nature there exists a corresponding fermionic partner, and vice versa. The first motivation for using such a symmetry was to avoid the hierarchy problem; it has been shown that the Higgs mass is protected from quantum corrections by supersymmetry. However, supersymmetry transformations also introduce many new particles–sparticles– which have not been observed[4]. A partial success in unifying all fundamental forces was reached in 1976 by considering theories based on local supersymmetry. Such theories are called supergravities, extensions of GR theory that behaved better at high energies, namely the infinities *partially* canceled. The spin 2 gauge boson responsible for mediating the gravitational force is called the graviton. Its supersymmetric partner is the gravitino[5].

String theory is the most promising proposal that can deal with quantum gravity. String theory replaces particles by the oscillation modes of relativistic strings[6]. Remarkably, the graviton and (non)abelian gauge fields are necessarily part of the spectrum. Thus string theory naturally unifies the gravitational interaction with Yang-Mills theory (nonabelian version of Maxwell theory). In addition, string theory provides a discrete but infinite tower of massive vibration modes. Their mass scale is of the order of the Planck mass. In supersymmetric versions of string theory (superstring theories), the graviton is at the massless level accompanied by the supergravity field content. Indeed, it was found that the *low-energy limit* of superstring theory is given by supergravity. There is an intuitive reason why superstring theory is free of infinities. These infinities usually appear at singular points, but a string moves in a spacetime tracing out a two dimensional surface. This fact exactly causes the

[4] If supersymmetry exists, it must therefore be spontaneously broken, yielding super-particles of higher mass. It is strongly hoped that these will be discovered at LHC.

[5] In order to measure these particles energies would be needed that are way out of the range of our present accelerators.

[6] Note that string has a typical length l_s of the order of the Planck length l_p.

interactions not to occur at one single point, but to be spread out over a small area. It turns out that the perturbative string interactions are UV *finite*[7].

String theory needs besides supersymmetry six extra dimensions to be set up consistently. One can take this as a virtue rather than a vice. It has been known for a long time that higher-dimensional theories have a number of attractive features. In the 1920's, Kaluza and Klein [2, 3] tried to unify Einstein and Maxwell theories by embedding four-dimensional gravity and electromagnetism in five-dimensional spacetime. By the same token, in string theory, we take the internal six dimensions to be very small and therefore invisible to the present-day experiments. This procedure is called Kaluza-Klein *dimensional reduction*.

Unfortunately, string theory has also its disadvantages. It is only defined perturbatively, namely scattering amplitudes are expressed as an infinite expansion in powers of the string coupling g_s, associated with the Feynman-diagrams of string theory. The main setback however became apparent where there seemed to be five different superstring theories, whereas we hoped to obtain one unique theory of quantum gravity. This means that perturbative string theories only provide part of the whole picture.

Fortunately, a lot of progress has been made on this point. The major step forward was the discovery of dualities, that are symmetry transformations that link the different string theories. They relate in some cases weak coupling regime to strong coupling regime so that perturbative calculations in the first theory provide non-perturbative information on the second theory (called S-duality). In addition, string theories on different backgrounds were found to be equivalent (called T-duality). An important role was played by the so-called "brane" solutions of string theory. These are solitonic objects that can be seen as higher-dimensional generalizations of strings [8]. An important class of branes are Dirichlet branes, D-branes. These are special, since they arise on one hand as hyperplanes on which strings can end, and on the other hand as stationary solutions of (super)gravity theories. There is another class of brane solutions, S-branes (spacelike branes) which are time-dependent solutions of (super)gravities. The five apparently distinct theories and their brane solutions are related by a web of dualities. During the 1990's, it became gradually understood that these five theories all represented different limits in the parameter space of a single eleven-dimensional underlying theory, called M-theory. The fundamental degrees of freedom of M-theory remain largely unknown. Rather than being a completed theory, M-theory remains very much work in progress.

Thus we have gained some insights and had a better understanding of perturbative and non-perturbative string theory. However, there are still many interesting open issues. First there is the lack of experimental evidence. Indeed, despite all of the

[7]There is no need for introducing an ultraviolet cut-off and the theory is consistent up to high energy scales and hence is fundamental.

[8]Branes can also be considered as higher-dimensional generalizations of black holes.

promises string theory does not make a single hard verifiable prediction. Neither do rival theories of quantum gravity. It is possible to construct configurations in string theory that resemble to a high extent the SM, for instance by using intersecting D-branes. However, as of yet there is no way to single out these models as a preferred vacuum. In addition, since string theory leans heavily on supersymmetry, and supersymmetry is shared with many other theories, most notably the supersymmetric SM, the experimental discovery of supersymmetry would hardly be a full confirmation of string theory. Because of the extremely high energies involved, perhaps the future of experimental verification lies not in particle accelerators but in astrophysical and cosmological developments. Note that string theory has already passed an important test in partially solving a problem that arises when describing a typical general relativistic object, a black hole, in a quantum mechanical way: it succeeds in computing the semiclassically predicted entropy of a supersymmetric black hole by counting its microstates. Unfortunately, many tough nuts remain to be cracked in these domains such as the explanation of the observed small positive cosmological constant and the construction of string theory in time-dependent backgrounds (e.g. S-brane solutions).

However, the discussion so far left out that string theory is sometimes an incredibly powerful tool in other fields of physics and mathematics. In this small space we can only give a few examples. Most successfully there is the connection with gauge theories. It turns out that many properties of gauge theory have a geometric interpretation in terms of D-branes. Some time ago 't Hooft argued that the large N limit of gauge theories [4] very much looked like a string theory. A first concrete realization of such a connection was the AdS/CFT duality[9]. Other examples are the incorporation of Montonen-Olive duality [6] of gauge theory in the larger S-duality of string theory and the recent advances in the non-perturbative calculation of the chiral sector of $N = 1$ Super-Yang-Mills theory [7]. Nonetheless, many of these links are not established in a strict mathematical sense. Indeed, for instance the AdS/CFT correspondence and S-duality are in fact conjectures, but meanwhile an impressive amount of indirect evidence has been found.

The low-energy limit (field theory limit) of string theory, remains an important tool to study the different phenomena in string theory. Many features of string and M-theory are also present in its low-energy limit, such as D-branes and dualities, and therefore it is interesting to study this effective description.

In this thesis we will study first the low-energy limit of string theory. In particular we will show how the strings manifest themselves as a gauge theory that lives on the D-brane. We will see how the corrections to the leading order of the Maxwell action provide interesting information about the 'stringy' aspects of D-brane physics. We will then try to constrain these corrections, using the *electromagnetic duality* symmetry. Also supergravity actions will be presented in this thesis as the low energy

[9] This correspondence states that $N = 4$ Super-Yang-Mills theory is dual to string theory on $\mathrm{AdS}^5 \times \mathrm{S}^5$ [5].

tree-level effective action of string theory for slowly varying curvature. Derivative corrections, in particular corrections of order α' to heterotic string[10], will be studied.

Back to brane solutions. The second goal of this thesis is to study branes that are solutions of (super)gravity theories. As we will see, the dimensions of the extended object form the worldvolume of the brane. The remaining spacetime dimensions form the transverse space. We distinguish between two types of branes: if time is part of the worldvolume the brane is called "timelike" p-brane. Here the p stands for the number of spatial wordvolume directions. The total number of dimensions of the worldvolume is $p+1$. If time is not included in the worldvolume the brane is called "spacelike" Sp-brane. For such a brane the total number of dimensions is $p+1$ which are all spatial. Thus in both cases p refers to a $p+1$-dimensional worldvolume.

Investigating brane solutions by directly solving the equations of motion that follow from the (super)gravity action is highly non-trivial. Instead we are going to look at brane solutions whose dynamics depend only on one parameter (particle-like solution). We will see that this parameter is one of the coordinates of the transverse space. This means that the worldvolume coordinates will not explicitly enter in the solutions. This implies that one can effectively dimensionally reduce the solution over the worldvolume[11]. This maps a p-brane to a (-1)-brane solution. If we reduce over an Euclidean torus, the resulting lower-dimensional theory is a Minkowskian theory and the corresponding solution is a S(-1)-brane. If the reduction is over a Minkowskian torus (having a timelike direction), the lower-dimensional theory lives in an Euclidean spacetime, and has a (-1)-brane (instanton) as a solution[12].

The number of global symmetries becomes larger and larger as one goes down in dimensions. This can be used to simplify further our quest for brane solutions. The (-1)-brane solutions of the lower-dimensional theory are carried by the metric and the scalar fields. We will show that one can decouple the gravity field equations from the scalar field equations. As a result, one can solve for the metric and the scalar fields independently. The solution-generating technique will enable us to find the most general scalar field solutions.

[10] Heterotic string is one of the five perturbative superstring theories mentioned before.

[11] The reduction over the worldvolume of the brane gives rise to a massless lower dimensional theory, while the reduction over the transverse directions of the brane will generate a scalar potential in the lower dimensional theory. If the lower dimensional massive theory lives in a Minkowski spacetime, one then has two distinct solutions: time-dependent solutions (cosmology) and time-independent solutions (domain-walls).

[12] In this analysis we only consider consistent reductions. This means that one can always undo the steps of the reduction in such a way that we are guaranteed that we also have a solution of the action we started with. Thus one might construct a higher-dimensional solution via uplifting (oxidation) a lower-dimensional solution.

1.2 Outline

We have structured the material into several chapters:
Chapter 2 opens with an overview of string theory. It glances over perturbative string theories first and then turns to non-perturbative effects with an emphasis on D-branes. The last part of the chapter introduces the low-energy effective action of a single D-brane (abelian). It gives an overview of past attempts and successes in constructing this effective action.

Chapter 3 can be considered as a natural extension of chapter two. It starts with an introduction of supergravity theory, outlining various approaches that have been pursued to construct the corresponding low-energy effective action and its derivative corrections. The second part summarizes paper [C], where the effective action of the heterotic string to order α' has been analyzed, establishing that the supersymmetric R^2 effective action, computed from the supersymmetrization of the Lorentz Chern-Simons term, is equivalent modulo field redefinitions to heterotic string effective actions obtained by different methods.

Chapter 4 treats the approach of [B] to constrain the derivative corrections of the 4-dimensional abelian Born-Infeld action from the requirement that those terms together with the Born-Infeld action should admit *electromagnetic duality* symmetry. In the rest of the chapter we review the properties of interacting field theories which are invariant under electromagnetic duality rotations (selfduality) which transform a vector field strength into its dual. The focus will be on introducing the relevant ingredients one needs for formulating a theory as a nonlinear realization of the duality group. We show that the invariance of the equations of motion requires that the Lagrangian changes in a particular way under duality. We use this property in the general construction of the supergravity Lagrangian.

In chapter 5 the concept of nonlinear σ- models is introduced. We exhibit how such models arise in Kaluza-Klein theories and extended supergravities. We restrict to reductions over tori that are relevant for studying brane solutions in (super)gravity.

Chapter 6 starts with reviewing brane solutions that appear in (super)gravity theories. In particular we spend some time on p- and Sp-brane solutions. Then we illustrate the power of nonlinear sigma-model techniques in finding brane solutions in a purely algebraic way. We look for time-dependent Sp-branes via reducing over their worldvolumes. By means of the solution-generating technique and the coset symmetry we will be able to construct the most general S-brane vacuum solution (pure Einstein-gravity solution) with deformed worldvolume. We also consider solutions for an Euclidean theory, i.e. instantons. We show that those solutions can be obtained from timelike dimensional reductions of ordinary Lorentzian (super)gravities. The focus will mainly be on finding the generating Euclidean brane solutions that can be seen as a generating geodesic on non-Riemannian moduli spaces.

Note that all papers that I co-authored are indicated by capital letters in contrast to all other referred articles which are labeled by numbers.

Chapter 2

String Theory, D-brane and Derivative Corrections

In this chapter the basic definitions and foundations of string theory and D-brane physics will be given. We will discuss the equally famous dualities that exist between different string theories and in some cases relate strong coupling to weak coupling so that the scope of the perturbative analysis can be extended to strong coupling regimes through calculations in the dual weak coupling regimes. Then we proceed discussing the low-energy approximation of string theory, i.e., effective description. This explains how Born-Infeld theory together with its derivative corrections arise as the tree-level low-energy effective action of an underlying theory, namely string theory. Also we will give a definition and outline the possible ways of calculating it. The chapter will be closed with a small illustration showing how the extensions to Maxwell theory smear out the singularity of the point-charge at the origin.

2.1 Introduction to String Theory

This section introduces the simplest string theory, called the bosonic string. We start by describing the free bosonic string for both classical and quantum levels. We also talk about the superstring and its corresponding spectrum. Then we briefly discuss some aspects of interacting strings. For an account that really does justice to the subject of string and superstring theories, the reader is referred to [8–12].

2.1.1 Actions of Free Strings

Perturbative string theories are field theories on the worldsheet of the string, which is the two-dimensional surface swept out when the string moves in D-dimensional spacetime[1]. Thinking about the analogous situation of a particle in relativistic mechanics, the most natural Poincaré-invariant action that springs to mind is to take the area of the worldsheet. This leads to the *Nambu-Gato action*

$$S_{NG} = -T \int d\sigma d\tau \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}, \tag{2.1.1}$$

where $X^\mu(\sigma, \tau)$ represents the embedding of the worldsheet into the spacetime, and $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$ and $X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$. This action is cumbersome to quantize as it contains a square-root. Fortunately, by introducing an auxiliary worldsheet metric h_{ab}, it is possible to construct quadratic action, *the Polyakov action*[2] [15, 16]. Integrating out the auxiliary metric, one recovers the Nambu-Goto action. The polyakov action reads

$$S_P = -\frac{T}{2} \int d\sigma d\tau \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu, \tag{2.1.2}$$

where $h = \det h_{ab}$ and the metric h_{ab} has a negative eigenvalue along the timelike direction. T is the string tension given by $1/2\pi\alpha'$ with $\alpha' = l_s^2$ the so-called Regge-slope.

Let's now write down and solve the equations of motion of the Polyakov action 2.1.2. Varying the string embeddings X^μ, we obtain

$$\begin{aligned} \delta S_P = & -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \partial_a [\sqrt{-h} h^{ab} \partial_b X^\mu] \delta X_\mu \\ & + \frac{1}{4\pi\alpha'} \int d\tau \sqrt{-h} \partial_\sigma X^\mu \delta X_\mu |_{\sigma=0}^{\sigma=l}. \end{aligned} \tag{2.1.3}$$

To make this variation zero both terms must vanish independently. The first term gives the 2-dimensional X^μ equations of motion to satisfy. The vanishing of the second term results in three possibilities for boundary conditions:

- Open string Neumann boundary condition; $\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, l) = 0$.
  These boundary conditions imply that no momentum flows in or out through the string endpoints, and hence that these move freely.

- Open string Dirichlet Boundary condition: $\delta X^\mu(\tau, 0) = \delta X^\mu(\tau, l) = 0$.
  These conditions mean that we are fixing the string endpoints and no longer consider them as dynamical. This means that the string endpoints are stuck to an hyperplane. The hyperplanes on which open string can end are called

[1] Throughout this chapter we adopt the Greek indices $\mu, \nu = 0 \cdots D-1$ for spacetime and $a, b \cdots$ for the 2-dimensional worldsheet. The worldsheet is parameterizing by the spacelike variable $\sigma (0 \leq \sigma \leq l_s)$, the coordinate along the string of length l_s, and timelike variable τ.

[2] This action found first by Deser and Zumino [13] and by Brink, Di Vecchia and Howe [14].

D-branes[3] [17]. D-branes are an important class of extended objects in string theory as they appear as excitations in non-perturbative spectrum of string theory. Therefore they are presently the cornerstone in understanding the non-perturbative structure of string theory. Extended objects are in general called p-branes, where “p” stands for the number of spatial directions in the worldvolume of such objects. For instance 0-brane is a particle and string is 1-brane. From the point of view of an open string ending on Dp-brane, there are $D-p-1$ directions that satisfy the Dirichlet boundary conditions and $p+1$ directions are subject to Newmann boundary conditions. In fact the Newmann boundary conditions are the only conditions that are consistent with Poincaré invariance, whereas the Dirichlet ones explicitly break it.

- Closed string periodic boundary condition: $X^{\mu}(\tau, 0) = X^{\mu}(\tau, l)$.
  This is the requirement that the string be closed, namely that has no endpoints.

Classically strings wave. For closed strings the left- and right-moving waves are independent while for open strings the boundary conditions force the left- and right-moving modes to combine into standing waves. Projecting onto states that are invariant under the worldwheet parity introduces two more types of strings: *unoriented* open and closed strings.

Symmetries of the Action

The Polyakov action is not only invariant under worldsheet general coordinate transformations, but also under a rescaling of the worldsheet metric $h_{ab} \to \Lambda h_{ab}$, called *Weyl transformation*. The latter property is unique to two worldsheet dimensions, i.e. it would not be true for higher-dimensional membranes. These three local symmetries, reparameterizations of the two coordinates and Weyl symmetry, can, barring topological subtleties, be used to put $h_{ab} = \eta_{ab}$. Even after choosing this *conformal gauge*, a combination of Weyl and reparameterization invariance remains as a classical symmetry: the *conformal symmetry*, which is generated by the infinite dimensional *Virasoro algebra* . As such, perturbative string theory belongs to the class of conformally invariant field theories CFT in two dimensions, which are in some cases exactly solvable because of their high amount of symmetries. Due to subtleties of normal ordering the quantum Virasoro will in general differ from its classical counterpart by the introduction of a *central charge* (for a review on conformal field theory with applications to string theory see [18]).

[3] “D” in D-brane stands for for Dirichlet and “brane” generalizes the notion of membrane. Note that imposing a Dirichlet condition on the time direction X^0 lead to what is called spacelike D-brane; for short S-brane; or one can even goes further and impose it on all directions, thereby one obtain the D-instaton: the analogue of Yang-Mills instanton in string theory.

State	Physical Conditions	$(\text{Mass})^2$	Field
$\|0;k\rangle$		$M^2 = -k^2 = -\frac{1}{\alpha'}$	T_0
$\|\zeta;k\rangle$	$\zeta_\mu k^\mu = 0$ $\zeta_\mu \cong \zeta_\mu + k_\mu$	$M^2 = -k^2 = 0$	A_μ

Table 2.1.1: In the critical dimension the ground state $|0,k\rangle$ is a tachyon, it can be represented by a scalar field T_0. The first excited state can be represented by a vector field A_μ.

2.1.2 Bosonic Spectrum

A first interesting result of quantizing the Polyakov action is the spectrum of physical states and their masses. Indeed, the quantized oscillation modes of the string determine the particle content. As usual in theories with local symmetries the spectrum contains unphysical states, which have to be separated from the physical ones. Actually this requires some technical machinery (light-cone gauge, old covariant quantization or BRST quantization) that we will not delve into. Accordingly, it has been found that bosonic string theory is living in 26 dimensions.

The lowest mass levels for the open bosonic string are summarized in table 2.1.1. To each of the particles in the table a field is associated in the low-energy limit. In the first line one finds a scalar particle with negative mass-squared, the *tachyon*. The presence of such a particle indicates, just as in the case of Higgs field, that one is perturbing around an unstable vacuum. If there is no other -stable- vacuum, the energy is unbounded from below and the theory is inconsistent. For the open bosonic string theory it is conjectured that a true vacuum exists. Indeed, in modern langauge, there is a space-filling D-brane present on which open string can end. We will elaborate on D-branes in section 2.3. The tachyon indicates that this D-brane is unstable: it can decay such that a closed string vacuum results. The difference in energy density between the two vacua is thus equal the brane tension. In any case, in supersymmetric string (superstring) theory the tachyon can consistently be removed from the spectrum by Gliozzi-Scherk-Olive (GSO)-projection [19].

Of more interest to us is the second particle, which has all the properties of an abelian gauge boson A_μ, i.e. it is massless and gauge symmetry is implemented as in the Gupta-Bleuder treatment of electrodynamics: $\zeta_\mu k^\mu = 0$ is the Lorentz gauge and $\zeta_\mu \to \zeta_\mu + k_\mu$ is the residual symmetry. In fact, the simplest way to handle the local symmetries of the Polyakov action and obtain these conditions for physical states, old covariant quantization, is very similar to Gupta-Bleuder quantization.

By assigning n different labels to the endpoints of open strings, Chan-Paton factors [20], it is possible to introduce a $\mathrm{U}(n)$ gauge group. Indeed, labelling the end-

State	Physical Conditions	(Mass)^2	Field(s)
$\|0;k\rangle$		$M^2 = -k^2 = -\frac{4}{\alpha'}$	T_0
$\|\zeta;k\rangle$	$\zeta_{\mu\nu}k^\mu = \zeta_{\mu\nu}k^\nu = 0$ $\zeta_{\mu\nu} \cong \zeta_{\mu\nu} + m_\mu k_\nu + k_\mu n_\nu$ $m_\mu k^\mu = n_\mu k^\mu = 0$	$M^2 = -k^2 = 0$	$G_{\mu\nu}, B_{\mu\nu}, \Phi$

Table 2.1.2: The lowest mass states for the closed bosonic string. We obtain a tachyon with field T_0 and a reducible tensor representation resulting in the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. $\zeta_{\mu\nu}$ is the polarization tensor.

points of open strings that way, one introduces n^2 different types of string filling out the adjoint representation of U(n). In modern language this amounts to introducing n D-branes (see section 2.3). Note that besides the tachyon and the gauge field the open string spectrum comprises an infinite tower of massive states with high spin.

For completeness we state the bosonic closed string spectrum in table 2.1.2. The first particle is again a tachyon T_0, but this time there is no other stable vacuum known. The second line contains a scalar and the three massless fields, of which the scalar Φ called the *dilaton*. The two other massless states are realized by fields of spin higher than zero, which therefore have an associated gauge invariance. The antisymmetric 2-tensor $B_{\mu\nu}$ of the little group SO$(D-2)$ can be obtained from a 2-form field $B_{\mu\nu}$, often called the *Kalb-Ramond* field, with gauge transformation

$$B_{\mu\nu} \to B_{\mu\nu} + 2\partial_{[\mu}\kappa_{\nu]}. \tag{2.1.4}$$

It contains $(D-2)(D-3)/2$ on-shell degrees of freedom. The symmetric traceless tensor is obtained from a symmetric field $G_{\mu\nu}$ which transforms as

$$G_{\mu\nu} \to G_{\mu\nu} + 2\partial_{(\mu}\eta_{\nu)}. \tag{2.1.5}$$

and contains $D(D-3)/2$ on-shell degrees of freedom. This field is the *graviton* mediating the gravitational force, and is of course what got this whole business started in the first place.

2.1.3 Supersymmetric Spectrum

It is natural that any unified theory of elementary particle physics should contain fermions. It turns out that including fermions in our theory will provide us with a way, as formerly mentioned, to eliminate the tachyon, and also the consistency of the theory will restrict the number of dimensions to ten.

We can add fermions to Polyakov action by again choosing conformal gauge and adding a kinetic term for a two-component Majorana spinor $\psi_\mu = \begin{pmatrix}\psi^\mu_+ \\ \psi^\mu_-\end{pmatrix}$ transforming as vectors under the spacetime Lorentz group, giving [21]

$$S = -\frac{T}{2}\int d\sigma d\tau(\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu \gamma^a \partial_a \psi_\mu), \tag{2.1.6}$$

where γ^a is a two dimensional representation of the Clifford algebra. This action turns out to have a worldsheet symmetry termed *supersymmetry*, mapping the fermions to bosons and vice versa. Just as in the bosonic case we can have two types of boundary conditions for the open string:

- Ramond Boundary condition (R):

$$\psi^\mu_+(0,\tau) = \psi^\mu_-(0,\tau), \qquad \psi^\mu_+(l_s,\tau) = \psi^\mu_-(l_s,\tau). \tag{2.1.7}$$

- Neveu-Schwarz (NS) boundary condition:

$$\psi^\mu_+(0,\tau) = \psi^\mu_-(0,\tau), \qquad \psi^\mu_+(l_s,\tau) = -\psi^\mu_-(l_s,\tau). \tag{2.1.8}$$

For the closed string the periodic Ramond or anti-periodic Neveu- Schwarz boundary conditions for left and right moving modes can be chosen independently, resulting in four different sectors; R-R, NS-NS, R-NS and NS-R. This theory, having manifest worldsheet supersymmetry, is called the Neveu-Schwarz-Ramond (NSR) formalism. A GSO projection is needed to obtain the spacetime supersymmetry. Note that the GSO projection plays a crucial role in preserving the spin-statistics theorem.

There is another formulation of superstring theory, the Green-Schwarz (GS) formalism. The advantage of working in such a formulation is to have a theory with manifest spacetime supersymmetry. Nevertheless, quantization of this theory, till the time of writing, was only possible in light-cone gauge [22,23]. Using either the (NSR) or (GS) formalism, and choosing various combinations of the boundary conditions in the open and closed string case turns out to yield five different supersymmetric string theories: type IIA, type IIB, type I, Heterotic $\mathrm{E}_8\times\mathrm{E}_8$ and Heterotic SO(32). They all live in ten spacetime dimensions. More precisely, choosing NS boundary conditions in both the right-moving and the left-moving sector (NS-NS) leads for type II string theories to the same spectrum as the bosonic closed string (see table 2.1.3): $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. Choosing R-R boundary conditions, one can obtain antisymmetric tensors of different dimensions; $C_{(n)}$, where the sub-index represents the rank of the tensors, n is even in type IIB and odd in type IIA. As an unoriented theory does not contain $B_{\mu\nu}$, but adds an antisymmetric 2-tensor $C_{(2)}$ from the R-R sector instead. It introduces the gauge boson A^{so}_μ with gauge group SO(32) via Chan-Patton factors. Heterotic string theories [24, 25] combine the left-moving side from bosonic string theory with the right moving side from supersymmetry string theory. Notice that both heterotic

Name	Type of Strings	Bosonic Spectrum	Supercharges
Type I	closed+unoriented open	$G_{\mu\nu}, \Phi, A_\mu^{so}, C_{\mu\nu}$	$N = 1$
Type IIA	closed strings	$G_{\mu\nu}, B_{\mu\nu}, \Phi, C_\mu, C_{\mu\nu\rho}$	$N = 2$
Type IIB	closed strings	$G_{\mu\nu}, B_{\mu\nu}, \Phi, C_0, C_{\mu\nu}, C^+_{\mu\nu\rho\sigma}$	$N = 2$
Type HE	heterotic closed strings	$G_{\mu\nu}, B_{\mu\nu}, \Phi, A_\mu^{E_8}$	$N = 1$
Type HSO	heterotic closed strings	$G_{\mu\nu}, B_{\mu\nu}, \Phi, A_\mu^{so}$	$N = 1$

Table 2.1.3: The bosonic particle spectrum of the five perturbative superstring theories.

theories have $N = 1$ and differ by their gauge groups, under which the massless vector transform. We will elaborate a bit more on the fields of table 2.1.3 in the context of supergravity in chapter 3.

Spacetime fermions originate from the mixed sector, NS-R or R-NS, of the closed string or from the R sector of open string. The heterotic and type I theories comprise one gravitino and one spinor, while type II theories have two gravitinos, two spinors and a double amount of supersymmetry.

2.1.4 Interacting Strings

Apart from the Polyakov action which describes a single string, there is no action which governs the dynamics of interacting strings. Of course there are some tentative approaches pursued by number of string field theorists, endeavoring to find a string theory action analog of the quantum field theory QFT action. The situation turns out to be more involved than one expects. Finding such an action requires an age to identify the physical degrees of freedom. However, without knowing the underlying action one can still obtain the spectrum of the string, and therefore one can write down all the excited states in terms of a perturbation series of string S-matrix.

The attractiveness of string theory lies in the fact that all the string amplitudes are at each order UV finite [26]. The amplitudes will contain divergences but they can be related to poles of intermediate particles going on-shell, which means that these particles propagate over long distances. So these divergences are in fact IR effects. Only UV divergences would signify a break-down of the theory while IR divergences can be dealt with precisely as in QFT.

As in QFT there are Feynman diagram-like graphs in string theory, where the expansion of the S-matrix is expressed in terms of compact punctured surfaces. In other words, one can organize the series in terms of the topology of the diagrams which are controlled by a coupling constant λ in the sense that for each diagram we assign a factor $e^{-\lambda\chi}$, where χ is the so-called Euler characteristic of the worldsheet defined below. In addition, with each string state momentum k_μ we associate the vertex operator $V(k)$ constructed from 2-dimensional CFT of the worldsheet. For example in bosonic string theory, the vertex operator corresponding to the tachyon field T_0 reads $V_{T_0} \sim \int_{\partial\Sigma} ds e^{ik\cdot X}$. Therefore the N-point amplitude is given schematically by the following path integral over the coordinates $X(\tau,\sigma)$ and the worldsheet metric $h(\tau,\sigma)$[4]

$$S_{m_1\cdots m_2}(k_1,\cdots,k_n) = \sum_{\text{topologies}} \int \frac{[dXdh]}{V_{\text{diff}\times Weyl}} \exp(-S_P[h,X]-\lambda\chi) V_{m_1}(k_1)\cdots V_{m_N}(k_N), \tag{2.1.9}$$

where χ depends on the topology of the world sheet. It is the so-called Euler number given by

$$\chi = 2 - 2a - b, \tag{2.1.10}$$

where a is the number of handles and b the number of holes in the worldsheet. In 2.1.9 we have to divide out the local symmetries to avoid overcounting equivalent configurations. Taking the string coupling constant to be $g_s = e^\lambda$, the different topologies are weighted with g_s^{2a+b-2}. In other words, each closed string loop-a handle-introduces a factor g_s^2 and each open string loop-a hole-introduces a factor of g_s. It follows that $g_c \sim g_o^2 \sim g_s$ where g_c is the closed string coupling constant and g_o the open string coupling constant. From the last term of the action 3.2.1 we see that the constant part of the dilaton, Φ_0, sets $g_s = \exp\Phi_0$. Therefore, the string coupling g_s is not a fundamental constant, but instead set by the vacuum expectation value of the dilaton.

Perturbative string theories suffer from some problems [10]. We only mention the most important two. The first problem could be summarized by saying that we lack a *background independent formulation.* the second problem has to do with the perturbative definition of string theory: the most severe objection actually is that for a quantum field theory a perturbative definition is hardly enough. Indeed, for string theories it can be shown that the perturbative expansion looks like [27]

$$\sum_{l=0}^{\infty} g_s^l \mathcal{O}(l!), \tag{2.1.11}$$

and diverges. This means that the sum cannot be unambiguously evaluated without non-perturbative information. One expects non-perturbative effects of the order $\exp(-\mathcal{O}(1/g_s))$. Obviously, the perturbative expansion is not a good description anymore when $g_s \to \infty$. This kind of behaviour is in fact very common in quantum field theory where effects of the order $\exp(-\mathcal{O}(1/g_s^2))$ are typically encountered. There the perturbative approach can be saved by introducing classical instanton backgrounds

[4] $S_P[h,X]$ is the Euclideanized version of the 2.1.2 where we have Wick rotated the time direction of the worldsheet.

around which to perturb and then summing over all possible instantons.

The first problem, together with the second one strongly suggest that perturbative string theory is only a tool that can probe aspects of a more fundamental theory.

2.2 T-duality

Surprisingly, it is possible that two string theories of different perturbative type and/or in different backgrounds are completely equivalent. These transformations between equivalent theories that generally define a discrete group are called *dualities*. A prominent example is T-duality [28], which relates spacetime geometries possessing a compact isometry group. T-duality is a perturbative duality in the sense that it is valid order per order in the loop expansion in g_s.

We consider the simplest case, bosonic theory in flat space with one dimension, say the rth, compactified on a circle with radius R,

$$X^r \cong X^r + 2\pi R. \tag{2.2.1}$$

The compact isometry is of course translation along X^r. The mass-shell condition for the open string now reads

$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{\omega R}{\alpha'}\right)^2 + \text{contribution from the oscillators.} \tag{2.2.2}$$

As seen before, when we calculated the spectrum the oscillation contribution depends on which quantum state of the string is excited, namely which particle it represents. As for the particle there is moreover a contribution from the center of mass momentum, which is discretized because of the periodic boundary conditions. Indeed, the operator $e^{2\pi i R p_r}$ translates the string once around the periodic dimension and must leave the states invariant so that

$$p_r = \frac{n}{R}. \tag{2.2.3}$$

The states labelled by n are called the *Kaluza-Klein* states. Since strings have a tension, there is an additional contribution proportional to the length of the string. Closed strings can wrap around the compactified dimension. The states labelled by ω, the *winding number*, correspond to strings winding ω times around the compact direction. The winding number is conserved as the closed string can not unwind without breaking. The mass-shell condition 2.2.2 is inert under the following actions

$$R \longleftrightarrow \frac{\alpha'}{R}, \qquad n \leftrightarrow \omega. \tag{2.2.4}$$

It is quite remarkable that the invariance of 2.2.2 continues to hold even when one adds the oscillations modes. Therefore we found a duality of the full worldsheet CFT, which means that the theory on R and the one on $R' = \frac{\alpha'}{R}$ are entirely equivalent. This means that the closed string can not probe distances smaller than $\alpha'^{1/2}$.

For open strings the situation is slightly different. In this case the momentum in the compactified dimension is still quantized. On the other hand, because of their tension open strings will contract as much as possible and unlike wrapped closed strings, there is nothing to prevent that. Thus there are Kaluza Klein states, but there is only zero winding. Performing T-duality as for the closed string, interchanging winding and momentum, one finds winding states with zero momentum. Since open strings differ only at the endpoints from closed strings, we want to explain this using only the endpoints: fixing the endpoints at a certain position X^r, the string can not have center of mass momentum in this direction anymore. On the other hand, fixing the endpoints prevents the open string from contracting and winding is possible. In this interpretation the endpoints of the open string are in the T-dual picture constrained to a hyperplane $X^r = x^r$. Put it another way, T-duality interchanges Neumann boundary conditions, allowing the ends to move freely, with Dirichlet boundary conditions, constraining the ends. The hyperplanes on which strings end were given the name D(irichlet)-branes in [17].

In order to have better understanding we discuss in more detail the case of open strings in a constant diagonal $M \times M$ background:

$$A_r = \mathrm{diag}_A\Big[-\frac{\theta_A}{2\pi R}\Big] = -i\lambda^{-1}\frac{\partial\lambda}{\partial X^r}, \qquad \lambda(X^r) = \mathrm{diag}_A\Big[\exp\Big(-\frac{i\theta_A}{2\pi R}\Big)\Big], \quad (2.2.5)$$

where θ_A are constants and diag_A indicates a diagonal matrix of which the element at the position AA is given in square brackets and A runs from 1 to M. Locally this is pure gauge, but not globally since the gauge parameter picks up a phase

$$\mathrm{diag}_A[-i\theta_A], \qquad \text{under } X^r \to X^r + 2\pi R. \quad (2.2.6)$$

The canonical momentum p_r conjugate to the center of mass position of the string is given by

$$p_r = \int_0^\pi d\sigma \frac{\partial\mathcal{L}}{\partial\dot{X}^r} = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\,\partial_\tau X_r + A_r|_\pi - A_r|_0, \quad (2.2.7)$$

where we have used the Polyakov action with the boundary term. The momentum is quantized as before. Now consider in the T-dual picture a string stretching from D-brane A to D-brane B. The momentum k_r appearing in the mode expansion of the string reads

$$k_r = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\,\partial_\tau X_r, \quad (2.2.8)$$

so that

$$k_r = \frac{n}{R} + A_{AA} - A_{BB} = \frac{1}{2\pi\alpha'}(2\pi n + \theta_B - \theta_A)R'. \quad (2.2.9)$$

The distance in the rth direction between the endpoints is

$$X'(\pi)-X'(0)=\int_0^\pi d\sigma\,\partial_\sigma X'_r=\int_0^\pi d\sigma\,\partial_\tau X_r=2\pi\alpha' k_r=(2\pi n+\theta_B-\theta_A)R', \quad (2.2.10)$$

where the prime denotes the quantities in the T-dual picture and we have made use of the fact that T-dualizing interchanges Neumann and Dirichlet boundary conditions:

$$\partial_\sigma X'_r=\partial_\tau X_r. \quad (2.2.11)$$

Actually the distance between the two endpoints A and B is fixed; they are really stuck to hyperplanes at positions θ_A and θ_B respectively. In the T-dual picture n becomes the winding number. The open string analog of eq. 2.2.2 reads

$$M^2=k_r^2+(\text{oscillator contr.})=\frac{(2\pi n+\theta_B-\theta_A)^2R'^2}{(2\pi\alpha')^2}+(\text{oscillator contr.}). \quad (2.2.12)$$

If the oscillators do not contribute, i.e. for the lowest lying modes, the mass is proportional to the string length. The minimal mass is attained when $n=0$ and is then proportional to the distance between the branes.

2.3 D-brane

In order to obtain a p-dimensional D-brane, we perform T-duality in $9-p$ dimensions. Consequently, one has next to Neumann boundary conditions in $p+1$ dimensions, Dirichlet boundary conditions in the remaining $9-p$ dimensions

$$\begin{aligned}\partial_\sigma X^\mu&=0 \qquad \mu=0,\cdots,p\\ X^i&=x^i, \qquad i=p+1,\cdots,9.\end{aligned} \quad (2.3.1)$$

Since T-duality interchanges Neumann and Dirichlet boundary conditions, it is now obvious that T-duality in a direction perpendicular to a Dp-brane results in a D(p+1)-brane, while T-duality in a longitudinal direction results in D($p-1$)-brane. The presence of Dp-brane will break translation invariance in the $9-p$ transversal directions and it will break Lorentz invariance SO(9,1) to SO(p,1)×SO($9-p$). As a consequence the spectrum of massless states from table 2.1.1 is deformed as

$$A_\mu, \qquad \mu=0,\cdots,9\rightarrow\begin{cases}A_a, & a=0,\cdots,p\\ \Phi_i, & i=p+1,\cdots,9\end{cases} \quad (2.3.2)$$

A_a describes a p-dimensional gauge theory on the brane, while the Φ_i are the Goldstone bosons associated with breaking of translational symmetry. They are collective cooredinates describing the position of the branes consistent with the fact that D-branes are in fact dynamical objects. Indeed, we have already seen that background value A_μ has in the T-dual picture the interpretation of the position of the D-brane.

Although D-branes were already discovered from the above T-duality argument, in [17], it was not until [29] that it was realized that D-branes were in fact a missing link as they can act as sources for the R-R fields $C_{(n)}$, which fundamental string ($p = 1$) can not. Indeed, the worldvolume of a p-brane couples in a natural way to $(p + 1)$-form potential as follows

$$\int_{V_{p+1}} C_{(p+1)}, \tag{2.3.3}$$

where the integral is over the D-brane worldvolume V_{p+1}. In fact, this is an *electric* coupling, but the same potential $C_{(p+1)}$ can couple *magnetically* to a D$(6 - p)$-brane as follows: the Hodge dual of the field strength $F_{(p+2)} = dC_{(p+1)}$ in 10 dimensions is a $(8 - p)$-form $(\tilde{F})_{(8-p)}$, which has a $(7 - p)$-form $C_{(7-p)}$ as potential; the latter $(7 - p)$-form is suitable for coupling to a D$(6 - p)$ brane.

Considering the possible R-R forms in table 2.1.3, we see that IIB theories should contain D(-1) (a.k.a D-instanton) D1-, D3-, D5- and D7-branes, while IIA theories involve D0-, D2-, D4- and D6-branes. Since T-duality sends a Dp-brane to D$(p + 1)$- or a D$(p-1)$-brane, we must conclude that it also interchanges type IIA and type IIB theories, which indeed the case. We learn in addition that there must be a D9-brane in IIB, and a D8-brane in IIA. These are not associated to propagating states, so they do not appear in the spectrum.

Labelling the endpoints of open strings by introducing n Chan-Paton labels translates in the new language into introducing n D-branes. For each brane, we have a copy of the gauge sector originating in strings beginning and ending on that brane. In this way we end up with the gauge group U$(1)^n$. As seen in eq.2.2.12 the low lying modes of a string have masses proportional to the length of the string so that if the D-branes coincide there are extra massless sates coming from strings beginning and ending on different coinciding D-branes. Keeping track of the orientation of strings we count a total of n^2 massless states making up the adjoint representation of U(n), which will be the new gauge group. All this is pictured in figure 2.3.1 for $n = 2$. Giving a vacuum expectation value (vev) to some or all of the diagonal element of Φ_i lets the branes move apart and breaks the U(n) gauge group. This provides us with a geometrical picture of the Higgs effect. This shows the power of string theory and D-branes as a tool to study (non)-abelian gauge theories.

From our discussion about T-duality and D-brane, it becomes clear that introducing open strings with free endpoints into the theory is actually equivalent to inserting a D9-brane. Obviously, when there are n space-filling D9-branes, they are always coinciding.

The tension τ_{D_p} of a D-brane is of order $e^{-\Phi_0} = g_s^{-1}$ as could be seen from the way how it couples to graviton. Consequently, plugging in a D-brane background in the path integral will lead to non-perturbative effects of the order $\exp(-1/g_s)$. To have

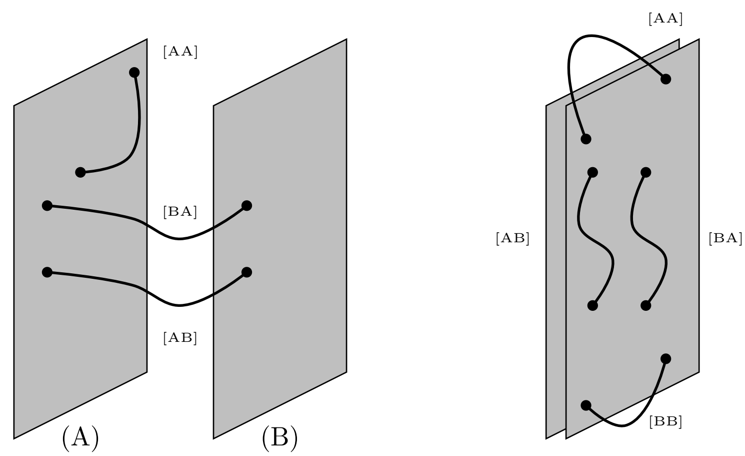

Figure 2.3.1: We have two parallel D-branes. The spectrum of [AA] and [BB] always comprise a U(1) gauge field. When the D-branes coincide, The [AB] and [BA] strings give rise to additional gauge fields which enhance the gauge symmetry from $U(1)^2$ to U(2).

a non-zero contribution these D-branes have to be localized in both time and space: D(-1)-branes, namely D-instanton or wrapped Dp-branes in Euclidean spacetime. In the end we point out that besides the microscopic description of D-branes as hyperplanes on which strings can end, they also possess a description as solutions of the equations of motion of the low-energy effective action of string theory: IIA or IIB supergravity (see chapters 3 and 6), where they look like higher-dimensional extensions of charged black holes, black p-branes [30]. It is worth noting that a D-brane breaks half of the supersymmetry. The fact that it does not break all suppersymmetry makes it a BPS (Bogomolny-Prasad-Sommerfield)-state (more about BPS states in chapter 6).

2.4 Non-perturbative Duality: S-duality

Up to now we have encountered a perturbative duality, to wit, T-duality which relates type IIA to type IIB. Actually T-duality also relates the two heterotic string theories. Still these are perturbative dualities. Nonetheless, there are also non-perturbative dualities relating a strongly coupled theory to a weakly coupled theory [31, 32]. One example is electromagnetic duality in Yang-Mills theory, which interchanges light electric charges with heavy magnetic monopoles [6]. In fact, this only works out well if there is enough supersymmetry, namely in $N = 4$ Super-Yang-Mills [33]. In that case it turns out that such a duality is a part of the discrete $\mathrm{SL}(2, \mathbb{Z})$ group dualities, S-duality.

The type IIB string theory is self-dual under the duality group $\mathrm{SL}(2, \mathbb{Z})$. Under this duality strings and D1-branes are interchanged. One of the nice features of type IIB is that it contains a non-abelian gauge theory-$N = 4$ Super-Yang-Mills-on a D3-brane. Thus, it enjoys the S-duality of Yang-Mills [34, 35] (see subsection 4.3.2 for the abelian case). Type I SO(32) and heterotic SO(32) also turn out to be related by S-duality.

Special cases are the strong coupling limit of type IIA and heterotic $E_8 \times E_8$ theory. Sticking to type IIA, one can take $g_s \to \infty$ and realize that the lowest mass states are actually D0-branes with tension τ_0. If one considers n bound D0-branes, the mass of the system is

$$m = n\tau_0 = \frac{n}{g_s\sqrt{\alpha'}}. \tag{2.4.1}$$

This evenly spaced spectrum looks like a Kaluza-Klein spectrum with periodic dimension $R_{10} = g_s\sqrt{\alpha'}$. So one could make the bold assumption that as $g_s \to 0$ this dimension is decompactified and one ends up with a 11-dimensional theory. As we will see in chapter 3 the low-energy effective theory would be then nothing but 11-dimensional supergravity. Eleven dimensions is the maximum number of dimensions allowing a locally supersymmetric field theory and this theory is unique. We have seen that string in 11 dimensions is not consistent, but in fact the strong coupling limit of type IIA is not a string theory anymore! In fact little is known about this theory and its fundamental degrees of freedom beyond the low-energy limit and its ties to type IIA. It has been given the name M-theory. From considerations about D-branes physics it became clear that all the five string theories can be understood as perturbative descriptions around five different points in the moduli space of M-theory, see figure 2.4.1.

After all we have learned, over last three decades, about non-perturbative string theory the question arises whether we have made any progress in solving the two problems stated in the end of section 2.1.4. As for the second problem we know now the nature of some of non-perturbative instanton effects: these are the D-instantons or wrapped Euclidean D-branes. Moreover, we have become more convinced that there indeed exists an underlying description, M-theory although we do not know the details. The background problem still stands. We know now some backgrounds are related by perturbative or non-perturbative dualities to other backgrounds, but the multitude of different backgrounds remains.

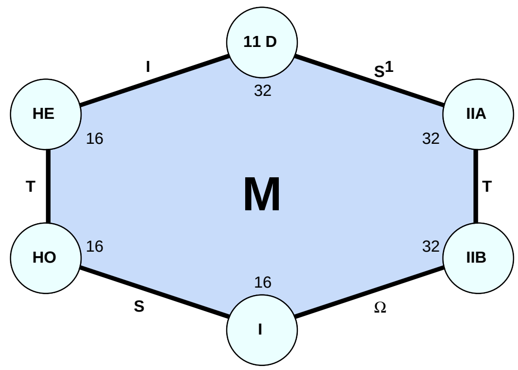

Figure 2.4.1: The M-theory moduli space.

2.5 Born-Infeld Theory and Derivative Corrections

In this section we will sketch the definition of the low-energy effective action, showing that the Born-Infeld theory arises as the low-energy approximation of open superstring theory, the worldvolume action of D9-brane, for slowly varying fields. Then we turn to the derivative corrections terms of the Born-Infeld action, listing some powerful methods for constructing such terms.

2.5.1 Effective Theory in Words: Low-Energy Approximation

One can think of string theory as a field theory with an infinite number of degrees of freedom. Indeed, every particle in the spectrum, massless or massive, corresponds to a field. Massive particles can only be detected at high energies, characterized by the string scale l_s^{-1}. For phenomenology we will be exclusively interested in the massless particles of which, fortunately, there is a finite number. For instance, for the open superstring we have the massless particles A_μ and their fermionic superpartners. Also we have seen in section 2.1.4 that the S-matrix perturbative series captures most of the perturbative effects of string theory. However, we would like to know more about the non-perturbative structure of string theories. In particular, we want to know what kind of solitons a given string theory contains. This implies that we need solutions to the classical equations of motion of the string. We do not have these equations, but we *do* know how these equations look like at low energies (energies below l_s^{-1}); they are field theory equations.

Accordingly, a *low-energy effective action* is defined as the result of integrating out all the massive and massless modes circulating in loops. Furthermore, we only allow the massless modes as external states. This means that the effective action should generate at tree-level an S-matrix, which reproduces the string S-matrix for massless external states. This action is called the *Wilsonian* effective action (WEA) (we refer the reader to [36,37] for elementary treatments, and [38] for a non-technical review). We can expect the effective action to be nonlocal[5], i.e. it will contain an infinite number of derivatives, and we can expect it to be highly complicated.

The WEA action is not the only "effective action" that occurs in quantum field theory. There also exists an object called *quantum* effective action (QEA) [39], which is by definition the generating functional of the amputated one particle irreducible diagram (1PI) of a generic field theory,

$$W[\phi] = \sum_s \frac{1}{s!} \int d^d x_1 \cdots d^d x_s W^{(s)}(x_1, \cdots, x_s) \phi(x_1) \cdots \phi(x_s). \tag{2.5.1}$$

The QEA $W[\phi]$ can therefore be viewed as a *classical* field theory that encodes all the quantum information of the underlying field theory[6]. Note that this object is *not* equal to the Wilsonian action! Indeed, the QEA incorporates the effects of loops of *all* fields (and is hence infrared divergent due to the massless particle loops), whereas the WEA only contains the effects of loops involving massive particles. However, the moment we stick to the tree-level or classical approximation, we do not see the effects of loops. Therefore the tree-approximation of the Wilsonian effective action *is* equal to the low-energy approximation of the tree-level quantum effective action for massless modes. We recommend the recent review of Burgess [40] for a thorough explanation on various effective theories.

An inconvenient property of the effective is its ambiguity. Indeed, in field theory the following equivalence theorem exists: different Lagrangians lead to the same on-shell S-matrix, in our case equal to string S-matrix, if there exists a *field redefinition*

$$\phi = \phi' + R(\phi') \tag{2.5.2}$$

transforming these Lagrangians into each other. Hence one should be prudent when comparing results that look at first sight different. Field redefinitions will be discussed in more detail, in the context of supergravity, in chapter 3.

2.5.2 D-brane Action

We restrict ourselves to the abelian case and take the limit of constant gauge field strengths. Under these conditions the bosonic part of the effective action for the fields

[5]Since string theories are extended objects, their interactions are intrinsically nonlocal. This nonlocality manifests itself in the infinite tower of massive string excitations, an phenomenon that does not occur in local field theories.

[6]If we consider a generic theory with an action $I(\phi)$, the interaction vertices stemmed from 2.5.1 are the 1PI diagrams of $I[\phi]$, reproducing thus already at tree-level *all* the amplitudes of $I[\phi]$.

coupling to D-brane can be found to all orders in α' and consists of two parts: The Dirac-Born-Infeld term and the Wess-zumino term,

$$S_{Dp-\text{brane}} = S_{DBI} + S_{WZ}. \tag{2.5.3}$$

The Dirac-Born-Infeld action [41, 42] takes on the form

$$S_{DBI} = -\tau_{D_p} \int d^{p+1}\sigma \exp(-\Phi)\sqrt{-\det(\jmath^*[G+B]_{ab} + 2\pi\alpha' F_{ab})}, \tag{2.5.4}$$

where τ_{D_p} is the tension of D-brane, G the metric, B the NS-NS 2-form, Φ is the dilaton (see table 2.1.2) and F is the field strength of the gauge potential A, $F = dA$, existing in table 2.1.1. In addition one has in 2.5.4 the pullback $\jmath^*$, from the target space to the worldvolume of the brane, which acts on an arbitrary tensor L as

$$\jmath^*[L]_{a_1\cdots a_n} = \frac{\partial X^{\mu_1}}{\partial \sigma^{a_1}} \cdots \frac{\partial X^{\mu_n}}{\partial \sigma^{a_n}} L_{\mu_1\cdots\mu_n}. \tag{2.5.5}$$

But we have also seen that D-brane couples to R-R fields $C_{(n)}$. These interactions are described by the Wess-Zumino term[7] which has been introduced first in [44]:

$$S_{WZ} = \varsigma_{Dp} \sum_n \int \jmath^*[C_{(n)} e^B] \wedge e^{2\pi\alpha' F}, \tag{2.5.6}$$

where ς_{Dp} is the charge of the Dp-brane. Here all the multiplications are in fact wedges of forms (appendix A.2.2). The formula 2.5.6 should be interpreted as follows: take allowed R-R fields $C_{(n)}$, i.e. even for type IIB and odd for type IIA, then select from the expansions of the exponentials a form with the appropriate dimension $p + 1 - n$ so that we can integrate over the worldvolume of the D-brane. In this way, one finds the coupling considered in 2.3.3 as the leading term.

The Dp-brane action is invariant under a number of local symmetries. First of all, there is the freedom to reparameterize worldvolume as well as spacetime coordinates, there is not only the gauge symmetry of A_a but also the gauge symmetries of B and $C_{(n)}$. The latter are realized as follows. The bulk field and the boundary field A appear in the combination $\mathcal{F} = \jmath^*[B] + 2\pi\alpha' F$ in the worldsheet action. From the string worldsheet action, one can easily see that the tensor gauge symmetry associated to B

$$B \to B + d\chi, \tag{2.5.7}$$

where χ is a 1-form, must because of the boundary be completed with

$$A \to A + \frac{\jmath^*[\chi]}{2\pi\alpha'}. \tag{2.5.8}$$

[7]The reason behind the appearance of such a term is traced back to the fact that this term has been necessary to cancel the anomaly of chiral fermions on the intersection of branes [43].

Indeed, the combination $\mathcal{F}$ is then invariant under the tensor gauge symmetry and it is this combination that must appear in the action as well. In addition, the Wess-Zumino action is invariant under the following collective gauge transformations of the R-R fields [45]

$$C \rightarrow C + d\kappa + H \wedge \kappa + \theta e^{B}, \tag{2.5.9}$$

where $C = \sum_n C_{(n)}$, $H = dB$ and both $\kappa = \sum_n \kappa_{[n-1]}$ and the scalar θ generate the gauge transformation. It is understood that forms of the appropriate dimensions are selected to match the dimensions on both sides.

In the upcoming sections the assumption of slowly varying fields will be loosened and we try to discuss the effective D-brane action in the abelian case with derivative corrections and say some words on non-abelian case. We emphasize here that in the non-abelian case there is no analogue of the slowly varying field strength approximation. Therefore the inclusion of derivative corrections must be mandatory.

As the dependence on B can be found from the dependence on F, we will put $B = 0$. Moreover, we will work in a flat background $G_{\mu\nu} = \eta_{\mu\nu}$ and we will not study corrections involving derivatives of the pullbacks of the bulk fields (for more about this subject we refer the reader to [46].) Next, we first employ spacetime diffeomorphism to align the worldvolume of the D-brane along $X^i = 0$ with $i = p+1, \cdots, 9$. and then the worldvolume diffeomorphisms to match the worldvolume coordinates with the remaining spacetime coordinates, $X^a = x^a$. This gauge is called the *static* gauge. Consequently, it follows that the induced metric is expressed as

$$\jmath^*[G]_{ab} = \eta_{ab} + \partial_a \Phi^i \partial_b \Phi_i, \tag{2.5.10}$$

where the scalars Φ^i are equal X^i, describing the transverse position of the D-brane. Moreover, we will usually deal with D9-brane or Dp-branes, setting all Φ^i to zero. Thus, the pullbacks become trivial and hence Dirac-Born-Infeld action reduces to the Born-Infeld action. The expression of lower-dimensional D-branes including the Φ_i can be derived by dimensionally reducing expression for the D9-brane.

2.5.3 Open Superstring Effective Action: D9-brane Action

Remarkably enough, for the abelian open superstring-in modern language the spacetime filling D9-brane action[8]- there exists a relatively simple closed expression valid to all orders in α', the Born-Infeld action. The catch is that this expression is only valid in the *slowly varying field strength limit.* This action was first obtained in [47] and [48]. The supersymmetric version of D9-brane Born-Infeld action has been achieved in [49–53]. In flat background, The Born-Infeld action is expressed as

$$S_{D9} = -\tau_{D9} \int d^{10}x \sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab})}, \tag{2.5.11}$$

[8]In [29] it was realized that D-branes take a prominent place as non-perturbative solitons, which led to a renewed interest in D-brane effective action.

where τ_{D9} is the D9-brane tension related to the string coupling constant g_s and string length l_s via $\tau_{D9} = 1/g_s(2\pi)^9 l_s^{10}$, and F_{ab} is the field strength of the gauge potential A_a living on the worldvolume of $D9$-brane. The α'-expansion of the Born-Infeld effective action reads

$$S = \frac{1}{g^2}\int d^{10}x\Big[-\frac{1}{4}F_{ab}F^{ab} + \frac{(2\pi\alpha')^2}{8}\left(\mathrm{tr}F^4 - \frac{1}{4}(\mathrm{tr}F^2)^2\right) + \cdots, \qquad (2.5.12)$$

where for D9-brane the coupling constant is $g^2 = g_s(2\pi)^7 l_s^6$.

However, as has been argued in [54], physically it is very hard to say that one is working in the limit of large and slowly varying fields; the restriction to slowly varying fields actually implies that gravitational effects are large, invalidating the restriction to flat backgrounds. The reasoning of [54] goes roughly as follows: negligible derivative implies that the field stay large over a wide region of spacetime. An estimate of total energy indicates that gravitational effects can no longer be neglected and the system is at risk of collapsing to a black hole. Thus the derivative corrections should be taken into account.

In discussing the derivative corrections, it is useful to introduce some notation. The generic term in the effective action can be written schematically as

$$\mathcal{L}_{\text{eff}} = \frac{1}{g^2}\sum_{m,n}\mathcal{L}_{(m,n)}, \qquad \text{with } \mathcal{L}_{(m,n)} = \alpha'^m(\partial^n F^p), \qquad (2.5.13)$$

the powers in 2.5.13 are related, by dimensional analysis, $2p - 2m + n - 4 = 0$. The terms at order α'^m with n derivative are denoted by $\mathcal{L}_{(m,n)}$. $\mathcal{L}_{\text{eff}}$ enjoys two symmetries which are inherited from the underlying string theory: Poincaré symmetry, from which follows that n is *even*, and the U(1) gauge invariance of massless fields. The term $\mathcal{L}_{(m,0)}$ is the Born-Infeld term mentioned above. It is well-known from string S-matrix calculation that the contributions with an *odd* number of massless fields vanish and hence terms with p odd in 2.5.13 are absent in the open string effective action (there are some terms which can be removed a by field redefinition).

Derivative corrections were first studied in [55]. In this article it has been demonstrated that for the sector $\mathcal{L}_{(m,2)} = 0,\ \forall m$ so that the first correction has four derivatives. Note that the contribution $\mathcal{L}_{(4,4)}$ to the Born-Infeld part was also calculated in the same article. Only much later all terms with four derivatives, $\sum_{m=4}^{\infty}\mathcal{L}_{(m,4)}$, were constructed in [56] and a conjecture has been made for terms with more derivatives [57]. Wyllard in this conjecture gives the recipe to calculate derivative corrections to the Born-Infeld sector from the derivative corrections of the Wess-Zumino sector. However, the conjecture is suffering from ordering ambiguities, therefore it is not well-defined since the terms following from Wess-Zumino term with $n > 4$, which have been found in [56], are not complete. Hence only partial results originated from the conjecture for the Born-Infeld term. The terms $\mathcal{L}_{(m,2m-4)}$, i.e. terms with $p = 4$, were found in [58] and the prescription therein has been generalized to non-abelian case in [59], again there with only partial results. Although the conjecture in [56] has some limitations, the results predicted by this conjecture regarding the terms with six derivatives and four fields agree with [58].

Let's say some words about non-abelian version of open superstring effective action. The coincidence of a stack of D-branes results in a non-abelian gauge theory where the zeroth order in α' is the Yang-Mills action. In order to obtain the effective action for a stack of D9-branes, we calculate α'-corrections to the effective action for open superstring with Neumann boundary conditions in all directions and non-trivial Chan-Paton factors.

Compared to the abelian case, much less progress has been made for non-abelian effective action. Up to now there is no an all-orders result in α' at hand. Firstly, it is not known how to order-now non-commuting- field strengths and covariant derivatives and secondly new identities of the form

$$[D_a, D_b]F_{cd} = [F_{ab}, F_{cd}], \tag{2.5.14}$$

appear, relating commutators of covariant derivatives to commutators of field strengths. Consequently, there is no clear way how to deal with the limit of slowly varying field strength without considering at the same time the abelian case. Some results, up to order α'^2, have been found for non-abelian effective action through the abelian one by symmetrizing over the gauge indices [60, 61]. Tseytlin in [62] has made a proposal, the *symmetrized trace* proposal, where a truncation of the non-abelian effective action has been performed, and only terms that come from symmetrizing over the gauge indices have been kept. Unfortunately, soon it was found out that this proposal missed essential physics. The problem starts at α'^4. In fact, in [63] it was revealed that already at order α'^3 there is a contradiction with the symmetrized trace prescription that only produces terms at even order in α' due to the antisymmetric property of F_{ab}.

2.5.4 Obtaining Open Superstring Effective Action

After this brief overview of the developments, let us now list some approaches to construct the abelian and non-abelian open superstring effective actions. We have first the partition function approach. This method was developed in [64–66] where it was realized that the Polyakov path-integral with background fields produced the effective action. The boundary state formalism was used in [56] to construct all terms with four derivatives in both the Born-Infeld and the Wess-Zumino part in the abelian case. Another method is based on requiring Weyl invariance of the nonlinear sigma model; here one looks at the action for open string in curved backgrounds. Then one requires the Weyl anomaly of the σ-model to vanish, which amounts to putting the β-function to zero. The resulting equations are equivalent to equations of motion derived from an effective action (see for example [48, 67] and also a quite recent paper [68] and references therein). We have been particularly interested in the string S-matrix method [69–71]. This approach follows immediately from the definition of the effective action. The idea is to calculate N-point scattering amplitudes in perturbative string theory. Then the most general gauge-invariant Lagrangian at the appropriate order is constructed or an appropriate Ansatz is made. Next its unknown coefficients are fixed by comparing the on-shell scattering amplitudes with the result of string theory. Since an N-point amplitude can only probe terms in the effective action involving up to N gauge potential A_μ and thus N field strengths $F_{\mu\nu}$, the method is perturbative in the number of field strengths although with a good Ansatz it is possible to construct an infinite series of derivative corrections. With an Ansatz based on supersymmetry, in [58] the maximum information was extracted from the 4-point amplitude. In view of its perturbative nature in F, the method is not powerful enough even to produce the complete abelian effective action. A general property for an amplitude with p external massless open string fields can be derived as follows:

$$A(1,2,\cdots,p-1,p) = (-1)^p A(p,p-1,\cdots,2,1). \tag{2.5.15}$$

The complete amplitude

$$\mathcal{A}(1,2,\cdots,p) = \sum_\sigma A(1,\sigma(2),\cdots,\sigma(p)) = (-1)^p \mathcal{A}(1,2,\cdots,p), \tag{2.5.16}$$

where the latter equality follows from 2.5.15 and cyclic invariance. Here we want to point out the consequences of the invariance of string theory under the worldsheet twist operation Ω [8]. It is known that Ω changes the orientation of the worldsheet, thus it reverses the order of the vertex operators on the boundary of the disk amplitude. In addition, Ω acts on the vertex operators, giving extra factor of $(-1)^N$, where N counts the number of oscillators involved. We conclude that indeed the S-matrix elements involving an odd number of massless fields vanish. Note that in non-abelian case the S-matrix approach has been more successful mostly because in that case the other available methods are not as powerful either.

There are some other indirect methods which use a symmetry or other property the action would have. The disadvantage of the indirect methods is that in most cases the action is not completely fixed by requiring the desired property and typically unknown coefficients remain. In particular we mention; the *Noether method* [72] which is an iterative method based on supersymmetry. Also the existence of certain BPS solutions used in [73, 74] helps to a great extent with fixing some coefficients of the non-abelian effective action. A method which is of our interest is the one we shall discuss in chapter 4, namely using *electromagnetic duality* invariance as a condition on the abelian effective action in the hope that we can constrain the derivative corrections terms [B].

2.5.5 The 4-point Function of Open Superstring Effective Action

The matching procedure based on the calculation of S-matrix has been used in [58] to derive the terms $\mathcal{L}_{(m,2m-4)}$, i.e. the terms with $p = 4$. The authors of [58] are able to write down a closed form for the effective action because the open string four-point function factorizes in a product of two terms: the first term (K) depending on polarization vectors and wave functions, the second term ($\mathcal{G}$) depending only on the momenta. The first term determines how the fields should appear in the effective action. The second term expands into an infinite series in α', and determines how derivatives should be distributed over the fields. The 4-point function is given by [75]

$$\mathcal{A}(1,2,3,4) = -16ig^{-2}\alpha'^2(2\pi)^{10}\delta^{10}(k_1+k_2+k_3+k_4)\times \mathcal{G}(k_1,k_2,k_3,k_4)K(1,2,3,4), \tag{2.5.17}$$

where $\mathcal{G}$ contains the α' dependence and behaves as

$$\begin{aligned} \mathcal{G}(k_1,k_2,k_3,k_4) &= G(s,t)+G(t,u)+G(u,s) \\ &= \frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(1-\alpha's-\alpha't)}+\frac{\Gamma(-\alpha't)\Gamma(-\alpha'u)}{\Gamma(1-\alpha't-\alpha'u)}+\frac{\Gamma(-\alpha's)\Gamma(-\alpha'u)}{\Gamma(1-\alpha's-\alpha'u)}. \end{aligned} \tag{2.5.18}$$

The Mandelstam variables s, t, and u are defined, up to momentum conservation and the mass-shell condition, in such a way that $\mathcal{G}$ is manifestly symmetric in the momentum k_i. They have been chosen to be

$$s = -k_1\cdot k_2 - k_3\cdot k_4, \qquad t = -k_1\cdot k_3 - k_2\cdot k_4 \tag{2.5.19}$$

$$u = -k_1\cdot k_4 - k_2\cdot k_3, \qquad \text{with } s+t+u=0. \tag{2.5.20}$$

It is easy to see by expanding in α' 2.5.18 that $\mathcal{G}$ is regular. The leading term of this expansion is written as

$$\mathcal{G}(k_1,k_2,k_3,k_4) = -\frac{\pi^2}{2}+\mathcal{O}(\alpha'^2). \tag{2.5.21}$$

The momenta and the wave function dependence are encoded in K which, for 4-massless vector fields amplitude, has been found to be elegantly expressed in terms of k_i, the polarization vector ζ_i of the ith incoming field, and a tensor of rank eight, i.e. t_8. Thus we have

$$K(1,2,3,4) = t_{abcdefgh}k_1^a\zeta_1^b k_2^c\zeta_2^d k_3^e\zeta_3^f k_4^g\zeta_4^h, \tag{2.5.22}$$

with t_8 satisfying[9]

$$t_{abcdefgh}M_1^{ab}M_2^{cd}M_3^{ef}M_4^{gh} = -2(\mathrm{tr}M_1M_2\mathrm{tr}M_3M_4 + \mathrm{tr}M_1M_3\mathrm{tr}M_2M_4 + \mathrm{tr}M_1M_4\mathrm{tr}M_2M_3) + 8(\mathrm{tr}M_1M_2M_3M_4 + \mathrm{tr}M_1M_3M_2M_4 + \mathrm{tr}M_1M_3M_4M_2), \quad (2.5.23)$$

where the M_i are antisymmetric tensors. $t_{abcdefgh}$ is antisymmetric in the pairs (ab), (cd), etc., and is symmetric under the exchange of such pairs. Now, we will summarize the technique.

The leading order contribution to the amplitude is just the expression of K 2.5.22 times a constant. This term is reproduced by $\mathcal{L}_{(2,0)}$ contribution to the effective action, namely

$$\begin{aligned}\mathcal{L}_{(2,0)} &= \frac{a_{(2,0)}\alpha'^2}{8}\left(\frac{1}{24}\right)t_{abcdefgh}F^{ab}F^{cd}F^{ef}F^{gh} \\ &= \frac{a_{(2,0)}\alpha'^2}{8}\left(\mathrm{tr}F^4 - \frac{1}{4}(\mathrm{tr}F^2)^2\right), \end{aligned} \quad (2.5.24)$$

where $a_{(2,0)}$ has been fixed to be $(2\pi)^2$ which agrees with the expansion of the Born-Infeld action in 2.5.12.

In [58] it has been observed that every factor of momentum k_i in K is reproduced by a derivative acting on the appropriate field in 2.5.24. Notice that the complete amplitude 2.5.17 differs from the leading order contribution by extra factors of momentum, i.e. by $\mathcal{G}$. Therefore in order to reproduce these factors, one simply needs to act with derivatives on the appropriate fields. This can be done by first defining the four fields at different points in spacetime, giving rise to non-local action. That is, one considers the fields $A_a(x_i)$, where $i = 1, \cdots, 4$, and then replace the momenta k_i in the amplitude by differentiations with respect to the appropriate coordinate in the effective action, i.e. $k_{i,a} \to -i\partial/\partial x_i^a$. Certainly, one needs to multiply the resulting expression by delta functions and then integrate over the x_i to render the action local, which we denote by $S_{eff}[A_a]$[10]. Now, one can expand $\mathcal{G}$ in α', and then by substituting derivatives for momenta in this expansion and inserting the resulting expression in $S_{eff}[A_a]$, one can straightforwardly construct the contribution to the 4-point effective action at any desired order in α'. For example, for $m = 4$, i.e. at

[9] Explicit expression of t_8 is given first in [75].

[10] A detailed derivation for the complete 4-fields effective action which reproduces the 4-point amplitude can be found in [58].

order α'^4, one obtains

$$\begin{aligned}\mathcal{L}_{(4,4)} &= a_{(4,4)}\alpha'^4 t_{abcdefgh}\partial_k F^{ab}\partial^k F^{cd}\partial_l F^{ef}\partial^l F^{gh}\\ &= 8a_{(4,4)}\alpha'^4\Big[(\partial_k F_{ab}\partial_l F_{bc}\partial^k F_{cd}\partial^l F_{da} + 2\partial_k F_{ab}\partial^k F_{bc}\partial_l F_{cd}\partial^l F_{da})\\ &- \frac{1}{4}(\partial_k F_{ab}\partial^k F_{ab}\partial_l F_{cd}\partial^l F^{cd} + 2\partial_k F_{ab}\partial_l F_{ab}\partial^k F_{cd}\partial^l F_{cd})\Big],\end{aligned} \tag{2.5.25}$$

where $a_{(4,4)}$ is determined to be $\frac{\pi^4}{288}$ by string theory methods.

2.6 Illustration: Regularity of the Point-charge

This section is mainly based on [76]. There is a wide belief that α'-corrections are responsible for smearing out some singularities. The example of the open string that we will discuss below suggests that α'-corrections smooth out singularity of the leading order solution. In other words, we shall show that the point-charge singularity of Maxwell theory is absent in the open string theory. As seen before, the tree-level abelian vector field effective action of the open superstring theory takes schematically the form

$$\mathcal{L}_{\text{eff}} \sim \sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab})} + \text{derivative terms}, \tag{2.6.1}$$

where we ignore all the field strengths derivative terms and consider only the Born-Infeld sector. Since the open string is charged at its ends, a charged open string source can be added to the action 2.6.1, namely a point-particle source proportional to $QA_0(x)\delta^{D-1}(x)$ where Q is the charge. Therefore the corresponding electric field might be calculated.

One of the nice features of the Born-Infeld Lagrangian is that while in the Maxwell theory the point-like charge is singular at the origin and has an infinite self-energy, in the nonlinear Born-Infeld case the field is regular at $r = 0$ [41] (in spherical coordinates) where the field strength takes its maximal value and its total energy is finite. For example, in $D = 4$ and with only electric fields, the Born-Infeld Lagrangian recasts into

$$\mathcal{L}_{BI} \sim \sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab})} = \sqrt{1 - (2\pi\alpha' E)^2}. \tag{2.6.2}$$

The analogue of the Maxwell equation is

$$\frac{\partial(r^2 D)}{\partial r} \sim Q\delta(r), \qquad \text{with } D = \frac{E}{\sqrt{1-(2\pi\alpha' E)^2}}. \tag{2.6.3}$$

The solution is

$$D = \frac{Q}{r^2}, \qquad \text{namely } E \equiv F_{rt} = \frac{Q}{\sqrt{r^4 + r_0^4}}, \tag{2.6.4}$$

with $r_0^2 = 2\pi\alpha' Q$. The distribution of the electric field, i.e. $\mathrm{div}\,\mathrm{E} = 2\pi\rho$, gives rise to the fact that the source is no longer point-like, but has an effective radius $r_0 \sim \sqrt{\alpha' Q}$. To illustrate the effective description discussed above we distinguish:

- In the region $0 \leq r < r_0$, the electric field is approximately constant, i.e.,

$$E \sim \frac{Q}{r_0^2} \sim \frac{1}{\alpha'}. \tag{2.6.5}$$

- At $r = 0$, the derivative vanishes.
- Near $r \sim r_0$, the derivative is suppressed by a power of Q

$$\frac{\partial E}{\partial r} \sim \frac{Q}{r_0^3} \sim \alpha'^{-\frac{3}{2}} Q^{-1}. \tag{2.6.6}$$

That means that the effect of the derivative terms in 2.6.1 must be small, i.e., the conclusion about the regularity of the static spherically symmetric point source solution applies to the *full effective action* of the open string theory. We close this section with the observation that if Q is large the derivative corrections to the effective action do not affect significantly the form of the Born-Infeld solution 2.6.4.

Chapter 3

Heterotic Supergravity, Chern-Simons Terms and Field Redefinitions

In this chapter we will introduce the supergravity action as the low-energy effective action of superstring theories. We shall also outline various approaches that have been used to construct such an action and the corresponding higher derivative corrections. Field redefinitions and equivalent effective actions will be studied for the heterotic string to order α', having the Chern-Simons terms included. Also some comments on higher order terms in α' will be made.

3.1 Supergravity Theory

3.1.1 Preliminary

Supergravity theories were first presented as extensions of general relativity with fermionic and bosonic matter fields [77]. Such extensions have been performed in a way that the theory has a local supersymmetry which can be considered as an extension of Poincaré symmetry of general relativity. The Poincaré Lie algebra is formulated as a semi direct product of spacetime translations with generators P_m and Lorentz rotations J_{mn} such that

$$[P_m, P_n] = 0, \qquad [P_m, J_{nk}] = \eta_{mn}P_k - \eta_{mk}P_n, \tag{3.1.1a}$$

$$[J_{mn}, J_{kl}] = \eta_{nk}J_{ml} - \eta_{mk}J_{nl} - \eta_{nl}J_{mk} + \eta_{ml}J_{nk}. \tag{3.1.1b}$$

Adding extra fermionic generators Q_α to the Poincaré algebra leads to the well-known super Poincaré algebra. The generators Q_α are fermionic in the sense that they transform in spinor representations of the Lorentz group where α is the spinor index. Therefore the super Poincaré algebra must contain anti-commutation relations along with the commutation relations. We consider the example of the minimal super algebra[1]

$$[J_{mn}, Q_\alpha] = -\frac{1}{4}(\gamma_{mn})_\alpha{}^\beta Q_\beta, \qquad [Q_\alpha, P_m] = 0, \tag{3.1.2a}$$

$$\{Q_\alpha, Q_\beta\} \sim \gamma^m_{\alpha\beta} P_m. \tag{3.1.2b}$$

It is clear from these relations that the job of the Q_α-generator is to rotate fermion and boson fields to each other.

Local supersymetric invariant equations of motion and a set of fields that lie in irreducible representations of a super algebra lead to a supergravity theory. In order to construct the supergravity multiplet one has to associate to every generator a vector field. In other words, the gauge fields that correspond to P_m are the vielbein e^m_μ and for J_{mn} the gauge fields are the spin connection 1-form ω^{mn}_μ which due to the equations of motion is considered as a variable dependent on the vielbein, i.e. $\omega^{mn}_\mu(e)$. On the other hand, the gauge field corresponding to a supersymmetry generator is the gravitino denoted by ψ^α_μ, spin 3/2-field. The supergravity multiplet is then defined as the smallest set of fields involving the vielbein and the gravitino that form an irreducible representation of the super algebra[2]. Note that the number of boson and fermion degrees of freedom in any multiplet should be equal. This can be seen from the fact that $Q|Boson\rangle = |Fermion\rangle$. Acting again with the operator Q, one finds from the algebra that $Q^2|Boson\rangle \sim P|Boson\rangle = |Boson'\rangle$, where $|Boson'\rangle$ is a translated boson. Now if translations are invertible the dimension of the bosonic space is equal that of the fermionic.

Besides the supergravity multiplet one can out of the representations of super algebra construct multiplets that do not describe gravity, namely do not contain graviton and gravitino. These multiplets are representations of rigid supersymmetry, and are called the scalar, vector and tensor multiplets. It is worth noting that a rigid supersymmetry can be converted to local by coupling the multiplet to the supergravity multiplet.

So far we have just mentioned what we have called minimal super algebra, i.e., super algebra with one spinor Q_α. One can also generalize this to more supersymmetry generators Q^I_α, with I runs from $1 \cdots N$. The supergravity theories that have been

[1] The Gamma matrices $(\gamma_m)_\alpha{}^\beta$ obey the relation $\{\gamma_m, \gamma_n\} = 2\eta_{mn}$. The matrices γ_{mn} are the antisymmetric products $\gamma_{mn} = \gamma_{[m}\gamma_{n]}$. The charge conjugation matrix is playing the role of the metric on spinor space; It satisfies the relations $C^T = \kappa C$ and $\gamma^T_m = \epsilon C \gamma_m C^{-1}$ with κ and ϵ take the values +1 or -1.

[2] Often one has to add scalars such as dilaton, vectors, e.g. graviphoton, and fermions (dilatini) etc...

Superstring theory	Low-energy approximation
Type IIA	$N = 2$ type IIA supergravity
Type IIB	$N = 2$ type IIB supergravity
Type I	$N = 1$ supergravity coupled to SO(32) YM multiplet
Heterotic SO(32)	$N = 1$ supergravity coupled to SO(32) YM multiplet
Heterotic $E_8 \times E_8$	$N = 1$ supergravity coupled to $E_8 \times E_8$ YM multiplet

Table 3.1.1: Superstring theories and their low-energy limits.

found so far are labelled by the number of supersymmetries N and the spacetime dimension D where they live. The number of components of the irreducible spinors Q^I_α is known as supercharges. The maximal number of supercharges that a field theory (theory that does not contain fields with spin higher than two) can have is 32 or less. For example in $D = 4$, a spinor has four real components, then the maximal number of supersymmetries is $N = 8$, e.g. maximal supergravity in $D = 4$. One special example is 11 dimensional supergravity where the spinor has 32 supercharges and hence $N = 1$. In 11 dimensions, the supergravity theory is unique and there is only one supergravity multiplet consisting out of the 11-bein e^m_μ, gravitino ψ^α_μ and the 3-form gauge potential $A_{\mu\nu\rho}$.

3.1.2 Supergravity Effective Actions

Although supergravity theory was not shown to be a finite perturbation theory to all orders, their effective actions are still crucial for many applications, especially because of the remarkable fact that they turned out to describe the low-energy effective behavior of superstring theories see table 3.1.1. Several different methods can be used to formulate supergravity theories and their derivative corrections.

One straightforward approach is to directly gauge the supersymmetry algebra the way we described above. In addition, most of the methods that have been pursued to construct the low-energy effective actions $\mathcal{L}_{\rm eff}$ of superstring theories containing closed strings- supergravity actions with derivative corrections- are to some extent the same approaches mentioned in chapter two for constructing the open superstring effective actions. The first method, already outlined in chapter 2, is to simply construct the first quantized string theory in a background field (see for example [78–82]). The consistency requirements on the string theory then lead to constraints on the background fields, which can be promoted to be the equations of motion. In other words a consistent string theory can be constructed whenever the corresponding σ-model[3]

[3] To remind the reader, the nonlinear sigma model is a scalar field theory in which the scalar field takes values in some non-trivial manifold M, the target space.

is conformally invariant and the requirement of conformal invariance yields the classical equations of motion- which are identified with the β function, renormalization group coefficients- for the background fields. This method has the advantage that the 10-dimensional symmetries of superstring theory can be made explicitly and that one may have the ability to get results that are valid to all orders in perturbation theory. This method has the drawback of demanding an n-loop computation of the β functions in order to obtain the nth order term in the effective action $\mathcal{L}_{\text{eff}}$.

We have also seen in chapter 2 that there exists another method, which is in practice somewhat simpler, for constructing the effective action for open superstring theory. Here one can also use closed string theory to calculate the scattering amplitudes of its massless particles in the tree-level approximation. One then constructs an effective Lagrangian which reproduces the closed superstring S-matrix [83, 84]. Practically, S-matrix method, can be implemented as well in a perturbative fashion (in analogy with open string case) in a sense that one first constructs a 2-point function $\mathcal{L}_2$ that encodes the massless free particle of the closed superstring theory. We then incorporate cubic terms, i.e. the 3-point interactions, thus yielding $\mathcal{L}_3$. The 4-point function string scattering amplitudes can then be added[4]. The pole corresponding to the intermediate massive particles having no singularities for small values of momentum and can therefore be expanded in a power in α'. On the other hand, each term in this expansion can be reproduced by the local vertex operator, defined in section 2.1.4, namely the 4-point vertex operator V_4 which actually starts out quartic in the massless fields. Thus the 4-point sector $\mathcal{L}_4$ is constructed, the effective action for theories with closed superstring correct through quartic order. This machinery can be repeated for higher point amplitude, e.g. five, six and so forth, thereby yielding, in principle to all orders. In fact, by exploiting the local and global symmetries of the theory, the task of constructing the effective action $\mathcal{L}_{\text{eff}}$ can be greatly simplified. Roughly speaking, these symmetries help with generating terms at a given order that must appear in higher orders as a result of such symmetries.

3.1.3 Field Redefinitions Ambiguity

The effective action constructed this way, namely following either of the methods outlined above, will not be *unique*. That is because the scattering amplitude is unaffected by a field redefinition. In other words if we construct an action $\mathcal{L}[\Phi_a]$ to yield the S-matrix for particles represented by the fields Φ_a, the Lagrangian

$$\mathcal{L}[\Phi_a(\Phi')] \equiv \mathcal{L}'[\Phi'_a] \tag{3.1.3}$$

[4] Through unitarity one might guarantee that the massless poles will be those follow from the tree diagrams of $\mathcal{L}_3$.

will give the same S-matrix. Field redefinition can be performed order by order in perturbation theory provided that the field redefinition transformation

$$\Phi' \to \Phi(\Phi') = \Phi' + a_2\Phi'^2 + a_3\Phi'^3 + \cdots \tag{3.1.4}$$

is nonsingular.

To illustrate the field redefinition ambiguity in S-matrix method, let us imagine that we have calculated the 3-point sector $\mathcal{L}_3$ for one of the string theories. Now, we wish to find $\mathcal{L}_4$. This will involve new 4-point terms, to account for the pieces of the 4-point function which are not implied by $\mathcal{L}_3$. In order to obtain these we first denote by $\mathcal{L}_s$ the Lagrangian which reproduces all string theory 4-point amplitudes to desired order, i.e., $\mathcal{L}_s$ encodes a set of quartic terms. Then one can find a similar set of terms which we call $\mathcal{L}_f$, reproducing all the 4-point amplitudes coming from $\mathcal{L}_3$. Subtracting $\mathcal{L}_f$ from $\mathcal{L}_s$, one then obtain the terms that should be added to $\mathcal{L}_3$ to yield $\mathcal{L}_4$. For the sake of simplicity, let's calculate $\mathcal{L}_4$ for a toy model having $\mathcal{L}_s = 0$ and

$$\mathcal{L}_3 = -\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \kappa(\partial_\mu\partial_\nu\Phi\partial^\mu\Phi\partial^\nu\Phi). \tag{3.1.5}$$

The only contribution to the 4-point function following from $\mathcal{L}_3$ is the diagram consisting of the exchange of Φ. Therefore the vertex to which Φ couples can be obtained by varying the action $\mathcal{L}_3$ w.r.t the field, i.e.,

$$V_2 = \kappa\frac{\delta}{\delta\Phi}\left(\partial_\mu\partial_\nu\Phi\partial^\mu\Phi\partial^\nu\Phi\right) = -\kappa(\partial_\mu\partial_\nu\Phi\partial^\mu\partial^\nu\Phi) + \kappa(\partial^2\Phi\partial^2\Phi), \tag{3.1.6}$$

where we have made use of momentum conservation to move the derivatives from the Φ of intermediate state (virtual) to the physical ones. The expression 3.1.6 is evaluated on-shell. Therefore it is allowed to add terms to 3.1.6 that vanish on-shell. The term that we should add is

$$V_2' = -\kappa\left(2\partial^\mu\Phi\partial_\mu\partial^2\Phi + \frac{3}{2}(\partial^2\Phi)^2 + \frac{1}{2}\Phi(\partial^2)^2\Phi\right). \tag{3.1.7}$$

The expression 3.1.6 becomes

$$V_2 = -\frac{1}{4}\kappa(\partial^2)^2(\Phi^2). \tag{3.1.8}$$

We have done this in order to have the momenta in the vertex operator V_2 emerge in the form of an inverse of propagator, that cancel with the propagators to which it is attached.

The Lagrangian $\mathcal{L}_f$ representing the scattering amplitude behaves as

$$\begin{aligned}\mathcal{L}_f &= V_2 P V_2 \\ &= -\frac{1}{8}\kappa^2[(\partial^2)^2(\Phi^2)]\frac{-1}{\partial^2}[(\partial^2)^2(\Phi^2)],\end{aligned} \tag{3.1.9}$$

where P is the Φ propagator. Again, one can add terms involving $\partial^2\Phi$, that vanish on-shell, to obtain the following expression[5]

$$\mathcal{L}_f = -\frac{1}{4}\kappa^2 \partial^\rho\partial^\sigma\Phi\partial_\rho\partial_\sigma\Phi\partial^\delta\Phi\partial_\delta\Phi. \tag{3.1.10}$$

Since $\mathcal{L}_s$ vanishes, we have $\mathcal{L}_4 = \mathcal{L}_3 - \mathcal{L}_f$. Then the 4-point effective Lagrangian, after integrating by parts, takes on the form

$$\begin{aligned}\mathcal{L}_4 = &-\frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi + \kappa\partial^\mu\partial^\nu\Phi\partial_\mu\Phi\partial_\nu\Phi \\ &-\frac{1}{2}\kappa^2(\partial^\rho\partial^\sigma\Phi\partial_\rho\Phi\partial^\delta\partial_\sigma\Phi\partial_\delta\Phi).\end{aligned} \tag{3.1.11}$$

Performing the following field redefinition

$$\Phi' = \Phi - \frac{1}{2}\kappa\partial^\mu\Phi\partial_\mu\Phi, \tag{3.1.12}$$

one can then realize that 3.1.11 is *equivalent* to a free theory with

$$\mathcal{L}_4 = -\frac{1}{2}\partial^\mu\Phi'\partial_\mu\Phi'. \tag{3.1.13}$$

This agrees with the fact that the 3 and 4-point scattering amplitudes for our model and for the free field theory are identical; they are all zero, and that the S-matrix does not change under field redefinitions.

The same ambiguity exists in the previously discussed σ-model approach to the string equations of motion. Indeed, the β-functions of a renormalizable field theory with couplings Φ_a are not *unique*. They depend upon the definition of the coupling constant and the renormalization prescription. Using the definition 3.1.3 and the transformation 3.1.4, we find that the equations of motion have the same content since the extrema of $\mathcal{L}$ and $\mathcal{L}'$ are equal

$$\frac{\delta\mathcal{L}[\Phi]}{\delta\Phi_a} = \frac{\delta}{\delta\Phi_a}\mathcal{L}'[\Phi'(\Phi)] = \sum_b \frac{\delta\mathcal{L}'}{\delta\Phi'_b}\frac{\delta\Phi'_b}{\delta\Phi_a}, \tag{3.1.14}$$

as long as the Jacobian $\delta\Phi'_b/\delta\Phi_a$, is nonsingular. Now, if we redefine the couplings, namely the fields, $\Phi \to \Phi(\Phi')$, the β-functions

$$\beta_a(\Phi) = \mu\left(\frac{\partial\Phi}{\partial\mu}\right), \tag{3.1.15}$$

[5]Naively $\mathcal{L}_f$ seemed to have a pole due to the propagator, however this pole cancelled by the inverse propagator in the vertex, leaving a contact term.

transform under a field redefinition as

$$\beta_a(\Phi) = \mu \frac{\partial}{\partial \mu} \Phi_a(\Phi') = \beta'_a(\Phi') \frac{\partial \Phi_a}{\partial \Phi'_b}. \tag{3.1.16}$$

Nonetheless, The zeroes of β_a, identified with the equations of motion, are invariant under a non-singular field redefinition. In order to avoid having the fields overconstrained, both sets of equations β_a and $\frac{\delta \mathcal{L}}{\delta \Phi_a} = 0$ have to be satisfied and coincide. The properties of 3.1.4 suggests that they are related by a metric in the field space

$$\beta_a(\Phi) = G_{ab} \frac{\delta \mathcal{L}}{\delta \Phi_b}. \tag{3.1.17}$$

A direct connection between the β functions and the equations of motion is argued for in [85].

3.2 Strings in Background Fields: Nonlinear Sigma Model

Let's now make use of the σ-model approach and derive the bosonic sector of the supergravity action. We restrict ourselves to the bososnic string and try to describe a string moving in a more general spacetime than the Minkowski space we have considered in chapter 2. The most general covariant action we can write down with two worldsheet derivatives and appropriate symmetries, i.e. gauge invariance and local Weyl invariance, is the *nonlinear sigma model action*

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma d\tau d\sigma (\sqrt{-h} h^{ab} G_{\mu\nu}(X) - \varepsilon^{ab} B_{\mu\nu}(X)) \partial_a X^\mu \partial_b X^\nu + S[\Phi], \tag{3.2.1}$$

where ε^{ab} is the fully antisymmetric tensor in two dimensions, and the integral[6] is over the worldsheet Σ.
Actually, one can think of this action as a string moving in coherent backgrounds, $G_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ called an axion, and a scalar field Φ, i.e. the dilaton. The sector $S[\Phi]$ of the action represents the coupling of string to the dilaton

$$S[\Phi] = -\frac{1}{4\pi} \int_\Sigma d\sigma \mathrm{d}\tau \sqrt{-h} \mathcal{R}^{(2)} \Phi(X) - \frac{1}{2\pi} \int_{\partial\Sigma} ds \mathcal{K} \Phi(X), \tag{3.2.2}$$

Where $\mathcal{R}^{(2)}$ is the two-dimensional Ricci scalar of the two-dimensional worldsheet metric h_{ab}, and $\mathcal{K}$ is an extrinsic curvature and is added to cancel the total derivative

[6]The integral over Σ reflects the fact that closed string vertex operators are inserted in the bulk of Σ.

that is obtained by varying $\mathcal{R}^{(2)}$ [86]. Note that by setting the dilaton to a constant mode Φ_0, the first term of 3.2.2 is proportional to a topological invariant quantity from the worldsheet viewpoint, the Euler characteristic

$$\chi = \frac{1}{4\pi}\int_\Sigma d\sigma d\tau \sqrt{-h}\mathcal{R}^{(2)} = 2 - 2b, \tag{3.2.3}$$

where b is the genus, i.e. number of holes, of the Riemann surface Σ. This means that the first term in 3.2.2 provides us with information about the number of loops in string S-matrix. Therefore one can easily notice that the different topologies in the path integral representation of Euclideanized version of S are weighted with g_s^{2-2b} where the string coupling constant identified with the vev value of e^Φ. We know that the symmetries of the free field theory action 2.1.2 are crucial in obtaining a consistent quantization of the string since they are actually responsible for the decoupling of unphysical degrees of freedom. However, now we are dealing with an interacting field theory which does not turn into the Polyakov action in the conformal gauge $h_{ab} = \Lambda\eta_{ab}$, which makes it a non-trivial 2-dimensional field theory. As a result, if we want to do quantum calculations we are forced to a perturbation expansion in α'. In other words, the Weyl symmetry is ruined at quantum level unless the renormalization group β-functions for the field dependent couplings $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ vanish. At first non-trivial order in α' and tree-level in the loop expansion one obtains

$$\begin{aligned}
\beta^G_{\mu\nu} &= R_{\mu\nu} - 2\nabla_\mu\partial_\nu\Phi + \frac{9}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \mathcal{O}(\alpha') = 0 \\
\beta^B_{\mu\nu} &= \nabla_\rho H^\rho{}_{\mu\nu} - 2H^\rho{}_{\mu\nu}\partial_\rho\Phi + \mathcal{O}(\alpha') = 0 \\
\beta^\Phi &= (D-26) + 3\alpha'(R + 4(\partial\Phi)^2 - 4\nabla^2\Phi + \frac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho}) + \mathcal{O}(\alpha') = 0,
\end{aligned} \tag{3.2.4}$$

where $R_{\mu\nu}$ and R are respectively the Ricci tensor and Ricci scalar associated to the background metric $G_{\mu\nu}$, and ∇_μ is the spacetime covariant derivative. $H_{\mu\nu\rho}$ is the field strength of the Kalb-Ramond background $B_{\mu\nu}$ defined by

$$H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]} = \frac{1}{3}(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}). \tag{3.2.5}$$

Note that H is invariant under the gauge transformation assigned to $B_{\mu\nu}$.

The constraints 3.2.4 can be interpreted as target spacetime equations of motion and one can wonder whether they could be derived from an action principle. Indeed, the action has been found to be

$$S = \frac{1}{2}\int d^Dx\sqrt{G}e^{-2\Phi}\Big[-\frac{(D-26)}{3\alpha'} - R + 4(\partial\Phi)^2 - \frac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big] + \mathcal{O}(\alpha'). \tag{3.2.6}$$

The action 3.2.6 is known as the low-energy effective action; it describes the massless modes of slowly varying embedding coordinates X^μ in the target space in which the string moves. In the critical dimensions, i.e. $D = 26$, the dimension dependent terms in 3.2.4 and 3.2.6 drop out.

The appearance of $G_{\mu\nu}$ in the low-energy action 3.2.6 adds additional evidence to the argument that string theory would be the theory of quantum gravity. Of course, one can alternatively view the action 3.2.6 differently; it can be seen as the action of 26-dimensional gravity coupled to tensor and scalar fields. The action 3.2.6 can receive higher order terms in α' or string loops, namely stringy corrections to general relativity. This is a good place to point out to the reader that two coupling constants for two entirely different quantum theories were actually introduced:

- α' coupling constant: it controls the auxiliary two dimensional theory living on the worldsheet. For some special backgrounds, flat and some curved ones, the 2-dimensional theory can be found completely to all orders in α'. While for a generic background this is not plausible anymore. One has to evaluate β-function order by order in α'. This gives rise to higher derivative terms in the effective action.

- The string coupling g_s : dictates the loop expansion in the underlying target space theory.

The analysis of higher orders corrections in α', particularly to heterotic string, will come later.

This mechanism of calculating the low-energy effective action might be applied as well for supersymmetric string, i.e. superstring theory. Indeed, it has been shown that the low-energy approximation of superstring is the 10-dimensional supergravity, locally supersymmetric quantum field theory. As mentioned in chapter 2, the $N = 1$ worldsheet supersymmetry induces $N = 2$ spacetime supersymmetry. The way that these supersymmetries enter in the theory determines the types of superstring theories and the corresponding low-energy effective actions see table 3.1.1.

Type I

Although this theory is a theory of open strings, closed strings are also involved in type I and that is due to the fact that a closed string can split up into two interacting open strings. For such a theory, the boundary conditions of open string break the original $N = 2$ to $N = 1$ supersymmetry. We recall from chapter 2 that there is a non-abelian group (Yang-Mills) with charges attached at the endpoints of open string. The gauge group which is allowed by the consistency at the quantum level, is SO(32). According to [87–89] the bosonic part of $N = 1$, $D = 10$ supergravity reads

$$S_{\text{type I}} = \frac{1}{2}\int d^{10}x\sqrt{G}\Big[e^{-2\Phi}(-R + 4(\partial\Phi)^2) - \frac{3}{4}H_{(3)}^2 + \frac{1}{4}e^{-\Phi}F_2^I F_2^I\Big], \qquad (3.2.7)$$

where the subscripts (3) and (2) are to indicate the rank of the field strength. The gauge field associated to the gauge group SO(32) is represented by the field strength of the vector field which lies in the adjoint representation of SO(32).

Type IIA

The type IIA theory contains only closed strings. The theory is non-chiral in the sense that the two spacetime supersymmetries of the theory show up with two opposite chiralities. Contrary to type I, type IIA does not have a gauge group. The bosonic field content of this theory comprises, besides the metric, axion and dilaton of type I, a one-form field $C_{(1)}$ and 3-form gauge field $C_{(3)}$ (see table 2.1.3 in chapter 2). The type IIA supergravity action [90–92] behaves as

$$\begin{aligned} S_{type\,IIA} = \frac{1}{2}\int d^{10}x\sqrt{G}\Big[e^{-2\Phi}\left(-R+4(\partial\Phi)^2-\frac{3}{4}H^2_{(3)}\right) \\ \frac{1}{4}G^2_{(2)}+\frac{3}{4}G^2_{(4)}+\frac{1}{64}(G)^{-\frac{1}{2}}\epsilon_{10}\partial C_3\partial C_3 B_{(2)}\Big], \end{aligned} \tag{3.2.8}$$

with $G_{(2)}$ and $G_{(4)}$ are the field strengths of the R-R gauge fields $C_{(1)}$ and $C_{(3)}$ respectively, and ϵ_{10} is the 10-dimensional fully antisymmetric tensor. Notice that the fields of NS-NS sector have an explicit dilaton coupling via the factor $e^{-2\Phi}$, whereas the R-R fields are not multiplied by this factor. The appearance of the coupling as such reflects the fact that R-R fields correspond to a higher order in string coupling constant. The existence of R-R fields, bosonic fields, in type IIA action 3.2.8 is necessitated by the extension of supersymmetry from $N = 1$ to $N = 2$. It is worth recalling that the solutions-p-brane- that couple to these R-R fields belong to non-perturbative spectrum.

Type IIB

The type IIB theory is a theory of closed strings as well, having $N = 2$ supersymmetry, though for this theory the two-supersymmetries have the same chirality, i.e., it is a chiral theory. Similarly to type IIA, there is no possibility for non-abelian gauge groups, and besides the NS-NS fields, one has R-R sector consisting of a scalar C_0, 2-form field $C_{(2)}$ and a selfdual 4-form gauge field $C^{+}_{\nu\mu\rho\lambda}$, table 2.1.3. The selfduality property of the 4-form prohibits writing down an effective action of type IIB in a covariant way. An action has been found in [93] wherein there has not been made use of the selduality condition, but is added as an extra condition on the 4-form. The type IIB supergravity action is

$$
\begin{aligned}
S_{\text{type}\,IIB} = \frac{1}{2}\int d^{10}x\sqrt{G}\Big[& e^{-2\Phi}\left(-R + 4(\partial\Phi)^2 - \frac{3}{4}H_{(3)}^2\right) \\
& - \frac{1}{2}(\partial C_0)^2 - \frac{3}{4}(G_{(3)} - C_0 H_{(3)})^2 - \frac{5}{6}G_{(5)}^2 \\
& - \frac{\epsilon_{10}}{96\sqrt{G}} C_{(4)} \wedge G_{(3)} \wedge H_{(3)}\Big],
\end{aligned} \tag{3.2.9}
$$

with

$$
H_{(3)} = dB_{(2)}, \qquad G_{(1)} = dC_0, \qquad G_{(3)} = dC_{(2)}. \tag{3.2.10}
$$

The above IIB action is called the non-selfdual action, as we pointed out before the selfduality condition of 4-form does not follow from the action. The equations of motion have to be supplemented by

$$
G_{(5)\,\mu_1\cdots\mu_5} = \frac{1}{5!\sqrt{G}}\epsilon_{\mu_1\cdots\mu_{10}} G_{(5)}^{\mu_6\cdots\mu_{10}}. \tag{3.2.11}
$$

Heterotic String

The structure of heterotic string theory rests upon the fact that closed strings which form this theory have independent the right and left moving sectors. In heterotic string, one sector is supersymmetric, namely the theory has $N = 1$ supersymmetry (which is enough to remove the tachyon from the spectrum). This can be seen from the fact that the left moving sector can coincide with a purely bosonic strings, contrasting with a right moving sector which consists of modes of a superstring. In heterotic string theory we do have a non-abelian (Yang-Mills) gauge theory which results from the compactification of the bosonic sector on a 16-dimensional compact internal space, yielding 10-dimensional superstring theory. Due to quantum consistency, the gauge group turns out to be SO(32) or $E_8 \times E_8$. Therefore the bosonic part of the low-energy effective action, i.e., the bosonic sector of heterotic supergravity [94] is written as

$$
S_{\text{Het}} = \frac{1}{2}\int d^{10}x\sqrt{G}e^{-2\Phi}\Big[-R + 4(\partial\Phi)^2 - \frac{3}{4}H_{(3)}^2 + \frac{1}{4}F_{(2)}^{I}F_{(2)\,I}\Big]. \tag{3.2.12}
$$

Note that the metric G, the dilaton Φ and the axion B appear in the same way in all string theories, except type I. This has been referred to as the *common sector* in supergravity.

11-dimensional Supergravity

We pointed out in chapter 2 that in spite of the fact that superstring theory lives in $D = 10$, there is also a supergravity theory living in 11 dimensions. Despite the intimate relation between superstring and supergravity theories, the 11-dimensional supergravity does not follow from a low-energy effective action of superstring theory. However, 11-dimensional supergravity is still interesting by itself. It plays a crucial role in unifying the above five superstring theories. It is well-known that the higher number of dimensions that supergravity can live in is eleven[7]. Therefore 11-dimensional supergravity is a unique theory with $N = 1$ supersymmetry. The bosonic sector of 11-dimensional supergravity action [96] is expressed as

$$S_{11-\text{sup.}} = \frac{1}{2}\int d^{11}x\sqrt{G}\Big[-R + \frac{3}{4}G^2_{(4)} + \frac{1}{384\sqrt{G}}\epsilon_{11}C\partial C\partial C\Big], \tag{3.2.13}$$

where the field contents of eleven dimensional supergravity are the metric G and the 3-form gauge field $C_{\mu\nu\rho}$ with $G_{(4)} = dC$. $\epsilon_{(11)}$ is a fully anti-symmetric tensor in 11 dimensions.

3.3 String Effective Action and Chern-Simons Terms

The low-energy effective action of string theory often involves Chern-Simons forms, which are totally antisymmetric tensors $\mathcal{O}_{\mu_1\cdots\mu_n}$. They depend on one or more lower rank gauge fields or spin connections/Christoffel symbols rather than just the field strength. Consequently, $\mathcal{O}$ is not invariant under the gauge transformation associated with these lower rank gauge fields. However $\mathcal{O}$ has a peculiar property that the variations of $\mathcal{O}$ under various gauge transformations are exact forms:

$$\delta\mathcal{O}_{\mu_1\cdots\mu_n} = \partial_{[\mu_1}\varphi_{\mu_2\cdots\mu_n]}, \tag{3.3.1}$$

for some quantity φ. Therefore the curvature $\partial_{[\mu_1}\mathcal{O}_{\mu_2\cdots\mu_{n+1}]}$ is a covariant tensor. Let us give an example of such a Chern-Simons term. Assume the theory has a r-form gauge field $B^{\mu_1\cdots\mu_r}_{(1)}$ and a s-form gauge field $B^{\mu_1\cdots\mu_s}_{(2)}$ with associated gauge transformations of the form

$$\delta A_{\mu_1\cdots\mu_r} = \partial_{[\mu_1}\beta_{\mu_2\cdots\mu_r]}, \qquad \delta B_{\mu_1\cdots\mu_s} = \partial_{[\mu_1}\gamma_{\mu_2\cdots\mu_s]}. \tag{3.3.2}$$

Then the $r + s + 1$-form

$$\mathcal{O}_{\mu_1\cdots\mu_{r+s+1}} = A_{[\mu_1\cdots\mu_r}\partial_{\mu_{r+1}}B_{\mu_{r+2}\cdots\mu_{r+s+1}]} \tag{3.3.3}$$

transforms by a total derivative of the form 3.3.1 under the gauge transformation induced by β. Thus $\mathcal{O}_{\mu_1\cdots\mu_{r+s+1}}$ defined in 3.3.3 is a Chern-Simons $(r + s + 1)$-form.

It may happen that the Chern-Simons terms show up in the expression of low-energy effective action of string theory in two different ways:

[7]For supergravity theories in dimensions higher than eleven, fields with spin greater than two appear [95], and it is not clear how to deal with these higher spin fields in an adequate way

- The action itself might contain a Chern-Simons term of the form

$$\int d^D x \epsilon^{\mu_1 \cdots \mu_D} \mathcal{O}_{\mu_1 \cdots \mu_D}. \tag{3.3.4}$$

  Since $\delta\mathcal{O}$ is total derivative, an action of this form is gauge invariant up to surface terms.

- In some theories the gauge invariant field strength associated with an antisymmetric tensor field $B_{\mu_1 \cdots \mu_{n-1}}$ is given by

$$H_{\mu_1 \cdots \mu_n} = \partial_{[\mu_1} B_{\mu_2 \cdots \mu_n]} + \mathcal{O}_{\mu_1 \cdots \mu_n} \tag{3.3.5}$$

  for some Chern-Simons n-form $\mathcal{O}$ constructed out of lower dimensional gauge fields and spin connection. Under the gauge transformation 3.3.1, $B_{\mu_1 \cdots \mu_{n-1}}$ is assigned the transformation

$$\delta B_{\mu_1 \cdots \mu_{n-1}} = -\varphi_{\mu_1 \cdots \mu_{n-1}}, \tag{3.3.6}$$

  such that

$$\delta H_{\mu_1 \cdots \mu_n} = 0. \tag{3.3.7}$$

  A typical example of such a term is the 3-form field strength associated with the NS sector 2-form gauge field of heterotic string theory. The definition of the three form field strength comprises both gauge and Lorentz Chern-Simons LCS 3-forms. In such cases the low-energy effective action being a function of $H_{\mu_1 \mu_2 \mu_3}$ is invariant under the gauge transformation 3.3.1 and 3.3.6 for $n = 3$.

3.4 α'-Corrections to Heterotic Supergravity

The heterotic supergravity action defined above has received higher curvature corrections as it is the low-energy effective actions of heterotic superstring theory. In this section we are going to clarify the relation between two formulations of the order α' heterotic string effective action. One formulation follows from the methods discussed in section 3.1, namely the string S-matrix calculations [84, 97] and the requirement of conformal symmetry of the corresponding sigma model to the appropriate order [97, 98], the other formulation [99, 100] is based on the supersymmetrization of Lorentz Chern-Simon forms. In [C] it has been argued that the bosonic expression for the order α' corrections constructed in [97] has to be part of a supersymmetric invariant. It has been proved a long time ago [99] that the heterotic string effective action is supersymmetric through order α'. A few months later, in [100], the supersymmetry of the action has been established to order α'^2 and α'^3. In [C] we have shown that to order α' [99] agrees with [97], demonstrating in a direct way that the action of [97] is indeed part of a supersymmetric invariant. The field redefinitions required to establish this correspondence generate additional terms at higher orders in α'.

In what follows we will try to establish that the two actions are equivalent [C]. We relegate the reader to appendix B for necessary material and conventions. Then we discuss the terms of order α'^2 and α'^3.

The heterotic string effective action to order α', as found in [97], is

$$\begin{aligned}\mathcal{L}_{\mathrm{MT}} &= -\frac{2}{\kappa^2} e e^{-2\Phi}\Big[R(\Gamma) - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi \qquad (3.4.1)\\ &+\frac{1}{8}\alpha'\{R_{\mu\nu ab}(\Gamma)R^{\mu\nu ab}(\Gamma) - \frac{1}{2}R_{\mu\nu ab}(\Gamma)H^{\mu\nu c}H^{abc}\\ &-\frac{1}{8}(H^2)_{ab}(H^2)^{ab} + \frac{1}{24}H^4\}\Big], \qquad (3.4.2)\end{aligned}$$

where we have

$$\begin{aligned} H_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]}, \qquad H^2 = H_{abc}H^{abc},\\ (H^2)_{ab} &= H_{acd}H_b{}^{cd}, \qquad H^4 = H^{abc}H_a{}^{df}H_b{}^{ef}H_c{}^{de}, \qquad (3.4.3)\end{aligned}$$

normalisations are as in [97].

On the other hand there is the result of supersymmetrising the LCS of [99,100]. In this section we only discuss the bosonic contributions to the effective action. Fermionic contributions can be found in [100]. Thus the bosonic terms take on the form

$$\begin{aligned}\mathcal{L}_{\mathrm{BR}} &= \frac{1}{2}ee^{-2\Phi}[\{-R(\omega) - \frac{1}{12}\widetilde{H}_{\mu\nu\rho}\widetilde{H}^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi\} \qquad (3.4.4)\\ &-\frac{1}{2}\alpha R_{\mu\nu ab}(\Omega_-)R^{\mu\nu ab}(\Omega_-)]. \qquad (3.4.5)\end{aligned}$$

With respect to [100] we have redefined the dilaton and the normalisation of $B_{\mu\nu}$ (see Appendix B.1). In 3.4.4 $\widetilde{H}$ contains the LCS terms with H-torsion:

$$\begin{aligned}\widetilde{H}_{\mu\nu\rho} &= H_{\mu\nu\rho} - 6\alpha\mathcal{O}_{\mu\nu\rho}(\Omega_-), \qquad (3.4.6)\\ \mathcal{O}_{3\,\mu\nu\rho}(\Omega_-) &= \Omega_{-[\mu}ab\partial_\nu\Omega_{-\rho]}{}^{ab} - \frac{2}{3}\Omega_{-[\mu}{}^{ab}\Omega_{-\nu}{}^{ac}\Omega_{-\rho]}{}^{cb}, \qquad (3.4.7)\\ \Omega_{-\mu}{}^{ab} &= \omega_\mu{}^{ab} - \frac{1}{2}\widetilde{H}_\mu{}^{ab}. \qquad (3.4.8)\end{aligned}$$

The coefficient α is proportional to α', notice that the relative normalization between the LCS term and the R^2 action is fixed.

In order to show that the two actions (3.4.1, 3.4.2) and (3.4.4, 3.4.5) are equivalent we expand $R(\Omega_-)$ in 3.4.5, perform the required field redefinitions and fix the normalisations.

To start with, we have

$$R_{\mu\nu}{}^{ab}(\Omega_-) = R_{\mu\nu}{}^{ab}(\omega) - \frac{1}{2}(\mathcal{D}_\mu \widetilde{H}_\nu{}^{ab} - \mathcal{D}_\nu \widetilde{H}_\mu{}^{ab}) - \frac{1}{8}(\widetilde{H}_\mu{}^{ac}\widetilde{H}_\nu{}^{cb} - \widetilde{H}_\nu{}^{ac}\widetilde{H}_\mu{}^{cb}). \quad (3.4.9)$$

where the derivatives $\mathcal{D}$ are covariant with respect to local Lorentz transformations. Obviously the substitution of 3.4.9 in 3.4.5 gives terms similar to those in 3.4.2, additional terms come from expanding $\widetilde{H}$ (see Appendix B.3) in 3.4.4. The effect of these substitutions is, to order α :

$$\begin{aligned}
\mathcal{L}_{\rm BR} \;=\; & \frac{1}{2}ee^{-2\Phi}[-R(\omega) - \frac{1}{12}\bar{H}_{\mu\nu\rho}\bar{H}^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi \\
& +\alpha\{\frac{1}{2}H^{\mu\nu\rho}\partial_\mu(\omega_\nu{}^{ab}H_\rho{}^{ab}) - \frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)H_\rho{}^{ab}H^{\mu\nu\rho} + \frac{1}{4}H^{\mu\nu\rho}H_\mu{}^{ab}\mathcal{D}_\nu H_\rho{}^{ab} \\
& -\frac{1}{12}H^4\} && (3.4.10) \\
& -\frac{1}{2}\alpha\{R_{\mu\nu}{}^{ab}(\omega)R^{\mu\nu ab}(\omega) && (3.4.11) \\
& -2R^{\mu\nu ab}(\omega)\mathcal{D}_\mu H_{\nu ab} && (3.4.12) \\
& +\frac{1}{2}(\mathcal{D}_\mu H_\nu{}^{ab} - \mathcal{D}_\nu H_\mu{}^{ab})\mathcal{D}^\mu H^{\nu ab} && (3.4.13) \\
& -R_{\mu\nu}{}^{ab}(\omega)H^{\mu ac}H^{\nu cb} && (3.4.14) \\
& +\frac{1}{2}(\mathcal{D}_\mu H_\nu{}^{ab} - \mathcal{D}_\nu H_\mu{}^{ab})H^{\mu ac}H^{\nu cb} && (3.4.15) \\
& +\frac{1}{8}((H^2)_{ab}(H^2)^{ab} - H^4)\}]. && (3.4.16)
\end{aligned}$$

Here $\bar{H}$ contains the LCS term without H-torsion:

$$\bar{H}_{\mu\nu\rho} = H_{\mu\nu\rho} - 6\alpha\mathcal{O}_{3\,\mu\nu\rho}(\omega). \quad (3.4.17)$$

We now rewrite the terms (3.4.11-3.4.16) in $\mathcal{L}_{\rm BR}$, see Appendix B.4 for details. The result, keeping only contributions to order α, is

$$\begin{aligned}
\mathcal{L}_{\rm BR} \;=\; & \frac{1}{2}ee^{-2\Phi}[-R(\omega) - \frac{1}{12}\bar{H}_{\mu\nu\rho}\bar{H}^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi \\
& -\frac{1}{2}\alpha\{R_{\mu\nu}{}^{ab}(\omega)R^{\mu\nu ab}(\omega) + \frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)H_\rho{}^{ab}H^{\mu\nu\rho} \\
& +\frac{1}{8}(H^2)_{ab}(H^2)^{ab} + \frac{1}{24}H^4\} && (3.4.18) \\
& -\frac{1}{2}\alpha\{R_\mu{}^c(\omega)H^{\mu ab}H_{abc} + e^\mu{}_c e^\nu{}_d\mathcal{D}_\nu H_{abd}\mathcal{D}_\mu H_{abc} \\
& +2\partial_c\Phi H_{abd}\mathcal{D}_d H_{abc} - 2\partial_d\Phi H_{abd}\mathcal{D}_c H_{abc}\}]. && (3.4.19)
\end{aligned}$$

The term proportional to the Ricci tensor in 3.4.19 contributes through a field redefinition to the terms quartic in H, and gives an additional contribution involving derivatives of Φ (see B.2.4). Making use of B.2.2 and integrating by parts all remaining terms can be made to cancel.

The final result is then

$$\begin{aligned}\mathcal{L}_{\mathrm{BR}} &= \frac{1}{2}ee^{-2\Phi}[-R(\omega) - \frac{1}{12}\bar{H}_{\mu\nu\rho}\bar{H}^{\mu\nu\rho} + 4\partial_\mu\phi\partial^\mu\Phi \\ &\quad -\frac{1}{2}\alpha\{R_{\mu\nu}{}^{ab}(\omega)R^{\mu\nu ab}(\omega) + \frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)H_\rho{}^{ab}H^{\mu\nu\rho} \\ &\quad -\frac{1}{8}(H^2)_{ab}(H^2)^{ab} + \frac{1}{24}H^4\}],\end{aligned} \tag{3.4.20}$$

in agreement with [97] if we set $R(\Gamma) = -R(\omega)$ and $\alpha = -\frac{1}{4}\alpha'$, and adjust the overall normalisation. Of course [97] also includes the LCS term in H^2 for the heterotic string effective action, see the footnote in [97], page 400.

3.4.1 Higher Orders and Field Redefinitions

It has been shown in [100] that the effective action to order α^2 consists of terms which are bilinear in the fermions (3.4.4, 3.4.5). This is no longer true when the effective action at order α is in the form 3.4.20.

Since the steps to go from (3.4.4, 3.4.5) to 3.4.20 have all been explicitly determined, the effective action at order α^2 can in principle be constructed. Let us identify the sources of bosonic terms of order α^2 that we have encountered:

1. From the action 3.4.4 there are contributions outlined in Appendix B.3. We should now expand $\widetilde{H}$ to order α^2, which means that in $\mathcal{A}$ B.3.2 also terms of order α should be considered. Then one should calculate $\widetilde{H}^2$.

2. $\bar{H}$ contains the LCS term of order α. These should now also be kept in the higher order contributions.

3. In a number of places we have used the identity B.4.1, the resulting R^2 terms contribute to order α^2.

4. We have used field redefinitions to modify the effective action at order α. A field redefinition is of the form

$$e_\mu{}^a \to e_\mu{}^a + \alpha\Delta^a_\mu, \tag{3.4.21}$$

and is applied to the order α^0 action. This has the effect of giving an extra contribution

$$\alpha\Delta^a_\mu\mathcal{E}^\mu{}_a \tag{3.4.22}$$

to the action, where $\mathcal{E}^{\mu}{}_a$ is the Einstein equation at order α^0. Thus one can eliminate a term

$$-\alpha\Delta^a_\mu\mathcal{E}^\mu{}_a, \quad (3.4.23)$$

at order α. Contributions of order α^2 arise because the transformation should also be applied to the order α action.

Accordingly, the bosonic part of terms with six derivatives in the effective action at order α^2, corresponding to the order α action 3.4.2, can be obtained, including the complete dependence on H.

At order α^3 the situation is different. In [100] an invariant related to the supersymmetrisation of the LCS terms was constructed. The status of R^4 invariants was discussed in [101], with extensive reference to the earlier work.

3.5 Conclusion

We have devoted this chapter to introduce a supergravity action as the low-energy effective action of a superstring theory, outlining the most powerful methods that have been pursued for constructing such an action and the derivative corrections (α' corrections) contributions to them. We found out that the heterotic string actions (with Chern-Simons forms) which follow from the σ-model approach and the string S-matrix calculation- note that it has been established in [97] that those two actions are equivalent to order α' modulo field redefinitions- are equivalent to order α' to the heterotic string action constructed in [100], i.e. through the supersymmetrisation of LCS. Actually, our interest in the relation between these results was triggered by a remark in a paper of Sahoo and Sen [102] in which the entropy of a supersymmetric black hole was obtained using the method of [103], with [97] for the derivative corrections to the action. The result was found to agree with that obtained by several other methods, which was taken by [102] as an indirect indication that the bosonic expression for the order α' corrections given [97] must be part of a supersymmetric invariant.

Chapter 4

Duality Symmetries for Interacting Fields

In chapter 2 we have pointed out that one of the indirect methods for constructing the higher derivative terms of Born-Infeld theory is to maximize the usefulness of duality symmetries. It has been shown a long time ago that most of the four dimensional nonlinear electrodynamics models, including Born-Infeld theory, are electromagnetic dual invariant (selfdual) theories. In this chapter we will start our discussion with introducing and defining electromagnetic duality in (non)-linear electromagnetism. Then we will try to make contact with the previous chapter by including higher derivative corrections of the field strength F_{ab}. The investigation of electromagnetic duality invariance in the presence of scalar fields is of great interest for us, since such symmetries are going to play a crucial role in constructing the nonlinear sigma models that we will study in chapter 5. We also find it useful to derive the Noether charges associated with the duality symmetries since those charges will form the cornerstone in chapter 6 for classifying solutions in (super)gravity theories. Selfdual invariant quantities will be briefly mentioned, and the chapter will be closed with a discussion.

4.1 Electro-Magnetic Duality: Overview

It has been established a long time ago that the Hodge duality operation $\sim$ (defined in A.2.2) is the duality symmetry of the four dimensional Maxwell theory. In terms of the electric field E and magnetic field B, Hodge duality symmetry is a SO(2)[1]

[1] All the group theory notions and definitions that will be used in this thesis are reviewed in appendix C.

rotation written in matrix form as

$$\begin{pmatrix} E \\ B \end{pmatrix} \to \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix}. \tag{4.1.1}$$

In differential form notation on Minkowski spacetime[2], the transformation is expressed in terms of a field strength $F_{ab} = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a$ so that the duality symmetry $\sim$ takes on the form

$$F_{ab} \to \cos\alpha F_{ab} + \sin\alpha \widetilde{F}_{ab}. \tag{4.1.2}$$

The sourceless linear Maxwell's equations read

$$\nabla \cdot B = 0, \qquad \nabla \times E = -\frac{\partial B}{\partial t} \tag{4.1.3}$$

$$\nabla \cdot D = 0, \qquad \nabla \times H = +\frac{\partial D}{\partial t}, \tag{4.1.4}$$

where D is the electric induction and H the magnetic intensity, which are simply equivalent in forms notation to the combined field equations system of the Bianchi identity and the equations of motion

$$\partial_a F^{ab} = 0, \qquad \partial_a \widetilde{F}^{ab} = 0. \tag{4.1.5}$$

What is *meant* by electro-magnetic E-M duality symmetry is the symmetry of the Maxwell equations not of their corresponding Lagrangian which is expressed by

$$\mathcal{L}_{Maxwell.} = -\frac{1}{4} F_{ab} F^{ab} = -\frac{1}{2}(B^2 - E^2). \tag{4.1.6}$$

Notice that only for the case of Maxwell theory in vacuum for which $E = D$ and $B = H$, E-M duality breaks down to Hodge duality. For more general cases, namely the nonlinear electrodynamics models, E-M duality has a slightly different interpretation which is the topic of the next section.

4.2 (Self)duality Rotations: the Gaillard-Zumino Model

In this section we review some results of Gaillard and Zumino [104–106] from the early 80's and developed further by Gibbons and Rasheed later in the 90's [107, 108].

First of all, it is well-known that E-M duality transformation is implemented via

[2] We adopt in this chapter the following conventions; Minkowski metric η_{ab} with diag$(-,+,+,+)$ signature, where $\widetilde{\widetilde{F}} = -F$, and often we make use of the notation tr$FG = -F_{ab}G^{ab}$.

the transformation of the field strength F_{ab} rather than the fundamental variable, the vector gauge potential $\mathcal{A}_a$. Therefore E-M duality can be realized only on the level of equations of motion. The reason is that the duality transformations are *consistent* only on-shell. Since the independent variable of the theory is $\mathcal{A}_a$, δF_{ab} in eq.4.1.2 should be derived from $\delta\mathcal{A}_a$;

$$\partial_a \delta\mathcal{A}_b - \partial_b \delta\mathcal{A}_a = \cos\alpha F_{ab} + \sin\alpha \widetilde{F}_{ab}. \tag{4.2.1}$$

The integrability of this equation requires $\partial_a(\cos\alpha\widetilde{F} + \sin\alpha\widetilde{\widetilde{F}}) = 0$, i.e., $\partial_a F^{ab} = 0$. Thus the equations of motion must be satisfied. Even if we ignore this point and formally consider the transformation 4.1.2 off-shell, the lagrangian 4.1.6 is not invariant. Therefore in order to construct theories invariant under E-M duality transformations, it is easier to study the covariance of equations of motion.

The off-shell realization of manifestly E-M duality invariance on the level of the action has been exhaustively investigated by a handful of researchers. We mention in particular results by [109], in which they found a formalism that helps with uplifting duality invariance to the action. In such avenues, E-M duality transformations are basically defined through the off-shell gauge potential $\mathcal{A}_a$. This thus generates non-local terms in the action in the light of the relation $F = d\mathcal{A}$. However, in order to circumvent the locality violation, all one needs is to double the number of gauge fields [109](and references therein)- Sen-Schwarz model- in the sense that the gauge fields and their duals appear on a par in the action. In doing so, there is a price one should pay; doubling gauge fields actually ruins Lorentz covariance which Pasti et al. [110] have managed to restore afterwards. However, the manifest E-M duality invariant actions are out of the scope of this thesis. In what follows we are going to focus on some aspects of E-M dualities which are *only* symmetries of the equations of motions.

To end this short introduction we remind the reader that all our considerations are classical. The systems we study should be regarded as effective theories, in accordance with the appearance of the Born-Infeld action as the worldvolume action of the D-brane (see chapter 2).

4.2.1 Gaillard-Zumino Condition: Selfduality Condition

In order to gain some insights into the significance of duality invariance[3] (selfduality), we start our analysis by considering nonlinear extensions of Maxwell theory, i.e., nonlinear electrodynamics models. The nonlinearity of such a model can be obtained by adding polynomial higher order terms of the field strength F_{ab} to the Lagrangian of the Maxwell theory. For physical reasons we restrict ourselves to nonlinear models

[3]Throughout this chapter notions like duality invariance, selfduality and duality symmetry all refer to the same concept.

which coincide with the Maxwell model at weak field limits, i.e. $\mathcal{L}(F) = -\frac{1}{4}F_{ab}F^{ab} + \mathcal{O}(F^4)$. The equations of motion and the Bianchi identity are

$$\partial_a G^{ab} = 0, \qquad \partial_a \widetilde{F}^{ab} = 0, \tag{4.2.2}$$

where G is the dual[4] of F defined by the following constitutive relation

$$G^{ab}(F) = -\frac{\partial \mathcal{L}(F)}{\partial F_{ab}}, \qquad \frac{\partial F_{ab}}{\partial F_{cd}} = (\delta_a^c \delta_b^d - \delta_b^c \delta_a^d). \tag{4.2.3}$$

Note that F_{ab} and G_{ab} are not independent of one another but nonlinearly related by 4.2.3. A pair (G, F) can be mapped to (G', F') via transformations S such that

$$\begin{pmatrix} G'(F') \\ \widetilde{F}' \end{pmatrix} = S \begin{pmatrix} G(F) \\ \widetilde{F} \end{pmatrix}, \tag{4.2.4}$$

with $S \in \mathrm{GL}(2, \mathbb{R})$. This means that one can solve 4.2.4 for (G, F) in terms of (G', F') and then use the equations 4.2.2-4.2.3 to find the transformed version of the field equations and the Lagrangian. Thus, the transformed Lagrangian $\mathcal{L}'$ does exist and must satisfy $G'^{ab}(F') = -\frac{\partial \mathcal{L}'(F')}{\partial F'_{ab}}$. In general, the Lagrangian $\mathcal{L}'(F)$ differs from $-\frac{1}{4}F_{ab}F^{ab} + \mathcal{O}(F^4)$. Therefore the shift in the functional form of the Lagrangian behaves as

$$\Delta\mathcal{L}(F) = \mathcal{L}'(F) - \mathcal{L}(F) = \delta\mathcal{L}(F) - \mathrm{tr}\Big(\delta F \frac{\partial \mathcal{L}}{\partial F}\Big), \tag{4.2.5}$$

where δ is the infinitesimal form of 4.2.4, and the variation of the Lagrangian under such a transformation is defined by $\delta\mathcal{L}(F) = \mathcal{L}'(F') - \mathcal{L}(F)$.

The requirement of selfduality or the E-M duality invariance of the equations of motion comes down to setting Δ to zero. Consequently, the possible functional form of $\mathcal{L}$ is severely constrained

$$\mathcal{L}'(F) = \mathcal{L}(F). \tag{4.2.6}$$

Models which obey 4.2.6 are called *selfdual* models.

The selfduality requirement has many implications which follow from 4.2.6:

- The invariance of the constitutive relation 4.2.3.
- The Lagrangian $\mathcal{L}$ solves a second order partial differential equation in the six variables of F_{ab}

$$\mathrm{tr} G\widetilde{G} = \mathrm{tr} F\widetilde{F}. \tag{4.2.7}$$

[4] From the viewpoint of Hamiltonian formalism, there is a geometric interpretation for G. F together with G form respectively the coordinates and the dual coordinates of the 2-form space $V = \Lambda^2(\mathbb{R}^4)$ and its dual V^*. From this one can obtain the symplectic phase space $V \oplus V^*$ where the Lagrangian $\mathcal{L}$ plays the role of the generating functional. That is where the word *dual* comes from.

This is the necessary and sufficient condition on the Lagrangian $\mathcal{L}(F)$ so that its corresponding field equations admit E-M duality invariance. For the derivation of 4.2.7 we refer the reader to the ensuing sections where we shall do the calculations for more general cases.

- For purely nonlinear electrodynamics models, the E-M duality symmetry is described and represented by the compact abelian group $\mathrm{U}(1) \cong \mathrm{SO}(2)$.
- The combination $\mathcal{L}(F) - \frac{1}{4}\mathrm{tr}FG$ is duality invariant.

As mentioned before, we do not impose the invariance of the Lagrangian itself; we shall see that the system of the equations of motion can be invariant only if $\delta\mathcal{L}$ does not vanish. Instead the variation of $\mathcal{L}$ is required to have a specific form.

4.2.2 Solutions of Selfduality Condition

An interesting solution of 4.2.7 is the Born-Infeld Lagrangian defined in chapter 2. It is given in four dimensions by the Lagrangian[5]

$$\mathcal{L}(F)_{BI} = (-\mathrm{D}^{\frac{1}{2}}(F) + 1), \tag{4.2.8}$$

where

$$\begin{aligned} D(F) = -\det(\eta_{ab} + F_{ab}) &= 1 - \frac{1}{2}\mathrm{tr}F^2 - \frac{1}{16}(\mathrm{tr}F\widetilde{F})^2 \\ &= 1 + \frac{1}{2}P - \frac{1}{16}Q^2, \end{aligned} \tag{4.2.9}$$

with $P = -\mathrm{tr}FF$ and $Q = -\mathrm{tr}F\widetilde{F}$ are the only two independent Lorentz invariants of electromagnetism in four dimensions.
Then

$$\frac{\partial D}{\partial F} = 2F - \frac{1}{2}\widetilde{F}, \qquad G = -\frac{\partial\mathcal{L}}{\partial F} = D^{-\frac{1}{2}}(-F + \frac{1}{4}\widetilde{F}). \tag{4.2.10}$$

Using 4.2.10, one can verify that $\mathrm{tr}G\widetilde{G} = \mathrm{tr}F\widetilde{F}$, and hence the Born-Infeld theory is duality invariant [111].

It is quite natural to ask whether the Born-Infeld theory is the most general physically acceptable solution of 4.2.7. This has been investigated intensively in [104, 105, 108, 112] where a negative answer has been reached.

More general solutions of the differential equation 4.2.7 have been studied in [105, 113] and a prescription to obtain solutions, corresponding to selfdual models, has been

[5] We set the fundamental (scale)^2, $T^{-1} = 2\pi\alpha'$ in string theory context, equal to one.

presented. The derivation of such solutions goes as follows: first we consider $\mathcal{L}$ to be a function of (P, Q), namely $\mathcal{L}(P, Q)$. It follows that the dual tensor G is written as

$$G = -4(\mathcal{L}_P F + \mathcal{L}_Q \widetilde{F}), \qquad \widetilde{G} = -4(\mathcal{L}_P \widetilde{F} - \mathcal{L}_Q F) \tag{4.2.11}$$

where $\mathcal{L}_P$ and $\mathcal{L}_Q$ are the derivatives of $\mathcal{L}$ w.r.t P and Q respectively. Substituting 4.2.11 into 4.2.7 leads to

$$\left[16((\mathcal{L}_Q)^2 - (\mathcal{L}_P)^2) + 1\right] Q + 32 P \mathcal{L}_P \mathcal{L}_Q = 0. \tag{4.2.12}$$

This may be simplified further by considering another change of variables

$$U = \frac{1}{8}(P + \sqrt{P^2 + Q^2}), \qquad V = \frac{1}{8}(P - \sqrt{P^2 + Q^2}). \tag{4.2.13}$$

Then 4.2.12 is reduced to [108]

$$\mathcal{L}_U \mathcal{L}_V = 16. \tag{4.2.14}$$

This is a familiar nonlinear differential equation which has been studied extensively in mathematics. In our case we must also impose the boundary condition which makes $\mathcal{L}(U, V)$ approaches the Maxwell Lagrangian $\mathcal{L}_M(U, V) = -P/4 = -U - V$ when the field strength F is small. According to [114], the general solution solving 4.2.14 is expressed explicitly in terms of an arbitrary function $\mathcal{F}(T)$ determined by the initial values $\mathcal{L}(0, V) = \mathcal{F}(V)$ and $\mathcal{L}_U(0, V) = 1/\mathcal{F}'(V)$, where the prime is the derivative of $\mathcal{F}$ with respect to T. The general solution thus reads

$$\frac{1}{4}\mathcal{L}(U, V) = \frac{2U}{\mathcal{F}'(T)} + \mathcal{F}(T), \qquad \text{with } V = \frac{U}{[\mathcal{F}'(T)]^2} + T. \tag{4.2.15}$$

Solving the second equation of 4.2.15 for $T(U, V)$ results in determining the corresponding $\mathcal{L}(U, V)$. It is worth noting that in [115] it has been verified that indeed 4.2.15 solves 4.2.14 with $\mathcal{L}_U = 4/\mathcal{F}'$ and $\mathcal{L}_V = 4\mathcal{F}'$, and moreover the condition that $\mathcal{L}$ should approach Maxwell Lagrangian for a small field strength implies that $\mathcal{F}(T) = \mathcal{L}(U = 0, T) \cong -T$ for a small T.

There have been a few explicit and exact solutions of the selfduality condition [113]. As a crosscheck one might reconsider the Born-Infeld theory example given above. In terms of (U, V), 4.2.8 takes on the form

$$\mathcal{L}_{BI}(U, V) = -\sqrt{(1 + 2U)(1 + 2V)} + 1. \tag{4.2.16}$$

The associated function $\mathcal{F}(T)$ to $\mathcal{L}_{BI}$ is determined by setting $U = 0$. This yields

$$\mathcal{F}(T) = -(1 + 2T)^{1/2} + 1, \qquad \text{with } T(U, V) = \frac{V - 4U}{1 + 8U}. \tag{4.2.17}$$

Thus roughly speaking, there are as many selfdual deformations of Maxwell theory as there are real analytic functions $\mathcal{F}(T)$ of one real argument.

In order to have a better understanding of what is exactly going on, we present the following illustration:
We actually know that approximate solutions to Eq.4.2.7 can also be obtained by a power series expansion in F of Lagrangians whose leading order is Maxwell term $-1/4\mathrm{tr}F^2$. It is convenient to write down the Lagrangian in terms of P and Q. Up to fourth order in F, using the identities

$$(\widetilde{F}F)_a{}^b = \frac{1}{4}Q\delta_a^b, \qquad (\widetilde{F}^2 - F^2)_a{}^b = -\frac{1}{2}P\delta_a^b, \tag{4.2.18}$$

the Lagrangian can be expressed as

$$\mathcal{L}(P,Q) = \frac{1}{4}P + a_1P^2 + a_2Q^2 + a_3PQ + \mathcal{O}(P^3,Q^3), \tag{4.2.19}$$

where a_1, a_2 and a_3 are arbitrary coefficients. If we substitute 4.2.19 in 4.2.7, then one can find, up to the same order, that $a_1 = 0$ and a free parameter $a = a_2 = a_3$. However, the coefficient a can be yet appropriately fixed by rescaling the field strength F. It turns out that, up to order four in F, $\mathcal{L}$ coincides with $\mathcal{L}_{BI}$. One can push this calculation an order further by adding the most general Lagrangian terms through order six in F, and again requiring that the resulting Lagrangian solves 4.2.7. What comes out is that, up to this given order, all the coefficients but one are determined. The question now is how many free parameters one has in the power expansion of duality invariant Lagrangians. The power expansion for arbitrarily large m behaves as

$$\mathcal{L} = \frac{P}{4} + a\mathcal{L}_1 + a^2\mathcal{L}_2 + \cdots + a^{2m-1}\mathcal{L}_{2m-1} + a^{2m}\mathcal{L}_{2m}, \tag{4.2.20}$$

where $\mathcal{L}_{2m-1}$ and $\mathcal{L}_{2m}$ are written as:

$$\mathcal{L}_{2m-1} = \sum_{n=0}^{m} a_nP^{2(m-n)}Q^{2n} \tag{4.2.21a}$$

$$\mathcal{L}_{2m} = \sum_{n=0}^{m} a_nP^{2(m-n)+1}Q^{2n}. \tag{4.2.21b}$$

In [116] a theorem has been reached which states that in the power expansion 4.2.20 of duality invariant Lagrangians, i.e. Lagrangians that satisfy 4.2.7, there will occur a free parameter in each $\mathcal{L}_{2m-1}$ as defined in 4.2.21a and there will occur no free parameter in each $\mathcal{L}_{2m}$ as defined in 4.2.21b.

We conclude that for specific choices of the free parameters, the power expansion 4.2.20, satisfying duality condition, coincides with the power expansion of the

Born-Infeld Lagrangian. Actually there are infinitely many free parameters, since for arbitrary large odd powers in the expansion there will still occur new free parameters in the general expansion. So, there are indeed more Lagrangians which admit selfduality next to the Born-Infeld theory.

4.2.3 Selfduality of Nonlinear Models with Derivative Corrections

The extension of the E-M selfduality principle to models which involve higher derivative corrections is of great interest. In particular, this is relevant for applications in string theory, where it is known (chapter 2) that the open superstring effective action, which for slowly varying fields coincides with the Born-Infeld theory, also comprises derivative corrections. It is natural to ask what happens if the Lagrangian also depends on derivatives of the field strength, i.e., $\mathcal{L}(F, \partial F)$. At first sight, it seems that the analysis presented in 4.2.1 is no more valid, and hence should be modified. However, most of the discussion in 4.2.1 can be taken over if one works with the action S rather than the Lagrangian, and uses differentiation of functionals [112][B].

Derivation of Selfduality Condition

We start out with the definitions

$$G_{ab} = -\frac{\delta}{\delta F^{ab}} S[F], \qquad \frac{\delta}{\delta F_{ab}(x)} F_{cd}(y) = 2\delta^{ab}_{cd}\delta(y-x). \tag{4.2.22}$$

The selfduality rotations 4.2.4 have the same structure, yet G' must be consistently obtained from an action S'. The selfduality condition, written in the integral form, then reads

$$S'[F] = S[F] \Rightarrow \int d^4x \mathrm{tr} G\widetilde{G} = \int d^4x \mathrm{tr} F\widetilde{F}. \tag{4.2.23}$$

■ Proof:
We resort to the infinitesimal version of 4.2.4 associated to SO(2), then one has

$$G'^{ab}[F'] = G^{ab}[F] + \lambda \widetilde{F}^{ab}, \qquad \widetilde{F}'^{ab} = \widetilde{F}^{ab} - \lambda G^{ab}[F], \tag{4.2.24}$$

where λ is the infinitesimal parameter of 4.2.4, and $G'^{ab}[F'] = \delta S'[F']/\delta F'_{ab}(x)$. Using the selfduality condition $S'[F] = S[F]$ one finds

$$G'^{ab}[F', x] = -\frac{\delta S[F']}{\delta F'_{ab}(x)} = -\Big(\frac{\delta S[F]}{\delta F'_{ab}(x)} + \frac{\delta}{\delta F_{ab}(x)}\delta S[F]\Big), \tag{4.2.25}$$

where we have made use of

$$\delta S[F] = S[F'] - S[F], \qquad \text{and} \quad \frac{\delta}{\delta F'_{ab}(x)}\delta S[F] = \frac{\delta}{\delta F_{ab}(x)}\delta S[F] + \mathcal{O}(\lambda^2). \tag{4.2.26}$$

The evaluation of $\delta S[F]/\delta F'_{ab}$ results in

$$\frac{\delta S[F]}{\delta F'_{ab}(x)} = -G^{ab}[F,x] + \frac{\lambda}{4}\frac{\delta}{\delta F_{ab}(x)}\Big(\int d^4y G^{cd}[F,y]\widetilde{G}_{cd}[F,y]\Big). \tag{4.2.27}$$

The substitution of 4.2.27 into 4.2.25 gives rise to

$$G'^{ab}[F',x] = G^{ab}[F,x] - \frac{\delta}{F_{ab}(x)}\Big(\delta S[F] + \frac{\lambda}{4}\int d^4y G^{cd}[F,y]\widetilde{G}_{cd}[F,y]\Big). \tag{4.2.28}$$

On the other hand, from the variation 4.2.24 of G it follows

$$G'^{ab}[F',x] = G^{ab}[F,x] - \frac{\delta}{F_{ab}}\Big(- \frac{\lambda}{4}\int d^4y\, F^{cd}(y)\widetilde{F}_{cd}(y)\Big). \tag{4.2.29}$$

Inserting the variation $\delta S = -\frac{\lambda}{2}\int d^4y G^{cd}[F,y]\widetilde{G}_{cd}[F,y]$ into 4.2.28, and comparing the resulting expression to 4.2.29, one obtains the integrated form of the consistency condition 4.2.23. Here is the end of the proof.

Application

The terms we will consider are the terms 2.5.13 discussed in chapter 2

$$\mathcal{L}_{(m,n)} = \alpha'^m \partial^n F^p, \qquad \text{for } p = m + 2 - n/2, \tag{4.2.30}$$

$$\mathcal{L}_m = \alpha'^m F^{m+2}, \qquad \text{for } n = 0. \tag{4.2.31}$$

Now, we need to establish the E-M duality invariance of the $p = 4$ terms. Of course, superstring corrections are obtained in $d = 10$, while E-M selfduality, that has been discussed so far in this chapter, is realized in $d = 4$. Therefore we will examine the validity of E-M selfduality of the contributions of the type 4.2.30, setting all other ten-dimensional fields to zero, and truncating the resulting expression to $d = 4$, by restricting the Lorentz index to run from one to four. Moreover, the result can hold order-by-order in α' so that for each order of α' the corresponding $p = 4$ contribution to the Lagrangian satisfies, together with Maxwell $m = 0$ term, E-M selfduality to order m in α'.

We start the analysis with the $m = 4$ terms. Then the four-derivative terms 2.5.25 stated in chapter 2 are

$$\mathcal{L}_{(4,4)} = a_{(4,4)}\alpha'^4 t_8^{abcdefgh}\partial_k F_{ab}\partial^k F_{cd}\partial^l F_{ef}\partial_l F_{gh}, \tag{4.2.32}$$

where $t_8^{abcdefgh}$ has been defined by 2.5.23. The combination $\mathcal{L}_0 + \mathcal{L}_{(4,4)}$ generates

$$G^{ab} = F^{ab} + G^{ab}_{(4,4)}, \qquad G^{ab}_{(4,4)} = \alpha'^4 t_8^{abcdefgh}\partial_k(\partial^k F_{cd}\partial^l\partial F_{ef}\partial_l F_{gh}). \tag{4.2.33}$$

To establish E-M selfduality we have to verify whether $\mathcal{L}_0 + \mathcal{L}_{(4,4)}$ solves 4.2.23. The verification only makes sense to order four in α', since in higher orders other contributions to the effective action would interfere. Since the zero order term in 4.2.23 cancel it remains to verify that

$$I = \int d^4x \mathrm{tr} \widetilde{F} G_{(4,4)} = 0. \tag{4.2.34}$$

Integrating 4.2.34 by parts implies

$$I = \int d^4x t_8^{abcdefgh} \partial_k \widetilde{F}_{ab} \partial^k F_{cd} \partial_l F_{ef} \partial^l F_{gh}. \tag{4.2.35}$$

The crucial property, which in fact holds to all orders in α', is that in $\mathcal{L}_{(m,2m-4)}$ the indices of the field strengths F are all contracted amongst each other, and therefore also the derivatives are contracted (see section 2.5.5). The complete symmetry of t_8 in combination with

$$(\widetilde{F}_k F_l + \widetilde{F}_l F_k)_a{}^b = -\frac{1}{2}\delta_a^b \mathrm{tr} \widetilde{F}_k F_l; \qquad \text{with } F_k = \partial_k F, \tag{4.2.36}$$

allows us to express all the traces over the four matrices resulting from the expansion of 4.2.35 in terms of products of traces over two matrices. Thus the cancellation of 4.2.34 has been verified [B].

For higher orders in α' the p = 4 terms contain more derivatives, but again these are all contracted, while the tensor structure of the field strengths remains the same. Essentially one has to show that

$$t_{8\,abcdefgh}\Big(\widetilde{F}_1^{ab} F_2^{cd} F_3^{ef} F_4^{gh} + F_1^{ab} \widetilde{F}_2^{cd} F_3^{ef} F_4^{gh} + F_1^{ab} F_2^{cd} \widetilde{F}_3^{ef} F_4^{gh} + F_1^{ab} F_2^{cd} F_3^{ef} \widetilde{F}_4^{gh}\Big), \tag{4.2.37}$$

where the subscripts $1,2,3,4$ indicate the derivative structure, vanishes. Using again 4.2.36 and the symmetry of t_8 one establishes that 4.2.37 vanishes independently of the precise way the derivatives are contracted.

This gives the desired result: E-M selfduality survives, up to this order in α', the addition of derivative corrections.

Selfduality and Higher Orders Terms

It would be of interest to use electromagnetic selfduality to constrain, or to determine, the derivative corrections to the Born-Infeld action that are not known explicitly. However, it is well-known that already the Born-Infeld action itself is not the only selfdual deformation of the Maxwell action, the ambiguity can be parametrized by a real function of one variable. From what precedes it is clear that $\mathcal{L}_{(m,2m-4)}$ is not the only $p = 4$ action with derivative corrections that satisfies 4.2.23 to order α'^4. Indeed, we found that the result depends only on the presence of the tensor t_8 and on the fact that there are no contractions between derivatives and field strengths. The result is independent of the precise way the derivatives are placed.

Given these ambiguities, it is clear that E-M selfduality can only constrain but not determine the derivative corrections to the terms related to the six-point function, $p = 6$. For the four-derivative terms n = 4 we do have the result of [57]. The method used above is however not applicable, because the property of having no contractions between field strengths and derivatives no longer holds. Nevertheless, it would be interesting to extend the analysis of selfduality to those terms.

Selfduality and Field Redefinitions

It could happen that some of the Lagrangians in question contain terms which are proportional to the lowest order equations of motion $\partial_a G_0^{ab}$; for instance, the Lagrangian 4.2.32 is equivalent modulo lowest order equations of motion to the terms found in [56], for $m = 4$ and $p = 4$, written in a different basis. Now, we argue that those terms, found in [56], satisfy 4.2.23 modulo terms proportional to the lowest orders equations of motion. For the sake of simplicity, we shall prove this statement for a more general case:
Given an action S written as

$$S = S_0 + S_1, \qquad S_1 = \int d^4x V_b[F,x]\partial_a G_0^{ab}. \tag{4.2.38}$$

The equations of motion derived from 4.2.38 contain

$$G^{ab}(x) = G_0^{ab}(x) - \int d^4y\Big(\frac{\delta V_d(y)}{\delta F_{ab}(x)}\partial_c G_0^{cd}(y) - \partial_c V_d(y)\frac{\delta G_0^{cd}(y)}{\delta F_{ab}(x)}\Big). \tag{4.2.39}$$

Using the fact that S_0 is selfdual, the only remaining terms in the selfduality condition are

$$0 = \int d^4x d^4y\Big(\widetilde{G}_{0\,ab}(x)\frac{\delta V_d(y)}{F_{ab}(x)}\partial_c G_0^{cd}(y) - \partial_c V_d(y)\frac{\delta G_0^{cd}(y)}{\delta F_{ab}(x)}\widetilde{G}_{0\,ab}(x)\Big). \tag{4.2.40}$$

By virtue of the identity $\frac{\delta G_0^{cd}(y)}{\delta F_{ab}(x)} = \frac{\delta G_0^{ab}(x)}{\delta F_{cd}(y)}$ and again selfduality of S_0, the second term in 4.2.40 can be expressed in terms of a combination involving $\partial_a \widetilde{F}^{ab}$ which vanishes due to the Bianchi identity. The remaining term is proportional to $\partial_c G_0^{cd}(y)$ which must disappear as a result of a field redefinition on the vector potential. This implies that we should allow 4.2.23 to hold up to terms containing $\partial_a G_0^{ab}$, the equation of motion of S_0.

So, if we have an action S_0 satisfying the condition of selfduality, then of course any action related to that action by a field redefinition should also be considered to be electromagnetically selfdual.

4.3 Selfduality of Nonlinear Models Coupling to Matter Fields

A natural step forward is to couple the model of the previous section, being generalized to n abelian gauge fields, to matter fields, e.g. scalars, fermions and forms [117]. For these models the consistency of selfduality requires, besides the invariance of the field equations 4.2.2, the covariance of the matter fields equations of motion under selfduality transformations. Therefore the selfduality condition 4.2.23 has to be amended to include the effect of the matter fields couplings.

For the sake of generality, we denote the action of such a model by[6]

$$S[F^i, \Phi^\alpha] = \int d^4x \mathcal{L}(F^i, \partial_a F^i, \Phi^\alpha, \partial_a \Phi^\alpha), \tag{4.3.1}$$

where i and α label the number of gauge fields and matter fields, respectively. Then we consider a linear infinitesimal transformation of the form

$$G'^i_{ab}[F,\Phi] = (A+1)G^i_{ab}[F,\Phi] + B\widetilde{F}^i_{ab}, \tag{4.3.2a}$$

$$\widetilde{F}'^i_{ab} = CG^i_{ab}[F,\Phi] + (1+D)\widetilde{F}^i_{ab}, \tag{4.3.2b}$$

$$\Phi'^\alpha = \Phi^\alpha + \zeta^\alpha[\Phi], \tag{4.3.2c}$$

where A,B,C and D are arbitrary real $n \times n$ matrices, and ζ^α some unspecified functions of the matter fields.

The derivation of a selfduality consistency condition for those models is straightforward, and it is to some extent similar to the derivation which has been done in section 4.2.3. Using the fact that 4.3.1 satisfies $S'[F,\Phi] = S[F,\Phi]$, the transformed dual tensor G', arising in the equations of motion of the gauge fields, is expressed as

$$\begin{aligned} G'^i_{ab}[F',\Phi',x] &= -\frac{\delta}{\delta F'^{i\,ab}(x)} S'[F',\Phi'] \\ &= -\frac{\delta}{\delta F'^{i\,ab}(x)} S[F,\Phi] - \frac{\delta}{\delta F^{i\,ab}(x)} \delta S[F,\Phi], \end{aligned} \tag{4.3.3}$$

where $\delta S[F,\Phi] = S[F',\Phi'] - S[F,\Phi]$. By means of definitions 4.3.2b and 4.3.2c, one can express 4.3.3 solely in terms of the original fields. The variation δG then reads

$$\delta G^i_{ab}[F,\Phi,x] = \int d^4y C^{jk} G^j_{cd}(y) \frac{\delta \widetilde{G}^{k\,cd}(y)}{\delta F^{i\,ab}(x)} + D^{ji} G^j_{ab}[F,\Phi,x] - \frac{\delta}{\delta F^{i\,ab}(x)} \delta S[F,\Phi]. \tag{4.3.4}$$

[6] Cases, where there are no derivatives for the field strength, have been thoroughly worked out in [104, 106, 107, 112].

Now, Eq.4.3.4 must coincide with the variation of G following from 4.3.2. One therefore obtains the following constraints on the parameters of the transformation

$$D^{ij} + A^{ji} = \epsilon\delta^{ij}, \qquad B^{ij} = B^{ji}, \qquad C^{ij} = C^{ji}, \tag{4.3.5}$$

from which it follows that[7]

$$\frac{\delta}{\delta F^{i\,ab}(x)} \left[\delta S + \epsilon S + \frac{1}{4}\int d^4y \Big(B^{kj}\widetilde{F}^k_{cd}(y)F^{j\,cd}(y) - C^{jk}G^j_{cd}(y)\widetilde{G}^{k\,cd}(y)\Big)\right] = 0. \tag{4.3.6}$$

This is the condition that S should satisfy in order that the equations of motion of the gauge fields, combined with the Bianchi identity, are invariant under 4.3.2.

It is clear that the parameter ϵ might still be determined. We have not actually used up all the selfduality requirements. One still has the demand that for selfdual models the matter field equations of motion should transform covariantly under 4.3.2. Given the matter equations of motion

$$\Sigma^\alpha[F,\Phi,x] = \frac{\delta}{\delta\Phi^\alpha(x)}S[F,\Phi] = 0, \tag{4.3.7}$$

the transformed equations read

$$\begin{aligned}\Sigma'^\alpha[F',\Phi',x] = \frac{\delta}{\delta\Phi^\alpha(x)}\delta S[F,\Phi] + \int d^4y \frac{\delta S}{\delta\Phi^\beta(y)}\frac{\delta\Phi^\beta(y)}{\delta\Phi'^\alpha(x)} + \\ + \int d^4y \frac{\delta S}{\delta F^i_{ab}(y)}\frac{\delta F^{i\,ab}(y)}{\delta\Phi'^\alpha(x)}.\end{aligned} \tag{4.3.8}$$

Again, one might make use of 4.3.2 so as to obtain the variation $\delta\Sigma$ in terms of the original fields

$$\delta\Sigma^\alpha[F,\Phi,x] = \frac{\delta}{\delta\Phi^\alpha(x)}\left[\delta S + \frac{1}{4}\int C^{ij}d^4y\widetilde{G}^i_{ab}(y)G^{j\,ab}(y)\right] - \frac{\partial\zeta^\beta}{\partial\Phi^\alpha}\Sigma_\beta[F,\Phi]. \tag{4.3.9}$$

The requirement that 4.3.7 is covariant under 4.3.2 results in

$$\frac{\delta}{\delta\Phi^\alpha(x)}\left[\delta S + \frac{1}{4}\int C^{ij}d^4y\widetilde{G}^i_{ab}(y)G^{j\,ab}(y)\right] = 0. \tag{4.3.10}$$

The two relations 4.3.6 and 4.3.10 are compatible with one another provided that $\epsilon = 0$, and therefore for a selfdual model the action should vary in a specific way under 4.3.2

$$\delta S[F,\Phi] = \frac{1}{4}\int d^4y\Big(-B^{kj}\widetilde{F}^k_{cd}(y)F^{j\,cd}(y) + C^{jk}G^j_{cd}(y)\widetilde{G}^{k\,cd}(y)\Big). \tag{4.3.11}$$

[7]Notice that in the absence of ∂F, the functional form of expression 4.3.6 breaks to the normal form.

Moreover, one can easily realize that

$$\delta\left(S-\frac{1}{4}\int d^4y \mathrm{tr} F^i(y)G^i(y)\right)=0. \tag{4.3.12}$$

Finally, the selfduality consistency equation is summarized as

$$\begin{aligned}\delta_\Phi S[F,\Phi] &= \Sigma^\alpha[F,\Phi]\delta\Phi^\alpha \\ &= -\frac{1}{4}\int d^4y\left(B^{ij}F^i_{ab}(y)\widetilde{F}^{j\,ab}(y)+C^{ij}G^i_{ab}(y)\widetilde{G}^{j\,ab}(y)\right. \\ &\left.+2D^{ij}G^i_{ab}(y)F^{j\,ab}(y)\right).\end{aligned} \tag{4.3.13}$$

The combination of 4.3.5 and the condition $\epsilon=0$ lead to the fact that the selfduality transformations 4.3.2 are described by the real non-compact group $\mathrm{Sp}(2n,\mathbb{R})$; non-compact real form or slice of the complex $\mathrm{Sp}(2n)$ in a real basis of the $2n$-dimensional representation[8]. The group $\mathrm{Sp}(2n,\mathbb{R})$ is the maximal group of duality transformations, although in specific models the group of selfduality transformations G, leaving the field equations invariant, may be actually smaller. It should be pointed out that $\mathrm{Sp}(2n,\mathbb{R})$ or its subgroup G may appear as the group of duality symmetries if the set of matter fields Φ^α include scalar fields parameterizing the coset space G/H, with H is the maximal compact subgroup of U (see chapter 5 for more detailed discussion). Any selfdual model without matter fields, $\mathcal{L}(F)$, can be viewed as a selfdual model $\mathcal{L}(F,\Phi,\partial\Phi)$ with the matter fields frozen, $\Phi^\alpha(x)=\Phi^\alpha_0\in G/H$. The duality transformation preserving this background has thus to be a subgroup of $\mathrm{U}(n)$, the maximal compact subgroup of $\mathrm{Sp}(2n,\mathbb{R})$. Strictly speaking, in the absence of matter fields the $\mathrm{Sp}(2n,\mathbb{R})$ breaks down to it is maximal subgroup $\mathrm{U}(n)$[9]. If one treats the matter fields Φ^α as coupling constants, then non-compact duality transformations relate models with different coupling constants. We stress in the end that the formalism that has been developed may be applied directly to cases in which the fields F and Φ interact with an external gravitational field, described by g_{ab} or by a vierbein; that is actually because those fields are inert under the action of duality symmetries.

It is worth mentioning that the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ preserves an antisymmetric bilinear form $\eta=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ such that

$$\begin{pmatrix}L & \widetilde{K}\end{pmatrix}\begin{pmatrix}0 & -\mathbb{1}_{n\times n}\\ \mathbb{1}_{n\times n} & 0\end{pmatrix}\begin{pmatrix}G\\ \widetilde{F}\end{pmatrix}=\text{invariant}. \tag{4.3.14}$$

Therefore the only duality invariant which can be constructed from vectors in the fundamental $2n$-dimensional representation is an antisymmetric bilinear 4.3.14.

[8] The symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is isomorphic to $\mathrm{SL}(2,\mathbb{R})$ for $n=1$.

[9] For the maximal compact subgroup $\mathrm{U}(n)$, the relations 4.3.5 are reduced to $D=A,\ C=-B,\ A^T=-A,\ B^T=B$.

4.3.1 Coupling to Axion and Dilaton

As a well-known particular example [104, 105, 107], we consider a model $\mathcal{L}(F,\chi,,\partial\chi)$ in which one gauge field ($n=1$) is coupling to a complex scalar field $\chi=\chi_1+i\chi_2$ (i.e. $\Phi^\alpha=(\chi,\overline{\chi})$), with $\overline{\chi}$ is the complex conjugate of χ. Consequently, the compact duality symmetry SO(2) (for pure electrodynamics) should be enhanced to a duality symmetry under a larger non-compact group $\mathrm{SL}(2,\mathbb{R})$ whose finite realization on the fields reads

$$\begin{pmatrix} G' \\ \widetilde{F}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} G \\ \widetilde{F} \end{pmatrix}, \qquad \chi'=\frac{a\chi+b}{c\chi+d}. \tag{4.3.15}$$

The corresponding infinitesimal transformations 4.3.2 become

$$G'_{ab}=(A+1)G_{ab}+B\widetilde{F}_{ab}, \tag{4.3.16a}$$

$$\widetilde{F}'_{ab}=CG_{ab}+(1-A)\widetilde{F}_{ab}, \tag{4.3.16b}$$

$$\delta\chi=B+2A\chi-C\chi^2, \tag{4.3.16c}$$

where the scalar field χ transforms *nonlinearly* in consistency with the shift invariance. The only manifest $\mathrm{SL}(2,\mathbb{R})$ invariant term of $\mathcal{L}(F,\chi,\partial\chi)$ is the kinetic term of the scalar field

$$\mathcal{L}_S(\chi,\partial\chi)=\frac{\partial\chi\partial\overline{\chi}}{(\chi-\overline{\chi})^2}. \tag{4.3.17}$$

Accordingly, we assume that the total Lagrangian decomposes into two parts

$$\mathcal{L}(F,\chi,\partial\chi)=\mathcal{L}_S(\chi,\partial\chi)+\widehat{\mathcal{L}}(F,\chi). \tag{4.3.18}$$

Thus the problem of finding the most general Lagrangian $\mathcal{L}$, whose equations of motion are invariant under $\mathrm{SL}(2,\mathbb{R})$, boils down to a problem of solving equation 4.3.13 for $\mathcal{L}$. For this case the consistency condition 4.3.13 becomes

$$\frac{1}{4}B\mathrm{tr}F\widetilde{F}+\frac{1}{4}C\mathrm{tr}G\widetilde{G}-\frac{1}{2}A\mathrm{tr}GF=\frac{\partial\widehat{\mathcal{L}}}{\partial\chi}\delta\chi+\frac{\partial\widehat{\mathcal{L}}}{\partial\overline{\chi}}\delta\overline{\chi}. \tag{4.3.19}$$

Now, substitute 4.3.16c in 4.3.19 and then make use of the fact that A,B and C are arbitrary parameters, the selfduality equation 4.3.19 leads to the following three equations

$$\frac{\partial\widehat{\mathcal{L}}}{\partial\chi}+\frac{\partial\widehat{\mathcal{L}}}{\partial\overline{\chi}} = \frac{1}{4}\mathrm{tr}F\widetilde{F}, \tag{4.3.20}$$

$$\chi\frac{\partial\widehat{\mathcal{L}}}{\partial\chi}+\overline{\chi}\frac{\partial\widehat{\mathcal{L}}}{\partial\overline{\chi}} = \frac{1}{4}\mathrm{tr}GF, \tag{4.3.21}$$

$$\chi^2\frac{\partial\widehat{\mathcal{L}}}{\partial\chi}+\overline{\chi}^2\frac{\partial\widehat{\mathcal{L}}}{\partial\overline{\chi}} = -\frac{1}{4}\mathrm{tr}G\widetilde{G}. \tag{4.3.22}$$

It has been shown in [107, 108] that the solution of these equations is

$$\widehat{\mathcal{L}}(F,\chi) = \mathcal{L}_0(\sqrt{\chi_2}F) - \frac{1}{4}\chi_1 \mathrm{tr} F\widetilde{F}, \tag{4.3.23}$$

where $\mathcal{L}_0(\sqrt{\chi_2}F)$ solves the selfduality equation 4.2.7 of purely (non)-linear electrodynamics models. In other words, if we redefine F by $\mathcal{F} = \sqrt{\chi_2}F$, one then sees that $\mathcal{F}$ and its dual $\mathcal{G} = -\partial\mathcal{L}/\partial\mathcal{F}$ are scale invariant and have the very simple transformation law

$$\delta\widetilde{\mathcal{F}} = -\chi_2 C\mathcal{G}, \qquad \delta\mathcal{G} = \chi_2 C\widetilde{\mathcal{F}}, \tag{4.3.24}$$

i.e., they transform according to the SO(2) transformation whose infinitesimal parameter λ redefined as $\lambda = \chi_2 C$.

If we replace χ_1 and χ_2 respectively by axion and dilaton, namely $\chi = a + ie^{-\varphi}$, then the most general selfdual Lagrangian $\mathcal{L}$ is schematically written as

$$\mathcal{L}(F, a, \varphi, \partial a, \partial\varphi) \sim -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}e^{2\varphi}(\partial a)^2 - a\mathrm{tr}F\widetilde{F} + \mathcal{L}_0(e^{-\frac{1}{2}\varphi}F). \tag{4.3.25}$$

But wait there is more! in [107] it has been established that there is a unique generalization of the Born-Infeld theory, i.e. $\mathcal{L}_0 = \mathcal{L}_{BI}$, admitting $\mathrm{SL}(2,\mathbb{R})$ invariant equations of motion. The reader should not be confused with an analogous effective Lagrangian [108] stemming from closed superstring theory, which also contains the Born-Infeld sector. For such a Lagrangian the $\mathrm{SL}(2,\mathbb{R})$ selfduality is completely lost. However it continues to hold once one truncates the Born-Infeld sector to its low-energy limit, i.e. keeping only the quadratic terms in F.

4.3.2 Coupling to Type IIB Supergravity Backgrounds

As seen in chapter 2 one of the nice features of type IIB string theory is that it contains a four-dimensional effective gauge field theory living on the D3-brane, which actually motivates the study of selfduality. Indeed, in [35, 118] it has been proven that the worldvolume theory of the D3-brane admits $\mathrm{SL}(2,\mathbb{R})$ as a duality group symmetry. It has actually been widely believed that selfduality of D3-brane is inherited from the $\mathrm{SL}(2,\mathbb{R})$ symmetry of type IIB supergravity. Alternatively, we are going to invoke the machinery that has been developed in this chapter in order to establish the slefduality of D3-brane worldvolume theory [34, 117, 119].

We know from section 2.5.2 that the D3-brane wolrdvolume action is divided into two pieces, namely the Dirac-Born Infeld sector and the Wess-Zumino sector. In type IIB on-shell supergravity backgrounds the action is

$$\mathcal{L} = \mathcal{L}_{DBI} + \mathcal{L}_{WZ}. \tag{4.3.26}$$

The two pieces of 4.3.26 are written in terms of component fields as

$$\mathcal{L}_{DBI} = -\sqrt{-g}\sqrt{1 + \frac{e^{-\varphi}}{2}\mathcal{F}_{ab}\mathcal{F}^{ab} - \frac{e^{-2\varphi}}{16}(\mathcal{F}_{ab}\widetilde{\mathcal{F}}^{ab})^2}, \tag{4.3.27}$$

$$\mathcal{L}_{WZ} = \epsilon^{abcd}\left(\frac{1}{24}C_{abcd} + \frac{1}{4}C_{ab}\mathcal{F}_{cd} + \frac{1}{8}a\mathcal{F}_{ab}\mathcal{F}_{cd}\right), \tag{4.3.28}$$

where $g = \det g_{ab}$, and $\Phi^\alpha = (a, \varphi, \mathcal{B}, C_2, C_4)$ are the possible bosonic background fields of type IIB with $\mathcal{F}_{ab} = F_{ab} - \mathcal{B}_{ab}$.

Now let us see whether the selfduality condition 4.3.13 is satisfied for the above action. First of all, the dual tensor G following from 4.3.26 can be derived as

$$G^{ab} = -\frac{\mathcal{L}_{DBI}}{\partial F_{ab}} - \widetilde{C}_2^{ab} - a\widetilde{\mathcal{F}}^{ab}, \tag{4.3.29}$$

$$\widetilde{G}^{ab} = (\widetilde{\frac{\mathcal{L}_{DBI}}{\partial F}})^{ab} + C_2^{ab} + a\mathcal{F}^{ab}. \tag{4.3.30}$$

The SL$(2,\mathbb{R})$ infinitesimal transformations of various fields in our theory are given by 4.3.16a, 4.3.16b and

$$\delta a = 2Aa - Ca^2 + B + Ce^{-2\varphi}; \qquad \delta\varphi = -2A + 2aC, \tag{4.3.31a}$$

$$\delta C_2^{ab} = AC_2^{ab} + B\mathcal{B}^{ab}; \qquad \delta\mathcal{B}^{ab} = CC_2^{ab} - A\mathcal{B}^{ab}, \tag{4.3.31b}$$

$$\delta C_4^{abcd} = \frac{B}{2}\mathcal{B}^{ab}\mathcal{B}^{cd} + \frac{C}{2}C_2^{ab}C_2^{cd}. \tag{4.3.31c}$$

The argument that the transformation of C_4 provides a nonlinear representation of SL$(2,\mathbb{R})$, is traced back to the consistency that one should maintain between the duality transformations and the standard gauge transformations associated with forms $\mathcal{B}$, C_2 and C_4.

As a next step we evaluate the variation of the Lagrangian $\mathcal{L}$ w.r.t its argument Φ in the left-hand side of 4.3.13

$$\delta_\Phi\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial a}\delta a + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{B}^{ab}}\delta\mathcal{B}^{ab} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial C_2^{ab}}\delta C_2^{ab} + \frac{1}{4}\frac{\partial\mathcal{L}}{\partial C_4^{abcd}}\delta C_4^{abcd}, \tag{4.3.32}$$

then substitute 4.3.31a-4.3.31c into 4.3.32 and the resulting expression into 4.3.13. The demand that the equation 4.3.13 must hold for arbitrary values of A, B and C leads to the fact that all the coefficients of those parameters should vanish identically. The vanishing of those coefficients results in the following three equations

$$-\frac{1}{2}G_{ab}F^{ab} - 2\frac{\partial\mathcal{L}}{\partial\varphi} + 2a\frac{\partial\mathcal{L}}{\partial a} - \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{B}^{ab}}\mathcal{B}^{ab} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial C_2^{ab}}C_2^{ab} = 0, \tag{4.3.33a}$$

$$\frac{1}{4}F_{ab}\widetilde{F}^{ab} + \frac{\partial\mathcal{L}}{\partial a} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial C_2^{ab}}\mathcal{B}^{ab} + \frac{1}{8}\epsilon_{abcd}\mathcal{B}^{ab}\mathcal{B}^{cd} = 0, \tag{4.3.33b}$$

$$\frac{1}{4}G_{ab}\widetilde{G}^{ab} + 2a\frac{\partial\mathcal{L}}{\partial\varphi} + \frac{\partial\mathcal{L}}{\partial a}(e^{-2\varphi} - a^2) + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\mathcal{B}^{ab}}C_2^{ab} + \frac{1}{8}\epsilon_{abcd}C_2^{ab}C_2^{cd} = 0. \tag{4.3.33c}$$

The fact that the $\mathcal{L}_{DBI}$ depends upon the dilaton field φ in the form of $e^{-\frac{\varphi}{2}}\mathcal{F}$ implies

$$\frac{\partial\mathcal{L}_{DBI}}{\partial\varphi} = -\frac{1}{4}\frac{\partial\mathcal{L}_{DBI}}{\partial F^{ab}}\mathcal{F}^{ab}, \tag{4.3.34}$$

which comes in handy for verifying 4.3.33a. Notice that 4.3.33b is almost trivially satisfied.

The most intricate equation is 4.3.33c. After somewhat tedious but straightforward calculations it can be reduced to the following equation

$$\epsilon^{abcd}\left(\frac{\partial\mathcal{L}_{DBI}}{\partial F^{ab}}\frac{\partial\mathcal{L}_{DBI}}{\partial F^{ab}} + e^{-2\varphi}\mathcal{F}_{ab}\mathcal{F}_{cd}\right) = 0. \tag{4.3.35}$$

Plugging the explicit definition 4.3.27 of DBI Lagrangian into 4.3.35, it is easily shown that 4.3.33c is satisfied. We have thereby demonstrated that the D3-brane action 4.3.26 in type IIB supergravity backgrounds indeed satisfies the selfduality condition 4.3.13. Therefore the D3-brane worldvolume theory is selfdual.

The generalization of this analysis to the super D3-brane is somehow feasible and has been worked out in [118, 120, 121]. It is worthwhile also to note that in [120, 121] there have been achieved more than that. They have basically succeeded in uplifting the selfduality transformations from a symmetry of the field equations to a symmetry of the action: there has been argued that an off-shell duality transformation might be realized through the gauge potential $\mathcal{A}_a$, and by imposing the Gaillard-Zumino condition, namely selfduality condition 4.3.13, one can easily obtain the invariance of the action up to surface terms. This method can be regarded as an alternative approach, in some specific cases, to the PST formalism formerly mentioned.

4.4 Gaillard-Zumino Model: Selfduality in Supergravity

One might wonder whether the analysis of selfduality that we have developed so far may be extended to higher dimensions. We should admit that working in four dimensions makes life easy, however, there is no obstruction to go higher in dimensions, specifically even dimensions $d = 2p$, and hence higher in the field strength rank [106, 122, 123]. Such a generalization of the Gaillard-Zumino model comes in handy when one wants to turn on duality in supergavity.

We consider theories of n $(p-1)$-th rank antisymmetric tensor fields $A^i_{a_1\cdots a_{p-1}}(x)$

$(i = 1, \cdots, n)$ interacting with matter fields $\Phi^\alpha(x)$. The field strengths and its Hodge duals are defined by

$$F^i_{a_1\cdots a_p} = p\,\partial_{[a_1} A^i_{a_2\cdots a_{p-1}]} \tag{4.4.1}$$

$$\widetilde{F}^{i\,a_1\cdots a_p} = \frac{1}{p!}\epsilon^{a_1\cdots a_p b_1\cdots b_p} F^i_{b_1\cdots b_p}. \tag{4.4.2}$$

The Hodge duality operation has a peculiar property in d dimensions

$$\widetilde{\widetilde{F}} = \epsilon F, \qquad \epsilon = \begin{cases} +1 & \text{for } d = 4r+2 \\ -1 & \text{for } d = 4r. \end{cases} \tag{4.4.3}$$

Given a Lagrangian $\mathcal{L}(F, \Phi, \partial\Phi)$ which governs the dynamics of interacting theories, the equations of motion and the Bianchi identity read

$$\partial_{a_1} G^{a_1\cdots a_p} = 0, \qquad \partial_{a_1}\widetilde{F}^{i\,a_1\cdots a_p} = 0, \tag{4.4.4}$$

where the dual tensors $G_i^{a_1\cdots a_p}$ generated by $\mathcal{L}$ are defined as

$$G^i_{a_1\cdots a_p} = -p!\,\frac{\partial\mathcal{L}}{\partial F^i_{a_1\cdots a_p}}. \tag{4.4.5}$$

The selfduality requirement imposed on higher dimensional theories - the simultaneous covariance of 4.4.5 and of matter fields equations of motion under 4.3.2- gives rise to the following constraints on the parameters

$$A_i{}^j = -D_j{}^i, \qquad B_{ij} = \epsilon B_{ji}, \qquad C_{ij} = -\epsilon C_{ji}, \tag{4.4.6}$$

which can be recast into

$$S^T\eta + \eta S = 0, \qquad \text{where } S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \eta = \begin{pmatrix} 0 & \epsilon\cdot\mathbb{1}_{n\times n} \\ \mathbb{1}_{n\times n} & 0 \end{pmatrix}. \tag{4.4.7}$$

Here one should distinguish two cases:

- If the dimension is $d = 4r$ $(\epsilon = -1)$, then η is an antisymmetric bilinear form and the above condition corresponds to a $\mathrm{Sp}(2n, \mathbb{R})$ duality group or its subgroup.
- For $d = 4r+2$ case corresponding to $\epsilon = 1$, η is symmetric bilinear form which by an appropriate change of basis can be brought into its diagonal form $\mathrm{diag}(1, -1)$. Then the duality symmetry associated to this case is $\mathrm{SO}(n, n)$ or its subgroup.

As usual, we are not seeking the duality invariance of the Lagrangian, however the variation of the Lagrangian must take on a very definite form under the duality transformation

$$\delta\mathcal{L} = \frac{1}{2p!}(B^{ij}\mathrm{tr}F^i\widetilde{F}^j - C^{ij}\mathrm{tr}G^i\widetilde{G}^j) = -\delta(\frac{1}{2p!}\mathrm{tr}F^iG^j). \tag{4.4.8}$$

The variation 4.4.8 of the Lagrangian $\mathcal{L}$ is rather indicative and suggestive. As a consequence one can obtain an explicit form of the Lagrangian $\mathcal{L}$ satisfying 4.4.8

$$\mathcal{L} = \frac{1}{2p!}\mathrm{tr}F^i G_i + \frac{1}{2p!}\mathrm{tr}(\widetilde{K}^i G_i + \epsilon L^i \widetilde{F}_i) + \mathcal{L}_{\mathrm{inv}}(\Phi, \partial\Phi), \tag{4.4.9}$$

where p-th antisymmetric tensors $(L_i(\Phi, \partial\Phi), \widetilde{K}^i(\Phi, \partial\Phi))$ transform the same way as $(G^i, \widetilde{F}^i)$ see 4.3.14, and $\mathcal{L}_{\mathrm{inv}}$ is a manifestly dual invariant (selfdual) sector[10].

One still have the possibility to eliminate G in favor of the other fields. Substituting 4.4.9 into 4.4.5 we obtain the following differential equation[11]

$$(G - \epsilon\widetilde{L})_i = (F - \widetilde{K})^j \frac{\partial}{\partial F^i}(G - \epsilon\widetilde{L})_j. \tag{4.4.10}$$

In order to solve this equation, one might introduce the so-called j-operation

$$\widetilde{F} = jF, \qquad \text{with } j^2 = \epsilon. \tag{4.4.11}$$

The solution of this equation has then been found to be

$$G_i = \epsilon\widetilde{L}_i - M_{ij}(\Phi)(F - \widetilde{K})^j, \tag{4.4.12}$$

where $M_{ij}(\Phi)$ is a posteriori an arbitrary $n \times n$ symmetric matrix function of the matter fields[12].

The determination of the matrix M_{ij} in terms of Φ rests upon two physical observations:

- The transformation law of M follows from the covariance of Eq.4.4.12, therefore the matrix M must transform nonlinearly as

$$\delta M = -jB + AM - MD + \epsilon jMCM. \tag{4.4.13}$$

- The form of the kinetic energy term for the vector fields requires

$$M_{ij}(\Phi) = \delta_{ij} + N_{ij}(\Phi). \tag{4.4.14}$$

[10] We have assumed that we are dealing with the maximal duality group, i.e., $\mathrm{Sp}(2n, \mathbb{R})$ in $d = 4r$ or $\mathrm{SO}(n, n)$ in $d = 4r+2$; any dual invariant quantity associated to this group should be proportional to $\widetilde{K}G + \epsilon L\widetilde{F}$. However, had we considered models that admit subgroups of them as duality symmetries, there have been other invariants than $\widetilde{K}G + \epsilon L\widetilde{F}$.

[11] For simplicity, we drop spacetime indices in equation 4.4.10.

[12] Note that the matrix M_{ij}, restricted only to scalar fields, is in some sense the scalar matrix which will parameterize the target space of the $4D$ nonlinear sigma-model in chapter 5.

Thus, if we can find out functions $K^i(\Phi, \partial\Phi)$, $L^i(\Phi, \partial\Phi)$ and $M_{ij}(\Phi)$ with appropriate transformations properties, we have then an explicit form of the Lagrangian

$$\mathcal{L} = -\frac{1}{2p!}\mathrm{tr}F^i M_{ij} F^j + \epsilon\frac{1}{p!}\mathrm{tr}F^i(\widetilde{L}_i - M_{ij}\widetilde{K}^j) + \epsilon\frac{1}{2p!}\mathrm{tr}K^i(L_i - M_{ij}K^j) + \mathcal{L}_{\mathrm{inv}}(\Phi, \partial\Phi). \tag{4.4.15}$$

Note that the Lagrangians of supergravities, whose equations of motion are invariant under selfduality transformations, are often of this type.

4.4.1 Special Case: Compact Selfduality

We now consider the case where $M_{ij} = \delta_{ij}$, reproducing some of the results we have obtained in section 4.3. In this special case we will see that the duality symmetry group must be demoted to a compact group in the absence of the matter fields. From 4.4.13 we see that the parameters of the transformations have to obey

$$A_{ij} = D_{ij}, \qquad B = \epsilon C. \tag{4.4.16}$$

Again, two cases must be distinguished. For $d = 4r$, the condition 4.4.16 implies

$$S = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad A_{ij} = -A_{ji}, \qquad B_{ij} = B_{ji}. \tag{4.4.17}$$

In an appropriate complex basis the transformation law becomes

$$\delta\begin{pmatrix} G + i\widetilde{F} \\ G + i\widetilde{F} \end{pmatrix} = \begin{pmatrix} \overline{U} & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} G + i\widetilde{F} \\ G + i\widetilde{F} \end{pmatrix}, \qquad \text{with } U = A + iB. \tag{4.4.18}$$

Since U is anti-hermitian, i.e. $U = -U^\dagger$, the duality symmetry group is $\mathrm{U}(n)$, the maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{R})$. For $d = 4k + 2$ the constraints 4.4.16 becomes

$$S = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \qquad A_{ij} = -A_{ji}, \qquad B_{ij} = -B_{ji}. \tag{4.4.19}$$

and

$$\delta\begin{pmatrix} G + \widetilde{F} \\ G + \widetilde{F} \end{pmatrix} = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}\begin{pmatrix} G + \widetilde{F} \\ G + \widetilde{F} \end{pmatrix}. \tag{4.4.20}$$

with $U_+ = A + B$, and $U_- = A - B$ are real and antisymmetric matrices. The duality symmetry group is hence $\mathrm{SO}(n) \times \mathrm{SO}(n)$, the maximal compact subgroup of $\mathrm{SO}(n, n)$.

For a non-compact duality group, one may construct a $M_{ij}(\Phi)$, with Φ confined to scalar fields, by resorting to the well-know description of nonlinear sigma models for scalars valued in the coset spaces G/H of a group by a subgroup. Then G is a non-compact semisimple duality group but the subgroup H is its maximal compact subgroup. This nonlinear realization of G and others will be discussed in chapter 5.

4.5 Charges and Selfdual Invariant Quantities

4.5.1 Charges of Duality Symmetry

The duality symmetries that have been examined in this chapter are global symmetries, i.e. A, B and C are constant parameters. Although we are dealing with 4-dimensional models whose actions are not manifestly invariant under selfduality 4.3.2, one can still *partially* perform the usual Noether's argument and find the conservation laws. It is known from Noether's procedures that the conserved currents should be constructed in terms of the basic fields of the theory. But we know from section 4.2 that the action of the duality transformation on the gauge field is through the field strength F, therefore the realization of duality symmetry on the basic field $\mathcal{A}$ is nonlocal as $F = d\mathcal{A}$. The way out of this problem, namely locality violation, is to construct the current so that the basic fields $\mathcal{A}$ show up on equal footing with their duals $\widehat{\mathcal{A}}$ in the final expression.

By contrast, the covariance of matter fields equations of motion under selfduality is locally realized and via the basic fields Φ's. Therefore all one requires so as to apply consistently the Noether's theorem is to extract $\delta_\Phi \mathcal{L}$ out of $\delta\mathcal{L}$. Then we get

$$\delta_\Phi \mathcal{L}(F^i, \Phi^\alpha, \partial\phi^\alpha) = -\frac{1}{4}\left(B^{ij} F^i_{ab} \widetilde{F}^{j\,ab} + C^{ij} G^i_{ab} \widetilde{G}^{j\,ab} + 2D^{ij} G^i_{ab} F^{j\,ab}\right). \qquad (4.5.1)$$

Invoking field equations 4.2.2, we are able to re-express 4.5.1 as a divergence of a quantity J_1^a

$$\delta_\Phi \mathcal{L} = -\partial_a J_1^a, \qquad (4.5.2)$$

where

$$J_1^a = \frac{1}{2}(B^{ij} F^{i\,ab} \mathcal{A}^j_b + C^{ij} G^{i\,ab} \widehat{\mathcal{A}}^j_b + D^{ij} G^{i\,ab} \mathcal{A}^j_b - A^{ij} F^{i\,ab} \widehat{\mathcal{A}}^j_b) \qquad (4.5.3)$$

with $G^i_{ab} = \partial_a \widehat{\mathcal{A}}^i_b - \partial_b \widehat{\mathcal{A}}^i_a$ follows from the equations of motion $\partial_a G^{ab} = 0$.
Now the standard argument due to Emmy Noether, applying for $\delta_\Phi \mathcal{L}$, comes into play

$$\delta_\Phi \mathcal{L} = \partial_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_a \Phi^\alpha)} \delta\Phi^\alpha\right) = \partial_a J_2^a, \qquad (4.5.4)$$

where we have made use of the equations of motion of the matter fields. Then one can easily infer that $J_2^a = \zeta^\alpha \frac{\partial\mathcal{L}}{\partial(\partial_a \Phi^\alpha)}$, with ζ^α are the infinitesimal parameters defined in 4.3.2.

Equation 4.5.2 together with 4.5.4, using all equations of motion, lead to the result that selfduality symmetries imply the existence of conserved currents

$$J^a = J_1^a + J_2^a, \qquad \text{with } \partial_a J^a = 0. \qquad (4.5.5)$$

Notice that the current 4.5.5 is not invariant under the gauge transformation associated with the gauge fields

$$\mathcal{A}_a^i \to \mathcal{A}_a^i + \partial_a \kappa^i, \qquad \widehat{\mathcal{A}}_a^i \to \widehat{\mathcal{A}}_a^i + \partial_a \pi^i, \tag{4.5.6}$$

it still changes by the divergence of antisymmetric tensor

$$J^a \to J^a + \frac{1}{2}\partial_b (B^{ij} F^{i\,ab} \kappa^j + C^{ij} G^{i\,ab} \pi^j + D^{ij} G^{i\,ab} \kappa^j - A^{ij} F^{i\,ab} \pi^j). \tag{4.5.7}$$

Therefore the corresponding charge $Q = \int J^0 d^3x$ is gauge invariant and turns out actually to be the *generator*[13]of the selfduality symmetry. It is worth pointing out that the charges of global symmetries in general and duality symmetries in particular will play an essential role in classifying solutions of (super)gravity theories in chapter 6.

4.5.2 Selfdual Invariant Quantities

Although the Lagrangian is not invariant under the duality transformation, a suitably defined derivative of the Lagrangian with respect to an invariant parameter is invariant. Assume that $\mathcal{L}$ depends upon an invariant parameter ω. If $\zeta^\alpha(\Phi)$ is independent of ω, we differentiate 4.3.11 with respect to ω, then obtain

$$\frac{\partial}{\partial \omega} \delta \mathcal{L} = \frac{1}{2} \frac{\partial G^i_{ab}}{\partial \omega} C^{ij} \widetilde{G}^{j\,ab}. \tag{4.5.8}$$

On the other hand, the derivative of

$$\delta \mathcal{L} = \delta_\Phi \mathcal{L} + \frac{1}{2} \delta F_{ab} \frac{\partial \mathcal{L}}{\partial F_{ab}} \tag{4.5.9}$$

in terms of ω yields

$$\frac{\partial}{\partial \omega} \delta \mathcal{L} = \delta \left(\frac{\partial \mathcal{L}}{\partial \omega} \right) + \frac{1}{2} G^i_{ab} C^{ji} \frac{\partial G^{i\,ab}}{\partial \omega}. \tag{4.5.10}$$

Comparing 4.5.8 and 4.5.10, we find

$$\delta \frac{\partial \mathcal{L}}{\partial \omega} = 0. \tag{4.5.11}$$

The result 4.5.11 provides a way of checking that a theory admits selfduality or of constructing the Lagrangian for such a theory, by switching on couplings in an invariant way. The case when ζ^α depend on ω is a little more delicate (see [106]).

As an example, if ω represents an external gravitational field, 4.5.11 implies that the energy-momentum tensor, which is the variational derivative of the Lagrangian w.r.t to the gravitational field, is invariant under selfduality transformations.

[13]By using Coulomb-like gauge and developing the appropriate canonical formalism.

4.6 Summary

The main purpose behind this chapter was first to present E-M duality symmetries as a physical requirement for constraining the higher derivative structure for a given nonlinear electromagnetic theory, in particular the Born-Infeld theory. It's well-known that already the Born-Infeld theory is *not* the only selfdual extension of the Maxwell theory. As shown in [108] the ambiguity is labelled by a real function of one real argument. In this chapter we have moreover demonstrated that electromagnetic selfduality can *only* constrain but not determine the higher derivative corrections terms. This has been established for derivative corrections to the terms related to the 4-point function.

The rest of this chapter has been devoted to show how a general theory invariant under dual rotations may be constructed. We tried to clarify the structure of the theories admitting both compact and non-compact duality. A non-compact duality invariance is possible only when there are matter fields such as scalar fields. It has been moreover exhibited that the scalar fields must transform nonlinearly under the action of the duality group G. We derived the transformation property of the Lagrangian which is required for the equations of motion to be duality invariant, and show that this property implies the existence of conserved currents and the invariance of the energy-momentum tensor. We further exploited this property for explicit construction of the selfdual supergravity Lagrangians, which we illustrated by specializing to the compact case. We found that these theories still have a considerable degree of arbitrariness, namely in the choice of the matter Lagrangian $\mathcal{L}_{\rm inv}(\Phi, \partial\Phi)$. In supergravity theories this quantity, and in fact the field content itself, are fixed by supersymmetry. It appears that duality invariance of supergravity theories is implied by supersymmetry.

Motivated and inspired by our study of selfdual theories we will try in the next chapter to formulate Kaluza-Klein theories resp. extended supergravities as a *nonlinear realization* of the duality group G.

Chapter 5

Nonlinear σ-models and Toroidal Reductions

In this chapter we shall study some aspects of nonlinear sigma models on Riemannian and pseudo-Riemannian symmetric spaces. Definition of the nonlinear sigma model will be given, exhibiting how such models arise in Kaluza-Klein theories and extended supergravity theories.

It is known that dimensional reduction has been used to make a connection to lower dimensional theories, in particular to our four-dimensional universe. But dimensional reduction will be studied here not only for this reason, but also to show in the next chapter that reducing a theory over some of its dimensions leads to a theory which is easier to solve than the original one. Via uplifting back to the original dimensions we have generated a solution of the higher dimensional system. In chapter 6 we will be also interested in reducing brane solutions over their worldvolumes. Therefore we will restrict in this chapter to torus reductions, distinguishing between what we will call spacelike and timelike reductions. In the end of this chapter we will outline the reduction of maximally extended supergravity theories over a torus.

5.1 Introduction

In chapter 4 we have found that the most general duality group that can be realized with a 4-dimensional theory describing among the other fields a set of n abelian vector fields and m scalar fields is the non-compact real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$, which has $\mathrm{U}(n)$ as its maximal compact subgroup. In addition we have seen that in the absence of scalar matter fields, $\mathrm{U}(n)$ is the largest group of duality transformations. In specific examples the actual group of duality transformations can be smaller, namely a non-compact subgroup G of $\mathrm{Sp}(2n,\mathbb{R})$, having H as its maximal compact subgroup. In those examples the scalar sector is described by a 4-dimensional *nonlinear sigma model*, which means that the scalar fields are local coordinates of a non-compact Riemannian manifold G/H and the scalar action is required to be invariant with respect to the isometries of G/H. For example, in $N=8$, $D=4$ supergravity there are 28 field strengths. The non-compact invariance is the E_7 subgroup of $\mathrm{Sp}(56,\mathbb{R})$ and its maximal compact subgroup is the SU(8) subgroup of U(28). The scalar fields thus take values in the target space $E_{7(7)}/\mathrm{SU}(8)$, describing $E_{7(7)}/\mathrm{SU}(8)$ nonlinear σ-model.

Nonlinear σ-models of our interest are those typically arise through dimensional reduction of gravitational theories ((super)gravity theories), where the scalar fields form coset manifolds G/H exhibiting explicitly larger and larger symmetries as one goes down in dimensions. In the case of eleven-dimensional supergravity for example, reduction on a n-torus, $\mathbb{T}^n$, reveals a chain of exceptional symmetries $G=E_{n(n)}$. In these theories, generically the fields transform as follows. The metric is invariant, the (abelian) gauge fields transform linearly under G and the fermions transform linearly under the group H. However, in some dimensions the G-invariance is not realized at the level of the action, but at the level of the combined field equations and Bianchi identities. For example, in the 4-dimensional example given above the 28 vector fields do not constitute a representation of the group $E_{7(7)}$. As we have seen in chapter 4 the group G in this case is realized by electromagnetic duality and acts on the field strengths, rather on the vector fields.

We begin our study by describing some general aspects of nonlinear sigma models for finite-dimensional coset spaces. In this chapter and the one that follows we will try to minimize the geometrical and group theoretical technicality, which we review in appendices A and C.

5.2 Nonlinear Sigma Model Based on Symmetric Spaces

A nonlinear sigma model describes maps Φ from one (pseudo)-Riemannian space Σ equipped with a metric g to another (pseudo)-Riemannian space, "the target space" M, with metric G_{ij}. Let x^μ ($\mu=1,\cdots D$) be coordinates on Σ and Φ^i ($i=1,\cdots \mathrm{dim}M$) be coordinates on M. Then the standard action for this sigma model is

$$S=\int_\Sigma d^Dx\sqrt{|g|}g^{\mu\nu}\partial_\mu\Phi^i(x)\partial_\nu\Phi^j(x)G_{ij}(\Phi(x)). \tag{5.2.1}$$

Solutions to the equations of motion resulting from this action will describe the maps Φ^i as functions of x^μ.

In what follows, we shall be concerned with sigma models on non-compact Riemannian symmetric spaces $M = G/H$ where G is a non-compact Lie group generated by the semi-simple real Lie algebra $\mathfrak{G}$ and H is its maximal compact subgroup generated by the real Lie algebra $\mathfrak{H}$. Since elements of the coset are obtained by quotienting out H, this subgroup is referred to as the " local gauge symmetry" (see below). Our aim is to provide an algebraic construction of the metric G_{ij} on the coset and of the Lagrangian [106, 124–126].

5.2.1 Symmetric Spaces and Nonlinear Realizations

Suppose G is a group and H is a subgroup of G. The coset space G/H is defined as the set of equivalence classes $[g]$ of G defined by the equivalence relation

$$g \sim g' \qquad \text{iff} \qquad gg'^{-1} \in H, \text{ and } g, g' \in G, \tag{5.2.2}$$

i.e.,

$$[g] = \{hg | h \in H\}. \tag{5.2.3}$$

If G is a Lie group and H is any Lie subgroup of G, the coset G/H is a manifold and thus can be described by local coordinates. Since any two points p and p' on G/H can by construction be connected by an action of G, the manifold is a *homogeneous space* with G being the isometry group and H the isotropy group.

We have investigated the non-compact real forms G of a complex semi-simple group $G^{\mathbb{C}}$ in appendix C.2.2 and have found that the involutive automorphism θ of G induces a *Cartan decomposition* of $\mathfrak{G}$ into even and odd eigenspaces:

$$\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{F}, \tag{5.2.4}$$

where

$$\mathfrak{H} = \{\mathfrak{j} \in \mathfrak{G} |\, \theta(\mathfrak{j}) = \mathfrak{j}\}, \qquad \mathfrak{F} = \{\mathfrak{k} \in \mathfrak{G} |\, \theta(\mathfrak{k}) = -\mathfrak{k}\} \tag{5.2.5}$$

play central roles. The decomposition 5.2.4 is orthogonal, in the sense that $\mathfrak{F}$ is the orthogonal complement of $\mathfrak{H}$ with respect to the invariant inner product $\langle\, . \,|\, . \,\rangle$ induced by the Killing metric $B(\, . \,, \, . \,)$,

$$\mathfrak{F} = \{\mathfrak{k} \in \mathfrak{G} |\, \forall \mathfrak{j} \in \mathfrak{H} : \langle \mathfrak{k} | \mathfrak{j} \rangle = 0\}. \tag{5.2.6}$$

The commutator relations split in a way characteristic for *symmetric spaces*,

$$[\mathfrak{H}, \mathfrak{H}] \subset \mathfrak{H}, \qquad [\mathfrak{H}, \mathfrak{F}] \subset \mathfrak{F}, \qquad [\mathfrak{F}, \mathfrak{F}] \subset \mathfrak{H}. \tag{5.2.7}$$

The subspace $\mathfrak{F}$ is not a subalgebra. Elements $\mathfrak{F}$ transform in some representation of $\mathfrak{H}$, which depends on the Lie algebra $\mathfrak{G}$. We stress that if the commutator $[\mathfrak{F}, \mathfrak{F}]$ had also contained elements in $\mathfrak{F}$ itself, this would not have have been a symmetric space.

If the so-called involution θ taken to be the involution θ_c defined in appendix C.2.2, then it will have the effect of reversing the sign of every non-compact generator in the Lie algebra $\mathfrak{G}$, while leaving the sign of every compact generator unchanged (see appendix C.2.2). If we denote the positive root generators, negative root generators and Cartan generators of $\mathfrak{G}$ by $(E_{\vec{\alpha}}, E_{-\vec{\alpha}}, \vec{H})$, where $\vec{\alpha}$ ranges over all the positive roots, then for our algebra θ_c effects the mapping

$$\theta_c : \qquad (E_{\vec{\alpha}}, E_{-\vec{\alpha}}, \vec{H}) \longrightarrow (-E_{-\vec{\alpha}}, -E_{\vec{\alpha}}, -\vec{H}). \tag{5.2.8}$$

The real form corresponding to θ_c is called the *maximally non-compact* (split) real form G_{θ_c} of $G^{\mathbb{C}}$. It is always, in any real Lie group G_{θ_c}, the generator combinations $(E_\alpha - E_{-\alpha})$ are compact while the combinations $(E_\alpha + E_{-\alpha})$ are non-compact.

The Cartan split of the generators into compact and non-compact parts reveals the fact that the Killing metric of a real form G may have an indefinite signature. However, the metric G_{ij} on the symmetric space G/H has a definite sign. This can be seen from the fact that all the generators that span G/H are non-compact, therefore the Killing metric restricted to $\mathfrak{F}$ is positive definite, and hence the symmetric space G/H is *Riemannian.* The restriction of the Killing metric to $\mathfrak{H}$ gives rise to a negative definite metric. Note that had we started with a different real form of $G^{\mathbb{C}}$ with a different involutive automorphism, we would have obtained a different symmetric space with a different signature for the metric. For more about the classification of real forms of a complex Lie algebra using involutive automorphisms we refer to [127].

The group G naturally acts through (here, right) multiplication on the coset space G/H as

$$[k] \mapsto [kg]. \tag{5.2.9}$$

This definition makes sense because if $k \sim k'$, i.e. $k' = hk$ for some $h \in H$, then $k'g \sim kg$ since $k'g = (hk)g = h(kg)$. Note that left and right multiplications commute.

In order to describe a dynamical theory on the coset space G/H, it is convenient to introduce as dynamical variable the group element $L(x) = L(\Phi(x)) \in G$ (Φ^i are the local coordinates parameterizing G/H) and to construct the action for $L(x)$ in such a way that the equivalence relation

$$\forall\ h(x) \in H\ : \qquad L(x) \sim h(x)L(x) \tag{5.2.10}$$

corresponds to a gauge symmetry. The physical (gauge invariant) degrees of freedom are then parameterized indeed by points of the coset spaces. We want also to impose 5.2.9 as a rigid symmetry. Thus, the action should be invariant under

$$L(x) \longmapsto h(x)L(x)g, \qquad h(x) \in H,\ g \in G. \tag{5.2.11}$$

One may develop the formalism without fixing the H-gauge symmetry, or one may instead fix the gauge symmetry by choosing a specific coset representative $L(x) \in G/H$. When H is a maximal compact subgroup of G, i.e. G/H is a Riemannian symmetric space, there are no topological obstructions, and a standard choice which is always available, is to take $L(x)$ to be upper triangular form as allowed by Iwasawa decomposition (discussed in section 5.2.3). This gauge choice is called the *solvable gauge*. Given such a gauge choice (or any other one), the global action of an arbitrary element $g \in G$, on $L(x)$ will lead to a non-standard representative $L'(x)$ and must therefore be accompanied by a compensating local transformation $h(L(x), g) \in H$ in order to maintain the gauge choice. The total transformation is thus

$$L(x) \longmapsto L''(x) = h(L(x), g)L(x)g, \qquad h(L(x), g) \in H,\ g \in G. \tag{5.2.12}$$

where $L''(x)$ is again in the upper triangular gauge. Since now $h(L(x), g)$ depends nonlinearly on $L(x)$, this is called a *nonlinear realization* of G.

5.2.2 Nonlinear Sigma Model Coupled to Gravity

Given the field $L(x)$, we can form the Lie algebra valued one-form

$$dLL^{-1} = dx^\mu \partial_\mu LL^{-1}. \tag{5.2.13}$$

Under the Cartan decomposition, this element splits according to 5.2.4,

$$dL(x)L^{-1}(x) = (\Omega_\mu(x) + E_\mu(x))dx^\mu, \tag{5.2.14}$$

where $\Omega_\mu \in \mathfrak{H}$ and $E_\mu \in \mathfrak{F}$. In virtue of the involutive automorphism θ one can write these explicitly as projections onto the odd and even eigenspaces as follows:

$$\begin{aligned} E &= \frac{1}{2}[dLL^{-1} - \theta(dLL^{-1})], \\ \Omega &= \frac{1}{2}[dLL^{-1} + \theta(dLL^{-1})]. \end{aligned} \tag{5.2.15}$$

Now, if we define a *generalized transpose* $\sharp$ [128] (see also [126]) by

$$(\ )^\sharp = -\theta(\ ), \tag{5.2.16}$$

then E and Ω correspond to symmetric and antisymmetric elements, respectively,

$$E^\sharp(x) = E(x), \qquad \Omega^\sharp(x) = -\Omega(x). \tag{5.2.17}$$

The Lie algebra valued one-forms with components dLL^{-1}, Ω and E are invariant under the rigid right multiplication, $L(x) \mapsto L(x)g$.

Being an element of the Lie algebra of the gauge group, $\Omega_\mu(x)$ can be interpreted as a gauge connection for the local symmetry H. Under local transformation $h(x) \in H$, $\Omega_\mu(x)$ transforms as

$$H\ :\ \Omega_\mu(x) \longmapsto h(x)\Omega_\mu(x)h^{-1}(x) + \partial_\mu h(x)h^{-1}(x), \tag{5.2.18}$$

which indeed is the characteristic transformation property of a gauge connection. On the other hand, $E_\mu(x)$ transforms covariantly,

$$H \ :\ E_\mu(x) \longrightarrow h(x)E_\mu(x)h^{-1}(x). \tag{5.2.19}$$

Making use of the scalar product $\langle\,.\,|\,.\,\rangle$, induced by the Killing metric B written in a R representation of the group G, we can now form a manifestly H and G invariant expression by simply squared $E_\mu(x)$. Thus the D-dimensional nonlinear sigma model action coupled to gravity takes the form

$$S_{G/H} = \int_\Sigma d^Dx\sqrt{|g|}\Big[\mathcal{R} - g^{\mu\nu}\langle E_\mu(x)|E_\nu(x)\rangle\Big] \tag{5.2.20}$$

$$= \int_\Sigma d^Dx\sqrt{|g|}\Big[\mathcal{R} - C_R g^{\mu\nu}\mathrm{tr}[E_\mu(x)E_\nu(x)]\Big]. \tag{5.2.21}$$

$$\equiv \int_\Sigma d^Dx\sqrt{|g|}\Big[\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi^i(x)\partial_\nu\Phi^j(x)G_{ij}(\Phi(x))\Big], \tag{5.2.22}$$

where $\mathcal{R}$ is the Ricci scalar and C_R is a positive constant depending on the representation R of G with $\langle\,.\,|\,.\,\rangle = C_R\mathrm{tr}[\ ]$.

There is an alternative way of writing the above action. We can also form a generalized "metric"[1] $\mathcal{M}$ that does not transform at all under the local symmetry, but only transforms under the rigid G-transformations. This can be done by using the generalized transpose, extended from the algebra to the group through the exponential map. Actually the main advantage behind using the automorphism θ is that it provides us with an embedding of G/H in G [129]

$$L \longmapsto \mathcal{M} = \theta(L^{-1})L = L^\sharp L, \qquad \theta(\mathcal{M}) = \mathcal{M}^{-1}, \tag{5.2.23}$$

where $\theta(L^{-1}) = L^\sharp$. The matrix $\mathcal{M}$ transforms as follows under global transformations on $L(x)$ from the right

$$G \ :\ \mathcal{M}(x) \longmapsto g^\sharp\mathcal{M}(x)g, \qquad g \in G. \tag{5.2.24}$$

A short calculation shows that the relation between $\mathcal{M}(x) \in G$ and $L(x) \in G/H = \exp\mathcal{F}$ is given by

$$\frac{1}{2}\mathcal{M}^{-1}(x)\partial_\mu\mathcal{M}(x) = L^{-1}(x)E_\mu(x)L(x). \tag{5.2.25}$$

[1] We call $\mathcal{M}$ a "generalized metric" because in the Kaluza-Klein reduction of Einstein gravity over a torus, it corresponds to the metric of the torus, the field $L(x)$ being the "vielbein" (see section 5.3.2).

This relation provides another way to write the G- and H-invariant action, completely in terms of the generalized metric $\mathcal{M}(x)$,

$$S_{G/H} = \int_\Sigma d^D x \sqrt{|g|} \Big[\mathcal{R} - \frac{1}{4} g^{\mu\nu} \langle \mathcal{M}^{-1}\partial_\mu \mathcal{M} | \mathcal{M}^{-1}\partial_\nu \mathcal{M} \rangle \Big] \tag{5.2.26}$$

$$= \int_\Sigma d^D x \sqrt{|g|} \Big[\mathcal{R} + \frac{C_R}{4} \mathrm{tr}[\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}] \Big]. \tag{5.2.27}$$

The equations of motion following from the nonlinear σ-model coupled to gravity action read

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} \partial_\mu \Phi^i \partial_\nu \Phi^j G_{ij}(\Phi) = 0, \tag{5.2.28}$$

$$D^\mu \partial_\mu \Phi^i(x) = 0, \tag{5.2.29}$$

where D is the covariant derivative associated with the local transformation H. The second equation is equivalent to

$$D^\mu E_\mu(x) = 0, \tag{5.2.30}$$

with $D_\mu E_\nu(x) = \nabla_\mu E_\nu(x) - [\Omega_\mu(x), E_\nu(x)]$, where ∇^μ is a covariant derivative on Σ compatible with the Levi-Civita connection. Equations 5.2.30 simply express the covariant conservation of $E_\mu(x)$.

It is also interesting to examine the dynamics in terms of the generalized metric $\mathcal{M}$. The equations of motion for $\mathcal{M}$ are

$$\nabla^\mu (\mathcal{M}^{-1} \partial_\mu \mathcal{M}) = 0. \tag{5.2.31}$$

These equations ensure the conservation of the current

$$J_\mu = \frac{1}{2} \mathcal{M}^{-1} \partial_\mu \mathcal{M} = L^{-1} E_\mu L, \tag{5.2.32}$$

i.e., $\nabla^\mu J_\mu = 0$. This is the conserved Noether current associated with the rigid G-invariance of the action. Using $\theta(\mathcal{M}) = \mathcal{M}^{-1}$, one can derive that the currents J obey the identity $\theta(J) = -\mathcal{M} J \mathcal{M}^{-1}$.

The incorporation of form fields into the action 5.2.26 is straightforward as long as one does not want to turn on duality invariance (selfduality of section 4.4). For example in $D = 4$ the addition of vectors fields might be accompanied by the duality action of G exactly the way we saw in the previous chapter. In this case the vector fields should be added to gravity and scalar fields forming a G/H σ-model in such a way that the resulting equations of motion are invariant under the action of G if the field strengths together with their duals are transformed suitably with a R representation of G. In [130] it has been shown that the matrix $\mathcal{M}$ parameterized by the scalar Φ^i is intimately related to the symmetric matrix defined in section 4.4. Furthermore it has been argued that the coupling of the vector fields should be such that the resulting energy density is positive, and hence the symmetric matrix $\mathcal{M}$ is positive definite.

The action describing a nonlinear σ-model coupled to gravity and vector fields arises typically in Kaluza-Klein theories and extended supergravity theories, in particular from the torus reduction of pure Einstein gravity[2] to D dimensions which will be discussed in section 5.3.

5.2.3 Iwasawa Decomposition: Borel Gauge

We have seen above that there exists a nice gauge or coordinate frame, namely the solvable gauge. It has been shown that the idea behind such a gauge originates from the *Iwasawa decomposition.* Given a real form G of a semisimple complex Lie group $G^{\mathbb{C}}$, the associated Lie algebra $\mathfrak{G}$ can be decomposed as $\mathfrak{G} = \mathfrak{H} + \mathfrak{s}$ where $\mathfrak{H}$ is the maximal compact subalgebra, and $\mathfrak{s}$ is the solvable subalgebra of $\mathfrak{G}$. The solvable Lie algebra $\mathfrak{s}$ can itself split up as $\mathfrak{s} = \mathfrak{c} \oplus \mathfrak{n}$ where $\mathfrak{c}$ is the maximal set of commuting non-compact generators, i.e. the non-compact part of the Cartan subalgebra CSA $\mathfrak{h}$ of G, and $\mathfrak{n}$ is the lie algebra consists of the generators which have the positive roots with respect to $\mathfrak{c}$. This is the Iwasawa decomposition, a description of which can be found in reference [131]. One of the nice properties of a solvable Lie algebra is that the matrix representation can be chosen such that all elements of $\mathfrak{s}$ are upper triangular. Due to this decomposition one can easily notice that the coset space G/H is globally isometrical to a group G_s, a subgroup of G associated to the solvable Lie algebra, namely

$$\frac{G}{H} \cong G_s, \qquad \text{with } G_s = \exp[\mathfrak{s}]. \tag{5.2.33}$$

This reflects the fact that the solvable parametrization of G/H holds *globally.*

We denote the Cartan subalgebra generators by H_I with $I = 1, \cdots, r = \text{rank}(G)$ and the positive root generators with $E_{\vec{\alpha}}$. The commutation relations read

$$[H_I, H_J] = 0, \qquad [H_I, E_{\vec{\alpha}}] = \alpha_I E_{\vec{\alpha}}, \tag{5.2.34a}$$

$$[E_{\vec{\alpha}}, E_{\vec{\beta}}] = \begin{cases} 0 & \text{if } \alpha + \beta \text{ is not a root} \\ N(\alpha,\beta) E_{\vec{\alpha}+\vec{\beta}} & \text{otherwise,} \end{cases} \tag{5.2.34b}$$

where $\vec{\alpha}$ is the root vector of the Lie algebra $\mathfrak{G}$.

Now let us be a little more precise and distinguish two cases:

- If $\mathfrak{c}$ coincides with the whole Cartan subalgebra $\mathfrak{h}$ and $\mathfrak{n}$ is the subspace generated by all the positive root generators, then $\mathfrak{G}$ is $\mathfrak{G}_{\theta_c}$ the *maximally non-compact*

[2]From now on we call the Kaluza-Klein theories, which stem from the dimensional reduction of pure Einstein gravity theory, pure Kaluza-Klein theory.

real form of a complex semisimple algebra $\mathfrak{G}^{\mathbb{C}}$ (see C.2.2). This means that the difference between the number of non-compact generators and the number of compact ones is the rank $r = \text{rank}G_{\theta_c}$. In this case the solvable algebra $\mathfrak{s}$ is isomorphic to the Borel subalgebra of $\mathfrak{G}_{\theta_c}$. The Borel subalgebra of any Lie algebra $\mathfrak{G}_{\theta_c}$ is the subalgebra generated by the positive root generators and the Cartan generators, namely

$$L \in \frac{G_{\theta_c}}{H} \cong \exp[\text{Borel}\mathfrak{G}_{\theta_c}], \qquad L = \Pi_I \exp[\frac{1}{2}\phi^I H_I]\Pi_\alpha \exp[\chi^\alpha E_\alpha]. \qquad (5.2.35)$$

In fact for the nonlinear sigma model action of the type discussed above, the scalars might come in two disguises: either they appear with derivatives or they also appear in exponential couplings to other fields. The scalars of the first kind are called *axions* χ^α and the scalars of the second kind are called *dilatons* ϕ^I. This means that the coset representative L is written in the *Borel* coordinate system 5.2.35[3]. The isometry group G is always maximally non-compact in the case of maximal supergravity, e.g. $N = 8$, $D = 4$, and some less-extended supergavities [132, 133].

- If $\mathfrak{G}$ is one of the non-maximally non-compact real forms (non-split real form) of $\mathfrak{G}^{\mathbb{C}}$, then $\mathfrak{c}$ is the subspace of non-compact generators of the Cartan subalgebra and $\mathfrak{n}$ is the set of the positive *restricted roots* generators (more about such roots see [126]). In the case $\mathfrak{G}$ is not maximally non-compact, the supergravity theories with G/H nonlinear σ-model are less-extended supergravities. See [A] for discussion of this in the case of dimensionally reduced heterotic supergravity.

Henceforth we will restrict to the real form $G \equiv G_{\theta_c}$, the maximally non-compact real form of $G^{\mathbb{C}}$.

5.2.4 Example: SL$(n,\mathbb{R})$/SO(n) Nonlinear σ-model

As an example we discuss the coset SL$(n,\mathbb{R})$/SO(n). The group SL$(n,\mathbb{R})$ has a rank $r = n - 1$ and its maximal compact subgroup is SO(n). There will therefore be $n - 1$ dilaton fields and $n(n-1)/2$ axion fields χ^α. The Cartan generators are given in terms of the weights $\vec{\beta}$ of SL$(n,\mathbb{R})$ in the fundamental represenation

$$(\vec{H})_{ij} = (\vec{\beta}_i)\delta_{ij}. \qquad (5.2.36)$$

The tracelessness of the special linear group SL generators implies $\sum_i \beta_{iI} = 0$. Moreover, a convenient normalization of the Cartan killing form (metric of the Lie algebra

[3] Apart from the fact that these coordinates can be useful for practical purposes, it is also the coordinate system that is naturally obtained after torus dimensional reduction defined later in this chapter. The dilatons ϕ^I then correspond to the radii of the internal tori and the axions χ^α are the various off-diagonal internal gravitational degrees of freedom and the internal components of the p-form gauge potentials.

in the fundamental representation) leads to two additional identities obeyed by the weight vectors

$$\sum_i \beta_{iI}\beta_{iJ} = 2\delta_{IJ}, \qquad \vec{\beta}_i \cdot \vec{\beta}_j = 2\delta_{ij} - \frac{2}{n}. \tag{5.2.37}$$

The positive step operators E_{ij} are all upper triangular and a handy basis is one in which they have only non-zero entry $(E_{ij})_{ij} = 1$. Note that the negative step operators are the transpose of the positive ones.

The SO(n) subalgebra is spanned by the combinations

$$\frac{1}{\sqrt{2}}(E_\beta - E_{-\beta}). \tag{5.2.38}$$

If we work in the fundamental representation of SL, the Lie algebra of SO(n) is then the vectorspace of anti-symmetric matrices. Now, use the fact that for SL the generalised transpose $\sharp$ defined above coincides with the ordinary matrix transpose "T", namely $\theta(L) = (L^\sharp)^{-1} = (L^T)^{-1}$, the relations 5.2.15 become

$$E = \frac{1}{2}[dLL^{-1} + (dLL^{-1})^T] \qquad \Omega = \frac{1}{2}[dLL^{-1} - (dLL^{-1})^T]. \tag{5.2.39}$$

Thus the hermatian positive definite matrix $\mathcal{M}$ becomes $\mathcal{M} = L^T\delta L = L^T L$, i.e. symmetric matrix, with δ is the Euclidean metric invariant under the action of SO(n).

A calculation exhibits that the SL($n,\mathbb{R}$)/SO(n) nonlinear σ-model Lagrangian reads

$$\mathcal{L}_{\text{scalar}} = -\sqrt{|g|}\text{tr}[E^2] = +\frac{1}{4}\sqrt{|g|}\text{tr}[\partial\mathcal{M}\partial\mathcal{M}^{-1}]. \tag{5.2.40}$$

The action will generically look complicated but when all axions are set to zero, L is diagonal $L = \text{diag}[\exp(\frac{1}{2}\vec{\beta}_i \cdot \vec{\phi})]$ and the action becomes

$$\frac{1}{4}\text{tr}\partial\mathcal{M}\partial\mathcal{M}^{-1} = -\frac{1}{4}(\sum_i \beta_{iI}\beta_{iJ})\partial\phi^I\partial\phi^J = -\frac{1}{2}\delta_{IJ}\partial\phi^I\partial\phi^J. \tag{5.2.41}$$

This action describes a truncated $\mathbb{R}^{n-1}$ nonlinear σ-model, where the $n-1$ dilatons are parameterizing the flat scalar space $\mathbb{R}^{n-1}$.

This brings us to the issue of consistent truncations. According to [130], a nonlinear σ-model with target space G_1/H_1 is a consistent truncation of another σ-model with target space G_2/H_2 if $G_1/H_1 \subset G_2/H_2$ and if every solution of the field equations for the G_1/H_1 σ-model is a solution of the field equations for the G_2/H_2 σ-model as well. In other words, the truncation is consistent if and only if G_1/H_1 is totally geodesic[4] subspace of G_2/H_2. Note that if G/H is a Riemannian symmetric

[4] Totally geodesic submanifold is a submanifold such that all geodesics in the submanifold are also geodesics of the surrounding manifold.

space then every totally geodesic sub-space of G/H is again a Riemannian symmetric space. Similar statements hold true if we are dealing with a pseudo-Riemannian G/H^* symmetric space that will be defined in a minute. Thus in the above example, putting all the axions to zero turns out to be consistent with the equations of motion

$$\partial_\mu(\sqrt{|g|}\mathcal{M}^{-1}\partial^\mu\mathcal{M}) = 0. \tag{5.2.42}$$

Let us consider the example of the coset space for $n = 2$, which, although very simple, is nonetheless quite illustrative. The Lie algebra of $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ has the following standard commutation relations

$$[H, E_2] = 2E_2, \qquad [H, E_{-2}] = -2E_{-2}, \qquad [E_2, E_{-2}] = H. \tag{5.2.43}$$

The $\mathrm{SL}(2,\mathbb{R})$ fundamental realization takes on the form

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{-2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.2.44}$$

The coset representative, written in the Borel gauge (upper-triangular), behaves as

$$L = \exp[\frac{1}{2}\phi H]\exp[\chi E_2] = \begin{pmatrix} e^{\frac{\phi}{2}} & e^{-\frac{\phi}{2}}\chi \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix}, \tag{5.2.45}$$

where $\phi(x)$ and $\chi(x)$ represent coordinates on the coset space, i.e. they describe the sigma model map

$$x \in \Sigma \longmapsto (\phi, \chi) \in \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2). \tag{5.2.46}$$

We saw that an arbitrary transformation on $L(x)$ behaves as

$$L(x) \longmapsto h(x)L(x)g, \qquad h(x) \in \mathrm{SO}(2),\ g \in \mathrm{SL}(2,\mathbb{R}), \tag{5.2.47}$$

which infinitesimally becomes

$$\delta_{h,g}L = \delta h(x)L + L\delta g, \tag{5.2.48}$$

where $\delta h(x)$, and δg are respectively elements of the $\mathrm{SO}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ Lie algebras. Let us now check how $L(x)$ transforms under the generators $\delta g = \{H, E_2, E_{-2}\}$. As expected, the Borel generators H and E_2 preserve the upper triangular structure

$$\delta_{E_2}L = LE_2 = \begin{pmatrix} 0 & e^{\phi/2} \\ 0 & 0 \end{pmatrix}, \qquad \delta_H L = LH = \begin{pmatrix} e^{\phi/2} & -e^{\phi/2}\chi \\ 0 & -e^{\phi/2} \end{pmatrix}, \tag{5.2.49}$$

whereas the negative root generator E_{-2} does not respect the form of $L(x)$,

$$\delta_{E_{-2}}L = LE_{-2} = \begin{pmatrix} e^{\phi/2}\chi & 0 \\ e^{-\phi/2} & 0 \end{pmatrix}. \tag{5.2.50}$$

Thus in this case we need a compensating transformation to restore the upper triangular form. This transformation needs to switch off the entry $e^{-\phi/2}$ in the lower left corner of the matrix 5.2.50 and therefore it must necessarily depend on $\phi(x)$. The transformation that can do this job is a SO(2) Lie algebra element

$$\delta h(x) = \begin{pmatrix} 0 & e^{-\phi} \\ -e^{-\phi} & 0 \end{pmatrix}, \tag{5.2.51}$$

and we find

$$\delta_{h,E_{-2}} L = \delta h(x) L + L E_{-2} = \begin{pmatrix} e^{\phi/2}\chi & e^{-3\phi/2} \\ 0 & -\chi e^{-\phi/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2). \tag{5.2.52}$$

Finally, from 5.2.45 one can calculate $\mathcal{M} = L^T L$ and hence the $\mathrm{SL}(2,\mathbb{R})/SO(2)$ nonlinear σ-model Lagrangian is

$$\mathcal{L}_{\mathrm{scalar}} = -\frac{1}{2}\sqrt{|g|}\Big[(\partial\phi)^2 + e^{2\phi}(\partial\chi)^2\Big]. \tag{5.2.53}$$

In type IIB supergavity the scalar coset is $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ where the Borel gauge has the interpretation that the dilaton ϕ is in the NS sector and determines the string coupling and χ is the RR field $C_{(0)}$ that couples electrically to D-instanton (-1-brane) and magnetically to 7-brane.

This is somewhat a trivial example so let us consider $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. The Borel algebra is

$$\begin{aligned} &[H_1, E_{12}] = 2E_{12}, \qquad [H_1, E_{13}] = E_{13}, \qquad [H_1, E_{23}] = -E_{23}, \\ &[H_2, E_{12}] = 0, \qquad [H_2, E_{13}] = \sqrt{3}E_{13}, \qquad [H_2, E_{23}] = \sqrt{3}E_{23}, \\ &[E_{12}, E_{13}] = 0, \qquad [E_{13}, E_{23}] = 0, \qquad [E_{12}, E_{23}] = E_{13}. \end{aligned} \tag{5.2.54}$$

The Cartan generators in the 3-dimensional fundamental representation are written as

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{5.2.55}$$

and the three positive step operators are

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.2.56}$$

The three negative step operators can be found by taking the transpose of the positive ones. The coset representative is expressed as

$$L = \exp[\chi_{12}E_{12}]\exp[\chi_{13}E_{13}]\exp[\chi_{23}E_{23}]\exp[\frac{1}{2}\phi_1 H_1 + \frac{1}{2}\phi_2 H_2]$$
$$= \begin{pmatrix} e^{\frac{1}{2}(\phi_1+\frac{\phi_2}{\sqrt{3}})} & e^{\frac{1}{2}(-\phi_1+\frac{\phi_2}{\sqrt{3}})}\chi_{12} & e^{\frac{-\phi_2}{\sqrt{3}}}(\chi_{12}\chi_{23}+\chi_{13}) \\ 0 & e^{\frac{1}{2}(-\phi_1+\frac{\phi_2}{\sqrt{3}})} & e^{-\frac{\phi_2}{\sqrt{3}}}\chi_{23} \\ 0 & 0 & e^{-\frac{\phi_2}{\sqrt{3}}} \end{pmatrix}. \qquad (5.2.57)$$

The SL(3, ℝ)/SO(3) nonlinear σ-model Lagrangian is expressed as

$$\mathcal{L}_{\text{scalar}} = -\sqrt{|g|}[-\frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2}\{e^{-2\phi_1} + e^{-\phi_1-\sqrt{3}\phi_2}\}(\partial\chi_{12})^2$$
$$- \frac{1}{2}e^{-\phi_1-\sqrt{3}\phi_2}(\partial\chi_{13})^2 - \frac{1}{2}e^{\phi_1-\sqrt{3}\phi_2}(\partial\chi_{23})^2 - e^{-\phi_1-\sqrt{3}\phi_2}\chi_{23}\partial\chi_{12}\partial_{13}]. \qquad (5.2.58)$$

5.2.5 σ-model on Pseudo-Riemannian Symmetric Spaces G/H^*

The theories that we have seen so far are Minkowskian theories in D dimensions where the scalars form G/H nonlinear σ-model with G/H is a Riemannian target space. However, there exist also non linear σ-models which might arise from *Euclidean* Kaluza-Klein (reduction over timelike killing vector) and extended supergravity theories where the scalars take values in a pseudo-Riemannian space G/H^*, with G is the maximally non-compact real form of a complex semisimple Lie algebra and H^* is a non-compact version of H. The Borel coordinates ϕ^I , χ^α are no longer valid for non-Riemannian symmetric spaces since this coordinate system does not cover the whole manifold[5] (see [134]). Nevertheless, one can still work on the level of the H^*-invariant matrix, $\mathcal{M}$, for which the choice of the gauge does not play any role.

The basis for a representation R will be chosen such that

$$\theta(\mathfrak{j}) = -\eta(\mathfrak{j})^\sharp\eta, \qquad \forall\, \mathfrak{j} \in \mathfrak{G}. \qquad (5.2.59)$$

The matrix η is H^*-invariant matrix having a Lorentzian signature. It can be restricted to the metric on G/H^* manifold or to the Killing metric of H^*. If we define $\mathcal{M} = \eta\tilde{\mathcal{M}} = L^\sharp\eta L$, the matrix $\mathcal{M}$ will again be hermitian (symmetric) and $\tilde{\mathcal{M}} = \eta L^\sharp\eta L$ is an element of G.

Due to the non-compactness of H^* the matrix η and thus $\mathcal{M}$ and G_{ij} will not

[5] The solvable (Borel) parametrization for G/H^*, in contrast to the G/H case in which H is the maximal compact subgroup of G, holds only *locally*. To understand this issue, one can think of the simple case of $dS_2 = \mathrm{SO}(1,2)/\mathrm{SO}(1,1)$, in which the solvable parametrization describes the *stationary univers* and thus covers only half of the hyperboloid.

be positive definite. Therefore there exist kinetic terms with the 'wrong sign' in the G/H^* nonlinear σ-model Lagrangians. Scalar fields with the wrong sign of kinetic terms are sometimes called *ghosts* in analogy with ghosts in the quantization of Yang-Mills theory. For example the nonlinear σ-model of Euclidean type IIB supergravity has the following Lagrangian

$$\mathcal{L}_{\rm scalar} = \sqrt{|g|}[-\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}e^{-2\phi}(\partial\chi)^2]. \qquad (5.2.60)$$

The only difference with Lorentzian IIB is the sign of $(\partial\chi)^2$. An explanation of this sign difference can for instance be found in [135]. The different sign does not ruin the $\mathrm{SL}(2,\mathbb{R})$-invariance but does change the scalar coset to $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$. Thus the metric on the scalar coset is indefinite, and hence as we expected the metric G_{ij} fixes the kinetic terms of the scalar fields. As we have mentioned before for such a case the isotropy group $\mathrm{SO}(1,1) = H^*$ is not the maximal compact subgroup, therefore the Iwasawa gauge (Borel) fails. This can be seen as follows. Let $k \in \mathrm{SO}(1,1)$ (the 1+1 Lorentz group, consisting only of boots), and $V \in \mathrm{SL}(2,\mathbb{R})$, then the gauge action of $\mathrm{SO}(1,1)$ $k \cdot V$ is a boost of both columns. If a column is lightlike then you can not boost it to become spacelike or timelike. In other words for V one can not find an upper-triangular representative. There are however other general gauges one can think of. One can see for instance [134] for a "good gauge".

It is worth recalling that the G/H and G/H^* nonlinear σ-models arising in the pure Kaluza-Klein theories and maximally extended supergravities follow basically from the dimensional reduction over tori of pure Einstein-gravity theory and higher dimensional supergravities, respectively. Therefore the coming sections will be devoted to studying the dimensional reduction over tori, focusing mainly on the reduction of pure Einstein gravity over Euclidean and Lorentzian tori.

5.3 Dimensional Reduction

Dimensional reduction of a theory, by definition, consists of an expansion over an internal space and subsequent truncation to the lightest modes. In order to see that let's have an instructive illustration:

We consider complex scalar field $\hat{\phi}$ living in $\hat{D}$ dimensional spacetime parameterized by the coordinates $x^{\hat{\mu}} = (x^\mu, y)$. The fourier transformation of $\hat{\phi}$ with respect to the coordinate y can be performed as

$$\hat{\phi}(x,y) = \int dk\, \phi_k(x) e^{iky}, \qquad (5.3.1)$$

where k represents the momentum of modes ϕ_k. Now one can first compactify the y direction to have the length $2\pi L$, then impose the boundary condition

$$\hat{\phi}(x,0) = \hat{\phi}(x, 2\pi L). \qquad (5.3.2)$$

It implies that the integral is converted to a sum due to the fact that over compact direction the momentum takes discrete values

$$\hat{\phi}(x,y) = \sum_n \phi_n(x) e^{i\frac{n}{L}y}. \tag{5.3.3}$$

Assume the scalar field $\hat{\phi}$ solves the Klein-Gordon equation such that

$$\Box_{\hat{D}}\hat{\phi} = 0, \qquad \text{with } \Box_{\hat{D}} = \partial_\mu \partial^\mu + \partial_y \partial^y. \tag{5.3.4}$$

In order to identify the fields of $D = \hat{D} - 1$ dimensions, we insert 5.3.3 in 5.3.4, implying infinite number of separate equations for every mode ϕ_n with different mass

$$\Box_D \phi_n(x) - \left(\frac{n}{L}\right)^2 \phi_n(x) = 0, \tag{5.3.5}$$

where $\Box_D = \partial_\mu \partial^\mu$. As a result a spectrum of fields, which are called the Kaluza-Klein fields, arises so that ϕ_0 is effectively a massless field and ϕ_n are massive fields with mass $m = \frac{|n|}{L}$.

In fact, this holds for a general compact internal space and fields $\hat{\phi}$. From the D-dimensional viewpoint we always have a massless sector and a massive sector with mass inversely proportional to the size of the extra dimensions (internal directions). Since we live in an effectively four dimensional world, we take the radius of the internal direction to be small in order for it to be unobservable; the fields ϕ_n become extremely massive. Therefore these modes are too massive to be physically important and are usually decoupled, namely keep only ϕ_0 and truncate the other modes. When the massive fields are truncated the field $\hat{\phi}$ is independent of the internal dimensions. However, the lower dimensional degrees of freedom are not always massless. It may happen that the D-dimensional spectrum does not contain massless fields. In this case we truncate to the lightest modes of the fields.

Consistency of Dimensional Reduction

One can always obtain lower dimensional theories through a dimensional reduction over compact internal spaces. However one can not reduce over any compact space since there are consistency conditions. Consistency of dimensional reduction is nothing else than that every lower dimensional solution can be uplifted to a higher dimensional solution. In practice, consistency of dimensional reduction is the consistency of the truncation of the massive modes. To check consistency, one can consider a general internal space and a set of eigenfunction E^{λ_j} of the Laplacian operator defined above, namely $\Box E^{\lambda_j} = \lambda_j E^{\lambda_j}$. Again the Fourier decomposition over the general internal space gives

$$\hat{\phi}(x,y) = \sum_j \phi_{\lambda_j}(x) E^{\lambda_j}. \tag{5.3.6}$$

As before we have massless fields for which $\lambda_0 = 0$ and massive fields. Substituting 5.3.6 into higher dimensional equations leads to

$$\Box_D \phi_{\lambda_j}(x) = S_{\lambda_j}. \quad (5.3.7)$$

S_{λ_j} is a function which depends on both massless and massive fields. If S is such that it vanishes when all massive ϕ_{λ_j} are constant, then the equation 5.3.7 is solved, hence the truncation to the massless fields are consistent. In other words, we can truncate the massive modes if the the massless fields do not form a source for massive ones. This implies that one has a restrictive number of internal spaces where one can reduce. However the non-consistency of the truncation of the massive fields does not mean that reduction is useless. From a physical point of view it is probable that the massive modes have negligible interactions with the massless sector because they are heavy. So even if the massive modes can not be truncated consistently, leaving them out will not be too much problem at low energies. However, dimensional reduction is not only used here for obtaining effective lower-dimensional theories. The exact result often matters, for example if a dimensional reduction is used as a solution generating technique (see chapter 6) when lower-dimensional solutions are lifted to higher-dimensional solutions or vice-versa, the reduction has to be consistent.

This way of performing dimensional reduction-an expansion over an internal space and truncation to the lightest sector- is unrealistic. Actually this procedure amounts to writing down an Ansatz which relates the higher dimensional fields to lower dimensional ones, i.e. to the lightest modes of the expansion. Dimensional reduction then consists in substituting the reduction Ansatz in the field equations or the action (Lagrangian). In most cases the reduction Ansatz depends on the internal space coordinates. This dependence should cancel at the end of the day in order to get the field equations corresponding to lower dimensional theory. This requirement is equivalent to the consistency of the truncation to a finite number of lower dimensional fields discussed above. From now on, we can just make the higher dimensional fields independent of the internal dimensions in order to perform dimensional reduction. Note that the number of degrees of freedom is unchanged under dimensional reduction.

The compactness of the internal manifold is not a necessary demand for having a dimensional reduction, what does really matter is to have a consistent truncation. It might happen that the internal space is non-compact, in this case the resulting Kaluza-Klein spectrum is continuous, and nonetheless the truncation to massless fields is still plausible. Such a reduction is called non-compactification while the reduction over compact spaces is called *compactification.*

5.3.1 Circle Reduction of Gravity

We will now consider the dimensional reduction of Einstein gravity in $\hat{D}$ dimensions over a circle to $D = \widehat{D} - 1$ dimensions. In general, bosonic solution in $\hat{D}$ dimensions is specified by the metric, describing the geometry and other fields such as scalar and general p-forms. Regarding the metric field, the solution and the geometric structure must be compatible in a sense that the $\hat{D}$-dimensional space $\mathcal{M}_{\hat{D}}$ can be split up in a D-dimensional space and a S^1 as an internal manifold, that is $\mathcal{M}_{\hat{D}} = \mathcal{M}_D \times S_1$. The coordinates are split up according to $x^{\hat{\mu}} = (x^\mu, y)$. In this way we write the metric

$$ds^2_{\hat{D}} = ds^2_D + e^\sigma dy^2. \tag{5.3.8}$$

In fact we should specify which metric we take on a circle as there is one-parameter family of metrics in the light of the function e^σ. The parameter e^σ in some sense describes the size of S^1. There is nothing against making σ depends on x. In general, the Einstein equations tell us that it has to depend on x! This alleviates the status of σ from a parameter to a physical field $\sigma = \phi(x)$ with its own field equations, found by substituting the Ansatz 5.3.8 in the higher dimensional field equations.

We have seen above that one of the main outcomes of dimensional reduction is that the higher dimensional field $\hat{\phi}$ is independent of the extra direction. It implies that for a scalar, vector field and metric, we have schematically:

$$\hat{\phi} = \phi(x), \qquad \hat{A}_{\hat{\mu}} = (A_\mu(x), \chi(x)), \qquad \hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & A_\mu \\ A_\nu & \varphi \end{pmatrix}. \tag{5.3.9}$$

Thus the vector field $\hat{A}_\mu$ gives rise to a vector field and a scalar field (axion) in lower dimensions, and the reduction of the metric generates a metric $g_{\mu\nu}$, scalar φ (dilaton) and the so-called Kaluza-Klein A_μ. This example reflects the fact that dimensional reduction of a higher-dimensional theory generates a lower-dimensional theory with more fields. This means that a 4-dimensional theory that looks complex can have a simple higher dimensional origin.

The Ansatz 5.3.9 for the metric is a correct Ansatz, but if we plug it into $\hat{D}$-dimensional Einstein equations, then the D-dimensional equations would take on an odd form which rather unfamiliar. For instance, the lower dimensional theory would not have the standard Einstein-Hilbert terms. Therefore the D-dimensional metric can not be interpreted as a solution of the standard Einstein equation. In order to get around this problem, one may redefine the metric $(g_D)_{\mu\nu} \to e^{2\alpha\varphi}(g_D)_{\mu\nu}$ so that the metric $(g_D)_{\mu\nu}$ solves the lower dimensional Einstein equation. The Ansatz becomes

$$ds^2_{\hat{D}} = e^{2\alpha\varphi} ds^2_D + e^{2\beta\varphi}(dy + A)^2, \tag{5.3.10}$$

where we have redefined the Kaluza-Klein vector such that its corresponding kinetic term is as close to the standard Maxwell-form $1/4F^2$. The α and β that parameterize the metric must be

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}, \qquad \beta = -(D-2)\alpha. \tag{5.3.11}$$

The D-dimensional Einstein-Hilbert action can be obtained by inserting the Ansatz 5.3.10 into $\hat{D}$-dimensional equations, i.e. in form notation we get

$$\mathcal{L} = \star\mathcal{R}_D - \frac{1}{2}\star d\varphi\wedge d\varphi - \frac{1}{2}e^{-2(D-1)\alpha\varphi}\star dA\wedge dA. \tag{5.3.12}$$

The coupling between φ and A_μ indicates that one cannot truncate φ and maintain A in order to obtain Einstein-Maxwell theory. If this would have been possible it would have implied the unification of electromagnetism with gravity[6]. A truncation of the vector without truncation of the scalar is possible.

The consistency of the metric Ansatz can be understood as follows: imagine we take the size of S^1, e^σ defined above, to be non-dynamical; namely the function σ is constant. Therefore one would not be able to find a solution to the equations of motion. This can be seen from the fact that the more general and correct Ansatz 5.3.9 gives equation for φ that does not have constant $\varphi = e^\sigma$ as a solution. In other words, the details of the interactions between various lower-dimensional fields prevent the truncation of the scalar φ.

5.3.2 Torus Reduction of Gravity

The circle reduction explained above can be repeated on a series of circles leading to what is called reduction over a torus $\mathbb{T}^n = \underbrace{S^1\times S^1\cdots\times S^1}_{n}$.

The reduction of gravity over $\mathbb{T}^n$ generates n vectors fields A^m with $m = 1,\cdots,n$, n dilatons ϕ^m (they correspond to the radii of the circles), and $n(n-1)/2$ axions χ_α. The dilatons and axions scalar fields parameterize the coset $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n) = \mathbb{R}^+\times\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. Thus the reduction Ansatz of $\hat{D}$-dimensional gravity over n-torus to $D = \hat{D} - n$ dimensions reads (with coordinates $x^{\hat{\mu}} = (x^\mu, y^m)$)

$$ds^2_{\hat{D}} = e^{2\alpha\varphi}ds^2_D + e^{2\beta\varphi}\mathcal{M}_{mn}(dy^m + A^m_\mu dx^\mu)(dy^n + A^n_\mu dx^\mu). \tag{5.3.13}$$

The kinetic term for $\mathcal{M}_{mn}$ reveals that $\mathcal{M}$ parameterizes the coset $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ and since the "breathing mode" φ decouples from the scalars in $\mathcal{M}_{mn}$ in the kinetic term, the scalar manifold gets an extra factor[7] $\mathbb{R}^+$ and thus becomes the coset $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n)$. The resulting scalar manifold plays the role of the *moduli space*[8] of the torus $\mathbb{T}^n$, where the scalars in $\mathcal{M}$ can be interpreted as shape-moduli of the torus. The scalar matrix $\mathcal{M}_{mn}$ is a strictly positive definite symmetric matrix defined

[6] The original motivation for dimensional reduction was unification of the forces in nature from the dimensional reduction of pure gravity in some higher dimension. Unfortunately this is not possible since the dilaton field can not be stabilized.

[7] This is associated with the constant shift symmetry $\mathrm{SO}(1,1)\cong\mathbb{R}^+$ of the dilaton φ.

[8] The moduli are parameters that change the shape of the torus, at fixed volume, while keeping it flat. One can see, as an example, for $\mathbb{T}^2$ that as ϕ (not the breathing one) varies, the relative radii of the two circles change, while as χ varies, the angle between the two circles changes.

by $\mathcal{M} = L^T L$, ($\det\mathcal{M} = 1$) and plays the role of an internal metric of the torus $\mathbb{T}^n$, where L is the corresponding vielbein. We call φ the breathing mode since it describes the overall volume of the torus, and GL($n, \mathbb{R}$) is the symmetry of $\mathbb{T}^n$. Now if we plug 5.3.13 into the $\hat{D}$-dimensional Einstein-Hilbert Lagrangian, it yields

$$\mathcal{L} = \star\mathcal{R}_D - \frac{1}{2}\star d\varphi\wedge d\varphi + \frac{1}{4}\star d\mathcal{M}_{mn}\wedge d(\mathcal{M}^{-1})^{mn} - \frac{1}{2}e^{2(\beta-\alpha)\varphi}\mathcal{M}_{mn}\star dA^n\wedge dA^m, \quad (5.3.14)$$

with

$$\alpha^2 = \frac{n}{2(D+n-2)(D-2)}, \qquad \beta = -\frac{(D-2)\alpha}{n}. \quad (5.3.15)$$

The Lagrangian 5.3.14 is the Lagrangian of a pure Kaluza-Klein theory in D dimensions, which describes the coupling of GL($n, \mathbb{R}$)/SO(n) nonlinear σ-model to gravity and n vector gauge fields. For future use it is worthy noting that the matrix $\mathcal{M}$ might be combined with the breathing φ into a matrix $\hat{\mathcal{M}}$ which parameterizes the coset GL($n, \mathbb{R}$)/SO(n), namely

$$\hat{\mathcal{M}} = (|\det\hat{\mathcal{M}}|)^{\frac{1}{n}}\mathcal{M}, \qquad |\det\hat{\mathcal{M}}| = \exp\sqrt{2n}\varphi. \quad (5.3.16)$$

We close this part with a word of caution: in the above reduction we reduced the action in order to find a new lower-dimensional action. This is not without danger since it is known that filling in on-shell information (the Ansatz) in an action and then performing Euler-Lagrange variation with respect to the remaining unfixed degrees of freedom can be inconsistent. To avoid this problem one should actually do more work and reduce the field equations instead of the action. But we are lucky as this problem will not arise in the reductions in this thesis.

5.3.3 Spacelike and Timelike Toroidal Reductions

So far we have considered dimensional reductions of pure Einstein gravity over Euclidean tori, giving rise to Minkowskian pure Kaluza-Klein theories in D dimensions. We call such reductions *spacelike reductions*, reductions over spacelike isometries (the killing vectors are spacelike). However, one can also reduce over a n-torus with a Lorentzian signature $\mathbb{T}^{n-1,1}$ and obtain an Euclidean theory. This means that the time is included in the dimensional reductions, and hence the reduction over the time-circle is named *timelike reduction*.

What we have done for spacelike reductions turns out to be valid for timelike reductions. For instance the reduction Ansatz 5.3.13 continues to hold for timelike case. The only difference between the two reductions is encoded in the definition of the internal metric $\mathcal{M}$ of the Lorentzian torus $\mathbb{T}^{n-1,1}$, i.e.

$$\mathcal{M} = L^T\eta L, \qquad \det\mathcal{M} = -1, \qquad \eta = \text{diag}(-1, 1, \cdots), \quad (5.3.17)$$

where η here is the tangent space metric of the torus. Surprisingly enough, timelike reductions give rise to the Lagrangian 5.3.14 but now for Euclidean Kaluza-Klein theories where the scalars form $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n-1,1)$ nonlinear σ-models. That is indeed an example of pseudo-Riemannian σ-models that we have discussed in section 5.2.5.

Reducing to three dimensions makes things look special. One can dualize all the gauge potentials A^m to scalars as follows: consider in $D=3$ the last term in 5.3.14

$$S_A \sim \int -\mathcal{M}_{mn} \star F^m \wedge F^n. \tag{5.3.18}$$

To dualize the vectors we need to be sure that the F^m are closed and therefore locally exact, i.e. $F^m = dA^m$. Therefore we enforce this using the Lagrange multipliers χ_m

$$S_{F,\chi} \sim \int -\mathcal{M}_{mn} \star F^m \wedge F^n - \chi_m dF^m. \tag{5.3.19}$$

Variation with respect to χ indeed gives us that F is closed two-form. Now we can treat the action as if F is a fundamental field. The equation of motion for F is

$$\mathcal{M}_{mn} \star F^n = d\chi_m \Rightarrow F^n = (-)^s (\mathcal{M}^{-1})^{mn} \star d\chi_m, \tag{5.3.20}$$

where we have made use of the relation A.2.32 for a p-form. If we plug the result for F, perform partial integration and reshuffle the terms, then we get

$$S_\chi \sim \int (-1)^s (\mathcal{M}^{-1})^{mn} \star d\chi_m \wedge d\chi_n - (-)^s d(\chi_n (\mathcal{M}^{-1})^{mn} \star d\chi_m), \tag{5.3.21}$$

where the total derivative can of course be dropped. We therefore notice that in the case of the three-dimensional Euclidean theory (reduction over a timelike isometry) ($s=0$) the axion kinetic terms appear with the opposite sign of their related vector kinetic terms. Consequently, there is a symmetry enhancement in $D=3$ since it can be shown that the extra scalars combine with the existing scalars into the coset $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n-1,2)$[9]. In this case there is no decoupled $\mathbb{R}$. Note that for the reduction over an Euclidean torus ($s=1$) from $3+n$ to three dimensions we obtain the coset $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$.

5.4 Torus Reductions of Maximal Supergravities

We start our discussion with emphasizing the fact that torus reductions do not break supersymmetry. Therefore the dimensional reductions of type IIB and type IIA on a n-torus and 11-dimensional supergravity on $n+1$-torus lead to the unique maximal

[9] This means that $\mathrm{SO}(n-1,2)$-invariant matrix η has as a signature $\mathrm{diag}(-1,-1,+1,+1,\cdots,+1)$.

	G/H-Minkowskian SUGRA	G/H^*-Euclidean SUGRA
$D = 10$	SO(1,1)	SO(1,1)
$D = 9$	$\frac{\mathrm{GL}(2,\mathbb{R})}{\mathrm{SO}(2)}$	$\frac{\mathrm{GL}(2,\mathbb{R})}{\mathrm{SO}(1,1)}$
$D = 8$	$\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)} \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$	$\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(2,1)} \times \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(1,1)}$
$D = 7$	$\frac{\mathrm{SL}(5,\mathbb{R})}{\mathrm{SO}(5)}$	$\frac{\mathrm{SL}(5,\mathbb{R})}{\mathrm{SO}(3,2)}$
$D = 6$	$\frac{\mathrm{SO}(5,5)}{\mathrm{S[O(5)\times O(5)]}}$	$\frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5,C)}$
$D = 5$	$\frac{\mathrm{E}_{6(+6)}}{\mathrm{USp}(8)}$	$\frac{\mathrm{E}_{6(+6)}}{\mathrm{USp}(4,4)}$
$D = 4$	$\frac{\mathrm{E}_{7(+7)}}{\mathrm{SU}(8)}$	$\frac{\mathrm{E}_{7(+7)}}{\mathrm{SU}^*(8)}$
$D = 3$	$\frac{\mathrm{E}_{8(+8)}}{\mathrm{SO}(16)}$	$\frac{\mathrm{E}_{8(+8)}}{\mathrm{SO}^*(16)}$

Table 5.4.1: Cosets for maximal supergravities in Minkowskian and Euclidean signatures.

supergavities in $D < 11$.

The (maximal) supergavity theories can be classified according to the nonlinear sigma models describing the scalar field interactions. This means that the geometry of (pseudo)-Riemannian symmetric spaces fixes the scalar field interactions terms in supergavity Lagrangian. We summarize the scalar manifolds of the maximally extended supergavities that appear after dimensional reduction of 11-dimensional supergavity on a torus in table 5.4.1 [136]. For future use we show it both for Minkowskian and Euclidean maximal supergavities. The cosets G/H in the left column are all maximally non-compact since G is the maximally non-compact real form of a semisimple complex Lie group and H is its maximal compact subgroup. Since H is compact the metric is strictly positive definite and the coset is Riemannian. The cosets G/H^* in the right column only differ in the isotropy group H^* which is some non-compact version of H and as a result G/H^* is pseudo-Riemannian. In appendix D of [G] we perform the reductions of type II theories and we give in particular the precise group theoretical characterization of the ten-dimensional origin of the bosonic fields in $D = 3$.

5.5 From G/H to G/H^*: the Wick Rotation

Let us end this chapter by defining a generalized Wick rotation which maps a geodesic on G/H in a Minkowskian theory, into geodesic on G/H^* in its Euclidean version [G].

In order to map a compactification on a spatial circle into one on a time-circle, we need to analytically continue the internal radius: $R_0 \to iR_0$. This transformation can be viewed as the action of a *complexified* $\mathrm{O}(1,1)$ transformation

$$\mathcal{O} = i^{H_0}, \tag{5.5.1}$$

on the Minkowskian D-dimensional theory in which the scalar fields span the target space G/H. The generator H_0 is a suitable combination of the Cartan generators H_I of $\mathfrak{G}$ with $I = 1, \cdots \mathrm{rank}\mathfrak{G}$. Consider the following action on the generators t_n of $\mathfrak{G}$ taking values in some representation of $\mathfrak{G}$:

$$t_n \to \mathcal{O}_n{}^m[\mathcal{O}t_m\mathcal{O}^{-1}]. \tag{5.5.2}$$

The action of $\mathcal{O}$ on the Cartan generators is trivial $\mathcal{O}^I_J = \delta^I_J$, while it has the following action on the shift generators:

$$\mathcal{O}_\alpha{}^\sigma = i^{-\alpha(H_0)}\delta^\sigma_\alpha, \tag{5.5.3}$$

where $\alpha(H_0)$ is the scalar product defined via the commutation relation $[H_0, E_\alpha] = \alpha(H_0)E_\alpha$. We see that a generic shift generator E_α is mapped into itself by 5.5.2.

$$E_\alpha \to \mathcal{O}_\alpha{}^{\alpha'}[i^{H_0}E_{\alpha'}i^{-H_0}] = \mathcal{O}_\alpha{}^{\alpha'}i^{\alpha'(H_0)}E_{\alpha'} = E_\alpha. \tag{5.5.4}$$

Therefore the transformation 5.5.2 maps $\mathfrak{G}$ into itself. According to the Cartan decomposition 5.2.4 the compact subalgebra $\mathfrak{H}$ and the non-compact space $\tilde{\mathfrak{F}}$ are generated by

$$\begin{aligned} \mathfrak{H} &= \{\tilde{J}_\alpha\} = \{E_\alpha - E_{-\alpha}\}, \\ \tilde{\mathfrak{F}} &= \{H_I, \tilde{K}_\alpha\} = \{H_I, E_\alpha + E_{-\alpha}\}. \end{aligned} \tag{5.5.5}$$

On a generic element g of G the above transformation amounts to a combination of a change of basis for the matrix representation and a redefinition of the group parameters. Indeed if we write an element of G as the product of a coset representative $\tilde{L} \in \exp(\tilde{\mathfrak{F}})$ times an element $\tilde{h}$ of H we have

$$g = \tilde{L}(\varphi,\phi)\tilde{h}(\xi) = e^{\phi^\alpha \tilde{K}_\alpha}e^{\varphi^I H_I}e^{\xi^\alpha \tilde{J}_\alpha} \longrightarrow \mathcal{O}\, e^{\phi'^\alpha \tilde{K}_\alpha}e^{\varphi'^I H_I}e^{\xi'^\alpha \tilde{J}_\alpha}\mathcal{O}^{-1}, \tag{5.5.6}$$

where the redefined parameters are

$$\varphi'^I = \varphi^I; \qquad \phi'^\alpha = \phi^\sigma \mathcal{O}_\sigma{}^\alpha; \qquad \xi'^\alpha = \xi^\sigma \mathcal{O}_\sigma{}^\alpha. \tag{5.5.7}$$

Let us consider the effect of this transformation on the generators of the coset representative and on the compact factor

$$\begin{aligned} \phi'^\alpha\,[\mathcal{O}\,\tilde{K}_\alpha\,\mathcal{O}^{-1}] &= \phi^\alpha\, i^{-\alpha(H_0)}[i^{\alpha(H_0)}E_\alpha + i^{-\alpha(H_0)}\,E_{-\alpha}] = \\ &= \phi^\alpha\,(E_\alpha + (-1)^{\alpha(H_0)}\,E_{-\alpha}) = \phi^\alpha\,K_\alpha\,, \\ \xi'^\alpha\,[\mathcal{O}\,\tilde{J}_\alpha\,\mathcal{O}^{-1}] &= \xi^\alpha\, i^{-\alpha(H_0)}[i^{\alpha(H_0)}E_\alpha - i^{-\alpha(H_0)}\,E_{-\alpha}] = \\ &= \xi^\alpha\,(E_\alpha - (-1)^{\alpha(H_0)}\,E_{-\alpha}) = \xi^\alpha\,J_\alpha\,, \end{aligned} \tag{5.5.8}$$

where J_α and K_α differ from $\tilde{J}_\alpha$ and $\tilde{K}_\alpha$ only for $\alpha = \gamma$, for which $J_\gamma = E_\gamma + E_{-\gamma}$ and $K_\gamma = E_\gamma - E_{-\gamma}$. J_α are therefore generators of H^* and K_α, together with H_I are in $\mathfrak{G}/\mathfrak{H}^*$. The Wick rotation defines therefore a mapping between two different representations of a same element g of G: one as the product of a coset representative $\tilde{L}$ in G/H and an element $\tilde{h}$ of H and the other as a product of a coset representative L in G/H^* times an element h in H^*. The matrix $\tilde{M}(\varphi^I, \phi^\alpha) = \tilde{L}^\sharp \tilde{L}$, defined in subsection 5.2.2, which describes the scalar fields on G/H transforms as follows:

$$\tilde{\mathcal{M}}(\varphi^I, \phi^\alpha) \quad \rightarrow \quad \mathcal{O}^\sharp \tilde{\mathcal{M}}(\varphi'^I, \phi'^\alpha)\mathcal{O} = L^\sharp \eta L = \mathcal{M}(\varphi^I, \phi^\alpha)\,, \tag{5.5.9}$$

where $\eta = \mathcal{O}^\sharp \mathcal{O}$ and $\mathcal{M}$ is the matrix describing the scalars on G/H^*. For example in $D = 3$ maximal supergravity, the effect of the transformation $\mathcal{O}$ is to map the $E_{8(8)}/SO(16)$ coset in the last row of table 5.4.1 to $E_{8(8)}/SO^*(16)$ coset of the same row. In other words, 56 compact generators of SO(16) are mapped into 56 non-compact generators J_γ in $SO^*(16)$.

Chapter 6

Brane Solutions and Generating Geodesic Flows

The main goal behind this chapter is to illustrate the power of the sigma-model technique in constructing solutions in a purely algebraic way.

We show first that via torus reduction over the worldvolume of a brane we obtain a link between timelike p-branes, e.g. Dp-branes, and instantons (D(-1)-instanton), and similarly between Sp-branes and S(-1)-branes[1]. The wordvolume reduction of an Sp-brane over a Euclidean torus (spacelike reductions) leads to a nonlinear σ-model coupled to gravity, describing the dynamics of an S(-1)-brane. We will restrict to σ-models whose target space (moduli space) metrics G_{ij} belong to Riemannian maximally non-compact symmetric spaces G/H with H the maximal compact subgroup of G. By the same token, worldvolume reduction of a p-brane over a Lorentzian torus results in a nonlinear σ-model based on pseudo-Riemannian symmetric spaces G/H^* with H^* is a non-compact version of H.

Next we shall show that the branes are described by a geodesic motion on moduli spaces. Our approach is based on the construction of the *the minimal generating geodesic solution*: a geodesic with the minimal number of free parameters such that all the geodesics are generated by isometry transformations of the moduli space. We will mainly focus on the Kaluza-Klein theory that follows from the reduction of pure Einstein gravity, where G is $\mathrm{GL}(p+1,\mathbb{R})$ in $D > 3$ and $\mathrm{SL}(p+2,\mathbb{R})$ in $D = 3$. In the case of S-branes this approach allows the construction of new vacuum time-dependent solutions (fluxless Sp-brane solutions). In the case of timelike branes we obtain some stationary solutions of pure gravity which still have to be identified.

This chapter is mainly based on work done in [E], [F] and [G].

[1]S-branes have been briefly introduced in chapter 1. We shall define them at greater length in section 6.1.

6.1 Solutions in (Super)gravity

Many (super)gravity solutions of equations of motion have the structure of a p-brane. These solutions have played an essential role in strengthening our belief in dualities in the non-perturbative limit.

6.1.1 p-brane Solutions

We have seen in chapter 2 that the spectrum of string theory contains higher rank gauge fields. Therefore, a further generalization of strings is possible, namely p-branes, $(p+1)$-dimensional extended objects in d-dimensional spacetime of supergravity theory. The p-brane electrically couples to a $(p+1)$-rank gauge field $A_{(p+1)}$, or magnetically to a $(d-p-3)$-form gauge field $A_{(d-p-3)}$. Another characteristic of brane solutions is that the geometry has a flat $(p+1)$-dimensional worldvolume. Generically, we discriminate between two kinds of p-brane solutions:

Timelike p-branes

They are related to D-branes and M-branes. A Dp-brane is a static supersymmetric object appearing in string theory and has three descriptions. The first description is the one of chapter 2, namely the Dp-brane which can be viewed as $(9-p)$ spacelike boundary conditions for the open string in its perturbative limit. In the supergravity picture Dp-branes are supersymmetric solitonic solutions extended in p spacelike dimensions and one timelike dimension. We say that the Dp-brane has a $(p+1)$-dimensional worldvolume containing time and a $(9-p)$-dimensional transverse space. They can roughly be seen as higher-dimensional extensions of Reissner-Nordström black hole of four-dimensional Einstein-Maxwell theory. The third picture is less evident and considers stable Dp-branes as tachyonic kink solutions on the worldvolume of unstable D$(p+1)$-branes [137].

Such branes couple electrically to the gauge fields according to

$$\int d^{p+1}\sigma A_{\mu_1\cdots\mu_{p+1}}\partial_{a_1}X^{\mu_1}\cdots\partial_{a_{p+1}}X^{\mu_{p+1}}\epsilon^{a_1\cdots a_{p+1}}, \tag{6.1.1}$$

similarly to the point particle ($p=0$), which couples to a 1-form gauge field, and the NS-NS 2-form $B_{\mu\nu}$ which couples to a string worldsheet. Thus the electric charge of such an object can be determined through a generalization of Gauss' law:

$$Q_e \sim \int_{S^{d-p-2}} \star F_{(p+2)}, \tag{6.1.2}$$

where $\star F_{(p+2)}$ represents the Hodge dual of $A_{(p+1)}$ field strength, and the S^{d-p-2} is a sphere surrounding the p-brane. This charge is conserved due to the equations of motion of the gauge field. Moreover there is the dual magnetic $(d-p-2)$-brane which couples to $A_{(d-p-3)}$ dual to $A_{(p+1)}$. Its topological magnetic charge is found to be

$$Q_m \sim \int_{S^{p+2}} F_{(p+2)}, \tag{6.1.3}$$

which is conserved due to the Bianchi identity. Here we perform the integration over the transversal directions of p-brane. The charges satisfy the following Dirac's quantization condition for electric and magnetic monopoles

$$Q_e \cdot Q_m = 2\pi n, \qquad n \in \mathbb{Z}. \tag{6.1.4}$$

The consistent bosonic truncation of the supergravity action that one needs to find the solitonic p-brane solutions reads

$$S = \frac{1}{2\kappa^2}\int d^dx\sqrt{|g|}(\mathcal{R} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2(p+2)!}e^{a\phi}F^2_{(p+2)}), \tag{6.1.5}$$

i.e., brane solutions are supported by the metric, possible the dilaton and a $(p+1)$-form gauge potential.

Naively, one can think about deriving the equations of motion which follow from 6.1.5, and then solving them for a solitonic p-brane solution. But this is highly non-trivial. Instead one can write down a convenient Ansatz for such a solution which is given by

$$ds^2_d = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}dr^2 + e^{2C(r)}d\Sigma^2_k, \tag{6.1.6}$$

where A, B and C are arbitrary functions, η is the usual Minkowski metric in $p+1$ dimensions $\eta = \text{diag}(-,+,\cdots,+)$ and $d\Sigma^2_k$ is the metric of a maximally symmetric space with unit curvature $k=-1,0,1$ (see appendix A.4). We refer to such p-branes as *timelike* branes, stationary solutions (time-independent solutions). If we assume that the above Ansatz is consistent with the $\text{ISO}(p,1)\times\text{SO}(d-p-1)$ symmetry of spacetime, with the ISO along the worldvolume directions, the Ansatz 6.1.6 becomes

$$\begin{aligned} ds^2_d &= e^{2A(r)}dx^\mu dx^\nu\eta_{\mu\nu} + e^{2B(r)}dy^a dy^b\delta_{ab}, \qquad \phi = \phi(r), \\ \mu,\nu &= 0,\cdots,p \qquad a,b = p+1,\cdots,d-1, \end{aligned} \tag{6.1.7}$$

with $r \equiv \sqrt{y^a y^b \delta_{ab}}$ the isotropic radial distance in the transverse space. There are two possible solutions of the equations of motion, resulting in an electric or magnetic p-brane [138, 139]. The metric of those solutions are given by

$$ds^2_d = h^{\frac{-4(d-p-3)}{\Delta}(d-2)}dx^2_{(p+1)} + h^{\frac{4(p+1)}{\Delta}(d-2)}dy^2_{(d-p-1)}, \tag{6.1.8}$$

where the harmonic function h satisfies $\nabla^2 h = 0$, and the parameter Δ is given by

$$\Delta = a^2 + \frac{2(p+1)(d-p-3)}{d-2}. \tag{6.1.9}$$

For $d-p-1>2$, h can be written as $h(r)=1+(\frac{r_0}{r})^{d-p-3}$, where r_0 is an integration constant related to the charge in the magnetic case. For the electric-brane solution the scalar and gauge fields read

$$e^{\phi}=h^{\frac{2a}{\Delta}}, \qquad F_{a\mu_1\cdots\mu_{p+1}}=\frac{2}{\sqrt{\Delta}}\epsilon_{\mu_1\cdots\mu_{p+1}}\partial_a(h^{-1}). \tag{6.1.10}$$

Similarly, the magnetic solution has

$$e^{\phi}=h^{-\frac{2a}{\Delta}}, \qquad F_{a_1\cdots a_{p+2}}=-\frac{2}{\sqrt{\Delta}}\epsilon_{a_1\cdots a_{p+2}r}\partial_r(h). \tag{6.1.11}$$

The simplest example in type II theories is the electric 1-brane, coupling to the NS-NS 2-form $B_{\mu\nu}$, called the fundamental string F1. This solution can be obtained from the solution above by setting $p=1$, $a=-1$, and $d=10$. Its magnetic dual is called the Neveu-Schwarz 5-brane NS5. As seen in chapter 2, type II also contain RR-gauge fields allowing for a separate class of solutions, D-branes solutions (see chapter 2).

The eleven dimensional supergravity theory only involves one 3-form gauge field, and no dilaton. The only sources one can associate to a 3-form are 2-brane or 5-brane solutions, so we take $\Delta=4$. The resulting solutions are called the electric M2-brane [140] and magnetic M5-brane [141]. Note that the compactification of the M2-brane along its spatial directions yields exactly the F1 solution of type IIA supergravity. The NS5 solution can be obtained by compactifying the M5-brane along a transverse direction.

A special case of p-brane is the so-called domain wall, a $(d-2)$-brane with one transverse direction, separating space into two regions.

It has been shown that all those brane solutions preserve half of the supersymmetries of the supergravity theories. This implies that such solutions have to satisfy some first-order differential equations which arise from demanding that the supersymmetry variation of the fermion vanishes. These first order-equations are now referred to as Bogomol'nyi, Prasad and Sommerfeld or BPS equations [142, 143]. In [144] Witten and Olive gave the condition to preserve supersymmetry of solitons in supersymmetric theories. The term BPS equation is now generically used for equations of motion that are inferred by rewriting the action as a sum of squares. Supersymmetric solutions, in general, belong to this class[2]. In the literature these supersymmetric branes are also called *extremal*. The word extremal originates from the fact the branes are subject to a relation between the mass and the charge of Dp-branes. In other words, when the mass equals the charge a brane is called extremal [146], otherwise the brane is called non-extremal.

[2]Stationary non-extremal and time-dependent solutions (discussed later) cannot preserve supersymmetry in ordinary supergravity theories. However, it has been argued in [E] and [145] that these solutions often can be found from first-order equations called *fake*-or *pseudo*-BPS equations.

Spacelike p-branes

There is another kind of p-brane solutions where the time direction belongs to the transversal space and hence they have Euclidean worldvolume. This means that one chooses a Dirichlet condition for the time-direction. Such p-branes are called *spacelike* branes or Sp-branes for short, and they are explicitly time-dependent.

Similarly to Dp-brane, a Sp-brane solution is carried by a metric, a dilaton and a $(p+1)$-form gauge potential. The metric, which describes a time-dependent geometry, is schematically given by[3]

$$ds_d^2 = e^{2A(t)}\delta_{\mu\nu}dx^\mu dx^\nu - e^{2B(t)}dt^2 + e^{2C(t)}d\Sigma_k^2, \tag{6.1.12}$$

where δ is the usual flat Euclidean metric in $p+1$ dimensions $\delta_{\mu\nu} = \mathrm{diag}(+,+,\cdots,+)$. The transverse space consists of time and $(d-p-2)$ dimensions. Ansatz 6.1.12 has $\mathrm{ISO}(p+1)$ worldvolume symmetry, and Lorentzian $\mathrm{SO}(d-p-2,1)$ transversal symmetry in the $k=-1$ case and can be asymptotically flat (in contrast to $k=+1$ solutions). Those solutions are the spacelike branes introduced by Gutperle and Strominger [147], who conjectured that such branes correspond to specific time-dependent processes in string theory. Nonetheless, in this chapter we will also define S-brane in the generalized sense, i.e. for all the other possible slicings.

Due to the time-dependence the S-brane solutions belonging to type II supergravities are not supersymmetric. Consequently, the solutions are more complicated to write down. Hence we prefer to focus on S(-1)-branes of type IIB supergravity. This brane can be viewed as the time-dependent twin of the Euclidean D(-1)-instanton. The action of the S(-1)-brane follows from the truncation of type IIB supergravity $(d=10)$ to its scalar sector

$$S = \int d^d\sqrt{|g|}x\Big[\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2\Big], \tag{6.1.13}$$

where ϕ is the dilaton and we denote the axion with χ instead of $C_{(0)}$ defined in chapter 2. As we will see in the ensuing sections, the axion and the dilaton form a $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ nonlinear σ-model, and also the equations of motion derived from 6.1.13 tells us that the scalar fields trace out a geodesic on the target space.

For $p=-1$ all space is transverse, so the part involving A is not present in the Ansatz 6.1.12. We choose the gauge where $e^{2C}=t^2$ and the Ansatz becomes

$$ds^2 = -f(t)dt^2 + t^2\Big[\frac{1}{1-kr^2}dr^2 + r^2 d\Omega_{d-2}^2\Big]. \tag{6.1.14}$$

For $k=0$ we have flat space, for $k=+1$ a sphere and finally for $k=-1$ a hyperboloid. This follows from the fact that when t goes to infinity the metric describes a flat

[3] In this chapter a Sp-brane has a $p+1$-dimensional Euclidean worldvolume just like a Dp-brane has a $(p+1)$-dimensional Lorentzian worldvolume.

Minkwoski spacetime only for $k = -1$ (if $f \to 1$). Only for $k = -1$ there is an expected string theory interpretation. The two scalars depend only on t.

The metric solution of the equations of motion following from 6.1.13 behaves as

$$ds^2 = -\frac{dt^2}{\frac{||v||^2}{2(d-1)(d-2)}t^{-2(d-2)} - k} + t^2\left(\frac{1}{1-kr^2}dr^2 + r^2 d\Omega_{d-2}^2\right), \tag{6.1.15}$$

where $||v||^2$ is a strictly positive number which turns out to be the affine velocity[4] labelling the geodesics traced out by the scalar fields on the targetspace (scalar manifold). The scalar fields solution are given by

$$\phi(t) = \log\Big[C_1 \cosh(||v||h(t) + C_2)\Big], \qquad \chi(t) = \pm\frac{1}{C_1}\tanh\Big[||v||h(t) + C_2\Big] + C_3. \tag{6.1.16}$$

The C_i's are constants of integrations and the harmonic function h is

$$h(t) = \frac{1}{\sqrt{a}(2-d)}\ln|\sqrt{a}t^{2-d} + \sqrt{at^{2(2-d)} - k}| + c, \tag{6.1.17}$$

with

$$a = \frac{||v||^2}{2(d-1)(d-2)}. \tag{6.1.18}$$

The two scalar fields and the geodesic are related via the following relation

$$||v||^2 = (\partial_h\phi)^2 + e^{2\phi}(\partial_h\chi)^2. \tag{6.1.19}$$

More about this later.

For the three different values of k we have

- For $k = -1$ one has the S(−1)-brane of type IIB supergravity [148].
- For $k = 0$ the brane describes a so-called power-law universe in the FLRW-coordinates.
- For $k = +1$ the solution is not really an S(−1)-brane. Actually it describes a transition from a Big-Bang to a Big Crunch for a closed universe. We recommend [149] for a nice explanation.

In fact, time-dependent backgrounds (solutions), e.g. S-branes, are badly understood in string theory because string theory is not yet formulated in such a background. One of the reasons is, as previously mentioned, that (most) time-dependent backgrounds are not supersymmetric. Similarly to the timelike p-brane solutions, e.g. D-branes, there have been some conjectures about the open string picture of S-branes

[4] The geodesic motion and the affine parameter are defined in appendix A.3.2.

as timelike tachyonic kinks. D-branes play a central role in the holographic description of string theory such as the AdS/CFT correspondence [5]. Similarly the S-branes are conjectured to play a similar role as D-branes in the context of holography. However, in this case the holographic duality would be a dS/CFT correspondence [150]. In order to understand this correspondence better one has to improve the understanding of S-branes. Quite recently there was a proposal to understand cosmological solutions as Wick-rotations of supersymmetric domain-walls (DW) [151]. This suggests a relation between DW/QFT correspondence [152] (this is a non-conformal extension of the AdS/CFT correspondence) and a hypothetical COSM/QFT correspondence. Central in that discussion is the concept of pseudo-supersymmetry [151]. It is interesting to formulate pseudo supersymmetry in 10- or 11-dimensional supergravity and interpret S-branes as pseudo-BPS objects. Such a formulation of S-branes would improve the understanding of the dS/CFT correspondence. For more about S-brane solutions we refer the reader to [148, 149, 153–157].

6.2 From p-brane to (-1)-brane

Since the worldvolume of p-brane solutions (timelike and spacelike) is translation invariant ($x^a \to x^a + c^a$), all these solutions have the property that the worldvolume directions correspond to Killing directions. In order to fulfill this property, the matter fields that carry the solutions must also be translation invariant. This implies that one can effectively dimensionally reduce the solution over the worldvolume. Thus a p-brane can be mapped to a (-1)-brane solution in $D = d - p - 1$ dimensions whose equations of motion can be derived from the action:

$$S = \int d^D x \sqrt{|g|} \Big[\mathcal{R} - \frac{1}{2} G_{ij}(\Phi) \partial_\mu \Phi^i \partial^\mu \Phi^j \Big], \tag{6.2.1}$$

where G_{ij} is the metric defined in eq.5.2.1 on the moduli space that appears after dimensional reduction over a torus. For timelike branes, time is included in the reduction and the corresponding moduli spaces are pseudo-Riemmanian G/H^*, in contrast to moduli spaces (Riemannian G/H) appear after spacelike reduction. If we compare 6.1.12 and 6.1.6 with 5.3.13, we realize that one has to set the Kaluza-Klein vectors to zero. In addition, the worldvolume of the theory is identified with $\mathcal{M}_{mn}$. In general, the metric of the torus $\mathcal{M}_{mn}$ breaks the worldvolume symmetries (ISO$(p, 1)$ and ISO$(p + 1)$), since we will obtain extra terms multiplying the $dxdx$-terms on the worldvolume. If we reduce to $D = 3$ one can dualize all Kaluza-Klein vectors to scalars (see section 5.3.3). These Kaluza-Klein vectors will lead to off-diagonal terms that mix worldvolume directions with transversal directions (product of x-direction and angular direction).

The metric Ansatz for the (-1)-brane is

$$ds_D^2 = \epsilon f^2(\rho) d\rho^2 + g^2(\rho) g_{ab}^{D-1} dx^a dx^b, \qquad \Phi^i = \Phi^i(\rho), \tag{6.2.2}$$

where the function f corresponds to gauge freedom of reparameterizing the coordinate ρ.

Now we have two cases

- ϵ is -1, then the radial coordinate ρ corresponds to time, i.e. $\rho = t$, and g_{ab} is a metric on a Euclidean maximally symmetric space (see A.4), the three possible FLRW-geometries.
- For $\epsilon = 1$, 6.2.2 describes an *instanton* geometry with $\rho = r$ the direction of the tunnelling process. Timelike D(-1)-branes are solutions of Euclidean supergravities.

If we reparameterize the coordinate ρ to $h(\rho)$ via

$$dh(\rho) = g^{1-D} f d\rho, \tag{6.2.3}$$

then the equations of motion for the scalars are derived from the one dimensional action

$$S = \int G_{ij}\partial_h \Phi^i \partial_h \Phi^j dh. \tag{6.2.4}$$

This action demonstrates that the solutions describe geodesic motion on the moduli space with $h(\rho)$ as an affine parameter (see appendix A.3.2). From equation 6.2.3 we read off that $h(r)$ is a harmonic function on the (-1)-brane geometry. In terms of the affine parameter the velocity $\|v\|$ is constant such that

$$\|v\|^2 = G_{ij}\partial_h \Phi^i \partial_h \Phi^j. \tag{6.2.5}$$

The Ricci tensor following from the metric 6.2.2 is given by

$$\mathcal{R}_{\rho\rho} = (D-1)[-\frac{\ddot{g}}{g} + \frac{\dot{g}\dot{f}}{gf}], \tag{6.2.6}$$

$$\mathcal{R}_{ab} = -\epsilon\Big[\frac{\ddot{g}g}{f^2} - \frac{g\dot{g}\dot{f}}{f^3} + (D-2)\frac{\dot{g}^2}{f^2}\Big] g^{D-1}_{ab} + \mathcal{R}^{D-1}_{ab}, \tag{6.2.7}$$

where a dot denotes differentiation with respect to ρ. The Einstein equation for (-1)-branes is expressed as

$$\mathcal{R}_{\rho\rho} = \frac{1}{2} G_{ij}\partial_\rho \Phi^i \partial_\rho \Phi^j = \frac{1}{2}||v||^2(\partial_\rho h(\rho))^2, \qquad \mathcal{R}_{ab} = 0. \tag{6.2.8}$$

The combination of the Einstein equation together with 6.2.5 leads to the following first-order equation

$$\dot{g}^2 = \frac{||v||^2}{2(D-2)(D-1)} f^2 g^{4-2D} + \epsilon k f^2. \tag{6.2.9}$$

This equation gives rise to a solution only when the right-hand side remains positive. Note that there is no equation of motion for f due to the fact that it corresponds to the reparametrization freedom that one has of ρ. Interestingly enough, the scalar field solution has no influence on solving the metric.

To summarize, one has two kinds of worldvolume reduction (WR) of branes:

- Worldvolume reduction of spacelike branes: the resulting moduli space is Riemannian G/H with a compact isotropy group. Therefore the metric G_{ij} is positive definite, and then $||v||^2 > 0$. Thus the scalar fields trace out spacelike geodesics on the moduli space G/H. We have seen also that Sp-branes reduced over their worldvolumes lead to a system containing gravity and scalars fields only. Generically, a solution which is carried by a metric and scalar fields alone has a simpler mathematical structure than those solutions that are carried by non-trivial p-form potentials. When we have solved the lower-dimensional (scalar) equations of motion we can lift up the solution to the original fields. This way one might obtain a solution carried by a non-trivial p-form potential as well. We will see later in this chapter that via a worldvolume uplifting (WU) the S(-1)-brane of a pure Kaluza-Klein theory becomes a fluxless Sp-brane. We refer to [158, 159] for a description of spacelike branes, in maximal supergravity, in terms of a geodesic motion. We thus have the following map

$$\text{S}p\text{-brane} \xrightarrow{WR} \text{S}(-1)\text{-brane} \xrightarrow{WU} \text{S}p\text{-brane}. \tag{6.2.10}$$

- Worldvolume reduction of timelike branes: for this case the reduction gives rise to a pseudo-Riemannain moduli space G/H^* with a non compact isotropy group. Hence the metric G_{ij} has an indefinite signature and as a result $||v||^2$ can be zero, positive or negative. Therefore the geodesic curves traced out by the scalar fields on G/H^* are labeled according to the sign of $||v||^2$, i.e. null–like, spacelike and timelike. As an example of a geodesic motion on the moduli space, we consider the supersymmetric IIB instanton [160]. That solution corresponds to the lighlike geodesics on $\text{SL}(2,\mathbb{R})/\text{SO}(1,1)$ (the Euclidean axion-dilaton system) whereas the non-supersymmetric IIB instantons correspond to spacelike and timelike geodesics [161] on $\text{SL}(2,\mathbb{R})/\text{SO}(1,1)$. This way we obtain

$$p\text{-brane} \xrightarrow{WR} (-1)\text{-brane} \xrightarrow{WU} p\text{-brane}. \tag{6.2.11}$$

  In the case of the reduction of timelike branes, the correspondence between geodesics and branes is probably best known in terms of four-dimensional black holes (0-branes) and the three-dimensional instantons [130, 162–164].

6.2.1 (-1)-brane Geometries

Here we want to look for metric solutions belonging to the action 6.2.1 which only depend on the coordinate ρ.

S(−1)-brane Geometries

We first consider the spacelike (−1)-branes ($\rho = t$). For this case the target space is Riemannian and all geodesics have strictly positive affine velocity squared $||v||^2 > 0$. The solution to the Einstein equations 6.2.9 gives the following D-dimensional metric

$$ds_D^2 = -\frac{dt^2}{at^{-2(D-2)} - k} + t^2 d\Sigma_k^2, \qquad a = \frac{||v||^2}{2(D-1)(D-2)}, \tag{6.2.12}$$

while the scalar fields trace out geodesics curves with the harmonic function $h(t)$ as affine parameter. The harmonic function h is given by

$$h(t) = \frac{1}{\sqrt{a}(2-D)} \ln|\sqrt{a}t^{2-D} + \sqrt{at^{2(2-D)} - k}| + b. \tag{6.2.13}$$

We take b=0 in what follows since b just corresponds to a shift in the affine parameter h. For $k = 1$ the metric 6.2.12 has a coordinate singularity.

Timelike (−1)-brane Geometries: D(-1)-instanton

As mentioned above the timelike (−1) brane ($\rho = r$) is an *instanton*. Its geometry entirely depends on the character of the geodesic curve (spacelike, nulllike or timelike). Some of these solutions have appeared in the literature before [130, 149, 161, 165, 166].

- $||v||^2 > 0$

  For this class of instantons we will be using the gauge $f = g$. In the table below we present the conformal factor f that determines the metric and the radial harmonic function ρ. Note that for all three values of k the solutions have singularities.

- $||v||^2 = 0$

  We take the Euclidean "FLRW gauge" for which $f = 1$. It is clear from (6.2.9) that for $k = -1$ we do not find a solution and that for $k = 0$ we find flat space in Cartesian coordinates ($g = 1$) and for $k = +1$ we find flat space in spherical coordinates ($g = r$). This makes sense since a lightlike geodesic motion comes with zero "energy-momentum"[5]. The harmonic function is

$$\begin{aligned} k &= 0 \qquad h(r) = c\,r + b\,, \\ k &= 1 \qquad h(r) = \frac{c}{r^{D-2}} + b\,, \end{aligned} \tag{6.2.14}$$

[5] The fact that the $k = -1$ solution does not exists reflects that there does not exist a hyperbolic slicing of the Euclidean plane.

$k=-1$	$f(r) = (\frac{\|\|v\|\|^2}{2(D-1)(D-2)})^{\frac{1}{2D-4}} \cos^{\frac{1}{D-2}}[(D-2)r]$ $h(r) = \sqrt{\frac{8(D-1)}{(D-2)\|\|v\|\|^2}} \,\mathrm{arctanh}[\tan(\frac{D-2}{2}r)] + b$
$k=0$	$f(r) = \Big(\sqrt{\frac{(D-2)\|\|v\|\|^2}{2(D-1)}}\, r\Big)^{\frac{1}{D-2}}$ $h(r) = \sqrt{\frac{2(D-1)}{(D-2)\|\|v\|\|^2}} \log r + b$
$k=+1$	$f(r) = (\frac{\|\|v\|\|^2}{2(D-1)(D-2)})^{\frac{1}{2D-4}} \sinh^{\frac{1}{D-2}}[(D-2)r]$ $h(r) = \sqrt{\frac{2(D-1)}{\|\|v\|\|^2(D-2)}} \,\log[\tanh(\frac{D-2}{2}r)] + b$

Table 6.2.1: *The Euclidean geometries with* $||v^2|| > 0$ *in the gauge* $f = g$*. The real number* b *is an integration constant.*

where c is a constant. In Euclidean IIB supergravity the axion-dilaton parameterize $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ and for $||v||^2 = 0$ and $k = 1$ we have the standard half-supersymmetric D-instanton [160].

- $||v||^2 < 0$

For $k = 0$ and $k = -1$ we clearly have no solutions since the right-hand side of (6.2.9) is always negative. For $k = +1$ a solution does exist, and in the conformal gauge ($g = fr$) it is given by

$$f(r) = \Big(1 - \tfrac{||v||^2}{8(D-1)(D-2)} r^{-2(D-2)}\Big)^{\frac{1}{D-2}}\,, \tag{6.2.15}$$

where indeed only $||v||^2 < 0$ is valid. This geometry is smooth everywhere and describes a wormhole, since there is a $\mathbb{Z}_2$-symmetry that acts as follows

$$r^{D-2} \to \tfrac{-||v||^2}{8(D-1)(D-2)} r^{-(D-2)}\,, \tag{6.2.16}$$

and interchanges the two asymptotic regions. The harmonic function is given by

$$h(r) = \sqrt{-\tfrac{8(D-1)}{(D-2)||v||^2}}\,\arctan\Big(\sqrt{\tfrac{-||v||^2}{8(D-1)(D-2)}}\; r^{-(D-2)}\Big) + b\,. \tag{6.2.17}$$

For a very nice definition of a wormhole and its geometrical structure we refer the reader to [161].

6.3 Geodesic Curves

Our method to understanding *all* geodesic curves, traced out by scalars falling into G/H or G/H^* nonlinear σ-models which arise in pure Kaluza-Klein theories and maximally exetended supergravities, is by constructing the *generating solution.* This is what we call a solution-generating technique. By definition, a generating solution is a geodesic with the minimal number of arbitrary integration constants so that the action of the isometry group G generates all other geodesics from the generating solution. In [E] we proved a theorem states that for G/H, where G is a maximally non-compact real form of a complex semi-simple group and H is the maximal compact subgroup, the generating solution can be taken to be the straight line through the origin carried only by the dilaton fields.

■-Proof
We have seen in chapter 5 that the Riemannian symmetric spaces G/H can be parameterized by the Borel coordinate system, namely by scalar fields which are divided into dilatons ϕ^I and axions χ^α. This can be done by choosing the coset representative to be

$$L = \Pi_I \exp[\frac{1}{2}\phi^I H_I]\Pi_\alpha \exp[\chi^\alpha E_\alpha], \tag{6.3.1}$$

where H_I are the generators spanning the Cartan subalgebra CSA of the Lie algebra of G, and the E_α are the positive root generators. We know that the dimension of the CSA is the rank r of G and for the symmetric spaces G/H (moduli spaces) listed in the table 5.4.1, the rank is $r = 11 - D$. Since the Lie algebra of H is spanned by the combinations $E_\alpha - E_{-\alpha}$, the number of axions equals the dimensions of H.

In the Borel gauge the geodesic equation takes on the form

$$\ddot{\phi}^I + \Gamma^I{}_{JK}\dot{\phi}^J\dot{\phi}^K + \Gamma^I{}_{\alpha J}\dot{\chi}^\alpha\dot{\phi}^J + \Gamma^I{}_{\alpha\beta}\dot{\chi}^\alpha\dot{\chi}^\beta = 0, \tag{6.3.2}$$

$$\ddot{\chi}^\alpha + \Gamma^\alpha{}_{JK}\dot{\phi}^J\dot{\phi}^K + \Gamma^\alpha{}_{\beta J}\dot{\chi}^\beta\dot{\phi}^J + \Gamma^\alpha{}_{\beta\gamma}\dot{\chi}^\beta\dot{\chi}^\gamma = 0. \tag{6.3.3}$$

At points for which $\chi^\alpha = 0$, the components $\Gamma^I{}_{JK}$ and $\Gamma^\alpha{}_{JK}$ of the Christoffel symbol on the moduli space G/H vanish. Therefore a trivial solution can be found

$$\phi^I = v^I t, \qquad \chi^\alpha = 0, \tag{6.3.4}$$

for some parameter t. How many other solutions are there? A first thing we notice is that every global G-transformation $\Phi \to \tilde{\Phi}$ brings us from one solution to another solution. Since G generically mixes dilatons and axions we can construct solutions with non-trivial axions in this way.

Let us now prove that in this way *all* geodesics are obtained and that depends on the fact that G is maximally non-compact with H the maximal compact subgroup of G. Consider an arbitrary geodesic curve $\Phi(t)$ on G/H. The point $\Phi(0)$ can be mapped to the origin $L[\Phi(t)] = \mathbb{1}$ using G-transformation (in the fundamental representation), since we can identify $\Phi(0)$ with an element of G and then we multiply the geodesic curve $\Phi(t)$ with $\Phi(0)^{-1}$, generating a new geodesic curve $\Phi_2(t) = \Phi(0)^{-1}\Phi(t)$ that goes through the origin. The origin is invariant under H-rotations but the tangent space at the origin transforms under the adjoint representation (Adj) of H. One can prove that there always exists an element $k \in H$, such that $Adj_k\dot{\Phi}_2(0) \in CSA$ [167]. Therefore $\dot{\chi}^\alpha = 0$ and this solution must be straight line. In short, we started out with a general curve $\Phi(t)$ and proved that the curve $\Phi_3(t) = k\Phi(0)^{-1}\Phi(t)$ is a straight line

$$\phi^I(t) = v^I h(t), \qquad \chi^\alpha = 0, \tag{6.3.5}$$

where $h(t)$ is the harmonic function obtained by the reparametrization 6.2.3 for $\rho = t$ (time-dependent S-brane solutions are carried by the scalars $\Phi(t)$). This is the end of the proof. ■

For the sake of illustration, let us give a counting argument to advocate the theorem. The number of integration constants in the geodesic equation of motion equals $2\times$ (dim[Borelalgebra]) since for every scalar field ϕ_I , χ_α we have to specify the initial speed and place. If we classify geodesic τ by the couple

$$\tau = (\vec{v}, g), \qquad \vec{v} \in CSA, \qquad g \in G, \tag{6.3.6}$$

then the number of parameters is indeed dimCSA+dim$G = 2\times$(dim[Borel] under the conditions on G/H stated. Thus the number of dilatons is given by $r = \text{rank}G$, and the number of axions by dim$H$. Nonetheless, one can not forget about the theorem proved above and consider this counting as a proof since it may be that the action of G does not create independent integration constants.

Unfortunately, due to the fact that the Borel gauge is not a valid gauge on pseudo-Riemannian spaces G/H^*, the derivation of the generating geodesic is no longer feasible by working on the level of the coset representative L (unless one can find a "good" gauge choice). Instead one can work on the level of the matrix $\mathcal{M}$ (see section 5.2.5) so that one does not need to be bothered with the subtleties regarding the gauge choice. Note that the counting argument given above is not a trustful proof for G/H^* either since it might happen that the solutions lie in disconnected areas of the moduli spaces. There the straight line solution is not generating since the affine velocity is positive:

$$||v||^2 = \sum_I (v^I)^2 > 0. \tag{6.3.7}$$

The affine velocity is invariant under G-transformation and by transforming the straight line we only generate spacelike geodesics ($||v||^2 > 0$). But cosets with non-compact H^* (see table 5.4.1 for examples in maximal supergravity) have metrics with an indefinite signature and therefore allow for lightlike, spacelike and timelike geodesics.

In the following we will work directly with the matrix $\mathcal{M}$ or $\hat{\mathcal{M}}$ (defined in 5.3.16), restricting to pure Kaluza-Klein theories, i.e. G will be the group $\mathrm{GL}(n,\mathbb{R})$ for $D > 3$ or $\mathrm{SL}(n+1,\mathbb{R})$ for $D = 3$ [G]. The approach that we will follow allows us to rederive the straight line generating geodesic of $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n)$ moduli space but also allows for the generalisation to $\mathrm{GL}(s+r=n,\mathbb{R})/\mathrm{SO}(r,s)$. The extension of our approach to cover the symmetric spaces G/H^* of the right column of table 5.4.1, namely of those of the maximally-extended Euclidean supergravities, has been worked out in [G].

6.3.1 The Geodesic Curves of Pure Kaluza-Klein Theory

As mentioned before the (-1)-brane solutions are carried by the metric and the scalar fields and therefore we truncate the Kaluza-Klein vectors in $D > 3$ and dualize them in $D = 3$. The (-1)-brane geometries have just been described above. Let us therefore focus on the geodesic motion that comes about.

The $\mathrm{SL}(n,\mathbb{R})/\tilde{H}$ nonlinear σ-model action of the geodesic curves can be compactly written in terms of the symmetric coset matrix $\mathcal{M}$

$$S = \int dh\, \mathrm{tr}[\partial_h \mathcal{M} \partial_h \mathcal{M}^{-1}], \qquad \mathcal{M} = L^T \eta L, \tag{6.3.8}$$

where $\tilde{H}$ can be $H = \mathrm{SO}(n)$ or $H^* = \mathrm{SO}(n-1,1)$, $\mathrm{SO}(n-2,2)$. The matrix η is the $\tilde{H}$-invariant matrix. The corresponding equations of motion can compactly be written as

$$[\mathcal{M}^{-1}\mathcal{M}']' = 0, \tag{6.3.9}$$

where the prime denotes the derivative with respect to the affine parameter $h(\rho)$. This implies that $\mathcal{M}^{-1}\mathcal{M}' = K$ with K a constant matrix, which can be seen as the matrix of Noether charges of the group $\mathrm{SL}(n,\mathbb{R})$ (see section 5.2.2). The affine velocity squared of the geodesic curves is $||v||^2 = \frac{1}{2}\mathrm{tr}[K^2]$. Because of the identity $\mathcal{M}^{-1}\mathcal{M}' = K$, the problem is integrable and a general solution is found to be

$$\mathcal{M}(h) = \mathcal{M}(0) e^{K h(\rho)}. \tag{6.3.10}$$

By virtue of the transitive action of the isometry group $\mathrm{SL}(n,\mathbb{R})$ on the symmetric space, we can restrict ourselves to geodesics that go through the origin. Since one has the freedom of affine reparametrization of h we can assume that $L(0) = \mathbb{1}$. The matrix of Noether charges is not completely arbitrary, it is actually subject to a constraint which can be derived from the properties of $\mathcal{M}$

$$\eta K = K^T \eta, \qquad \mathrm{tr}[K] = 0. \tag{6.3.11}$$

K is an element of the Lie algebra of $\mathrm{SL}(n,\mathbb{R})$ and accordingly it transforms in the adjoint of $\mathrm{SL}(n,\mathbb{R})$

$$K \to \Omega K \Omega^{-1}, \tag{6.3.12}$$

under which the $(n-1)$ quantities

$$\mathcal{I}_k = \text{tr}[K^{k+1}], \qquad \text{with } k = 1, \cdots, n-1 \tag{6.3.13}$$

are invariant, i.e. are Casimirs. Notice that the constraints 6.3.11 are not invariant under the total isometry group but under the smaller isotropy group $\tilde{H}$.

We here stress that for a pure Kaluza-Klein theory in $D > 3$ all geodesics that are related through a $\text{GL}(n, \mathbb{R})$-transformation lift up to exactly the same vacuum (pure gravity) solution in $D + n$ dimensions since the $\text{GL}(n, \mathbb{R})$ corresponds to rigid coordinate transformations from a $(D + n)$-dimensional point of view. So, in this sense it is absolutely necessary to understand the generating geodesic since it classifies higher dimensional solutions modulo coordinate transformations. Of course, this is not true for $D = 3$ where $\text{SL}(n + 1, \mathbb{R})$ maps higher dimensional solutions to each other that are not necessarily related by coordinate transformations.

Due to the non-compactness of the isotropy group such as $\text{SO}(n-1, 1)$, the theory will contain ghosts. A ghost, as previously mentioned, is an axion field with the opposite sign for the kinetic term in the Lagrangian. For the sake of generality, let us discuss the ghost content for a theory with scalar coset $\text{GL}(r + s, \mathbb{R})/\text{SO}(r, s)$.

For a general symmetric space $\text{GL}(s+r)/\text{SO}(r, s)$ the number of axion fields with the opposite sign for the kinetic term (ghosts) is $r \times s$. For the pure Kaluza-Klein moduli spaces this can be seen as follows. When one considers a reduction over time then there are two possible origins for ghosts. Ghost fields χ^Λ appear as the zero-component of a one-form $\hat{A}^\Lambda$ in the higher dimension, that is $\hat{A}^\Lambda = \chi^\Lambda dt + A^\Lambda$. Alternatively, extra ghost fields appear in three dimensions upon dualisation of the one-form[6]. Therefore, imagine we reduce Einstein gravity in $D + n$ to dimensions to $D+1$ dimensions over a spacelike torus and then perform a subsequent reduction over a timelike circle, then the $n-1$ Kaluza-Klein vectors in $D+1$ dimensions give $n-1$ ghostlike axions. This fits with the fact that the scalar coset is $\text{GL}(n, \mathbb{R})/\text{SO}(n-1, 1)$. If $D = 3$ then we can further dualise those $n - 1$ descendants of the Kaluza-Klein vectors to $n-1$ ghostlike axions, thereby doubling the number of ghosts. The Kaluza-Klein vector that appears from the last timelike reduction does not dualise to a ghost but to a normal axion since that vector appeared with a wrong sign in three dimensions. This indeed explains why there are $2(n-1)$ ghosts in $\text{SL}(n+1)/\text{SO}(n-1, 2)$.

Since the matrix K determines all geodesics through the origin, and by transitivity all geodesics on $G/\tilde{H}$ we will look for the *normal form* K_N of K under $(\tilde{H} \subset G)$-transformations. As a result the geodesics will be determined by the "integration constants" in K_N, generating *all* geodesics through a rigid G-transformation[7]. A matrix normal form or matrix canonical form describes the transformation of a matrix

[6] This is due to the fact that the three-dimensional theory is Euclidean.

[7] In the discussion section we will briefly mention another approach to generate geodesic solutions, this approach uses the local isotropy H.

to another with special properties. For instance the normal form of a symmetric matrix is the diagonal matrix which can be realized by the action of the orthogonal groups. Since we have restricted to geodesics that go through the origin- a point which is invariant under the action of the isotropy $\tilde{H}$- we see that one can always find matrices K_N having all possible combinations of values of the invariants $\mathcal{I}_k$, namely one can restrict the $n-1$ invariants of K on K_N and hence any $\tilde{H}$-orbit has one element of the form K_N. In other words we can always transform a generic K into K_N through an $\tilde{H}$ transformation. Below we derive the normal forms of the matrix K associated to the generating geodesic curves on the GL$(n,\mathbb{R})$/SO(n), GL$(n,\mathbb{R})$/SO$(n-1,1)$, and GL$(n+1,\mathbb{R})$/SO$(n-1,2)$ moduli spaces[8]. We are able to give a compact proof for the general case GL$(r+s,\mathbb{R})$/SO(r,s), instead of a case by case discussion. It is worth recalling that GL-cosets are parameterized by the matrix $\hat{\mathcal{M}}$ (see section 5.3.16) rather than $\mathcal{M}$.

The Normal Form of $\mathfrak{gl}(r+s)/\mathfrak{so}(r,s)$

Consider $K \in \mathfrak{gl}(r+s)/\mathfrak{so}(r,s)$, where $\mathfrak{gl}$ and $\mathfrak{so}$ are respectively the Lie algebras of GL and SO. By definition K obeys 6.3.11 (tr$[K] \neq 0$)

$$\eta K = K^T \eta\,, \quad \text{with} \quad \eta = (-\mathbb{1}_r, +\mathbb{1}_s)\,. \tag{6.3.14}$$

Two eigenvectors of K, $\mathbf{v_1}$ and $\mathbf{v_2}$, that belong to different eigenvalues λ_1 and λ_2 are necessarily orthogonal with respect to the inner product $(\,,\,)$ defined with the bilinear form η, because $(\mathbf{v_2}, K\mathbf{v_1}) = (K\mathbf{v_2}, \mathbf{v_1})$ and thus $\lambda_1(\mathbf{v_1}, \mathbf{v_2}) = \lambda_2(\mathbf{v_1}, \mathbf{v_2})$. Now if $\lambda_1 \neq \lambda_2$ then this is only consistent if $(\mathbf{v_1}, \mathbf{v_2}) = 0$. We will say that $\mathbf{v_1}$ and $\mathbf{v_2}$ are *pseudo-orthogonal*. If two eigenvectors have the same eigenvalue we can always perform a generalized Gramm–Schmidt procedure so that they become pseudo-orthogonal with respect to η. In general K may not be diagonalizable. This is the case for instance if K is nilpotent, namely if $K^k = 0$ for some $k > 1$. Or proof applies for diagonalizable matrices only. By definition $\mathcal{M}$, see eq. 6.3.10, and therefore K, should always be real matrices and thus if λ is a complex eigenvalue of K also $\bar{\lambda}$ is. Let $\mathbf{v}$ and $\bar{\mathbf{v}}$ be the corresponding eigenvectors. If we write $\mathbf{v} = \mathbf{v_1} + \mathbf{i}\mathbf{v_2}$ and $\lambda = \lambda_1 + i\lambda_2$ then this means that

$$K\mathbf{v_1} = \lambda_1 \mathbf{v_1} - \lambda_2 \mathbf{v_2}\,, \qquad K\mathbf{v_2} = \lambda_2 \mathbf{v_1} + \lambda_1 \mathbf{v_2}\,, \tag{6.3.15}$$

pseudo-orthogonality between $\mathbf{v}$ and $\bar{\mathbf{v}}$ implies $(\mathbf{v_1}, \mathbf{v_1}) = -(\mathbf{v_2}, \mathbf{v_2})$.

In what follows we shall consider the cases in which K is nilpotent as singular limits of diagonalizable matrices, construct a normal form K_N of K and then show that the resulting normal form also encodes the most general nilpotent case[9]. In a

[8] Here the GL$(n,\mathbb{R}) = \mathbb{R}^+ \times$ SL$(n,\mathbb{R})$ with $\mathbb{R}^+$ is associated with the breathing mode.

[9] A nilpotent matrix is an $n \times n$ square matrix K such that $K^m = 0$ for some positive integer m.

suitable basis constructed out of the real and imaginary parts of $\mathbf{v}$, the matrix K is represented by a 2×2 real block. We shall construct a basis out of the eigenvectors of a diagonalizable K in which K has a block diagonal form K_N, with a real 2×2 block for each couple of λ, $\bar{\lambda}$ eigenvalues, and single diagonal entry for each real eigenvalue. K_N will satisfy the same property 6.3.14 as K and can be obtained from it through an $\mathrm{SO}(r,s)$ conjugation. The general form of K_N is thus characterized by the maximal number of complex eigenvalues. In the following we shall construct K_N for the coset $\mathrm{GL}(r+s,\mathbb{R})/\mathrm{SO}(r,s)$ and show that the maximal number of complex eigenvalues is $\min(r,s)$[10].

Construction of K_N

Let $K\in \mathfrak{gl}(n,\mathbb{R})/\mathfrak{so}(r,s)$, with $n=r+s$ and $r\le s$, be an $n\times n$ real matrix satisfying eq. 6.3.14. We want to show that K *can have at most r complex distinct eigenvalues.*

Let $\mathbf{v}=(\vec{v},\vec{w})$ denote a vector in $\mathbb{R}^{r,s}$ with $\vec{v}\in\mathbb{R}^r$ and $\vec{w}\in\mathbb{R}^s$. We start showing that a set $\mathbf{v}^{(i)}=(\vec{v}^{(i)},\vec{w}^{(i)})$, $i=1,\dots,\ell+1$, of mutually pseudo-orthogonal, null-vectors, are linearly independent iff the vectors $(\vec{v}^{(i)})$ are. Suppose $(\vec{v}^{(i)})_{i=1,\dots,\ell}$ are linearly independent but that $\vec{v}^{(\ell+1)}$ can be expressed as a linear combination of the first ℓ: $\vec{v}^{(\ell+1)}=\sum_{i=1}^{\ell}a^i\,\vec{v}^{(i)}$. Let us show that $\mathbf{v}^{(\ell+1)}=\sum_{i=1}^{\ell}a^i\,\mathbf{v}^{(i)}$. By hypothesis $(\mathbf{v}^{(i)})_{i=1,\dots,\ell+1}$ are light-like and mutually pseudo-orthogonal:

$$(\mathbf{v}^{(i)},\,\mathbf{v}^{(j)}) \;=\; 0 \;\Leftrightarrow\; \vec{v}^{(i)}\cdot\vec{v}^{(j)}=\vec{w}^{(i)}\cdot\vec{w}^{(j)}\;,\;\;\forall i,j=1,\dots,\ell+1\,. \tag{6.3.16}$$

Being $(\vec{v}^{(i)})_{i=1,\dots,\ell}$ linearly independent, we can define the following non singular matrix h_{ij}:

$$h_{ij} \;=\; \vec{v}^{(i)}\cdot\vec{v}^{(j)}=\vec{w}^{(i)}\cdot\vec{w}^{(j)}\;,\;\;i,j=1,\dots,\ell\,. \tag{6.3.17}$$

The vector $\vec{w}^{(\ell+1)}$ will have the general form: $\vec{w}^{(\ell+1)}=\sum_{i=1}^{\ell}c^i\,\vec{w}^{(i)}+\vec{w}_\perp$, where $\vec{w}^{(i)}\cdot\vec{w}_\perp=0$, $\forall i=1,\dots,\ell$. Then from the pseudo-orthogonality condition 6.3.16 we find that $c^j\,h_{ij}=\vec{w}^{(i)}\cdot\vec{w}^{(\ell+1)}=\vec{v}^{(i)}\cdot\vec{v}^{(\ell+1)}=a^j\,h_{ij}$, from which we conclude that $c^i=a^i$, $i=1,\dots,\ell$. From this it follows that $\mathbf{v}^{(\ell+1)}=\sum_{i=1}^{\ell}a^i\,\mathbf{v}^{(i)}+(\vec{0},\,\vec{w}_\perp)$. Requiring $\mathbf{v}^{(\ell+1)}$ to be null, we find $\vec{w}_\perp\cdot\vec{w}_\perp=0$, which implies $\vec{w}_\perp=\vec{0}$ and thus $\mathbf{v}^{(\ell+1)}=\sum_{i=1}^{\ell}a^i\,\mathbf{v}^{(i)}$. Therefore if $(\vec{v}^{(i)})_{i=1,\dots,\ell+1}$ are not linearly independent, the same is true for $(\mathbf{v}^{(i)})_{i=1,\dots,\ell+1}$. It is straightforward to show the reverse, namely that if $(\vec{v}^{(i)})_{i=1,\dots,\ell+1}$ are linearly independent, also $(\mathbf{v}^{(i)})_{i=1,\dots,\ell+1}$ are.

Let us now suppose that K has $r+1$ distinct complex eigenvalues $\lambda_i\neq\bar{\lambda}_i$, $i=1,\dots,r+1$, and let $\mathbf{v}^{(i)}=\mathbf{v}_1^{(i)}+i\,\mathbf{v}_2^{(i)}$ be the corresponding linearly independent eigenvectors. Let us show that $\mathbf{v}^{(r+1)}$ is either zero or a linear combination of the

[10] This number of distinct complex eigenvalues of K_N will turn out to be equal to the number of ghostlike scalar fields that will be existing in the geodesics generating solution $\hat{\mathcal{M}}_N$.

first r eigenvectors, contradicting thus the hypothesis. From the general analysis we know that

$$(\mathbf{v}_{1,2}^{(i)},\,\mathbf{v}_{1,2}^{(j)}) \;=\; 0\;,\;\; i\neq j\;\;,\quad (\mathbf{v}_{1}^{(i)},\,\mathbf{v}_{1}^{(i)})=-(\mathbf{v}_{2}^{(i)},\,\mathbf{v}_{2}^{(i)})\,. \tag{6.3.18}$$

We shall consider the case in which if $\mathbf{v}_1^{(i)}$ is null for some i, $\mathbf{v}_1^{(i)}\cdot\mathbf{v}_2^{(i)}\neq 0$ [11]. Under these assumptions the linear independence of $\mathbf{v}^{(i)}$ implies the linear independence of $\mathbf{v}_1^{(i)}$. If all $\mathbf{v}_1^{(i)}$ (and thus $\mathbf{v}_2^{(i)}$) were null-vectors, then we would have $r+1$ independent, mutually pseudo-orthogonal null vectors. This would imply that the $r+1$ components $\vec{v}_1^{(i)}$ were independent, which can not be, being them r-dimensional vectors. Consider now the case in which some of the $\mathbf{v}_1^{(i)}$ are timelike, or spacelike. We can define the eigenvectors in such a way that $\mathbf{v}_1^{(a)}=(\vec{v}^{(a)},\,\vec{w}^{(a)})$, $a=1,\dots,\ell$, are null-vectors, while $\mathbf{v}_1^{(r)}$, $r=\ell+1,\dots,r+1$, are timelike. Moreover we can use $\mathrm{SO}(r,s)$ to write the timelike vectors in the form $\mathbf{v}_1^{(r)}=(\vec{v}^{(r)},\,\vec{0})$, $(\vec{v}^{(r)})$ being mutually orthogonal. Therefore the matrices $h_{rs}=\vec{v}^{(r)}\cdot\vec{v}^{(s)}$ and $h_{ab}=\vec{v}^{(a)}\cdot\vec{v}^{(b)}$ are non–singular matrices. Suppose $\mathbf{v}_1^{(r+1)}=(\vec{v}^{(r+1)},\,\vec{0})$, and thus $\vec{v}^{(r+1)}$, depends linearly on the first r vectors:

$$\vec{v}^{(r+1)} \;=\; c^a\,\vec{v}^{(a)}+c^r\,\vec{v}^{(r)}\,, \tag{6.3.19}$$

from the pseudo–orthogonality condition we find:

$$\begin{aligned} 0 &= \mathbf{v}_1^{(r)}\cdot\mathbf{v}_1^{(r+1)}=\vec{v}^{(r)}\cdot\vec{v}^{(r+1)}=h_{rs}\,c^s \;\Rightarrow\; c^r=0\,,\\ 0 &= \mathbf{v}_1^{(a)}\cdot\mathbf{v}_1^{(r+1)}=\vec{v}^{(a)}\cdot\vec{v}^{(r+1)}=h_{ab}\,c^b \;\Rightarrow\; c^b=0\,, \end{aligned} \tag{6.3.20}$$

which implies that $\mathbf{v}_1^{(r+1)}=0$.

Consider then a matrix with $2r$ distinct complex eigenvalues and $s-r$ real. We can construct the following basis of pseudo-orthonormal vectors $\mathbf{u}_{1,2}^{(i)}$ defined as follows

$$\begin{aligned} \mathbf{v}_1^{(i)}\cdot\mathbf{v}_1^{(i)} &= 1\;:\quad \mathbf{u}_1^{(i)}=\mathbf{v}_1^{(i)}\;\;,\;\; \mathbf{u}_2^{(i)}=\sin\alpha_i\,\mathbf{v}_1^{(i)}+\cos\alpha_i\,\mathbf{v}_2^{(i)}\,,\\ |\mathbf{v}_1^{(i)}|^2 &= 0\;:\quad \mathbf{u}_1^{(i)}=\frac{\mathbf{v}_1^{(i)}\pm\mathbf{v}_2^{(i)}}{\sqrt{2\,|\mathbf{v}_1^{(i)}\cdot\mathbf{v}_2^{(i)}|}}\;\;,\;\; \mathbf{u}_2^{(i)}=\frac{\mathbf{v}_1^{(i)}\mp\mathbf{v}_2^{(i)}}{\sqrt{2\,|\mathbf{v}_1^{(i)}\cdot\mathbf{v}_2^{(i)}|}}\,, \end{aligned} \tag{6.3.21}$$

where we have denoted by $\tan(\alpha_i)=-\mathbf{v}_1^{(i)}\cdot\mathbf{v}_2^{(i)}$. In the last of the above equations the upper and lower signs refer to the cases in which $\mathbf{v}_1^{(i)}\cdot\mathbf{v}_2^{(i)}$ is positive and negative respectively. Define now the following matrix

$$T \;=\; (\mathbf{u}_1^{(1)},\,\mathbf{u}_2^{(1)},\dots,\mathbf{u}_1^{(r)},\,\mathbf{u}_2^{(r)},\mathbf{u}^{(k)})\,, \tag{6.3.22}$$

[11] If this were not the case one can show that the corresponding eigenvalue would be degenerate and the matrix not diagonalizable.

where $\mathbf{u}^{(k)}$ are the $s-r$ spacelike (normalized) eigenvectors corresponding to the real eigenvalues λ^k. The matrix T is in $\mathrm{SO}(r,s)$:

$$T^T\,\eta'\,T \;=\; \eta' \;,\quad \eta' = \mathrm{diag}(\overbrace{+,-,\dots,+,-}^{2r},\overbrace{-,\dots,-}^{s-r})\,. \tag{6.3.23}$$

Upon action of T, the matrix K will acquire the following normal form K_N:

$$K_N \;=\; T^{-1}\,K\,T = \begin{pmatrix} A_1 & 0 & \dots & & & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & A_r & 0 & & \\ & & 0 & \lambda_{r+1} & & \\ & & & & \ddots & \vdots \\ 0 & \dots & & & \dots & \lambda_s \end{pmatrix}\,, \tag{6.3.24}$$

where $A_1,\dots,A_r$ are 2×2 blocks corresponding to each complex eigenvalue $\lambda_1,\dots,\lambda_r$ whose for is

$$\mathbf{v}_1^{(i)}\cdot\mathbf{v}_1^{(i)} \;=\; 1 \;:\quad A_i = \begin{pmatrix} \lambda_1^{(i)} + \lambda_2^{(i)}\tan(\alpha_i) & -\lambda_2^{(i)}\cos^{-1}(\alpha_i) \\ \lambda_2^{(i)}\cos^{-1}(\alpha_i) & -\lambda_2^{(i)}\tan(\alpha_i) + \lambda_1^{(i)} \end{pmatrix}\,,$$

$$\mathbf{v}_1^{(i)}\cdot\mathbf{v}_1^{(i)} \;=\; 0 \;:\quad A_i = \begin{pmatrix} \lambda_1^{(i)} & \pm\lambda_2^{(i)} \\ \mp\lambda_2^{(i)} & \lambda_1^{(i)} \end{pmatrix}\,. \tag{6.3.25}$$

The remaining eigenvalues $\lambda_{r+1},\dots,\lambda_s$ in K_N are real. On each block A_i we can still act by means of r-independent $\O(1,1)$ transformations which may set α_i either to 0 or to π. Although K_N also describes non diagonalizable matrices, like for instance in the case $\lambda_1 = 0$, $\lambda_2 = a\cos(\alpha)$ and $\alpha = \pi/2$, in which case the block A is nilpotent, we the above theorem holds for diagonalizable matrices only. If K is not diagonalizable, its normal form can be expressed by the normal form $K_N^{(0)}$ of the diagonalizable matrix $K^{(0)}$ having the same spectrum as K, plus a *constant nilpotent matrix* which interpolates between the blocks corresponding to the same degenerate eigenvalue, the spectrum being encoded in $KK^{(0)}$. For instance an example of the normal form on a non-diagonalizable matrix in $\mathfrak{sl}(4,\mathbb{R})/\mathfrak{so}(2,2)$ with a 2 times degenerate complex eigenvalue $\lambda \neq \bar{\lambda}$ is:

$$K_N \;=\; K_N^{(0)} + Nil\,,$$

$$K_N^{(0)} \;=\; \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\,,\quad Nil = \begin{pmatrix} \mathrm{Id}_2 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & -\mathrm{Id}_2 \end{pmatrix}\,,\quad A = \begin{pmatrix} \lambda_1 & \lambda_2 \\ -\lambda_2 & \lambda_1 \end{pmatrix}\,. \tag{6.3.26}$$

We come to the conclusion that there is a subset of K's (of smaller dimension than the whole space of K matrices) that is not "diagonalizable" when the eigenvalues of the charge matrix are degenerate. In the following, we will restrict to the diagonalizable K : the absence of the constant nilpotent part Nil.

6.4 Uplift to Einstein Vacuum Solutions

In order to uplift the solutions from $D > 3$ dimensions to $D + n$ (n=p+1, the word-volume dimensions of p-brane) dimensions one uses the Kaluza–Klein Ansatz 5.3.13 where with Kaluza–Klein vectors put to zero

$$\mathrm{d}s^2_{D+p+1} = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s^2_D + \mathrm{e}^{2\beta\varphi}\mathcal{M}_{mn}\mathrm{d}z^m\mathrm{d}z^n\,. \tag{6.4.1}$$

Consider the symmetric coset matrix $\hat{\mathcal{M}}(h) = \eta \exp K_N h$ with K_N the normal form of K that generates all other geodesics and h the harmonic function defined in 6.2.3. The relation between $\hat{\mathcal{M}}$ and the moduli φ and $\mathcal{M}$ is as follows

$$\hat{\mathcal{M}} = (|\mathrm{det}\hat{\mathcal{M}}|)^{\frac{1}{n}}\mathcal{M}\,, \qquad |\mathrm{det}\hat{\mathcal{M}}| = \exp\sqrt{2n}\varphi\,. \tag{6.4.2}$$

In the following we will present the vacuum solutions obtained from uplifting the (-1)-brane solutions of pure Kaluza–Klein theory. This is a nice illustration of the power of the sigma-model technique since we construct the solutions in a purely algebraic way. Solving the second-order differential equations for such a vacuum Ansatz with this degree of complexity is highly non-trivial. It is worth recalling that we will not make use of a coordinate system on the cosets, so in the case of a non-compact isotropy group H^* we do not need to be bothered with subtleties regarding the Borel gauge. Note that for uplifting in $D = 3$ one has to take into account the Kaluza-Klein vectors.

6.4.1 Time-dependent Solutions

Here we consider the time-dependent (-1)-brane solutions in D dimensions and their uplifts over a $p + 1$-torus to Sp-brane solutions of $D + p + 1 = d$-dimensional pure gravity.

Sp-brane from GL$(p + 1 = n, \mathbb{R})$/SO(n)

In section 6.3 we showed that the most general geodesic on $\mathrm{GL}(p + 1 = n, \mathbb{R})/\mathrm{SO}(n)$ is given by the most general $\mathrm{GL}(p + 1, \mathbb{R})$-transformation of the generating straight line solution through the origin. Alternatively, by using the relation 6.3.10 and the normal form K_N (defined above) one can obtain the same result, namely

$$\hat{\mathcal{M}}(h) = \begin{pmatrix} \mathrm{e}^{\lambda_1 h} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathrm{e}^{\lambda_n h} \end{pmatrix}\,, \tag{6.4.3}$$

with h given by 6.2.13.

The scalar field matrix transforms as $\mathcal{M} \to \Omega^T\mathcal{M}\Omega$ with $\Omega \in \mathrm{SL}(p + 1, \mathbb{R})$.

Therefore all we need is to uplift the straight line geodesic since all the other geodesics are just Ω-transformations which can be absorbed by redefining the torus coordinates $d\vec{z}' = \Omega d\vec{z}$. The higher-dimensional geometries we find depend on the curvature k of the lower-dimensional FLRW-space.

If we take the (-1)-brane geometry with $k = 0$, then the generating solution lifts up to the Kasner solutions with [E]

$$\mathrm{d}s^2 = -\tau^{2p_0}\mathrm{d}\tau^2 + \sum_b \tau^{2p_b}\mathrm{d}x_b^2\,, \qquad b = 1, \ldots, D+n-1\,. \tag{6.4.4}$$

where the power-laws are defined by

$$p_0 = (D-2) + \frac{\alpha \sum_i \lambda_i}{\sqrt{2an}}\,, \qquad p_1 = \ldots = p_{D-1} = 1 + \frac{\alpha \sum_i \lambda_i}{\sqrt{2an}}\,, \tag{6.4.5}$$

$$p_{D+i-1} = \frac{\sum_i \lambda_i}{2\sqrt{a}}\left(\frac{2\beta}{\sqrt{2n}} - \frac{1}{n}\right) + \frac{\lambda_i}{2\sqrt{a}}\,, \tag{6.4.6}$$

where we defined a in equation 6.2.12 and use that $||v||^2 = \frac{1}{2}\sum_i \lambda_i^2$. The numbers p obey the Kasner constraints

$$p_0 + 1 = \sum_{b>0} p_b\,, \quad (p_0+1)^2 = \sum_{b>0} p_b^2\,. \tag{6.4.7}$$

When we take slicing with $k = -1$, we obtain a generalization of the fluxless S-brane solutions [147,153,168]. For $k = +1$ the solutions are not given a special name. Uplifting the straight line gives a generalization of the fluxless solutions considered in for instance [169]. Those solutions are the familiar Kasner solutions. We present the solutions with $k = \pm 1$.

$$\mathrm{d}s^2 = W^{p_0}\Big(-\frac{\mathrm{d}t^2}{at^{-2(D-2)} - k} + t^2\mathrm{d}\Sigma_k^2\Big) + \sum_{i=1}^{n} W^{p_i}(\mathrm{d}z^i)^2\,, \tag{6.4.8}$$

where the function $W(t)$ is defined as

$$W(t) = \sqrt{a}t^{2-D} + \sqrt{at^{2(D-2)} - k}\,, \tag{6.4.9}$$

and the various constants p_0, p_i are defined as

$$p_0 = -\frac{\sum_i \lambda_i}{||v||(D-2)}\sqrt{\tfrac{2(D-1)}{(D+n-2)}}\,, \qquad p_i = -\frac{D-2}{n}p_0 + \frac{(\sum_j \lambda_j - n\lambda_i)}{n||v||}\sqrt{\tfrac{2(D-1)}{D-2}}\,, \tag{6.4.10}$$

and the affine velocity is given by $||v||^2 = \frac{1}{2}\sum_i \lambda_i^2$.

Note that the $k = -1$ solutions (S-branes) approach flat Minkowski space in Milne coordinates for $t \to \infty$.

Time-dependent Solutions from SL$(n+1,\mathbb{R})$/SO$(n+1)$

If we reduce to three dimensions a symmetry-enhancement of the coset takes place. The dualisation of the three-dimensional KK vectors generate the coset SL$(n+1,\mathbb{R})/\mathrm{SO}(n+1)$ instead of the expected GL$(n,\mathbb{R})/\mathrm{SO}(n)$. However the generating solution of the SL$(n+1,\mathbb{R})/\mathrm{SO}(n+1)$-coset has only non-trivial dilatons and is therefore the same as the generating solution of GL$(n,\mathbb{R})/\mathrm{SO}(n)$. Nonetheless, there is an important difference with the time-dependent solutions from GL$(n,\mathbb{R})/\mathrm{SO}(n)$. In that case a solution-generating transformation $\in \mathrm{GL}(n,\mathbb{R})$ can be interpreted as a coordinate transformation in $D+n$ dimensions and therefore maps the vacuum solution to the same vacuum solution in different coordinates. In the case of symmetry enhancement to SL$(n+1,\mathbb{R})$ a solution-generating transformation is not necessarily a coordinate transformation in $D+n$ dimensions. Instead the time-dependent vacuum solution transforms into a "twisted" vacuum solution. Where the twist indicates off-diagonal terms that cannot be redefined away. Such twisted solutions with $k=-1$ have received considerable interest since they can be regular [156, 157].

6.4.2 Stationary Solutions from GL$(n,\mathbb{R})$/SO$(n-1,1)$

The normal form is given by

$$K_N = \begin{pmatrix} \lambda_a & \omega & 0 & \dots & 0 \\ -\omega & -\lambda_a & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} + \begin{pmatrix} \lambda_b & 0 & 0 & \dots & 0 \\ 0 & \lambda_b & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}. \qquad (6.4.11)$$

We exponentiate this to $\hat{\mathcal{M}}(h(r)) = \eta \mathrm{e}^{K_N h(r)} =$

$$\begin{pmatrix} -\mathrm{e}^{\lambda_b h(r)} f_+(r) & -\omega \mathrm{e}^{\lambda_b h(r)} \Lambda^{-1} \sinh(\Lambda h(r)) & 0 & \dots & 0 \\ -\omega \mathrm{e}^{\lambda_b h(r)} \Lambda^{-1} \sinh(\Lambda h(r)) & \mathrm{e}^{\lambda_b h(r)} f_-(r) & 0 & \dots & 0 \\ 0 & 0 & \mathrm{e}^{\lambda_3 h} & \dots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathrm{e}^{\lambda_n h} \end{pmatrix}, \qquad (6.4.12)$$

with

$$f_\pm(r) = \mathrm{e}^{\lambda_b h(r)} \Big(\cosh(\Lambda h(r)) \pm \lambda_a \frac{\sinh(\Lambda h(r))}{\Lambda} \Big), \qquad (6.4.13)$$

and where we define the SO$(1,1)$ invariant quantity Λ as

$$\Lambda = \sqrt{\lambda_a^2 - \omega^2}\,. \qquad (6.4.14)$$

There exist three distinctive cases depending on the character of Λ. If Λ is real the above expression does not need rewriting but we can put ω to zero using a SO(1,1)-boost and then the generating solution is just the straight line solution. If $\Lambda = i\tilde{\Lambda}$ with $\tilde{\Lambda}$ real then the terms with $\cosh(\Lambda h)$ become $\cos\tilde{\Lambda}$ and $\Lambda^{-1}\sinh\Lambda h$ become $\tilde{\Lambda}^{-1}\sin\tilde{\Lambda}h$. Finally, if $\Lambda = 0$ then the term $\Lambda^{-1}\sinh\Lambda h$ becomes just h and the term with $\cosh\Lambda h$ becomes equal to one.

To discuss the zoo of solutions one should make a classification in terms of the different signs for k, $||v||^2$ and Λ^2. We restrict to solutions in spherical coordinates which have $k = +1$. The other solutions can similarly be found. The solutions with spherical symmetry have the more interesting properties that they lift up to vacuum solutions that can be asymptotically flat. These solutions can be found in [G].

6.5 Discussion

In this chapter we studied the (-1)-brane solutions of the Kaluza-Klein theory (KK vectors truncated) that can be obtained from reducing pure gravity over a torus. We introduced the concept of a generating solution. A generating solution is a geodesic with the minimal number of arbitrary integration constants such that the action of isometry group G generates all other geodesics from it. We then presented the theorem that for maximally non-compact Riemannian cosets G/H, with H its maximal compact subgroup, the generating solution can be constructed from the Cartan subalgebra CSA only. In [G] we have shown that the analysis of the generating solution can be extended to G/H^*, where H^* is a non-compact version of H. In this chapter we focused only on the $\mathrm{GL}(r+s,\mathbb{R})/\mathrm{SO}(r,s)$ coset and showed that the number of complex eigenvalues that one needs in order to construct the normal form K_N of the generating solution is at most $\min(r,s)$.

We first illustrated this technique for Sp-branes. That is we studied a Lagrangian contains only Einstein gravity. Reducing this over the worldvolume of the Sp-brane gives the moduli space $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p+1)$. Using the above mentioned theorem we presented the generating geodesic solution, a straight line, for the S(-1) belonging to this coset. We oxidized the time-dependent geodesic solution back to the original higher dimensional theory where we obtained a fluxless Sp-brane. This led to S(-1)-brane /Sp-brane map. If we reduce to three dimensions a symmetry enhancement of the coset takes place. The dualisation of the three-dimensional Kaluza-Klein vectors generates the coset $\mathrm{SL}(p+2,\mathbb{R})/\mathrm{SO}(p+2)$. The uplifting back to pure gravity leads to time-dependent vacuum solution transforming into a twisted vacuum solution.

We like to mention another closely related way to classify geodesics on symmetric spaces G/H when G and H obey the same conditions as above. This mechanism is called the compensator algorithm and is developed by Fré et al. [158,170]. The compensator algorithm offers a way to write down exact solutions for the different scalar fields in an iterative manner that illustrates nicely the integrability of the geodesic equations of motion. More precisely, this method constructs geodesic curves from the straight line by performing a H-local transformation on the tangent space at each point on the straight curve.

Similarly, we used the same technique for the timelike p-brane case. This actually has led to different classes of instantons, labelled by the sign of $||v||^2$. Reducing gravity over the worldvolume of timelike p-brane gives rise to $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p,1)$ σ-model for $D>3$ and $\mathrm{SL}(p+2,\mathbb{R})/\mathrm{SO}(p+1,1)$ for $D=3$, which can be considered as two extensions of the prototype Lorentzian scalar coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$ of Euclidean type IIB supergravity. The geodesic generating solution on $\mathrm{GL}(p+1,\mathbb{R})/\mathrm{SO}(p,1)$ is constructed using our approach. Uplifting this stationary solution back to higher dimensional theory provided us with new solutions which still have to be analyzed. The generating solution for $\mathrm{SL}(p+2,\mathbb{R})/\mathrm{SO}(p+1,1)$ case and its uplifting to vacuum solutions are still under investigation. However, the upliftings of $3D$ instanton solutions to intermediate dimensions, in particular to four-dimensional extremal black hole solutions, have been pointed out first by Breitenlohner et al. [130], and worked out recently by Gaiotto et al. [163]. This is known as *instanton /black hole correspondence*. The authors of [163] have actually derived extremal solutions for a variety of four-dimensional models which, after Kaluza-Klein reduction, admit a description in terms of $3D$ gravity coupled to a σ-model with symmetric target space. The solutions are found to be in correspondence with certain nilpotent generators K of the isometry group G. In particular, they provide the exact solution for a non-BPS black hole with generic charges and asymptotic moduli in $N=2$ supergravity coupled to two vector fields (one vector multiplet). In [G] we extend the analysis of [163], applying our generating solution approach, to obtain extremal and non-extremal black holes in $D=4$, $N=8$ supergravity.

Although in this chapter our analysis has been restricted to the cosets of pure Kaluza-Klein theory following from the reduction of Einstein-gravity over a torus, our mechanism for obtaining the generating solution can be straightforwardly extended to theories with $\mathrm{GL}(r+s=n,\mathbb{R})/\mathrm{SO}(r,s)$ cosets and also to the cosets in the right column of table 5.4.1.

Let us finish the discussion by making one surprising observation. It is known that a geodesic motion sometimes occurs in the presence of a scalar potential and for time dependent solutions, this can happen for scaling cosmologies. In [E] we studied such a solution in the context of fake/pseudo-supersymmetry (see for definition [151]) for multi-field systems whose first-order equations we derived using the Bogomol'nyi-like method. In particular we showed that scaling solutions that are pseudo-BPS must describe geodesic curves.

Chapter 7

Summary and Outlook

This thesis has covered in general two separate topics: the string effective actions and the geodesic motion of brane solutions.

The main theme of the first topic, i.e., the string effective actions, is the construction of the D-brane effective action and supergravity actions. For the D-brane effective action, in the abelian case and in the limit of constant field strengths this action has been already known for a long time to all orders in α': it is the Born-Infeld action. The introductory chapter 2 gives an overview of past attempts and successes in constructing in the general case. In the chapter 4 we proposed a new method for constraining the four dimensional D-brane effective action and applied to the abelian case with derivative corrections. The method is based on the electromagnetic duality invariance. We have shown that selfduality requirement can only constrain the derivative corrections terms to the Born-Infeld theory but *not* determine them. It is quite interesting to think about possible application of the α'-corrections that we considered in chapter 2, in particular to the non-abelian Born-infeld theory. For example it remains to be seen how the α'-corrections modify the behavior of classical solutions of the Yang-Mills equations of motion (zeroth order equations) that are more complicated than the flat background. One may expect for instance, that the instanton solutions of $N = 4, D = 4$ Yang-Mills theory receive α'-corrections. One can easily see from our discussion in section 2.6 that the upper limit for the field strength F_{ab} on a D-brane could only be obtained by using the complete infinite collection of α'-corrections of the Born-Infeld action. In addition, it turns out that the Born-Infeld action also provides a finite self-energy for the electric point particle solution. In the nonabelian case, answers to these and other questions that are related to the resolution of classical singularities by α'-corrections, will have to wait until we have an all-order result. We have also seen in chapter 3 that string theory at low-energy describes Einstein gravity coupled to certain matter fields, together with an infinite number of higher derivative corrections. Recently, study of the black hole physics in string theory involves study of black holes in higher derivative theories of gravity, in particular in the tree-level heterotic string theory. In chapter 3, we have established the equivalence between the heterotic effective actions of [100] and [97] to order α. This indicates that the result of [102] might indeed be a consequence of supersymmetry.

The second topic of this thesis was concerned with showing that Dp-branes and Sp-branes can be linked to lower dimensional theories whose solutions are respectively given by instantons or S(-1)-branes if we reduce over the worldvolume of the brane. In the lower dimensional action the gravity part decouples and can be solved independently, while the σ- model sector, obtained after a worldvolume reduction, leads to a geodesic motion. Then we turned to deriving the generating solution associated with the geodesic motion traced out by the scalars carrying the brane solutions. This applies both to instantons and to S(-1)-branes. We introduced the generating geodesic solution as a solution with the minimal number of arbitrary integration constants so that the action of the isometry group G actually generates all other geodesics from the generating one. This way we found the most general fluxless Sp-brane of Einstein gravity with (deformed) worldvolume via the reduction over an Euclidean torus. In case we reduce over a Lorentzian torus, the target space becomes a pseudo-Riemannian G/H^* with H^* is a non-compact real form of H. Correspondingly, the geodesic solutions on G/H^* are labeled by the sign of the affine velocity $||v||^2$. We derived the generating solution for cosets $\mathrm{GL}(r+s)/\mathrm{SO}(r,s)$, and gave the Einstein vacuum solutions that can be obtained from uplifting a $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n-1,1)$ stationary (-1)-brane solution.

The generating solutions that we have considered in chapter 6 are all restricted to theories which are based on symmetric spaces G/H or G/H^*, where G is the maximally non-compact real form (split form). The derivation of the generating solution can be extended to Euclidean theories in which G is a non-split isometry group (G is not a maximally non-compact real form), which typically occur in non-maximally extended supergravities. In [G] we give the results for the half- and quarter-maximal supergravity theories, e.g., $N=4$, $D=3$ symmetric Euclidean models.

We would like to conclude this final chapter with addressing some questions for future research:

1. We have seen in chapter 2 that the abelian Born-Infeld action has a closed expression. One would definitely like to have a closed expression for the non-abelian D-brane effective action.

2. In principle the method of the black hole entropy calculation of [103] can be extended to corrections with any number of derivatives. Supersymmetry provides the derivative contributions at order α'^2, at α'^3 only partial results are known.

It will be interesting to extend the analysis of [102] to include the next order.

3. We have seen in chapter 4 that the addition of higher derivative corrections of the field strength survives selfduality symmetry. Another extension would be to add derivative corrections to the $\mathrm{SL}(2,\mathbb{R})$-invariant extension of the Born-Infeld thoery. This problem is currently under investigation.

4. In all the Euclidean theories that we have studied so far the scalar manifold is a symmetric space. A natural extension is to look for the generating solutions of non-symmetric supergravity models where the scalar manifold is only homogenous.

5. In our analysis we restricted to uplifts to pure gravity. It would be nice to extend this with a $(p+1)$-potential as well. This would enable us to write down the most general p-brane with deformed worldvolume. This would apply both to timelike and spacelike branes.

Appendix A

Elementary Differential Geometry

In this appendix we will bring together the ideas of differential geometry, which are required throughout the thesis. There are several books, e.g, [171, 172], written for physicists, which explore the subject at greater length and greater depth.

A.1 Convention

Apart from chapter 3 which has its own convention, we take the following metric signature in the main text to be

$$g = \mathrm{diag}(- \cdots -, + \cdots +), \tag{A.1.1}$$

with (-) occurring t times and (+1) occurring s times. The pair (s,t) is called the signature of the metric g.

A.2 Introductory Concepts

A.2.1 Manifolds

A D-dimensional manifold is a topological space together with a family of open sets M_i that cover it, i.e, $M = \bigcup_i M_i$. M_i's are called coordinate patches. Within one patch one may defines a 1:1 map ϕ_i, called the chart, from $M_i \rightarrow \mathbb{R}$. Concretely speaking, a point $p \in M_i \subset M$ is mapped to $\phi_i(p) = (x^1, x^2, \cdots, x^D)$. We say that the set $(x^1, x^2, \cdots, x^D)$ are the local coordinates of the point p in the patch M_i.
If $p \in M_i \cap M_j$, then the map $\phi_j(x'^1, x'^2, \cdots, x'^D)$ provides a second set of coordinates

for the point p. The composite map

$$\phi_i \circ \phi_j : \qquad \mathbb{R}^D \to \mathbb{R}^D \tag{A.2.1}$$

is then specified by the set of functions $x'^\mu(x^\nu)$. These functions, and there inverses $x^\nu(x'^\mu)$ are required to be smooth, usually C^∞.

A.2.2 Tensor Fields

Scalar, Vector Fields and 1-Forms

The simplest object to define on a manifold M are scalar functions f that map $M \to \mathbb{R}$. We say that the point p maps to $f(p) = z$. On each coordinate patch M_i we can define the compound map $f \circ \phi_i^{-1}$ from $\mathbb{R}^D \to \mathbb{R}$ as $f_i(x^\mu) \equiv f \circ \phi_i^{-1}(x^\mu) = z$. On the overlap $M_i \cap M_j$ of two patches with local coordinate x^μ and x'^ν of the point p, the two descriptions of f must agree. Thus $f_i(x^\mu) = f_j(x'^\nu)$.

Vectors on a manifold M always describe tangents vectors to a curve in M. Let $p(t)$ be some curve. The coordinates of this curve are $x^i(p(t))$, $i = 1 \cdots D$ and the tangent vector to the curve is given by $\frac{d}{dt}x^i(p(t))$. Defining the differential operator

$$X = X^i \frac{\partial}{\partial x^i}, \qquad \text{with } X^i = \frac{dx^i(p(t))}{dt} \tag{A.2.2}$$

we obtain

$$\frac{d}{dt} f(p(t)) = Xf, \tag{A.2.3}$$

where f is a function on M. The tangent space to the manifold M at p, the space of all possible tangents at p, is denoted by $T_p(M)$.

To the contravariant vectors, which we have considered up to now, there also exist their duals - the covariant vectors. The dual space to $T_p(M)$ is the cotangent space $T_p^*(M)$ where duality is defined via the inner product $(dx^i, \frac{\partial}{\partial x^j}) = \delta^i{}_j$.
An element of $T_p^*(M)$ is given by the so-called 1-form

$$w = w_i dx^i \in T_p^*(M), \tag{A.2.4}$$

where $\{dx^i\}$ represents the dual basis in $T_p^*(M)$.

Tensor Algebra

We can now construct tensors of type (a, b) by mapping a elements of $T_p^*(M)$ and b elements of $T_p(M)$ into $\mathbb{R}$. So the space of these tensors is defined by

$$T^a{}_b = \underbrace{T_p(M) \otimes \cdots \otimes T_p(M)}_{a \text{ factors}} \otimes \underbrace{T_p^*(M) \otimes \cdots \otimes T_p^*(M)}_{b \text{ factors}}. \tag{A.2.5}$$

In terms of local coordinates, it reads[1]

$$T(x) = T^{i_1 \cdots i_a}_{j_1 \cdots j_b} \partial_{i_1} \cdots \partial_{i_a} dx^{j_1} \cdots dx^{j^b} \in T^a{}_b. \tag{A.2.6}$$

The action of T on 1-forms $w_1, \cdots, w_a$ and vectors $X_1, \cdots, X_b$ gives the number

$$T(w_1, \cdots, w_a, X_1, \cdots, X_b) = T^{i_1 \cdots i_a}_{j_1 \cdots j_b} w_{1 i_1} \cdots w_{a i_a} X_1^{j_1} \cdots X_b^{j_b}. \tag{A.2.7}$$

Allowing the point p to vary smoothly over the whole manifold, the vectors and tensors also vary smoothly over M, and one achieves so-called **vector fields** and **tensor fields** on M.

We now introduce the additional structure of a **metric** on a manifold. A metric or an inner product on a real vector space V is a non-degenerate bilinear map on each $V \otimes V \to \mathbb{R}$. The inner product of two vectors $u, v \in V$ is a real number denoted by (u, v). The inner product must satisfy the following properties:

(i) bilinearity -$(u, c_1 v_1 + c_2 v_2) = c_1(u, v_1) + c_2(u, v_2)$.

(ii) non-degeneracy- if $(u, v) = 0$ for all $v \in V$, then $u = 0$.

(iii) symmetry-$(u, v) = (v, u)$.

A metric on a manifold is a smooth assignment of a inner product on each $T_p(M) \otimes T_p(M) \to \mathbb{R}$. In a local coordinates the metric is specified by a covariant second rank tensor field $g_{\mu\nu}$ whose determinant denoted by g, and the inner product of two vectors fields U^μ and V^μ is $g_{\mu\nu} U^\mu V^\nu$, which is a scalar field. In particular the metric gives the length τ of a curve $x^\mu(t)$ with tangent vector dx^μ/dt.

Specifying a metric on a manifold, it will help with the classification of manifolds. In other words, the manifold is said to be **Riemannian** if its metric satisfies the following axioms at each point $p \in M$;

(i) $g(U, V) = g(V, U)$,

(ii) $g(U, U) \geq 0$, equality only for $U = 0$.

This means the metric evaluated at point p is a symmetric positive definite bilinear form. A **pseudo-Riemannian** manifold is a manifold endowed with a metric which obeys, beside axiom (i), the axiom (ii') states that if $g(U, V) = 0$ for all $U \in T_p M$, then $V = 0$, i.e., the manifold has an indefinite signature.

Differential forms: With the help of the wedge product

$$dx^\mu \wedge dx^\nu := dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu, \tag{A.2.8}$$

[1] The Einstein convention is used throughout the text; any index that appears twice in an expression is summed over if it appears once as upper index and once as a lower index.

one can define now several differential forms

$$0-\text{form} \qquad w = w(x) \tag{A.2.9}$$

$$1-\text{form} \qquad w = w_\mu dx^\mu \tag{A.2.10}$$

$$p-\text{form} \qquad w = \frac{1}{p!} w_{\mu_1 \cdots \mu_p}(x) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu p}. \tag{A.2.11}$$

We denote the set of all p-forms by Λ^p. This is a vector space of dimension[2]

$$\dim \Lambda^p = \binom{D}{p} = \frac{D!}{p!(D-p)!}. \tag{A.2.12}$$

One can therefore construct $(p+q)$-forms out of p-forms and q-forms in a straightforward manner by means of the wedge product $\alpha_p \wedge \beta_q \in \Lambda^{p+q}$, in such a way that

$$\alpha_p \wedge \beta_q = \chi_{p+q} \Rightarrow \chi_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!q!} \alpha_{[\mu_1 \cdots \mu_p} \beta_{\mu_{p+1} \cdots \mu_{p+q}]}. \tag{A.2.13}$$

Commuting the forms α_p and β_q, one also obtains

$$\alpha_p \wedge \beta_q = (-)^{pq} \beta_q \wedge \alpha_p. \tag{A.2.14}$$

All forms belong to the space

$$\Lambda^* = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \cdots \Lambda^D, \tag{A.2.15}$$

which is closed under the wedge product operation (or exterior product). Λ^* is a graded algebra, also named Cartan's exterior algebra (Grassmann algebra).

One differentiates the forms by introducing the **exterior derivative**, namely $d = \partial_\mu dx^\mu$, acting on a p-form in the following way

$$dw = \frac{1}{p!} \partial_\nu w_{\mu_1 \cdots \mu_p}(x) dx^\nu \wedge \cdots \wedge dx^{\mu_p}. \tag{A.2.16}$$

In fact, the exterior product is a map d: $\Lambda^p \to \Lambda^{p+1}$ which transforms p-forms into (p+1)-forms, satisfying nilpotency condition $d^2 = 0$ as well as obeying the antiderivation rule

$$d(\alpha_p \wedge \beta_q) = (d\alpha_p \wedge \beta_q) + (-)^p \alpha_p \wedge d\beta_q. \tag{A.2.17}$$

A p-form that satisfies $d\alpha_p$=0 is called **closed**. A p form α_p that can be expressed as $\alpha_p = d\alpha_{p-1}$ is called **exact**. Poincaré's lemma implies that locally any closed p-form can be expressed as $d\alpha_{p-1}$, but α_{p-1} may not be well defined globally on M.

[2]Note that Λ^p and Λ^{D-p} have the same dimensions.

Trace, (anti)-commutator: Next we define the trace, the commutator and the anti-commutator of differential forms.

Let $\alpha_p \in V \otimes \Lambda^p$, $\beta_q \in V \otimes \Lambda^q$ be forms which are V-valued, where V is actually a linear vector space consisting of vectors, e.g., Lie algebra or matrices. One says V-valued $\alpha_{\mu_1 \cdots \mu_p}, \beta_{\nu_1 \cdots \nu_q} \in V$

$$\alpha_{\mu_1 \cdots \mu_p} = \alpha^i_{\mu_1 \cdots \mu_p} T_i \tag{A.2.18}$$

$$\beta_{\mu_1 \cdots \mu_q} = \beta^i_{\mu_1 \cdots \mu_q} T_i, \tag{A.2.19}$$

with T_i the vectors (generators, matrices) of a vector space V. This definition actually means the direct product, like $\alpha = T_i \otimes \alpha^i \in V \otimes \Lambda^p$, between the basis $\{T_i\}$ of V and the wedge of differential forms.

Since, for instance, T^i matrices satisfy $[T_i, T_j] = f_{ijk} T^k$, where f^{ijk} are the anti-symmetric structure constants, one can derive the following rules;

$$[\alpha_p, \beta_q] = \alpha_p \wedge \beta_q - (-)^{pq} \beta_q \wedge \alpha_p = -(-)^{pq} [\beta_q, \alpha_p] \tag{A.2.20}$$

$$\{\alpha_p, \beta_q\} = \alpha_p \wedge \beta_q + (-)^{pq} \beta_q \wedge \alpha_p = (-)^{pq} \{\beta_q, \alpha_p\} \tag{A.2.21}$$

$$[\alpha_p \wedge \beta_q, \gamma_r] = \alpha_p \wedge [\beta_q, w_r] + (-)^{pr} [\alpha_p, w_r] \wedge \beta_q \tag{A.2.22}$$

$$\mathrm{tr}(\alpha_p \wedge \beta_q) = (-)^{pq} \mathrm{tr}(\beta_q \wedge \alpha_p), \qquad \text{also } \mathrm{tr}[\alpha_p, \beta_q] = 0. \tag{A.2.23}$$

Hodge $\star$ operation: The fact that the space of all p-forms Λ^p and the space Λ^{D-p} have the same dimensions, implies a duality between 2 spaces, an isomorphism given by the Hodge $\star$ operation; $\Lambda^p \xrightarrow{\star} \Lambda^{D-p}$. In other words, the star $\star$ transforms p-forms into $(D-p)$-forms and its action is defined by

$$\alpha_{\mu_1 \cdots \mu_{D-p}} = \frac{1}{p!} \epsilon_{\mu_1 \cdots \mu_{D-p}}{}^{\nu_1 \cdots \nu_p} \beta_{\nu_1 \cdots \nu_p}, \tag{A.2.24}$$

and denoted by $\star\beta_p$. The natural choice of ϵ is specified up to sign, i.e. up to a choice of the orientation, by the condition

$$\epsilon^{\mu_1 \cdots \mu_D} \epsilon_{\mu_1 \cdots \mu_D} = (-)^s D!, \tag{A.2.25}$$

with s the number of minuses appearing in the signature of the metric $g_{\mu\nu}$. It is also worth noting that

$$\epsilon^{\mu_1 \cdots \mu_D} \epsilon_{\nu_1 \cdots \nu_D} = (-)^s D! \delta^{[\mu_1}{}_{\nu_1} \delta^{\mu_2}{}_{\nu_2} \cdots \delta^{\mu_D]}{}_{\nu_D}. \tag{A.2.26}$$

Contraction of equation A.2.25 over j indices yields

$$\epsilon^{\mu_1 \cdots \mu_j \mu_{j+1} \cdots \mu_D} \epsilon_{\mu_1 \cdot \mu_j \nu_{j+1} \cdots \nu_D} = (-)^s j! (D-j)! \delta^{[\mu_{j+1}}{}_{\nu_{j+1}} \delta^{\mu_2}{}_{\nu_2} \cdots \delta^{\mu_D]}{}_{\nu_D}. \tag{A.2.27}$$

The totally antisymmetric ϵ-tensor or **Levi-Cevita tensor** is precisely defined by

$$\epsilon_{\mu_1 \cdots \mu_D} = \begin{cases} (-)^\sigma & \text{if all } \mu_i \text{ are distinct} \\ 0 & \text{otherwise,} \end{cases} \tag{A.2.28}$$

where σ is the signature of the permutation $(1, \cdots, D) \to (\mu_1, \cdots, \mu_D)$.
Note that

$$\epsilon^{\mu_1 \cdots \mu_D} = g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \cdots g^{\mu_D \nu_D} \epsilon_{\nu_1 \nu_2 \cdots \nu_D} = g^{-1} \epsilon_{\mu_1 \mu_2 \cdots \mu_D}, \tag{A.2.29}$$

and

$$\epsilon_{12 \cdots D} = [(-1)^s \det(g_{\mu\nu})]^{1/2} = \sqrt{|g|}. \tag{A.2.30}$$

The inner product associated with the star $\star$ operation can, up to some integral over M, be written as follows

$$\alpha_p \wedge \star \beta_p = \frac{1}{p!} \alpha_{\mu_1 \cdots \mu_p} \beta^{\mu_1 \cdots \mu_p} \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^D, \tag{A.2.31}$$

with $\epsilon = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^D$ is the natural volume element of M. The action of star on $\star\beta_p$ yields

$$\star \star \beta_p = (-)^{p(D-p)+s} \beta_p. \tag{A.2.32}$$

A.3 Homogeneous Spaces, Isometries and Geodesic Flow

This section is based on a section in the book by Nakahara [171]. We will assume that the reader is familiar with Lie groups.

A.3.1 Homogeneous Spaces

Let us start by defining the action of a group on a manifold.
Definition: Given a Lie Group G and differentiable manifold M, we define an **action** of G on M to be a differentiable map σ: $G \times M \to M$, which satisfies the following conditions:

(i) $\sigma(e, p) = p$ for any $p \in M$,

(ii) $\sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p)$ for any $g_1 g_2 \in G$ and any $p \in M$,

where e is the identity element of the group.

We also need to define the following properties of the group actions:
Definition: Let G be a Lie group that acts on a manifold M by $\sigma : G \times M \to M$. The action σ is said to be

(a) **transitive** if, for any p_1, $p_2 \in M$, there exists an element $g \in G$ such that $\sigma(g, p_1) = p_2$;

(b) **free** if every non-trivial element $g \neq e$ of G has no fixed points in M. In other words, given an element $g \in G$, if there exists an element $p \in M$ such that $\sigma(g,p) = p$, then g must be the identity element e.

Now we are ready to define a homogeneous space. A manifold M is said to be **homogeneous**, if there exists a Lie group G that acts *transitively* on M. The n-sphere is homogeneous because its group $\mathrm{SO}(n+1)$ acts transitively on it.

Definition: Let G be a Lie group that acts on a manifold M. The **isotropy** group of $p \in M$ is a subgroup of G defined by

$$H(p) = [g \in G | \sigma(g,p) = p]. \tag{A.3.1}$$

This means that $H(p) \subset G$ is the group of elements that leave p fixed. This is called the *little group or stabilizer*. If G acts transitively on M, one can show that isotropy groups of all points in M are isomorphic to each other.
Theorem: Under certain conditions, if one has a homogeneous manifold M with the group acting on it with isotropy group H, then the coset space G/H is a manifold (i.e. it has a differentiable structure), and it is diffeomorphic to M, i.e. $G/H \cong M$. As an example we have $\mathrm{SO}(n+1)/\mathrm{SO}(n) \cong S^n$.

M is said to be isotropic at p if all tangent vectors at p can be rotated into each other by elements of the isotropy group of p. This matches our intuition that isotropy means that space 'looks' the same in every direction. Spaces that are homogeneous and isotropic are said to be *maximally symmetric*.

A.3.2 Isometries, Geodesic Flow

Isometry

An isometry of a manifold (M, g) is a diffeomorphism[3] $f : M \to M$ which preserves the metric

$$f^* g_{f(p)} = g_p \qquad \text{or} \;\; g_{f(p)}(f_* U, f_* V) = g_p(U, V), \tag{A.3.2}$$

where f^* and f_* are respectively the pullbacks and the push-forwards of f. In components one can write A.3.2

$$g_{\mu\nu}(p) = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)), \tag{A.3.3}$$

where x^μ and y^α are respectively the coordinates of p and $f(p)$. If we take the infinitesimal isometry to be generated by ϵX, the vector field X is called the *Killing vector field*. This leads to the following *Killing equation*

$$X^\rho \partial_\rho g_{\mu\nu} + \partial_\mu X^\sigma g_{\sigma\nu} + \partial_\nu X^\lambda g_{\mu\lambda} = 0. \tag{A.3.4}$$

[3] Diffeomorphism is an invertible function that maps one manifold to another, such that both the function and its inverse are smooth.

As an example we consider D-dimensional Minkowski spacetime ($D \geq 2$) there exist $D(D+1)/2$ Killing vector fields, D of which generate the translations, $(D-1)$ boosts and $(D-1)(D-2)/2$ space rotations. Such spaces which admit $D(D+1)/2$ Killing vector fields are example of the maximally symmetric spaces defined above.

Geodesic Flow

A vector field on a manifold M describes, quite naturally, a **flow** in M. We consider the integral curve $\sigma(t, x)$ of a vector field $U \in T_x(M)$ passing through x at a time $t = 0$. In a given patch one has

$$\frac{d\sigma^\mu(t,x)}{dt} = U^\mu(\sigma(t,x)) \qquad \text{with } \sigma(0, x^\mu) = x^\mu. \tag{A.3.5}$$

Such an integral curve representing a map $\sigma : \mathbb{R} \times M \to M$, is termed a flow generated by the vector U.

A geodesic defined with respect to a connection on a manifold M gives the local extremum of the length of an integral curve connecting two points. Let $c : (a, b) \to M$ be a curve in M. If the tangent vector $U(t)$ on $c(t)$ is parallel transported along $c(t)$, namely if

$$\nabla_U U = 0 \tag{A.3.6}$$

the integral curve $c(t)$ is called **geodesic**, i.e. the straightest as well as the shortest possible curve, where ∇ is the covariant derivative defined below. In components, the geodesic equation A.3.6 becomes

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} = 0, \tag{A.3.7}$$

where $\{x^\mu\}$ are the local coordinate of $c(t)$ and $\Gamma^\mu{}_{\nu\rho}$ is the connection coefficients.

The parameter t typically represents time for a timelike curve, or distance for a spacelike curve. This parameter cannot be chosen arbitrarily. Rather, it must be chosen so that the tangent vector dx^μ/dt has a constant magnitude. This is referred to as an **affine parametrization**. Any two affine parameters are linearly related. That is, if r and t are affine parameters, then there exist constants a and b such that $r = at + b$.

A.4 Connections, Curvatures and Covariant Derivatives

The (pseudo)-Riemannian manifold (M, g) that physicists use in General Relativity is a D-dimensional spacetime endowed with a bilinear form, $(2, 0)$ tensor with signature

$(-\cdots-,+\cdots+)$, taking on the form

$$ds^2 = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu, \qquad \mu,\nu = 1,\cdots D. \tag{A.4.1}$$

The Levi-Cevita connection following from this metric is

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} - \partial_\nu g_{\mu\sigma} + \partial_\sigma g_{\mu\nu}\right), \tag{A.4.2}$$

from which one obtain the Riemann tensor

$$\mathcal{R}^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\gamma_{\nu\sigma}\Gamma^\mu_{\gamma\rho} - \Gamma^\gamma_{\nu\rho}\Gamma^\mu_{\gamma\sigma}. \tag{A.4.3}$$

The Ricci tensors $\mathcal{R}_{\nu\rho}$ and the Ricci scalar $\mathcal{R}$ are defined via the contractions as follows

$$\mathcal{R}_{\nu\sigma} \equiv \mathcal{R}^\mu{}_{\nu\mu\sigma}, \qquad \mathcal{R} \equiv \mathcal{R}^\mu_\mu. \tag{A.4.4}$$

In addition, the Einstein tensor $G_{\mu\nu}$ takes on the form

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}. \tag{A.4.5}$$

The action of the covariant derivative ∇, associated to the general coordinate transformation, on a (p,q) tensor is defined by

$$\begin{aligned}\nabla_\alpha T^{\nu_1\cdots\nu_q}_{\mu_1\cdots\mu_p} = {}& \partial_\alpha T^{\nu_1\cdots\nu_q}_{\mu_1\cdots\mu_p} - \Gamma^\rho_{\alpha\mu_1}T^{\nu_1\cdots\nu_q}_{\rho\mu_2\cdots\mu_p}\cdots - \Gamma^\rho_{\alpha\mu_p}T^{\nu_1\cdots\nu_q}_{\mu_1\cdots\cdots\mu_{p-1}\rho} \\ & + \Gamma^{\nu_1}_{\rho\alpha}T^{\rho\nu_2\cdots\nu_q}_{\mu_1\cdots\mu_p} + \cdots + \Gamma^{\nu_q}_{\rho\alpha}T^{\nu_1\cdots\nu_{q-1}\rho}_{\mu_1\cdots\mu_p}.\end{aligned} \tag{A.4.6}$$

The action of the box operator $\Box$ on a scalar field Φ is given by

$$\Box\Phi = \nabla_\mu\partial^\mu\Phi = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\Phi\right), \tag{A.4.7}$$

where g is the determinant of $g_{\mu\nu}$.

One can prove that for maximally symmetric space the Riemann tensor is expressed as

$$\mathcal{R}_{\rho\sigma\mu\nu} = C(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}), \tag{A.4.8}$$

where C is a constant. In the metric Ansatz 6.2.2, g_{ab} often describes an Euclidean maximally symmetric space. This means we have the sphere S^n for $k=1$, the hyperboloid $\mathbb{H}^n$ for $k=-1$ or flat space $\mathbb{E}^n$ for $k=0$. Then we have

$$ds^2 = \frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2_{n-1}, \tag{A.4.9}$$

where $d\Omega^2_m$ is the metric on the S^m sphere defined by

$$d\Omega^2_m = d\theta_1^2 + \sin^2(\theta_1)d\theta_2^2 + \cdots + \sin^2(\theta_1)\cdots\sin^2(\theta_{m-1})d\theta_m^2. \tag{A.4.10}$$

Performing the following redefinition

$$\frac{1}{1-kr^2}dr^2 = d\eta^2, \qquad \text{(A.4.11)}$$

one obtain the metrics

$$\begin{aligned} k=-1: &\qquad ds^2 = d\eta^2 + \sinh^2\eta d\Omega_{n-1}^2, \\ k=0 \quad : &\qquad ds^2 = d\eta^2 + \eta^2\, d\Omega_{n-1}^2, \\ k=+1: &\qquad ds^2 = d\eta^2 + \sin^2\eta\, d\Omega_{n-1}^2. \end{aligned} \qquad \text{(A.4.12)}$$

The Ricci tensor corresponding to these metrics can be obtained by having $C = k$, namely $\mathcal{R}_n = kn(n-1)$.

Appendix B

Some Calculational Details for Chapter 3

In this appendix we will give some calculational details related to section 3.4 of chapter 3 . Most of the conventions and notations will be used in this appendix are the same of [100]. The parameter α which will appear throughout this appendix is a free parameter proportional to α', the inverse of string tension.

B.1 Lagrangian Density and Redefinitions

In [100] the Lagrangian density behaves

$$\mathcal{L}_R = \frac{1}{2} e\phi^{-3} \left(-R(\omega) - \frac{3}{2} \widetilde{H}_{\mu\nu\rho} \widetilde{H}^{\mu\nu\rho} + 9(\phi^{-1}\partial_\mu \phi)^2 \right), \tag{B.1.1}$$

with the following definitions:

$$\begin{aligned}
\widetilde{H}_{\mu\nu\rho} &= \partial_{[\mu} B_{\nu\rho]} - \alpha\sqrt{2}\mathcal{O}_{3\,\mu\nu\rho}(\Omega_-), \\
\mathcal{O}_{3\,\mu\nu\rho} &= \Omega_{-[\mu}{}^{ab} \partial_\nu \Omega_{-\rho]}{}^{ab} - \frac{2}{3} \Omega_{-[\mu}{}^{ab} \Omega_{-\nu}{}^{ac} \Omega_{-\rho]}{}^{cb}, \\
\Omega_{-\mu}{}^{ab} &= \omega_\mu^{ab} - \frac{3}{2}\sqrt{2}\widetilde{H}_\mu{}^{ab}.
\end{aligned} \tag{B.1.2}$$

Antisymmetrasation brackets are with weight 1.

First we redefine the field in order to make the comparison of the actions tractable. The redefinitions are:

1. The dilaton changes as

$$\phi^{-3} \to e^{-2\Phi}, \qquad (\phi^{-}\partial\phi) \to \frac{2}{3}\partial\Phi. \tag{B.1.3}$$

2. For the two and three-form, one can set

$$\widetilde{H} \to \frac{1}{3\sqrt{2}}\widetilde{H}, \qquad B \to \frac{1}{\sqrt{2}}B. \tag{B.1.4}$$

The Lagrangian $\mathcal{L}_R$ then becomes

$$\mathcal{L}_R = \frac{1}{2}ee^{-2\Phi}\left(-R(\omega) - \frac{1}{12}\widetilde{H}_{\mu\nu\rho}\widetilde{H}^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi\right) \tag{B.1.5}$$

as in 3.4.4.

The spin connections $\omega(e)$ solve the equations

$$\mathcal{D}_\mu e_\nu{}^a - \mathcal{D}_\nu e_\mu^a = 0, \qquad \text{with } \mathcal{D}_\mu e_\nu{}^a \equiv \partial_\mu e_{\nu\,a} - \omega_\mu{}^{ac} e_{\nu\,c}. \tag{B.1.6}$$

The Riemann tensor and related quantities are defined as

$$\begin{aligned} R_{\mu\nu}{}^{ab}(\omega) &= \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\omega_{\nu\,c}{}^b + \omega_\nu{}^{ac}\omega_{\mu\,c}{}^b\,, && \text{(B.1.7)}\\ R_\mu{}^a(\omega) &= e^\nu{}_b R_{\mu\nu}{}^{ab}(\omega)\,, && \text{(B.1.8)}\\ R(\omega) &= e^\mu{}_a R_\mu{}^a(\omega)\,. && \text{(B.1.9)} \end{aligned}$$

B.2 Equations of Motion

The lowest order equations of motion, i.e., at order α'^0 are:

$$\begin{aligned} \mathcal{S} &= ee^{-2\Phi}[R(\omega) - 4\mathcal{D}_a\partial^a\Phi + 4(\partial_a\Phi)^2 + \frac{1}{2}H^{abc}H_{abc}] = 0, && \text{(B.2.1)}\\ \mathcal{B}^{\nu\rho} &= \frac{1}{4}\partial_\mu(ee^{-2\Phi}H^{\mu\nu\rho}) = 0, && \text{(B.2.2)}\\ \mathcal{E}^\lambda{}_c &= -\frac{1}{2}e^\lambda{}_c\mathcal{S} + ee^{-2\Phi}(R^\lambda{}_c(\omega) + \frac{1}{4}(H^2)^\lambda{}_c - 2e^\lambda{}_d\mathcal{D}_c\Phi\partial^d\Phi) = 0. && \text{(B.2.3)} \end{aligned}$$

In the main text of section 3.4 we use a field redefinition to eliminate any contribution proportional to the Ricci tensor. The required equation is, modulo $\mathcal{E}$ and $\mathcal{S}$:

$$R_\mu{}^a(\omega) = 2\mathcal{D}_\mu\partial^a\Phi - \frac{1}{4}(H^2)_\mu{}^a. \tag{B.2.4}$$

B.3 Expanding $\mathcal{L}_R$ in Powers of α

The 3-form field $\widetilde{H}$ is defined recursively by (3.4.6, 3.4.7, 3.4.8). We find

$$\widetilde{H}_{\mu\nu\rho} = H_{\mu\nu\rho} - 6\alpha(\mathcal{O}_{3\,\mu\nu\rho}(\omega) + \mathcal{A}_{\mu\nu\rho}) = \bar{H}_{\mu\nu\rho} - 6\alpha\mathcal{A}_{\mu\nu\rho}, \tag{B.3.1}$$

where $\mathcal{O}_{3\,\mu\nu\rho}$ is the gravitational contribution (order α^0) of the Lorentz Chern-Simons term, and

$$\begin{aligned}\mathcal{A}_{\mu\nu\rho} = &\frac{1}{2}\partial_{[\mu}(\omega_\nu{}^{ab}\widetilde{H}_{\rho]}{}^{ab}) - \frac{1}{2}R^{ab}_{[\mu\nu}(\omega)\widetilde{H}_{\rho]}{}^{ab} + \frac{1}{4}\widetilde{H}_{[\mu}{}^{ab}\mathcal{D}_\nu\widetilde{H}_{\rho]}{}^{ab} \\ &+ \frac{1}{12}\widetilde{H}_{[\mu}{}^{ab}\widetilde{H}_\nu{}^{ac}\widetilde{H}_{\rho]}{}^{cb}.\end{aligned} \tag{B.3.2}$$

To order α $\mathcal{L}_R$ B.1.5 can be expressed as

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}ee^{-2\Phi}[-R(\omega) - \frac{1}{12}\bar{H}_{\mu\nu\rho}\bar{H}^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi \\ &+ \alpha\{\frac{1}{2}H^{\mu\nu\rho}\partial_\mu(\omega_\nu{}^{ab}H_\rho{}^{ab}) - \frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)H_\rho{}^{ab}H^{\mu\nu\rho} + \frac{1}{4}H^{\mu\nu\rho}H_\mu{}^{ab}\mathcal{D}_\nu H_\rho{}^{ab} + \\ &+ \frac{1}{12}H^{\mu\nu\rho}H_\mu{}^{ab}H_\nu{}^{ac}H_\rho{}^{cb}\}].\end{aligned} \tag{B.3.3}$$

The term with $H\partial(\omega H)$ is after partial integration, proportional to B.2.2 and can be eliminated by a field redefinition.

B.4 Simplification of $\mathcal{L}_{R^2}$ Terms

We often use the identity

$$\mathcal{D}_{[a}(\Omega_-)\widetilde{H}_{bcd]} = -\frac{3}{2}\alpha R_{[ab}{}^{ef}(\Omega_-)R_{cd]}{}^{ef}(\Omega_-), \tag{B.4.1}$$

to isolate terms that are of higher order terms in α. The term 3.4.14 can be simplified by using the cyclic identity for the Riemann tensor:

$$R_{\mu\nu}{}^{ab}(\omega)H^{\mu\,ac}H^{\nu\,cb} = -\frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)H^{\mu\nu\,c}H^{abc}. \tag{B.4.2}$$

Now we consider 3.4.15. Note that the two terms written in 3.4.15 are actually the same. Then we have

$$\begin{aligned}\frac{1}{2}(\mathcal{D}_\mu H_\nu{}^{ab} - \mathcal{D}_\nu H_\mu{}^{ab})H^{\mu ac}H^{\nu cb} &= -\mathcal{D}_\mu H_\nu{}^{ab}H^{\mu\,ac}H^{\nu bc} \\ &= -\mathcal{D}_{[e}H_{fab]}H^{eac}H^{fbc}.\end{aligned} \tag{B.4.3}$$

The term is completely of order α'^2. Finally we consider 3.4.13. This can be rewritten as

$$\frac{1}{2}ee^{-2\Phi}(\mathcal{D}_\mu H_\nu{}^{ab} - \mathcal{D}_\nu H_\mu{}^{ab})\mathcal{D}^\mu H^{\nu ab} = ee^{-2\Phi}(2R_{\mu\nu}{}^{ab}H^{\mu ac}H^{\nu cb} + R_\mu{}^c H^{\mu ab}H_{abc}$$
$$+ e^\mu{}_c e^\nu{}_d \mathcal{D}_\nu H_{abd}\mathcal{D}_\mu H_{abc} + 2\partial_c\Phi H_{abd}\mathcal{D}_d H_{abc} - 2\partial_d\Phi H_{abd}\mathcal{D}_c H_{abc}$$
$$+ 2\mathcal{D}_c H_{abd}\mathcal{D}_{[c}H_{abd]}). \tag{B.4.4}$$

The last term is of order α'^2.

Appendix C

Some Lie Algebra Theory

In this introductory appendix we define the basic concepts relating to Lie group and Lie algebra [131, 173, 174].

C.1 Classical Lie Groups

Consider a group G acting on a space V over a field F, e.g. ($\mathbb{R}$ or $\mathbb{C}$). We can think of G as being matrices, and of V as a vector space on which these matrices act. A group element $g \in G$ transforms the vector $v \in V$ into $g \cdot v = v'$.

Once an additional structure, in the form of a metric, has been imposed on an N-dimensional vector space over a field F, one would be able to classify the classical (matrix) groups acting on V. Recall the definition of the metric

$$(v_1, v_2) = f \qquad v_1,\, v_2 \in V,\, f \in F \tag{C.1.1}$$

obeying the following conditions:

$$(v_1, av_2 + bv_3) = a(v_1, v_2) + b(v_1, v_3) \tag{C.1.2a}$$

and

$$(av_1 + bv_2, v_3) = (v_1, v_3)a + (v_2, v_3)b \tag{C.1.2b}$$

or

$$(av_1 + bv_2, v_3) = (v_1, v_3)a^* + (v_2, v_3)b^* \tag{C.1.2c}$$

Metrics obeying conditions C.1.2a and C.1.2b are called *bilinear* metrics; those obeying C.1.2a and C.1.2c are called sesquilinear. The groups metric-preserving are then classified as follows[1]

[1] Orthogonal groups preserving metric (p, q) in ($\mathbb{R}$ or $\mathbb{C}$) are denoted by $\mathrm{O}(p, q, \mathbb{R})$, $\mathrm{O}(p, q, \mathbb{C})$.

(a) Groups preserving *bilinear* symmetric metrics are called **orthogonal**.

(b) Groups preserving *bilinear* antisymmetric metrics are called **symplectic.**

(c) Groups preserving *sesquilinear* symmetric metrics are called **unitary**.

The metric preserving group which are in addition volume-preserving are called the special metric-preserving groups and are denoted by an additional S., e.g., SO(n), Sp(n), and SU(n).

In addition, we have five isolated groups, which are called

$$E_6, \qquad E_7, \qquad E_8, \qquad G_2, \qquad F_4. \tag{C.1.3}$$

In all groups the subscript denotes the rank of the group. Those five isolated groups are referred to as the **exceptional** Lie groups.

C.2 Structure of Simple Lie algebra

C.2.1 The basics

A complex Lie algebra $\mathfrak{G}$ is a vector space over F endowed with a binary operation which is called a Lie bracket commutator

$$[\,,] : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}. \tag{C.2.1}$$

The two defining properties of $[\,,]$ read

$$[X, X] = 0 \qquad \forall\, X \in \mathfrak{G} \tag{C.2.2}$$

and

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \qquad \forall\, X,\, Y,\, Z \in \mathfrak{G}. \tag{C.2.3}$$

The identity C.2.3 is the so-called **Jacobi identity.**
A Lie algebra is specified by a set of generators $\{T^a\}$ and their commutator relations

$$[T^a, T^b] = f^{bc}{}_d T^d, \tag{C.2.4}$$

where $f^{bc}{}_d$ are the structure constants. The *dimension* d of the lie algebra $\mathfrak{G}$ is thus the dimension of the underlying vector space spanned by the basis

$$\mathcal{B} = \{T^a | a = 1, \cdots, d\}. \tag{C.2.5}$$

Simple Lie algebras are Lie algebras which contain no proper[2] ideal and which is not abelian. An ideal or invariant subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ is a subspace satisfying simultaneously $[\mathfrak{H},\mathfrak{H}] \subseteq \mathfrak{H}$ and $[\mathfrak{H},\mathfrak{G}] \subseteq \mathfrak{H}$. An **abelian** Lie algebra is a Lie algebra which satisfies $[\mathfrak{G},\mathfrak{G}] = 0$. A direct sum of simple Lie algebras forms the so-called **semi-simple** Lie algebra.
Levi Theorem: Every Lie algebra can be decomposed into the direct sum of simple Lie algebra and solvable algebras; solvable Lie algebra can be defined iteratively by the series $\mathfrak{s}^0 = \mathfrak{s}$, $\mathfrak{s}^1 = [\mathfrak{s}^0, \mathfrak{s}^0]$, $\mathfrak{s}^i = [\mathfrak{s}^{i-1}, \mathfrak{s}^{i-1}]$, for a finite number of steps, it ends up with zero.

In general the action of a Lie algebra $\mathfrak{G}$ on a vector space V is carried out via a linear **representation** of $\mathfrak{G}$

$$R:\ \mathfrak{G} \to \mathfrak{gl}(\mathfrak{G}):\ X \to R(X), \qquad R(X):\ V \to V:\ v \to R(X)\cdot v. \tag{C.2.6}$$

It is possible to represent $\mathfrak{G}$ on itself; thereby one obtains the **adjoint** representation: for any $T \in \mathfrak{G}$

$$\mathrm{ad}T(T_a) \equiv [T, T_a], \qquad \Rightarrow [\mathrm{ad}T_a]^c{}_b = -f_{ab}{}^c. \tag{C.2.7}$$

Exponentiating the generators of the Lie algebra $\mathfrak{G}$ in the adjoint representation, we get the adjoint representation of the corresponding group G

$$\mathrm{Ad}(g) = \exp[\tau^a \mathrm{ad}(T_a)], \qquad \text{with } T'_b = T_a[\mathrm{Ad}(g)]^a{}_b, \tag{C.2.8}$$

where $g \in G$ and τ is the group parameter. Actually, in any representation R, the adjoint action of G on $\mathfrak{G}$ is given by

$$R(g)R(T_a)R(g^{-1}) = R(T_b)[\mathrm{Ad}(g)]^b{}_a. \tag{C.2.9}$$

The Killing metric $B(.\,,.)$ is a symmetric bilinear form defined by

$$B(T_a, T_b) \equiv \mathrm{tr}(\mathrm{ad}T_a \mathrm{ad}T_b) = f_{ac}{}^d f_{bd}{}^c. \tag{C.2.10}$$

Suppose the Lie algebra is semisimple[3]. According to Cartan's criterion, *the Killing metric is non-degenerate for a semisimple algebra*. This means $\det B_{ab} \neq 0$, so that the inverse of B_{ab}, denoted by B^{ab}, exists. Since the Killing metric is also real and symmetric, it can be reduced, choosing an orthonormal basis, to canonical form $B_{ab} = \mathrm{diag}(-1, \cdots, -1, 1, \cdots, 1)$ with p (-1's) and $(d-p)$ (+1's) are respectively the number of compact and non-compact generators (see next section), where d is the dimension of $\mathfrak{G}$. When thinking about a real form, that will be discussed in the next section, its convenient to visualize it in terms of the signature of its metric.

In any (semi)-simple Lie algebra $\mathfrak{G}$ there are two kinds of generators: there is a

[2] Any Lie algebra has two subalgebras, namely $\mathfrak{G}$ itself and zero. These subalgebras are called trivial subalgebras; any other subalgebra of $\mathfrak{G}$ is called proper subalgebra of $\mathfrak{G}$.

[3] This is true for all classical Lie algebras except for the Lie algebras $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{u}(n, \mathbb{C})$.

maximal abelian subalgebra, called the **Cartan subalgebra** CSA $\mathfrak{h} = H_1, \cdots, H_r$, $[H_I, H_J] = 0$ for two elements of CSA. There are shift operators denoted by E_α. α is an r-dimensional vector $\alpha = (\alpha_1, \cdots, \alpha_r)$ and r is the **rank** of $\mathfrak{G}$. The latter are eigenoperators of the H_I in the adjoint represenation belonging to α_I: $[H_I, E_\alpha] = \alpha_I E^\alpha$. For each eigenvalue, or **roots** α^I, there is another eigenvalue $-\alpha_I$ and a corresponding eigenoperator $E_{-\alpha}$ under the action of H_I.

Suppose we represent each element of the Lie algebra by an $n \times n$ matrix. Then $[H_I, H_J] = 0$ means that the matrices H_I can all be diagonalized simultaneously. The eigenvalues β_I are given by $H_I|\beta\rangle = \beta_I|\beta\rangle$, where the eigenvectors are labelled by the **weight vector** $\beta = (\beta_1, \cdots, \beta_r)$. The canonical commutation relations are summarized by :

$$[H_I, H_J] = 0, \qquad [H_I, E_\alpha] = \alpha_I E_\alpha, \qquad [E_\alpha, E_{-\alpha}] = \alpha_I H_I. \tag{C.2.11}$$

C.2.2 Real Forms

Let us recall some definitions,
$V^{\mathbb{C}}$: Let V be a vector space over $\mathbb{R}$. $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ is called the *complexification* of V. One has $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{\mathbb{C}}$.
$W^{\mathbb{R}}$: Let W be a vector space over $\mathbb{C}$. Restricting the definition of scalars to $\mathbb{R}$ then leads to a vector space $W^{\mathbb{R}}$ over $\mathbb{R}$ and $\dim_{\mathbb{C}} W = 1/2 \dim_{\mathbb{R}} W^{\mathbb{R}}$.

Real form of $\mathfrak{G}^{\mathbb{C}}$: Let $\mathfrak{G}^{\mathbb{C}}$ be a Lie algebra over $\mathbb{C}$. A **real form** of $\mathfrak{G}^{\mathbb{C}}$ is a subalgebra $\mathfrak{G}$ of the real Lie algebra $(\mathfrak{G}^{\mathbb{C}})_{\mathbb{R}}$ such that

$$(\mathfrak{G}^{\mathbb{C}})_{\mathbb{R}} = \mathfrak{G} \oplus_{\mathbb{R}} i\mathfrak{G} \qquad \text{direct sum of vector spaces.} \tag{C.2.12}$$

In other words, A real form of a Lie algebra is just a choice of generators for which the structure constants are real. For example, The complex algebra $\mathfrak{sl}(2, \mathbb{C})$ of the complex group $\mathrm{SL}(2, \mathbb{C})$ has two real forms; the compact $\mathfrak{su}(2)$ algebra and the non-compact $\mathfrak{sl}(2, \mathbb{R})$ algebra. The possible third real form $\mathfrak{su}(1, 1)$ is included as it is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Any finite dimensional $\mathfrak{G}^{\mathbb{C}}$ possesses a unique real form in which all the generators are **compact**. Compact means that the scalar product of the generators, defined by the Killing metric, is negative definite. It is given by taking the generators[4]

$$\hat{U}_\alpha = i(E_\alpha + E_{-\alpha}), \ \hat{V}_\alpha = (E_\alpha - E_{-\alpha}), \ \hat{H}_I = iH_I. \tag{C.2.13}$$

We refer to this compact algebra as $\mathfrak{G}^{cp}$.

Definition: An **involution** is a map which is an automorphism defined by

$$\theta(T_a T_b) = \theta(T_a)\theta(T_b) \qquad \forall\, T_a, T_b \in \mathfrak{G}, \qquad \theta^2 = 1. \tag{C.2.14}$$

[4]The compact nature of the generators follows in obvious way from the fact that the only non zero Killing metric between E_α and $E_{-\alpha}$ is $B(E_\alpha, E_{-\alpha}) = 1$ and $B(H_I, H_J) = -(\alpha_I, \alpha_J) < 0$.

By considering all involutions of the unique compact real form $\mathfrak{G}^{cp}$ one can construct all other real forms of $\mathfrak{G}^{\mathbb{C}}$. In particular, the real forms are in one to one correspondence with all those involutive automorphisms of the compact real form [127, 175].

Given an involutive θ we can divide the generators of the compact real form $\mathfrak{G}^{cp}$ into those which possess +1 and -1 eigenvalues of θ. We denote these eigenspaces by

$$\mathfrak{G} = \mathfrak{H} \oplus \hat{\mathfrak{F}} \tag{C.2.15}$$

respectively. Since θ is an automorphism it preserves the structure of the algebra and as a result the algebra when written in terms of this split must take the generic form

$$[\mathfrak{H}, \mathfrak{H}] \subset \mathfrak{H}, \qquad [\mathfrak{H}, \hat{\mathfrak{F}}] \subset \hat{\mathfrak{F}}, \qquad [\hat{\mathfrak{F}}, \hat{\mathfrak{F}}] \subset \mathfrak{H}. \tag{C.2.16}$$

Now, from the generators $\hat{\mathfrak{F}}$ we define new generators $\mathfrak{F} = -i\hat{\mathfrak{F}}$, whereupon the algebra now takes the generic form

$$[\mathfrak{H}, \mathfrak{H}] \subset \mathfrak{H}, \qquad [\mathfrak{H}, \mathfrak{F}] \subset \mathfrak{F}] \subset \mathfrak{F}, \qquad [\mathfrak{F}, \mathfrak{F}] \subset (-1)\mathfrak{H}. \tag{C.2.17}$$

Thus we find a *new* real form of $\mathfrak{G}^{\mathbb{C}}$ in which the generators $\mathfrak{H}$ are compact while the generators $\mathfrak{F}$ are **non-compact**[5]. Clearly, the new real form has **maximal compact subalgebra** $\mathfrak{H}$ and this is just the part of the algebra invariant under θ.

As each real form corresponds to an involutive θ we can write the corresponding real form as $\mathfrak{G}_\theta$[6]. The number of compact generators is dim$\mathfrak{H}$ and the number of non-compact generators is dim$\mathfrak{G}$ − dim$\mathfrak{H}$.

Definition: The **character** σ of the real form is the number of non-compact minus the number of compact generators and so $\sigma = \dim\mathfrak{G} - 2\dim\mathfrak{H}$.

If the involutive θ is taken to be θ_c which is a linear operator that takes $E_\alpha \leftrightarrow -E_{-\alpha}$ and $H_I \to -H_I$, an important real form can be constructed. Accordingly, the generators of the compact real form transform as $\hat{V}_\alpha \to \hat{V}_\alpha$, $\hat{U}_\alpha \to -\hat{U}_\alpha$, and $\hat{H}_I \to -\hat{H}_I$ where $\hat{V}_\alpha = E_\alpha - E_{-\alpha}$, $\hat{U}_\alpha = E_\alpha + E_{-\alpha}$ and $\hat{H}_I = H_I$. Using θ_c we find a real form with generators

$$V_\alpha = \hat{V}_\alpha, \qquad U_\alpha = -i\hat{U}_\alpha, \qquad H_I = -i\hat{H}_I. \tag{C.2.18}$$

The V_α remain compact generators while U_α and H_I become non-compact[7]. Clearly, the non-compact part of the real form of the algebra found in this way contains all the Cartan subalgebra CSA and it turns out that it has the maximal number of

[5] This follows from the fact that all the generators in the original algebra are compact and so have negative definite Killing metric and as a result of the change all the generators $\mathfrak{F}$ will have positive definite B.

[6] For the compact real form the involution is just the identity map Id on all the generators and so we may write $\mathfrak{G}^{cp} = \mathfrak{G}_I$.

[7] We are denoting with H_I both the Cartan generators of $G^{\mathbb{C}}$ and the Cartan generators in this particular real form. The maximal compact subalgebra is just that invariant under θ_c.

non-compact generators of all real forms one can construct. It is therefore called the **maximally non-compact** real form or split real form[8] denoted by $\mathfrak{G}_{\theta_c}$. Let's consider two examples:

(1) The complex Lie algebra $\mathfrak{G}^{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C})$ has $\mathfrak{su}(n,\mathbb{C}) = \mathfrak{G}^{cp}$ as its unique compact real form and $\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{G}_{\theta_c}$ as its maximally non-compact real form.

(2) For $\mathfrak{e}_8$ algebra of group E_8 the maximally non-compact real form is denoted by $\mathfrak{e}_{8(8)} = \mathfrak{G}_{\theta_c}$ and its maximal compact subalgebra is $\mathfrak{so}(16)$ of group SO(16). The character of $\mathfrak{e}_{8(8)}$ is $\sigma = 248 - 2.120 = 8 = \text{rank}(E_8)$. This notation may use for all the exceptional groups.

Taking different non-trivial involutions we find different real forms. For example, for the real form of E_8 denoted by $\mathfrak{e}_{8(-24)}$ the maximal compact subalgebra is $\mathfrak{e}_7 \otimes \mathfrak{su}(2)$.

As the involution θ is an automorphism it preserves the Killing metric and as a result

$$B(\theta(X),\theta(Y)) = B(X,Y) = -B(X,Y) = 0 \text{ if } X \in \mathfrak{H},\ Y \in \mathfrak{F}. \tag{C.2.19}$$

Thus the spaces $\mathfrak{H}$ and $\mathfrak{F}$ are **orthogonal**[9]. As one can realize from the previous discussion the Cartan subalgebra CSA $\mathfrak{h}$ of $\mathfrak{G}_\theta$ can be split between compact generators $\mathfrak{H}$ and non-compact generators of $\mathfrak{F}$. Let us denote the Cartan subalgebra elements in $\mathfrak{F}$ by $\mathfrak{c} = \mathfrak{h} \cap \mathfrak{F}$. The real rank r_θ of $\mathfrak{G}_\theta$ is the dimension of $\mathfrak{c}$. Clearly, it takes its maximal value for maximally non-compact case where it equals the rank r of $\mathfrak{G}_\theta$.

[8] In some literatures the involution θ is called the Cartan involution, and the involution corresponds to the split form $\mathfrak{G}_{\theta_c}$ is called the Chevalley involution θ_c.

[9] It also follows from this discussion that $B(X,\theta(Y))$ is negative definite. In fact one can define a Cartan involution for which this true.

Appendix D

Publications

- [A] D. B. Westra and W. Chemissany, *Coset symmetries in dimensionally reduced heterotic supergravity*, JHEP **0602** (2006) 004 [hep-th/0510137].
- [B] W. Chemissany, Joost de Jong and M. de Roo, *Selfduality of non-linear electrodynamics with derivative corrections*, JHEP **0611** (2006) 086 [hep-th/0610060].
- [C] W. Chemissany, M. de Roo and S. Panda, *α'-corrections of heterotic superstring effective action revisited*, JHEP **0708** (2007) 037 [arXiv:0706.3636].
- [D] W. Chemissany, M. de Roo and S. Panda, *Thermodynamics of Born-Infeld Black Holes*, arXiv:0806.3348 [hep-th].
- [E] W. Chemissany, A. Ploegh and T. Van Riet *Scaling cosmologies, geodesic motion and pseudo-susy*, Class. Quantum Grav.**24** (2007) [arXiv:0704.1653].
- [F] E. Bergshoeff, W. Chemissany, A. Ploegh and T. Van Riet, *Geodesic Flows in Cosmology* , Journal of Physics: Conference Series, Ref.: Ms. No. EPSHEPP156.
- [G] E. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante and T. Van Riet, *Generating Geodesic Flows and Supergravity Solutions*, arXiv:0806.2310 [hep-th].

Bibliography

[1] G. t Hooft and M. J. G. Veltman, *Regularization and Renormalization of Gauge Fields*, Nucl. Phys. **B44** (1972) 189–213

[2] T. Kaluza, *On the Problem of Unity in Physics*, in *Unified Field Theories of ¿4 Dimensions*, pp. 427–+. 1983.

[3] O. Klein, *Quantum Theory and Five-Dimensional Relativity Theory (translated from the German)*, pp. 67–+. The Oskar Klein Memorial Lectures, Vol. 1, 1991.

[4] G. 't Hooft, *a planar diagram theory for strong interactions*, Nucl. Phys. **B72** (1974) 461

[5] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231–252, `hep-th/9711200`

[6] C. Montonen and D. I. Olive, *Magnetic Monopoles as Gauge Particles?*, Phys. Lett. **B72** (1977) 117

[7] N. Seiberg, *Electric - magnetic duality in supersymmetric nonAbelian gauge theories*, Nucl. Phys. **B435** (1995) 129–146, `hep-th/9411149`

[8] M. Green, J. Schwarz and E. Witten, *Superstring Theory. Vol. 1: Introduction*, Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics)

[9] M. Green, J. Schwarz and E. Witten, *String Theory. Vol. 1: An introduction to the bososnic string*, Cambridge, Uk: Univ. Pr. (1998) 402 P.

[10] J. Polchinski, *Superstring Theory. Vol. 2: Introduction*, Cambridge, Uk: Univ. Pr. (2007) 739 P.

[11] K. Becker, M. Becker and J. Schwarz, *String theory and M-theory: Modern introduction*, Cambridge, Uk: Univ. Pr. (1987) 469 P.

[12] M. Eenink, *Dirichlet branes, effective actions and supersymmetry.* PhD thesis, Rijksuniversiteit Groningen, 2005.

[13] S. Deser and B. Zumino, *A complete action for the spinning string*, Phys.Lett. **B65** (1976) 369

[14] P. D. V. L. Brink and P. Howe, *A locally supersymmetric and reparametrization invariant action for the spinning string*, Phys.Lett. **B65** (1976) 471

[15] A. Polyakov, *Quantum geometry of bosonic strings*, Phys.Lett. **B103** (1981) 207

[16] A. Polyakov, *Quantum geometry of fermionic strings*, Phys.Lett. **B103** (1981) 211

[17] J. Dai, R. G. Leigh and J. Polchinski, *New Connections Between String Theories*, Mod. Phys. Lett. **A4** (1989) 2073–2083

[18] D. Friedan, *Notes on string theory and two-dimensional conformal field theory*, Appeared in Proc. of Workshop on Unified String Theories, Santa Barbara, CA, Jul 29 - Aug 16, 1985

[19] F. Gliozzi, J. Scherk and D. I. Olive, *Supersymmetry, Supergravity Theories and the Dual Spinor Model*, Nucl. Phys. **B122** (1977) 253–290

[20] J. E. Paton and H.-M. Chan, *Generalized veneziano model with isospin*, Nucl. Phys. **B10** (1969) 516–520

[21] A. M. Polyakov, *Quantum geometry of fermionic strings*, Phys. Lett. **B103** (1981) 211–213

[22] N. Berkovits, *Super-Poincare covariant quantization of the superstring*, JHEP **04** (2000) 018, `hep-th/0001035`

[23] N. Berkovits, *Covariant quantization of the superstring*, Nucl. Phys. Proc. Suppl. **127** (2004) 23–29, `hep-th/0008145`

[24] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *Heterotic String Theory. 1. The Free Heterotic String*, Nucl. Phys. **B256** (1985) 253

[25] D. J. Gross, J. A. Harvey, E. Martinec and R. Rohm, *Heterotic String*, Phys. Rev. Lett. **54** (Feb, 1985) 502–505

[26] S. Mandelstam, *The n loop string amplitude: Explicit formulas, finiteness and absence of ambiguities*, Phys. Lett. **B277** (1992) 82–88

[27] D. J. Gross and V. Periwal, *String Perturbation Theory Diverges*, Phys. Rev. Lett. **60** (May, 1988) 2105–2108

[28] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, Phys. Rept. **244** (1994) 77–202, `hep-th/9401139`

[29] J. Polchinski, *Dirichlet-Branes and Ramond-Ramond Charges*, Phys. Rev. Lett. **75** (1995) 4724–4727, `hep-th/9510017`

[30] G. T. Horowitz and A. Strominger, *Black strings and P-branes*, Nucl. Phys. **B360** (1991) 197–209

[31] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. **B443** (1995) 85–126, `hep-th/9503124`

[32] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. Phys. **B438** (1995) 109–137, `hep-th/9410167`

[33] H. Osborn, *Topological Charges for N=4 Supersymmetric Gauge Theories and Monopoles of Spin 1*, Phys. Lett. **B83** (1979) 321

[34] M. B. Green and M. Gutperle, *Comments on Three-Branes*, Phys. Lett. **B377** (1996) 28–35, `hep-th/9602077`

[35] A. A. Tseytlin, *Self-duality of Born-Infeld action and Dirichlet 3-brane of type IIB superstring theory*, Nucl. Phys. **B469** (1996) 51–67, `hep-th/9602064`

[36] J. Polchinski, *Effective field theory and the Fermi surface*, `hep-th/9210046`

[37] C. P. Burgess, *An ode to effective Lagrangians*, `hep-ph/9812470`

[38] S. Weinberg, *What is quantum field theory, and what did we think it was?*, `hep-th/9702027`

[39] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications*, Cambridge, UK: Univ. Pr. (1996) 489 p

[40] C. P. Burgess, *Introduction to effective field theory*, `hep-th/0701053`

[41] M. Born and L. Infeld, *Foundations of the new field theory*, Proc. Roy. Soc. Lond. **A144** (1934) 425–451

[42] P. A. M. Dirac, *An Extensible model of the electron*, Proc. Roy. Soc. Lond. **A268** (1962) 57–67

[43] Y.-K. E. Cheung and Z. Yin, *Anomalies, branes, and currents*, Nucl. Phys. **B517** (1998) 69–91, `hep-th/9710206`

[44] M. R. Douglas, *Branes within branes*, `hep-th/9512077`

[45] M. B. Green, C. M. Hull and P. K. Townsend, *D-Brane Wess-Zumino Actions, T-Duality and the Cosmological Constant*, Phys. Lett. **B382** (1996) 65–72, `hep-th/9604119`

[46] C. P. Bachas, P. Bain and M. B. Green, *Curvature terms in D-brane actions and their M-theory origin*, JHEP **05** (1999) 011, `hep-th/9903210`

[47] E. S. Fradkin and A. A. Tseytlin, *Nonlinear Electrodynamics from Quantized Strings*, Phys. Lett. **B163** (1985) 123

[48] A. Abouelsaood, J. Callan, Curtis G., C. R. Nappi and S. A. Yost, *Open Strings in Background Gauge Fields*, Nucl. Phys. **B280** (1987) 599

[49] M. Cederwall, A. von Gussich, B. E. W. Nilsson and A. Westerberg, *The Dirichlet super-three-brane in ten-dimensional type IIB supergravity*, Nucl. Phys. **B490** (1997) 163–178, `hep-th/9610148`

[50] M. Cederwall, A. von Gussich, B. E. W. Nilsson, P. Sundell and A. Westerberg, *The Dirichlet super-p-branes in ten-dimensional type IIA and IIB supergravity*, Nucl. Phys. **B490** (1997) 179–201, `hep-th/9611159`

[51] M. Aganagic, C. Popescu and J. H. Schwarz, *Gauge-invariant and gauge-fixed D-brane actions*, Nucl. Phys. **B495** (1997) 99–126, `hep-th/9612080`

[52] M. Aganagic, C. Popescu and J. H. Schwarz, *D-brane actions with local kappa symmetry*, Phys. Lett. **B393** (1997) 311–315, `hep-th/9610249`

[53] E. Bergshoeff and P. K. Townsend, *Super D-branes*, Nucl. Phys. **B490** (1997) 145–162, `hep-th/9611173`

[54] A. Bilal, *Higher-derivative corrections to the non-abelian Born- Infeld action*, Nucl. Phys. **B618** (2001) 21–49, `hep-th/0106062`

[55] O. D. Andreev and A. A. Tseytlin, *Partition Function Representation for the Open Superstring Effective Action: Cancellation of Mobius Infinities and Derivative Corrections to Born-Infeld Lagrangian*, Nucl. Phys. **B311** (1988) 205

[56] N. Wyllard, *Derivative corrections to D-brane actions with constant background fields*, Nucl. Phys. **B598** (2001) 247–275, `hep-th/0008125`

[57] N. Wyllard, *Derivative corrections to the D-brane Born-Infeld action: Non-geodesic embeddings and the Seiberg-Witten map*, JHEP **08** (2001) 027, `hep-th/0107185`

[58] M. de Roo and M. G. C. Eenink, *The effective action for the 4-point functions in abelian open superstring theory*, JHEP **08** (2003) 036, `hep-th/0307211`

[59] O. Chandia and R. Medina, *4-point effective actions in open and closed superstring theory*, JHEP **11** (2003) 003, `hep-th/0310015`

[60] D. J. Gross and E. Witten, *Superstring Modifications of Einstein's Equations*, Nucl. Phys. **B277** (1986) 1

[61] A. A. Tseytlin, *Vector Field Effective Action in the Open Superstring Theory*, Nucl. Phys. **B276** (1986) 391

[62] A. A. Tseytlin, *On non-abelian generalisation of the Born-Infeld action in string theory*, Nucl. Phys. **B501** (1997) 41–52, `hep-th/9701125`

[63] Y. Kitazawa, *Effective Lagrangian for Open Superstring from Five Point Function*, Nucl. Phys. **B289** (1987) 599

[64] E. S. Fradkin and A. A. Tseytlin, *Quantum String Theory Effective Action*, Nucl. Phys. **B261** (1985) 1–27

[65] E. S. Fradkin and A. A. Tseytlin, *Effective Action Approach to Superstring Theory*, Phys. Lett. **B160** (1985) 69–76

[66] E. S. Fradkin and A. A. Tseytlin, *Effective field theory from quantized strings*, Physics Letters B **158** (Aug., 1985) 316–322

[67] R. G. Leigh, *Dirac-Born-Infeld Action from Dirichlet Sigma Model*, Mod. Phys. Lett. **A4** (1989) 2767

[68] S. Nevens, A. Sevrin, W. Troost and A. Wijns, *Derivative corrections to the Born-Infeld action through beta-function calculations in $N = 2$ boundary superspace*, JHEP **08** (2006) 086, `hep-th/0606255`

[69] A. Neveu and J. Scherk, *Connection between Yang-Mills fields and dual models*, Nuclear Physics B **36** (Jan., 1972) 155–161

[70] J. Scherk and J. H. Schwarz, *Dual Models for Nonhadrons*, Nucl. Phys. **B81** (1974) 118–144

[71] T. Yoneya, *Connection of Dual Models to Electrodynamics and Gravidynamics*, Prog. Theor. Phys. **51** (1974) 1907–1920

[72] E. Bergshoeff, M. Rakowski and E. Sezgin, *Higher derivative super Yang-Mills theories*, Physics Letters B **185** (Feb., 1987) 371–376

[73] P. Koerber and A. Sevrin, *The non-Abelian Born-Infeld action through order alpha'**3*, JHEP **10** (2001) 003, `hep-th/0108169`

[74] P. Koerber and A. Sevrin, *The non-abelian D-brane effective action through order alpha'**4*, JHEP **10** (2002) 046, `hep-th/0208044`

[75] J. H. Schwarz, *Superstring Theory*, Phys. Rept. **89** (1982) 223–322

[76] B. Zwiebach, *A first course in string theory*, Cambridge, UK: Univ. Pr. (2004) 558 p

[77] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, *Progress Toward a Theory of Supergravity*, Phys. Rev. **D13** (1976) 3214–3218

[78] J. Callan, Curtis G., E. J. Martinec, M. J. Perry and D. Friedan, *Strings in Background Fields*, Nucl. Phys. **B262** (1985) 593

[79] A. Sen, *The Heterotic String in Arbitrary Background Field*, Phys. Rev. **D32** (1985) 2102

[80] A. Sen, *Non-linear sigma-models and string theories*, NASA STI/Recon Technical Report N **87** (Oct., 1986) 28479–+

[81] S. P. de Alwis, *Strings in background fields, beta functions and vertex operators*, Phys. Rev. **D34** (1986) 3760

[82] E. S. Fradkin and A. A. Tseytlin, *Fields as excitations of quantizaed coordinates*, JETP Lett. **41** (1985) 206–209

[83] H. Kawai, D. C. Lewellen and S. H. H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, Nucl. Phys. **B269** (1986) 1

[84] D. J. Gross and J. H. Sloan, *The Quartic Effective Action for the Heterotic String*, Nucl. Phys. **B291** (1987) 41

[85] B. E. Fridling, *On the String - Nonlinear Sigma Model Connection*, UMI 86-17563

[86] T. Ortin, *Gravity and strings*, Cambridge Unversity, Cambridge University Press, 2004

[87] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, *Ten-dimensional Maxwell-Einstein supergravity, its currents and the issue of its auxiliary fields.*, Nuclear Physics B **195** (1982) 97–136

[88] A. Chamseddine, *N=4 supergravity coupled to N=4 matter and hidden symmetry*, Nucl. Phys. **B185** (1981) 403

[89] G. F. Chapline and N. S. Manton, *Unification of Yang-Mills Theory and Supergravity in Ten- Dimensions*, Phys. Lett. **B120** (1983) 105–109

[90] I. C. G. Campbell and P. C. West, *N=2 D=10 Nonchiral Supergravity and Its Spontaneous Compactification*, Nucl. Phys. **B243** (1984) 112

[91] F. Giani and M. Pernici, *N=2 supergravity in ten dimensions*, Phys. Rev. **D30** (1984) 325–333

[92] M. Huq and M. A. Namazie, *Kaluza-Klein supergravity in ten dimensions*, Classical and Quantum Gravity **2** (May, 1985) 293–308

[93] E. Bergshoeff, H. J. Boonstra and T. Ortin, *S duality and dyonic p-brane solutions in type II string theory*, Phys. Rev. **D53** (1996) 7206–7212, `hep-th/9508091`

[94] Gross, D. J. and Harvey, J. A. and Martinec, E. and Rohm, R. , *Heterotic string theory (II). The interacting heterotic string*, Nuclear Physics B **267** (Mar., 1986) 75–124

[95] W. Nahm, *Supersymmetries and their representations*, Nucl. Phys. **B135** (1978) 149

[96] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, Phys. Lett. **B76** (1978) 409–412

[97] R. R. Metsaev and A. A. Tseytlin, *Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the Sigma Model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor*, Nucl. Phys. **B293** (1987) 385

[98] C. M. Hull and P. K. Townsend, *The loop beta function for sigma models with torsion*, Phys. Lett. **B191** (1987) 115

[99] E. Bergshoeff and M. de Roo, *Supersymmetric Chern-Simons terms in ten-dimensions*, Phys. Lett. **B218** (1989) 210

[100] E. A. Bergshoeff and M. de Roo, *The quartic effective action of the heterotic string and supersymmetry*, Nucl. Phys. **B328** (1989) 439

[101] A. A. Tseytlin, *Heterotic - type I superstring duality and low-energy effective actions*, Nucl. Phys. **B467** (1996) 383–398, `hep-th/9512081`

[102] B. Sahoo and A. Sen, *alpha'-Corrections to Extremal Dyonic Black Holes in Heterotic String Theory*, JHEP **01** (2007) 010, `hep-th/0608182`

[103] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, JHEP **09** (2005) 038, `hep-th/0506177`

[104] M. K. Gaillard and B. Zumino, *Self-duality in nonlinear electromagnetism*, `hep-th/9705226`

[105] M. K. Gaillard and B. Zumino, *Nonlinear electromagnetic self-duality and Legendre transformations*, `hep-th/9712103`

[106] M. K. Gaillard and B. Zumino, *Duality Rotations for Interacting Fields*, Nucl. Phys. **B193** (1981) 221

[107] G. W. Gibbons and D. A. Rasheed, *SL(2,R) Invariance of Non-Linear Electrodynamics Coupled to An Axion and a Dilaton*, Phys. Lett. **B365** (1996) 46–50, `hep-th/9509141`

[108] G. W. Gibbons and D. A. Rasheed, *Electric - magnetic duality rotations in nonlinear electrodynamics*, Nucl. Phys. **B454** (1995) 185–206, `hep-th/9506035`

[109] J. H. Schwarz and A. Sen, *Duality symmetric actions*, Nucl. Phys. **B411** (1994) 35–63, hep-th/9304154

[110] P. Pasti, D. P. Sorokin and M. Tonin, *Note on manifest Lorentz and general coordinate invariance in duality symmetric models*, Phys. Lett. **B352** (1995) 59–63, hep-th/9503182

[111] E. Schrodinger, *Contributions to Born's New Theory of the Electromagnetic Field*, Proc.Roy.Soc. **A150** (1935) 465

[112] S. M. Kuzenko and S. Theisen, *Nonlinear self-duality and supersymmetry*, Fortsch. Phys. **49** (2001) 273–309, hep-th/0007231

[113] M. Hatsuda, K. Kamimura and S. Sekiya, *Electric-magnetic duality invariant Lagrangians*, Nucl. Phys. **B561** (1999) 341–353, hep-th/9906103

[114] G. Whitman, *Linear and non-linear waves*, Jhon Wiley and sons (1974)

[115] M. Perry and J. H. Schwarz, *Interacting chiral gauge fields in six dimensions and Born-Infeld theory*, Nucl. Phys. **B489** (1997) 47–64, hep-th/9611065

[116] H. Van der Heinden, *Non-linear electrodynamics and duality*, www.strings.fmns.rug.nl/postscripts/vanderheiden140706.pdf

[117] W. Chemissany, *Selfduality and derivative corrections*, (in progress)

[118] T. Kimura and I. Oda, *Duality of super D-brane actions in general type II supergravity background*, Int. J. Mod. Phys. **A16** (2001) 503–528, hep-th/9904019

[119] T. Kimura and I. Oda, *Duality of super D-brane actions in type II supergravity background*, Nucl. Phys. **B551** (1999) 183–203, hep-th/9811134

[120] Y. Igarashi, K. Itoh and K. Kamimura, *Electric-magnetic duality rotations and invariance of actions*, Nucl. Phys. **B536** (1998) 454–468, hep-th/9806160

[121] Y. Igarashi, K. Itoh and K. Kamimura, *Self-duality in super D3-brane action*, Nucl. Phys. **B536** (1998) 469–482, hep-th/9806161

[122] P. Aschieri, D. Brace, B. Morariu and B. Zumino, *Nonlinear self-duality in even dimensions*, Nucl. Phys. **B574** (2000) 551–570, hep-th/9909021

[123] D. Brace, B. Morariu and B. Zumino, *Duality invariant Born-Infeld theory*, hep-th/9905218

[124] P. Fre, *Gaugings and other supergravity tools of p-brane physics*, hep-th/0102114

[125] B. de Wit, *Supergravity*, hep-th/0212245

[126] D. Westra, *Symmetries in $N = 4$ supergavities.* PhD thesis, Rijksuniversiteit Groningen, 2006.

[127] U. Magnea, *An introduction to symmetric spaces*, cond-mat/0205288

[128] J. Maharana and J. H. Schwarz, *Noncompact symmetries in string theory*, Nucl. Phys. **B390** (1993) 3–32, hep-th/9207016

[129] H. Eichenherr and M. Forger, *On the Dual Symmetry of the Nonlinear Sigma Models*, Nucl. Phys. **B155** (1979) 381

[130] P. Breitenlohner, D. Maison and G. W. Gibbons, *Four-Dimensional Black Holes from Kaluza-Klein Theories*, Commun. Math. Phys. **120** (1988) 295

[131] R. Gilmore, *Lie groups, Lie algebra, and some of their applications*, Wiley Interscience (1974)

[132] P. Fre', F. Gargiulo and K. Rulik, *Cosmic billiards with painted walls in non-maximal supergravities: A worked out example*, Nucl. Phys. **B737** (2006) 1–48, hep-th/0507256

[133] P. Fre *et al.*, *Tits-Satake projections of homogeneous special geometries*, Class. Quant. Grav. **24** (2007) 27–78, hep-th/0606173

[134] A. Keurentjes, *Poincare duality and G+++ algebras*, Commun. Math. Phys. **275** (2007) 491–527, `hep-th/0510212`

[135] A. Collinucci, *Instantons and Cosmologies in String Theory.* PhD thesis, Rijksuniversiteit Groningen, 2005.

[136] C. M. Hull and B. Julia, *Duality and moduli spaces for time-like reductions*, Nucl. Phys. **B534** (1998) 250–260, `hep-th/9803239`

[137] A. Sen, *Non-BPS states and branes in string theory*, `hep-th/9904207`

[138] M. J. Duff, R. R. Khuri and J. X. Lu, *String solitons*, Phys. Rept. **259** (1995) 213–326, `hep-th/9412184`

[139] H. Lu, C. N. Pope, E. Sezgin and K. S. Stelle, *Stainless super p-branes*, Nucl. Phys. **B456** (1995) 669–698, `hep-th/9508042`

[140] M. J. Duff and K. S. Stelle, *Multi-membrane solutions of $D = 11$ supergravity*, Phys. Lett. **B253** (1991) 113–118

[141] R. Güven, *Black p-brane solutions of $D = 11$ supergravity theory.*, Physics Letters B **276** (Feb., 1992) 49–55

[142] E. B. Bogomolny, *Stability of Classical Solutions*, Sov. J. Nucl. Phys. **24** (1976) 449

[143] M. K. Prasad and C. M. Sommerfield, *Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon*, Phys. Rev. Lett. **35** (Sep, 1975) 760–762

[144] E. Witten and D. I. Olive, *Supersymmetry Algebras That Include Topological Charges*, Phys. Lett. **B78** (1978) 97

[145] B. Janssen, P. Smyth, T. Van Riet and B. Vercnocke, *A first-order formalism for timelike and spacelike brane solutions*, `arXiv:0712.2808 [hep-th]`

[146] A. Dabholkar, *Microstates of non-supersymmetric black holes*, Phys. Lett. **B402** (1997) 53–58, `hep-th/9702050`

[147] M. Gutperle and A. Strominger, *Spacelike branes*, JHEP **04** (2002) 018, `hep-th/0202210`

[148] M. Kruczenski, R. C. Myers and A. W. Peet, *Supergravity S-branes*, JHEP **05** (2002) 039, `hep-th/0204144`

[149] E. Bergshoeff, A. Collinucci, A. Ploegh, S. Vandoren and T. Van Riet, *Non-extremal D-instantons and the AdS/CFT correspondence*, JHEP **01** (2006) 061, `hep-th/0510048`

[150] A. Strominger, *The dS/CFT correspondence*, JHEP **10** (2001) 034, `hep-th/0106113`

[151] K. Skenderis and P. K. Townsend, *Hidden supersymmetry of domain walls and cosmologies*, Phys. Rev. Lett. **96** (2006) 191301, `hep-th/0602260`

[152] H. J. Boonstra, K. Skenderis and P. K. Townsend, *The domain wall/QFT correspondence*, JHEP **01** (1999) 003, `hep-th/9807137`

[153] C.-M. Chen, D. V. Gal'tsov and M. Gutperle, *S-brane solutions in supergravity theories*, Phys. Rev. **D66** (2002) 024043, `hep-th/0204071`

[154] S. Arapoglu, N. S. Deger, A. Kaya, E. Sezgin and P. Sundell, *Multi-spin giants*, Phys. Rev. **D69** (2004) 106006, `hep-th/0312191`

[155] N. Ohta, *Intersection rules for S-branes*, Phys. Lett. **B558** (2003) 213–220, `hep-th/0301095`

[156] G. Jones, A. Maloney and A. Strominger, *Non-singular solutions for S-branes*, Phys. Rev. **D69** (2004) 126008, `hep-th/0403050`

[157] J. E. Wang, *Twisting S-branes*, JHEP **05** (2004) 066, `hep-th/0403094`

[158] P. Fre *et al.*, *Cosmological backgrounds of superstring theory and solvable algebras: Oxidation and branes*, Nucl. Phys. **B685** (2004) 3–64, hep-th/0309237

[159] A. Kleinschmidt and H. Nicolai, *E(10) cosmology*, JHEP **01** (2006) 137, hep-th/0511290

[160] G. W. Gibbons, M. B. Green and M. J. Perry, *Instantons and Seven-Branes in Type IIB Superstring Theory*, Phys. Lett. **B370** (1996) 37–44, hep-th/9511080

[161] E. Bergshoeff, A. Collinucci, U. Gran, D. Roest and S. Vandoren, *Non-extremal D-instantons*, JHEP **10** (2004) 031, hep-th/0406038

[162] D. V. Gal'tsov and O. A. Rytchkov, *Generating branes via sigma-models*, Phys. Rev. **D58** (1998) 122001, hep-th/9801160

[163] D. Gaiotto, W. W. Li and M. Padi, *Non-Supersymmetric Attractor Flow in Symmetric Spaces*, arXiv:0710.1638 [hep-th]

[164] M. Gutperle and M. Spalinski, *Supergravity instantons for $N = 2$ hypermultiplets*, Nucl. Phys. **B598** (2001) 509–529, hep-th/0010192

[165] N. Arkani-Hamed, J. Orgera and J. Polchinski, *Euclidean Wormholes in String Theory*, arXiv:0705.2768 [hep-th]

[166] M. Gutperle and W. Sabra, *Instantons and wormholes in Minkowski and (A)dS spaces*, Nucl. Phys. **B647** (2002) 344–356, hep-th/0206153

[167] A. Knapp, *Lie group beyond an introduction*, Birkh

[168] P. K. Townsend and M. N. R. Wohlfarth, *Accelerating cosmologies from compactification*, Phys. Rev. Lett. **91** (2003) 061302, hep-th/0303097

[169] H. Lu, S. Mukherji and C. N. Pope, *From p-branes to cosmology*, Int. J. Mod. Phys. **A14** (1999) 4121–4142, hep-th/9612224

[170] P. Fre and A. Sorin, *Integrability of supergravity billiards and the generalized Toda lattice equation*, Nucl. Phys. **B733** (2006) 334–355, hep-th/0510156

[171] M. Nakahara, *Geometry, topology and physics*, IOP Publishing (2003)

[172] T. Frankel, *The geometry of physics: an introduction*, Cambridge Univ. Pr. (1997)

[173] J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representation*,

[174] H. Georgi, *Lie Algebras in Particle Physics. From Isospin to Unified Theories*, Front. Phys. **54** (1982) 1–255

[175] S. Helgason, *Differential geometry and symmetric spaces*, Academic, New York, 1969

Nederlandse Samenvatting

Het eerste gedeelte van dit proefschrift behandelt de constructie van de ijktheorie die op een enkel D-braan leeft alsmede superzwaartekrachtstheorieën die kunnen ontstaan als laag-energetische effectieve beschrijvingen van snaartheorie en M-theorie. Van deze laatste wordt gedacht dat ze consistente quantumzwaartekrachtstheorieën zijn, welke de vier verschillende krachten unificeren en zo quantumveldentheorie (QFT) en algemene relativiteitstheorie (GR) met elkaar verenigen. Het tweede gedeelte van het proefschrift is gewijd aan het afleiden van braanoplossingen van (super)zwaartekrachtstheorieën, welke een essentiële rol blijken te spelen in het versterken van ons geloof in dualiteiten in de niet-perturbatieve limiet. Om de verschijning en de verdienste van snaar- en M-theorie en ook de ontdekking van dualiteiten volledig te kunnen waarderen, zullen we eerst de historische ontwikkeling van de deeltjes- en de hoge-energie fysica weergeven.

Het grootste gedeelte van de theoretische natuurkunde van de twintigste eeuw wordt gedomineerd door twee grote mijlpalen die allebei een radicale verandering veroorzaakten: quantummechanica en algemene relativiteitstheorie.

In de twintiger en dertiger jaren van die eeuw werd quantummechanica geformuleerd als de theorie die het gedrag van deeltjes op (sub)atomaire schaal beschrijft, het is daarom de theorie die van toepassing is als men met elementaire deeltjes van doen heeft. Gebaseerd op experimenten werd opgemerkt dat alle deeltjes in de natuur een fundamentele eigenschap, genaamd *spin*, hebben, waarvan de waarde deze deeltjes verdeelt in twee categorieën: bosonen en fermionen. De fermionische sector bevat alle materie en bestaat uit drie generaties, die elk weer twee quarks en twee leptonen (een electron en een neutrino) bevatten. Bijna alle ons bekende materie bestaat uit de lichtste variant van deze drie generaties.

Tussen 1905 en 1916 stelde Einstein zijn relativiteitstheorie voor. Hij stelt dat de wetten van de natuur voor iedere observator hetzelfde moeten zijn en dat deze daarom in een observator-onafhankelijke manier geformuleerd dienen te worden (covariant). De relativiteitstheorie bestaat uit twee delen: de speciale relativiteitstheorie welke onze noties van tijd en ruimte radicaal veranderde en laat zien hoe deze concepten op een gecompliceerde manier verbonden zijn, en de algemene relativiteitstheorie (GR) welke de ruimtetijd als dynamische entiteit beschrijft: het metrische veld. In GR manifesteert de zwaartekracht zich als kromming van de ruimtetijd, veroorzaakt door de aanwezigheid van massa en energie.

Een combinatie van de speciale relativiteitstheorie en quantummechanica leidde uiteindelijk rond 1970 tot het standaardmodel (SM), welke vrij succesvol de interacties tussen elementaire deeltjes beschrijft. Het standaard model is een specifieke quantumveldentheorie (QFT) uit oneindig veel mogelijke quantumveldentheorieën. Hierbij speelt het concept van ijktheorie een belangrijke rol. Door symmetrietransformaties lokaal te maken, dat wil zeggen door coordinaat-afhankelijke transformatieparameters te introduceren, ontstaan spin 1 ijkbosonen welke de krachten tussen deeltjes dragen. Eigenlijk hebben bovengenoemde materiedeeltjes hun interactie middels het uitwisselen van bosonen: de electromagnetische, de zwakke en de sterke krachten worden beschreven door het uitwisselen van respectievelijk fotonen, W/Z intermediaire vectorbosonen en gluonen. De groep van het SM is $SU(3) \times SU(2) \times U(1)$. De experimentele bevestiging van het SM is verbijsterend goed tot energieën van 10^2GeV. Echter, er resteren enkele problemen. Ten eerste is de Higgs-sector, verantwoordelijk voor het geven van massa aan de andere elementaire deeltjes, tot dusver nog niet waargenomen[1]. Ten tweede zijn er dwingende theoretische argumenten om de zoektocht naar verdere uitbreiding voort te zetten: allereerst bevat het SM negentien nauwkeurig afgestemde parameters[2] die niet voorspeld kunnen worden en dientengevolge is het geen fundamentele theorie. Verder is het moeilijk om te verklaren waarom de Higgs-massa zo ontzettend klein is (met $m_H \leq 1\text{TeV}/c^2$), dit wordt ook wel het hiërarchie-probleem genoemd. Ook is het nog niet begrepen waarom er drie generaties materiedeeltjes bestaan. Bovendien blijkt dat de drie 'running coupling constants' die worden geassociëerd met de SM-ijkgroep ongeveer gelijke waarden aannemen bij de enorm hoge energie van 10^{15}GeV. Dit suggereert dat de drie krachten bij deze energie geünificeerd geraken in een enkele 'grote geünificeerde theorie' (GUT) die is gebaseerd op een simpele ijkgroep. Merk op dat het SM de vierde fundamentele kracht, de zwaartekracht, niet bevat omdat de andere drie krachten veel sterker zijn dan de zwaartekracht.

Laten we eens teruggaan naar GR. De experimentele en theoretische successen van GR zijn net zo ontzagwekkend als die van SM. Ter illustratie, GR verklaart het buigen van licht door massieve voorwerpen zoals onze zon. Ook voorspelt GR het bestaan van ruimtetijd singulariteiten binnen zwarte gaten[3]. Tevens speelt GR een sleutelrol in de hedendaagse kosmologie, waar het bijvoorbeeld de waargenomen kosmologische roodverschuiving van licht dat afkomstig is van verre sterrenstelsels verklaart, tengevolge van de uitbreiding van het heelal. Tot dusver wordt GR gebruikt als een

[1]Dit is een van de belangrijkste doelen van de nieuwe LHC versneller van CERN, die de experimentele energieschaal verhoogt tot $\sim 10^4$GeV.

[2]Bijvoorbeeld de parameters die corresponderen met de massa's van elementaire deeltjes.

[3]Zwarte gaten zijn objecten die zo massief zijn, dat ze achter een waarnemingshorizon verscholen zijn, een oppervlak waardoor zelfs licht niet kan ontsnappen (tenminste volgens de klassieke theorie).

klassieke veldentheorie. Een poging om de zwaartekracht te quantiseren met gelijksoortige quantisatietechnieken zoals gebruikt voor het SM is mislukt. De theorie gaat gebukt onder singulariteiten welke, in tegenstelling tot bij SM ('t Hooft en Veltman [1]), niet in de hand gehouden kunnen worden. Dit kan worden ingezien vanwege het feit dat de koppelingsconstante van de zwaartekracht $\kappa = 8\pi G/c^4$ niet dimensieloos is, deze is daarom ongeschikt om te gebruiken in een storingsreeks. De schaal waarop quantumzwaartekracht belangrijk wordt, is de Planck schaal, gegeven door

$$l_{\text{Planck}} = \sqrt{\frac{8\pi G h}{c^3}} \sim 4.1 \cdot 10^{-35}\text{m}, \quad M_{\text{Planck}} = \left(\frac{hc^5}{8\pi G}\right)^{1/2} \sim 10^{18}\text{GeV}, \tag{D.0.1}$$

met h Planck's constante. Zoals men kan zien, ligt de Planck schaal erg dicht bij de GUT schaal (10^{15}GeV). Deze observatie laat zien dat er een "quantumzwaartekrachtstheorieïs vereist die alle vier fundamentele krachten gelijktijdig kan behandelen.

Als een eerste poging dachten fysici aan een theoretische verbetering van SM middels de introductie van een ander type symmetrie, genaamd supersymmetrie. Dit is een symmetrie tussen bosonen en fermionen die voorspelt dat er bij ieder boson in de natuur een fermionische partner bestaat, zo ook vice versa. De eerste motivatie om zulk een symmetrie te gebruiken is dat het hiërarchie-probleem ermee omzeild wordt; het is aangetoond dat de Higgs-massa wordt afgeschermd van quantumcorrecties door supersymmetrie. Echter, supersymmetrie transformaties introduceren ook vele nieuwe deeltjes-'sdeeltjes' die niet zijn geobserveerd[4]. Een gedeeltelijk succes in het samenbrengen van alle fundamentele krachten werd bereikt in 1976 door theorieën te beschouwen die gebaseerd zijn op lokale supersymmetrie. Zulke theorieën worden superzwaartekrachtstheorieën genoemd, uitbreidingen van GR theorie die zich beter gedragen bij hoge energieën; de oneindigheden worden namelijk *gedeeltelijk* opgeheven. Het spin 2 ijkboson dat verantwoordelijk is voor de dracht van de zwaartekracht wordt het graviton genoemd. Zijn supersymmetrische partner is het gravitino[5].

Snaartheorie is het meest veelbelovende voorstel waarmee de quantumzwaartekracht beschreven kan worden. Snaartheorie vervangt deeltjes door de trillingstoestanden van relativistische snaren[6]. Opmerkelijk is dat het graviton en niet-Abelse ijkvelden noodzakelijkerwijs deel van het spectrum uitmaken. Zo verenigt snaartheorie op een natuurlijke manier de zwaartekrachtsinteractie met Yang-Mills theorie (niet-Abelse versie van Maxwell theorie). Daarbij levert snaartheorie een discrete maar oneindige toren van massieve trillingstoestanden. Hun massa-schaal is van de orde van de Planck-massa. In supersymmetrische versies van snaartheorie (supers-

[4] Als supersymmetrie bestaat, dan moet die spontaan gebroken zijn opdat er superdeeltjes met een hogere massa uit voortkomen. Er bestaat een sterke hoop dat deze zullen worden ondekt in de LHC.

[5] Om deze deeltjes waar te nemen zouden energieën nodig zijn die ver buiten het bereik van onze huidige versnellers vallen.

[6] Merk op dat de snaar een typische lengte l_s van de orde van de Planck l_p lengte heeft.

naartheorieën) wordt het graviton op het massaloze niveau vergezeld door de superzwaartekrachtsveld sector. Inderdaad werd gevonden dat de *laag-energetische limiet* van supersnaartheorie wordt gegeven door superzwaartekracht. Er bestaat een intuitieve reden waarom supersnaartheorie vrij van oneindigheden is. Deze oneindigheden verschijnen meestal in singuliere punten, maar een snaar beweegt in een ruimtetijd over een twee-dimensionaal oppervlak. Dit feit zorgt er precies voor dat interacties niet in een enkel punt plaatsvinden, maar zijn uitgespreid over een kleine oppervlakte. Het blijkt dat de perturbatieve snaarinteracties UV-eindig zijn[7].

Snaartheorie heeft naast supersymmetrie zes extra dimensies nodig om consistent te kunnen worden opgezet. Dit kan men eerder als deugd opvatten dan als een kwaad. Het is al lange tijd bekend dat hoger-dimensionale theorieën een aantal aantrekkelijke eigenschappen hebben. In de twintiger jaren van de twintigste eeuw hebben Kaluza en Klein [2,3] geprobeerd om Einstein en Maxwelltheorie te verenigen door vier-dimensionale zwaartekracht en electromagnetisme in een vijf-dimensionale ruimte in te bedden. Precies zo nemen we in snaartheorie de zes interne dimensies erg klein zodat ze niet waar te nemen zijn in hedendaagse experimenten. Deze procedure heet Kaluza-Klein *dimensionele reductie*.

Helaas heeft snaartheorie ook zo zijn nadelen. Ze is slechts perturbatief gedefiniëerd; verstrooiingsamplitudi worden uitgedrukt als een oneindige ontwikkeling in machten van de snaarkoppelconstante g_s die is geassocieerd met de Feynmandiagrammen van snaartheorie. Het grootste minpunt werd echter duidelijk toen er vijf verschillende supersnaartheorieën leken te zijn terwijl er werd gehoopt op een enkele, unieke theorie van quantumzwaartekracht. Dit betekent dat perturbatieve snaartheorieën slechts een gedeelte van het gehele plaatje verschaffen.

Gelukkig is er veel progressie geboekt op dit punt. De grootste stap voorwaarts was de ontdekking van dualiteiten, symmetrietransformaties die de verschillende snaartheorieën verbinden. Ze relateren in sommige gevallen het regime van zwakke koppeling met dat van sterke koppeling zo dat perturbatieve berekeningen in de eerste theorie niet-perturbatieve informatie verschaffen over de tweede theorie (genaamd S-dualiteit). Daarnaast werden snaartheorieën op verschillende achtergronden equivalent bevonden (T-dualiteit). Een belangrijke rol werd gespeeld door de zogenaamde braanoplossingen van snaartheorie. Dit zijn solitonische objecten die kunnen worden gezien als hoger-dimensionale generalisaties van snaren[8]. Een belangrijke klasse van branen zijn Dirichlet-branen, kortweg D-branen. Deze zijn speciaal omdat ze aan enerzijds verschijnen als hypervlakken waarop snaren kunnen eindigen en aan de andere kant als stationaire oplossingen van (super)zwaartekrachtstheorieën. Er is nog een klasse van braanoplossingen, S-branen (ruimtelijke branen) welke tijdsafhankelijke oplossingen zijn van (super)zwaartekrachtstheorieën. De vijf ogenschijnlijk

[7]Er is geen noodzaak om een ultraviolette cut-off te introduceren en de theorie is consistent tot op hoge energieschalen, daarom is ze fundamenteel.

[8]Branen kunnen ook worden beschouwd als hoger-dimensionale generalisaties van zwarte gaten.

verschillende theorieën en hun braanoplossingen worden verbonden middels een web van dualiteiten. Gedurende de negentiger jaren van de vorige eeuw werd geleidelijk duidelijk dat deze vijf theorieën allen verschillende limieten in de parameterruimte van een enkele onderliggende theorie, genaamd M-theorie, representeerden. De fundamentele vrijheidsgraden van M-theorie blijven grotendeels onbekend. In plaats van een voltooide theorie, blijft M-theorie vooral werk in progressie.

Aldus hebben we wat inzicht verkregen en een wat beter begrip opgedaan van perturbatieve en niet-perturbatieve snaartheorie. Echter, er zijn nog veel interessante open stukken. Ten eerste is er het gebrek aan experimentele ondersteuning. Inderdaad, alle beloften ten spijt levert snaartheorie geen enkele harde, controleerbare voorspelling. De rivaliserende quantumzwaartekrachtstheorieën doen dit trouwens ook niet. Het is mogelijk om configuraties in snaartheorie te construeren die erg op het SM lijken, bijvoorbeeld door snijdende D-branen te gebruiken. Echter, tot op heden is er nog geen manier gevonden om deze modellen uit te lichten als voorkeursvacua. Daarnaast, omdat snaartheorie zwaar leunt op supersymmetrie en supersymmetrie wordt gedeeld met veel andere theorieën, vooral met het supersymmetrische SM, zou de experimentele ontdekking van supersymmetrie nauwelijks een volledige bevestiging van snaartheorie zijn. Vanwege de extreem hoge energieën die hierbij betrokken zijn, ligt de toekomst van experimentele verificatie wellicht niet bij deeltjesversnellers maar in astrofysische en cosmologische ontwikkelingen. Merk op dat snaartheorie al een belangrijke toets heeft doorstaan door op een quantummechanische manier een probleem deels op te lossen dat verschijnt bij het beschrijven van een typisch algemeen relativistisch object, een zwart gat: ze berekent succesvol de semi-klassiek voorspelde entropie van een supersymmetrisch zwart gat middels het optellen van zijn micro-toestanden. Helaas blijven er nog vele harde noten te kraken in deze domeinen zoals de verklaring voor de geobserveerde kleine positieve cosmologische constante en de constructie van snaartheorie in tijdsafhankelijke achtergronden (bijv. S-braanoplossingen).

Echter, in de discussie is tot dusver achtergehouden dat snaartheorie soms een ongeloofelijk krachtig gereedschap is in andere velden van de natuur- en wiskunde. In deze beperkte ruimte kunnen we slechts een paar voorbeelden geven. Meest succesvol is de connectie met ijktheorieën. Het blijkt dat veel eigenschappen van ijktheorie een geometrische interpretatie hebben in termen van D-branen. Enige tijd geleden beargumenteerde 't Hooft dat ijktheorieën in de limiet van N groot [4] erg op een snaartheorie lijken. Een eerste concrete realisatie van zo'n verband was de AdS/CFT dualiteit[9]. Andere voorbeelden zijn de inlijving van Montonen-Olive dualiteit [6] van ijktheorie in de grotere S-dualiteit van snaartheorie en de recente progressies in de niet-perturbatieve berekening van de chirale sector van $N = 1$ Super-Yang-Mills theorie. Hoe dan ook, vele van deze verbanden zijn niet bevestigd op een stricte mathematische manier. Zo zijn bijvoorbeeld de AdS/CFT correspondentie en S-

[9]Deze betrekking stelt dat $N = 4$ Super-Yang-Mills theorie duaal is aan snaartheorie op $AdS^5 \times S^5$ [5].

dualiteit feitelijk vermoedens waarvoor inmiddels wel een indrukwekkende hoeveelheid indirecte aanduidingen zijn gevonden.

De lage-energie limiet (veldentheorie limiet) van snaartheorie blijft een belangrijk gereedschap om de verschillende verschijnselen in de snaartheorie te bestuderen. Veel kenmerken van snaar- en M-theorie zijn ook aanwezig in de lage-energie limiet, zoals D-branen en dualiteiten, en daarom is het interessant om deze effectieve beschrijving te bestuderen.

In dit proefschrift zullen we eerst de lage-energie limiet van de snaartheorie bestuderen. In het bijzonder zullen we laten zien, hoe de snaren zichzelf manifesteren als een ijktheorie die leeft op het D-braan. We zullen zien hoe de correcties op de leidende orde van de Maxwell actie interessante informatie verschaffen over de 'snaar-achtige' aspecten van D-braan fysica. We zullen vervolgens proberen om deze correcties in te perken, gebruikmakend van de *electromagnetische dualiteitssymmetrie*. Ook zullen superzwaartekrachts-acties in dit proefschrift worden gepresenteerd als de lage-energie boom-niveau effectieve actie van snaartheorie voor langzaam variërende kromming. Afgeleide-correcties, in het bijzonder correcties van de orde α' op heterotische snaartheorie[10], zullen worden onderzocht.

Terug naar de braanoplossingen. Het tweede doel van dit proefschrift is om branen te bestuderen die oplossingen zijn van (super)zwaartekrachtstheorieën. Zoals we zullen zien, vormen de dimensies van het uitgebreide object het wereldvolume van de braan. De overige ruimtetijd dimensies vormen de transversale ruimte. We maken onderscheid tussen twee soorten branen: indien de tijd deel uitmaakt van het wereldvolume wordt de braan "tijdachtige"p-braan genoemd. De p staat hier voor het aantal ruimtelijke richtingen van het wereldvolume. Het totale aantal dimensies van het wereldvolume is $p+1$. Als de tijd niet is bevat in het wereldvolume wordt de braan "ruimtelijke"Sp-braan genoemd. Voor zulke branen is het totale aantal dimensies $p+1$, welke allemaal ruimtelijk zijn. Zo refereert in beide gevallen p naar een $p+1$-dimensionaal wereldvolume.

Het onderzoeken van braanoplossingen middels het direct oplossen van de bewegingsvergelijkingen die volgen uit de (super)zwaartekrachtsactie is verre van triviaal. In plaats daarvan gaan we kijken naar braanoplossingen van welke de dynamica slechts van één parameter afhangen (deeltjes-achtige oplossingen). We zullen zien dat deze parameter een van de coordinaten van de transversale ruimte is. Dit betekent dat de wereldvolume coordinaten niet expliciet in de oplossingen zullen voorkomen. Dit impliceert dat men de oplossing effectief dimensioneel kan reduceren over het wereldvolume[11]. Dit projecteert een p-braan op een (-1)-braanoplossing. Indien we

[10]Heterotische snaartheorie is één van de eerdergenoemde vijf perturbatieve supersnaartheorieën.

[11]De reductie over het wereldvolume van de braan geeft aanleiding tot een massaloze lager-dimensionale theorie, terwijl de reductie over de transversale richtingen van de braan een scalar-potentiaal zal genereren in de lager-dimensionale theorie. Indien de lager-dimensionale massieve theorie leeft in een Minkowski-ruimtetijd, dan zijn er twee afzonderlijke oplossingen: tijdsafhankelijke oplossingen (kosmologie) en tijdsonafhankelijke oplossingen (domeinmuren).

reduceren over een Euclidische torus reduceren, is de resulterende lager-dimensionale theorie een Minkowski-theorie en de corresponderende oplossing is een S(-1)-braan. Als de reductie geschiedt over een Minkowski-torus (welke een tijdachtige richting bevat) dan leeft de lager-dimensionale theorie in een Euclidische ruimtetijd en heeft deze een (-1)-braan (instanton) als oplossing[12].

Het aantal globale symmetrieën wordt groter en groter als men het aantal dimensies kleiner maakt. Dit kan worden gebruikt om onze zoektocht naar braanoplossingen verder te vereenvoudigen. De (-1)-braanoplossingen van de lager-dimensionale theorie worden gedragen door de metriek en de scalaire velden. We zullen laten zien dat men de zwaartekrachtsveldvergelijkingen kan ontkoppelen van de scalarveldvergelijkingen. Daardoor kan men onafhankelijk voor de metriek en voor het scalarveld oplossen. De oplossing-genererende techniek zal ons in staat stellen om de meest algemene scalarveld-oplossingen te vinden.

[12] In deze analyse beschouwen we enkel consistente reducties. Dit houdt in dat we altijd de reductiestappen ongedaan kunnen maken zodanig dat we ervan verzekerd zijn dat we ook een oplossing hebben van de actie waarmee we begonnen zijn. Zo zou men een hoger-dimensionale oplossing kunnen verkrijgen via het opliften (oxidatie) van een lager-dimensionale oplossing.

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