Relativistic generalization of the Hubble law

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Abstract. A generalization of the Hubble law in the framework of the special theory of relativity received. The relativistic Doppler effect allows you to find a nonlinear relationship between redshift and distance.

1. About Hubble's law

The law of Edwin Hubble (1929), independently discovered by Georges Lemaître (1927), is considered [1, 2]. In standard terms, the law has the form:

$$v = H \cdot R, \;\; 0 \leq R \leq R_* = \frac{c}{H}.$$

It is known that the Hubble law (or the Hubble–Lemaître law) is invariant under Galilean transformations. Let us prove the opposite: the velocity distribution of galaxies, which is invariant with respect to the Galilean transformations or the Lorentz transformations, leads only to the Hubble law or to its generalization.

2. The Hubble law as a consequence of the invariance of the velocities distribution law with respect to Galilean transformations

According to the cosmological principle, we choose a homogeneous and isotropic model of the Universe, and write the dependence of speed on distance with the formula $v = f(R)$. Take three galaxies I, J, K, moving along one straight line. The distance between I and J is equal x, the distance between J and K is equal y, the distance between I and K is equal z. The galaxy J moves away from the galaxy I with speed U, the galaxy K moves away from the galaxy J with speed V, the galaxy K moves away from the galaxy I with speed W. The following equalities hold:

$$U = f(x), \; V = f(y), \; W = f(z), \; x + y = z, \; U + V = W.$$

The last velocities addition formula given by Newton [3] is a consequence of the Galilean transformations. Five equalities give a functional equation

$$f(x) + f(y) = f(x + y).$$

Cauchy found a solution to this equation in the class of continuous functions [4]:

$$f(x) = a \cdot x, \quad a - \text{constant}.$$

Equating the constant to the Hubble constant, we get the Hubble law $v = H \cdot R$.

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3. Relativistic generalization of the Hubble law as a consequence of the invariance of the velocities distribution law with respect to Lorentz transformations

We take the same three galaxies that move at high speeds and write five equalities for them

$$U = f(x), \quad V = f(y), \quad W = f(z), \quad x + y = z, \quad \frac{U+V}{1+\frac{U \cdot V}{c^2}} = W.$$

Here c is the speed of light. The last velocities addition formula given by Einstein [5] is a consequence of the Lorentz transformations. Equalities give a functional equation

$$\frac{f(x)+f(y)}{1+\frac{f(x)\cdot f(y)}{c^2}} = f(x+y).$$

The hyperbolic tangent gives the solution to this equation, in the class of continuously differentiable functions:

$$f(x) = c\tanh\frac{a \cdot x}{c}, \quad a - \text{constant}.$$

We get a generalization of the Hubble law:

$$v = c\tanh\frac{H \cdot R}{c}, \quad 0 \leq R < \infty.$$

According to the correspondence principle, the generalization of the law goes over into a linear law for small values R. Therefore, $a = H$. For large values R we obtain the asymptotic law $v = c$.

If $R = R_* = \frac{c}{H}$ then $v = \tanh(1) \cdot c \approx 0,76 \cdot c$.

4. Red shift and distance to galaxies. Relativistic theory

In this section, z is the redshift, the remaining notation is standard. Einstein found the formula for the relativistic Doppler effect [5]:

$$1 + z = 1 + \frac{\Delta\lambda}{\lambda} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}.$$

The Hubble law new formula gives equality

$$1 + z = \sqrt{\frac{1+\tanh\frac{H \cdot R}{c}}{1-\tanh\frac{H \cdot R}{c}}}.$$

Thanks to identity

$$\exp(x) = \sqrt{\frac{1+\tanh x}{1-\tanh x}},$$

we get the functional relationship of redshift and distance:

$$1 + z = \exp\left(\frac{H \cdot R}{c}\right) \quad \text{or} \quad R = \frac{c}{H}\ln(1+z).$$

We obtain the following correspondence of quantities:

$$z = e - 1 \approx 1,71828 \;\Rightarrow\; R = R_*;$$
$$z = 100 \;\Rightarrow\; R \approx 4,615 \cdot R_*;$$
$$z = 1000000 \;\Rightarrow\; R \approx 13,8 \cdot R_*.$$

For small values of R, we arrive at the well-known relation

$$z = \frac{H \cdot R}{c} = \frac{v}{c}, \quad 0 \leq v \ll c.$$

Expanding the exponent in a power series, we obtain:

$$z = \frac{H \cdot R}{c} + \frac{1}{2} \cdot \left(\frac{H \cdot R}{c}\right)^2 + O\left(\left(\frac{R}{R_*}\right)^3\right).$$

The last formula is close to empirical formulas for redshift.

5. Hubble's law and rapidity

Rapidity is determined by the formula

$$\theta = c \cdot \mathrm{arctanh}\frac{v}{c}.$$

The croatian physicist Varićak proposed the variable θ [6]. A generalization of the Hubble law for rapidity gives the formula

$$\theta = H \cdot R.$$

The transition from the Hubble linear law to its generalization is associated with the replacement of speed by rapidity.

References

[1] Lemaître G 1927 Un Univers homogène de masse constante et de rayon croissant rendant compte de la vitesse radiale des nébuleuses extra-galactiques *Annales de la Société Scientifique de Bruxelles* **A47** pp 49-59
[2] Hubble E 1929 A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae *Proceedings of the National Academy of Sciences of the United States of America* **15** pp 168–73
[3] Newton I 1687 *Philosophiae naturalis principia mathematica* (Londini)
[4] Cauchy A L 1821 *Analise algébrique* (Paris)
[5] Einstein A 1905 Zur Elektrodynamik bewekter Körper *Annalen der Physik* **17** pp 891-921
[6] Varićak V 1910 Anwendung der Lobatschefskijschen Geometrie in der Relativtheorie *Physicalische Zeitschrift* **11** pp 93-6