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Generalized Pauli Fibonacci Polynomial Quaternions

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Abstract: Since Hamilton proposed quaternions as a system of numbers that does not satisfy the ordinary commutative rule of multiplication, quaternion algebras have played an important role in many mathematical and physical studies. This paper introduces the generalized notion of Pauli Fibonacci polynomial quaternions, a definition that incorporates the advantages of the Fibonacci number system augmented by the Pauli matrix structure. With the concept presented in the study, it aims to provide material that can be used in a more in-depth understanding of the principles of coding theory and quantum physics, which contribute to the confidentiality needed by the digital world, with the help of quaternions. In this study, an approach has been developed by integrating the advantageous and consistent structure of quaternions used to solve the problem of system lock-up and unresponsiveness during rotational movements in robot programming, the mathematically compact and functional form of Pauli matrices, and a generalized version of the Fibonacci sequence, which is an application of aesthetic patterns in nature.

Keywords: generalized Fibonacci polynomials; Pauli matrices; quaternions; Pauli Fibonacci polynomial quaternions

MSC: 47S10; 46S10; 11B39

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1. Introduction

Particle physics, sometimes called high energy physics, is the area of natural science that investigates the ultrastructure of matter. This research deals with two issues: The first is the search for elementary particles, the ultimate constituents of matter at its smallest scales. The second is to explain what interactions take place between them to create matter as we view it [1]. While knowledge about electricity goes back much further, the fact that the electron is a particle has been proven more recently. In this sense, the first experimental evidence for the existence of the electron as a charged particle dates back to Thomas Alva Edison's experiments in the 1880s [2]. For the development of the elementary particle, the dates in [1] will be taken as a basis, as given below: After years in which the fundamental particle was considered to be the atom, J. J. Thomson brought a different perspective to the situation when he extracted electrons from matter in the form of cathode rays in 1897 [3]. In 1932, Chadwick discovered that the nucleus, the core of the atom, is composed of protons and neutrons [4]. In 1934, Fermi established the theory of weak interactions [5]. In 1935, Yukawa developed the meson theory to explain the nuclear force acting between them [6]. Thus protons and neutrons, two of the three grains of baryons, were considered fundamental particles until the 1960s. Quarks and leptons are now recognized as the essential components of matter [1].

Currently, we are dealing with spin, which is of great importance for the physics of elementary particles and is also a complex concept. Eliot Leader has beautifully summarized the place of the concept of spin in the field with an analogy: Spin acts a dramatic Jekyll and Hyde role in the theater of elementary particle physics, at times heralding the collapse of an existing theory and at other times serving as a strong device in confirming another theory [7].

Spin, which requires a deep understanding of quantum mechanics to understand how it actually arises, first appeared when Paul Andrew Maurice Dirac derived the Dirac Equation, the cornerstone of relativistic quantum mechanics. Richard Feynman wrote the following explanation about the concept of spin [8]: "It appears to be one of the few places in physics where there is a rule which can be stated very simply, but for which no one has found a simple and easy explanation [9]". In their classic textbook on quantum mechanics, Landau and Lifshitz wrote the following about the spin property [8]: "The spin property of elementary particles is peculiar to quantum theory... It has no classical interpretation... It would be wholly meaningless to imagine the 'intrinsic' angular momentum of an elementary particle as being the result of its rotation about its own axis [10]". In [7], it is stated that Dirac's (1927) work demonstrated that spin arises automatically in a relativistic theory and can no more be considered as an autonomous additional degree of independence. Particles with full spin satisfy the Bose–Einstein statistics, and particles with half spin comply with the Fermi–Dirac statistics [11].

The concept of spin appears in quantum mechanics books through an idealized Stern–Gerlach experiment in which a non-relativistic ray of silver atoms crosses an inhomogeneous magnetic field [7]. In non-relativistic quantum mechanics, Bargmann showed that it is possible to describe the particle spin by performing the central extension of the Galilei group [12]. The spin of a particle in non-relativistic quantum mechanics is given as an additional rotational degree of freedom [7].

There is also an approach that does not accept the Stern–Gerlach experiment, which is considered the first measurement of spin, as a measurement of electron spin. Bohr and Pauli are researchers who claimed that it is impossible to measure the spin of an electron. Despite this idea, spin has played a major role in the development of modern science both theoretically and technologically. In the last 30 years, studies that have gained great momentum have been carried out after the characterization of some metals using electron spin resonance, and studies have been carried out to detect radio frequency spin signals with scanning tunneling microscopy, and thus a new class of studies has emerged by taking advantage of the chemical sensitivity of spin resonance methods below the nanometer scale [13].

Spinors are theoretically used to model particles with half-integer spin, such as the massive electron and the massive or massless neutrino [14]. In fact, the spinor theory was developed independently by physicists and mathematicians. As a result, there is no single definition of the spinor concept in the literature [15]. According to Figueiredo et al., spinors have three different definitions, each showing a different perspective [15,16]. First is the covariant definition [17,18], the second is the algebraic definition [19–21], and, finally, the third is the operatorial definition [22]. Spinors have performed an essential role in physics and mathematics for the past eighty years [2]. In theoretical physics, one of the long-enduring problems is the combination of space–time and internal symmetries. The space–time symmetries produced by the Poincare group (including the space reversal P, time reversal T, and their combination PT) are treated as completely certain transformations of the space–time continuum. The first transformation, charge conjugation C, which is associated intimately with a complex conjugation of Lorentz group representations but is not a space–time symmetry, can be treated as an internal symmetry. There are also a wide variety of internal symmetries that come from quark phenomenology from $SU(N)$-theories [23].

As is well known, elementary particles might be grouped into multiplets satisfying nonreducible representations of the so-called algebras of internal symmetries (e.g., the multiplet of the isospin algebra $su(2)$ or the multiplet of the algebra $su(3)$) [23]. Gell-Mann [24] and Ne'eman [25] argue that the charge multiplicities of the group $SU(2)$ can be combined into a larger group, such as the group $SU(3)$. Thus the isospin group $SU(2)$ is considered as a subgroup of $SU(3)$ such that $SU(2) \subset SU(3)$ [23].

Looking back on the nineteenth century, there was a need for a mathematical theory in which physical laws could be described and their universality checked. The conditions were in place for the construction of such a theory. The two leading names in developing this theory were Hamilton and Grassmann. Although Hamilton is thought to have done this work to find a suitable mathematical tool to apply Newtonian mechanics to various aspects of astronomy and physics, Grassmann's real intention was to develop a theoretical algebraic structure on which the geometry of any number of dimensions could be based. From a purely mathematical point of view, Hamilton perhaps wanted to introduce a binary operation that could be physically interpreted in terms of a rotation in space [26].

The algebra of two-component spinors and Pauli matrices, part of the $SU(2)$ group, enables a more compact and elegant formalism for defining classical rotations in real three-dimensional space. While the spinor representation is normally related with quantum mechanics, it is very intimately associated with quaternions, which were used a century ago by Hamilton to describe the inner degree of freedom of the electron, recognized as spin [27,28].

Quaternions were defined by Hamilton in 1843 [29]. Hamilton expressed the quaternion by means of a certain quadruple, which he called the real part of the first term and the imaginary part of the structure formed by the other three terms, and some equalities regarding the squares and products of the symbols in the imaginary part are valid [15]. Hamilton introduced a non-commutative binary algebra based on four basic units, such as $(1, i, j, k)$, in which many aspects of mechanics could be treated. James C. Maxwell argued that quaternions could be valuable in the theoretical advancement of electricity and magnetism. The theory of quaternions did not survive in its original form, despite the numerous applications Hamilton pointed out and the insistence of many researchers, including the Scottish mathematician Tait [26].

Despite the great value of Grassmann's work, the importance of neither the original nor the extended work has been adequately recognized. The work needed to bring formal clarity and simplicity to the subject, and to demonstrate its important contributions, has been done by subsequent researchers instead of Hamilton and Grassmann. John Willard Gibbs, one of the most important mathematical physicists, was highly influential in developing the form of vector analysis. His need for a simpler mathematical framework for topics such as electromagnetics and thermodynamics was satisfied by selecting those aspects of the subject that could best be applied to theoretical physics, thanks to his insight into the work of Hamilton and Grassmann [26]. Quaternions found application especially in physics in the late 19th and early 20th centuries [30].

In robotics and computer-aided design, it is very important to be capable of defining geometric relationships clearly and unambiguously. This is done through coordinate systems and geometric transformations in the form of translation, scaling, symmetry, and rotation [31]. Euler angles were often used to formulate the rotational motion of objects. One of the disadvantages of utilizing Euler angles to simulate rotation is a condition known as gimbal lock [32]. Gimbal lock has been a highly visible problem in spacecraft control after the Apollo mission suffered from it [33]. In systems using three separate rotations, gimbal lock is a situation where two rotation axes align at certain angles, resulting in the loss of control of one axis. With quaternions, which eliminate this limitation and have many other advantages, research on a wide range of applications, especially in 3D space, is still ongoing. Programs such as Python SciPy 1.2.0 a popular language for robot programming, have added commands to their libraries to convert from Euler angles to quaternions [31].

In 1927, Pauli introduced the idea that the wave function of the electron could be described by a vector with two complex components; this vector is a spinor in three-dimensional Euclidean space. A year later, Dirac, in line with the needs of the relativistic equation, defined a wave function for the electron represented by a vector with four complex components, a spinor of four-dimensional pre-Euclidean space–time. In fact, physicists had unwittingly rediscovered the mathematical objects created by Cartan in 1913 while studying linear representations of groups. Moreover, the spinors studied by Cartan are examples of spinors that can be defined in a very general way, starting only from certain axioms. Additionally, vector spaces whose elements are spinors are related to the general theory of Clifford spaces introduced by Clifford in 1876 [34].

The term spin was first used by physicists to describe certain properties of quantum particles that emerged during various experiments. To quantify these properties, some new mathematical concepts called spinors were defined. These are vectors of a space whose transformations are related in a certain way to rotations in physical space. The analysis of the geometrical characteristics of spinors is necessary to better reveal the relationship between rotations in the space of spinors and rotations in three-dimensional physical space [34].

Pauli matrices, also referred to as Pauli spin matrices, are mathematical instruments of quantum mechanics. They took this name from the famous physicist Wolfgang Pauli, who made great advances in physics. Pauli matrices can be represented by using complex numbers as follows: $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Together with these three matrices, the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is also included as a fourth element in order to represent observables. The state of an electron's spin along orthogonal axes can be represented using the corresponding σ_1, σ_2, σ_3. One of the interesting properties of Pauli matrices is that these three matrices form a basis for the space of 2×2 matrices. That is, any 2×2 matrix can be written as a linear combination of σ_1, σ_2, σ_3 and the identity matrix. The Pauli matrices are orthogonal. Also, Pauli matrices are Hermitian. That is, these matrices are equal to their conjugate transpose. The traces of Pauli matrices are equal to zero. That is, the sum of the elements on the principal diagonal of σ_1, σ_2, σ_3 is zero. On the other hand, the determinant of σ_1, σ_2, σ_3 is one. This property means that Pauli matrices represent rotations in the quantum analogue of the phase space of classical mechanics.

Let the unit matrix be denoted by σ_0. Some important properties of Pauli matrices can be given as follows [34] (σ_0 is obtained by taking the squares of the Pauli matrices):

For $m \neq n$ and $m, n \in \{1, 2, 3\}, \sigma_m \sigma_n = -\sigma_n \sigma_m$. Also,

$$\sigma_1 \sigma_2 = i\sigma_3, \ \sigma_2 \sigma_3 = i\sigma_1 \ \text{and} \ \ \sigma_3 \sigma_1 = i\sigma_2. \tag{1}$$

The following relations are valid for Pauli matrices:

$$\sigma_1 \sigma_2 - \sigma_2 \sigma_1 = 2i\sigma_3, \sigma_2 \sigma_3 - \sigma_3 \sigma_2 = 2i\sigma_1 \ \text{and} \ \ \sigma_3 \sigma_1 - \sigma_1 \sigma_3 = 2i\sigma_2.$$

Using the linearly independent of unit matrix σ_0 and Pauli matrices, the following relation is written:

$$\lambda_0\sigma_0 + \lambda_1\sigma_1 + \lambda_2\sigma_2 + \lambda_3\sigma_3 = \mathbf{0}.$$

For $m \in \{0,1,2,3\}$, σ_m matrices are a base of the vector space of 2×2 matrices over real or complex numbers. Also, each matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as

$$A = \alpha\sigma_0 + \beta\sigma_1 + \gamma\sigma_2 + \delta\sigma_3 = \mathbf{0}$$

where

$$\alpha = \frac{1}{2}(a+d), \beta = \frac{1}{2}(b+c), \gamma = \frac{1}{2}i(b-c), \delta = \frac{1}{2}(a-d).$$

The Pauli matrices used to describe the behavior of quantum bits (qubits) can play important roles in quantum computers and artificial intelligence processes that are still under development.

Pauli matrices are introduced into the theory of spinors in the following way:

$$\begin{aligned} x &= \psi\phi^* + \psi^*\phi, \\ y &= i(\psi\phi^* - \psi^*\phi), \\ z &= \psi\psi^* - \phi\phi^*. \end{aligned}$$

The quaternion can be represented using the complex numbers ψ and ϕ, with the associated spinor components (ψ, ϕ) and (ϕ^*, ψ^*) expressed in matrix form. With this information, take $\chi = \begin{bmatrix} \psi \\ \phi \end{bmatrix}$ and $\chi^\dagger = \begin{bmatrix} \psi^* & \phi^* \end{bmatrix}$ using Pauli matrices, the x, y, z components of the vector can be written as follows: $x = \chi^\dagger\sigma_1\zeta$, $y = \chi^\dagger\sigma_2\chi$ and $z = \chi^\dagger\sigma_3\chi$.

Every matrix corresponds to a component along one of the axes of the reference trihedron. These are infinitesimal rotation matrices around the axes Ox, Oy, Oz multiplied by a factor [34].

Pauli used spinors, thought to be elements of C^2, to reveal the behavior of an electron by taking the spin of the electron into account in quantum mechanics. In physics, spinors arose as a product of Pauli's theory of non-relativistic quantum mechanics (1926) and Dirac's theory of relativistic quantum mechanics (1928) [35,36]. These matrices, which appear in the Pauli equation, which takes into account the interaction of a particle's spin with an external electromagnetic field, are named after the physicist Wolfgang Pauli (1928) [36]. These matrices have a very important place in nuclear physics studies. Dirac gave important formulas about Pauli matrices [35]. Less than two years after the 1926 discovery of the Schrödinger equation, Dirac derived a first-order wave equation for a four-component spinor field describing relativistic spin-1/2 particles such as electrons [35]. For detailed information on this subject, reference [37] can be reviewed. Vivarelli [38] was involved in this area from the geometrical aspect. He showed an injective and linear correspondence between spinors and quaternions, and, in three-dimensional Euclidean space, he gave spinor representations of rotations. Thus, a more concise and simpler depiction of quaternions can be reached by the concept of spinors. Quaternions, being applied to the fields of mathematics, physics, robotics, engineering, and chemistry, can be worked through spinors with the help of the correspondence given by Vivarelli [38].

We will now examine in more detail the quaternions mentioned above, which Hamilton constructed using the three symbols i, j, k to generalize the concept of complex numbers, with the following properties:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = -k,\ \ jk = -kj = -i,\ ki = -ik = -j.$$

If q is written using these three components and **1**,

$$q = u1 + xi + yj + zk$$

q formed by this method are called quaternions and they form a four-dimensional vector space over real numbers. The structure shown by **H** represents the quaternion algebra. The moment he discovered these equations, Hamilton carved them into a bridge. He spent the rest of his life working on quaternions, which is why this algebra is now represented by **H**, after him [39]. Also, the quaternion algebra is a Clifford algebra and the generating elements are $e_1 = i, e_2 = j$ and for $i \neq j$ $e_i e_j = -e_j e_i$, $e_j^2 = -1$. Here, $e_3 = e_1 e_2 = ij = -k$.

A matrix representation of the quaternions is then the following:

$$q = u\sigma_0 + xi\sigma_1 + yi\sigma_2 + zi\sigma_3.$$

With the help of the generating symbols of quaternions, an isomorphism can be established between the quaternion algebra and other four-dimensional algebras. A classical representation of quaternions is given by the following generator elements and their outputs [34]:

$$e_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},\ \ e_1 e_2 = e_3 = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, we get for the matrix representation of a quaternion

$$q = u\mathbf{1} + xe_1 + ye_2 + ze_3,$$

where $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

A generalized non-commutative quaternion q is a vector in 4-dimensional vector space of the form $q = u\mathbf{1} + xe_1 + ye_2 + ze_3$, where quaternionic units e_1, e_2, e_3 satisfy the next equalities for $c_1, c_2 \in \mathbb{R}$.

$$\begin{aligned} e_1^2 &= -c_1,\ e_2^2 = -c_2,\ e_3^2 = -c_1 c_2, \\ e_1 e_2 &= -e_2 e_1 = e_3,\ e_2 e_3 = -e_3 e_2 = c_2 e_1 \text{ and } e_3 e_1 = -e_1 e_3 = c_1 e_2. \end{aligned}$$

For special values of c_1 and c_2, we obtain well-known subclasses of non-commutative quaternions. Specific choices of c_1 and c_2 yield well-established types of non-commutative quaternions:

$c_1 = c_2 = 1$, real quaternions,
$c_1 = 1, c_2 = -1$, split quaternions,
$c_1 = 1, c_2 = 0$, semi quaternions,
$c_1 = 0, c_2 = 0$, $\frac{1}{4}$ quaternions [40].

A generalized commutative quaternion q is a vector in 4-dimensional vector space of the form $q = u\mathbf{1} + xe_1 + ye_2 + ze_3$, where quaternionic units e_1, e_2, e_3 satisfy the next equalities for $c_1, c_2 \in \mathbb{R}$.

$$\begin{aligned} e_1^2 &= c_1,\ e_2^2 = c_2,\ e_3^2 = c_1 c_2, \\ e_1 e_2 &= e_2 e_1 = e_3,\ e_2 e_3 = e_3 e_2 = c_2 e_1 \text{ and } e_3 e_1 = e_1 e_3 = c_1 e_2. \end{aligned}$$

For special values of c_1 and c_2 we obtain well-known subclasses of non-commutative quaternions. Specific choices of c_1 and c_2 yield well-established types of non-commutative quaternions:

$c_1 < 0, c_2 = 1$, elliptic quaternions,
$c_1 = 0, c_2 = -1$, parabolic quaternions,
$c_1 > 0, c_2 = 1$, hyperbolic quaternions [41].

The real linear span of $\{I, \sigma_1, \sigma_2, \sigma_3\}$ is isomorphic to the real algebra of $\mathbb{H}$ quaternions, and the Pauli quaternions are defined by this basis [42]. A Pauli quaternion is given as

$$p = x_0\mathbf{1} + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$$

and a set of Pauli quaternions is denoted by $\mathbb{H}_p$. The conjugate of a Pauli quaternion, represented by p^*, in [43], is shown as

$$p^* = x_0\mathbf{1} - x_1\sigma_1 - x_2\sigma_2 - x_3\sigma_3.$$

Isomorphism from $\mathbb{H}$ to this set is given by the following transformation, which has the opposite sign for Pauli matrices: $1 \to I, i \to -i\sigma, 1, j \to -i\sigma_2, k \to -i\sigma_3$ [44]. Optionally, the isomorphism may be realized by a transformation that uses the Pauli matrices in inverse order, such that $1 \to I, i \to i\sigma_3, j \to i\sigma_2, k \to i\sigma_1$ [42].

For any Pauli quaternion, the product is defined as follows:

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & -ix_3 & ix_2 \\ x_2 & ix_3 & x_0 & -ix_1 \\ x_3 & -ix_2 & ix_1 & x_0 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

For all $p, q \in \mathbb{H}_p$ Pauli quaternion product is given by

$$\begin{aligned} pq &= (x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3)\mathbf{1} + ((x_0y_1 + x_1y_0) + i(x_2y_3 - x_3y_2))\sigma_1 \\ &\quad + ((x_0y_2 + x_2y_0) + i(x_3y_1 - x_1y_3))\sigma_2 + ((x_0y_3 + x_3y_0) + i(x_1y_2 - x_2y_1))\sigma_2. \end{aligned}$$

Then, $pp^* = p^*p = (x_0^2 - x_1^2 - x_2^2 - x_3^2)\mathbf{1}$ [45].

Another concept that is important for our study is Fibonacci numbers. The literature on Fibonacci numbers, which has been analyzed in many remarkable ways for quite some time, is extensive. Fibonacci numbers are numbers that originated when Leonardo Pisano, who introduced Hindu-Arabic numerals to Europe in the early 1200s, presented a problem involving the increase in the rabbit population in his book Liber Abaci, which later led Pisano to be known as Fibonacci. These numbers go on as $0, 1, 1, 1, 2, 3, 5, 8, 13, \ldots$ and are called the Fibonacci sequence and are expressed by the following recurrence relation: For $n \geq 0$, $F_n = F_{n-1} + F_{n-2}$, here F_n is the nth Fibonacci number, $F_0 = 0$, $F_1 = 1$.

After Horadam 's study of a generalized version of the Fibonacci sequence in [46], many different generalizations were made about these fascinating numbers, and their mathematical properties and practical applications were studied.

One of the most popular generalizations of Fibonacci numbers is the sequence of Fibonacci polynomials : $F_n(x) = F_{n-1}(x) + F_{n-2}(x)$ for $n \geq 2$, with $F_0(x) = 0$, $F_1(x) = 1$.

Some cryptographic algorithms use complex sequences of numbers and polynomials. With the idea that a structure more general than Fibonacci polynomials can provide certain advantages in key generation and encryption methods, the following generalizations, which will be employed in this study, are summarized below.

Since quantum theory can be formulated using Hilbert spaces on any of the three relational normed division algebras, namely real numbers, complex numbers, and quaternions [39], we will now briefly talk about Fibonacci polynomials in our study on the combination of quaternions, which we have touched upon in detail, with number sequences, which are important topics in the mathematical field. The generalized Fibonacci polynomials, $\{W_n(x)\}$ are given as follows [47]:

$$\begin{aligned} W_n(x) &= r(x)W_{n-1}(x) + s(x)W_{n-2}(x), \\ W_0(x) &= a(x), W_1(x) = b(x), n \geq 2. \end{aligned}$$

Binet's formula of generalized Fibonacci polynomials can be calculated using the characteristic equation, which is given as

$$t^2 - r(x)t - s(x) = 0.$$

The roots of the characteristic equation are

$$\alpha(x) = \alpha = \frac{r(x) + \sqrt{r^2(x) + 4s(x)}}{2}, \beta(x) = \beta = \frac{r(x) - \sqrt{r^2(x) + 4s(x)}}{2}.$$

These equalities are consistent with the formulas provided in [48].

Note: For the sake of simplicity throughout the rest of the paper, we use

$$W_n, r, s, W_0, W_1, \alpha, \beta, G_n, H_n, G_0, G_1, H_0, H_1$$

instead of

$$W_n(x), r(x), s(x), W_0(x), W_1(x), \alpha(x), \beta(x), G_n(x), H_n(x), G_0(x), G_1(x), H_0(x), H_1(x).$$

In the next theorem, we recall Binet's formula of generalized Fibonacci polynomials.

Theorem 1.

(a) *(Distinct Roots Case: $\alpha \neq \beta$) Binet's formula of generalized Fibonacci (Horadam) polynomials is*

$$W_n = \frac{p_1\alpha^n - p_2\beta^n}{\alpha - \beta} \tag{2}$$

where

$$p_1 = W_1 - \beta W_0, \; p_2 = W_1 - \alpha W_0.$$

(b) *(Single Root Case: $\alpha = \beta$) Binet's formula of generalized Fibonacci (Horadam) polynomials is*

$$W_n = (D_1 + D_2 n)\alpha^n \tag{3}$$

where

$$\begin{aligned} D_1 &= W_0, \\ D_2 &= \frac{1}{\alpha}(W_1 - \alpha W_0). \end{aligned}$$

For more detail, see Soykan [47].

There are some studies in the literature that bring together the classical Fibonacci sequence, which is a special case of a more general concept employed in this study, and the concept of quaternions, which are integrated with the Pauli matrix structure. There are not enough studies covering all these concepts. The fundamental studies of Fibonacci quaternions are [45,49,50]. A few important works inspired by these studies in the following periods are as follows. In [51], Fibonacci quaternions were generalized to define higher order Fibonacci quaternions. In [52], which establishes a relationship between spinors and Fibonacci polynomials, a new sequence family of Generalized Fibonacci polynomial spinors is introduced; the matrix structure of the sequences related to these defined polynomials and some special relations are given. Fibonacci type quaternions with evaluations from different perspectives can be found in [53–58]. Generalized commutative quaternions of the Fibonacci type were defined in [41]. When it comes to studies that associate the concept of Pauli quaternion with number sequences, we can say that the first of these studies is [59]. Azak studied [60] the Gaussian version of Torunbalci's work. In [61], Pauli–Leonardo quaternions are introduced and various equalities are obtained. İşbilir and others are working on incomplete generalized (p, q, r)-Tribonacci Pauli quaternion polynomials [62].

A Fibonacci quaternion is generally defined in [50] as

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

where Q_n is the nth Fibonacci quaternion, F_n denotes the nth Fibonacci number, and i, j, k are the classical quaternion units.

In [49], the author, by defining $Q_n = F_n + u$, for $u = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$, presented the following properties regarding to the isomorphic structure of Fibonacci quaternions. For

$$\mathbb{H} = \{Q_n : Q_n = (F_n, F_{n+1}, F_{n+2}, F_{n+3})\}$$

and

$$\mathbb{H}' = \{P_n : P_n = \begin{pmatrix} w & -z \\ \overline{z} & \overline{w} \end{pmatrix}, \text{ w,z} \in \mathbb{C}\},$$

there exists an isomorphism between $\mathbb{H}$ and $\mathbb{H}'$.

$$Q_n = (F_n, F_{n+1}, F_{n+2}, F_{n+3}) \rightarrow P_n = \begin{pmatrix} F_n + iF_{n+1} & -F_{n+2} - iF_{n+3} \\ F_{n+2} - iF_{n+3} & F_n - iF_{n+1} \end{pmatrix}.$$

Also,

$$P_n = F_n + F_{n+1}E + F_{n+2}I + F_{n+3}K$$

where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$.

In this paper, we introduce a notion based on Pauli matrices in the framework of quaternions using the structure of generalized Fibonacci polynomials given in [47]: Generalized Pauli Fibonacci polynomial quaternions. Important equalities have been obtained with the help of Pauli (r, s) Fibonacci polynomial quaternion and Lucas versions of these polynomials. Also, the relation of the defined concept with matrices is given. Thus, the defined concept will be carried to the broad axis of matrix theory. The main idea behind the development of generalized Pauli Fibonacci polynomial quaternions is, first of all, to provide a transition from the usual Euler approach, which has disadvantages especially for rapidly changing angles in robotic coding and 3D animation system in the developing digital world, to a more useful quaternion-based structure, and then to take advantage of the Fibonacci sequence and polynomials associated with the balance and aesthetic approach in nature (golden ratio).

Previous studies on Fibonacci quaternions [39,45,63] and spinors [51] have established a foundational understanding of algebraic structures involving Fibonacci numbers and quaternionic representations. These works mainly focused on the properties and identities of Fibonacci quaternions and their connections to spinorial objects in classical settings. In contrast, the present study introduces a novel generalization by incorporating Pauli matrices into the structure of generalized Pauli Fibonacci polynomial quaternions (GPFPQs). This integration enriches the algebraic framework by combining the recursive nature of Fibonacci polynomials with the intrinsic non-commutative and anti-commutative properties of Pauli matrices. As a result, the proposed structure is better suited to modeling quantum mechanical systems, particularly in areas involving spin dynamics, entanglement, and unitary transformations.

Moreover, the Pauli-based extension of Fibonacci quaternions opens potential applications in both quantum information theory (e.g., quantum state representation and encoding) and classical cryptography, where matrix-based operations and structural complexity are desirable features. This approach not only generalizes existing quaternion models but also establishes a bridge between number theory, operator theory, and quantum computation, offering a richer mathematical toolset for future research.

The incorporation of Pauli matrices brings several novel aspects that were not explored in the prior quaternion models:

- Non-commutative Structure Enhancement:

  Pauli matrices inherently possess non-commutative and anti-commutative properties, which, when combined with Fibonacci polynomials, yield richer algebraic structures than those presented in classical Fibonacci quaternion models. This allows for a more nuanced representation of quantum mechanical symmetries and transformations.

- Quantum Mechanical Relevance:

  Pauli matrices play a foundational role in quantum mechanics, particularly in spin and quantum state representations. By embedding these matrices into the GPFPQ framework, our approach creates a bridge between number-theoretic constructs (Fibonacci polynomials) and quantum operators, facilitating more natural modeling of quantum systems, such as entanglement, quantum rotations, and spin states.

- Potential for Quantum and Classical Applications:

  The new structure opens up applications not just in theoretical algebra, but also in quantum information (e.g., state encoding) and cryptographic schemes where Pauli-based operations are commonly used in quantum error correction and secure communications.

- Analytical Generalization:

  Our approach also generalizes previous models by defining an extended algebra that encompasses both classical quaternion identities and matrix-based operations, thus allowing for further exploration in operator theory, matrix representation, and coding theory.

The concept, which we have defined with the help of general polynomials used in the context of Pauli matrices and quaternions, is presented for the evaluation of researchers, considering that it will be used together with the below studies in the literature.

In article [64], a cryptographic framework is proposed that employs Pauli spin-1/2 matrices in conjunction with finite state machines to encrypt data streams, aiming to enhance the robustness and security of data transmission processes. In [65], a secure cryptosystem is presented based on the braiding and entanglement of Pauli 3/2 matrices, aiming to ensure the secure transmission of sensitive information over the Internet. A solution for maximal dense coding with symmetric quantum states, based on unitary operators derived from the Pauli group, is proposed in [66], where quantum communication and coding theory are integrated. In [67], a fast construction of quantum codes based on the residues of Pauli block matrices is explored, with significant discussions on quantum error correction and stabilizer codes. In [68], the authors propose an approach to quantum fully homomorphic encryption (FHE) by integrating Pauli one-time pad encryption with quaternion algebra. The integration of these two techniques enables fully homomorphic operations on encrypted quantum data without compromising privacy.

We can list the contributions that the concept of "Generalized Pauli Fibonacci Polynomial Quaternions" can provide to the literature and its features that may attract the attention of researchers looking for a study topic in different fields as follows:

- By combining the time-dependent growth properties of Fibonacci polynomials with quantum transformations of Pauli matrices, more flexible and accurate modeling of temporally evolving quantum systems can be achieved.
- With the help of Fibonacci polynomial-based quaternion structure, more accurate modeling of systems such as biological signals and financial fluctuations involving nonlinear dynamics can be achieved.
- Pauli matrices are already fundamental components in quantum error-correcting codes (see [69]). Non-commutative Pauli Fibonacci quaternions could inspire new code systems with pattern-based coding logic to design error-correcting and corruption-tolerant systems in quantum information theory.
- After the encryption systems based on Fibonacci numbers, which have been previously investigated in classical cryptography (see [70]), quaternions and non-commutative structures can offer potential solutions for post-quantum cryptography. Combining the recurrence property of Fibonacci sequences and the orientation structure of quaternions, the presented concept can help to develop a new generation of secure algorithms with asymmetric and quantum-resistant encryption methods.

The structure of the sections in the manuscript is as follows: The introduction presents a comprehensive overview of Fibonacci quaternions and explores their connection to Pauli matrices through the quaternion–spinor relationship. Furthermore, the discussion highlights the theoretical significance and practical applicability of these mathematical constructs in addressing complex and evolving problems in various fields, including physics, engineering, and computational sciences.

In the second section, the mathematical structure of the generalized Pauli Fibonacci polynomial quaternions is formally introduced. To support practical implementation and demonstrate the applicability of the proposed formulation, detailed tables are presented for both positive and negative indices, providing insight into the behavior and numerical characteristics of the quaternion components across varying index values. Moreover, the (r, s)-Fibonacci and Lucas extensions of the definition are presented to illustrate the algebraic flexibility and generalization capacity of the proposed structure. These formulations not only expand the classical Fibonacci and Lucas sequences within a quaternionic framework, but also demonstrate that the study aligns with the expected generalization procedures in the literature, thereby reinforcing its theoretical soundness and potential for broader algebraic applications. Furthermore, by deriving the Binet formula and the corresponding generating function for the generalized Pauli Fibonacci polynomial quaternions, the study provides readers with a closed-form expression and a systematic computational tool. These formulations not only facilitate the efficient calculation of higher-order terms but also align with standard methodologies in the literature aimed at revealing the analytical structure and recurrence behavior of generalized number sequences within algebraic systems.

In Section 3, the concepts of Pauli (r, s)-Fibonacci polynomial quaternions and their Lucas versions are further extended through the derivation of important summation formulas. These formulas are obtained using recurrence relations that are commonly employed to generate polynomial sequences, thereby providing closed-form expressions for the sums of the defined polynomials. Through this approach, the study aims to establish the necessary theoretical framework to support further algebraic exploration and computational application of these generalized quaternionic structures.

In Section 4, several findings are presented to demonstrate that the matrix-based methodological framework, which is widely used in the literature for generalized Fibonacci polynomials, can be successfully extended to the newly defined structure that integrates Pauli matrix properties with quaternionic algebra. These results emphasize the adaptability of classical recursive techniques to more complex algebraic systems.

The final section of the manuscript is dedicated to presenting the conclusions drawn from the study, along with a set of recommendations for future research and potential applications.

2. Main Results

This section is devoted to the exploration of various mathematical properties and identities associated with generalized Pauli Fibonacci polynomial quaternions. We begin by presenting the structure and definition of the Pauli (r,s)-Fibonacci polynomial quaternion, followed by a detailed version of Binet's formula adapted to this generalized framework. Additionally, we examine several classical identities in the context of these quaternions, including Catalan's identity, the Simpson formula, and the Cassini identity, each of which provides deeper insight into the algebraic and analytical nature of these constructs. The results obtained not only extend well-known Fibonacci-related identities to a broader setting but also offer new perspectives relevant to both pure mathematics and theoretical physics.

The generalized Pauli Fibonacci polynomial quaternions can be defined by the basis $\{1, i\sigma_1, i\sigma_2, i\sigma_3\}$ where $i\sigma_s = 1,2,3$ satisfy the conditions as follows:

$$Q_pW_n = W_n\mathbf{1} + W_{n+1}\sigma_1 + W_{n+2}\sigma_2 + W_{n+3}\sigma_3. \tag{4}$$

Note that the identity (4) can be written as

$$Q_pW_n = \begin{pmatrix} W_n + W_{n+3} & W_{n+1} - iW_{n+2} \\ W_{n+1} + iW_{n+2} & W_n - W_{n+3} \end{pmatrix}. \tag{5}$$

The transformation defined as

$$(W_n\mathbf{1}, W_{n+1}\sigma_1, W_{n+2}\sigma_2, W_{n+3}\sigma_3) \to Q_pW_n = \begin{pmatrix} W_n + W_{n+3} & W_{n+1} - iW_{n+2} \\ W_{n+1} + iW_{n+2} & W_n - W_{n+3} \end{pmatrix}$$

is a linear isomorphism.

Lemma 1. *For nonnegative integer n, the generalized Pauli Fibonacci polynomial quaternions sequence $\{Q_pW_n\}$ are defined by second order recurrence relation as follows:*

$$Q_pW_{n+2} = rQ_pW_{n+1} + sQ_pW_n \tag{6}$$

with the initial conditions Q_pW_0 and Q_pW_1.

Proof. Using (4) and recurrence relation $W_n = rW_{n-1} + sW_{n-2}$, we obtain

$$\begin{aligned} Q_pW_{n+2} &= W_{n+2}\mathbf{1} + W_{n+3}\sigma_1 + W_{n+4}\sigma_2 + W_{n+5}\sigma_3 \\ &= (rW_{n+1} + sW_n)\mathbf{1} + (rW_{n+2} + sW_{n+1})\sigma_1 \\ &\quad +(rW_{n+3} + sW_{n+2})\sigma_2 + (rW_{n+4} + sW_{n+3})\sigma_3 \\ &= r(W_{n+1}\mathbf{1} + W_{n+2}\sigma_1 + W_{n+3}\sigma_2 + W_{n+4}\sigma_3) \\ &\quad +s(W_n\mathbf{1} + W_{n+1}\sigma_1 + W_{n+2}\sigma_2 + W_{n+3}\sigma_3) \end{aligned}$$

□

Lemma 2. *For negative integers n, we have the following identity:*

$$Q_pW_{-n-2} = -\frac{r}{s}Q_pW_{-n-1} + \frac{1}{s}Q_pW_{-n}. \quad (7)$$

Proof. From (6), we obtain

$$Q_pW_{-n} = rQ_pW_{-n-1} + sQ_pW_{-n-2}$$

Hence, we have

$$Q_pW_{-n-2} = -\frac{r}{s}Q_pW_{-n-1} + \frac{1}{s}Q_pW_{-n}$$

□

Following this, Table 1 presents the first few generalized Pauli Fibonacci polynomial quaternions with positive subscripts.

Table 1. The first few generalized Pauli Fibonacci polynomial quaternions with positive subscripts.

n	Q_pW_n
0	$W_0(\mathbf{1} + s\sigma_2 + rs\sigma_3)$ $+W_1(\sigma_1 + r\sigma_2 + s\sigma_3 + r^2\sigma_3)$
1	$sW_0(\sigma_1 + r\sigma_2 + (s+r^2)\sigma_3)$ $+W_1(\mathbf{1} + r\sigma_1 + (s+r^2)\sigma_2 + (r^3+2rs)\sigma_3)$
2	$sW_0(\mathbf{1} + r\sigma_1 + (s+r^2)\sigma_2 + (r^3+2rs)\sigma_3)$ $+W_1(r\mathbf{1} + (s+r^2)\sigma_1 + (r^3+2rs)\sigma_2 + (r^4+s^2+3r^2s)\sigma_3)$
3	$sW_0(r\mathbf{1} + (s+r^2)\sigma_1 + (r^3+2rs)\sigma_2 + (r^4+s^2+3r^2s)\sigma_3)$ $+W_1((s+r^2)\mathbf{1} + (r^3+2rs)\sigma_1 + (r^4+3r^2s+s^2)\sigma_2 + (r^5+3rs^2+4r^3s)\sigma_3)$
4	$sW_0((s+r^2)\mathbf{1} + (r^3+2rs)\sigma_1 + (r^4+s^2+3r^2s)\sigma_2 + (r^5+3rs^2+4r^3s)\sigma_3)$ $+W_1((r^3+2rs)\mathbf{1} + (r^4+s^2+3r^2s)\sigma_1 + (r^5+3rs^2+4r^3s)\sigma_2 + (s^3+r^6+6r^2s^2+5r^4s)\sigma_3)$
5	$sW_0((r^3+2rs)\mathbf{1} + (r^4+s^2+3r^2s)\sigma_1 + (r^5+3rs^2+4r^3s)\sigma_2 + (s^3+r^6+6r^2s^2+5r^4s)\sigma_3)$ $+W_1((r^4+s^2+3r^2s)\mathbf{1} + (3rs^2+r^5+4r^3s)\sigma_1 + (s^3+r^6+6r^2s^2+5r^4s)\sigma_2$ $+(r^7+10r^3s^2+4rs^3+6r^5s)\sigma_3)$

As a special case of Q_pW_n, taking $r=2$ and $s=1$, we have the following table (Table 2).

Table 2. The first few generalized Pauli Fibonacci polynomial quaternions with positive subscripts for the case $r=2$ and $s=1$.

n	Q_pW_n
0	$W_0(\mathbf{1} + \sigma_2 + 2\sigma_3) + W_1(\sigma_1 + 2\sigma_2 + \sigma_3 + 4\sigma_3)$
1	$W_0(\sigma_1 + 2\sigma_2 + 5\sigma_3) + W_1(\mathbf{1} + 2\sigma_1 + 5\sigma_2 + 12\sigma_3)$
2	$W_0(\mathbf{1} + 2\sigma_1 + 5\sigma_2 + 12\sigma_3) + W_1(2.\mathbf{1} + 5\sigma_1 + 12\sigma_2 + 29\sigma_3)$
3	$W_0(2.\mathbf{1} + 5\sigma_1 + 12\sigma_2 + 29\sigma_3) + W_1((5.\mathbf{1} + 12\sigma_1 + 29\sigma_2 + (r^5+3rs^2+4r^3s)\sigma_3)$
4	$W_0(5.\mathbf{1} + 12\sigma_1 + 29\sigma_2 + 70\sigma_3) + W_1(12.\mathbf{1} + 29\sigma_1 + 70\sigma_2 + 169\sigma_3)$
5	$W_0(12.\mathbf{1} + 29\sigma_1 + 70\sigma_2 + 169\sigma_3) + W_1(29.\mathbf{1} + 70\sigma_1 + 169\sigma_2 + 408\sigma_3)$

Next, the first few generalized Pauli Fibonacci polynomial quaternions with negative subscripts are presented in Table 3.

Table 3. The first few generalized Pauli Fibonacci polynomial quaternions Q_pW_n with negative subscripts.

n	Q_pW_n
0	$W_0(\mathbf{1}+s\sigma_2+rs\sigma_3)$ $+W_1(\sigma_1+r\sigma_2+s\sigma_3+r^2\sigma_3)$
-1	$(-\frac{r}{s}\mathbf{1}+\sigma_1+s\sigma_3)W_0+(\frac{1}{s}\mathbf{1}+\sigma_2+r\sigma_3)W_1$
-2	$((\frac{1}{s}+\frac{r^2}{s^2})\mathbf{1}-\frac{r}{s}\sigma_1+\sigma_2)W_0+(-\frac{r}{s^2}\mathbf{1}+\frac{1}{s}\sigma_1+\sigma_3)W_1$
-3	$((-\frac{r^3}{s^3}-2\frac{r}{s^2})\mathbf{1}+(\frac{1}{s}+\frac{r^2}{s^2})\sigma_1-\frac{r}{s}\sigma_2+\sigma_3)W_0$ $+((\frac{1}{s^2}+\frac{r^2}{s^3})\mathbf{1}-\frac{r}{s^2}\sigma_1+\frac{1}{s}\sigma_2)W_1$
-4	$((3\frac{r^2}{s^3}+\frac{r^4}{s^4}+\frac{1}{s^2})\mathbf{1}+(-\frac{r^3}{s^3}-2\frac{r}{s^2})\sigma_1+(\frac{1}{s}+\frac{r^2}{s^2})\sigma_2-\frac{r}{s}\sigma_3)W_0$ $+((-\frac{r^3}{s^4}-2\frac{r}{s^3})\mathbf{1}+(\frac{1}{s^2}+\frac{r^2}{s^3})\sigma_1-\frac{r}{s^2}\sigma_2+\frac{1}{s}\sigma_3)W_1$
-5	$((-4\frac{r^3}{s^4}-\frac{r^5}{s^5}-3\frac{r}{s^3})\mathbf{1}+(\frac{1}{s^2}+3\frac{r^2}{s^3}+\frac{r^4}{s^4})\sigma_1+(-\frac{r^3}{s^3}-2\frac{r}{s^2})\sigma_2$ $+(\frac{1}{s}+\frac{r^2}{s^2})\sigma_3)W_0+((\frac{1}{s^3}+3\frac{r^2}{s^4}+\frac{r^4}{s^5})\mathbf{1}+(-\frac{r^3}{s^4}-2\frac{r}{s^3})\sigma_1$ $+(\frac{1}{s^2}+\frac{r^2}{s^3})\sigma_2+(-\frac{r}{s^2})\sigma_3)W_1$

As a special case of Q_pW_n, taking $r=2$ and $s=1$, we have the following table (Table 4).

Table 4. The first few generalized Pauli Fibonacci polynomial quaternions Q_pW_n with negative subscripts for the case $r=2$ and $s=1$.

n	Q_pW_n
0	$W_0(\mathbf{1}+\sigma_2+2\sigma_3)+W_1(\sigma_1+r\sigma_2+s\sigma_3+r^2\sigma_3)$
-1	$(-2.\mathbf{1}+\sigma_1+\sigma_3)W_0+(\mathbf{1}+\sigma_2+2\sigma_3)W_1$
-2	$(5.\mathbf{1}-2\sigma_1+\sigma_2)W_0+(-2.\mathbf{1}+\sigma_1+\sigma_3)W_1$
-3	$(-12.\mathbf{1}+5\sigma_1-2\sigma_2+\sigma_3)W_0+(5.\mathbf{1}-2\sigma_1+\sigma_2)W_1$
-4	$(29.\mathbf{1}+-12\sigma_1+5\sigma_2-2\sigma_3)W_0+(-12.\mathbf{1}+5\sigma_1-2\sigma_2+\sigma_3)W_1$
-5	$(-70.\mathbf{1}+29\sigma_1-12\sigma_2+5\sigma_3)W_0+(29.\mathbf{1}-12\sigma_1+5\sigma_2-2\sigma_3)W_1$

Now, we define two special cases of the generalized Pauli Fibonacci polynomial quaternions Q_pW_n, denoted by Pauli $(r,s)-$ Fibonacci polynomial quaternion Q_pG_n and Pauli (r,s)-Lucas polynomial quaternion Q_pH_n.

For all integers n, the nth Pauli (r,s)-Fibonacci polynomial quaternions Q_pG_n are defined by

$$Q_pG_n = G_n\mathbf{1}+G_{n+1}\sigma_1+G_{n+2}\sigma_2+G_{n+3}\sigma_3 \tag{8}$$

with $Q_pG_0 = G_0\mathbf{1}+G_1\sigma_1+G_2\sigma_2+G_3\sigma_3$ and $Q_pG_1 = G_1\mathbf{1}+G_2\sigma_1+G_3\sigma_2+G_4\sigma_3$, and the Pauli (r,s)-Lucas polynomial quaternions Q_pH_n are defined by

$$Q_pH_n = H_n\mathbf{1}+H_{n+1}\sigma_1+H_{n+2}\sigma_2+H_{n+3}\sigma_3 \tag{9}$$

with $Q_pH_0 = H_0\mathbf{1}+H_1\sigma_1+H_2\sigma_2+H_3\sigma_3$ and $Q_pH_1 = H_1\mathbf{1}+H_2\sigma_1+H_3\sigma_2+H_4\sigma_3$.

Lemma 3. *For all integers n, we have the following identities:*

(a)

$$Q_pG_{n+2} = rQ_pG_{n+1}+sQ_pG_n$$

with

$$\begin{aligned} Q_pG_0 &= \sigma_1+r\sigma_2+(s+r^2)\sigma_3, \\ Q_pG_1 &= \mathbf{1}+r\sigma_1+(s+r^2)\sigma_2+r(2s+r^2)\sigma_3. \end{aligned}$$

(b)

$$Q_pH_{n+2} = rQ_pH_{n+1} + sQ_pH_n$$

with

$$\begin{aligned} Q_pH_0 &= 2.\mathbf{1} + r\sigma_1 + (2s + r^2)\sigma_2 + r(3s + r^2)\sigma_3, \\ Q_pH_1 &= r\mathbf{1} + (2s + r^2)\sigma_1 + r(3s + r^2)\sigma_2 + (4r^2s + r^4 + 2s^2)\sigma_3. \end{aligned}$$

Proof. Taking $Q_pW_n = Q_pG_n$ and $Q_pW_n = Q_pH_n$ in Lemma 1, (a) and (b) follow. □

In the next tables below (Tables 5–12), we give some terms of the Pauli (r,s)-Fibonacci polynomial quaternion Q_pG_n and Pauli (r,s)-Lucas polynomial quaternion Q_pH_n with negative and positive subscripts.

Table 5. The first few Pauli (r,s)-Fibonacci polynomial quaternions Q_pG_n with positive subscripts.

n	Q_pG_n
0	$\sigma_1 + \sigma_2 r + \sigma_3(r^2 + s)$
1	$\mathbf{1} + r\sigma_1 + \sigma_2(r^2 + s) + \sigma_3(r^3 + 2sr)$
2	$r\mathbf{1} + \sigma_1(r^2 + s) + \sigma_2(r^3 + 2sr) + \sigma_3(r^4 + 3r^2s + s^2)$
3	$(r^2 + s)\mathbf{1} + \sigma_1(r^3 + 2sr) + \sigma_2(r^4 + 3r^2s + s^2) + \sigma_3(r^5 + 4r^3s + 3rs^2)$
4	$(r^3 + 2sr)\mathbf{1} + \sigma_1(r^4 + 3r^2s + s^2) + \sigma_2(r^5 + 4r^3s + 3rs^2) + \sigma_3(r^6 + 5r^4s + 6r^2s^2 + s^3)$
5	$(r^4 + 3r^2s + s^2)\mathbf{1} + \sigma_2(r^6 + 5r^4s + 6r^2s^2 + s^3) + \sigma_1(r^5 + 4r^3s + 3rs^2) + \sigma_3(r^7 + 6r^5s + 10r^3s^2 + 4rs^3)$

As a special case of Q_pG_n, taking $r = 2$ and $s = 1$, we have the following tables (Table 6).

Table 6. The first few Pauli (r,s)-Fibonacci polynomial quaternions Q_pG_n with positive subscripts for the case $r = 2$ and $s = 1$.

n	Q_pG_n
0	$\sigma_1 + 2\sigma_2 + 5\sigma_3$
1	$\mathbf{1} + 2\sigma_1 + 5\sigma_2 + 12\sigma_3$
2	$2.\mathbf{1} + 5\sigma_1 + 12\sigma_2 + 29\sigma_3$
3	$5.\mathbf{1} + 12\sigma_1 + 29\sigma_2 + 70\sigma_3$
4	$12.\mathbf{1} + 29\sigma_1 + 70\sigma_2 + 169\sigma_3$
5	$29.\mathbf{1} + 70\sigma_1 + 169\sigma_2 + 408\sigma_3$

Table 7. The first few Pauli (r,s)-Lucas polynomial quaternions Q_pH_n with positive subscripts.

n	Q_pH_n
0	$2.\mathbf{1} + r\sigma_1 + \sigma_2(2s + r^2) + \sigma_3(r^3 + 3rs)$
1	$r.\mathbf{1} + \sigma_1(r^2 + 2s) + \sigma_2(r^3 + 3rs) + \sigma_3(r^4 + 4r^2s + 2s^2)$
2	$r.\mathbf{1} + \sigma_1(r^2 + s) + \sigma_2(r^3 + 2sr) + \sigma_3(r^4 + 3r^2s + s^2)$
3	$(2r + r^3 + rs).\mathbf{1} + \sigma_1(2r^2 + r^4 + 2s + 2r^2s) + \sigma_2(2r^3 + r^5 + rs^2 + 3r^3s + 4rs)$ $+\sigma_3(6r^2s + 4r^4s + 3r^2s^2 + 2r^4 + 2s^2 + r^6)$
4	$(r^4 + 4r^2s + 2s^2).\mathbf{1} + \sigma_1(r^5 + 5r^3s + 5rs^2) + \sigma_2(9r^2s^2 + 2s^3 + 6r^4s + r^6)$ $+\sigma_3(r^7 + 7r^5s + 14r^3s^2 + 7rs^3)$
5	$(r^5 + 5r^3s + 5rs^2).\mathbf{1} + \sigma_1(r^6 + 6r^4s + 9r^2s^2 + 2s^3) + \sigma_2(r^7 + 7r^5s + 14r^3s^2 + 7rs^3)$ $+\sigma_3(r^8 + 8r^6s + 20r^4s^2 + 16r^2s^3 + 2s^4)$

As a special case of Q_pH_n, taking $r = 2$ and $s = 1$, we have the following tables (Table 8).

Table 8. The first few Pauli (r,s)-Fibonacci polynomial quaternions Q_pH_n with positive subscripts for the case $r = 2$ and $s = 1$.

n	Q_pH_n
0	$2.\mathbf{1} + 2\sigma_1 + 6\sigma_2 + 14\sigma_3$
1	$r.\mathbf{1} + 6\sigma_1 + 14\sigma_2 + 34\sigma_3$
2	$r.\mathbf{1} + 5\sigma_1 + 12\sigma_2 + 29\sigma_3$
3	$14.\mathbf{1} + 34\sigma_1 + 82\sigma_2 + 198\sigma_3$
4	$34.\mathbf{1} + 82\sigma_1 + 198\sigma_2 + 478\sigma_3$
5	$82.\mathbf{1} + 198\sigma_1 + 478\sigma_2 + 1154\sigma_3$

Table 9. The first few generalized Pauli (r,s)-Fibonacci polynomial quaternions Q_pG_n with negative subscripts.

n	Q_pG_n
0	$\sigma_1 + \sigma_2 r + \sigma_3(r^2 + s)$
−1	$\frac{1}{s}.\mathbf{1} + \sigma_2 + r\sigma_3$
−2	$-\frac{r}{s^2}.\mathbf{1} + \frac{1}{s}\sigma_1 + \sigma_3$
−3	$(\frac{1}{s^2} + \frac{r^2}{s^3}).\mathbf{1} - \frac{r}{s^2}\sigma_1 + \frac{1}{s}\sigma_2$
−4	$(-\frac{r^3}{s^4} - 2\frac{r}{s^3}).\mathbf{1} + (\frac{1}{s^2} + \frac{r^2}{s^3})\sigma_1 - \frac{r}{s^2}\sigma_2 + \frac{1}{s}\sigma_3$
−5	$(\frac{1}{s^3} + 3\frac{r^2}{s^4} + \frac{r^4}{s^5}).\mathbf{1} + (-\frac{r^3}{s^4} - 2\frac{r}{s^3})\sigma_1 + (\frac{1}{s^2} + \frac{r^2}{s^3})\sigma_2 + (-\frac{r}{s^2})\sigma_3$

As a special case of Q_pG_n, taking $r = 2$ and $s = 1$, we have the following tables (Table 10).

Table 10. The first few generalized Pauli (r,s)-Fibonacci polynomial quaternions Q_pG_n with negative subscripts for the case $r = 2$ and $s = 1$.

n	Q_pG_n
0	$\sigma_1 + 2\sigma_2 + 5\sigma_3$
−1	$1.\mathbf{1} + \sigma_2 + 2\sigma_3$
−2	$-2.\mathbf{1} + \sigma_1 + \sigma_3$
−3	$5.\mathbf{1} - 2\sigma_1 + \sigma_2$
−4	$-12.\mathbf{1} + 5\sigma_1 - 2\sigma_2 + \sigma_3$
−5	$29.\mathbf{1} - 12\sigma_1 + 5\sigma_2 - 2\sigma_3$

Table 11. The first few Pauli (r,s)-Lucas polynomial quaternions Q_pH_n with negative subscripts.

n	Q_pH_n
0	$2.\mathbf{1} + r\sigma_1 + \sigma_2(2s + r^2) + \sigma_3(r^3 + 3rs)$
−1	$-\frac{r}{s}\mathbf{1} + 2\sigma_1 + r\sigma_2 + \sigma_3(2s + r^2)$
−2	$(\frac{2}{s} + \frac{r^2}{s^2}).\mathbf{1} - \frac{r}{s}\sigma_1 + 2\sigma_2 + r\sigma_3$
−3	$(-\frac{r^3}{s^3} - 3\frac{r}{s^2}).\mathbf{1} + \sigma_1(\frac{2}{s} + \frac{r^2}{s^2}) - \frac{r}{s}\sigma_2 + 2\sigma_3$
−4	$(4\frac{r^2}{s^3} + \frac{r^4}{s^4} + \frac{2}{s^2}).\mathbf{1} + \frac{2}{s}\sigma_2 + \sigma_1(-\frac{r^3}{s^3} - 3\frac{r}{s^2}) + \frac{r^2}{s^2}\sigma_2 - \frac{r}{s}\sigma_3$
−5	$(-5\frac{r^3}{s^4} - \frac{r^5}{s^5} - 5\frac{r}{s^3}).\mathbf{1} + \sigma_1(\frac{2}{s^2} + 4\frac{r^2}{s^3} + \frac{r^4}{s^4}) + \sigma_2(-\frac{r^3}{s^3} - 3\frac{r}{s^2}) + \sigma_3(\frac{2}{s} + \frac{r^2}{s^2})$

As a special case of Q_pH_n, taking $r = 2$ and $s = 1$, we have the following table (Table 12).

Table 12. The first few Pauli (r,s)-Lucas polynomial quaternions Q_pH_n with negative subscripts for the case $r=2$ and $s=1$.

n	Q_pH_n
0	$2.\mathbf{1}+2\sigma_1+6\sigma_2+14\sigma_3$
−1	$-2\mathbf{1}+2\sigma_1+2\sigma_2+6\sigma_3$
−2	$6.\mathbf{1}-2\sigma_1+2\sigma_2+2\sigma_3$
−3	$14.\mathbf{1}+6\sigma_1-2\sigma_2+2\sigma_3$
−4	$34.\mathbf{1}-14\sigma_1+6\sigma_2-2\sigma_3$
−5	$-82.\mathbf{1}+34\sigma_1-14\sigma_2+6\sigma_3$

The addition, subtraction, and multiplication by real scalars of two generalized Pauli Fibonacci polynomial quaternions gives the generalized Pauli Fibonacci polynomial quaternions. Then, the addition and subtraction of two generalized Pauli Fibonacci polynomial quaternions are defined by

$$Q_pW_n \pm Q_pW_m = (W_n \pm W_m)\mathbf{1} + (W_{n+1} \pm W_{m+1})\sigma_1 + (W_{n+2} \pm W_{m+2})\sigma_2 + (W_{n+3} \pm W_{m+3})\sigma_3.$$

The multiplication of a generalized Pauli Fibonacci polynomial quaternions by a real scalar λ is defined as follows:

$$\lambda Q_pW_n = \lambda W_n\mathbf{1} + \lambda W_{n+1}\sigma_1 + \lambda W_{n+2}\sigma_2 + \lambda W_{n+3}\sigma_3.$$

By using (1), multiplication of two generalized Pauli Fibonacci polynomial quaternions is formulated as

$$\begin{aligned}
Q_pW_n \times Q_pW_m &= (W_n\mathbf{1} + W_{n+1}\sigma_1 + W_{n+2}\sigma_2 + W_{n+3}\sigma_3) \times (W_m\mathbf{1} + W_{m+1}\sigma_1 + W_{m+2}\sigma_2 + W_{m+3}\sigma_3) \\
&= W_n\mathbf{1}W_m\mathbf{1} + W_n\mathbf{1}W_{m+1}\sigma_1 + W_n\mathbf{1}W_{m+2}\sigma_2 + W_n\mathbf{1}W_{m+3}\sigma_3 + W_{n+1}\sigma_1W_m\mathbf{1} \\
&\quad + W_{n+1}\sigma_1W_{m+1}\sigma_1 + W_{n+1}\sigma_1W_{m+2}\sigma_2 + W_{n+1}\sigma_1W_{m+3}\sigma_3 \\
&\quad + W_{n+2}\sigma_2W_m\mathbf{1} + W_{n+2}\sigma_2W_{m+1}\sigma_1 + W_{n+2}\sigma_2W_{m+2}\sigma_2 + W_{n+2}\sigma_2W_{m+3}\sigma_3 \\
&\quad + W_{n+3}\sigma_3W_m\mathbf{1} + W_{n+3}\sigma_3W_{m+1}\sigma_1 + W_{n+3}\sigma_3W_{m+2}\sigma_2 + W_{n+3}\sigma_3W_{m+3}\sigma_3. \\
&= W_nW_m\mathbf{1} + W_nW_{m+1}\sigma_1 + W_nW_{m+2}\sigma_2 + W_nW_{m+3}\sigma_3 + W_{n+1}W_m\sigma_1 \\
&\quad + W_{n+1}W_{m+1}\mathbf{1} + W_{n+1}W_{m+2}i\sigma_3 + W_{n+1}W_{m+3}(-i)\sigma_2 + W_{n+2}W_m\sigma_2 \\
&\quad + W_{n+2}W_{m+1}(-i)\sigma_3 + W_{n+2}W_{m+2}\mathbf{1} + W_{n+2}W_{m+3}i\sigma_1 + W_{n+3}W_m\sigma_3 \\
&\quad + W_{n+3}W_{m+1}i\sigma_2 + W_{n+3}W_{m+2}(-i)\sigma_1 + W_{n+3}W_{m+3}\mathbf{1}. \\
&= (W_nW_m + W_{n+1}W_{m+1} + W_{n+2}W_{m+2} + W_{n+3}W_{m+3})\mathbf{1} \\
&\quad + (W_nW_{m+1} + W_{n+1}W_m + W_{n+2}W_{m+3}i - W_{n+3}W_{m+2}i)\sigma_1 \\
&\quad + (W_nW_{m+2} + W_{n+2}W_m + W_{n+3}W_{m+1}i - W_{n+1}W_{m+3}i)\sigma_2 \\
&\quad + (W_nW_{m+3} + W_{n+3}W_m + W_{n+1}W_{m+2}i - W_{n+2}W_{m+1}i)\sigma_3.
\end{aligned}$$

The scalar and vector parts of Q_pW_n, which are the nth term of the generalized Pauli Fibonacci polynomial quaternions with (Q_pW_n), are denoted by

$$S_{Q_pW_n} = W_n\mathbf{1} \text{ and } V_{Q_pW_n} = W_{n+1}\sigma_1 + W_{n+2}\sigma_2 + W_{n+3}\sigma_3.$$

Thus, the generalized Pauli Fibonacci polynomial quaternions Q_pW_n is given by

$$Q_pW_n = S_{Q_pW_n} + V_{Q_pW_n}. \tag{10}$$

Then, the multiplication of two generalized Pauli Fibonacci polynomial quaternions is defined by

$$Q_pW_n \times Q_pW_m = S_{Q_pW_n}S_{Q_pW_m} + \left\langle V_{Q_pW_n}, V_{Q_pW_m} \right\rangle + S_{Q_pW_n}.V_{Q_pW_m} + S_{Q_qW_n}.V_{Q_pW_m} + V_{Q_pW_n}\Lambda V_{Q_pW_m}.$$

For more details, see Jafarı and Yaylı [40].

Also, the generalized Pauli Fibonacci polynomial quaternions product may be obtained as follows:

$$Q_pW_n \times Q_pW_m = \begin{pmatrix} W_n & W_{n+1} & W_{n+2} & W_{n+3} \\ W_{n+1} & W_n & -iW_{n+3} & iW_{n+2} \\ W_{n+2} & iW_{n+3} & W_n & -iW_{n+1} \\ W_{n+3} & -iW_{n+2} & iW_{n+1} & W_n \end{pmatrix} \begin{pmatrix} W_m \\ W_{m+1} \\ W_{m+2} \\ W_{m+3} \end{pmatrix}.$$

The conjugate of generalized Pauli Fibonacci polynomial quaternions Q_pW_n is denoted by $\overline{Q_pW_n}$, and it is

$$\overline{Q_pW_n} = W_n\mathbf{1} - W_{n+1}\sigma_1 - W_{n+2}\sigma_2 - W_{n+3}\sigma_3.$$

The norm of Q_pW_n is defined as follows:

$$\left\|Q_pW_n\right\|^2 = Q_pW_n\overline{Q_pW_n} = \left|W_n^2 - W_{n+1}^2 - W_{n+2}^2 - W_{n+3}^2\right|.$$

Now, using (10), we introduce Binet's formula for generalized Pauli Fibonacci polynomial quaternions Q_pW_n.

Binet's formula for generalized Pauli Fibonacci polynomial quaternions can be calculated using its characteristic equation, which is given as

$$t^2 - rt - s = 0 \tag{11}$$

where the roots of this equation are

$$\alpha = \frac{r + \sqrt{r^2 + 4s}}{2}, \ \beta = \frac{r - \sqrt{r^2 + 4s}}{2}.$$

In the next Theorem, we present Binet's formula for the Pauli generalized Fibonacci polynomial quaternions.

Theorem 2. *For all integers n, we have the following formulas:*

(a) *(Distinct Roots Case $\alpha \neq \beta$) Binet's formula of generalized Pauli Fibonacci polynomial quaternions is*

$$Q_pW_n = \frac{\widehat{p}_1\alpha^n - \widehat{p}_2\beta^n}{\alpha - \beta} \tag{12}$$

$$= \frac{\widetilde{p}_1\alpha^n - \widetilde{p}_2\beta^n}{\alpha - \beta}, \tag{13}$$

where

$$\begin{aligned} \widehat{p}_1 &= Q_pW_1 - \beta Q_pW_0, \\ \widehat{p}_2 &= Q_pW_1 - \alpha Q_pW_0, \\ \widetilde{p}_1 &= p_1\mathbf{1} + p_1\alpha\sigma_1 + p_1\alpha^2\sigma_2 + p_1\alpha^3\sigma_3, \\ \widetilde{p}_2 &= p_2\mathbf{1} + p_2\beta\sigma_1 + p_2\beta^2\sigma_2 + p_2\beta^3\sigma_3. \end{aligned}$$

Note that p_1 and p_2 are as stated in Theorem 1.

(b) *(Single Root Case $\alpha = \beta$) Binet's formula for generalized Pauli Fibonacci polynomial quaternions is*

$$Q_pW_n = (\widehat{d_1} + \widehat{d_2}n)\alpha^n \quad (14)$$
$$= (\widetilde{d_1} + \widetilde{d_2}n)\alpha^n, \quad (15)$$

where

$$\widehat{d_1} = Q_pW_0,$$
$$\widehat{d_2} = \frac{1}{\alpha}(Q_pW_1 - \alpha Q_pW_0),$$
$$\widetilde{d_1} = D_1\mathbf{1} + (D_1 + D_2)\alpha\sigma_1 + (D_1 + 2D_2)\alpha^2\sigma_2 + (D_1 + 3D_2)\alpha^3\sigma_3,$$
$$\widetilde{d_2} = D_2\mathbf{1} + D_2\alpha\sigma_1 + D_2\alpha^2\sigma_2 + D_2\alpha^3\sigma_3.$$

Note that D_1 and D_2 are as stated in Theorem 1.

Proof.

(a) If the roots of (11) are distinct, then Binet's formula of Q_pW_n is given below:

$$Q_pW_n = C_1\alpha^n + C_2\beta^n.$$

Taking $n = 0$ and $n = 1$, respectively, we have the following system of linear equations:

$$Q_pW_0 = C_1 + C_2$$
$$Q_pW_1 = C_1\alpha + C_2\beta.$$

Hence, solving these two equations, we obtain

$$C_1 = \frac{Q_pW_1 - \beta Q_pW_0}{\alpha - \beta},$$
$$C_2 = \frac{Q_pW_1 - \alpha Q_pW_0}{\alpha - \beta}.$$

Then, (13) can be proved by (4).

(b) If the roots of (11) are equal, then Binet's formula for Q_pW_n is given below:

$$Q_pW_n = (\widehat{d_1} + \widehat{d_2}n)\alpha^n.$$

Taking $n = 0$ and $n = 1$, respectively, we have the following system of linear equations:

$$Q_pW_0 = (\widehat{d_1} + \widehat{d_2}0)\alpha^0$$
$$Q_pW_1 = (\widehat{d_1} + \widehat{d_2}1)\alpha^1.$$

Hence, solving these two equations, we obtain

$$\widehat{d_1} = Q_pW_0, \ \widehat{d_2} = \frac{1}{\alpha}(Q_pW_1 - \alpha Q_pW_0).$$

Then, (15) can be proved easily by (4).

□

Theorem 2 gives us the following results as particular examples of Binet's formula for the Pauli (r,s)-Fibonacci polynomial quaternion Q_pG_n and Binet's formula for the Pauli (r,s)-Lucas polynomial quaternion Q_pH_n.

Corollary 1. *For all integers n, we have the following formulas:*

(a) *(Distinct Roots Case $\alpha \neq \beta$) Binet's formula for the Pauli (r,s)-Fibonacci polynomial quaternion Q_pG_n is*
$$Q_pG_n = \frac{\widehat{p}_1\alpha^n - \widehat{p}_2\beta^n}{\alpha-\beta},$$
where
$$\begin{aligned}\widehat{p}_1 &= \mathbf{1} + (r-\beta)\sigma_1 + (r^2+s-\beta r)\sigma_2 + \Big(r^3+2sr-\beta(r^2+s)\Big)\sigma_3,\\ \widehat{p}_2 &= \mathbf{1} + (r-\alpha)\sigma_1 + (r^2+s-\alpha r)\sigma_2 + \Big(r^3+2sr-\alpha(r^2+s)\Big)\sigma_3.\end{aligned}$$

(b) *(Distinct Roots Case $\alpha \neq \beta$) Binet's formula for the Pauli (r,s)-Lucas polynomial quaternion Q_pH_n is*
$$Q_pH_n = \frac{\widehat{p}_1\alpha^n - \widehat{p}_2\beta^n}{\alpha-\beta},$$
where
$$\begin{aligned}\widehat{p}_1 &= (r-2\beta)\mathbf{1} + (r^2+2s-\beta r)\sigma_1 + (r^3+3rs-\beta(2s+r^2))\sigma_2\\ &\quad + \Big(r^4+4r^2s+2s^2-\beta(r^3+3rs)\Big)\sigma_3,\end{aligned}$$
$$\begin{aligned}\widehat{p}_2 &= (r-2\alpha)\mathbf{1} + (r^2+2s-\alpha r)\sigma_1 + (r^3+3rs-\alpha(2s+r^2))\sigma_2\\ &\quad + \Big(r^4+4r^2s+2s^2-\alpha(r^3+3rs)\Big)\sigma_3.\end{aligned}$$

(c) *(Distinct Roots Case $\alpha = \beta$) Binet's formula for the Pauli (r,s)-Fibonacci polynomial quaternion Q_pG_n is*
$$Q_pG_n = (\widehat{d}_1 + \widehat{d}_2 n)\alpha^n$$
where
$$\widehat{d}_1 = \sigma_1 + \sigma_2 r + \sigma_3(r^2+s),$$
$$\widehat{d}_2 = \frac{1}{\alpha} + \frac{r-\alpha}{\alpha}\sigma_1 + \frac{r^2+s-\alpha r}{\alpha}\sigma_2 + \frac{r^3+2sr-\alpha(r^2+s)}{\alpha}\sigma_3.$$

(d) *(Distinct Roots Case $\alpha = \beta$) Binet's formula for the Pauli (r,s)-Lucas polynomial quaternion Q_pH_n is*
$$Q_pH_n = (\widehat{d}_1 + \widehat{d}_2 n)\alpha^n$$
where
$$\widehat{d}_1 = 2.\mathbf{1} + r\sigma_1 + \sigma_2(2s+r^2) + \sigma_3(r^3+3rs),$$
$$\begin{aligned}\widehat{d}_2 &= \frac{r-2\alpha}{\alpha} + \frac{r^2+2s-\alpha r}{\alpha}\sigma_1 + \frac{r^3+3rs-\alpha(2s+r^2)}{\alpha}\sigma_2\\ &\quad + \frac{r^4+4r^2s+2s^2-\alpha(r^3+3rs)}{\alpha}\sigma_3.\end{aligned}$$

Proof.

(a) Taking $Q_pW_n = Q_pG_n$ in Theorem 2 (a), we obtain

$$Q_pG_n = \frac{\widehat{p}_1\alpha^n - \widehat{p}_2\beta^n}{\alpha - \beta}$$

where

$$\begin{aligned}
\widehat{p}_1 &= Q_pG_1 - \beta Q_pG_0 \\
&= \mathbf{1} + r\sigma_1 + \sigma_2\left(r^2 + s\right) + \sigma_3\left(r^3 + 2sr\right) - \beta(\sigma_1 + \sigma_2 r + \sigma_3(r^2 + s)) \\
&= \mathbf{1} + (r - \beta)\sigma_1 + (r^2 + s - \beta r)\sigma_2 + \left(r^3 + 2sr - \beta(r^2 + s)\right)\sigma_3, \\
\widehat{p}_2 &= Q_pG_1 - \alpha Q_pG_0 \\
&= \mathbf{1} + r\sigma_1 + \sigma_2\left(r^2 + s\right) + \sigma_3\left(r^3 + 2sr\right) - \alpha(\sigma_1 + \sigma_2 r + \sigma_3(r^2 + s)) \\
&= \mathbf{1} + (r - \alpha)\sigma_1 + (r^2 + s - \alpha r)\sigma_2 + \left(r^3 + 2sr - \alpha(r^2 + s)\right)\sigma_3.
\end{aligned}$$

(b) Taking $Q_pW_n = Q_pH_n$ in Theorem 2 (a), we obtain

$$Q_pH_n = \frac{\widehat{p}_1\alpha^n - \widehat{p}_2\beta^n}{\alpha - \beta}$$

where

$$\begin{aligned}
\widehat{p}_1 &= Q_pH_1 - \beta Q_pH_0 \\
&= r.\mathbf{1} + \sigma_1(r^2 + 2s) + \sigma_2(r^3 + 3rs) + \sigma_3(r^4 + 4r^2 s + 2s^2) \\
&\quad -\beta(2.\mathbf{1} + r\sigma_1 + \sigma_2(2s + r^2) + \sigma_3(r^3 + 3rs)) \\
&= (r - 2\beta)\mathbf{1} + (r^2 + 2s - \beta r)\sigma_1 + (r^3 + 3rs - \beta(2s + r^2))\sigma_2 \\
&\quad + \left(r^4 + 4r^2 s + 2s^2 - \beta(r^3 + 3rs)\right)\sigma_3, \\
\widehat{p}_2 &= Q_pH_1 - \alpha Q_pH_0 \\
&= r.\mathbf{1} + \sigma_1(r^2 + 2s) + \sigma_2(r^3 + 3rs) + \sigma_3(r^4 + 4r^2 s + 2s^2) \\
&\quad -\alpha(2.\mathbf{1} + r\sigma_1 + \sigma_2(2s + r^2) + \sigma_3(r^3 + 3rs)) \\
&= (r - 2\alpha)\mathbf{1} + (r^2 + 2s - \alpha r)\sigma_1 + (r^3 + 3rs - \alpha(2s + r^2))\sigma_2 \\
&\quad + \left(r^4 + 4r^2 s + 2s^2 - \alpha(r^3 + 3rs)\right)\sigma_3.
\end{aligned}$$

(c) Taking $Q_pW_n = Q_pG_n$ in Theorem 2 (b), we obtain

$$Q_pG_n = (\widehat{d}_1 + \widehat{d}_2 n)\alpha^n$$

where

$$\begin{aligned}
\widehat{d}_1 &= Q_pG_0 \\
&= \sigma_1 + \sigma_2 r + \sigma_3(r^2 + s),
\end{aligned}$$

$$\begin{aligned}
\widehat{d}_2 &= \frac{1}{\alpha}(Q_pG_1 - \alpha Q_pG_0) \\
&= \frac{1}{\alpha}\left(\mathbf{1} + r\sigma_1 + \sigma_2\left(r^2 + s\right) + \sigma_3\left(r^3 + 2sr\right) - \alpha(\sigma_1 + \sigma_2 r + \sigma_3(r^2 + s))\right) \\
&= \frac{1}{\alpha} + \frac{r - \alpha}{\alpha}\sigma_1 + \frac{r^2 + s - \alpha r}{\alpha}\sigma_2 + \frac{r^3 + 2sr - \alpha(r^2 + s)}{\alpha}\sigma_3.
\end{aligned}$$

(d) Taking $Q_pW_n = Q_pH_n$ in Theorem 2 (b), we obtain

$$Q_pH_n = (\widehat{d_1} + \widehat{d_2}n)\alpha^n$$

where

$$\begin{aligned} \widehat{d_1} &= Q_pH_0 \\ &= 2.\mathbf{1} + r\sigma_1 + \sigma_2(2s + r^2) + \sigma_3(r^3 + 3rs), \end{aligned}$$

$$\begin{aligned} \widehat{d_2} &= \frac{1}{\alpha}(Q_pH_1 - \alpha Q_pH_0) \\ &= \frac{1}{\alpha}(r.\mathbf{1} + \sigma_1(r^2 + 2s) + \sigma_2(r^3 + 3rs) + \sigma_3(r^4 + 4r^2s + 2s^2) \\ &\quad -\alpha(2.\mathbf{1} + r\sigma_1 + \sigma_2(2s + r^2) + \sigma_3(r^3 + 3rs)))) \\ &= \frac{r - 2\alpha}{\alpha} + \frac{r^2 + 2s - \alpha r}{\alpha}\sigma_1 + \frac{r^3 + 3rs - \alpha(2s + r^2)}{\alpha}\sigma_2 \\ &\quad + \frac{r^4 + 4r^2s + 2s^2 - \alpha(r^3 + 3rs)}{\alpha}\sigma_3. \end{aligned}$$

□

As a different method, Binet's formula of generalized Fibonacci polynomial quaternions formula can also be expressed as given in the theorem below.

Theorem 3. *For all integers n, we have the following formulas:*

(a) (Distinct Roots Case $\alpha \neq \beta$) Binet's formula for generalized Pauli Fibonacci polynomial quaternions is

$$Q_pW_n = \frac{1}{\alpha - \beta}\begin{pmatrix} \alpha^n p_1 - \beta^n p_2 + \alpha^{n+3}p_1 - \beta^{n+3}p_2 & \alpha^{n+1}p_1 - \beta^{n+1}p_2 - i\alpha^{n+2}p_1 + i\beta^{n+2}p_2 \\ (\alpha^{n+1}p_1 - \beta^{n+1}p_2 + i\alpha^{n+2}p_1 - i\beta^{n+2}p_2) & \alpha^n p_1 - \beta^n p_2 - \alpha^{n+3}p_1 + \beta^{n+3}p_2 \end{pmatrix}$$

(b) (Single Root Case $\alpha = \beta$) Binet's formula for generalized Pauli Fibonacci polynomial quaternions is

$$Q_pW_n = \alpha^n \begin{pmatrix} D_1 + D_2n + D_1\alpha^3 + D_2n\alpha^3 & D_1\alpha + D_2n\alpha - iD_1\alpha^2 - D_2n\alpha^2 \\ D_1\alpha + D_2n\alpha + iD_1\alpha^2 + D_2n\alpha^2 & D_1 + D_2n - D_1\alpha^3 - D_2n\alpha^3 \end{pmatrix}$$

where p_1, p_2, D_1,and D_2 are as stated in the Theorem 1.

Proof.

(a) Using Binet's formula for the Generalized Fibonacci (Horadam) polynomial given in Theorem 1 (a) together with identity (5), we obtain

$$\begin{aligned} Q_pW_n &= \begin{pmatrix} W_n + W_{n+3} & W_{n+1} - iW_{n+2} \\ W_{n+1} + iW_{n+2} & W_n - W_{n+3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{p_1\alpha^n - p_2\beta^n}{\alpha-\beta} + \frac{p_1\alpha^{n+3} - p_2\beta^{n+3}}{\alpha-\beta} & \frac{p_1\alpha^{n+1} - p_2\beta^{n+1}}{\alpha-\beta} - i(\frac{p_1\alpha^{n+2} - p_2\beta^{n+2}}{\alpha-\beta}) \\ \frac{p_1\alpha^{n+1} - p_2\beta^{n+1}}{\alpha-\beta} + i(\frac{p_1\alpha^{n+2} - p_2\beta^{n+2}}{\alpha-\beta}) & \frac{p_1\alpha^n - p_2\beta^n}{\alpha-\beta} - \frac{p_1\alpha^{n+3} - p_2\beta^{n+3}}{\alpha-\beta} \end{pmatrix} \\ &= \frac{1}{\alpha - \beta}\begin{pmatrix} \alpha^n p_1 - \beta^n p_2 + \alpha^{n+3}p_1 - \beta^{n+3}p_2 & \alpha^{n+1}p_1 - \beta^{n+1}p_2 - i\alpha^{n+2}p_1 + i\beta^{n+2}p_2 \\ \alpha^{n+1}p_1 - \beta^{n+1}p_2 + i\alpha^{n+2}p_1 - i\beta^{n+2}p_2 & \alpha^n p_1 - \beta^n p_2 - \alpha^{n+3}p_1 + \beta^{n+3}p_2 \end{pmatrix}. \end{aligned}$$

(b) From Theorem 1 (b), which provides Binet's formula for the generalized Fibonacci (Horadam) polynomial, and using identity (5), we obtain

$$
\begin{aligned}
Q_pW_n &= \begin{pmatrix} W_n + W_{n+3} & W_{n+1} - iW_{n+2} \\ W_{n+1} + iW_{n+2} & W_n - W_{n+3} \end{pmatrix} \\
&= \begin{pmatrix} (D_1 + D_2n)\alpha^n + (D_1 + D_2n)\alpha^{n+3} & (D_1 + D_2n)\alpha^{n+1} - i(D_1 + D_2n)\alpha^{n+2} \\ (D_1 + D_2n)\alpha^{n+1} + i(D_1 + D_2n)\alpha^{n+2} & (D_1 + D_2n)\alpha^n - (D_1 + D_2n)\alpha^{n+3} \end{pmatrix} \\
&= \alpha^n \begin{pmatrix} D_1 + D_2n + D_1\alpha^3 + D_2n\alpha^3 & D_1\alpha + D_2n\alpha - iD_1\alpha^2 - D_2n\alpha^2 \\ D_1\alpha + D_2n\alpha + iD_1\alpha^2 + D_2n\alpha^2 & D_1 + D_2n - D_1\alpha^3 - D_2n\alpha^3 \end{pmatrix}.
\end{aligned}
$$

□

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} z^n Q_pW_n$ of the sequence of generalized Pauli Fibonacci polynomial quaternions.

Theorem 4. *Suppose that $f_{Q_pW_n}(z) = \sum_{n=0}^{\infty} z^n Q_pW_n$ is the ordinary generating function of the sequence of generalized Pauli Fibonacci polynomial quaternions. Then $\sum_{n=0}^{\infty} z^n Q_pW_n$ is given by*

$$
\sum_{n=0}^{\infty} z^n Q_pW_n = \frac{Q_pW_0 + z(Q_pW_1 - sQ_pW_0)}{(1 - rz - sz^2)}. \tag{16}
$$

Proof. Using the definition of the generalized Pauli Fibonacci polynomial quaternions and subtracting $rz \sum_{n=0}^{\infty} z^n Q_pW_n$ and $sz^2 \sum_{n=0}^{\infty} z^n Q_pW_n$ from $\sum_{n=0}^{\infty} z^n Q_pW_n$, we obtain

$$
\begin{aligned}
(1 - rz - sz^2)\sum_{n=0}^{\infty} z^n Q_pW_n &= \sum_{n=0}^{\infty} z^n Q_pW_n - rz\sum_{n=0}^{\infty} z^n Q_pW_n - sz^2\sum_{n=0}^{\infty} z^n Q_pW_n \\
&= \sum_{n=0}^{\infty} z^n Q_pW_n - r\sum_{n=0}^{\infty} z^{n+1} Q_pW_n - s\sum_{n=0}^{\infty} z^{n+2} Q_pW_n \\
&= \sum_{n=0}^{\infty} z^n Q_pW_n - r\sum_{n=1}^{\infty} z^n Q_pW_{n-1} - s\sum_{n=2}^{\infty} z^n Q_pW_{n-2} \\
&= (Q_pW_0 + zQ_pW_1) - szQ_pW_0 \\
&\quad + \sum_{n=2}^{\infty} (Q_pW_n - rQ_pW_{n-1} - sQ_pW_{n-2}) \\
&= Q_pW_0 + z(Q_pW_1 - sQ_pW_0)
\end{aligned}
$$

By rearranging the above equation, we obtain (16). □

From Theorem 4, we have the following corollary.

Corollary 2. *For all integers n, we have the following formulas:*

(a) The generating function of the Pauli (r, s)-Fibonacci polynomial quaternion Q_pG_n is

$$
\begin{aligned}
\sum_{n=0}^{\infty} z^n Q_pG_n &= \frac{1}{(1 - rz - sz^2)}(\sigma_1 + \sigma_2 r + \sigma_3(r^2 + s) + z(\mathbf{1} + (r - s)\sigma_1 \\
&\quad + (r^2 + s - rs)\sigma_2 + (r^3 + 2rs - sr^2 - s^2)\sigma_3)).
\end{aligned}
$$

(b) *The generating function of the Pauli (r,s)-Lucas polynomial quaternion Q_pH_n is*

$$\begin{aligned}\sum_{n=0}^{\infty} z^n Q_p H_n &= \frac{1}{(1-rz-sz^2)}(2.\mathbf{1}+r\sigma_1+\sigma_2(2s+r^2)+\sigma_3(r^3+3rs)\\ &\quad +z((r-2s).\mathbf{1}+(r^2+2s-rs)\sigma_1+\sigma_2(r^3+3rs-2s^2-sr^2)\\ &\quad +\sigma_3(r^4+4r^2s+2s^2-sr^3-3rs^2))).\end{aligned}$$

Proof. By taking $Q_pW_n = Q_pG_n$ and $Q_pW_n = Q_pH_n$ in Theorem 4, parts (a) and (b) follow directly. □

Next, we present a theorem concerning Q_pW_n and G_n.

Theorem 5. *For all integers m,n, we have the following formulas:*

$$Q_pW_{n+m} = Q_pW_nG_{m+1} + sQ_pW_{n-1}G_m, \tag{17}$$

i.e.,

$$Q_pW_{n+m} = Q_pW_mG_{n-1} + sQ_pW_{m-1}G_n.$$

Proof. The proof can be carried out by induction on m. First, we assume that $m \geq 1$.

If $m = 1$, then Equation (17) holds. Since

$$\begin{aligned}Q_pW_{n+1} &= rQ_pW_n + sQ_pW_{n-1}\\ &= Q_pW_nG_2 + sQ_pW_{n-1}G_1\end{aligned}$$

where $G_2 = r$ and $G_1 = 1$. For $m = 2$, (17) is true. As

$$\begin{aligned}Q_pW_{n+2} &= rQ_pW_{n+1} + sQ_pW_n\\ &= r(rQ_pW_n + sQ_pW_{n-1}) + sQ_pW_n\\ &= (r^2+s)Q_pW_n + sQ_pW_{n-1}\\ &= Q_pW_nG_3 + sQ_pW_{n-1}G_2.\end{aligned}$$

Now, we assume that the equation holds for all m with $1 \leq m \leq k+1$. Thus, by our assumption, for $m = k$ and $m = k+1$, respectively, we have

$$Q_pW_{n+k} = Q_pW_nG_{k+1} + sQ_pW_{n-1}G_k, \tag{18}$$

$$Q_pW_{n+k+1} = Q_pW_nG_{k+2} + sQ_pW_{n-1}G_{k+1}. \tag{19}$$

By using (6), (18) and (19), we have

$$\begin{aligned}Q_pW_{n+k+2} &= rQ_pW_{n+k+1} + sQ_pW_{n+k}\\ &= r(Q_pW_nG_{k+2} + sQ_pW_{n-1}G_{k+1}) + s(Q_pW_nG_{k+1} + sQ_pW_{n-1}G_k)\\ &= Q_pW_n(rG_{k+2} + sG_{k+1}) + sQ_pW_{n-1}(rG_{k+1} + sG_k)\\ &= Q_pW_nG_{k+3} + sQ_pW_{n-1}G_{k+2}\end{aligned}$$

That means the equations hold for $m = k+2$. □

Next, we assume that $m \leq 0$; this implies $|m| = -m = v$. For $v = 0$, i.e., $m = 0$, Equation (17) is true. Since

$$Q_pW_n = Q_pW_nG_1 + sQ_pW_{n-1}G_0,$$

where $G_0 = 0$ and $G_1 = 1$. For $v = 1$, i.e., $m = -1$, Equation (17) is true:

$$Q_pW_{n-1} = Q_pW_nG_0 + sQ_pW_{n-1}G_{-1},$$

where $G_0 = 0$ and $G_{-1} = \frac{1}{s}$. Let Equation (17) hold for all $|m| = -m = v$ with $0 \leq v \leq k+1$. Thus, by our assumption, for $v = k$ and $v = k+1$, respectively, we have

$$Q_pW_{n-k} = Q_pW_nG_{-k+1} + sQ_pW_{n-1}G_{-k}, \tag{20}$$

$$Q_pW_{n-k-1} = Q_pW_nG_{-k} + sQ_pW_{n-1}G_{-k-1}. \tag{21}$$

By using (7), (20) and (21), we have

$$\begin{aligned}
Q_pW_{-n-k-2} &= -\frac{r}{s}Q_pW_{-n-k-1} + \frac{1}{s}Q_pW_{-n-k} \\
&= -\frac{r}{s}(Q_pW_nG_{-k} + sQ_pW_{n-1}G_{-k-1}) + \frac{1}{s}(Q_pW_nG_{-k+1} + sQ_pW_{n-1}G_{-k}) \\
&= Q_pW_n(-\frac{r}{s}G_{-k} + \frac{1}{s}G_{-k+1}) + sQ_pW_{n-1}(-\frac{r}{s}G_{-k-1} + \frac{1}{s}G_{-k}) \\
&= Q_pW_nG_{-k-1} + sQ_pW_{n-1}G_{-k-2}.
\end{aligned}$$

So means Equation (17) holds for $v = |m| = k+2$.

Note that, if we take $n = 1$ and $m = n-1$ in Theorem 5, we have the following identity:

$$Q_pW_n = Q_pW_1G_n + sQ_pW_0G_{n-1}. \tag{22}$$

Next, we give a theorem that provides some identities related to the Pauli (r,s)-Fibonacci polynomial quaternion Q_pG_n and the Pauli (r,s)-Lucas polynomial quaternion Q_pH_n.

Theorem 6. *For any integer n, the following equalities are true:*

$$\begin{aligned}
s^3Q_pH_n &= -(3rs + r^3)Q_pG_{n+4} + (4r^2s + r^4 + 2s^2)Q_pG_{n+3}, \\
s^2Q_pH_n &= (2s + r^2)Q_pG_{n+3} - (3rs + r^3)Q_pG_{n+2}, \\
sQ_pH_n &= -rQ_pG_{n+2} + (2s + r^2)Q_pG_{n+1}, \\
Q_pH_n &= 2Q_pG_{n+1} - rQ_pG_n, \\
Q_pH_n &= rQ_pG_n + 2sQ_pG_{n-1}, \\
(r^2s^3 + 4s^4)Q_pG_n &= -(3rs + r^3)Q_pH_{n+4} + (4r^2s + r^4 + 2s^2)Q_pH_{n+3}, \\
(r^2s^2 + 4s^3)Q_pG_n &= (2s + r^2)Q_pH_{n+3} - (3rs + r^3)Q_pH_{n+2}, \\
(r^2s + 4s^2)Q_pG_n &= -rQ_pH_{n+2} + (2s + r^2)Q_pH_{n+1}, \\
(r^2 + 4s)Q_pG_n &= 2Q_pH_{n+1} - rQ_pH_n, \\
(r^2 + 4s)Q_pG_n &= rQ_pH_n + 2sQ_pH_{n-1}.
\end{aligned}$$

Proof. Using (8) and (9), and $s^3H_n = -(3rs + r^3)G_{n+4} + (4r^2s + r^4 + 2s^2)G_{n+3}$ (for the proof, see Lemma 9 in [47], we have

$$
\begin{aligned}
s^3 Q_p H_n &= s^3 H_n \mathbf{1} + s^3 H_{n+1}\sigma_1 + s^3 H_{n+2}\sigma_2 + s^3 H_{n+3}\sigma_3 \\
&= (-(3rs+r^3)G_{n+4} + (4r^2s + r^4 + 2s^2)G_{n+3})\mathbf{1} \\
&\quad +(-(3rs+r^3)G_{n+5} + (4r^2s + r^4 + 2s^2)G_{n+4})\sigma_1 \\
&\quad +(-(3rs+r^3)G_{n+6} + (4r^2s + r^4 + 2s^2)G_{n+5})\sigma_2 \\
&\quad +(-(3rs+r^3)G_{n+6} + (4r^2s + r^4 + 2s^2)G_{n+6})\sigma_3 \\
&= -(3rs+r^3)(G_{n+4}\mathbf{1} + G_{n+5}\sigma_1 + G_{n+6}\sigma_2 + G_{n+7}\sigma_3) \\
&\quad +(4r^2s + r^4 + 2s^2)((G_{n+3}\mathbf{1} + G_{n+4}\sigma_1 + G_{n+5}\sigma_2 + G_{n+6}\sigma_3)) \\
&= -(3rs+r^3)Q_pG_{n+4} + (4r^2s + r^4 + 2s^2)Q_pG_{n+3}.
\end{aligned}
$$

The other cases of the theorem can be proved using (8) and (9), and the equalities that are given in Lemma 9 in [47]. □

Theorem 7. *For all integers n, we have the following identity:*

$$Q_pW_n - Q_pW_{n+1}\sigma_1 - Q_pW_{n+2}\sigma_2 - Q_pW_{n+3}\sigma_3 = (W_n - W_{n+2} - W_{n+4} - W_{n+6})\mathbf{1}$$

Proof. Using (1) and (4), we obtain

$$
\begin{aligned}
Q_pW_n - Q_pW_{n+1}\sigma_1 - Q_pW_{n+2}\sigma_2 - Q_pW_{n+3}\sigma_3 &= W_n\mathbf{1} + W_{n+1}\sigma_1 + W_{n+2}\sigma_2 + W_{n+3}\sigma_3 \\
&\quad -(Q_pW_{n+1} - Q_pW_{n+2}\sigma_1 - Q_pW_{n+3}\sigma_2 - Q_pW_{n+4}\sigma_3)\sigma_1 \\
&\quad -(Q_pW_{n+2} - Q_pW_{n+3}\sigma_1 - Q_pW_{n+4}\sigma_2 - Q_pW_{n+5}\sigma_3)\sigma_2 \\
&\quad -(Q_pW_{n+3} - Q_pW_{n+4}\sigma_1 - Q_pW_{n+5}\sigma_2 - Q_pW_{n+6}\sigma_3)\sigma_3 \\
&= W_n\mathbf{1} + W_{n+1}\sigma_1 + W_{n+2}\sigma_2 + W_{n+3}\sigma_3 \\
&\quad -\sigma_1Q_pW_{n+1} + \sigma_1^2Q_pW_{n+2} + \sigma_2\sigma_1Q_pW_{n+3} + \sigma_3\sigma_1Q_pW_{n+4} \\
&\quad -\sigma_2Q_pW_{n+2} + \sigma_2^2Q_pW_{n+4} + \sigma_1\sigma_2Q_pW_{n+3} + \sigma_3\sigma_2Q_pW_{n+5} \\
&\quad -\sigma_3Q_pW_{n+3} + \sigma_1\sigma_3Q_pW_{n+4} + \sigma_2\sigma_3Q_pW_{n+5} + \sigma_3^2Q_pW_{n+6} \\
&= (W_n - W_{n+2} - W_{n+4} - W_{n+6})\mathbf{1}
\end{aligned}
$$

Now, we give the Catalan's identity for the generalized Pauli Fibonacci polynomial quaternions Q_pW_n. □

Theorem 8 (Catalan's identity)**.** *For all integer n, we have the following formula:*

$$Q_pW_{n+m}Q_pW_{n-m} - Q_pW_nQ_pW_n = (-s)^n(Q_pW_mQ_pW_{-m} - Q_pW_0Q_pW_0).$$

Proof. For the proof, we use Binet's formula of the Q_pW_n and (5). We know that there are two cases for Binet's formula of the generalized Pauli Fibonacci polynomial quaternions Q_pW_n. Therefore, we investigate the distinct roots case $\alpha \neq \beta$, using Theorem 3 (a); we have

$$Q_pW_{n+m}Q_pW_{n-m} - Q_pW_nQ_pW_n = (\alpha\beta)^{n-m}p_1p_2\frac{\alpha^m - \beta^m}{(\alpha-\beta)^2}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$A_{11} = -(\alpha^m - \beta^m - \alpha\beta^{m+1} + \alpha^{m+1}\beta + i\alpha\beta^{m+2} - i\alpha^{m+2}\beta - \alpha^3\beta^m + \alpha^m\beta^3 + \alpha^{m+3} - \beta^{m+3} - i\alpha^2\beta^{m+1} + i\alpha^{m+1}\beta^2 - \alpha^2\beta^{m+2} + \alpha^{m+2}\beta^2 - \alpha^3\beta^{m+3} + \alpha^{m+3}\beta^3)$,

$A_{12} = (\alpha\beta^{m+3} - \alpha^{m+3}\beta - i\alpha^2\beta^m + i\alpha^m\beta^2 - \alpha^{m+1} + i\alpha^{m+2} + \beta^{m+1} - i\beta^{m+2} - \alpha^3\beta^{m+1} + \alpha^{m+1}\beta^3 - i\alpha^2\beta^{m+3} + i\alpha^3\beta^{m+2} - i\alpha^{m+2}\beta^3 + i\alpha^{m+3}\beta^2 + \alpha\beta^m - \alpha^m\beta)$,

$A_{21} = -(\alpha\beta^{m+3} - \alpha^{m+3}\beta - i\alpha^2\beta^m + i\alpha^m\beta^2 + \alpha^{m+1} + i\alpha^{m+2} - \beta^{m+1} - i\beta^{m+2} - \alpha^3\beta^{m+1} + \alpha^{m+1}\beta^3 + i\alpha^2\beta^{m+3} - i\alpha^3\beta^{m+2} + i\alpha^{m+2}\beta^3 - i\alpha^{m+3}\beta^2 - \alpha\beta^m + \alpha^m\beta)$,

$A_{22} = -(\alpha^m - \beta^m - \alpha\beta^{m+1} + \alpha^{m+1}\beta - i\alpha\beta^{m+2} + i\alpha^{m+2}\beta + \alpha^3\beta^m - \alpha^m\beta^3 - \alpha^{m+3}$
$+ \beta^{m+3} + i\alpha^2\beta^{m+1} - i\alpha^{m+1}\beta^2 - \alpha^2\beta^{m+2} + \alpha^{m+2}\beta^2 - \alpha^3\beta^{m+3} + \alpha^{m+3}\beta^3)$.

Similarly, we have

$$Q_pW_mQ_pW_{-m} - Q_pW_0Q_pW_0 = (\alpha\beta)^{-m}p_1p_2\frac{\alpha^m - \beta^m}{(\alpha-\beta)^2}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Note that p_1, p_2 are as stated in Theorem 1. Therefore, we have the result that we need. (Note that, using (11), we have $\alpha\beta = -s$.)

Now, we investigate the other root case. For the single root case $\alpha = \beta$, using Theorem 3 (b), we have

$$Q_pW_{n+m}Q_pW_{n-m} - Q_pW_nQ_pW_n = \alpha^{2n}mD_2^2\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where

$B_{11} = -\left(m - 2i\alpha^3 + m\alpha^2 + 2m\alpha^3 + m\alpha^4 + m\alpha^6\right)$,
$B_{12} = -2\alpha\left(-i\alpha^4 + 2\alpha^3 - im\alpha + m\right)$,
$B_{21} = -2\alpha\left(-i\alpha^4 - 2\alpha^3 + im\alpha + m\right)$,
$B_{22} = -\left(m + 2i\alpha^3 + m\alpha^2 - 2m\alpha^3 + m\alpha^4 + m\alpha^6\right)$.

Similarly, we have

$$Q_pW_mQ_pW_{-m} - Q_pW_0Q_pW_0 = mD_2^2\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where D_1, D_2 are as stated in Theorem 1. Therefore, we get the result that we need. (Note that, using (11), we have $\alpha^2 = -s$.) □

This equality is non-commutative, meaning that the order in which the operations are performed affects the result. In other words, the terms involved in the equation do not commute, and switching the order of operations will generally lead to a different outcome. This non-commutativity arises due to the properties of the quaternionic operations and the specific structure of the generalized Pauli Fibonacci polynomials involved in the identity.

Next, we give the Cassani's identity for the generalized Pauli Fibonacci polynomial quaternions Q_pW_n.

Theorem 9 (Cassini's identity)**.** *For all integers n, we have the following formula:*

$$Q_pW_{n+1}Q_pW_{n-1} - Q_pW_nQ_pW_n = (-s)^n(Q_pW_1Q_pW_{-1} - Q_pW_0Q_pW_0).$$

Proof. Taking $m = 1$ in Theorem 8, the proof can be completed. □

Next, we present the Simpson formula for the generalized Pauli Fibonacci polynomial quaternions Q_pW_n.

Theorem 10 (Simpson Formula)**.** *For all integer n, we have the following formula:*

$$\begin{vmatrix} Q_pW_{n+1} & Q_pW_n \\ Q_pW_n & Q_pW_{n-1} \end{vmatrix} = s^{2n}\begin{vmatrix} Q_pW_1 & Q_pW_0 \\ Q_pW_0 & Q_pW_{-1} \end{vmatrix}$$

Proof. Using Theorem 9, and properties of the 2×2 block matrix, we have

$$\begin{aligned} \begin{vmatrix} Q_pW_{n+1} & Q_pW_n \\ Q_pW_n & Q_pW_{n-1} \end{vmatrix} &= \det(Q_pW_{n+1}Q_pW_{n-1} - Q_pW_nQ_pW_n) \\ &= \det((-s)^n(Q_pW_1Q_pW_{-1} - Q_pW_0Q_pW_0)) \\ &= s^{2n}\det((Q_pW_1Q_pW_{-1} - Q_pW_0Q_pW_0)) \\ &= s^{2n}\begin{vmatrix} Q_pW_1 & Q_pW_0 \\ Q_pW_0 & Q_pW_{-1} \end{vmatrix}. \end{aligned} \quad (23)$$

□

3. Linear Sum Formulas

In this section, we present some sum formulas related to the generalized Pauli Fibonacci polynomial quaternions Q_pW_n. The following theorem presents some summation formulas of generalized Pauli Fibonacci polynomial quaternions Q_pW_n with positive subscripts.

Theorem 11. *Let x be a complex number. For $n \geq 0$, we have the following formulas:*

(a) *If $sx^2 + rx - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k Q_pW_k = \frac{x^{n+2}Q_pW_{n+2} + x^{n+1}(1-rx)Q_pW_{n+1} - xQ_pW_1 + (rx-1)Q_pW_0}{sx^2 + rx - 1}.$$

(b) *If $r^2x - s^2x^2 + 2sx - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k Q_pW_{2k} = \frac{-x^{n+1}(sx-1)Q_pW_{2n+2} + rsx^{n+2}Q_pW_{2n+1} - rxQ_pW_1 + (r^2x + sx - 1)Q_pW_0}{r^2x - s^2x^2 + 2sx - 1}.$$

(c) *If $r^2x - s^2x^2 + 2sx - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k Q_pW_{2k+1} = \frac{rx^{n+1}Q_pW_{2n+2} - sx^{n+1}(sx-1)Q_pW_{2n+1} + (sx-1)Q_pW_1 - rsxQ_pW_0}{r^2x - s^2x^2 + 2sx - 1}.$$

Proof. (a) Using the recurrence relation

$$Q_pW_n = rQ_pW_{n-1} + sQ_pW_{n-2},$$

i.e.,

$$sQ_pW_{n-2} = Q_pW_n - rQ_pW_{n-1},$$

we obtain

$$\begin{aligned} sx^1Q_pW_1 &= x^1Q_pW_3 - rx^1Q_pW_2 \\ sx^2Q_pW_2 &= x^2Q_pW_4 - rx^2Q_pW_3 \\ &\vdots \\ sx^{n-1}Q_pW_{n-1} &= x^{n-1}Q_pW_{n+1} - rx^{n-1}Q_pW_n \\ sx^nQ_pW_n &= x^nQ_pW_{n+2} - rx^nQ_pW_{n+1}. \end{aligned}$$

By adding the equations side by side, we obtain

$$\sum_{k=0}^{n} x^k Q_pW_k = \frac{x^{n+2}Q_pW_{n+2} + x^{n+1}(1-rx)Q_pW_{n+1} - xQ_pW_1 + (rx-1)Q_pW_0}{sx^2 + rx - 1}.$$

(b) and (c) Using the recurrence relation

$$Q_pW_n = rQ_pW_{n-1} + sQ_pW_{n-2}$$

i.e.,

$$rQ_pW_{n-1} = Q_pW_n - sQ_pW_{n-2}$$

we obtain

$$\begin{aligned}
rx^1Q_pW_3 &= x^1Q_pW_4 - sx^1Q_pW_2 \\
rx^2Q_pW_5 &= x^2Q_pW_6 - sx^2Q_pW_4 \\
rx^3Q_pW_7 &= x^3Q_pW_8 - sx^3Q_pW_6 \\
&\vdots \\
rx^{n-1}Q_pW_{2n-1} &= x^{n-1}Q_pW_{2n} - sx^{n-1}Q_pW_{2n-2} \\
rx^nQ_pW_{2n+1} &= x^nQ_pW_{2n+2} - sx^nQ_pW_{2n}.
\end{aligned}$$

Now, if we add the above equations side by side, we get

$$\begin{aligned}
r(-Q_pW_1 + \sum_{k=0}^{n} x^kQ_pW_{2k+1}) &= (x^nQ_pW_{2n+2} - Q_pW_2 - x^{-1}Q_pW_0 \\
&\quad + \sum_{k=0}^{n} x^{k-1}Q_pW_{2k}) - s(-Q_pW_0 + \sum_{k=0}^{n} x^kQ_pW_{2k}).
\end{aligned} \tag{24}$$

Similarly, using the recurrence relation

$$Q_pW_n = rQ_pW_{n-1} + sQ_pW_{n-2}$$

i.e.,

$$rQ_pW_{n-1} = Q_pW_n - sQ_pW_{n-2}$$

we write the following clear equations:

$$\begin{aligned}
rx^1Q_pW_2 &= x^1Q_pW_3 - sx^1Q_pW_1, \\
rx^2Q_pW_4 &= x^2Q_pW_5 - sx^2Q_pW_3, \\
rx^3Q_pW_6 &= x^3Q_pW_7 - sx^3Q_pW_5, \\
&\vdots \\
rx^{n-1}Q_pW_{2n-2} &= x^{n-1}Q_pW_{2n-1} - sx^{n-1}Q_pW_{2n-3}, \\
rx^nQ_pW_{2n} &= x^nQ_pW_{2n+1} - sx^nQ_pW_{2n-1}.
\end{aligned}$$

Now, if we add the above equations side by side, we obtain

$$r(-Q_pW_0 + \sum_{k=0}^{n} x^kQ_pW_{2k}) = (-Q_pW_1 + \sum_{k=0}^{n} x^kQ_pW_{2k+1}) - s(-x^{n+1}Q_pW_{2n+1} + \sum_{k=0}^{n} x^{k+1}Q_pW_{2k+1}). \tag{25}$$

Then, solving system (24) and (25), the required result of (b) and (c) follow. □

The following corollary gives the sum formulas related to the Pauli (r,s)-Fibonacci polynomial quaternion Q_pG_n and the Pauli (r,s)-Lucas polynomial quaternion Q_pH_n.

Corollary 3. *The following sum formulas are given:*

(a) *If $sx^2 + rx - 1 \neq 0$, then we have the following sum formula for the Pauli (r,s)-Fibonacci polynomial quaternion Q_pG_n :*

(1) $\sum_{k=0}^{n} x^k Q_pG_k = \frac{1}{sx^2+rx-1}(x^{n+2}Q_pG_{n+2} + x^{n+1}(1-rx)Q_pG_{n+1} - x\mathbf{1} - \sigma_1 + (-r - sx)\sigma_2 - (s + r^2 + rsx)\sigma_3)$.

(2) $\sum_{k=0}^{n} x^k Q_pG_{2k} = \frac{1}{r^2x-s^2x^2+2sx-1}(-x^{n+1}(sx-1)Q_pG_{2n+2} + rsx^{n+2}Q_pG_{2n+1} - rx\mathbf{1} + (sx-1)\sigma_1 - r\sigma_2 - (s - s^2x + r^2)\sigma_3)$.

(3) $\sum_{k=0}^{n} x^k Q_pG_{2k+1} = \frac{1}{r^2x-s^2x^2+2sx-1}(rx^{n+1}Q_pG_{2n+2} - sx^{n+1}(sx-1)Q_pG_{2n+1} + (sx-1)\mathbf{1} - r\sigma_1 - (s - s^2x + r^2)\sigma_2 - r(2s - s^2x + r^2)\sigma_3)$.

(b) *If $sx^2 + rx - 1 \neq 0$, then we have the following sum formula for the Pauli (r,s)-Lucas polynomial quaternion Q_pH_n :*

(1) $\sum_{k=0}^{n} x^k Q_pH_k = \frac{1}{sx^2+rx-1}(x^{n+2}Q_pH_{n+2} + x^{n+1}(1-rx)Q_pH_{n+1} + (rx-2)\mathbf{1} - (r+2sx)\sigma_1 - (2s + r^2 + rsx)\sigma_2 - (2s^2x + 3rs + r^3 + r^2sx)\sigma_3)$.

(2) $\sum_{k=0}^{n} x^k Q_pH_{2k} = \frac{1}{r^2x-s^2x^2+2sx-1}(-x^{n+1}(sx-1)Q_pH_{2n+2} + rsx^{n+2}Q_pH_{2n+1} + (r^2x + 2sx - 2)\mathbf{1} - r(sx+1)\sigma_1 - (2s - 2s^2x + r^2)\sigma_2 - r(3s - s^2x + r^2)\sigma_3)$.

(3) $\sum_{k=0}^{n} x^k Q_pH_{2k+1} = \frac{1}{r^2x-s^2x^2+2sx-1}(rx^{n+1}Q_pH_{2n+2} - sx^{n+1}(sx-1)Q_pH_{2n+1} - r(1 + sx)\mathbf{1} - (2s - 2s^2x + r^2)\sigma_1 - r(3s - s^2x + r^2)\sigma_2 - (4r^2s - 2s^3x + r^4 + 2s^2 - r^2s^2x)\sigma_3)$.

Proof. (a) and (b) Taking $Q_pW_n = Q_pG_n$ and $Q_pW_n = Q_pH_n$ in Theorem 11, the proof is completed. □

The Case $x = 1$

In this subsection, we investigate the case $x = 1, r = 1, s = 2$, i.e., $r^2x - s^2x^2 + 2sx - 1 = 0$.

Observe that setting $x = 1, r = 1, s = 2$ (i.e., for the generalized Pauli Jacobsthal quaternion that can be denoted the Q_pJ_n case) in Theorem 11 (b) and (c) makes the right-hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule, however, provides the evaluation of the sum formulas. If $r = 1, s = 2$, then we have the following theorem.

Theorem 12. *If $r = 1, s = 2$, then for $n \geq 0$, we have the following formulas:*

(a)

$$\sum_{k=0}^{n} Q_pJ_k = \frac{1}{2}(Q_pJ_{n+2} - Q_pJ_1).$$

(b)

$$\sum_{k=0}^{n} Q_pJ_{2k} = \frac{1}{3}((n+3)Q_pJ_{2n+2} - 2(n+2)Q_pJ_{2n+1} + Q_pJ_1 - 3Q_pJ_0).$$

(c)

$$\sum_{k=0}^{n} Q_pJ_{2k+1} = \frac{1}{3}(-(n+1)Q_pJ_{2n+2} + 2(n+3)Q_pJ_{2n+1} - 2Q_pJ_1 + 2Q_pJ_0).$$

Proof.

(a) Taking $x = 1, r = 1, s = 2$ in Theorem 11 (a), we obtain the desired identity.

(b) The proof of this part is carried out using Theorem 11 (b). Setting $r = 1, s = 2$ in Theorem 11 (b), we obtain

$$\sum_{k=0}^{n} x^k Q_pJ_{2k} = \frac{-x^{n+1}(2x-1)Q_pJ_{2n+2} + 2x^{n+2}Q_pJ_{2n+1} - xQ_pJ_1 + (3x-1)Q_pJ_0}{-4x^2 + 5x - 1}.$$

For $x = 1$, the right-hand side of the above summation formulas becomes an indeterminate form. Now, we apply L'Hospital rule. Then, we get

$$\sum_{k=0}^{n} Q_pJ_{2k} = \left.\frac{\frac{d}{dx}(-x^{n+1}(2x-1)Q_pJ_{2n+2}+2x^{n+2}Q_pJ_{2n+1}-xQ_pJ_1+(3x-1)Q_pJ_0)}{\frac{d}{dx}(-4x^2+5x-1)}\right|_{x=1}$$
$$= \frac{1}{3}((n+3)Q_pJ_{2n+2}-2(n+2)Q_pJ_{2n+1}+Q_pJ_1-3Q_pJ_0).$$

(c) We apply Theorem 11 (c). From Theorem 11 (c), with $r=1, s=2$, it follows that

$$\sum_{k=0}^{n} x^kQ_pJ_{2k+1} = \frac{x^{n+1}Q_pJ_{2n+2}-2x^{n+1}(2x-1)Q_pJ_{2n+1}+(2x-1)Q_pJ_1-2xQ_pJ_0}{-4x^2+5x-1}.$$

For $x=1$, the right-hand side of the above summation formulas results in an indeterminate form. Now, we can employ L'Hospital rule. Consequently, we obtain

$$\sum_{k=0}^{n} Q_pJ_{2k+1} = \left.\frac{\frac{d}{dx}(x^{n+1}Q_pJ_{2n+2}-2x^{n+1}(2x-1)Q_pJ_{2n+1}+(2x-1)Q_pJ_1-2xQ_pJ_0)}{\frac{d}{dx}(-4x^2+5x-1)}\right|_{x=1}$$
$$= \frac{1}{3}(-(n+1)Q_pJ_{2n+2}+2(n+3)Q_pJ_{2n+1}-2Q_pJ_1+2Q_pJ_0).$$

□

4. Matrices Associated with Generalized Pauli Fibonacci Polynomial Quaternions Q_pW_n

In this part of our study, we give some identities on some matrices linked to the Generalized Pauli Fibonacci polynomial quaternions Q_pW_n. First, we assume that $A = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}$ and $N_{Q_pW_n} = \begin{pmatrix} Q_pW_2 & Q_pW_1 \\ Q_pW_1 & Q_pW_0 \end{pmatrix}$. Hence, we have the following identity:

$$A^n = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix}.$$

For the proof, see Soykan [47].

Therefore, we obtain the following theorem associated with the matrix A and $N_{Q_pW_n}$.

Theorem 13. *For all integer n, we have the following identity:*

$$A^nN_{Q_pW_n} = \begin{pmatrix} Q_pW_{n+2} & Q_pW_{n+1} \\ Q_pW_{n+1} & Q_pW_n \end{pmatrix}.$$

Proof. Using identity (22) and properties of Generalized Pauli Fibonacci polynomial quaternions Q_pW_n, we obtain

$$A^nN_{Q_pW_n} = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix}\begin{pmatrix} Q_pW_2 & Q_pW_1 \\ Q_pW_1 & Q_pW_0 \end{pmatrix}$$
$$= \begin{pmatrix} W_2Q_pG_{n+1}+sQ_pW_1G_n & Q_pW_1G_{n+1}+sQ_pW_0G_n \\ Q_pW_2G_n+sQ_pW_1G_{n-1} & Q_pW_1G_n+sQ_pW_0G_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} Q_pW_{n+2} & Q_pW_{n+1} \\ Q_pW_{n+1} & Q_pW_n \end{pmatrix}.$$

□

Remark 1. *By taking the Generalized Pauli Fibonacci polynomial quaternions Q_pW_n as the Pauli (r,s)-Fibonacci polynomial quaternion Q_pG_n and the Pauli (r,s)-Lucas polynomial quaternion Q_pH_n, respectively, we obtain matrices $N_{Q_pG_n}$, $N_{Q_pH_n}$ as follows:*

$$N_{Q_pG_n} = \begin{pmatrix} Q_pG_2 & Q_pG_1 \\ Q_pG_1 & Q_pG_0 \end{pmatrix}.$$

$$N_{Q_pH_n} = \begin{pmatrix} Q_pH_2 & Q_pH_1 \\ Q_pH_1 & Q_pH_0 \end{pmatrix}.$$

From Theorem 13, we have the following corollary.

Corollary 4. *For all integer n, we have the following results:*

(a)

$$A^n N_{Q_pG_n} = \begin{pmatrix} Q_pG_{n+2} & Q_pG_{n+1} \\ Q_pG_{n+1} & Q_pG_n \end{pmatrix}$$

(b)

$$A^n N_{Q_pH_n} = \begin{pmatrix} Q_pH_{n+2} & Q_pH_{n+1} \\ Q_pH_{n+1} & Q_pH_n \end{pmatrix}$$

Proof. Taking $Q_pW_n = Q_pG_n$ and $Q_pW_n = Q_pH_n$ in Theorem 13, (a) and (b) follow.

Based on our theoretical perspective, we considered that the terms of the matrix we defined, being related to

$$Q_pW_n = W_n\mathbf{1} + W_{n+1}\sigma_1 + W_{n+2}\sigma_2 + W_{n+3}\sigma_3,$$

correspond to the Hamiltonian of a spin-1/2 particle in a magnetic field. In this context, if generalized Fibonacci polynomials, or more specifically Fibonacci numbers, are used in the following equation,

$$H_n = W_n\mathbf{1} + \sum_{i=1}^{3} W_{n+i}\sigma_i.$$

The terms $W_{n+i}\sigma_i$ may be interpreted as components of a magnetic field that vary with time or state, while the term $W_n\mathbf{1}$ can be viewed as a scalar contribution shifting the total energy. □

5. Conclusions

In this paper, a modification of generalized Fibonacci polynomials is established by combining the concepts of quaternion and Pauli matrix. It is expected that this study will attract the attention of experts who will conduct research in both physics and mathematics. The generalized Fibonacci properties and polynomials containing elements in the sense of Pauli matrix and quaternion introduced in this paper can be evaluated in terms of investigating the suitability of quaternion structures for studies in physics, such as the reformulation of Dirac and Maxwell equations. The (r,s)-extensions of the Fibonacci and Lucas sequences presented in this study were introduced to demonstrate the algebraic flexibility and generalization capacity of the proposed structure. These formulations not only extend the classical Fibonacci and Lucas sequences within a quaternionic framework but also align with established generalization procedures in the literature. In doing so, they reinforce the theoretical soundness of the approach and highlight its potential for broader algebraic applications. In conclusion, this study introduces a novel generalization by incorporating Pauli matrices into the framework of Generalized Pauli Fibonacci Polynomial Quaternions (GPFPQs). This integration enriches the algebraic structure by merging the recursive nature of Fibonacci polynomials with the non-commutative and anti-commutative properties of Pauli matrices. As a result, the proposed structure is particularly effective in modeling quantum mechanical systems, especially in the fields of spin dynamics, entanglement, and unitary transformations. Moreover, the Pauli-based extension opens up new potential applications in quantum information theory, such as quantum state representation and encoding, as well as in classical cryptography where matrix-based operations and structural complexity are highly valued. This approach not only generalizes existing quaternion models but also establishes a connection between number theory, operator theory, and quantum computation, providing a more comprehensive mathematical toolset for future research. In future work, one can try to relate matrix polynomials to the concept we have presented, and try to relate it to structures such as the pseudo Hermitian operator representation using [71].

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