Critical matter and geometric phase transitions

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ABSTRACT

A review of the fundamental nature of critical phenomena suggests that fluctuations of matter fields coupled with a topological transition are the signature elements of critical systems. These two elements are shown to induce a geometric phase transition from Riemannian geometry to a conformally invariant geometry.

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1 Introduction

Critical phenomena, which are synonymous with continuous or second-order phase transitions, have been studied for more than a century through a progression of developments that include [1]: van der Waal's equation; mean-field theory; Landau's expansion of the free energy in terms of an order parameter; Onsager's exact solution of the two-dimensional Ising model; critical exponents; universality classes; scale invariance; and the renormalization group. During this historical development there has been an increasing emphasis placed on the role that fluctuations play near the critical point. In the "classical" period of the theory, local fluctuations were ignored and average fields were assumed sufficient to determine the properties of the system. This eventually gave way to the recognition that strong fluctuations are an inherent feature of critical phenomena and, as such, must be taken into account. This has been done most effectively in recent years through the use of the renormalization group. In this approach, one identifies key pieces of information such as relations among the critical exponents by progressively scaling out to larger and larger distances. A remarkable result that has emerged from this analysis is the recognition that diverse physical systems share common characteristics near their critical points. This property, known as universality, suggests that it should be possible to characterize critical phenomena in terms of a more rudimentary language than is needed to account for the specific microscopic details of any one particular system. An attempt is made in the present paper to take a step in this direction by considering the topological nature of critical phenomena from a macroscopic perspective. It is proposed here that critical phenomena emerge when a projection map (described below) acquires a multi-valued nature. Furthermore, when this topological feature and the associated critical fluctuations are incorporated into a geometric theory of spacetime, a phase transition from Riemannian geometry to a conformally invariant geometry follows as a consequence, providing a geometric basis for the scale invariance associated with critical phenomena.

A cursory study of the literature dealing with critical phenomena reveals that there are several intertwined topics that are relevant. The key elements that define our current understanding of critical phenomena, which are briefly reviewed in Section 2, include: a foundation that is rooted in symmetry considerations; parameters of the physical system that experience large fluctuations; discontinuous (topological) changes that result when control parameters are varied continuously; universality classes that are governed by spatial dimension and symmetry; and the onset of scale invariance. This latter characteristic of critical matter triggering the emergence of scale invariance in a localized region of spacetime calls for a geometrical response. It is generally assumed[1] that the properties of the spacetime geometry should reflect the characteristics displayed by matter. Under this assumption, one is confronted with the challenge of understanding how the properties of the local geometry might change in step with matter when it passes from a system with a well-defined standard of length to one that is scale invariant at the critical point. What is needed is a framework that is capable of providing a link between critical matter and geometry. Einstein's gravitational field equations seem most appropriate for this task. The starting point of the present geometrical analysis is to consider fluctuations of the spacetime metric (Sec. 3). That is, rather than presupposing that critical matter will *not* interact in any significant way with spacetime geometry, this possibility is allowed for by incorporating conformal fluctuations into the gravitational field equations. By relating the topological properties of the critical system to that of the scalar field governing the metric fluctuations, it will be shown that the Riemannian geometry of general relativity undergoes a phase transition at a critical point to a geometry that is locally scale invariant. Scale invariance, as used within the context of critical phenomena, implies that one has the freedom to rescale lengths by a constant factor (the same factor at each spacetime point). Yet the physical invariance under re-scaling of lengths is actually valid only within the limited region of the critical system. What is needed is a locally defined scale invariance rather than global scale invariance. This type of symmetry, known as conformal invariance, has proven useful in the two-dimensional application of conformal field theory to critical phenomena [3].

[1]An exception to this view is Dirac's proposal of a dual metric model [2] where an atomic metric is introduced that is independent of the spacetime metric.

Some concluding remarks are provided in Section 4. An earlier version of this paper was presented at the Tenth Canadian Conference on General Relativity and Relativistic Astrophysics [4].

2 The topological nature of critical phenomena

Landau [5] emphasized the role that symmetry plays in governing the properties of phase transitions. The relevant symmetry might reflect a spatial or temporal invariance, or some invariance property in an internal space. In general, one of the two phases involved in a second-order phase transition will be in a higher symmetry state than the other phase; that is, it may contain the lower symmetry elements along with some additional ones. The transition from a higher symmetry state to a lower symmetry state is referred to as a symmetry-breaking process. In the case of a first-order phase transition, which is characterized thermodynamically by the release of latent heat and the discontinuous change in an order parameter (see below), the symmetries of the two phases may be completely unrelated. A necessary condition for a first-order phase transition line to terminate at a critical point in a phase diagram is that the symmetries of the two phases below the critical temperature differ at most quantitatively. It is for this reason that a critical point *can* exist for the liquid/gas phase(s) since, below the critical temperature, both contain a continuous translational symmetry, whereas a critical point *cannot* exist for the liquid/solid phases since the discrete translational symmetry of the solid phase differs qualitatively from that of the liquid. In this sense, symmetry considerations play a defining role in critical phenomena.

Landau's introduction of the order parameter concept, where the order parameter vanishes in the higher symmetry or disordered phase and is non-zero in the lower symmetry ordered phase, provided a means by which phase transitions in a variety of branches of science could be studied within a common framework. With the incorporation of group theory into the analysis, a powerful tool to explore symmetry breaking effects in fields such as particle physics emerged. This mathematical model for describing critical phenomena (known as the Landau-Ginzburg model [6]) is based on a potential that depends on external control parameters as well as the order parameter. This simple model affords a description of "spontaneous" symmetry breaking when the sign of the control parameter is changed, causing the single minimum of the potential to transition to degenerate minima. In terms of the order parameter, the critical point represents a bifurcation point where the single stable minimum above the critical point splits into two stable states below the critical point. Although the order parameter approach continues to be a useful tool in the study of symmetry breaking processes, it is well known that its quantitative predictions fail in regions close to the critical point: the theory gives the wrong values for the critical exponents that are associated with the power law behavior of systems near a critical point. Onsager demonstrated [7] that the singularity of the free energy at the critical point renders Landau's power series expansion of the order parameter invalid in the region of the critical point. The quantitative failure of Landau's approach near the critical point does not mean that the symmetry properties associated with critical phenomena outlined above do not hold; rather, it illustrates that, due to the singular nature of the critical point, the mathematical tools used to analyze this point and its corresponding properties must be chosen with care.

It eventually became evident that the physical reason for the failure of Landau's expansion was the neglect of strong fluctuations close to the critical point that violate the assumption held in any mean field theory that the fluctuating field experienced by an individual atom can be replaced by its average. In general, the macroscopic properties of a physical system that is made of up many atomic constituents will be governed by two competing effects: the interaction processes between the constituents of the system that tend to minimize the energy, and the thermal fluctuations that perturb the interaction processes. The distance over which a change at one atomic site can influence the properties of other sites is called the correlation length. As the system approaches a critical point, the regions that experience a common fluctuation grow anomalously large [6]. However, as the correlation length diverges, fluctuations continue to occur over the full range of smaller scales. This is most easily demonstrated with a fluid system. Smoluchowski and Einstein observed that density fluctuations in a fluid cause fluctuations in the refractive index, which, in turn, leads to the scattering of light. As the fluid approaches a critical point, it becomes cloudy due to strong scattering of light, a phenomenon known as critical opalescence. Between the point where the fluctuations are sufficient to initially cause the fluid to turn cloudy and the critical point, where the fluctuations become anomalously large, smaller fluctuations continue to occur and the cloudy state persists. With the fluctuations spanning all scales at the critical point, the system loses its intrinsic ability to distinguish lengths (at least above the molecular scale), and the system is said to be *scale invariant*. Given the predominance of fluctuations at the critical point, it is clear that any approach that is based strictly in terms of the mean values of fields neglects the fundamental nature of critical phenomena and will be unsuccessful in determining the properties of the system arbitrarily close to the critical point.

The challenge of probing the properties of physical systems near a critical point, where the fluctuations span all scales, can be addressed by using the renormalization group (RG) approach [8]. The strategy employed is to adopt an iterative method of characterizing a small region of the system in terms of its constituent properties and their couplings, then increasing the size of the region by a particular factor, re-characterizing the system at the new scale, and repeating the process. The process progressively averages out the smaller fluctuations and results in a mapping of the coupling constants, called the RG flow, in a parameter space defined by all possible coupling constants associated with the system. A critical point of the physical system is represented by a fixed point of the mapping in the parameter space, and the slope of the parameter surface at the fixed point yields the correct values of the critical exponents for the system. Hence, the RG group provides a method by which the properties of critical systems can be determined. It should be noted, however, that the RG approach does not provide a descriptive theory of critical phenomena. Nonetheless, it does offer some insight into the general features of critical phenomena. For example, the "zooming out" process [9] renders the microscopic details of the physical system irrelevant. Indeed, the RG flow of systems as diverse as fluids and uniaxial ferromagnets converge to the same fixed point and have the same critical exponents. It turns out that these two systems belong to the same *universality class*, where such classes are only dependent on the spatial dimension of the system and its symmetry properties. The fact that the RG approach predicts that systems which, under normal conditions differ in fundamental ways and are typically governed by distinct sets of Lagrangians and differential equations, but near the critical point assume the same characteristics, suggests that it may be possible to use a more fundamental mathematical language to describe critical phenomena than is offered by an approach based on differential equations. By employing a scaling procedure to determine the properties of the critical point, the RG approach also affirms the scale invariant nature of critical phenomena.

Efforts to probe the physical nature of critical phenomena must be formulated within some branch of mathematics. Historically, such attempts have tended to draw upon elements of topology. In a recent series of articles, Casetti *et al* (see [10] for a review) have used Morse theory [11] to argue that second order phase transitions are associated with topology changes that occur in the high-dimensional microscopic configuration space of the system. Morse theory allows one to relate the local properties of a smooth function on a manifold to the global properties of the manifold by considering the level sets of the function. To see how this works, consider the potential $V(\varphi_m; c)$ of a physical system that depends on the state variables or order parameters φ_m (the subscript m stands for matter) as well as a set of control parameters c. A familiar example that is associated with critical phenomena is the Landau-Ginzburg potential,

$$V(\varphi_m; a, b) = \frac{1}{4}\varphi_m^4 + \frac{1}{2}a\varphi_m^2 + b\varphi_m, \tag{1}$$

given here is its most general form [12]. The condition

$$\nabla V = 0, \tag{2}$$

where differentiation is with respect to φ_m, defines the equilibrium states (stable or unstable) of the system. In mathematical terms[2], the solutions to equation (2) define the critical points

[2]Note that the expression “critical point” generally has different meanings in physics and mathematics.

of $V(\varphi_m; a, b)$ which form the equilibrium surface or critical manifold in $(\varphi_m; a, b)$-space. A (mathematical) critical point is called non-degenerate, isolated, or Morse if the Hessian of the function has only non-zero eigenvalues at that point. The Morse lemma states that, in the neighborhood of a non-degenerate critical point, a smooth change of variables can be performed to express the potential locally as a quadratic form. Morse theory provides a link between the properties of the potential and the topology of the manifold as follows: if a non-degenerate critical point lies between two level surfaces of the potential defined by $V^{-1}(\alpha) = \{\varphi_m : V(\varphi_m) = \alpha\}$, then the topology of a level surface will change in accordance with the properties of the Hessian as it passes through the critical point; otherwise the two level sets of the potential will be diffeomorphic to one another. It is important to note that Morse theory applies directly[3] *only* to non-degenerate critical points. In particular, as the Landau-Ginzburg potential (1) does not satisfy the non-degeneracy requirement, an alternative approach will be used here to establish a topological connection with critical phenomena.

A degenerate or non-Morse critical point is one for which the Hessian of the potential has at least one vanishing eigenvalue. This could result, for example, if the potential took on a minimum value along a line rather than a single isolated point as in the Morse case. Near a Morse minimum, where the potential takes on a quadratic form, a perturbation of the system is represented well in terms of damped harmonic motion. As a consequence, the characteristically large fluctuations associated with critical phenomena cannot be modeled by a Morse minimum. The growth of oscillations as a system approaches a degenerate critical point suggests that critical phenomena should be associated with mathematical degenerate

[3]It is possible to apply Morse theory in cases where the potential has degenerate critical points by employing techniques that remove the degeneracies of the potential. While such techniques are needed when dealing with high dimensional configuration spaces (see, e.g., [13]), they tend to obscure the central role that degenerate critical points play in continuous phase transitions. For example, even after the problematic degeneracies have been dealt with in Ref. [13], the authors take care to track them and ultimately conjecture that a topology change entails a continuous phase transition only when it is associated with a degenerate critical point.

critical points. This is supported by the correspondence between the mathematical properties of critical points under perturbations and the observed nature of critical phenomena. It can be shown [11] that a critical point is structurally stable if and only if it is non-degenerate. A Morse critical point may shift slightly under a perturbation, but its type will not change. In contrast, a non-Morse critical point may be transformed under a perturbation into a number of isolated critical points at different locations. These types of change in the properties of the potential are what typically occur in examples of spontaneous symmetry breaking. This leads to the important observation that *critical phenomena correspond to mathematical degenerate critical points* [12, 14]. Hence, the branch of mathematics that will be used here to study critical phenomena is that which is associated with degenerate critical points. The relevant field of mathematics, which encompasses the topics of singularities, bifurcations, and catastrophes, emerged with the 1955 paper of Whitney [15]. René Thom [16] later developed a formalism to describe the canonical forms of singular mappings as well as their structural stability properties. While this formalism, known as *Catastrophe Theory*, has been criticized for its popularized applications [17], it nevertheless provides a framework to study discontinuous changes (catastrophes) that result from continuous ones. To facilitate the analysis of non-Morse functions, one embeds them into a smooth family of functions that are governed by parameters (e.g., the control parameters in eq. (1)) and studies their mapping properties. The critical or catastrophe manifold for systems with a degenerate critical point is smooth, but contains various "folds" or "cusps". In the case of the Landau-Ginzburg potential (1), the critical manifold is known as the cusp catastrophe. If one projects down from a catastrophe manifold (with the φ_m-axis vertical) onto the control parameter space, the projection map is single-valued in regions corresponding to Morse points and multi-valued in regions corresponding to non-Morse points [14]. Given that this multi-valued property is common to all non-Morse critical points, which are in turn associated with critical phenomena, the multi-valued property will be taken here as the crucial topological characteristic associated with the onset of critical phenomena.

3 Critical matter and spacetime geometry

It should be noted that the topic of gravitational critical phenomena has received considerable attention ever since Choptuik presented his self-similar solution for the spherical collapse of a scalar field [18]. As well, the RG approach has been applied to gravitational topics such as relativistic cosmology (see, e.g., [19]). These avenues of research have generally applied techniques of critical phenomena to gravitational problems. In the present work, the reverse approach is taken: geometric techniques of general relativity are used to probe the spacetime nature of critical phenomena. Indeed, while the framework of Riemannian geometry provides a starting point for the present analysis, a connection with gravitational phenomena is rather secondary. As such, it is conceivable that the topological interaction proposed herein could be relevant well outside the large scale domain of applicability normally associated with general relativity. While the gravitational interaction is negligible on the laboratory scale, this does not preclude the possibility that non-gravitational geometric effects may play a role at that scale.

As noted above, it is essential that the fluctuations of matter fields be included in any analysis of the properties of critical phenomena. This principle is extended here to include geometric fields by using Einstein's theory of gravitation to investigate the effect of fluctuations of matter on the geometry of spacetime. Given the local nature of the interaction between matter and geometry, it seems most natural to expect the metric to respond in kind to matter fluctuations. Other authors have considered fluctuations in general relativity (see, e.g., [20]); however, such work has generally been posed within the context of quantum or semi-classical gravity. In the case of critical phenomena, at least at the level where the microscopic details are irrelevant, it is sufficient to consider the properties of classical fields.

In order to preserve the causal structure of spacetime, it will be assumed that the induced geometric fluctuations are described by conformal fluctuations of the metric:

$$g_{\mu\nu}(x) = \varphi^2(x) < g_{\mu\nu}(x) >, \tag{3}$$

where $\varphi(x)$ is a classical fluctuating field and $< g_{\mu\nu} >$ is the average background metric.

Taking the average of (3), it follows that

$$< \varphi^2 >= 1. \tag{4}$$

At each spacetime point, the standard of length fluctuates about some mean value. Fluctuations in $\varphi(x)$ can be viewed as movements along a fiber over x, with the section over spacetime defined by the local values of φ forming a fluctuating surface. For a neighboring point $x + \delta x$, fluctuations will, in general, differ from those at x. These variations in the standard of length between neighboring spacetime points will produce a change in length of a parallel transported vector given by

$$\delta\ell = \ell\varphi_{,\mu}\delta x^\mu. \tag{5}$$

Regions characterized by $\delta\ell > 0$ or $\delta\ell < 0$ define a geometric correlation length. It follows from the derivative of eq. (4) that $< \delta\ell >= 0$. As such, the mean spacetime is clearly Riemannian and does not support the scale invariance associated with critical phenomena. However, the topological nature of critical phenomena has not, as yet, been taken into account in the geometric theory.

Recall that the topological character of critical phenomena was framed within the context of a mapping between the (φ_m, c)-space and the control parameter space. In order to establish a link between this topological property on the matter side with a corresponding one on the geometric side, the following parallel structure is employed. On the matter side, the order parameters φ_m represent "internal" or "state" variables that define the vertical axis over the space of "external" or control variables c [14]. The scalar field φ, which is an "internal" geometric field, and which has values on the vertical fiber over spacetime, is associated with the order parameter, while the spacetime manifold corresponds to the control parameter space. Within this framework, the emergence of critical phenomena is taken to correspond to φ acquiring a multi-valued character. In fact, it is well-known [21] that when φ is multi-valued, $\delta\ell \neq 0$ around a closed path with the result that absolute standards of length are lost and $\varphi_{,\mu}$ becomes a non-trivial vector field. Outside the critical region, $\delta\ell$ still vanishes around local closed loops and the average geometry is Riemannian. In this manner, the local character of matter is reflected in the geometry both in and outside the critical region.

It remains to determine how the two spaces can fit together. In the present approach, the conformal metric fluctuations (3) propagate through the Einstein tensor $G_{\mu\nu}$, transforming it to $< G_{\mu\nu} > -I_{\mu\nu}$, where

$$I_{\mu\nu} = \frac{2}{\varphi}(\varphi_{;\mu\nu} - < g_{\mu\nu} > \varphi^{,\alpha}{}_{;\alpha}) - \frac{1}{\varphi^2}(4\varphi_{,\mu}\varphi_{,\nu} - < g_{\mu\nu} > \varphi_{,\alpha}\varphi^{,\alpha}), \tag{6}$$

and where a semi-colon denotes the covariant derivative associated with $< g_{\mu\nu} >$. It has been shown elsewhere [22] that $I_{\mu\nu}$ can be used to construct a model that allows a conformally invariant (Weyl) space and a Riemannian space to coexist on either side of a thin shell, where the surface tension arises from the boundary conditions satisfied by $\varphi_{,\mu}$. In this manner, the conformal metric fluctuations coupled with the topological property of φ lead to a consistent representation of the observed metric properties of critical systems, without violating the Riemannian character of geometry associated with non-critical matter.

Some comments on the conformal nature of the geometry associated with critical matter are in order. Only a few years after Einstein proposed that gravitational phenomena can be described in geometric terms via the rotation of a vector under parallel transport, Weyl [23] proposed that a change in length of the transported vector could be used to give a geometric interpretation of electromagnetic phenomena. It was subsequently argued, however, that one could *either* have the conformal invariance of Weyl's geometry *or* the absolute standards of length provided by the constituents of matter, but not both. The rejection of Weyl's geometry was based on the conviction that the properties of the geometry should reflect the characteristics displayed by matter. Yet, various forms of conformal symmetry continue to find application. Forty years ago, Fulton, Rohrlich and Witten [24] published an excellent review of conformal invariance in physics. While their work continues to provide a benchmark, more recent developments in physics have led to new applications. The discussion here will be limited to a brief summary of their classification of conformal transformations (also see [25]) and its relevance to the study of critical phenomena. A more thorough survey of conformal symmetry is provided in [26].

The conformal transformations of the spacetime metric,

$$g^c_{\mu\nu}(x) = \sigma(x) g_{\mu\nu}(x), \tag{7}$$

where $\sigma(x) > 0$, form a group C_g with the set of all manifolds isomorphic under C_g constituting a conformal space. The transformation (7) preserves the angle between any two vectors as well as the ratio of their lengths, but not their individual lengths. The group of all coordinate transformations together with C_g form a larger group C, of which the former are proper subgroups. The space in which equations are invariant under C is called a Weyl space. When a vector is parallel transported around a closed path in Weyl space, it will not, in general, return with the same length that it had at the starting point. The subgroup of C obtained by imposing the condition that the transformation map a flat space into another flat space is called the *restricted* conformal group C_0. This restricted group is a 15-parameter Lie group on Minkowski space: four parameters for spacetime translations; six for homogenous Lorentz transformations; one for global scale transformations or dilatations; and four for "acceleration" transformations (more commonly known as *special* conformal transformations today). It is this latter restricted group C_0 that has found the greatest application in studies of critical phenomena. In a quantum field theory description of critical phenomena [3], scale invariance manifests itself through the covariance of the n-point correlation functions, as well as leading to the establishment of relations between the critical exponents. By extending dilatation invariance of the correlation functions to conformal invariance under the group C_0, considerably more information about the nature of a critical point can be obtained, but only in two dimensions. Specifically, one can classify the critical point partition functions and obtain exact values of the critical exponents [3]. The strength of conformal field theory in two dimensions is due to the fact that the restricted conformal group C_0 is infinite-dimensional in two, and only two, dimensions [27]. This has led to the commonly held perception that conformal invariance *itself* is only useful in the two-dimensional case. However, it should be noted that the flat space condition that under girds applications of C_0 is affected by the dimensionality of the space. This can been seen, for example, by noting that the conformal transformation of the scalar curvature and the Ricci tensor (which describe the gravitational content in two and three dimensions, respectively), contain terms proportional to $n-2$, where n is the dimension of the space [28]. It turns out that, for $n = 2$, the condition that flat space be mapped to flat space is trivially satisfied. This suggests that the uniqueness of the dimensionality of two is more significant in satisfying the flat space constraint than in governing the fundamental role that conformal invariance plays in nature. Turning this around, one may conclude that the full richness of conformal invariance can only be realized when curvature is included in the analysis.

4 Discussion

Critical phenomena offer a rich context to probe the subtle ways that matter and geometry interact. The singularities present at a physical critical point have made it difficult to formulate a comprehensive theory of critical phenomena; the current general options are to use a mean-field-theory approach which provides a descriptive theory, but ultimately leads to incorrect results, or to use the renormalization group approach which allows one to determine the correct critical exponents, but doesn't provide a descriptive theory of critical phenomena. The existence of universality classes suggests that the search for a description of critical phenomena should be conducted at a more basic level than is offered in terms of differential equations.

In the present paper, it has been argued that the fundamental nature of critical phenomena is rooted in the topological properties of the critical manifold. In particular, the onset of critical phenomena occurs when the order parameter acquires multiple values over the control parameter space. This topological signature of critical phenomena, coupled with the requirement that fluctuations not be ignored, was incorporated into a geometric theory of spacetime by allowing the metric to undergo conformal fluctuations involving the scalar field φ. The scale invariant nature of critical phenomena follows in the present geometric model when φ takes on the multi-valued nature of critical matter. While these results affirm the steps taken in the present work, it is recognized that a rigorous inclusion of the study of singularities, bifurcations and catastrophes in a geometric theory of spacetime warrants further analysis. As well, it is noted that the proposed relationship between critical matter and geometric phase transitions calls for further investigation in applications where phase transitions play a dominant role, such as in the study of the very early universe.

Finally, the proposed geometric phase transition provides a new perspective on the role that conformally invariant geometry might play in the physical world. The use of conformal symmetry, restricted to the group C_0, has proven to be a useful tool in two dimensions; the present geometric approach offers a more general context to explore the role of conformal invariance. The domain of applicability of this non-gravitational interaction between critical matter and geometry may extend to scales much less than the large scale domain of the gravitational interaction of general relativity, as did Weyl's proposal for a geometric interpretation of the electromagnetic interaction. There does not appear to be any direct relationship between the topologically induced vector field $\varphi_{,\mu}$ and Weyl's electromagnetic interpretation of conformal symmetry. However, there does appear to be a natural association between $\varphi_{,\mu}$ and the Casimir force. It is well-known [29] that any field that is fluctuating, and that is constrained to satisfy boundary conditions, forms the prerequisite conditions for the appearance of the Casimir effect. This suggests that, in the present context at least, Weyl's potential $\varphi_{,\mu}$ be associated with the Casimir force.

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