Antisymmetric Matrices Are Real Bivectors

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This paper briefly reviews the conventional method of obtaining the canonical form of an antisymmetric (skewsymmetric, alternating) matrix. Conventionally a vector space over the complex field has to be introduced. After a short introduction to the universal mathematical "language" Geometric Calculus, its fundamentals, i.e. its "grammar" Geometric Algebra (Clifford Algebra) is explained. This lays the groundwork for its real geometric and coordinate free application in order to obtain the canonical form of an antisymmetric matrix in terms of a bivector, which is isomorphic to the conventional canonical form. Then concrete applications to two, three and four dimensional antisymmetric square matrices follow. Because of the physical importance of the Minkowski metric, the canonical form of an antisymmetric matrix with respect to the Minkowski metric is derived as well. A final application to electromagnetic fields concludes the work.

Key Words : Geometric Calculus , Geometric Algebra, Clifford Algebra, Antisymmetric (Alternating, Skewsymmetric) Matrix, Real Geometry

1. Introduction

But the gate to life is narrow and the way that leads to it is hard, and there are few people who find it. ...I assure you that unless you change and become like children, you will never enter the Kingdom of heaven.

Jesus Christ[9]

... enter the teahouse.[2)] The sliding door is only thirty six inches high. Thus all who enter must bow their heads and crouch. This door points to the reality that all are equal in tea, irrespective of social status or social position.[21]

... for geometry, you know, is the gate of science, and the gate is so low and small that one can only enter it as a little child. *William K. Clifford*[4]

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2. Motivation

2.1 Antisymmetric matrices

We are all familiar with antisymmetric (alternating, skewsymmetric) matrices. Prominent examples are: the matrices describing infinitesimal rotations[1)] in mechanics the electromagnetic field tensor in Maxwell's electrodynamics, the three spatial Dirac matrices of quantum mechanics, the torsion tensor of space-time torsion, etc.

A matrix is said to be antisymmetric if interchanging rows and columns (transposing A->A^T) gives the negative of the original matrix

$$A^T = -A\,. \tag{2.1}$$

Or expressed in components

$$A_{kl} = -A_{lk}\,. \tag{2.2}$$

This naturally means that the main diagonal elements are all zero

$$A_{kk} = 0\,. \tag{2.3}$$

Every square matrix can be decomposed into its symmetric part with

$$A^T = A \qquad (2.4)$$

and antisymmetric part:

$$A = A_{(sy)} + A_{(as)} = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)\ . \qquad (2.5)$$

It is standard undergraduate textbook[1] knowledge, that symmetric matrices have a set of n orthonormal eigenvectors, n being the dimension of the space. Taking the n eigenvectors as basis, the symmetric matrix takes diagonal form

$$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}. \qquad (2.6)$$

$\lambda_1, \lambda_2, \dots \lambda_n$ are the corresponding real eigenvalues, some of which may be equal or zero.

Eq. (2.6) shows the socalled *canonical* form of a symmetric matrix.

Trying to do the same for antisymmetric matrices, we find[11] that a similar "canonical form" can be achieved, but not over the field of real numbers, only over the field of complex numbers. The eigenvalues will either be purely imaginary, occurring only in complex conjugate pairs or zero

$$j\nu_1, -j\nu_1, j\nu_2, -j\nu_2, \cdots, 0 \qquad (2.7)$$

where j stands for the usual imaginary unit with j^2=-1. The eigenvectors will also have complex valued components.

But since the imaginary unit j lacks a straightforward geometric interpretation, the questions for the canonical form restricted to real spaces arises.

In this work I will basically give two answers. The first is a classical answer as found in books on linear algebra[11]. This first answer arrives at a real canonical form using complex unitary spaces for the sake of proof.

The second answer will show how a redefinition of the product of vectors, the socalled *geometric product* of vectors can do without the complex field altogether and derive a more elegant real answer.

2.2 The 'classical' canonical form of antisymmetric matrices

Maltsev[11] states a theorem (p. 166, THEOREM 6&6a) about the canonical form of an antisymmetric matrix, representing a skewsymmetric transformation:

"In a real unitary space the matrix A of a skewsymmetric transformation, in a suitable orthonormal basis, assumes the form

$$A = \begin{pmatrix} 0 & \nu_1 & & & & & & \\ -\nu_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & \nu_m & & & \\ & & & -\nu_m & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}, \qquad (2.8)$$

Where O_k is the zero matrix of order *k(= n-2m).* "

$$O_k = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}. \qquad (2.9)$$

All matrix elements omitted in (2.8) and (2.9) are understood to be zero. Maltsev continues:

"For each real skewsymmetric matrix A there exists a real unitary matrix U such that $U^{-1}AU$ has the form (2.8)."

In order to show the zero or pure imaginary number property of the eigenvalues, Maltsev must resort to *Hermitian* skewsymmetric matrices. To prove his THEOREM 6, he basically invokes another theorem about the canonical form of matrices representing *normal* transformations ([11], p. 156). Skewsymmetric matrices are special normal matrices. To prove the theorem about normal transformations, Maltsev introduces complex vector coordinates and a (unitary) vector space over the field of complex numbers.

The reader is thus left with the impression, that there is no way to avoid complex numbers and complex vector spaces in order to arrive at the desired canonical form of an antisymmetric matrix.

The contrary is demonstrated in chapter 3.4 of Hestenes&Sobczyk's book "Clifford Algebra to Geometric Calculus."[7] But before showing what (purely real) Geometric Calculus can do for us in this issue, I will introduce Geometric Algebra for the readers, who are so far unfamiliar with this "unified language for mathematics and physics" [7], science and engineering.

3. First steps in Geometric Algebra

In this section I will first describe Geometric Algebra and Geometric Calculus. Understanding their relation, we can proceed to formulate the fundamentals of Geometric Algebra for two, three and higher dimensions.

3.1 Geometric Algebra and Geometric Calculus

Geometric Algebra (also known as Clifford Algebra) has been extended to a *universal* Geometric Calculus[12] including vector derivatives and directed integrals. It seamlessly integrates all fundamental interactions known in nature.[5,13]

Without using matrices Geometric Algebra already unifies projective geometry, linear algebra, Lie groups, and applies to computational geometry, robotics and computer graphics.

Geometric Calculus of multivectors and multilinear functions provides the foundation for *Clifford Analysis*, which is based upon a synthesis of Clifford algebra and differential forms, both of which owe much to the work of H. Grassmann (1844)[2]. This enables an elegant new formulation of Hamiltonian mechanics[22] and a complete, explicit description of Lie groups as spin groups.[15]

The relation of Geometric Algebra and Geometric Calculus may be compared to that of grammar and language.[7]

Diverse algebraic systems[16], such as

- coordinate geometry
- complex analysis
- quaternions (invented by Hamilton[14])
- matrix algebra
- Grassmann, Clifford and Lie algebras
- vector, tensor and spinor calculi
- differential forms
- twistors

and more appear now in a comprehensive unified system. The attribute *universal* is therefore justified.

3.2 Real two-dimensional Geometric Algebra

The theory developed in this section is not limited to two dimensions. In the case of higher dimensions we can always deal with the two-dimensional subspace spanned by two vectors involved, etc.

3.2.1 The geometric product

Let us start with the real two-dimensional vector space R^2. It is well known that vectors can be multiplied by the *inner* product which corresponds to a mutual projection of one vector **a** onto another vector **b** and yields a scalar: the projected length $a\cos\theta$ times the vector length b, i.e.

$$\mathbf{a}\cdot\mathbf{b} = (a\cos\vartheta)b = ab\cos\vartheta\,. \qquad (3.1)$$

The projected vector $\mathbf{a}_{\parallel}$ itself can then be written as

$$\mathbf{a}_{\parallel} = \mathbf{a}\cdot\mathbf{b}\mathbf{b}/b^2\,. \qquad (3.2)$$

In 1844 the German mathematician H. Grassmann [2] introduced another general (dimension independent) vector product: the antisymmetric *exterior* (outer) product. This product yields the size of the parallelogram area spanned by the two vectors together with an orientation, depending on the sense of following the contour line (e.g. clockwise and anticlockwise),

$$\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}\,. \qquad (3.3)$$

More formally, (3.3) is also called a bivector (2-blade). Grassmann later on unified the inner product and the exterior product to yield the *extensive* product, or how it was later called by W.K. Clifford, the *geometric* product[23,7] of vectors:

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\,. \qquad (3.4)$$

We further demand (nontrivial!) this geometric product to be associative,

$$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{bc}) \qquad (3.5)$$

and distributive,

$$\mathbf{a}(\mathbf{b}+\mathbf{c}) = \mathbf{ab} + \mathbf{ac}\,. \qquad (3.6)$$

Let us now work out the consequences of these definitions in the two-dimensional real vector space R^2. We choose an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. This means that

$$\mathbf{e}_1^2 = \mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_1\cdot\mathbf{e}_1 = \mathbf{1}\,, \mathbf{e}_2^2 = \mathbf{e}_2\mathbf{e}_2 = \mathbf{e}_2\cdot\mathbf{e}_2 = \mathbf{1}\,,$$

$$\mathbf{e}_1\cdot\mathbf{e}_2 = \mathbf{e}_2\cdot\mathbf{e}_1 = \mathbf{0}\,. \qquad (3.7)$$

Please note that e.g. in (3.7) we don't simply multiply the coordinate representations of the basis vectors, we multiply the vectors themselves. We are therefore still free to make a certain choice of the basis vectors, i.e. we work coordinate free! The product of the two basis vectors gives

$$\begin{aligned}\mathbf{e}_1\mathbf{e}_2 &= \mathbf{e}_1\cdot\mathbf{e}_2 + \mathbf{e}_1\wedge\mathbf{e}_2 = \mathbf{e}_1\wedge\mathbf{e}_2\\ &= -\mathbf{e}_2\wedge\mathbf{e}_1 = -\mathbf{e}_2\cdot\mathbf{e}_1 - \mathbf{e}_2\wedge\mathbf{e}_1 = -\mathbf{e}_2\mathbf{e}_1 \equiv \mathbf{i}\end{aligned} \qquad (3.8)$$

the real oriented area element, which I call **i**. It is important that you beware of confusing this real area element **i** with the conventional imaginary unit $j = \sqrt{-1}$.

But what is then the square of **i**?

$$\mathbf{i}^2 = \mathbf{ii} = (\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_2) = -(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_2\mathbf{e}_1) \\ = -\mathbf{e}_1(\mathbf{e}_2\mathbf{e}_2)\sigma_1 = -\mathbf{e}_1\mathbf{e}_1 = -1 \quad (3.9)$$

The square of the oriented real unit area element of **i** is therefore $\mathbf{i}^2 = -1$. This is the same value as the square of the imaginary unit *j*. The big difference however is, that *j* is postulated just so that the equation $j^2 = -1$ can be solved, whereas for **i** we followed a constructive approach: We just performed the geometric product repeatedly on the basis vectors of a real two-dimensional vector space.

So far we have geometrically multiplied vectors with vectors and area elements with area elements. But what happens when we multiply vectors and area elements geometrically?

3.2.2 Rotations, vector inverse and spinors

We demonstrate this by calculating both $\mathbf{e}_1\mathbf{i}$ and $\mathbf{e}_2\mathbf{i}$:

$$\mathbf{e}_1\mathbf{i} = \mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2) = (\mathbf{e}_1\mathbf{e}_1)\ \mathbf{e}_2 = \mathbf{e}_2 \quad (3.10)$$

$$\mathbf{e}_2\mathbf{i} = \mathbf{e}_2(-\mathbf{e}_2\mathbf{e}_1) = -(\mathbf{e}_2\mathbf{e}_2)\ \mathbf{e}_1 = -\mathbf{e}_1. \quad (3.11)$$

This is precisely a 90 degree anticlockwise (mathematically positive) rotation of the two basis vectors and therefore of all vectors by linearity. From this we immediately conclude that multiplying a vector twice with the oriented unit area element **i** constitutes a rotation by 180 degree. Consequently, the square $\mathbf{i}^2 = -1$ geometrically means just to rotate vectors by 180 degree. I emphasize again that *j* and **i** need to be thoroughly kept apart. *j* also generates a rotation by 90 degree, but this is in the plane of complex numbers commonly referred to as the Gaussian plane. It is not to be confused with the 90 degree real rotation **i** of real vectors in a real two-dimensional vector space.

i also generates all real rotations with arbitrary angles. To see this let **a** and **b** be unit vectors. Then I calculate:

$$\mathbf{a}(\mathbf{ab}) = (\mathbf{aa})\mathbf{b} = \mathbf{b} \quad (3.12)$$

Multiplying **a** with the product **ab** therefore rotates **a** into **b**. In *two* dimensions $R_{ab}=\mathbf{ab}$ is therefore the "rotor" that rotates (even all!) vectors by the angle between **a** and **b**. What does this have to do with **i**? Performing the geometric product **ab** explicitely yields:

$$\mathbf{ab} = \cos\theta + \mathbf{i}\sin\theta \quad (3.13)$$

(Please keep in mind that here $\mathbf{a}^2 = \mathbf{b}^2 = 1$ and that the area of the parallelogram spanned by **a** and **b** is precisely $\sin\theta_{ab}$, which explains the second term.) This can formally be written by using the exponential function as:

$$R_{ab} = \mathbf{ab} = \exp(\mathbf{i}\,\theta_{ab}) = \cos\theta_{ab} + \mathbf{i}\sin\theta_{ab} \quad (3.14)$$

We can therefore conclude that the oriented unit area element **i** generates indeed all rotations of vectors in the real two-dimensional vector space.

Another important facet of the geometric product is that it allows to universally define the inverse of a vector with respect to (geometric) multiplication as:

$$\mathbf{x}^{-1} = \frac{1}{\mathbf{x}} \overset{def}{=} \frac{\mathbf{x}}{\mathbf{x}^2},\ \ \mathbf{x}^2 = \mathbf{xx} = \mathbf{x}\cdot\mathbf{x}. \quad (3.15)$$

That this is indeed the inverse can be seen by calculating

$$\mathbf{xx}^{-1} = \mathbf{x}^{-1}\mathbf{x} = \frac{\mathbf{xx}}{\mathbf{x}^2} = 1. \quad (3.16)$$

Using the inverse $\mathbf{b}^{-1}$ of the vector **b**, we can rewrite the projection (3.2) of **a** unto **b** simply as

$$\mathbf{a}_{\|} = \mathbf{a}\cdot\mathbf{bb}^{-1}, \quad (3.17)$$

where I use the convention that inner (and outer) products have preference to geometric products.

It proves sometimes useful to also define an inverse for area elements $A = \pm|A|\mathbf{i}$:

$$A^{-1} = A/A^2 = A/(-|A|^2) = -A/|A|^2, \quad (3.18)$$

where |A| is the scalar size of the area as in (3.51) and one of the signs stands for the orientation of A relative to **i**. We can see that this is really the inverse by calculating

$$AA^{-1} = A^{-1}A = AA/A^2 = A^2/A^2 = -|A|^2/(-|A|^2) = 1. \quad (3.19)$$

By now we also know that performing the geometric product of vectors of a real two-dimensional vector space will only lead to (real) scalar multiples and linear combinations of scalars (grade 0), vectors (grade 1) and oriented area elements (grade 2). In algebraic theory one assigns grades to each of these. All these entities, which are such generated, form the real *geometric algebra* R_2 (note that the index is now a lower index) of a real two-dimensional vector space R^2. R_2 can be generated through (real scalar) linear combinations of the following list of 2^2=4 elements

$$\{1,\ \mathbf{e}_1, \mathbf{e}_2, \mathbf{i}\}. \quad (3.20)$$

This list is said to form the basis of R_2. When analyzing any algebra it is always very interesting to know if there are any subsets of an algebra which stay closed when performing both linear combinations and the geometric product. Indeed it is not difficult to see that the subset {1, **i**} is closed, because 1**i** = **i** and **ii** = −1. This sub-algebra is in one-to-one correspondence with the complex numbers C. We thus see that we can "transfer" all relationships of complex numbers, etc. to the real two-dimensional geometric algebra R_2. We suffer therefore no disadvantage by refraining from the use of complex numbers altogether. The important operation of complex conjugation (replacing *j* by −*j* in a complex number) corresponds to *reversion* in geometric algebra, that is the order of all vectors in a product is reversed:

$$(\mathbf{ab})^{\dagger} = \mathbf{ba} \quad (3.21)$$

and therefore

$$\mathbf{i}^{\dagger} = (\mathbf{e}_1\mathbf{e}_2)^{\dagger} = \mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2 = -\mathbf{i} . \qquad (3.22)$$

In mathematics the geometric product of two vectors [compare e.g. (3.4),(3.7),(3.8),(3.14)] is also termed a *spinor*. In physics use of spinors is frequently considered to be confined to quantum mechanics, but as we have just seen in (3.14), spinors naturally describe every elementary rotation in two dimensions. (Spinors describe rotations in higher dimensions as well, since rotations are always performed in plane two-dimensional subspaces, e.g. in three dimensions the planes perpendicular to the axis of rotation.)

3.3 Real three-dimensional Geometric Algebra

We begin with a real three-dimensional vector space R^3. In R^3 we introduce an orthonormal set of basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, that is $\mathbf{e}_m \cdot \mathbf{e}_m = 1$ and $\mathbf{e}_m \cdot \mathbf{e}_n = 0$ for n $\neq$ m, {n,m=1,2,3}. The basic $2^3 = 8$ geometric entities we can form with these basis vectors are:

1, scalar (grade 0)

$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, vectors (grade 1)

$\{\mathbf{i}_3 = \mathbf{e}_1\mathbf{e}_2, \mathbf{i}_1 = \mathbf{e}_2\mathbf{e}_3, \mathbf{i}_2 = \mathbf{e}_3\mathbf{e}_1\}$,

oriented unit real area elements (grade 2)

$i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, oriented real volume element (grade 3) (3.23)

We now have three real oriented unit area elements $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ corresponding to the three plane area elements of a cube oriented with its edges along $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. This set of eight elements $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, i\}$ forms basis of the real geometric algebra R_3 of the three dimensional vector space R^3. By looking at the subsets $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{i}_3\}$, $\{1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{i}_1\}$ and $\{1, \mathbf{e}_3, \mathbf{e}_1, \mathbf{i}_2\}$ we see that R_3 comprises three plane geometric sub-algebras, as we have studied them in section 3.2.2. In general, by taking any two unit vectors $\{\mathbf{u}, \mathbf{v}\}$ which are perpendicular to each other, we can generate new two-dimensional plane geometric sub-algebras of R_3 with the unit area element $\mathbf{i} = \mathbf{uv}$.

As in the two-dimensional case we have $\mathbf{i}_1^2 = \mathbf{i}_2^2 = \mathbf{i}_3^2 = \mathbf{i}^2 = -1$. And we have

$i^2 = i\,i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1(\mathbf{e}_3\mathbf{e}_2)$

$= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(\mathbf{e}_3\mathbf{e}_1)\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(\mathbf{e}_3\mathbf{e}_2)\mathbf{e}_1$

$= -\mathbf{e}_1\mathbf{e}_2(\mathbf{e}_3\mathbf{e}_3)\mathbf{e}_1\mathbf{e}_2 = \ldots = -1.$ (3.24)

Each permutation after the third, fourth and fifth equal sign introduced a factor of −1 as in (3.8). The square of the oriented three-dimensional volume element is therefore also $i^2 = -1$.

In three dimensions the vector **a** unto **b** projection formula (3.17) does not change, since it relates only entities in the **a**,**b** plane. But beyond that we can also project vectors **a** onto **i** planes, by characterizing a plane by its oriented unit area element **i**. In this context it proves useful to generalize the definition of scalar product to elements of higher grades [3]:

$$\mathbf{a} \cdot B_r \equiv \langle \mathbf{a} \cdot B_r \rangle_{r-1} = \frac{1}{2}\left(\mathbf{a}B_r + (-1)^{1+r} B_r\mathbf{a}\right), \qquad (3.25)$$

where r denotes the grade of the algebraic element B_r, and the bracket notation is explained in (3.36a). For $B_r = \mathbf{b}$ (r=1) we have as usual $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$, but for $B_r = \mathbf{i}$ (example with grade r=2) we have

$$\mathbf{a} \cdot \mathbf{i} = \frac{1}{2}(\mathbf{ai} - \mathbf{ia}). \qquad (3.26)$$

We can calculate for example

$$\mathbf{e}_1 \cdot \mathbf{i}_1 = \frac{1}{2}(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 - \mathbf{e}_2\mathbf{e}_3\mathbf{e}_1) = 0 \qquad (3.27)$$

$$\mathbf{e}_2 \cdot \mathbf{i}_1 = \frac{1}{2}(\mathbf{e}_2\mathbf{e}_2\mathbf{e}_3 - \mathbf{e}_2\mathbf{e}_3\mathbf{e}_2) = \mathbf{e}_3 \qquad (3.28)$$

$$\mathbf{e}_3 \cdot \mathbf{i}_1 = \frac{1}{2}(\mathbf{e}_3\mathbf{e}_2\mathbf{e}_3 - \mathbf{e}_2\mathbf{e}_3\mathbf{e}_3) = -\mathbf{e}_2 . \qquad (3.29)$$

If we now rotate $\mathbf{e}_3$ (and $-\mathbf{e}_2$) with $\mathbf{i}_1^{-1} = -\mathbf{i}_1$ from the right by −90 degree in the $\mathbf{e}_2$,$\mathbf{e}_3$ plane, we obtain

$$\mathbf{e}_2 \cdot \mathbf{i}_1\mathbf{i}_1^{-1} = -\mathbf{e}_3\mathbf{i}_1 = -\mathbf{e}_3\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_2 , \qquad (3.30)$$

$$\mathbf{e}_3 \cdot \mathbf{i}_1\mathbf{i}_1^{-1} = \mathbf{e}_2\mathbf{i}_1 = \mathbf{e}_2\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_3 , \qquad (3.31)$$

respectively.

The projection of any vector **a** unto the $\mathbf{e}_3$,$\mathbf{e}_2$ plane is therefore given by

$$\mathbf{a}_{\parallel} = \mathbf{a} \cdot \mathbf{i}_1\mathbf{i}_1^{-1} . \qquad (3.32)$$

We say therefore instead of $\mathbf{e}_3$,$\mathbf{e}_2$ plane also simply $\mathbf{i}_1$-plane. And in general the projection of a vector **a** unto any **i**-plane is then given by

$$\mathbf{a}_{\parallel} = \mathbf{a} \cdot \mathbf{i}\,\mathbf{i}^{-1} , \qquad (3.33)$$

which is in perfect analogy to the vector unto vector projection in formula (3.17).

There is more[3,4,5] to be said about R_3, and in the next section some more concepts of geometric algebra will be introduced for the geometric algebras of general *n*-dimensional real vector spaces.

3.4 Blades, magnitudes, orthogonality and duality

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$ be an orthonormal vector basis of the Euclidean vector space over the reals R^n. R_n denotes the geometric algebra generated by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$.

k-blades $\mathbf{B}_k$ are defined as multivectors that can be factorized by the outer product (3.4) into k vector factors

$$\mathbf{B}_k = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_k . \quad (3.34)$$

A general multivector element M of the geometric algebra R_n will consist of a sum of various elements of different grades

$$M=M_0+M_1+M_2+\ldots+M_n, \quad (3.35)$$

where some of the k-vectors M_k $(0 \le k \le n)$ may also be zero. Each k-vector M_k can in turn be written as a sum of pure blades of degree k as in (3.34). There are at most $\binom{n}{k}$ terms in such a sum, because the orthogonal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$ allows only to form $\binom{n}{k}$ different k-blades. Extracting the grade k part M_k from the multivector M (3.35) is sometimes denoted as

$$\langle M \rangle_k \equiv M_k . \quad (3.36a)$$

The grade 0 or scalar part of M is sometimes denoted as

$$\langle M \rangle \equiv \langle M \rangle_0 \equiv M_0 . \quad (3.36b)$$

Geometric algebra is very suitable to denote subspaces and perform direct algebraic operations with subspaces, because each pure blade also represents the subspace spanned by the vectors in its factorization (3.34). As shown in [7], p. 9, each inner product (3.25) of a vector $\mathbf{a}$ with a pure k-blade $\mathbf{B}_k$ has the expansion

$$\begin{aligned} &\mathbf{a} \cdot \mathbf{B}_k = \mathbf{a} \cdot (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_k) = \\ &\sum_{s=1}^{k} (-1)^{s+1} \mathbf{a} \cdot \mathbf{a}_s (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \breve{\mathbf{a}}_s \ldots \wedge \mathbf{a}_k) \end{aligned}, \quad (3.37)$$

where $\breve{\mathbf{a}}_s$ means that the vector $\mathbf{a}_s$ is to be omitted from the product. This means that if the vector $\mathbf{a}$ is perpendicular to all $\mathbf{a}_s$ (s=1...k), then the product (3.37) vanishes identical. If $\mathbf{a}$ is perpendicular to all $\mathbf{a}_s$ (s=1...k), it must also be orthogonal to all linear combinations of the $\mathbf{a}_s$ (s=1...k), and therefore to the k-dimensional subspace of R^n spanned by the $\mathbf{a}_s$ (s=1...k).

$$\mathbf{a} \cdot \mathbf{B}_k = 0 \quad (3.38)$$

tells us therefore that $\mathbf{a}$ is in the orthogonal complement of the k-dimensional subspace defined by $\mathbf{B}_k$.

Analogous to (3.25) the outer product can also be generalized to the outer product of vectors with k-vectors

$$\begin{aligned} \mathbf{a} \wedge \mathrm{M}_k &\equiv \frac{1}{2}(\mathbf{a}\mathrm{M}_k + (-1)^k \mathrm{M}_k \mathbf{a}) \\ &= \langle \mathbf{a}\mathrm{M}_k \rangle_{k+1} \end{aligned} \quad (3.39)$$

While an inner product with a vector as in (3.25) always reduces the grade by one, the outer product (3.39) increases the grade by one. The outer product can also serve to directly characterize subspaces of R^n related to k-blades $\mathbf{B}_k$. Suppose that $\mathbf{a}$ is linearly dependent on the vectors $\mathbf{a}_s$ (s=1...k), that define $\mathbf{B}_k$ as in (3.34). By linearity the outer product of $\mathbf{a}$ with $\mathbf{B}_k$ must vanish.

$$\mathbf{a} \wedge \mathrm{B}_k = 0 \quad (3.40)$$

tells us therefore that the vector $\mathbf{a}$ is contained in the k-dimensional subspace defined by $\mathbf{B}_k$.

By successive applications of (3.25) and (3.39) to all k-blades of the r-vector A_r and the s-vector B_s one can show ([7], p. 10) that the geometric product of them can be written as the graded multivector sum

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \ldots + \langle A_r B_s \rangle_{r+s} \quad (3.41)$$

Taking the lowest and hightest grade parts of (3.41) one defines the *scalar* product and the *outer* product of the r-vector A_r and the s-vector B_s to be

$$A_r \cdot B_s = \langle A_r B_s \rangle_{|r-s|} \quad (3.42)$$

and

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}, \quad (3.43)$$

respectively. But a warning is in place. As Dorst[17] showed, the definition (3.42) has some drawbacks, which can be improved by changing the definition of the scalar product.

If $\mathrm{A}_2 = \mathbf{a}_1 \wedge \mathbf{a}_2 = \mathbf{a}_1\mathbf{a}_2$ and s>1, then

$$\mathrm{A}_2 \cdot \mathrm{B}_s = \langle \mathbf{a}_1\mathbf{a}_2\mathrm{B}_s \rangle_{s-2} = \mathbf{a}_1 \cdot (\mathbf{a}_2 \cdot \mathrm{B}_s) \quad (3.44)$$

by repeatedly applying (3.25) and (3.39) and discarding the higher grade parts. If the grade s=2 then

$$\begin{aligned} \mathrm{A}_2 \cdot \mathrm{B}_2 &= \langle \mathbf{a}_1\mathbf{a}_2\mathrm{B}_2 \rangle_0 = \mathbf{a}_1 \cdot (\mathbf{a}_2 \cdot \mathrm{B}_2) \\ &= (\mathbf{a}_2 \cdot \mathrm{B}_2) \cdot \mathbf{a}_1 = -(\mathbf{a}_2\mathbf{a}_1) \cdot \mathrm{B}_2 \\ &= -\mathbf{a}_2 \cdot (\mathbf{a}_1 \cdot \mathrm{B}_2) = \mathbf{a}_2 \cdot (\mathrm{B}_2 \cdot \mathbf{a}_1) \end{aligned} \quad (3.45)$$

where the interchange of B_2 and a vector in the scalar product introduced always a negative sign. (3.45) shows, that the brackets are irrelevant and we can therefore drop them completely

$$\begin{aligned} (\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot \mathrm{B}_2 &= (\mathbf{a}_2 \cdot \mathrm{B}_2) \cdot \mathbf{a}_1 \\ &= \mathbf{a}_2 \cdot (\mathrm{B}_2 \cdot \mathbf{a}_1) = \mathbf{a}_2 \cdot \mathrm{B}_2 \cdot \mathbf{a}_1 \end{aligned} \quad (3.46)$$

Equation (3.46) makes it easy to prove, that if we define a new vector **b** as

$$\mathbf{b} \equiv \mathbf{a}^{-1} \cdot \mathbf{B}_2 \neq 0, \qquad (3.47)$$

it will be orthogonal to the original vector **a**. Remembering that $\mathbf{a}^{-1}=\mathbf{a}/\mathbf{a}^2$ ($\mathbf{a}=\mathbf{a}^{-1}$ if $\mathbf{a}^2=1$), we find that

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot(\frac{\mathbf{a}}{\mathbf{a}^2}\cdot\mathbf{B}_2) = \left\langle \mathbf{a}\frac{\mathbf{a}}{\mathbf{a}^2}\mathbf{B}_2 \right\rangle = \langle \mathbf{B}_2 \rangle = 0, \qquad (3.48)$$

where we have used definitions (3.15), (3.42) and (3.36b). If beyond that **a** is part of the subspace defined by the 2-blade $\mathbf{B}_2$ we can show that **B=ab**

$$\begin{aligned}\mathbf{ab} &= \mathbf{a}(\mathbf{a}^{-1}\cdot\mathbf{B}) = \mathbf{a}(\frac{\mathbf{a}}{\mathbf{a}^2}\cdot\mathbf{B}) = \frac{1}{\mathbf{a}^2}\mathbf{a}(\mathbf{a}\cdot\mathbf{B}) \\ &= \frac{1}{\mathbf{a}^2}\mathbf{a}(\mathbf{aB}) = \mathbf{B}\end{aligned}, \qquad (3.49)$$

where we used (3.15) and for the third equality that $\mathbf{aB} = \mathbf{a}\cdot\mathbf{B} + \mathbf{a}\wedge\mathbf{B} = \mathbf{a}\cdot\mathbf{B}$, and (3.40).

Concerning the outer product (3.43) of orthogonal 2-blades A_2 and B_2 (they consist of four mutually orthogonal vectors) it is easy to show the commutation properties

$$\begin{aligned} A_2B_2 &= \mathbf{a}_1\mathbf{a}_2\mathbf{b}_1\mathbf{b}_2 = \mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{b}_1\wedge\mathbf{b}_2 \\ &= (\mathbf{a}_1\wedge\mathbf{a}_2)\wedge(\mathbf{b}_1\wedge\mathbf{b}_2) = A_2\wedge B_2 \\ &= (\mathbf{b}_1\wedge\mathbf{b}_2)\wedge(\mathbf{a}_1\wedge\mathbf{a}_2) = B_2\wedge A_2 = B_2A_2 \end{aligned}. \qquad (3.50)$$

Every pure k-blade can be reduced to a product of orthogonal vectors by only retaining the part of each vector factor in the blade, which is orthogonal to the other (k-1) vectors. After this procedure, the outer product in (3.34) can be replaced by the geometric product, all without changing the value of the blade. Reversing the order of all vectors introduces a factor $(-1)^{k(k-1)/2}$. Multiplying a k-blade and its reverse gives the squares of all vector lengths

$$\mathbf{B}^\dagger\mathbf{B} = \mathbf{a}_1^2\mathbf{a}_2^2\ldots\mathbf{a}_k^2 = |\mathbf{B}|^2 = (-1)^{k(k-1)/2}\mathbf{BB}, \qquad (3.51)$$

which defines the magnitude $|\mathbf{B}|$ of a k-blade **B**.

Geometrically multiplying a k-vector $\mathbf{V}_k$ (understood to be a linear combination of k-blades) with a blade of maximum grade n (there is only one such maximum grade element in each geometric algebra R_n, all others are just scalar multiples) we obtain according to (3.41) a (n-k)-vector

$$\mathbf{V}_k\mathbf{I}_n = \langle \mathbf{V}_k\mathbf{I}_n \rangle_{|n-k|} = \langle \mathbf{V}_k\mathbf{I}_n \rangle_{n-k}. \qquad (3.52)$$

I_n is because of the maximum grade property a pure n-blade and its vector factors span the whole R^n. Each k-blade summand in the k-vector V_k represents therefore a subspace contained in the full I_n space R^n. This subspace relationship leaves according to (3.40) only (and with proper reshuffling in each k-blade under concern) successive inner products of I_n with the vector factors of the k-blade. Hence in (3.52) each k-blade summand will produce a *dual* (n-k)-blade and by linearity we get (3.52).

A consequence of (3.52) is that each vector will become a *dual* algebra element of grade (n-1) if multiplied with the maximum grade pseudoscalar. Each *2*-blade (bivector) will become a *dual* element of grace (n-*2*). A consequence of this is, that only in $n=3$ dimensions bivectors (*2*-blades, two dimensional area elements) are dual to vectors.

Comparing (3.41) and (3.52), we see that e.g. for $n=4$, the expressions

$$\mathbf{V}_2\mathbf{I}_4 = \mathbf{V}_2\cdot\mathbf{I}_4 = \langle \mathbf{V}_2\mathbf{I}_4 \rangle_{|4-2|} = \langle \mathbf{V}_2\mathbf{I}_4 \rangle_2 \qquad (3.53)$$

must all be equal.

Finally a remark about the squares of pseudoscalars. By induction one can prove the following formula for squares of Euclidean pseudoscalars

$$i_n^2 = (-1)^{n-1} i_{n-1}^2. \qquad (3.54)$$

Using this formula, the first 11 squares are (n=1,2,3,4,5,6,7,8,9,10,11) 1,-1,-1,1,1,-1-1,1,1,-1-1. The periodicity of two is evident. 11-dimensional spaces play a certain role in higher dimensional elementary particle field theories.

As is to be expected, the square of the pseudoscalar depends on the metric. If we choose therefore a (1,3) or (3,1) Minkowski metric, we end up with

$$i_4^2 = -1, \qquad (3.55)$$

and not with $i_4^2 = 1$ as in the Euclidean case.

This concludes our short tour of geometric algebra. It gives a taste of the huge potential for coordinate free algebraic calculations and provides most of the geometric algebra tools for the following sections. To someone just starting to get acquainted with geometric algebra, I strongly recommend reference [4].

4. Real treatment of antisymmetric matrices

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$ be an orthonormal vector basis of the Euclidean vector space over the reals R^n. R_n denotes the geometric algebra generated by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$. The set of bivectors $\{\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \ldots, \mathbf{e}_{n-1}\mathbf{e}_n\}$ forms a $n(n-1)/2$ dimensional linear space over the reals. This is the same dimension as for the space of antisymmetric square matrices in n dimensions. These two spaces are indeed isomorphic. Let **A** be a bivector in R_n and A the corresponding antisymmetric matrix with the components A_{kl}. Using (3.46) the isomorphism is given by:

$$A_{kl} = \mathbf{e_k} \cdot \mathbf{A} \cdot \mathbf{e_l}, \tag{4.1}$$

$$\mathbf{A} = \frac{1}{2}\sum_{\mathbf{k,l=1}}^{\mathbf{n}} A_{kl}\mathbf{e_k}\mathbf{e_l}. \tag{4.2}$$

Taking the transpose A^T on the left side of (4.1) corresponds to taking the reverse of the bivector **A**, because according to (3.21)

$$(\mathbf{e_k}\mathbf{e_l})^{\dagger} = \mathbf{e_l}\mathbf{e_k} = -\mathbf{e_k}\mathbf{e_l}. \tag{4.3}$$

The last equality is analogous to (3.22), because $\mathbf{e}_k$ and $\mathbf{e}_l$ are orthogonal to each other and anticommute therefore.

This isomorphism is very useful, because we can now seek the canonical form of the isomorphic bivector **A** of (4.2), using the powerful tools of geometric algebra. After establishing the canonical form in a completely real geometric way, we can simply map it back to the corresponding antisymmetric matrix A by applying (4.1).

4.1 The canonical form of a bivector A

The derivation of this canonical form is taken from chapter 3.4 of reference [7].

Every bivector **A** can be represented as a sum of distinct commuting plane (two dimensional) area elements [2-blades, k=2 in (3.34)]

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_m, \tag{4.4}$$

where in accordance with (3.50)

$$\mathbf{A}_k\mathbf{A}_l = \mathbf{A}_l\mathbf{A}_k = \mathbf{A}_l \wedge \mathbf{A}_k \tag{4.5}$$

and with (3.51) for $k \neq l$ and

$$\mathbf{A}_k^2 = -\nu_k^2 < 0,\ 0 < \nu_k \in \mathrm{R}. \tag{4.6}$$

According to (4.6) we can decompose every area element $\mathbf{A}_k$ into its scalar size ν_k times its unit area element $\mathbf{i}_k$

$$\mathbf{A}_k = \nu_k \mathbf{i}_k \tag{4.7}$$

with

$$\mathbf{i}_k^2 = \mathbf{i}_k\mathbf{i}_k = -1, \tag{4.8}$$

as for the unit area element in (3.9). The orthogonal decomposition is only unique if all ν_k are different.

The chief problem is to compute the $\mathbf{A}_k$ of (4.4) from the expression for **A** given by (4.2). To solve for the $\mathbf{A}_k$ it is convenient to introduce the multivectors $\mathbf{C}_k$ of grade $2k$ by the equation

$$\mathbf{C}_k \equiv \frac{1}{k!}\left\langle \mathbf{A}^k \right\rangle_{2k} = \sum_{l_1<l_2<\ldots<l_k} \mathbf{A}_{l_1}\mathbf{A}_{l_2}\ldots\mathbf{A}_{l_k}, \tag{4.9}$$

where k=1,2,...,m. $\mathbf{A}^k$=**AA**...**A** (k times) is evaluated from (4.2) and the right side of (4.9) is obtained by substituting (4.4) into the left side and applying (3.36a). Then (4.9) constitutes a set of m equations to be solved for each $\mathbf{A}_l$ in terms of the $\mathbf{C}_k$'s.

First the squares $\alpha_l = \mathbf{A}_l^2 = -\nu_l^2$ of each $\mathbf{A}_l$ can be found as the roots of the mth order polynomial

$$\sum_{k=0}^{m} \left\langle \mathbf{C}_k^2 \right\rangle (-\alpha)^{m-k}, \tag{4.10}$$

where the scalar coefficients $\left\langle \mathbf{C}_k^2 \right\rangle$ are calculated by taking the scalar part (3.36b) of the squares of the multivectors $\mathbf{C}_k$ defined in (4.9)

$$\left\langle \mathbf{C}_k^2 \right\rangle = \left\langle \mathbf{C}_k^2 \right\rangle_0 = \sum_{l_1<l_2<\ldots<l_k} \mathbf{A}_{l_1}^2\mathbf{A}_{l_2}^2\ldots\mathbf{A}_{l_k}^2. \tag{4.11}$$

(4.10) can be verified by identifying the $\left\langle \mathbf{C}_k^2 \right\rangle$ as the coefficients of the factored form of the polynomial

$$(\mathbf{A}_1^2 - \alpha)(\mathbf{A}_2^2 - \alpha)\ldots(\mathbf{A}_m^2 - \alpha). \tag{4.12}$$

After the roots $\alpha_l = \mathbf{A}_l^2$ of

$$\sum_{k=0}^{m} \left\langle \mathbf{C}_k^2 \right\rangle (-\alpha)^{m-k} = 0 \tag{4.13}$$

have been determined, equation (4.9) can be replaced by a set of m linear bivector equations for the m unknowns $\mathbf{A}_l$, which is given by

$$\sum_{\mathbf{k,l=1}}^{\mathbf{n}} A_{kl}\mathbf{e_k}\mathbf{e_l} = \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_m, \tag{4.14a}$$

$$\mathbf{C}_{k-1} \cdot \mathbf{C}_k = \sum_{l=1}^{m} \mathbf{A}_l \left(\sum_{\substack{l_1<l_2<\ldots<l_{k-1},\\ l_1,\ldots,l_{k-1}\neq l}} \mathbf{A}_{l_1}^2\mathbf{A}_{l_2}^2\ldots\mathbf{A}_{l_{k-1}}^2 \right) \tag{4.14b}$$

for k=2,...,m. In reference [7] {p. 81, Eqn. (4.15)} also the term $\mathbf{C}_0 \cdot \mathbf{C}_1$ occurs. $\mathbf{C}_0$ of grade zero should be a scalar $\in \mathrm{R}$, but as pointed out and remedied by Dorst[17] the scalar product in reference [7] is not well-defined if one of its factors is of grade zero (i.e. $\in \mathrm{R}$).

Equation (4.14) can be solved by standard procedure, if all α_l are distinct. But if e.g. $\alpha_l = \alpha_{l'}$, then all coefficients of $\mathbf{A}_l$ and $\mathbf{A}_{l'}$ will also be equal, and additional properties of **A** will be needed to determine the $\mathbf{A}_l$.

This 'orthogonalization' of a bivector produces via the isomorphism (4.1) the canonical form of the corresponding antisymmetric matrix.

Taking $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{A}$ as a linear (skewsymmetric) vector transformation, the square f^2 will be a symmetric transformation with the eigenvalues $\alpha_l = \mathbf{A}_l^2 = -\nu_l^2$ and the characteristic polynomial (4.10), which equals (4.12). To show this let $\mathbf{a}_l$ be a unit vector in the $\mathbf{i}_l$ plane, i.e.

$$\mathbf{a}_l \wedge \mathbf{i}_l = \mathbf{a}_l \wedge \mathbf{A}_l = 0, \qquad (4.15)$$

according to (3.40). By (4.5) we have

$$\mathbf{a}_l \cdot \mathbf{A}_k = 0 \quad \text{for } l \neq k, \qquad (4.16)$$

because the vectors that span the $\mathbf{i}_k$ must all be orthogonal to $\mathbf{a}_l \in \mathbf{i}_l$, since all $\mathbf{i}_k$ planes are orthogonal to each other by (4.5)-(4.7). So by using (4.4), (4.6), (3.25), (3.42) and (4.15) we see that

$$f^2(\mathbf{a}_l) = (\mathbf{a}_l \cdot \mathbf{A}) \cdot \mathbf{A} = \mathbf{A}_l^2 \mathbf{a}_l = -\nu_l^2 \mathbf{a}_l. \quad (4.17)$$

By finally defining the unit vector

$$\mathbf{b}_l \equiv \mathbf{a}_l \cdot \mathbf{i}_l \in \mathbf{i}_l\text{-plane}, \qquad (4.18)$$

which is by (3.48) orthogonal to $\mathbf{a}_l$, every plane area element $\mathbf{A}_l$ can be written as in (3.49)

$$\mathbf{A}_l = \nu_l \mathbf{i}_l = \nu_l \mathbf{a}_l \mathbf{b}_l, \qquad (4.19)$$

where

$$\{\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2, \ldots, \mathbf{a}_m, \mathbf{b}_m\} \qquad (4.20)$$

is a set of othonormal eigenvectors of f^2 with nonvanishing eigenvalues $-\nu_l^2 = \mathbf{A}_l^2$.

4.2 The canonical form of antisymmetric matrices by real geometry

Let us suppose for a moment, that the antisymmetric $n \times n$ square matrix A has maximum rank $r=n$. Then the number of distinct orthogonal area elements m in (4.4) will be $m=n/2$, and no zero eigenvalues will occur. The set of $n=2m$ orthogonal eigenvectors (4.20) will therefore form a basis of the vector space R^n.

Applying the bivector matrix isomorphism (4.1) with respect to the basis (4.20) we find with (3.46) that

$$\begin{aligned} A_{2k,2k+1} &= \mathbf{a}_k \cdot \mathbf{A} \cdot \mathbf{b}_k = \mathbf{a}_k \cdot \left(\sum_{l=1}^{m} \mathbf{A}_l \right) \cdot \mathbf{b}_k \\ &= \mathbf{a}_k \cdot \mathbf{A}_k \cdot \mathbf{b}_k = \mathbf{a}_k \cdot (\nu_k \mathbf{i}_k) \cdot \mathbf{b}_k \\ &= \nu_l (\mathbf{a}_k \cdot \mathbf{i}_k) \cdot \mathbf{b}_k = \nu_l \mathbf{b}_k \cdot \mathbf{b}_k = \nu_l . \end{aligned} \qquad (4.21)$$

The third equality holds because of (4.16), for the fourth (4.19) has been inserted, the sixth equality uses the definition of the vector $\mathbf{b}_k$ in (4.18), and the last just says that the $\mathbf{b}_k$'s are unit vectors. Because of the antisymmetry we have

$$A_{2k,2k+1} = -A_{2k+1,2k} = -\nu_l, \qquad (4.22)$$

and

$$A_{kk} = 0. \qquad (4.23)$$

Because of (4.16) all other matrix elements will necessarily be zero.

Summarizing the results in matrix form, we get

$$A = \begin{pmatrix} 0 & \nu_1 & & & \\ -\nu_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \nu_m \\ & & & -\nu_m & 0 \end{pmatrix}, \qquad (4.24)$$

where again all omitted elements are understood to be zero.

If we now drop the requirement of A to have maximum rank, the vectors in the kernel of the linear vector transformations f and f^2 (4.17) have to be added to our orthogonal basis (4.20). This are all the vectors, which are mapped to zero by f and f^2. They have therefore the eigenvalues zero. By Gram-Schmidt orthogonalization ([7], p.28), a set of orthonormal vectors, spanning the kernel, can therefore be added to (4.20), supplementing (4.20) to a full orthonormal basis of R^n.

Calculating the matrix components of A with respect to the supplemented basis according to the isomorphism (4.1) we end up with the full canoncical form of the antisymmetric matrix A as given in (2.8).

This completes the derivation of the canonical form of antisymmetric matrices by making use of the completely real geometric algebra. There was especially no need to introduce complex numbers, complex coordinates or complex vectors, Hermitian skewsymmetric matrices, or vector spaces over the field of complex numbers.

Beyond these strong formal and educational merits, we can now visualize the *geometric essence* of an antisymmetric matrix via the isomorphism (4.1), which allows us to switch freely back and forth between the matrix from and the bivector form: As proved in section 4.1, equations (4.5)-(4.7), the isomorphic bivector **A** represents a set of $m \leq n/2$ orthogonal (two dimensional) plane area elements $\mathbf{A}_l$ of size ν_l with oriented unit area elements $\mathbf{i}_l$ $(1 \leq l \leq m)$.

In the following section we will apply this new way of unraveling the geometry of an antisymmetric matrix for the examples of antisymmetric 2×2, 3×3 and 4×4 square matrices.

5. Application to antisymmetric 2×2, 3×3 and 4×4 square matrices

5.1 Application to antisymmetric 2×2 square matrix

A general two dimensional antisymmetric matrix A written with respect to any two dimensional orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ has already the canonical form

$$A = \begin{pmatrix} 0 & \nu \\ -\nu & 0 \end{pmatrix}. \tag{5.1}$$

Using the isomorphism (4.2) we can directly calculate the corresponding bivector **A** as

$$\begin{aligned} \mathbf{A} &= \frac{1}{2}\sum_{k,l=}^{2} A_{kl}\mathbf{e}_k\mathbf{e}_l = \frac{1}{2}(A_{12}\mathbf{e}_1\mathbf{e}_2 + A_{21}\mathbf{e}_2\mathbf{e}_1) \\ &= \frac{1}{2}(\nu\mathbf{e}_1\mathbf{e}_2 - \nu\mathbf{e}_2\mathbf{e}_1) = \frac{1}{2}(\nu\mathbf{e}_1\mathbf{e}_2 + \nu\mathbf{e}_1\mathbf{e}_2) \\ &= \nu\mathbf{e}_1\mathbf{e}_2 = \nu\mathbf{i}, \end{aligned} \tag{5.2}$$

where we have used the anticommutativity of $\mathbf{e}_1$ and $\mathbf{e}_2$ and the definition of the oriented (two dimensional) plane area element as in (3.8).

The set of all antisymmetric square matrices in two real dimensions, represents therefore nothing else but all possible plane oriented area elements only distinguished by their scalar size |ν| and their orientation encoded in **i**.

This is strongly reminiscent of Gibbs' cross product of vectors in three dimensions and indeed the antisymmetric (bivector) part of the geometric product of two vectors as in (3.4) is the generalization of the cross product independent of the dimension n of the vector space in which the vectors are situated. This is because in any dimension, we can restrict our considerations to the plane spanned by the two vectors to be multiplied.

If we were to calculate the eigenvalues of the matrix A, in the conventional way, we would find

$$\lambda_{1,2} = \pm j\nu, \quad j \equiv \sqrt{-1}. \tag{5.3}$$

But instead of sticking to the geometrically uninterpretable imaginary unit j we should rather take the eigen-bivectors

$$\boldsymbol{\lambda}_1 = \nu\mathbf{i}, \text{ and } \boldsymbol{\lambda}_2 = \nu\mathbf{i}^{\dagger} = -\nu\mathbf{i}. \tag{5.4}$$

It is not difficult to show along the lines of (3.10) and (3.11) that (5.4) leads indeed to a consistent real geometric interpretation of imaginary eigenvalues and the related complex eigenvectors as a 90 degree rotation operation on the vector doublet ($\mathbf{e}_1$, $\mathbf{e}_2$), respectively.[18,19] ν means an additional dilation of the doublet ($\mathbf{e}_1$, $\mathbf{e}_2$) by ν.

5.2 Application to antisymmetric 3×3 square matrix

A general three dimensional antisymmetric matrix A written with respect to a three dimensional orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ has the form

$$A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}. \tag{5.5}$$

Applying the isomorphism (4.2) we can directly calculate the corresponding bivector **A** as

$$\begin{aligned} \mathbf{A} &= \frac{1}{2}\sum_{k,l=}^{3} A_{kl}\mathbf{e}_k\mathbf{e}_l = \frac{1}{2}(A_{23}\mathbf{e}_2\mathbf{e}_3 + A_{32}\mathbf{e}_3\mathbf{e}_2 \\ &+ A_{31}\mathbf{e}_3\mathbf{e}_1 + A_{13}\mathbf{e}_1\mathbf{e}_3 + A_{12}\mathbf{e}_1\mathbf{e}_2 + A_{21}\mathbf{e}_2\mathbf{e}_1) \\ &= a\mathbf{e}_2\mathbf{e}_3 + b\mathbf{e}_3\mathbf{e}_1 + c\mathbf{e}_1\mathbf{e}_2 \end{aligned} \tag{5.6}$$

where we have used the anticommutativity (4.3)

$$\mathbf{e}_k\mathbf{e}_l = -\mathbf{e}_l\mathbf{e}_k \quad \text{for } l \neq k. \tag{5.7}$$

The square of **A** is

$$\mathbf{A}^2 = -a^2 - b^2 - c^2 = -\nu^2, \tag{5.8}$$

because all cross terms like

$$\begin{aligned} & a\mathbf{e}_2\mathbf{e}_3 b\mathbf{e}_3\mathbf{e}_1 + b\mathbf{e}_3\mathbf{e}_1 a\mathbf{e}_2\mathbf{e}_3 \\ &= ab(\mathbf{e}_2\mathbf{e}_3\mathbf{e}_3\mathbf{e}_1 + \mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) \\ &= ab(\mathbf{e}_2\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2) = 0 \end{aligned} \tag{5.9}$$

vanish according to (5.7). ν will only be zero if $a=b=c=0$, i.e. if the matrix A in (5.5) is the zero matrix. Defining

$$a' \equiv \frac{a}{\nu}, \quad b' \equiv \frac{b}{\nu}, \quad c' \equiv \frac{c}{\nu} \tag{5.10}$$

we get

$$\mathbf{A} = \nu(a'\mathbf{e}_2\mathbf{e}_3 + b'\mathbf{e}_3\mathbf{e}_1 + c'\mathbf{e}_1\mathbf{e}_2) = \nu\mathbf{i}, \tag{5.11}$$

with $\mathbf{i}^2=-1$. The unit vector

$$\mathbf{a} \equiv (-c'\mathbf{e}_2 + b'\mathbf{e}_3)/\sqrt{1-a'^2} \tag{5.12}$$

has the property (4.15)

$$\mathbf{a} \wedge \mathbf{i} = 0 \tag{5.13}$$

which is easy to show from (5.11) and (5.12) by explicit calculation. Defining the second unit vector **b** perpendicular to **a** we get according to (4.18)

$$\mathbf{b} \equiv \mathbf{a}\cdot\mathbf{i} = \left[(1-a'^2)\mathbf{e}_1 - a'b'\mathbf{e}_2 - a'c'\mathbf{e}_3\right]\Big/\sqrt{1-a'^2} \quad (5.14)$$

This gives the bivector **A** its final factorized form as in (4.19)

$$\mathbf{A} = \nu\mathbf{i} = \nu\mathbf{ab}\,. \quad (5.15)$$

The orthonormal vector pair {**a**, **b**} in the **i** plane is only unique up to an arbitrary common rotation, because any choice of unit vector **a** in the **i** plane will do.

The explicit form (5.15) of the bivector **A** shows that the sum of bivectors (4.4) will have precisely one term (m=1) in three dimensions. That means an antisymmetric matrix in three dimensions always specifies one corresponding oriented (two dimensional) area element **A**. Multiplying **A** with the three dimensional pseudoscalar i of (3.23) gives by (3.52) a vector **k** of length ν perpendicular to the **i** plane

$$\begin{aligned}\mathbf{k} &\equiv -i\mathbf{A} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 \\ &= \nu(a'\mathbf{e}_1 + b'\mathbf{e}_2 + c'\mathbf{e}_3) = \nu\mathbf{c}\end{aligned}, \quad (5.15)$$

where the minus sign is a convention, which ensures that $\mathbf{c} = (\mathbf{a}\times\mathbf{b})$, i.e. **c** is just Gibbs' cross product of the vectors **a** and **b**. This mapping of bivectors **A** and **i** to vectors **k** and **c**, respectively, works only in three dimensions, which is why Gibbs' cross product can only be defined in n=3 dimensions, as shown by (3.52). In contrast to this, the definition of the outer product of vectors (3.3) is completely dimension independent and is therefore to be preferred to the conventional cross product.

The fact that m=1 in n=3 dimensions means that the kernel of the linear transformations $f(\mathbf{x})$ and $f^2(\mathbf{x})$ of section 4.1 will have the dimension k=n-2m=1. This kernel consists of the vectors parallel to **k** or **c**.

{**a**, **b**, **c**} is therefore an orthonormal basis of R^3 with respect to which the matrix A takes its canonical form by applying the isomorphism (4.1)

$$A = \begin{pmatrix} 0 & \nu & 0 \\ -\nu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (5.16)$$

If we were again to calculate the eigenvalues of the matrix A in the conventional way we would find

$$\lambda_{1,2} = \pm j\nu,\quad j \equiv \sqrt{-1},\ \lambda_3 = 0. \quad (5.17)$$

As in (5.4) we should better replace the geometrically uninterpretable imaginary unit j by the eigen-bivectors

$$\boldsymbol{\lambda}_1 = \nu\mathbf{i},\ \text{and}\ \boldsymbol{\lambda}_2 = \nu\mathbf{i}^\dagger = -\nu\mathbf{i}, \quad (5.18)$$

where the bivector **i** is defined in equation (5.11). We could again show along the lines of (3.10) and (3.11) that (5.18) leads indeed to a consistent real geometric interpretation of imaginary eigenvalues and the related complex eigenvectors as a 90 degree rotation operation on the vector triplet ($\mathbf{e}_{1\|}$, $\mathbf{e}_{2\|}$, $\mathbf{e}_{3\|}$), respectively.[18,19] ν means an additional dilation of the triplet ($\mathbf{e}_{1\|}$, $\mathbf{e}_{2\|}$, $\mathbf{e}_{3\|}$) by ν. The rotation takes place in the **i** plane around the axis **c**. The index '||' means projection onto the **i** plane using (3.33).

5.3 Application to antisymmetric 4×4 square matrix

A general four dimensional antisymmetric matrix A written with respect to a four dimensional orthonormal Euclidean basis {$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$} has the form

$$A = \begin{pmatrix} 0 & c & -b & r \\ -c & 0 & a & s \\ b & -a & 0 & t \\ -r & -s & -t & 0 \end{pmatrix}. \quad (5.19)$$

Applying the isomorphism (4.2) we can directly calculate the corresponding bivector **A** as

$$\mathbf{A} = \frac{1}{2}\sum_{k,l=}^{3} A_{kl}\mathbf{e}_k\mathbf{e}_l = a\mathbf{e}_2\mathbf{e}_3 + b\mathbf{e}_3\mathbf{e}_1 + c\mathbf{e}_1\mathbf{e}_2 + r\mathbf{e}_1\mathbf{e}_4 + s\mathbf{e}_2\mathbf{e}_4 + t\mathbf{e}_3\mathbf{e}_4 \quad (5.20)$$

The two multivectors $\mathbf{C}_1$ and $\mathbf{C}_2$ of equation (4.9) are

$$\mathbf{C}_1 \equiv \frac{1}{1!}\left\langle \mathbf{A}^1\right\rangle_2 = \mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2, \quad (5.21)$$

$$\mathbf{C}_2 \equiv \frac{1}{2!}\left\langle \mathbf{A}^2\right\rangle_4 = \frac{1}{2}\mathbf{A}\wedge\mathbf{A} = \mathbf{A}_1\mathbf{A}_2. \quad (5.22)$$

The grade 4 part of the square of **A** yields

$$\begin{aligned}\frac{1}{2}\mathbf{A}\wedge\mathbf{A} &= ar\mathbf{e}_2\mathbf{e}_3\wedge\mathbf{e}_1\mathbf{e}_4 + bs\mathbf{e}_3\mathbf{e}_1\wedge\mathbf{e}_2\mathbf{e}_4 + ct\mathbf{e}_1\mathbf{e}_2\wedge\mathbf{e}_3\mathbf{e}_4 \\ &= (ar+bs+ct)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4 \\ &= (ar+bs+ct)i_4 = \frac{1}{2}|\mathbf{A}\wedge\mathbf{A}|i_4\end{aligned}, \quad (5.23)$$

because all other parts of the square of **A** have grades less than 4. i_4 is the oriented four dimensional unit volume element, also called pseudoscalar.

The polynomial (4.10), whose two roots are the squares of the two bivectors in (5.21)

$$\alpha_1 = \mathbf{A}_1^2 = -\nu_1^2,\quad \alpha_2 = \mathbf{A}_2^2 = -\nu_2^2 \quad (5.24)$$

becomes now

$$\sum_{k=0}^{2}\langle \mathbf{C}_k^2\rangle(-\alpha)^{2-k} = \alpha^2 - \langle \mathbf{A}^2\rangle\alpha + \left\langle\left(\frac{1}{2}\mathbf{A}\wedge\mathbf{A}\right)^2\right\rangle. \tag{5.25}$$

The two coefficients of (5.25) are

$$\langle \mathbf{C}_1^2\rangle = \langle \mathbf{A}^2\rangle = -(a^2+b^2+c^2+r^2+s^2+t^2) = -|\mathbf{A}|^2, \tag{5.26}$$

because only squares like $(\mathbf{e}_l\mathbf{e}_k)(\mathbf{e}_l\mathbf{e}_k)=-1$ with $l \neq k$ contribute, and

$$\langle \mathbf{C}_2^2\rangle = \left\langle\left(\frac{1}{2}\mathbf{A}\wedge\mathbf{A}\right)^2\right\rangle = (ar+bs+ct)^2 = \frac{1}{4}|\mathbf{A}\wedge\mathbf{A}|^2, \tag{5.27}$$

because $i_4^2=1$. To find α_1 and α_2 we therefore need to solve

$$\alpha^2 + (a^2+b^2+c^2+r^2+s^2+t^2)\alpha + (ar+bs+ct)^2 = 0. \tag{5.28}$$

(5.28) yields

$$\alpha_{1,2} = \mathbf{A}_{1,2}^2 = -v_{1,2}^2 = \frac{1}{2}\left[-|\mathbf{A}|^2 \pm \sqrt{|\mathbf{A}|^4 - |\mathbf{A}\wedge\mathbf{A}|^2}\right] = \frac{1}{2}[-(a^2+b^2+c^2+r^2+s^2+t^2) \pm \sqrt{(a^2+b^2+c^2+r^2+s^2+t^2)^2 - 4(ar+bs+ct)^2}]. \tag{5.29}$$

Next we write down the $m=2$ equations (4.14). Using (5.20) and (5.21) we get for (4.14a)

$$\mathbf{A} = a\mathbf{e}_2\mathbf{e}_3 + b\mathbf{e}_3\mathbf{e}_1 + c\mathbf{e}_1\mathbf{e}_2 + r\mathbf{e}_1\mathbf{e}_4 + s\mathbf{e}_2\mathbf{e}_4 + t\mathbf{e}_3\mathbf{e}_4 = \mathbf{A}_1 + \mathbf{A}_2 \tag{5.30a}$$

Using (5.21), (5.22), (5.23), (5.27) and (4.5) we can write for (4.14a)

$$\mathbf{C}_1\cdot\mathbf{C}_2 = \mathbf{C}_1\mathbf{C}_2 = \mathbf{A}\frac{1}{2}(\mathbf{A}\wedge\mathbf{A}) = \mathbf{A}i_4|\mathbf{A}\wedge\mathbf{A}| = (\mathbf{A}_1+\mathbf{A}_2)\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_1^2\mathbf{A}_2 + \mathbf{A}_2^2\mathbf{A}_1 \tag{5.30b}$$

The first equality in (5.30b) holds, because as in (3.53) $\mathbf{C}_2$ is already of maximum grade 4. Provided that α_1 and α_2 are distinct, the equations (5.30) can be solved to give

$$\mathbf{A}_1 = \frac{\mathbf{C}_1\mathbf{C}_2 - \alpha_1\mathbf{A}}{\alpha_2-\alpha_1} = \frac{i_4\frac{1}{2}|\mathbf{A}\wedge\mathbf{A}|\mathbf{A} - \alpha_1\mathbf{A}}{\alpha_2-\alpha_1}, \tag{5.31}$$

where we have used the fourth expression in (5.30b) to replace $\mathbf{C}_1\mathbf{C}_2$. The expression for $\mathbf{A}_2$ is obtained by interchanging the subscripts 1 and 2 in (5.31). Observing that

$$\alpha_1\alpha_2 = \left(\frac{1}{2}|\mathbf{A}\wedge\mathbf{A}|\right)^2, \tag{5.32}$$

and using v_1 and v_2 of (5.29) we finally get

$$\mathbf{A}_1 = v_1\mathbf{i}_1 = v_1\frac{v_1+i_4v_2}{v_1^2-v_2^2}\mathbf{A}, \tag{5.33}$$

and

$$\mathbf{A}_2 = v_2\mathbf{i}_2 = v_2\frac{v_2+i_4v_1}{v_2^2-v_1^2}\mathbf{A}. \tag{5.34}$$

It is easy to check that

$$\mathbf{A}=\mathbf{A}_1+\mathbf{A}_2=v_1\mathbf{i}_1+v_2\mathbf{i}_2,\quad \mathbf{i}_1\mathbf{i}_2=\mathbf{i}_2\mathbf{i}_1=i_4,\quad \mathbf{i}_1^2=\mathbf{i}_2^2=-1, \tag{5.35}$$

and that the two orthogonal unit bivectors $\mathbf{i}_1$ and $\mathbf{i}_2$ are related by

$$\mathbf{i}_2 = -i_4\mathbf{i}_1,\quad \mathbf{i}_1 = -i_4\mathbf{i}_2. \tag{5.36}$$

The explicit form (5.35) of the bivector $\mathbf{A}$ shows that in four dimensions the sum of bivectors (4.4) will have precisely two terms ($m=2$), provided that v_1 and v_2 are distinct. That means that an antisymmetric matrix in four dimensions always specifies two corresponding (two dimensional) distinct orthogonal area elements $\mathbf{A}_1$ and $\mathbf{A}_2$. The relations (5.36) show that the units $\mathbf{i}_1$, $\mathbf{i}_2$ of these two area elements are *dual* to each other by the multiplication with the four dimensional pseudoscalar i_4.

The duality of area elements (3.53) in four dimensions is also the reason, why Gibbs' cross product cannot be defined in four dimensions: The dual entity of an outer product $\mathbf{a}\wedge\mathbf{b}$ is not a vector (as in three dimensions), but because fo (3.53) again an area element.

In particular cases it will be no problem to find the orthonormal eigenvectors $\{\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2\}$ of the linear transformations $a(\mathbf{x})$ and $a^2(\mathbf{x})$ of section 4.1. A possible way of construction for the eigenvector $\mathbf{a}_1$ ($\in \mathbf{i}_1$ plane) is to simply take e.g. the basis vector $\mathbf{e}_1$, calculate its projection $\mathbf{e}_{1\|1}$ onto the $\mathbf{i}_1$ plane with (3.33) and divide it by its length.

$$\mathbf{a}_1 \equiv \frac{\mathbf{e}_{1\|1}}{|\mathbf{e}_{1\|1}|} = \frac{\mathbf{e}_1\cdot\mathbf{i}_1\mathbf{i}_1^{-1}}{|\mathbf{e}_1\cdot\mathbf{i}_1\mathbf{i}_1^{-1}|} = \frac{\mathbf{e}_1\cdot\mathbf{i}_1\mathbf{i}_1^{-1}}{|\mathbf{e}_1\cdot\mathbf{i}_1|}. \tag{5.37}$$

The second unit eigenvector $\mathbf{b}_1$ ($\in \mathbf{i}_1$ plane) can then be calculated with the help of (4.18) as

$$\mathbf{b}_1 \equiv \mathbf{a}_1 \cdot \mathbf{i}_1 = \mathbf{a}_1\mathbf{i}_1 = \frac{1}{|\mathbf{e}_1 \cdot \mathbf{i}_1|}\mathbf{e}_1 \cdot \mathbf{i}_1\mathbf{i}_1^{-1}\mathbf{i}_1 = \frac{\mathbf{e}_1 \cdot \mathbf{i}_1}{|\mathbf{e}_1 \cdot \mathbf{i}_1|}. \quad (5.38)$$

The second equality holds, because of $\mathbf{a}_1 \wedge \mathbf{i}_1 = 0$ (4.15). $\mathbf{a}_1$ and $\mathbf{b}_1$ will be unique up to a common rotation in the $\mathbf{i}_1$ plane. In the very same way we can calculate $\mathbf{a}_2$ and $\mathbf{b}_2$ ($\in \mathbf{i}_2$ plane) e.g. as

$$\mathbf{a}_2 \equiv \frac{\mathbf{e}_{1\|2}}{|\mathbf{e}_{1\|2}|}, \quad \mathbf{b}_2 \equiv \mathbf{a}_2 \cdot \mathbf{i}_2 = \frac{\mathbf{e}_1 \cdot \mathbf{i}_2}{|\mathbf{e}_1 \cdot \mathbf{i}_2|}, \quad (5.39)$$

which are again unique up to a common rotation in the $\mathbf{i}_2$ plane.

Applying the matrix ismormorphism (4.1) with respect to the orthonormal eigenvector basis $\{\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2\}$, we get the canonical form of the matrix A of (5.19) as

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & 0 \\ -\nu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu_2 \\ 0 & 0 & -\nu_2 & 0 \end{pmatrix}, \quad (5.40)$$

where the values ν_1, ν_2 are taken to be positive and defined by (5.29).

If we were again to calculate the eigenvalues of the matrix A in the conventional way we would find

$$\lambda_{1,2} = \pm j\nu_1, \quad \lambda_{3,4} = \pm j\nu_2, \quad j \equiv \sqrt{-1}. \quad (5.41)$$

As in (5.4) and (5.18) we should better replace the geometrically uninterpretable imaginary unit j by the eigen-bivectors

$$\boldsymbol{\lambda}_1 = \nu_1\mathbf{i}_1, \quad \boldsymbol{\lambda}_2 = \nu_1\mathbf{i}_1^{\dagger} = -\nu_1\mathbf{i}_1,$$

$$\boldsymbol{\lambda}_3 = \nu_2\mathbf{i}_2, \quad \boldsymbol{\lambda}_4 = \nu_2\mathbf{i}_2^{\dagger} = -\nu_2\mathbf{i}_2. \quad (5.42)$$

We could again show along the lines of (3.10) and (3.11) that (5.42) leads indeed to a consistent real geometric interpretation of imaginary eigenvalues of A and the related complex eigenvectors as 90 degree rotation operations on the vector.quadruplets ($\mathbf{e}_{1\|1}$, $\mathbf{e}_{2\|1}$, $\mathbf{e}_{3\|1}$, $\mathbf{e}_{4\|1}$), ($\mathbf{e}_{1\|2}$, $\mathbf{e}_{2\|2}$, $\mathbf{e}_{3\|2}$, $\mathbf{e}_{4\|2}$), respectively.[18,19,20] The rotations take place in the $\mathbf{i}_1$ and $\mathbf{i}_2$ planes, respectively. The factors ν_1, ν_2 result in additional dilations of the respective quadruplets by ν_1, ν_2. The indices '$\|1$' and '$\|2$' mean projections onto the $\mathbf{i}_1$ and $\mathbf{i}_2$ planes, respectively, using (3.33).

I did not discuss the case of one of the values ν_1, ν_2 equal to zero. This is an extension of the previous section, with a two dimensional kernel.

6. The canonical form of a bivector in Minkowski space

6.1 Derivation of the canonical form of a bivector in Minkowski space

In order to study important applications of the canonical form of antisymmetric matrices, we need to deal with the pseudoEuclidean (Minkowski) metric. This will open a wide field of applications in special relativity, e.g. to Lorentz transformation generators, to the field tensor of Maxwell's electrodynamics, relativistic quantum mechanics, etc. The Minkowski metric is defined by four orthogonal vectors satisfying

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -\mathbf{e}_4^2 = 1, \quad (6.1)$$

where the first three vectors span space and the fourth vector gives the time direction. The orthogonal decomposition in Minkowski space is used and alluded to (e.g. [7], pp. 10,11; [13], pp. 49,86) in the literature, yet I haven't seen any explicit proof so far.

The definition of the isomorphism formula (4.1) doesn't change. But in (4.2) the metric (6.1) must be taken into account, resulting in a minus, whenever the indices k or l take the value 4. For the matrix (5.19), this results in the isomorphic bivector to be

$$\mathbf{A} = \frac{1}{2}\sum_{k,l=}^{3} A_{kl}\mathbf{e}_k^{-1}\mathbf{e}_l^{-1} =$$

$$a\mathbf{e}_2\mathbf{e}_3 + b\mathbf{e}_3\mathbf{e}_1 + c\mathbf{e}_1\mathbf{e}_2 - r\mathbf{e}_1\mathbf{e}_4 - s\mathbf{e}_2\mathbf{e}_4 - t\mathbf{e}_3\mathbf{e}_4$$

because in the Euclidean case (4.2), we have $\mathbf{e}_1^{-1} = \mathbf{e}_1$, $\mathbf{e}_2^{-1} = \mathbf{e}_2$, $\mathbf{e}_3^{-1} = \mathbf{e}_3$, $\mathbf{e}_4^{-1} = \mathbf{e}_4$, but in the Minkowski case the inverse of $\mathbf{e}_4$ changes to $\mathbf{e}_4^{-1} = -\mathbf{e}_4$. The squares of the distinct commuting plane (two dimensional) area elements (2-blades) may now also become positive, so (4.6) has to be replaced by

$$\mathbf{A}_k^2 = \pm\nu_k^2 \neq 0, \quad 0 < \nu_k \in \mathbf{R}. \quad (6.2)$$

The factoring as in (4.7) will continue to be possible, but the squares of the unit area elements $\mathbf{i}_k$ may now also become positive. (4.8) has therefore to be replaced by

$$\mathbf{i}_k^2 = \mathbf{i}_k\mathbf{i}_k = \pm 1. \quad (6.3)$$

As an example for the positive sign in (6.3) we can e.g. calculate

$$(\mathbf{e}_1\mathbf{e}_4)(\mathbf{e}_1\mathbf{e}_4) = \mathbf{e}_1\mathbf{e}_4\mathbf{e}_1\mathbf{e}_4 = -\mathbf{e}_1\mathbf{e}_4\mathbf{e}_4\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_1 = 1.$$

After defining the multivectors $\mathbf{C}_k$ as in (4.9), the squares $\alpha_l = \mathbf{A}_l^2 = \pm\nu_l^2$ of each $\mathbf{A}_l$ can be calculated as the roots of (4.10). After the roots have been calculated (4.14) serves to find the actual bivectors.

Let me now turn to the antisymmetric 4×4 matrix (5.19), but use the Minkowski basis (6.1), instead of the four dimensional Euclidean basis of section 5.3. I will not repeat the whole argument of section 5.3, but I will point out all necessary modifications in detail.

First, the explicit expressions in (5.23) change to

$$\begin{aligned}&\frac{1}{2}\mathbf{A}\wedge\mathbf{A} = -ar\mathbf{e}_2\mathbf{e}_3\wedge\mathbf{e}_1\mathbf{e}_4\\ &-bs\mathbf{e}_3\mathbf{e}_1\wedge\mathbf{e}_2\mathbf{e}_4 - ct\mathbf{e}_1\mathbf{e}_2\wedge\mathbf{e}_3\mathbf{e}_4\\ &= -(ar+bs+ct)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\\ &= -(ar+bs+ct)i_4 = \frac{-1}{2}|\mathbf{A}\wedge\mathbf{A}|i_4\end{aligned} \tag{6.4}$$

As will soon be seen from the explicit expression, replacing (5.29) we now have instead of (5.24)

$$\alpha_1=\mathbf{A}_1^2=+\nu_1^2, \quad \alpha_2=\mathbf{A}_2^2=-\nu_2^2 \ . \tag{6.5}$$

The two coefficients in polynomial (5.25) change to

$$\begin{aligned}&\langle\mathbf{C}_1^2\rangle = \langle\mathbf{A}^2\rangle\\ &= -(a^2+b^2+c^2-r^2-s^2-t^2)\end{aligned}, \tag{6.6}$$

and to

$$\begin{aligned}&\langle\mathbf{C}_2^2\rangle = \left\langle\left(\frac{1}{2}\mathbf{A}\wedge\mathbf{A}\right)^2\right\rangle\\ &= -(ar+bs+ct)^2 = \frac{-1}{4}|\mathbf{A}\wedge\mathbf{A}|^2\end{aligned} \tag{6.7}$$

replacing (5.26) and (5.27), respectively. The sign in (6.7) changes, because in the basis (6.1) the pseudoscalar i_4 has the square (3.55) $i_4^2= -1$. With these new coefficients, the polynomial equation for the roots (6.5) now reads

$$\begin{aligned}&\alpha^2+(a^2+b^2+c^2-r^2-s^2-t^2)\alpha\\ &-(ar+bs+ct)^2=0\end{aligned} \tag{6.8}$$

instead of (5.28).

(6.8) yields

$$\begin{aligned}&\alpha_{1,2} = \mathbf{A}_{1,2}^2 = \pm\nu_{1,2}^2\\ &= \frac{1}{2}\left[-\langle\mathbf{A}^2\rangle \pm\sqrt{\langle\mathbf{A}^2\rangle^2+|\mathbf{A}\wedge\mathbf{A}|^2}\right]\\ &= \frac{1}{2}[-(a^2+b^2+c^2-r^2-s^2-t^2)\\ &\pm\sqrt{(a^2+b^2+c^2-r^2-s^2-t^2)^2+4(ar+bs+ct)^2}\,],\end{aligned} \tag{6.9}$$

where the plus sign stands for the index '1' and the minus sign for the index '2' respectively. (6.9) justifies (6.5).

Using the new α_1 and α_2 obtained from (6.9) the form (5.31) of the orthogonal bivectors $\mathbf{A}_1$ (and $\mathbf{A}_2$) will not change. Relation (5.32) changes its sign to

$$\alpha_1\alpha_2 = -\nu_1^2\nu_2^2 = -\left(\frac{1}{2}|\mathbf{A}\wedge\mathbf{A}|\right)^2 . \tag{6.10}$$

The explicit versions of $\mathbf{A}_1$ and $\mathbf{A}_2$ become therefore

$$\mathbf{A}_1 = \nu_1\mathbf{i}_1 = \nu_1\frac{\nu_1 - i_4\nu_2}{\nu_1^2+\nu_2^2}\mathbf{A}, \tag{6.11}$$

$$\mathbf{A}_2 = \nu_2\mathbf{i}_2 = \nu_2\frac{\nu_2 + i_4\nu_1}{\nu_1^2+\nu_2^2}\mathbf{A}. \tag{6.12}$$

In consequence of this relations (5.35) and (5.36) change to

$$\mathbf{A}=\mathbf{A}_1+\mathbf{A}_2=\nu_1\mathbf{i}_1+\nu_2\mathbf{i}_2,\ \mathbf{i}_1\mathbf{i}_2=\mathbf{i}_2\mathbf{i}_1=i_4,\ \mathbf{i}_1^2=-\mathbf{i}_2^2=1, \tag{6.13}$$

and

$$\mathbf{i}_2= i_4\mathbf{i}_1,\ \mathbf{i}_1= -i_4\mathbf{i}_2. \tag{6.14}$$

This concludes the derivation of the canonical form of a bivector **A**, where **A** is isomorphic to the antisymmetric matrix A of (5.19) supposing the Minkowski metric (6.1).

The question remains, what happens, if $\mathbf{i}_1$ or $\mathbf{i}_2$ would be null 2-blades, i.e. factoring them according to (3.49) would yield at least one of the vector factors **a** or **b** to be a null vector with zero square, e.g. $\mathbf{a}^2=0$. If we assume e.g. $\mathbf{i}_2$ to be a null 2-blade, with e.g. $\mathbf{a}_2^2=0$, then according to (3.51) we would also have $\mathbf{i}_2^2=0$. But this will not be the case, if $\alpha_2 \neq 0$ in (6.9), because $\alpha_2=\mathbf{A}_2^2=\nu_2\mathbf{i}_2^2$. $\mathbf{i}_2^2=0$ would therefore only be possible, if $\alpha_2 = 0$ in (6.9). In this case one has to check, whether $\mathbf{A}=\mathbf{A}_1$ with $\mathbf{A}_1$ defined according to (5.31) [and not (6.11)]. A null 2-blade $\mathbf{i}_2$ will exist, only if $\mathbf{A}\neq\mathbf{A}_1$.

6.2 Application to Maxwell's electromagnetic field tensor

The electromagnetic field tensor is given by (5.19), replacing the components of the electric field $\vec{E}=(r,s,t)$ and the magnetic field $\vec{B}=(a,b,c)$. And we need to remember that Maxwell's theory is special relativistic, i.e. invariant under Lorentz transformations in Minkowski space. The basis we have to use must therefore satisfy (6.1).

By (6.9) we obtain its two eigenvalues

$$\begin{aligned}&\nu_1 = \left\{\frac{1}{2}\left[\left(\vec{E}^2-\vec{B}^2\right)+\sqrt{\left(\vec{E}^2-\vec{B}^2\right)^2+4\left(\vec{E}\cdot\vec{B}\right)^2}\right]\right\}^{1/2}\\ &\nu_2 = \left\{\frac{1}{2}\left[-\left(\vec{E}^2-\vec{B}^2\right)+\sqrt{\left(\vec{E}^2-\vec{B}^2\right)^2+4\left(\vec{E}\cdot\vec{B}\right)^2}\right]\right\}^{1/2}\end{aligned} \tag{6.14}$$

which are not a space-time invariant, since the fields itself are not space-time invariant. (Only the energy density and the Pointing vector are invariants. Electromagnetic invariants are properly treated in [12,13].) But the term under the inner square roots of (6.14) is the trace of the Lorentz transformation invariant Maxwell energy momentum tensor (= energy density minus square of the Poynting vector) of the electromagnetic field.

In general the electromagnetic field tensor A is thus specified by two orthogonal bivectors

$$\mathbf{A}_1 = \nu_1 \mathbf{i}_1 = \nu_1 \frac{\nu_1 - i_4 \nu_2}{\sqrt{\left(\vec{E}^2 - \vec{B}^2\right)^2 + 4\left(\vec{E}\cdot\vec{B}\right)^2}} \mathbf{A} \quad (6.15)$$

$$\mathbf{A}_2 = \nu_2 \mathbf{i}_2 = \nu_2 \frac{\nu_2 + i_4 \nu_1}{\sqrt{\left(\vec{E}^2 - \vec{B}^2\right)^2 + 4\left(\vec{E}\cdot\vec{B}\right)^2}} \mathbf{A} \quad (6.16)$$

Let us now turn to the special case of plane electromagnetic waves, which mediate electromagnetic forces. They are light, warmth, radio transmission waves, transmit information to cellular phones, in short, our world would not be without electromagnetic interaction.

Maxwell's electrodynamics describes plane electromagnetic waves by oscillating, but always perpendicular, electric and magnetic vector fields.

The perpendicularity of $\vec{E}$ and $\vec{B}$ simply means that

$$ar + bs + ct = 0, \quad (6.17)$$

which results in great simplifications, because the coefficient (6.7) will therefore be zero as well. (6.8) will then have the form

$$\alpha^2 + (a^2 + b^2 + c^2 - r^2 - s^2 - t^2)\alpha = 0, \quad (6.18)$$

i.e. we have the two roots

$$\alpha_1 = -(a^2 + b^2 + c^2 - r^2 - s^2 - t^2), \alpha_2 = 0. \quad (6.19)$$

$\alpha_2 = 0$ in (6.19) means, that $0<\nu_k$ in (6.2) is no longer fulfilled. But that only means that we have a kernel of dimension two, otherwise the two α's are still distinct and the analysis of section 6.1 therefore still applies.

Inserting $\vec{E}$ and $\vec{B}$ in (6.19) gives

$$\alpha_1 = (\vec{E}^2 - \vec{B}^2) \quad (6.20)$$

Plane electromagnetic wave fields can therefore now be alternatively characterized in a new way. They have a degenerate field tensor, which becomes obvious in the canonical form. Only one eigenvalue $\nu_1 \neq 0$ is present

$$\mathbf{A}=\mathbf{A}_1. \quad (6.21)$$

7. Conclusion

This paper showed how the use of coordinates, matrices, complex numbers, and vector spaces over the complex numbers can be avoided in the analysis of antisymmetric matrices.

Utilizing real geometric algebra, i.e. the "grammar" of universal geometric calculus, antisymmetric matrices are found to best be treated via their isomorphism with real bivectors. The isomporphism allows to effortlessly switch back and forth between the antisymmetric matrix and the isomorphic bivector. Geometric algebra can easily yield the canonical form of this bivector, consisting of a decomposition into orthogonal plane area elements. These area elements can be interpreted both as plane two dimensional subspaces and as rotation operators in these subspaces.

It was explicitly demonstrated that the view "Antisymmetric Matrices are Real Bivectors" is consistent for both Euclidean spaces and (pseudoEuclidean) Minkowski space. This view has advantages for teaching, research and application of antisymmetric matrices, in whatever context they occur!

The calculations in this paper can be implemented both symbolically and numerically in commercial and freely available (stand alone) geometric algebra software packages and programs, e.g. in the Cambridge Geometric Algebra package[6] and others.

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"Soli Deo Gloria"[10]

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Notes

1) Antisymmetric matrices generate rotations in arbitrary high dimensions.

2) It is interesting to note this parallel in the Japanese tea ceremony. The present form of the tea ceremony was established by Sen Rikyu in the 16th century. His

wife is said to have been a secret Christian (Kirishitan), some think even Sen Rikyu was.

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