

Forces in Schwarzschild, Vaidya and generalized Vaidya spacetimes

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Abstract. In this paper we investigate how the leading term in the geodesic equation in Schwarzschild spacetime changes under the coordinate transformation to Eddington-Finkelstein coordinates. This term corresponds to the Newton force of attraction. Also we consider this term when we add the energy-momentum tensor of the form of the null dust and the null perfect fluid into right-hand side of the Einstein equation. We estimate the value of this force in Vaidya spacetime when the naked singularity formation occurs. Also we give conditions in generalized Vaidya spacetime when this force of attraction is replaced by the force of repulsion.

1. Introduction

In order to describe forces in general relativity one should study the geodesics [1, 2, 3]. In Schwarzschild metric the leading term in geodesic equation is the Newton force of attraction [4]. Under the notion 'the leading term' we mean the Γ_{00}^1 component of the Christoffel symbols. In Schwarzschild spacetime this component is given by:

$$\Gamma_{00}^1 = \left(1 - \frac{2M}{r}\right) \frac{M}{r^2}. \quad (1)$$

Christoffel symbols don't obey the tensor transformation law and if we do coordinate transformation then Christoffel symbols change and depending on the coordinate system, we obtain different inertial forces. However the main goal of this paper is to investigate how the term Γ_{00}^1 changes when one does coordinate transformation to Eddington-Finkelstein coordinates.

The Schwarzschild spacetime is the solution of the Einstein equations in the vacuum. However, if we consider the Schwarzschild metric in Eddington-Finkelstein coordinates with the energy-momentum tensor of the null dust in the right-hand side of the Einstein equations then one obtains the Vaidya spacetime. This spacetime is one of the earliest examples of cosmic censorship conjecture violation [5]. Moreover one could consider the mixture of two energy-momentum tensors of type-I (the null dust) and the type-II (the null perfect fluid). In this case one obtains the generalized Vaidya spacetime [6]. The only difference from Schwarzschild spacetime is that in the case of Vaidya spacetime the mass function is the function of the time and in the generalized Vaidya spacetime case the mass function depends upon both and the time and the radial coordinate. The generalized Vaidya spacetime also contains the naked singularities [7, 8, 9, 10].



We investigate the question how these energy-momentum tensors affect the term Γ_{00}^1 . We also consider the force expression in the case of the naked singularity formation in order to understand whether it is finite or infinite. Also it is interesting if the attraction can be replaced with repulsion.

The paper is organized as follows: in sec.2 we consider the Schwarzschild spacetime in Eddington-Finkelstein coordinates and see if this transformation changes the term Γ_{00}^1 . In sec.3 we consider the Vaidya spacetime and see how the null dust affects the forces and consider these forces in the naked singularity. In sec.4 we consider the generalized Vaidya spacetime and find the condition when the attraction can be replaced with repulsion. Section 5 is the conclusion.

The signature $-, +, +, +$ and the system of units $G = c = 1$ will be used throughout the paper. Also dash and over dot mean the partial derivative with respect to r and v respectively i.e. $M' = \frac{\partial M}{\partial r}$, $\dot{M} = \frac{\partial M}{\partial v}$.

2. Schwarzschild spacetime

Schwarzschild metric in coordinates $\{t, r, \theta, \varphi\}$ has the following form:

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2)$$

To obtain this metric in Eddington-Finkelstein Coordinates one should consider the radial null geodesic:

$$\left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{2M}{r}\right)^{-2}. \quad (3)$$

Integrating this equation and doing the following transformation from t to new time v :

$$t = v + r + 2M \ln(r - 2M), \quad (4)$$

we obtain the Schwarzschild metric in Eddington-Finkelstein Coordinates:

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (5)$$

The geodesic equation for the metric (5) is given by:

$$\begin{aligned} \frac{d^2 r}{d\tau^2} = & -\frac{M}{r^2} \left(1 - \frac{2M}{r}\right) \left(\frac{dv}{d\tau}\right)^2 + 2\frac{M}{r^2} \frac{dv}{d\tau} \frac{dr}{d\tau} + (r - 2M) \left(\frac{d\theta}{d\tau}\right)^2 + \\ & (r - 2M) \sin^2 \theta \left(\frac{d\varphi}{d\tau}\right)^2. \end{aligned} \quad (6)$$

Comparing the Γ_{00}^1 from (1) and (6) we see that it is the same in both coordinate systems. However in comparison to the old coordinates in Eddington-Finkelstein Coordinates we have new force which is proportional to the velocity $\frac{dr}{d\tau}$:

$$2\Gamma_{10}^1 \frac{dv}{d\tau} \frac{dr}{d\tau} = -2\frac{M}{r^2} \frac{dv}{d\tau} \frac{dr}{d\tau}. \quad (7)$$

As we noted above that Christoffel symbols don't obey the tensor transformation law and one might expect new inertial forces to appear in different frames.

We can conclude that in Eddington-Finkelstein Coordinates the leading term Γ_{00}^1 which corresponds to the Newton force of attraction doesn't change. It is useful to write down the

energy expression in this metric. For this purpose we should use the Lagrangian which has the following form:

$$\mathcal{L} = g_{ik} \frac{dx^i}{d\tau} \frac{dx^k}{d\tau} = - \left(1 - \frac{2M}{r}\right) \left(\frac{dv}{d\tau}\right)^2 + 2 \frac{dv}{d\tau} \frac{dr}{d\tau} + r^2 \left(\frac{d\theta}{d\tau}\right)^2 + r^2 \sin^2 \theta \left(\frac{d\varphi}{d\tau}\right)^2. \quad (8)$$

And we can easily obtain the energy expression from the (8):

$$\varepsilon = \left(1 - \frac{2M}{r}\right) \frac{dv}{d\tau} - \frac{dr}{d\tau} = \text{const}. \quad (9)$$

And the radial geodesic by using (9) and with $\varphi = \text{const}, \theta = \frac{\pi}{2}$ is given by:

$$\frac{d^2 r}{d\tau^2} = -\frac{M}{r^2} \left(\varepsilon \frac{dv}{d\tau} - \frac{dr}{d\tau} \frac{dv}{d\tau} \right) = -\frac{M}{r(r-2M)} \left[\varepsilon^2 - \left(\frac{dr}{d\tau} \right)^2 \right]. \quad (10)$$

3. Vaidya spacetime

If we consider Schwarzschild spacetime in Eddington-Finkelstein Coordinates (5) and assume that the mass function M is not a constant but depends on the time v :

$$M = M(v), \quad (11)$$

then one obtains the Vaidya spacetime:

$$ds^2 = - \left(1 - \frac{2M(v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \quad (12)$$

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2.$$

This is not the solution in empty spacetime anymore. On the right-hand side of the Einstein equation we have the following energy-momentum tensor:

$$T_{ik} = \mu \delta_i^0 \delta_k^0. \quad (13)$$

As we can see from (13) the only non-vanishing component of the energy-momentum tensor is T_{00} . This corresponds to the case of the null dust.

The energy conditions [11] demands that:

$$\dot{M}(v) > 0. \quad (14)$$

Now we can write down the radial geodesic in Vaidya metric:

$$\frac{d^2 r}{d\tau^2} = - \frac{\left(1 - \frac{2M(v)}{r}\right) M(v) + \dot{M}(v)r}{r^2} \left(\frac{dv}{d\tau}\right)^2 + 2 \frac{M(v)}{r^2} \frac{dv}{d\tau} \frac{dr}{d\tau} +$$

$$(r - 2M(v)) \left(\frac{d\theta}{d\tau}\right)^2 + (r - 2M(v)) \sin^2 \theta \left(\frac{d\varphi}{d\tau}\right)^2. \quad (15)$$

If we compare geodesic equations in Vaidya (15) and in Schwarzschild (6) spacetimes then we can see that only the leading term Γ_{00}^1 has difference i.e.:

$$\Gamma_{00 \text{ vaidya}}^1 = \Gamma_{00 \text{ sch}}^1(v) + \frac{\dot{M}(v)}{r}. \quad (16)$$

If we consider the leading term as the force of attraction then one can see that this force is bigger than in pure Schwarzschild spacetime. It is worth noticing that according to the energy condition (14) the term $\frac{\dot{M}(v)}{r} > 0$ and the leading term (16) doesn't change its sign.

The metric (12) depends upon time hence the energy is not conserved i.e.:

$$\varepsilon(v) = \left(1 - \frac{2M(v)}{r}\right) \frac{dv}{d\tau} - \frac{dr}{d\tau}. \quad (17)$$

Now differentiating ((17)) with respect to τ we obtain the second order geodesic equation which depends on the energy:

$$\begin{aligned} \varphi = const, \theta = \frac{\pi}{2}, \\ \frac{d^2 r}{d\tau^2} = - \frac{\left(1 - \frac{2M(v)}{r}\right) M(v) + 2\dot{M}(v)r}{r^2} \left(\frac{dv}{d\tau}\right)^2 + \\ 2 \frac{M(v)}{r^2} \frac{dr}{d\tau} \frac{dv}{d\tau} - \frac{d\varepsilon(v)}{dv} \frac{dv}{d\tau}. \end{aligned} \quad (18)$$

And comparing (18) to (15) one can find the energy change law:

$$\frac{\partial \varepsilon(v)}{\partial v} = - \frac{\dot{M}(v)}{r} \frac{dv}{d\tau}. \quad (19)$$

We can conclude that the particle loses its energy despite it falls onto a black hole.

As we mentioned above the Vaidya spacetime contains the naked singularity. It forms during the gravitational collapse at $v = 0, r = 0$. The case when the mass function is linear has been considered by P. Joshi [12]. Here we consider the mass function in the form:

$$M(v) = \lambda v^\beta, \beta \geq 1, \lambda > 0. \quad (20)$$

In this case the leading term is given by:

$$\Gamma_{00}^1 = - \frac{\left(1 - \frac{2\lambda v^\beta}{r}\right) \lambda v^\beta + \beta \lambda v^{\beta-1} r}{r^2}. \quad (21)$$

We are interested only in the case when $\beta \geq 1$ because only in this case the naked singularity might form. In the case $\beta < 1$ the result of the gravitational collapse is the black hole. When we have the naked singularity formation then the following condition must be held:

$$\lim_{v \rightarrow 0, r \rightarrow 0} \frac{dv}{dr} = X_0 > 0. \quad (22)$$

Moreover X_0 is finite.

Substituting this into (21) we can see that if $\beta \in [1, 2)$ then $\lim_{v \rightarrow 0, r \rightarrow 0} \Gamma_{00}^1 \rightarrow \infty$. When $\beta = 2$ it is a positive real constant and when $\beta > 2$ it is zero.

However we are interested only in gravitationally strong singularities. According to Tipler's definition [14] a singularity is termed to be gravitationally strong or simply strong if it destroys by stretching or crushing any object which falls into it. If it does not destroy any object this way then the singularity is termed to be gravitationally weak. Mathematically it is [13]:

$$\xi = \lim_{\tau \rightarrow 0} \tau^2 R_{ik} K^i K^k > 0, \quad (23)$$

where τ - is the affine parameter and K^i is the tangent vector to outgoing null geodesic congruence.

In our case (23) has the following form:

$$\xi = \lim_{v \rightarrow 0} 2\beta(\beta - 1)\lambda v^{\beta-1} X_0^2. \quad (24)$$

And we can see that the singularity is gravitationally strong only in the case $\beta = 1$. Hence we can conclude that when the naked singularity is gravitationally strong the force is infinite and when the force is finite the naked singularity is gravitationally weak.

4. Generalized Vaidya spacetime

When we consider the mixture of two types of the matter field i.e. type-I and type-II [15] then one can obtain the generalized Vaidya spacetime. Type-I is the null dust which has been considered in the previous section. Type-II is the null perfect fluid [6]. So one can write:

$$\begin{aligned} T_{ik}^{(I)} &= \mu L_i L_k, \\ T_{ik}^{(II)} &= (\rho + P)(L_i N_k + L_k N_i) + P g_{ik}, \end{aligned} \quad (25)$$

here μ is the energy density of the null dust and P and ρ are the pressure and energy density of the null perfect fluid:

$$\begin{aligned} \mu &= \frac{2\dot{M}}{r^2}, \quad \rho = \frac{2M'}{r^2}, \\ P &= -\frac{M''}{r}, \quad L_i = \delta_i^0, \end{aligned} \quad (26)$$

and l^i and n^i are two null vectors:

$$\begin{aligned} N_i &= \frac{1}{2} \left(1 - \frac{2M}{r} \right) \delta_i^0 - \delta_i^1, \\ L_i L^i &= N_i N^i = 0, \quad L_i N^i = -1. \end{aligned} \quad (27)$$

In this case the generalized Vaidya spacetime is given by:

$$ds^2 = - \left(1 - \frac{2M(v, r)}{r} \right) dv^2 + 2dvdr + r^2 d\omega^2. \quad (28)$$

Here the mass function not only the function of the time v but also radial coordinate r .

Weak, strong and dominant energy conditions demand:

$$\begin{aligned} \mu &\geq 0, \\ \rho &\geq P \geq 0. \end{aligned} \quad (29)$$

The radial geodesic equation in this metric has the following form:

$$\begin{aligned} \frac{d^2 r}{d\tau^2} &= - \frac{\left(1 - \frac{2M}{r} \right) (M - M'r) + \dot{M}r}{r^2} \left(\frac{dv}{d\tau} \right)^2 + 2 \frac{M - M'r}{r^2} \frac{dv}{d\tau} \frac{dr}{d\tau} + \\ &\quad (r - 2M) \left(\frac{d\theta}{d\tau} \right)^2 + (r - 2M) \sin^2 \theta \left(\frac{d\varphi}{d\tau} \right)^2. \end{aligned} \quad (30)$$

In comparison with (15) we have two changes: I) the inertial force which depends on the velocity $\frac{dr}{d\tau}$ linearly has the extra term $\frac{M'}{r} \frac{dv}{d\tau} \frac{dr}{d\tau}$. II) Our leading term has the following change:

$$\Gamma_{00}^1{}_{generalizedVaidya} = \Gamma_{00}^1{}_{Vaidya}(r, v) - \frac{M'}{r}. \quad (31)$$

From (31) one might expect the sign of the force to be changed. But first of all we should find out the explicit form of the mass function. For this purpose let's consider the null perfect fluid with the equation of the state $P = \alpha\rho$ where $\alpha \in [-1, 1]$. In this case the mass function has the following form [8]:

$$M(v, r) = C(v) + D(v)r^{1-2\alpha}, C(v) \geq 0, D(v) \geq 0. \quad (32)$$

From (30) we can easily see that the only term which can change the force sign is $\frac{(M-M'r)}{r^2}$. This term might change the sign for both forces i.e. and for leading term Γ_{00}^1 and the inertial force which depends linearly on the velocity $\frac{dr}{d\tau}$. So let's find when this factor can change its sign. For this purpose substituting the mass function (32) into this term:

$$\frac{M - M'r}{r^2} = \frac{C(v) + 2\alpha r^{1-2\alpha}}{r^2}. \quad (33)$$

$C(v) > 0$ and $D(v) > 0$ these are demanded by the dominant and weak energy condition. So if $\alpha \geq 0$ then (33) doesn't change its sign. The only option is the negative values of α . But in this case we have to violate the strong energy condition and consider the negative pressure.

So the inertial force which is linearly depends on the velocity $\frac{dr}{d\tau}$ changes its sign when the particle crosses the shell of the radius:

$$r = \left(-\frac{C(v)}{2\alpha D(v)} \right)^{\frac{1}{1-2\alpha}}. \quad (34)$$

Regarding the leading term Γ_{00}^1 things are not so clear. In general case when (33) changes its sign this term can be still positive. However it can change its sign and attraction will be replaced with repulsion if:

$$\left| \left(1 - \frac{2(C(v) + D(v)r^{1-2\alpha})}{r} \right) (C(v) + 2\alpha r^{1-2\alpha}) \right| > \dot{C}(v) + \dot{D}(v)r^{1-2\alpha}. \quad (35)$$

That the positive r exists in this case one can see if we consider the mass function which depends only on r . In this case $C(v) = \eta > 0, D(v) = \zeta > 0$ and attraction is replaced with repulsion at the radius:

$$r = \left(-\frac{\eta}{2\alpha\zeta} \right)^{\frac{1}{1-2\alpha}}. \quad (36)$$

So we can see that in generalized Vaidya spacetime when we consider the negative pressure of the null perfect fluid the force of attraction can be replaced with the force of repulsion.

Now let's find out the law of the energy change. The energy expression in the generalized Vaidya spacetime is:

$$\varepsilon(v) = \left(1 - \frac{2M}{r} \right) \frac{dv}{d\tau} - \frac{dr}{d\tau}. \quad (37)$$

Differentiating this with respect to τ one can find:

$$\frac{d^2 r}{d\tau^2} = -\frac{\left(1 - \frac{2M}{r}\right)(M - M'r) + 2\dot{M}r}{r^2} \left(\frac{dv}{d\tau}\right)^2 + 2\frac{M - M'r}{r^2} \frac{dv}{d\tau} \frac{dr}{d\tau} - \frac{d\varepsilon(v)}{dv} \frac{dv}{d\tau}, \quad (38)$$

and comparing this formula to (30) with $\varphi = \text{const}$, $\theta = \text{const}$ one can find the same law of the energy loss like in the previous section (19):

$$\frac{d\varepsilon(v)}{dv} = -\frac{\dot{M}}{r^2} \frac{dv}{d\tau} \quad (39)$$

5. Conclusion

In this paper we have considered the leading term Γ_{00}^1 in Schwarzschild spacetime which is the Newton force of attraction. First of all we have found out that this term doesn't change when we do the coordinate transformation to Eddington-Finkelstein Coordinates. Then the matter has been added to the right-hand side of the Einstein equation in order to track how this term will change. In the beginning we considered the energy-momentum tensor of the null dust - so-called Vaidya spacetime. In this case the force of attraction becomes bigger than in Schwarzschild metric and in the case of gravitationally strong naked singularity formation this force is infinite. This force becomes finite in the central naked singularity only in the case when it is gravitationally weak. When we consider the mixture of type-I and type-II of the matter fields (generalized Vaidya spacetime) then the force of attraction can be replaced with repulsion but only if we violate the strong energy condition and consider the negative pressure.

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