COHOMOLOGICAL FIELD THEORIES ON COMPLEX MANIFOLDS

JAE-SUK PARK

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{Samenvatting in het Nederlands}

Cohomologische veldentheorie was oorspronkelijk geintroduceerd als een getwiste versie van globale ruimte-tijd supersymmetrischequantum veldentheorie, specifiek in vier ruimte-tijd dimensies. Voor globale ruimte-tijd supersymmetrie is per definitie het bestaan van een spinor nodig, die overal op de ruimte-tijd varieteit constant is. Een spinor bestaat alleen op een spin-variëteit. Een spin-variëteit staat echter slechts zelden een constante spinor toe. De canonieke manier om dit probleem te omzeilen is het localiser maken van de supersymmetry; een procedure die haast op magische wijze (super)gravitatie introduceert.

Er is een tweede optie, genaamd twisten. Dit betekent dat men een nieuwe Lorentz symmetrie definieert als een geschikte combinatie van de originele Lorentz symmetrie met een interne globale symmetrie van de theorie. Dit resulteert in supersymmetrie generatoren die anders transformeren onder de nieuwe Lorentz symmetrie. Er zijn typisch componenten van de superladingen die transformeren als een scalar. Zo een scalar component Q, die nilpotent is, dat wil zeggen $Q^2 = 0$, wordt gezien als een supersymmetrie van de getwiste theorie. De resulterende theorie is goed gedefinieerd op een willekeurige variëteit, omdat er geen globale obstructies zijn voor een scalar. Verder is de theorie algemeen covariant, zonder de introductie van gravitatie. De padintegraal van de theorie hangt alleen af van de globale cohomologie van Q, onder voorbehoud dat men alleen Q-invariante observabele gebruikt. Dit is waarom de theorie cohomologisch wordt genoemd.

Een getwiste theorie is gerelateerd aan de onderliggende ruimte-tijd supersymmetrische theorie, doordat padintegraal van de getwiste theorie een zekere chirale (of BPS) sector van de fysische amplituden berekent. Dit is een gevolg van de triviale holonomie van de ruimte-tijd waar de theorie gewoonlijk is gedefinieerd. In dat geval is de operatie van twisten fysisch niet te zien. De typische fysische toepassing van een getwiste theorie is een niet-perturbatieve test van zekere dualiteiten, gebruik makend van de semi-klassieke exactheid van de theorie.

Twee beroemde voorbeelden van voor de tweede string revolutie zijnde toepassingen voor mirror symmetrie en S-dualiteit van N = 4 super-Yang-Mills in vier dimensies. De getwiste theorie van vier-dimensionale N = 2 supersymmetrische Yang-Mills theorie – de Donaldson-Witten theorie -- leverde ook cruciale hints voor de gevierde Seiberg-Witten oplossingen van de originele N = 2 theorie. De oplossingen van de onderliggende fysische theorie geven ons waardevolle inzichten in het wiskundige probleem gedefinieerd dor de getwiste theorie. Misschien wel de mooiste eigenschap van quantum veldentheorie is dat de theorie afhangt van een schaal. De even zo mooie eigenschap van cohomologische veldentheorie is dat de theorie niet van een schaal afhankelijk is. Hierdoor kan het wiskundige probleem gedefinieerd door de eerste theorie worden opgelost in termen van de laatste theorie bij een andere schaal, waar de relevante vrijheidsgraden in het algemeen compleet anders zijn van die van de originele -- microscopische -- theorie. Het historische voorbeeld is natuurlijk de Donaldson versus de Seiberg-Witten invariant.

Na de tweede string revolutie speelt cohomologische veldentheorie nog steeds een belangrijke rol. Voornamelijk in de fysica van D-branes. Een raison d'etre voor getwiste theorieën wordt gegeven door D-branes in een niet-triviale ruimte-tijd. Alle tellingen van BPS toestanden en hun toepassingen in de fysica van zwarte gaten en niet-perturbatieve testen van string-dualiteiten zijn gebaseerd op hetzelfde principe.

In het algemeen kunnen we de onderliggende fysische oorsprong van een cohomologische veldentheorie vergeten, en de theorie definiëren als een quantum veldentheorie met een globale fermionische symmetrie. Zo een theorie hoeft niet direct verkregen te kunnen worden als een getwiste versie van een supersymmetrische theorie. De meest fundamentele eigenschap van een quantum veldentheorie met een globale fermionische symmetrie is de vaste punten stelling van Witten. Bijna alle andere eigenschappen van een cohomologische veldentheorie kunnen worden verkregen als een

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Academisch Proefschrift

ter verkrijging van de graad van doctor
aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
prof. dr J.J.M. Franse
ten overstaan van een door het college voor promoties ingestelde
commissie in het openbaar te verdedigen in de Aula der Universiteit
op dinsdag 23 november 1999 te 11.00 uur

door

Jae-Suk Park

geboren te Jindo

Promotor: Prof. dr H.L. Verlinde

Promotiecommissie: Prof. dr P. van Baal
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Prof. dr E.P. Verlinde

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Instituut voor Theoretische Fysica

The research described in this thesis is supported
by a Pionier fund of NWO.

This thesis is based mainly on following papers:

- J.-S. Park,
  Cohomological Field Theories with a Kähler Structure,
  Lectures given at KIAS. [Ch2,3]
- R. Dijkgraaf, B.J. Schroers and J.-S. Park,
  N=4 Supersymmetric Yang-Mills Theory on a Four Manifold,
  `hep-th/9801066`. [Ch4]
- C. Hofman and J.-S. Park,
  Cohomological Yang-Mills Theory on a Kähler 3-Fold,
  `to appear`. [Ch5]
- C. Hofman and J.-S. Park,
  Holomorphic Casson, Floer and Donaldson-Witten Theories and Beyond/2,
  `to appear`. [Ch5,6]
- J.-S. Park,
  Higgs Bundle and Four Manifolds,
  `to appear`. [Ch7]

I also briefly used some results of the following papers:

- C. Hofman and J.-S. Park,
  Sigma Models for Bundles on Calabi-Yau: a proposal for matrix string compactification,
  Nucl. Phys. B, in press. `hep-th/9904150`.
- C. Hofman, H. Verlinde and J.-S. Park,
  Matrix String Theory of Five-Branes,
  `unpublished`.
- C. Hofman and J.-S. Park,
  Sigma Models for Quiver Variety,
  `unpublished`.
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- J.-S. Park
  N=2 Topological Yang-Mills Theory on Compact Kähler Surfaces,
  Commun. Math. Phys. 163 (1994) 113-140. `hep-th/9304060`.

Other papers:

- C. Hofman and J.-S. Park,
  Monads, Strings, and M Theory,
  Nucl. Phys. B520 (1998) 229-260. hep-th/9706130.

- J.-S. Park,
  Monads and D-instantons,
  Nucl. Phys. B493 (1997) 198-230. hep-th/9612096.

- S. Hyun, J. Park and J.-S. Park,
  N=2 Supersymmetric QCD and Four Manifolds; (I) the Donaldson and the Seiberg-Witten invariants,
  hep-th/9508162

- S. Hyun, J. Park and J.-S. Park,
  Topological QCD,
  Nucl. Phys. B453 (1995) 199-224. hep-th/9503201

- S. Hyun and J.-S. Park,
  Holomorphic Yang-Mills Theory and Variation of the Donaldson Invariants,
  hep-th/9503036.

- J.-S. Park,
  Holomorphic Yang-Mills Theory on Compact Kähler Manifolds,
  Nucl. Phys. B423 (1994) 559-579. hep-th/9305095

- H.J. Lee and J.-S. Park,
  Complex Geometrical Interpretation of BRST Algebra in Topological Yang-Mills Theory,
  Phys. Lett. B277 (1992) 119-122.

To my memories of
my Father
and Grandmother

Other papers:

- C. Hofman and J.-S. Park,
  Monads, Strings, and M Theory,
  Nucl. Phys. B539 (1999) 239-280. hep-th/9908180.

- J.-S. Park,
  Monads and D-instantons,
  Nucl. Phys. B493 (1997) 198-230. hep-th/9612096.

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Chapter 1

Introduction

Historically cohomological field theory first has been introduced as a *twisted* version of global space-time supersymmetric quantum field theory, specifically the $N = 2$ supersymmetric Yang-Mills theory in four dimensional space-time [1]. The global space-time supersymmetry, by definition, requires the existence of a spinor which is constant everywhere on the space-time manifold M. A spinor does exist on a spin manifold. A spin manifold, however, rarely admits a constant spinor. The canonical way overcoming the above difficulty is to localize the supersymmetry, a procedure that almost magically introduces (super-)gravity into the picture.

There is a second option called twisting, which means that one defines a new Lorentz symmetry group by a suitable combination of the original Lorentz symmetry with an internal global symmetry of the theory. As a result, the supercharges transform differently under the new Lorentz symmetry. Typically the supercharges include some components which transform as scalars. Such a scalar component Q, which is nilpotent, i.e. $Q^2 = 0$, is regarded as a supercharge of the twisted theory. The resulting theory is well-defined on an arbitrary space-time manifold since there are no global obstructions for a scalar, and enjoys general covariance without gravity. The path integral of the theory depends only on the global cohomology of Q, provided that one uses Q-invariant observables, which property coined the adjective cohomological [2].

A twisted theory is closely related to the underlying space-time supersymmetric theory. Namely the path integral of the twisted theory computes a certain chiral (or BPS) sector of physical amplitudes [3][4]. This is due to the trivial holonomy of flat space-time where the physical theory is usually defined. Then twisting is a physically invisible operation. The typical physical application of a twisted theory is a non-perturbative test of a certain duality utilizing the semi-classical exactness of the path integral. Two famous examples originating before the second string revolution are given by mirror symmetry [5][6] and S-duality of $N = 4$ supersymmetric Yang-Mills theory in four-dimensions [7]. The twisted version of four dimensional $N = 2$ supersymmetric Yang-Mills theory – the Donaldson-Witten theory [1][8]– also provided crucial hints [9][10] on the celebrated Seiberg-Witten solutions of the original $N = 2$ theory [11].

We must stress here that solutions of the underlying physical theory provide us with invaluable insights in the mathematical problem defined by the twisted theory. Perhaps one of the most beautiful properties of quantum field theory is that the theory depends on a scale. The equally beautiful property of cohomological field theory is that the theory does not depend on scale. Thus the mathematical problem defined by the latter theory can be solved in terms of the former theory at a different scale where its relevant degrees of freedom are, often completely different from the original microscopic one. The historical example is, of course, the Donaldson versus Seiberg-Witten invariant [12]. After the second string revolution [13][14][15], cohomological field theory still plays important roles, especially in D-brane physics [16]. A raison d'etre of twisted theory has been provided in terms of D-branes on non-trivial space-times [17]. All those countings of BPS states and their applications to blackhole physics and non-perturbative tests of string dualities are based on the same principle, see [18][19][20][21][22][23] etc.

In general we may forget about the underlying physical origin of a cohomological field theory and define the theory as a quantum field theory with a global fermionic symmetry. Such a theory may not be directly obtainable as a twisted version of an underlying space-time supersymmetric theory. The most fundamental property of a quantum field theory with a global fermionic symmetry is the fixed point theorem of Witten [3][24]. Almost all the other properties of cohomological field theory can be obtained as a certain lemma of the theorem. Thus it seems appropriate to quote the theorem here [3].

> Consider an arbitrary quantum field theory, with some function space X over which one wishes to integrate. Let F be a group of symmetries of the theory. Suppose F acts freely on X. Then one has a fibration $X \to X/F$, and by integrating first over the fibers of this fibration, one can reduce the integral over X to an integral over X/F. Provided one considers only F invariant observables $\mathcal{O}$, the integration over the fibers is particularly simple and just gives a factor of vol(F) (the volume of the group F):
>
> $$\int_{\mathcal{X}} e^{-S}\mathcal{O} = \mathrm{vol}(F) \cdot \int_{\mathcal{X}/F} e^{-S}\mathcal{O}. \qquad (1.0.1)$$
>
> Now we consider the case that F is a global fermionic symmetry generated by a supercharge Q. Then the volume of the group F is zero. It follows that if Q acts freely, the expectation value of any Q invariant operator vanishes. In general, F does not act freely, but has a fixed point locus X_0. If so, let $\mathcal{C}$ be an F-invariant neighborhood of X_0 and X' its complement. Then the path integral restricted to X' vanishes, by the above reasoning. So the entire contribution to the path integral comes from the integral over $\mathcal{C}$. Here $\mathcal{C}$ can be an arbitrarily small neighborhood, so the result is really a localization formula expressing the path integral as an integral on X_0. The details depend on the structure of Q near X_0. If the vanishing of Q near X_0 is a generic, simple zero, then the fixed point contribution

is simply an integral over X_0 weighted by the one loop determinants of the transverse degrees of freedom.

In Chapters 2 and 3 we will develop a general approach which identifies any cohomological field theory with a $0+0$-dimensional supersymmetric sigma model. Being in zero-dimensions the (space-time) supersymmetry simply means global fermionic symmetry. The target space of our sigma-model may be some function space X in the theorem quoted above. Such a space may be any (non-linear or linear and finite or infinite dimensional) space endowed with any of

$$\text{Riemannian} \supset \text{Kähler} \supset \text{hyper-Kähler} \tag{1.0.2}$$

structures. Actually the above structures may not be regarded as *a priori* notions. The cohomological field theory can be classified by the number $N_c = (N_c^+, N_c^-)$ of global supercharges, where we have $N_c^+ + N_c^-$ independent *mutually nilpotent* fermionic charges and $N_c^\pm$ denote the number of charges carrying fermionic (or ghost) numbers ± 1. Then we have the following sequence of fermionic symmetries

$$N_c^+ = 1 \supset N_c^+ = 2 \supset N_c^+ = 4, \tag{1.0.3}$$

which determines the sequence of geometrical structures (1.0.2).[1]

In this thesis we specialize to models with a Kähler structure. Those models are quite general and allow us to have very compact formulations. The initial data will be some function space X endowed with a complex structure compatible with the supersymmetry. Then most of the other structures of the models can be fixed. We will introduce three types of models, two with $N_c = (2,0)$ and one with $N_c = (2,2)$ symmetry, and establish general interrelations. For each type we will consider non-linear X and linear or non-linear X with a group $\mathcal{G}$ acting on X. Perhaps our definition of cohomological field theory as a zero dimensional sigma model might be confusing. If the target space X is the function space of certain fields on a manifold M we have a traditional cohomological field theory on M.

In due course the relation between our construction and two-dimensional space-time supersymmetric field theory will become obvious. This implies that we always have canonical string theoretic generalizations of those differential-topological invariants defined by cohomological field theory. One may also use the correspondence to define suitable matrix string theory [29]. We will not go into this direction in this thesis and refer to [30][31], as examples. The Chapters 2 and 3 may also be viewed, after slight modifications, as an unorthodox introduction to two-dimensional supersymmetric field theories. Our presentation for models with a group action will parallel the original literature on $N_{ws} = (2,2)$

[1] The above correspondence is originally due to supersymmetric sigma models in two-dimensions [25][26][27]. In certain respects, such a correspondence in zero-dimensional models is more striking since we do not need any underlying geometrical objects like the two-dimensional space-time. Actually the sequence (1.0.3) leads to more general geometrical structures including torsion [28]. However, the author is not aware of any examples of a traditional cohomological field theories with torsion in the space of fields.

and $N_{ws} = (2,0)$ gauged linear sigma-models in two dimensions [32][33]. We should also mention the influential paper of Witten on supersymmetry and Morse theory [34] dealing with $(0+1)$-dimensional supersymmetric sigma models, which can be regarded as the origin of cohomological field theory.[2]

For some general literature for cohomological field theory we refer to [2] and [35] for short but lucid introductions, and two review articles [36] and [37]. Those references are mostly about the Riemannian version of $\mathcal{N}_c = (2,0)$ models. For the Riemannian version of $\mathcal{N}_c = (2,2)$ models, called balanced cohomological field theory, we refer to [38]. For a mathematician the path integral of a cohomological field theory is the Mathai-Quillen formalism of the integral representation of the Thom class [39][35]. Though we will never refer to Mathai and Quillen, our (path) integral formula can be viewed as the Kähler version of the Mathai-Quillen formalism. More precisely our formulas should be viewed as a certain equivariant generalization of Fulton and MacPherson's intersection theory [40]. For a physicist a cohomological field theory is a supersymmetric gauged sigma model in $(0+0)$-dimensions. Though we will never use the superspace formalism our construction is equivalent to the $N = 2$ superspace formalism.

In the later chapters of this thesis we will apply our formalism to construct models with certain infinite dimensional target spaces. We will concentrate on two classes of examples whose target spaces are; (i) the space $\mathcal{A}$ of all gauge fields on complex 2, 3 and 4-dimensional Kähler or Calabi-Yau manifolds. (ii) the total space $T^*\mathcal{A}$ of cotangent bundle of $\mathcal{A}$ on complex 2-dimensional Kähler manifolds, as an example. We call the first and the second classes of the models cohomological Yang-Mills theory and cohomological Yang-Mills-Higgs theory, respectively. Those chapters will be devoted mainly to a detailed study of the physical and mathematical implications of those models. Cohomological Yang-Mills theory on a compact Calabi-Yau manifold or on a flat manifold is equivalent to global supersymmetric Yang-Mills theory on that manifold. One may regard such a theory as, *after being suitably interpreted*, effective world-volume theory of D-brane [16][41], or Matrix theory [42][43], or dual to supergravity/string/M theories [44]. On the other hand cohomological Yang-Mills-Higgs theory does not have corresponding global supersymmetric Yang-Mills theory. Nevertheless such a model is connected with physical theory by certain renormalization group flow. It is amusing to speculate that such a model may describe certain "unbroken phase" of supersymmetric Yang-Mills theory or "unbroken phase" of the theories in the same equivalence class.

[2]It is ironical since his construction can be regarded, from our viewpoint, as a generalized cohomological field theory.

Chapter 2

Standard Models of Cohomological Field Theory

This and the next chapters are devoted to an elementary and self-contained introduction to cohomological field theory. Though elementary, we will develop the most general construction of cohomological field theory involving Kähler geometry.

In this chapter we consider supersymmetric sigma models in $(0+0)$ dimensions, whose target space is a compact complex Kähler manifold X. Those models may be regarded as the quantum theory of single point-like "instanton" - the point-like event of X or point-like instanton probes of the classical geometry of X by means of the path integral. The space of all bosonic field will be the configuration space of the instanton, which is a copy of the manifold X. We will start from the simplest $N_c = (2,0)$ model as a toy model. A slightly more complicated $N_c = (2,2)$ model follows. Then we generalize it to another $N_c = (2,0)$ model. We will survey how those supersymmetric theories probe or give rise to the classical geometry of Kähler manifolds X, its tangent bundle TX and holomorphic Hermitian vector bundle $\mathbb{E}$ over X. The models to be covered here will be used as the prototypes of all the other more elaborated models to be introduced later. We refer to the models in this chapter as standard models since any cohomological field theory will reproduce to one of those models if it is "generic".

We follow a typical procedure of defining supersymmetric field theory, namely introducing bosonic fields, supercharges with their algebra, fermionic superpartners, supersymmetric action functional, and studying path integrals. Due to the triviality of the model everything can be made completely rigorous. Assuming existence of nil-potent supercharges, a simple application of Poincaré lemma leads to an appropriate supersymmetric action functional. All the other geometrical structures then naturally follow. We will also clarify the geometrical meaning of the supercharges.

2.1 A Toy Model

In this section we design perhaps the simplest path integral, which has many of the basic properties of cohomological field theory.

Consider a compact complex n-dimensional space X . We pick local coordinates x^I, $I = 1, \ldots, 2n$ on X. The local complex coordinates on X will be denoted as z^i, $i = 1, \ldots, n$; their complex conjugates are $z^{\bar{i}} = \overline{z^i}$. Let X^I be local coordinates fields describing the position of an instanton on X. More precisely, the X^I parameterize a map

$$X^I : point \to X. \tag{2.1.1}$$

We denote by X^i local complex coordinates fields and $X^{\bar{i}}$ be their complex conjugates. We call X^i and $X^{\bar{i}}$ bosonic fields. We introduce anti-commuting operators $\boldsymbol{s}$ and $\overline{\boldsymbol{s}}$ called supercharges satisfying the following anti-commutation relations,

$$\boldsymbol{s}^2 = 0, \qquad \{\boldsymbol{s}, \overline{\boldsymbol{s}}\} = 0, \qquad \overline{\boldsymbol{s}}^2 = 0. \tag{2.1.2}$$

We define a pair of graded quantum number (ghost numbers) (p, q) such that $\boldsymbol{s}$ and $\overline{\boldsymbol{s}}$ carry the following ghost numbers

$$\boldsymbol{s} : (1, 0), \qquad \overline{\boldsymbol{s}} : (0, 1). \tag{2.1.3}$$

We call the supersymmetry (2.1.2) of type $N_c = (2, 0)$, meaning that we have two supercharges both carrying positive ghost numbers.

We assume that the X^i are holomorphic fields, meaning that $\overline{\boldsymbol{s}}X^i = 0$, and their complex conjugate $X^{\bar{i}}$ are anti-holomorphic, $\boldsymbol{s}X^{\bar{i}} = 0$. Then we can postulate the following supersymmetry transformation laws

$$\begin{aligned}
\boldsymbol{s}X^i &= i\psi^i, & \boldsymbol{s}\psi^i &= 0, \\
\overline{\boldsymbol{s}}X^i &= 0, & \overline{\boldsymbol{s}}\psi^i &= 0, \\
\boldsymbol{s}X^{\bar{i}} &= 0, & \boldsymbol{s}\psi^{\bar{i}} &= 0, \\
\overline{\boldsymbol{s}}X^{\bar{i}} &= i\psi^{\bar{i}}, & \overline{\boldsymbol{s}}\psi^{\bar{i}} &= 0.
\end{aligned} \tag{2.1.4}$$

From the above we may write $\boldsymbol{s}$ and $\overline{\boldsymbol{s}}$ as follows

$$\boldsymbol{s} = i\psi^i \frac{\partial}{\partial X^i}, \qquad \overline{\boldsymbol{s}} = i\psi^{\bar{i}} \frac{\partial}{\partial X^{\bar{i}}}. \tag{2.1.5}$$

We call the anti-commuting superpartners ψ^i and $\psi^{\bar{i}}$ of X^i and $X^{\bar{i}}$, respectively, fermionic fields. They carry the ghost numbers $(1, 0)$ and $(0, 1)$, respectively. In general, a field with ghost number (p, q) is fermionic if $p + q$ is odd while, otherwise, it is bosonic.

Now we consider an action functional $S(X^i, X^{\bar{i}}, \psi^i, \psi^{\bar{i}})$ which is invariant under both of the supersymmetries with supercharges $\boldsymbol{s}$ and $\overline{\boldsymbol{s}}$. The conditions for supersymmetry $\boldsymbol{s}S = \overline{\boldsymbol{s}}S = 0$ together with the anti-commutation relations (2.1.2) imply, due to the Poincaré lemma, that S may be written as

$$S = i\boldsymbol{s}\overline{\boldsymbol{s}}\mathcal{K}(X^i, X^{\bar{i}}), \tag{2.1.6}$$

where $\mathcal{K}$ is a locally defined real functional of X^i and $X^{\bar{i}}$. Applying the transformation laws (2.1.4) we have

$$S = i\left(\frac{\partial^2 \mathcal{K}}{\partial X^i \partial X^{\bar{j}}}\right)\psi^i\psi^{\bar{j}} := -i\mathcal{K}_{i\bar{j}}\psi^i\psi^{\bar{j}}. \tag{2.1.7}$$

Now we consider the Feynman path integral of our model. The partition function is defined as integration over the space of all fields weighted by e^{-S},

$$Z = \int [\mathcal{D}X\mathcal{D}\overline{X}\mathcal{D}\psi\mathcal{D}\overline{\psi}]\, e^{-S}. \tag{2.1.8}$$

In everyday quantum field theory, we usually do not have a well-defined path integral measure though we have well-established rules of doing the path integral at least for the perturbative regime. For our trivial quantum field theory the path integral measure is perfectly well-defined. The space of all bosonic fields is a copy of X. Thus the path integral is an integral over X. We have

$$Z = \int_X \prod_{k,\bar{k}=1}^{n} dX^k dX^{\bar{k}} d\psi^k d\psi^{\bar{k}} \exp\left(i\mathcal{K}_{i\bar{j}}\psi^i\psi^{\bar{j}}\right). \tag{2.1.9}$$

Remark that the path integral measure carries ghost number (n, n), i.e., the ghost number anomaly. In the above evaluation we used the basic fact of integration over Grassmann numbers that the integrand should also carry the net ghost number (n, n) to have a non-vanishing integral. Performing the integral over ψ^i and $\psi^{\bar{i}}$, using the law of integral over Grassmannian number, we have

$$Z = \int_X \prod_{k,\bar{k}=1}^{n} dX^k dX^{\bar{k}} det(i\mathcal{K}_{i\bar{j}}). \tag{2.1.10}$$

Now we compare the properties of our model with the differential geometry of the Kähler manifold X. We denote the space of r-forms on X by $\Omega^r(X)$. We have the exterior derivative

$$d : \Omega^r(X) \to \Omega^{r+1}(X)$$

satisfying $d^2 = 0$. For any complex manifold we have decompositions

$$\Omega^r(X) = \bigoplus_{r=p+q} \Omega^{p,q}(X)$$

of r-forms into type (p, q)-forms with $p + q = r$. Similarly we have a decomposition $d = \partial + \overline{\partial}$ such that

$$\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \qquad \overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \tag{2.1.11}$$

and

$$\partial^2 = 0, \qquad \{\partial, \overline{\partial}\} = 0, \qquad \overline{\partial}^2 = 0. \tag{2.1.12}$$

In terms of the local complex coordinates z^i and $z^{\bar{i}}$ we have

$$\partial = dz^i \frac{\partial}{\partial z^i}, \qquad \overline{\partial} = dz^{\bar{i}} \frac{\partial}{\partial z^{\bar{i}}}. \tag{2.1.13}$$

A complex manifold is Kähler iff there exists a non-degenerated type $(1,1)$-form ϖ satisfying $d\varpi = 0$. A basic fact of the Kähler geometry is that the Kähler metric tensor $g_{i\bar{j}}$ can be written as

$$g_{i\bar{j}} = \frac{\partial^2 f}{\partial z^i \partial z^{\bar{j}}}, \tag{2.1.14}$$

where f is a Kähler potential. The Kähler form ϖ is given by

$$\varpi = \varpi_{i\bar{j}} dz^i \wedge dz^{\bar{j}} = i g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}, \tag{2.1.15}$$

where $\varpi_{i\bar{j}} = -\varpi_{\bar{j}i}$ while $g_{i\bar{j}} = g_{\bar{j}i}$.

A comparison with our supersymmetric theory leads to the following obvious dictionary

$$\begin{aligned} z^i &\to X^i, & dz^i &\to i\psi^i, \\ z^{\bar{i}} &\to X^{\bar{i}}, & dz^{\bar{i}} &\to i\psi^{\bar{i}}. \end{aligned} \tag{2.1.16}$$

Under the above isomorphism the relations (2.1.12) and (2.1.13) become (2.1.2) and (2.1.5), respectively, such that

$$\partial \to \boldsymbol{s}, \qquad \overline{\partial} \to \overline{\boldsymbol{s}}. \tag{2.1.17}$$

Also the Kähler form ϖ in (2.1.15), after identifying $\mathcal{K}$ with a Kähler potential f of X, i.e., $\mathcal{K}_{i\bar{j}} = g_{i\bar{j}}$, becomes (minus) our action functional S in (2.1.7). Now we examine the partition function Z defined by (2.1.9). It is obvious that, compare with (2.1.10)

$$Z = \int_X e^{\varpi} = \int_X \frac{\varpi^n}{n!} = \int_X \prod_{k,\bar{k}=1}^{n} dz^k dz^{\bar{k}} det(ig_{i\bar{j}}), \tag{2.1.18}$$

where the second identity follows from the fact that the integrand should be a top form and the third identity follows from the definition of ϖ. Thus the partition function of our first supersymmetric field theory is the symplectic volume of X. We remark that the second identity is equivalent to the condition of the ghost number anomaly cancellation.

One may formalize the above correspondence as follows. For the tangent bundle TX we define an associated superspace $\widehat{TX}$ where the hat symbol denotes the parity change of the fiber as in (2.1.16). Then the supercharges $\boldsymbol{s}$ and $\overline{\boldsymbol{s}}$ are odd vectors and the action S is a function on $\widehat{TX}$.

Now we move on to observables and correlation functions. A supersymmetric observable $\widehat{\alpha}$ is a quantity invariant under the symmetry of the theory and annihilated by supercharges. We consider the following polynomial function on $\widehat{TX}$,

$$\widehat{\alpha}^{p,q} = \alpha_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} \psi^{i_1} \ldots \psi^{i_p} \psi^{\bar{j}_1} \ldots \psi^{\bar{j}_p}, \tag{2.1.19}$$

carrying the ghost numbers (p,q). Due to the isomorphism (2.1.16) $\overline{s}\widehat{\alpha}^{p,q}=0$ iff $\overline{\partial}\alpha^{p,q}=0$ where $\alpha^{p,q}\in\Omega^{p,q}(X)$ is the (p,q)-form on X defined by

$$\alpha^{p,q}=\alpha_{i_1\ldots i_p\bar{j}_1\ldots\bar{j}_q}dz^{i_1}\wedge\ldots\wedge dz^{i_p}\wedge dz^{\bar{j}_1}\wedge\ldots\wedge dz^{\bar{j}_p}. \qquad (2.1.20)$$

Note that $\overline{s}$ defines a Dolbeault cohomology on the space of observables graded by the ghost numbers which correspond to the form degrees. In the above we showed that the $\overline{s}$ cohomology is isomorphic to the Dolbeault cohomology $(\overline{\partial},\Omega^{*,*}(X))$ on X.

The correlation function of observables or the expectation value is defined by

$$\left\langle\prod_{m=1}^{r}\widehat{\alpha}^{p_m,q_m}\right\rangle=\int[\mathcal{D}X\mathcal{D}\overline{X}\mathcal{D}\psi\mathcal{D}\overline{\psi}]\prod_{m=1}^{r}\widehat{\alpha}^{p_m,q_m}\cdot e^{-S}. \qquad (2.1.21)$$

For the present model we see that

$$\left\langle\prod_{m=1}^{r}\widehat{\alpha}^{p_m,q_m}\right\rangle=\int_X\alpha^{p_1,q_2}\wedge\ldots\wedge\alpha^{p_r,q_r}\wedge e^{\varpi}. \qquad (2.1.22)$$

Obviously we have non-vanishing correlation function if the observables satisfy the ghost number anomaly cancellation condition

$$\sum_{m=1}^{r}(p_m,q_m)=(\ell,\ell),\qquad \ell\leq n \qquad (2.1.23)$$

Then

$$\left\langle\prod_{m=1}^{r}\widehat{\alpha}^{p_m,q_m}\right\rangle=\frac{1}{(n-\ell)!}\int_X\alpha^{p_1,q_2}\wedge\ldots\wedge\alpha^{p_r,q_r}\wedge\varpi\wedge\ldots\wedge\varpi. \qquad (2.1.24)$$

It follows that correlation functions of supersymmetric observables depend only on the cohomology classes of observables and the Kähler form ϖ.[1] Thus the correlation function computes the classical cohomology ring of the target space X. Equivalently the correlation function computes intersection numbers of homology cycles dual to $\alpha^{p,q}\in H^{p,q}(X)$.

Using our toy model we illustrated many of the basic properties of cohomological field theory. In general, however, life is never as simple as in the idealized world. Typically we encounter an infinite dimensional space of certain set of fields on a manifold M as our target space X. Furthermore there usually exists an infinite dimensional group action on the target space. Nonetheless one is eventually interested in the subspace defined as the solution space of certain first order differential equations, modulo the gauge symmetry. Thus we will

[1] Consider the integral $\int_X\beta e^{\varpi}$ where β is a closed (ℓ,ℓ)-form. Let γ be homology cycle Poincaré dual to ϖ^{ℓ}. Then the integral reduces to $\frac{1}{(n-\ell)!}\int_\gamma\beta$. Let β' belongs to the same cohomology class as β, i.e., $\beta'=\beta+d\alpha$. We have, using Stokes' theorem, $\int_\gamma(\beta'-\beta)=\int_\gamma d\alpha=\int_{\partial\gamma}\alpha=0$.

need a machinery to reduce the path integral to such a subspace and to take care of the group action, as we will do later.

For the time being we ignore those things and assume that the path integral is eventually reduced to some finite dimensional moduli space. Then it may be equivalent to our toy model. We may call a quantum field theory on M with such a property a cohomological field theory. Usually the differential geometrical structures of the moduli space are induced from those of M. Such a field theory on M has global supersymmetry equivalent to $(0+0)$-dimensional supersymmetry. The cohomology of such a global supersymmetry is isomorphic to a certain cohomology of M. Consequently the correlation functions of supersymmetric observables are differential topological invariant of M. We refer to the original paper [1] of Witten for a lucid exposition of general properties of such a cohomological field theory. Here we repeated many of his arguments, perhaps in a slightly different context.

2.2 $N_c = (2,2)$ Model

In this section we consider a somewhat more interesting model by generalizing the toy model of the previous section. We introduce two copies $(\boldsymbol{s}_\pm, \overline{\boldsymbol{s}}_\pm)$ of the fermionic charges $(\boldsymbol{s}, \overline{\boldsymbol{s}})$. We regard the above doubling as a Z_2-grading in the sense that supercharges carry the following ghost numbers (p,q) introduced for the toy model,

$$\begin{aligned} \boldsymbol{s}_+ &: (+1,0), \qquad & \overline{\boldsymbol{s}}_+ &: (0,+1), \\ \boldsymbol{s}_- &: (-1,0), \qquad & \overline{\boldsymbol{s}}_- &: (0,-1). \end{aligned} \tag{2.2.1}$$

Thus the supercharges $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ can be identified with the original supercharges $\boldsymbol{s}$ and $\overline{\boldsymbol{s}}$ of the toy model. We want to define a supersymmetric theory invariant under all four supercharges. Obviously we will have a Z_2-symmetry exchanging the + and − indices. We will say that the resulting theory is of type $N_c = (2,2)$. We will see that such a model is related with the geometry of tangent bundle TX of a Kähler manifold X. The partition function of this model can be identified with the Euler characteristic of X.

2.2.1 Basic Structures

We postulate that the supercharges satisfy the following anti-commutation relations,

$$\boldsymbol{s}_\pm^2 = 0, \qquad \{\boldsymbol{s}_\pm, \overline{\boldsymbol{s}}_\pm\} = 0, \qquad \overline{\boldsymbol{s}}_\pm^2 = 0, \tag{2.2.2}$$

and

$$\{\boldsymbol{s}_+, \boldsymbol{s}_-\} = 0, \qquad \{\boldsymbol{s}_\pm, \overline{\boldsymbol{s}}_\mp\} = 0, \qquad \{\overline{\boldsymbol{s}}_+, \overline{\boldsymbol{s}}_-\} = 0, \tag{2.2.3}$$

which is an obvious generalization of (2.1.2). We will consider the same bosonic fields X^i and $X^{\bar{i}}$ as in our toy model. We demand X^i to be bi-holomorphic or *chiral*, meaning that $\overline{\boldsymbol{s}}_\pm X^i = 0$.[2] We call the complex conjugates $X^{\bar{i}}$ *anti-chiral*,

[2] Note that this choice is arbitrary. We may also demand twisted bi-holomorphicity or *twisted chirality* by imposing $\overline{\boldsymbol{s}}_+ X^i = \boldsymbol{s}_- X^i = 0$. A model with both chiral and twisted chiral multiplets has very interesting properties.

meaning that $s_+ X^{\bar{i}} = s_- X^{\bar{i}} = 0$. Now the anti-commutation relations among supercharges suggest that we have the following chiral multiplets

$$\begin{array}{ccccc} \psi^i_- & \xleftarrow{s_-} & X^i & \xrightarrow{s_+} & \psi^i_+ \\ & {\scriptstyle s_+}\searrow & & \swarrow{\scriptstyle s_-} & \\ & & H^i & & \end{array} \tag{2.2.4}$$

In the above H^i are called auxiliary fields, which are introduced due to the conditions

$$\{s_+, s_-\} X^i = i s_+ \psi^i_- + i s_- \psi^i_+ = 0, \tag{2.2.5}$$

can be solved as $s_\pm \psi^i = \pm H^i$ while they are indeterminate.[3] Denoting $\delta = s_+ \bar{\epsilon}_- + s_- \bar{\epsilon}_+ + \bar{s}_+ \epsilon_- + \bar{s}_- \epsilon_+$ we have the following transformation laws for chiral and anti-chiral multiplets,

$$\begin{aligned} \delta X^i &= i\bar{\epsilon}_- \psi^i_+ + i\bar{\epsilon}_+ \psi^i_-, & \delta X^{\bar{i}} &= i\epsilon_- \psi^{\bar{i}}_+ + i\epsilon_+ \psi^{\bar{i}}_-, \\ \delta \psi^i_+ &= +\bar{\epsilon}_+ H^i, & \delta \psi^{\bar{i}}_+ &= +\epsilon_+ H^{\bar{i}}, \\ \delta \psi^i_- &= -\bar{\epsilon}_- H^i, & \delta \psi^{\bar{i}}_- &= -\epsilon_- H^{\bar{i}}, \\ \delta H^i &= 0 & \delta H^{\bar{i}} &= 0. \end{aligned} \tag{2.2.6}$$

Now we define a natural supersymmetric action functional. The requirements $s_\pm S = \bar{s}_\pm S = 0$ for S to have $N_c = (2,2)$ supersymmetry and the anti-commutation relations (2.2.2) and (2.2.3) imply, by repeatedly applying the Poincaré lemma, that we can write S as follows,

$$S = s_+ \bar{s}_+ s_- \bar{s}_- \mathcal{K}(X^i, X^{\bar{i}}), \tag{2.2.7}$$

where $\mathcal{K}(X^i, X^{\bar{i}})$ is a locally defined real functional. Expanding the above we have

$$S = g_{i\bar{j}} H^i H^{\bar{j}} + i\partial_k g_{i\bar{j}}\, \psi^k_+ \psi^i_- H^{\bar{j}} + i\partial_{\bar{k}} g_{i\bar{j}}\, \psi^{\bar{k}}_+ H^i \psi^{\bar{j}}_- + \partial_\ell \partial_{\bar{k}} g_{i\bar{j}}\, \psi^\ell_+ \psi^{\bar{k}}_+ \psi^i_- \psi^{\bar{j}}_-, \tag{2.2.8}$$

where $\partial_i = \partial/\partial X^i$ and $\partial_{\bar{j}} = \partial/\partial X^{\bar{j}}$ and we set $g_{i\bar{j}} := \partial_i \partial_{\bar{j}} \mathcal{K}$. We can integrate out the auxiliary fields H^i and $H^{\bar{i}}$ by a Gaussian integral, or, equivalently, eliminate them by plugging in the algebraic equations of motions for H^i and $H^{\bar{i}}$;

$$\begin{aligned} H^i &= -i g^{i\bar{j}} \partial_k g_{\ell\bar{j}}\, \psi^k_+ \psi^\ell_-, \\ H^{\bar{j}} &= +i g^{i\bar{j}} \partial_{\bar{k}} g_{i\bar{\ell}}\, \psi^{\bar{k}}_+ \psi^{\bar{\ell}}_-, \end{aligned} \tag{2.2.9}$$

where $g^{i\bar{j}}$ is the inverse of $g_{i\bar{j}}$. Then we obtain the new action functional S',

$$S' = -R_{\ell\bar{k}i\bar{j}}\, \psi^\ell_+ \psi^{\bar{k}}_+ \psi^i_- \psi^{\bar{j}}_-, \tag{2.2.10}$$

[3] The equation might also be solved as $s_\pm \psi^i = 0$ without introducing H^i. However, the auxiliary fields are indispensable. The moral is that we better keep it whenever we encounter redundancy.

where

$$R_{\ell\bar{k}i\bar{j}} = -\partial_\ell \partial_{\bar{k}} g_{i\bar{j}} + g^{p\bar{q}} \partial_i g_{p\bar{k}} \partial_{\bar{j}} g_{\ell\bar{q}}, \tag{2.2.11}$$

which can be identified with the the Riemann curvature tensor of TX if $\mathcal{K}$ is a Kähler potential of X. Remark that the non-vanishing components of the Christoffel symbols in the Kähler geometry are

$$\Gamma^i_{k\ell} = g^{i\bar{j}} \partial_k g_{\ell\bar{j}}, \qquad \Gamma^{\bar{i}}_{\bar{k}\bar{\ell}} = g^{i\bar{j}} \partial_{\bar{k}} g_{i\bar{\ell}}. \tag{2.2.12}$$

The new action S' is invariant under the supersymmetry after modifying the transformation laws (2.2.6) by replacing H^i and $H^{\bar{i}}$ by their on-shell expressions (2.2.9).

Now we examine the path integral. The partition function is defined as usual,

$$\begin{aligned} Z &= \int [\mathcal{D}X \mathcal{D}\overline{X} \mathcal{D}\psi_\pm \mathcal{D}\overline{\psi}_\pm] e^{-S'}, \\ &= \left(\frac{1}{2\pi}\right)^n \int \prod_{k,\bar{k}=1}^{n} dX^k dX^{\bar{k}} d\psi^k_+ d\psi^{\bar{k}}_+ d\psi^k_- d\psi^{\bar{k}}_- \exp\left(R_{\ell\bar{k}i\bar{j}}\, \psi^\ell_+ \psi^{\bar{k}}_+ \psi^i_- \psi^{\bar{j}}_-\right), \end{aligned} \tag{2.2.13}$$

where the integration is over the space of all fields. The bosonic part of the path integral is an integration over a copy of X. We first perform the integral over ψ^i_+ and $\psi^{\bar{i}}_+$ which, as we saw earlier, is equivalent to replacing $R_{\ell\bar{k}i\bar{j}}\ \psi^\ell_+ \psi^{\bar{k}}_+ \psi^i_- \psi^{\bar{j}}_-$ by the (1,1)-form $\boldsymbol{R}_{i\bar{j}} := R_{k\bar{\ell}i\bar{j}} dz^k \wedge d\bar{z}^{\bar{\ell}}$ on X,

$$Z = \left(\frac{1}{2\pi}\right)^n \int_X \prod_{k,\bar{k}=1}^{n} d\psi^k_- d\psi^{\bar{k}}_- \exp\left(\boldsymbol{R}_{i\bar{j}}\ \psi^i_- \psi^{\bar{j}}_-\right). \tag{2.2.14}$$

Integration over ψ^i_- and $\psi^{\bar{i}}$ leads to

$$Z = \frac{1}{(2\pi)^n} \int_X det(\boldsymbol{R}_{i\bar{j}}) = \int_X e(TX) := \chi(TX). \tag{2.2.15}$$

The last identity is due to the Gauss-Bonnet theorem. Thus the partition function computes the Euler characteristic $\chi(X) = \chi(TX)$ of the manifold X.

2.2.2 Geometrical Interpretation of Supercharges

Now we examine the geometrical meaning of our supercharges. In Sect. 2.1.1 we already saw that the supercharges $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ are associated with the ∂ and $\overline{\partial}$ differential on the target space X. Our task is to understand the geometrical meaning of the remaining supercharges $\boldsymbol{s}_-$ and $\overline{\boldsymbol{s}}_-$.

We begin with discarding the obvious candidates for $\boldsymbol{s}_-$ and $\overline{\boldsymbol{s}}_-$, namely the operators ∂^* and $\overline{\partial}^*$ defined by

$$\begin{aligned} \partial^* &= -*\overline{\partial}* : \Omega^{p,q}(X) \to \Omega^{p-1,q}(X), \\ \overline{\partial}^* &= -*\partial* : \Omega^{p,q}(X) \to \Omega^{p,q-1}(X), \end{aligned} \tag{2.2.16}$$

where $*$ denote the Hodge star. They satisfy the following relations

$$\partial^{*2} = 0, \qquad \{\partial^*, \overline{\partial}^*\} = 0, \qquad \overline{\partial}^{*2} = 0, \tag{2.2.17}$$

and decrease the form degree by $(-1,0)$ and $(0,-1)$, respectively. We have, however, well-known relations in Kähler geometry

$$\{\partial, \partial^*\} = \{\overline{\partial}, \overline{\partial}^*\} = \frac{1}{2}\{d, d^*\} = \frac{1}{2}\nabla, \tag{2.2.18}$$

where ∇ is the Laplacian. On the other hand we have $\{s_+, s_-\} = \{\overline{s}_+, \overline{s}_-\} = 0$. We also have more obvious problem from $\partial^* X^i = 0$, while $s_- X^i = i\psi^i_- \neq 0$. Thus we have to seek an alternative set of operators.

We first consider the real symplectic case and then specialize to the Kähler case. Consider a symplectic manifold with symplectic form $\varpi = \varpi_{IJ} dx^I \wedge dx^J$. Since the matrix $\varpi_{IJ} = -\varpi_{JI}$ is non-degenerated we have a well-defined inverse matrix ϖ^{JI}. Using ϖ^{JI} we have a canonical map from a cotangent vector to a tangent vector.[4] Denoting $\alpha = \alpha_I dx^I$ and $\widetilde{\alpha} = \widetilde{\alpha}^I \frac{\partial}{\partial x^I}$ for a cotangent vector and its dual tangent vector, respectively, we have

$$\widetilde{\alpha}^I = \varpi^{IJ}\alpha_J. \tag{2.2.19}$$

One may define the corresponding operator $\sqcap$ as follows

$$\sqcap := \frac{\varpi^{IJ}}{2}\left(\left(\otimes\frac{\partial}{\partial x^I}\right)\frac{\partial}{\partial(dx^J)} - \left(\otimes\frac{\partial}{\partial x^J}\right)\frac{\partial}{\partial(dx^I)}\right), \tag{2.2.20}$$

where the symbol $\otimes\frac{\partial}{\partial x^I}$ means taking tensor product. For instance we have

$$\sqcap\alpha = \varpi^{IJ}\alpha_L\left(\otimes\frac{\partial}{\partial x^I}\right)\frac{\partial(dx^L)}{\partial(dx^J)} = \varpi^{IJ}\alpha_J\frac{\partial}{\partial x^I} = \widetilde{\alpha}^I\frac{\partial}{\partial x^I} = \widetilde{\alpha}. \tag{2.2.21}$$

Similarly $\sqcap$ induce an isomorphism

$$\sqcap : \Gamma(\wedge^p T^*X \otimes \wedge^q TX) \to \Gamma(\wedge^{p-1} T^*X \otimes \wedge^{q+1} TX), \tag{2.2.22}$$

where TX and T^*X are the tangent and cotangent vector, respectively, and Γ denotes the space of sections. Note that $\Gamma(\wedge^p T^*X \otimes \wedge^q TX) = \Omega^p(X, \wedge^q TX)$.

Now we can define a first order differential operator by taking the composition of $\sqcap$ and the exterior derivative d,

$$d : \Omega(\wedge^p T^*X \otimes \wedge^q TX) \to \Omega(\wedge^{p+1} T^*X \otimes \wedge^q TX), \tag{2.2.23}$$

as follows,

$$\widetilde{d} := (\sqcap d - d\sqcap) : \Omega(\wedge^p T^*X \otimes \wedge^q TX) \to \Omega(\wedge^p T^*X \otimes \wedge^{q+1} TX). \tag{2.2.24}$$

We will conveniently assign the form degree -1 to the operator $\widetilde{d}$. One can check

$$\widetilde{d}^2 = 0, \qquad \{d, \widetilde{d}\} = 0, \tag{2.2.25}$$

[4] We may also consider a Poisson manifold with a bi-vector ϖ^{IJ}.

after a direct computations. We also have the following obvious but important relation

$$d : x^I \to dx^I,$$
$$\widetilde{d} : x^I \to \varpi^{IJ}\frac{\partial}{\partial x^I}. \tag{2.2.26}$$

Thus for a symplectic manifold X with the symplectic form ϖ we have

$$(x^I, dx^I, \frac{\partial}{\partial x^I}; d, \widetilde{d}),$$

where dx^I and ∂/dx^I denote local coordinates in the fiber of TX and the fiber of T^*X, respectively. To relate with supersymmetry we perform the parity changes for both the fibers of TX and T^*X, i.e., $\widehat{TX}$ and $\widehat{T^*X}$. Then we have a map

$$(x^I, dx^I, \partial/\partial x^I; d, \widetilde{d}) \to (X^I, i\psi^I_+, i\chi_I; Q_+, Q_-), \tag{2.2.27}$$

where everything is in real coordinates, $\psi^I_- := \varpi^{IJ}\chi_J$ and $Q_\pm = s_\pm + \overline{s}_\pm$.

One may compare our operator $\widetilde{d}$ with the (different) operator Δ defined by Koszul [45]. The operator Δ is define as

$$\Delta := \sqcap_k d - d\sqcap_k, \tag{2.2.28}$$

where $\sqcap_k$ in the notation of (2.2.20) is given by

$$\sqcap_k := \frac{\varpi^{IJ}}{2}\left(\frac{\partial^2}{\partial(dx^I)\partial(dx^J)} - \frac{\partial^2}{\partial(dx^J)\partial(dx^I)}\right). \tag{2.2.29}$$

Thus Δ is a second order differential operator with degree -1 on $\Gamma(\wedge^* T^*X) = \Omega^*(X)$ and we have $\Delta x^I = 0$.[5]

Now we return to a Kähler manifold X with Kähler form $\varpi = \varpi_{i\bar{j}}dz^i \wedge dz^{\bar{j}}$ and show that the above interpretation is indeed the correct one. It is suffice to consider the holomorphic half, say s_+ and s_-. The operator $\sqcap$ is decomposed as $\sqcap = \sqcap' + \sqcap''$ where

$$\widehat{\sqcap}' = -\frac{1}{2}\varpi^{i\bar{j}}(X^\ell, X^{\bar{\ell}})\chi_{\bar{j}}\frac{\partial}{\partial\psi^i_+},$$
$$\widehat{\sqcap}'' = +\frac{1}{2}\varpi^{i\bar{j}}(X^\ell, X^{\bar{\ell}})\chi_i\frac{\partial}{\partial\psi^{\bar{j}}_+}, \tag{2.2.30}$$

where we did parity change $\sqcap \to \widehat{\sqcap}$ by

$$dz^i \to i\psi^i_+, \qquad \partial/\partial z^i \to i\chi_i,$$
$$dz^{\bar{i}} \to i\psi^{\bar{i}}_+, \qquad \partial/\partial z^{\bar{i}} \to i\chi_i. \tag{2.2.31}$$

[5] Koszul proved $\Delta^2 = \{d, \Delta\} = 0$ and defined a covariant Schouten-Nijenhuis bracket, $\alpha, \beta \in \Omega^*(X)$,

$$\{\alpha, \beta\}_{SN} = (\Delta\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \Delta\alpha - \Delta(\alpha \wedge \beta).$$

Now we define

$$s_- = \widehat{\sqcap}' s_+ - s_+ \widehat{\sqcap}'. \tag{2.2.32}$$

From

$$s_+ = i\psi_+^i \frac{\partial}{\partial X^i}, \tag{2.2.33}$$

we have

$$s_- = -\frac{i}{2}\varpi^{i\bar{j}}\chi_{\bar{j}}\frac{\partial}{\partial X^i} + \frac{i}{2}\frac{\partial \varpi^{i\bar{j}}}{\partial X^k}\psi_+^k \chi_{\bar{j}}\frac{\partial}{\partial \psi_+^i}. \tag{2.2.34}$$

Now we can check if the above identification of the supercharge s_- is the correct one. After direct computations we find the following relations

$$\begin{aligned}
& && s_+\psi_+^i = 0,\\
& s_+X^i = i\psi_+^i, && \\
& s_+X^{\bar{i}} = 0, && s_+\psi_-^i = +i\varpi_{\bar{j}\ell}\frac{\partial \varpi^{i\bar{j}}}{\partial X^k}\psi_+^k\psi_-^\ell,\\
& s_-X^{\bar{i}} = 0, && s_-\psi_+^i = -i\varpi_{\bar{j}\ell}\frac{\partial \varpi^{i\bar{j}}}{\partial X^k}\psi_+^k\psi_-^\ell,\\
& s_-X^i = i\psi_-^i, && \\
& && s_-\psi_-^i = 0,
\end{aligned} \tag{2.2.35}$$

where we defined

$$\psi_-^i = -\frac{1}{2}\varpi^{i\bar{j}}\chi_{\bar{j}}. \tag{2.2.36}$$

In checking $s_-\psi_-^i = 0$ we used the torsion-free condition of the Hermitian connection of TX, equivalent to the condition $d\varpi = 0$. Using the relation

$$\varpi_{i\bar{j}} = ig_{i\bar{j}} = -\varpi_{\bar{j}i} \tag{2.2.37}$$

we see that the above is exactly the supersymmetry algebra of s_+ and s_- in (2.2.6) after replacing the auxiliary fields H^i by their on-shell values given by (2.2.9).

Now we summarize. We have the following operators

$$\begin{aligned}
s_+ &= i\psi_+^i\frac{\partial}{\partial X^i}, & s_- &= \widehat{\sqcap}' s_+ - s_+\widehat{\sqcap}',\\
\bar{s}_+ &= i\psi_+^{\bar{i}}\frac{\partial}{\partial X^{\bar{i}}}, & \bar{s}_- &= \widehat{\sqcap}'' \bar{s}_+ - \bar{s}_+\widehat{\sqcap}'',
\end{aligned} \tag{2.2.38}$$

such that

$$\begin{aligned}
s_+ &: \widehat{\Omega}^{p,q}\left(\wedge^r \widehat{\mathcal{T}X} \otimes \wedge^s \widehat{\overline{\mathcal{T}}X}\right) \to \widehat{\Omega}^{p+1,q}\left(\wedge^r \widehat{\mathcal{T}X} \otimes \wedge^s \widehat{\overline{\mathcal{T}}X}\right),\\
\bar{s}_+ &: \widehat{\Omega}^{p,q}\left(\wedge^r \widehat{\mathcal{T}X} \otimes \wedge^s \widehat{\overline{\mathcal{T}}X}\right) \to \widehat{\Omega}^{p,q+1}\left(\wedge^r \widehat{\mathcal{T}X} \otimes \wedge^s \widehat{\overline{\mathcal{T}}X}\right),\\
s_- &: \widehat{\Omega}^{p,q}\left(\wedge^r \widehat{\mathcal{T}X} \otimes \wedge^s \widehat{\overline{\mathcal{T}}X}\right) \to \widehat{\Omega}^{p,q}\left(\wedge^{r+1} \widehat{\mathcal{T}X} \otimes \wedge^s \widehat{\overline{\mathcal{T}}X}\right),\\
\bar{s}_- &: \widehat{\Omega}^{p,q}\left(\wedge^r \widehat{\mathcal{T}X} \otimes \wedge^s \widehat{\overline{\mathcal{T}}X}\right) \to \widehat{\Omega}^{p,q}\left(\wedge^r \widehat{\mathcal{T}X} \otimes \wedge^{s+1} \widehat{\overline{\mathcal{T}}X}\right),
\end{aligned} \tag{2.2.39}$$

where $\mathcal{T}X$ denotes the holomorphic parts of the tangent bundle $TX = \mathcal{T}X \oplus \overline{\mathcal{T}}X$ of X.

2.2.3 Introducing a Holomorphic Potential

Now we consider a more general action functional. We pick a holomorphic function $\mathcal{W}(X^i)$ of the chiral fields X^i. Since $\overline{s}_\pm X^i = 0$ we have $\overline{s}_\pm \mathcal{W}(X^i) = 0$. It follows that we have the following more general $N = (2,2)$ supersymmetric action functional,

$$S(\lambda) = s_+\overline{s}_+s_-\overline{s}_-\mathcal{K}(X^i, X^{\bar{i}}) + \lambda s_+s_-\mathcal{W}(X^i) + \lambda\overline{s}_+\overline{s}_-\overline{\mathcal{W}}(X^i), \tag{2.2.40}$$

where λ is certain coupling constant introduced for convenience. Expanding $S(\lambda)$ we find

$$\begin{aligned} S(\lambda) =& g_{i\bar{j}}H^iH^{\bar{j}} + i\left(\partial_k g_{i\bar{j}}\psi_+^k\psi_-^i - \lambda V_{\bar{j}}\right)H^{\bar{j}} + iH^i\left(\partial_{\bar{k}}g_{i\bar{j}}\psi_+^{\bar{k}}\psi_-^{\bar{j}} - \lambda V_i\right) \\ &- \lambda\frac{\partial V_i}{\partial X^j}\psi_+^i\psi_-^j - \lambda\frac{\partial V_{\bar{i}}}{\partial X^{\bar{j}}}\psi_+^{\bar{i}}\psi_-^{\bar{j}} + \partial_\ell\partial_{\bar{k}}g_{i\bar{j}}\psi_+^\ell\psi_+^{\bar{k}}\psi_-^i\psi_-^{\bar{j}}. \end{aligned} \tag{2.2.41}$$

where we set $V_i := \partial\mathcal{W}/\partial X^i$. Now we integrate out the auxiliary fields by their algebraic equations of motions

$$\begin{aligned} H^i &= -i\Gamma^i_{k\ell}\psi_+^k\psi_-^\ell + i\lambda g^{i\bar{j}}V_{\bar{j}}, \\ H^{\bar{i}} &= +i\Gamma^{\bar{i}}_{\bar{k}\bar{\ell}}\psi_+^{\bar{k}}\psi_-^{\bar{\ell}} - i\lambda g^{j\bar{i}}V_j, \end{aligned} \tag{2.2.42}$$

where we used the notations in (2.2.12). We have

$$S'(\lambda) = \lambda^2 g^{i\bar{j}}V_iV_{\bar{j}} - \lambda\frac{DV_i}{DX^j}\psi_+^i\psi_-^j - \lambda\frac{DV_{\bar{i}}}{DX^{\bar{j}}}\psi_+^{\bar{i}}\psi_-^{\bar{j}} - R_{\ell\bar{k}i\bar{j}}\psi_+^\ell\psi_+^{\bar{k}}\psi_-^i\psi_-^{\bar{j}}. \tag{2.2.43}$$

where

$$\frac{DV_i}{DX^j} := \frac{\partial V_i}{\partial X^j} + \Gamma^\ell_{ij}V_\ell. \tag{2.2.44}$$

The Partition Function

The partition function is independent of λ since λ dependent term is $s_\pm$-exact deformation of S. In the limit $\lambda \to \infty$ the dominant contributions to the path integral are from the vanishing locus of holomorphic vector fields V_i. Or we may simply apply the fixed point theorem of Witten to reach the same conclusion; from the supersymmetry transformation laws (2.2.6) we see that the fixed point equations are $\psi_\pm^{\bar{j}} = H^{\bar{i}} = 0$. From the relations (2.2.17) the above implies $V_i = 0$.

For generic choices the vanishing locus will be zero dimensional and consists of isolated points. Then there are no fermionic zero-modes and the action functional evaluated at such a point is simply 0. Thus the partition function is just the sum of contributions of each point weighted by the one loop determinants of the transverse degrees of freedom. Due to the Bose-Fermi symmetry such a determinant is ± 1, depending on a certain orientation, due to supersymmetry and due to the ambiguity in taking the square root of the determinant. In our case they always can be set $+1$ since the ambiguities from holomorphic and anti-holomorphic contribution cancel each other. Thus the partition function is the number of zeros. If we turn off the potential we recover the original model. This gives rise to the Poincaré-Hopf theorem. We should mention that the usual derivation of Poincaré-Hopf theorem uses supersymmetric quantum mechanics, i.e., the $(0+1)$ dimensional sigma model [46][47][34], but with essentially the same arguments.

For a non-generic vector field V_i the vanishing locus can be a positive dimensional submanifold. One may try to perturb the vector field V_i, thus $\mathcal{W}(X^i)$, to a generic one or just evaluate the path integral. We will give a detailed analysis for this case in the next section in a more general context.

2.3 Generalization to $N_c = (2,0)$ Model

The model in the previous section enjoys a perfect symmetry between things with + and − indices. Now we want to relax such a symmetry. We shall see that such symmetry is due to the restriction of considering a very special Hermitian holomorphic vector bundle, namely the tangent bundle TX, over X. By maintaining only the $N_c = (2,0)$ supersymmetry generated by $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ we arrive at a more general model, which is related with a Hermitian holomorphic bundle $\mathbb{E}$ over X.

2.3.1 Basic Structures

First we write our action functional $S(\lambda)$ (2.2.40) in form such that only the $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ are manifest,

$$S(\lambda) = -\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\Big(g_{i\bar{j}}(X^i, X^{\bar{i}})\psi_-^i\psi_-^{\bar{j}}\Big) + i\lambda\boldsymbol{s}_+\big(\psi_-^i V_i(X^j)\big) + i\lambda\overline{\boldsymbol{s}}_+\Big(\psi_-^{\bar{i}} V_{\bar{i}}(X^{\bar{j}})\Big). \tag{2.3.1}$$

Similarly we disconnect the diagram (2.2.4) by removing the link $\boldsymbol{s}_-$

$$\begin{array}{ccccc} \psi_-^i & & X^i & \xrightarrow{\boldsymbol{s}_+} & \psi_+^i \\ & {\scriptstyle s_+}\searrow & & & \\ & & H^i & & \end{array} . \tag{2.3.2}$$

Now we can regard the above as two independent sets of multiplets. Then we rename various fields as follows

$$\begin{array}{llll} \psi_-^i \to \chi_-^\alpha, & H^i \to H^\alpha, & V_i \to \mathfrak{S}_\alpha(X^j), & \\ & & & g_{i\bar{j}} \to h_{\alpha\bar{\beta}}(X^i, X^{\bar{i}}), \\ \psi_-^{\bar{i}} \to \chi_-^{\bar{\alpha}}, & H^{\bar{i}} \to H^{\bar{\alpha}}, & V_{\bar{i}} \to \mathfrak{S}_{\bar{\alpha}}(X^{\bar{j}}), & \end{array} \tag{2.3.3}$$

where the new indices run as $\alpha, \beta = 1, \ldots, r$ and we maintain the Hermiticity of $h_{\alpha\bar{\beta}}$. The $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ transformation laws are

$$\begin{aligned} \delta X^i &= i\bar{\epsilon}_-\psi_+^i, & \delta\psi_+^i &= 0, \\ \delta X^{\bar{i}} &= i\epsilon_-\psi_+^{\bar{i}}, & \delta\psi_+^{\bar{i}} &= 0, \end{aligned} \tag{2.3.4}$$

and

$$\begin{aligned}\delta\chi_-^\alpha &= -\bar{\epsilon}_- H^\alpha, \qquad \delta H^\alpha = 0,\\ \delta\chi_-^{\overline{\alpha}} &= -\epsilon_- H^{\overline{\alpha}}, \qquad \delta H^{\overline{\alpha}} = 0.\end{aligned} \tag{2.3.5}$$

Now we have following new action functional

$$S = -\, s_+\bar{s}_+\left(h_{\alpha\overline{\beta}}(X^i, X^{\bar{i}})\chi_-^\alpha\chi_-^{\overline{\beta}}\right) + is_+\left(\chi_-^\alpha\mathfrak{S}_\alpha(X^i)\right) + i\bar{s}_+\left(\chi_-^{\overline{\alpha}}\mathfrak{S}_{\overline{\alpha}}(X^{\bar{i}})\right), \tag{2.3.6}$$

which is the general form of $N_c = (2,0)$ supersymmetric action functional.

Note that the above action functional may or may not have $N_c = (2,2)$ symmetry. Generically the model does not have $N_c = (2,2)$ supersymmetry. Note also that the model has the same supersymmetry as our toy model in Sect. 2.1.1. Thus the new model shares the same observables with the toy models, which are $\widehat{\alpha}^{p,q}$ obtained by an element $\alpha^{p,q} = H^{p,q}(X)$ of the cohomology group $H^{p,q}(X)$ after the parity change $TX \to \widehat{TX}$. The differences with the toy model are that we have additional Fermi multiplets $(\chi_-^\alpha, H^\alpha)$ with a different action functional. We call the multiplets $(\chi_-^\alpha, H^\alpha)$ Fermi multiplets. We call χ_-^α anti-ghosts. We remark that the action functional of the toy model may be regarded as zero by treating the Kähler form ϖ as an observables. Now we turn to examine the action functional.

Expanding S we have

$$\begin{aligned}S =& h_{\alpha\overline{\beta}}H^\alpha H^{\overline{\beta}} + i\left(\partial_i h_{\alpha\overline{\beta}}\psi_-^i\chi_+^\alpha - \mathfrak{S}_{\overline{\beta}}\right)H^{\overline{\beta}} + iH^\alpha\left(\partial_{\bar{j}}h_{\alpha\overline{\beta}}\psi_+^{\bar{j}}\chi_-^{\overline{\beta}} - \mathfrak{S}_\alpha\right)\\ &- \frac{\partial\mathfrak{S}_\alpha}{\partial X^j}\psi_+^j\chi_-^\alpha - \frac{\partial\mathfrak{S}_{\overline{\alpha}}}{\partial X^{\bar{j}}}\psi_+^{\bar{j}}\chi_-^{\overline{\alpha}} + (\partial_i\partial_{\bar{j}}h_{\alpha\overline{\beta}})\psi_+^i\psi_+^{\bar{j}}\chi_-^\alpha\chi_-^{\overline{\beta}}.\end{aligned} \tag{2.3.7}$$

After integrating out the auxiliary fields H^α and $H^{\overline{\beta}}$ by their algebraic equations of motion

$$\begin{aligned}H^\alpha &= -ih^{\alpha\overline{\beta}}\partial_k h_{\gamma\overline{\beta}}\psi_+^k\chi_-^\gamma + ih^{\alpha\overline{\beta}}\mathfrak{S}_{\overline{\beta}},\\ H^{\overline{\alpha}} &= +ih^{\alpha\overline{\beta}}\partial_{\bar{k}}h_{\alpha\bar{\gamma}}\psi_+^{\bar{k}}\chi_-^{\bar{\gamma}} - ih^{\beta\overline{\alpha}}\mathfrak{S}_\beta,\end{aligned} \tag{2.3.8}$$

we are left with

$$S' = h^{\alpha\overline{\beta}}\mathfrak{S}_\alpha\mathfrak{S}_{\overline{\beta}} - \frac{D\mathfrak{S}_\alpha}{DX^j}\psi_+^j\chi_-^\alpha - \frac{D\mathfrak{S}_{\overline{\alpha}}}{DX^{\bar{j}}}\psi_+^{\bar{j}}\chi_-^{\overline{\alpha}} - F_{\alpha\overline{\beta}i\bar{j}}\psi_+^i\psi_+^{\bar{j}}\chi_-^\alpha\chi_-^{\overline{\beta}}, \tag{2.3.9}$$

where

$$F_{\alpha\overline{\beta}i\bar{j}} = -\partial_{\bar{j}}\partial_i h_{\alpha\overline{\beta}} + h^{\gamma\bar{\rho}}(\partial_i h_{\alpha\bar{\rho}})(\partial_{\bar{j}}h_{\gamma\overline{\beta}}) \tag{2.3.10}$$

and

$$\frac{D\mathfrak{S}_\alpha}{DX^j} = \partial_j\mathfrak{S}_\alpha + h^{\beta\bar{\gamma}}(\partial_j h_{\alpha\bar{\gamma}})\mathfrak{S}_\beta. \tag{2.3.11}$$

Relations with Hermitian Holomorphic Vector Bundle

It turns out that we are describing a rank r Hermitian holomorphic vector bundle $\mathbb{E} \to X$ over a Kähler manifold X with Hermitian structure $h_{\alpha\overline{\beta}}$. Here we briefly summarize some properties of Hermitian holomorphic bundles [48].

Consider a rank r complex vector bundle $\mathbb{E}$ over X. Let $\Omega^{p,q}(X,\mathbb{E})$ denote the space of (p,q)-forms over X with values in $\mathbb{E}$. A connection (the covariant derivative) d_A can be decomposed as

$$d_A = \partial_A + \overline{\partial}_A : \Omega^{p,q}(X,\mathbb{E}) \to \Omega^{p+1,q}(X,E) \oplus \Omega^{p,q+1}(X,\mathbb{E}). \tag{2.3.12}$$

A connection d_A endows $\mathbb{E}$ with a structure of a holomorphic vector bundle if the $(0,2)$-component $F^{0,2} \in \Omega^2(X, End(\mathbb{E}))$ of its curvature F vanishes, i.e., $\overline{\partial}_A^2 = 0$. A complex vector bundle $\mathbb{E}$ is *Hermitian* if it has a fixed Hermitian structure h which is a C^∞ field of positive definite Hermitian inner products in the fibers of $\mathbb{E}$. Given a local frame field $s_U = (s_1, \ldots, s_r)$ of $\mathbb{E}$ over an open subset $U \subset X$ we set $h_{\alpha\overline{\beta}} = h(s_\alpha, s_\beta)$ where $\alpha, \beta = 1, \ldots, r$. Gluing them along different coordinate patches as usual we obtain $h_{\alpha\overline{\beta}}(z^i, z^{\bar{i}})$. A connection D in $(\mathbb{E}, h)$ is called an *h-connection* if $d(h(\xi,\eta)) = h(D\xi,\eta) + h(\xi, D\eta)$ for $\xi, \eta \in \Omega^0(\mathbb{E})$. The theorem is that given a Hermitian structure h in a holomorphic vector bundle $\mathbb{E}$, there is a unique h-connection d_A called *Hermitian connection* such that $\overline{\partial}_A = \overline{\partial}$. Finally the curvature two-form of a Hermitian connection is of type $(1,1)$, thus $F^{2,0}$ also vanishes. The curvature two-form is given by the formula

$$F_{\alpha\overline{\beta}} := F_{\alpha\overline{\beta}i\overline{j}} dz^i \wedge dz^{\bar{j}}, \tag{2.3.13}$$

where $F_{\alpha\overline{\beta}i\overline{j}}$ is defined as (2.3.10). We note that the Kähler metric $g_{i\bar{j}}$ on X is a Hermitian structure of TX.

We saw that our model describes a rank r Hermitian holomorphic vector bundle $\mathbb{E}$ with Hermitian structure $h_{\alpha\overline{\beta}}(z^i, z^{\bar{i}})$. Now $\mathfrak{S}_\alpha$ can be identified with a holomorphic section of $\mathbb{E}$. In summary a $N_c = (2,0)$ model is associated with a Hermitian holomorphic vector bundle $(\mathbb{E}, h)$ over a Kähler manifold X with holomorphic section. Associated with the base manifold X we have holomorphic multiplets (2.3.4), as in the toy model. Associated with the fiber space we have Fermi multiplets (2.3.5).

2.3.2 Path Integrals

Now we examine the path integral of our model in the various situations.

Turning Off the Holomorphic Section

To begin with we consider the case that $\mathfrak{S}_\alpha = 0$. The partition function Z is defined by

$$Z = \int \Big[\prod_{k,\overline{k}=1}^{n} \left(dX^k dX^{\overline{k}} d\psi_+^k d\psi_+^{\overline{k}} \right) \prod_{\gamma,\overline{\gamma}=1}^{r} \left(d\chi_-^\gamma d\chi_-^{\overline{\gamma}} \right) \Big] exp\left(F_{\alpha\overline{\beta}i\overline{j}} \psi_+^i \psi_+^{\bar{j}} \chi_-^\alpha \chi_-^{\overline{\beta}} \right). \tag{2.3.14}$$

The bosonic integral is an integral over X. As before the bosonic integral and integration over ψ_+^k and $\psi_+^{\overline{k}}$ combine into the integration of differential forms on

X by replacing $F_{\alpha\overline{\beta}i\overline{j}}\psi_+^i\psi_+^{\overline{j}}$ with the curvature two-form $F_{\alpha\overline{\beta}}$ defined in (2.3.13). Thus we have

$$Z = \int_X \prod_{\gamma,\overline{\gamma}=1}^{r} \left(d\chi_-^\gamma d\chi_-^{\overline{\gamma}}\right) exp\left(F_{\alpha\overline{\beta}}\chi_-^\alpha\chi_-^{\overline{\beta}}\right). \tag{2.3.15}$$

The fermionic integral of χ^α and $\chi^{\overline{\beta}}$ leads to the Pfaffian of the curvature two-form $F_{\alpha\overline{\beta}} \in \Omega^{1,1}(X, End(\mathbb{E})$. We immediately see that the integrand is not a top form on X unless $n = r$. For $n = r$ the partition function is the Euler character $\chi(\mathbb{E})$,

$$Z = \int_X e(\mathbb{E}) = \chi(\mathbb{E}), \tag{2.3.16}$$

otherwise, for $n \neq r$, the path integral vanishes. In the case $r < n$ we can insert a set of observables $\prod \widehat{\alpha}^{p_\ell,q_\ell}$ with the total ghost number $(n-r, n-r)$ and evaluate the correlation function

$$\left\langle \prod_{\ell=1}^{m} \widehat{\alpha}^{p_\ell,q_\ell} \right\rangle = \int_X e(\mathbb{E}) \wedge \alpha^{p_1,q_1} \wedge \ldots \wedge \alpha^{p_m,q_m} \tag{2.3.17}$$

The path integral always vanishes for $r > n$. We see that the vector bundle $\mathbb{E}$ after the parity change can be viewed as a bundle spanned by anti-ghosts χ_-^α over X.

Turning On the Holomorphic Section

Now we turn on the holomorphic section $\mathfrak{S}_\alpha$ of $E \to X$. Applying the fixed point theorem of Witten we see that the path integral is localized to an s_+ and $\overline{s}_+$ invariant neighborhood of the vanishing locus N of $\mathfrak{S}_\alpha(X^i)$ in X, where $\alpha = 1, \ldots, r$ and $i = 1, \ldots, n$. The condition $\mathfrak{S}_\alpha(X^i) = 0$ implies $s_+(S)_\alpha = 0$ in the s_+ invariant neighborhood of N. We have

$$\partial_j\mathfrak{S}_\alpha\psi_+^j = 0. \tag{2.3.18}$$

We call a non-trivial solution above a zero-modes of ψ_+, which is a degree of freedom tangent to the vanishing locus N. We call a non-trivial solution of the similar equations

$$\partial_j\mathfrak{S}_\alpha\chi_-^\alpha = 0 \tag{2.3.19}$$

a zero-mode of χ_-. For a generic choice of section $\mathfrak{S}_\alpha$ the equation $\mathfrak{S}_\alpha = 0$ cuts out a $(n-r)$ complex dimensional subspace of X. Then the equation (2.3.18) implies that we have exactly $(n-r)$ zero-modes of ψ_+, while the equation (2.3.19) implies that we do not have any zero-modes of χ_-, since $n \geq r$. Assume that the equations $\partial_j\mathfrak{S}_\alpha = 0$, only for a fixed α have common roots for all $j = 1, \ldots, n$. Then (2.3.18) for the fixed α do not impose any condition on the ψ_+^j and we may have $(n-r+1)$ zero-modes of ψ_+. Similarly the equations (2.3.19) do not impose any condition on the fixed component χ_-^α and we may have one zero-mode of χ_-. Thus we may draw two conclusions

1. For a generic choice of section we do not have any zero-modes of anti-ghosts. The vanishing locus $\mathfrak{S}^{-1}(0)$ of the section has the right complex $(n-r)$ dimensions and the zero-modes of ψ_+ span the tangent space of $\mathfrak{S}^{-1}(0)$.

2. For a non-generic choice of section we may have anti-ghost zero-modes. The vanishing locus $\mathfrak{S}^{-1}(0)$ of the section have dimension higher than the right one. In any cases we have

$$n - r = \#(\widetilde{\psi}_+) - \#(\widetilde{\chi}_-) \tag{2.3.20}$$

where $\#(\widetilde{fermi})$ denotes the number of fermionic zero-modes. We call the above the *formal* or *virtual* complex dimension of $\mathfrak{S}^{-1}(0)$. The space of anti-ghost zero-modes span a vector bundle $\mathbb{V}$ over $\mathfrak{S}^{-1}(0)$ called the anti-ghost bundle. The fiber dimension of $\mathbb{V}$ may jump when $\mathfrak{S}^{-1}(0)$ develops singularities.

We also see that our action functional S' (2.3.9) restricted to the s_+ and $\overline{s}_+$ invariant neighborhood $\mathcal{C}$ of the fixed point locus is given by

$$S'|_{\mathcal{C}} = -F_{\alpha'\overline{\beta}'i'\overline{j}'}\widetilde{\psi}^{i'}_+\widetilde{\psi}^{\overline{j}'}_+\widetilde{\chi}^{\alpha'}_-\widetilde{\chi}^{\overline{\beta}'}_-, \tag{2.3.21}$$

where it is understood all the fermions $(\psi^i_+, \psi^{\overline{i}}_+, \chi^\alpha_-, \chi^{\overline{\alpha}}_-)$ are replaced by their zero-modes $(\widetilde{\psi}^{i'}_+, \widetilde{\psi}^{\overline{i}'}_+, \widetilde{\chi}^{\alpha'}_-, \chi^{\overline{\alpha}'}_-)$ and the curvature above is the curvature of the anti-ghost bundle $\mathbb{V}$ over $\mathfrak{S}^{-1}(0)$.

Now we examine the path integral. For $n = r$ and with a generic section the vanishing locus $\mathfrak{S}^{-1}(0)$ is zero-dimensional and the path integral counts the number of zeros of the section. For $n = r$ and with a non-generic section the zeros of the section can be a positive dimensional submanifold $\mathfrak{S}^{-1}(0) \subset X$ of X. The path integral reduces to an integral over $\mathfrak{S}^{-1}(0)$ and over anti-ghost zero-modes. Note that the rank of the anti-ghost bundle $\mathbb{V}$ over $\mathfrak{S}^{-1}(0)$ is the same as the complex dimension of $\mathfrak{S}^{-1}(0)$. The path integral becomes $\chi(\mathbb{V})$

$$Z = \int_{\mathfrak{S}^{-1}(0)} e(\mathbb{V}) = \chi(\mathbb{V}), \tag{2.3.22}$$

which in turn can be identified with $\chi(\mathbb{E})$.

Now we consider the case $r < n$. The partition function still evaluates the Euler class $e(\mathbb{V})$ of the anti-ghost bundle $\mathbb{V}$ over $\mathfrak{S}^{-1}(0)$. Since, by the formula (2.3.20), the rank of $\mathbb{V}$ is smaller than the complex dimension of $\mathfrak{S}^{-1}(0)$. Thus the Euler class $e(\mathbb{V})$ is not a top form and the partition function vanishes. To get a non-trivial result we should insert a set of observables and evaluate the expectation value

$$\left\langle \prod_{\ell=1}^{m} \widehat{\alpha}^{p_\ell, q_\ell} \right\rangle = \int_{\mathfrak{S}^{-1}(0)} e(\mathbb{V}) \wedge \alpha^{p_1,q_1} \wedge \ldots \wedge \alpha^{p_m,q_m}, \tag{2.3.23}$$

where

$$\sum_{\ell=1}^{m} p_\ell = \sum_{\ell=1}^{m} q_\ell = n - r, \tag{2.3.24}$$

and otherwise the path integral vanishes. If there are no anti-ghost zero-modes we have $e(\mathbb{V}) = 1$ and the above correlation function reduces to the intersection number of homology cycles Poincaré dual to α^{p_ℓ, q_ℓ} in $\mathfrak{S}^{-1}(0)$. The selection rule above can be understood in more physical terms. The path integral measure contains a ghost number anomaly due to the fermionic zero-modes. The net ghost number violation of the path integral measure is $(n-r, n-r)$, which follows from the formula (2.3.20) and the ghost numbers of the fermions;

$$\begin{aligned} \psi_+^i &: (1,0), \qquad & \chi_-^\alpha &: (-1,0), \\ \psi_+^{\bar{i}} &: (0,1), \qquad & \chi_-^{\bar{\alpha}} &: (0,-1). \end{aligned} \tag{2.3.25}$$

To cancel the ghost number anomaly we have to insert observables according to the selection rule (2.3.24) to soak up the fermion zero-modes in the path integral measure.

Specializing to $N_c = (2,2)$ Model

Finally we consider a special case of $N_c = (2,0)$ model which actually has $N_c = (2,2)$ supersymmetry. We have the following properties

1. For a generic choice of holomorphic potential $\mathcal{W}(X^i)$ we do not have any anti-ghost zero-modes. The critical set $V_i^{-1}(0)$ where $V_i = \partial_i \mathcal{W}(X^j)$ consists of a collection of non-degenerate points. The partition function is the number of such points.

2. For a non-generic $\mathcal{W}(X^i)$ we may have anti-ghost zero-modes. The critical set $V_i^{-1}(0)$ may be a higher dimensional subvariety of X. The net ghost number violation in the path integral measure is always zero. Thus the rank of the anti-ghost bundle $\mathbb{V}$ is exactly the same as the complex dimension of $V_i^{-1}(0)$. Thus the partition function is well-defined and computes the Euler characteristic $\chi(\mathbb{V})$ of $\mathbb{V}$. We can identify $\mathbb{V}$ with the tangent bundle of $V_i^{-1}(0)$. Thus the partition function is the Euler characteristic of $V_i^{-1}(0)$. This, in turn, can be identified with the Euler characteristic of X.

2.3.3 An Infinite Dimensional Example

Here we present an infinite dimensional example - the topological sigma A-model in two dimensions. We consider a Riemann surface Σ and a compact complex d-dimensional Kähler manifold M. Now we let our infinite dimensional target space X be the space of all maps $\Sigma \to M$. Then we can introduce local holomorphic coordinate fields on X by $X^i(z, \bar{z})$ where $i = 1, \ldots, d$, leading to holomorphic multiplets (X^i, ψ_+^i) with the transformation laws (2.3.4). Now we consider the infinite dimensional vector bundle $\mathbb{E}$ over X whose fiber consist of

$\partial_{\bar{z}}X^i \oplus \partial_z X^{\bar{i}}$. Then we have natural complex and Hermitian structures on the fiber induced from the complex structure and Hermitian metric $g_{i\bar{j}}$ of M together with the integration over Σ. Since our holomorphic section is $\mathfrak{S}_\alpha(X^i) := \partial_{\bar{z}}X^i$ the associated Fermi multiplets are given by $(\chi^i_{\bar{z}-}, H^i_{\bar{z}})$ with the transformation laws (2.3.5).

Now the action functional (2.2.16) becomes

$$\begin{aligned} S =& i s_+ \int_\Sigma d^2z \left(\chi^{\bar{i}}_{z-}\partial_{\bar{z}}X^i g_{\bar{i}i}\right) + i\bar{s}_+ \int_\Sigma d^2z \left(\overline{\chi}^i_{\bar{z}-}\partial_z X^{\bar{i}} g_{\bar{i}i}\right) \\ & - s_+\bar{s}_+ \int_\Sigma d^2z \left(g_{i\bar{j}}\chi^{\bar{j}}_{z-}\overline{\chi}^i_{\bar{z}-}\right). \end{aligned} \tag{2.3.26}$$

The supercharges s_+ and $\bar{s}_+$ are scalars on both Σ and M. They are the ∂ and $\bar{\partial}$ operators, after the parity change, on the space X of all maps $\Sigma \to M$. The path integral is localized to the moduli space $\mathcal{M}$ of holomorphic maps $\Sigma \to M$. The resulting model is the topological sigma A model which can be obtained by a twisting of $N_{ws} = (2,2)$ two-dimensional space-time supersymmetric non-linear sigma-model whose target space is M [49][3].

$\partial_{\bar{z}}X^{i} \oplus \partial_{z}X^{\bar{i}}$. Then we have natural complex and Hermitian structures on the fiber induced from the complex structure and Hermitian metric $g_{i\bar{j}}$ of M together with the integration over Σ. Since our holomorphic section is $\mathfrak{S}_{z}(X^{i}) := \partial_{\bar{z}}X^{i}$ the associated Fermi multiplets are given by $(\chi_{\bar{z}}^{i}, H_{\bar{z}}^{i})$ with the transformation laws (2.2.5).

Now the action functional (2.2.10) becomes

$$S = \int_{\Sigma} d^2z \left([illegible] \right) + i\alpha \int_{\Sigma} d^2z \left([illegible] \right) \qquad (2.3.28)$$

[illegible]

The supercharges s_{+} and $\bar{s}_{+}$ are scalars on both Σ and M. They are the ∂ and $\bar{\partial}$ operators, after the parity change, on the space X of all maps $\Sigma \to M$. The path integral is localized to the moduli space $\mathcal{M}$ of holomorphic maps $\Sigma \to M$. The resulting model is the topological sigma-model which can be obtained by a twisting of $N_{c} = (2,0)$ two-dimensional chiral-type supersymmetric non-linear sigma-model whose target space is M [49][8].

Chapter 3

Equivariant Cohomological Field Theory

In the previous chapter we developed standard models of cohomological field theories associated with a Kähler manifold X, tangent bundle TX and Hermitian holomorphic vector bundle $\mathbb{E}$ over X. In this chapter we generalize those modes to the cases when there is a certain group $\mathcal{G}$ action. This generalization is relevant since most of field theory has a certain gauge symmetry. The models in the previous chapter are obviously empty if the target space X is linear. On the other hand models in this chapter have rich structures both for linear[1] and non-linear target spaces. This also allows us to consider more general classes of target spaces like the space of a certain set of matrices, the space of a certain set of fields on a manifold, etc.

The central tool will be the notion of equivariant cohomology and symplectic quotients. The only practical difference between the models in the previous chapter and their equivariant generalizations are that the later models further localize the path integrals to the vanishing locus of $\mathcal{G}$-momentum map, modulo the $\mathcal{G}$ symmetry. If the $\mathcal{G}$ acts freely on such locus we recover the standard models in the previous chapters now associated with the symplectic quotients. The momentum map is a generalization of the familiar angular momentum associated with a group of rotations in the classical mechanics.

3.1 Equivariant Toy Model

We return to our toy model in Sect. 2.1.1, where we considered a n-dimensional Kähler manifold (X, ϖ) with Kähler form ϖ as the target space. Now we assume that there is a group $\mathcal{G}$ action

$$\mathcal{G} \times X \to X, \tag{3.1.1}$$

[1]The relation between the previous section and the present section is best compared with that of non-linear sigma-models and linear gauged sigma models in two-dimensions.

preserving the complex and Kähler structures. We consider the toy model with the action functional S in (2.1.4). The action functional is invariant under $\mathcal{G}$ thus the path integral is degenerated. We want to remove the gauge degree of freedom as follows (compare with (2.1.8))

$$\begin{aligned} Z &= \frac{1}{vol(\mathcal{G})}\int_X [\mathcal{D}X\mathcal{D}\overline{X}\mathcal{D}\psi\mathcal{D}\overline{\psi}]\, e^{-S} \\ &= \frac{1}{\#(\mathcal{G})}\int_{X/\mathcal{G}} [\mathcal{D}X\mathcal{D}\overline{X}\mathcal{D}\psi\mathcal{D}\overline{\psi}]'\, e^{-S} \\ &= \frac{1}{\#(\mathcal{G})}\int_X [\mathcal{D}X\mathcal{D}\overline{X}\mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}(ghosts)]\, e^{-S-S_{gf}-S_{gh}}, \end{aligned} \tag{3.1.2}$$

where $\#(\mathcal{G})$ denotes the number of central elements of $\mathcal{G}$, S_{gf} and S_{gh} denote the gauge fixing and ghost terms. The above procedure is the well-known Faddeev-Popov-BRST quantization on which I do not want to review here.[2]

A general problem with the path integral above is that the quotient space $X/\mathcal{G}$ rarely has good topology and geometry. This means that it is difficult make sense out of our (even for finite dimensional) path integral. Furthermore the geometrical meaning of the $\boldsymbol{s}$ and $\overline{\boldsymbol{s}}$ supercharges on the quotient space is not quite obvious. This problem can be avoided by considering equivariant cohomology. For general references see [39][50][51].

3.1.1 Extending Our Toy Model

A nice route to introduce the equivariant cohomology is a simple generalization of our toy model in Sect. 2.1.1. Now we assume that there is a group $\mathcal{G}$ action $\mathcal{G}\times X \to X$ on our target space X preserving the complex and Kähler structures. Our goal is to extend our target space and supercharges $\boldsymbol{s}$ and $\overline{\boldsymbol{s}}$ by introducing extra fields such that

1. If $\mathcal{G}$ acts freely on X the degrees of freedom due to the extra fields disappear,
2. the supercharges become ∂ and $\overline{\partial}$ operators, after the parity change, on the *$\mathcal{G}$-invariant subspace.*

To implant the above idea we need the notion of Lie derivative. Consider a manifold with $\mathcal{G}$ action. Let $Lie(\mathcal{G})$ be the Lie algebra of $\mathcal{G}$. We will always assume that we have a bi-invariant inner product $<\ ,\ >$ on $Lie(\mathcal{G})$ such that we can identify $Lie(\mathcal{G})$ with its dual $Lie(\mathcal{G})^*$. Let X^I be the local coordinate fields on X. The $\mathcal{G}$ action induces a vector $V^I_a T^a$ such that an infinitesimal $\mathcal{G}$ action is represent by

$$X^I \to X^I + \varepsilon^a V^I_a. \tag{3.1.3}$$

We denote by j_a the interior derivative with respect to the vector V_a, i.e.,

$$\begin{aligned} j_a &: \Omega^r(X) \to \Omega^{r-1}(X), \\ (j_a\alpha)_{I_2 I_3\cdots I_r} &= rV^{I_1}\alpha_{I_1 I_2\cdots I_r}. \end{aligned} \tag{3.1.4}$$

[2] I only want to remark that it involves the Lie algebra cohomology with the parity change.

Let $\mathcal{L}_a$ be the Lie derivative with respect to the vector field V_a;

$$\mathcal{L}_a = dj_a + j_a d \tag{3.1.5}$$

Then the infinitesimal $\mathcal{G}$ action on $\alpha \in \Omega^*(X)$ is given by $\alpha \to \alpha + \varepsilon^a \mathcal{L}_a \alpha$. Thus a differential form α is $\mathcal{G}$-invariant if $\mathcal{L}_a \alpha = 0$. We note an obvious relation $\varepsilon^a \mathcal{L}_a X^I = \varepsilon^a V_a^I$.

Now we extend our target space X by introducing a $Lie(\mathcal{G})$-valued scalar $\phi = \phi^a T_a$ and modify the commutation relation (2.1.2) as[3]

$$\boldsymbol{s}^2 = 0, \qquad \{\boldsymbol{s}, \overline{\boldsymbol{s}}\} = -i\phi^a \mathcal{L}_a, \qquad \overline{\boldsymbol{s}}^2 = 0. \tag{3.1.6}$$

Thus $\{\boldsymbol{s}, \overline{\boldsymbol{s}}\} = 0$ on the $\mathcal{G}$-invariant subspace of X and the supercharges are related with the ∂ and $\overline{\partial}$ operators on the invariant subspace as in the case of our previous toy model. The ghost numbers of ϕ should be assigned (1, 1) to match the ghost numbers in the anti-commutation relations above. The above defines $\mathcal{G}$-equivariant Dolbeault cohomology [52][53].

By the new anti-commutation relations (3.1.6) the supersymmetry transformation laws (2.1.4) should be modified as follows

$$\begin{aligned}
&\boldsymbol{s}X^i = i\psi^i, && \boldsymbol{s}\psi^i = 0, && \\
&\overline{\boldsymbol{s}}X^i = 0, && \overline{\boldsymbol{s}}\psi^i = -\phi^a \mathcal{L}_a X^i, && \boldsymbol{s}\phi = 0, \\
&\boldsymbol{s}X^{\bar{i}} = 0, && \boldsymbol{s}\psi^{\bar{i}} = -\phi^a \mathcal{L}_a X^{\bar{i}}, && \overline{\boldsymbol{s}}\phi = 0, \\
&\overline{\boldsymbol{s}}X^{\bar{i}} = i\psi^{\bar{i}}, && \overline{\boldsymbol{s}}\psi^{\bar{i}} = 0. &&
\end{aligned} \tag{3.1.7}$$

where we obtained the conditions $\boldsymbol{s}\phi = \overline{\boldsymbol{s}}\phi = 0$ by demanding the algebra to be closed. Assume that we have a model with an action functional which is invariant under the supersymmetries generated by the above new supercharges. Then we can apply the fixed point theorem of Witten and we have the following fixed point equation, deduced from the above

$$\phi^a \mathcal{L}_a X^I = 0. \tag{3.1.8}$$

This equation tells us that $\phi^a = 0$ if $\mathcal{G}$ act freely on X while ϕ^a can be non-zero on a fixed point of the $\mathcal{G}$ action. Thus we achieved our initial two goals.

Now we consider a supersymmetric action functional S. Compare with the non-equivariant case in Sect. 2.1.1, an action functional should be invariant under $\mathcal{G}$ in addition to $\boldsymbol{s}S = \overline{\boldsymbol{s}}S = 0$. These conditions imply that one can also apply the Poincaré lemma since the new supercharges are also nilpotent if they are acting on $\mathcal{G}$ invariant quantities. Thus S can be written by the same form as the previous toy model

$$S = i\boldsymbol{s}\overline{\boldsymbol{s}}\mathcal{K}(X^i, X^{\bar{i}}), \tag{3.1.9}$$

where $\mathcal{K}$ should be $\mathcal{G}$ invariant. [4] Applying the transformation laws (3.1.7) we have

$$S = -i\langle \phi, \mu \rangle - ig_{i\bar{j}}\psi^i \psi^{\bar{j}}, \tag{3.1.10}$$

[3] where ϕ^a is ε^a in (3.1.3) incarnated as a field.

[4] Actually $\mathcal{K}$ only needs to satisfy a weaker condition that it should invariant under gauge transformations connected to the identity.

where

$$\mu_a = i\frac{\partial \mathcal{K}}{\partial X^{\bar{i}}}\left(\mathcal{L}_a X^{\bar{i}}\right). \tag{3.1.11}$$

Later we shall see that $\mu = \mu_a T^a$ is the equivariant $\mathcal{G}$ momentum map on X. Maintaining all the supersymmetry we consider the following more general action functional $S(\zeta)$

$$\begin{aligned} S(\zeta) &= is\bar{s}\mathcal{K} + i\phi^a\zeta_a \\ &= -i < \phi, \mu - \zeta > -ig_{i\bar{j}}\psi^i\psi^{\bar{j}}, \end{aligned} \tag{3.1.12}$$

where ζ belongs to the center of $\mathcal{G}$. We call the additional term a FI coupling.

Now we consider the partition function for the new action. We have

$$\begin{aligned} Z(\zeta) &= \frac{1}{vol(\mathcal{G})}\int [\mathcal{D}\phi\mathcal{D}X\mathcal{D}\overline{X}\mathcal{D}\psi\mathcal{D}\overline{\psi}]\, e^{-S(\zeta)} \\ &= \frac{1}{vol(\mathcal{G})}\int_X \delta(\mu-\zeta)\prod_{k,\bar{k}} dX^k dX^{\bar{k}} d\psi^k d\psi^{\bar{k}}\cdot e^{-ig_{i\bar{j}}\psi^i\psi^{\bar{j}}} \\ &= \frac{1}{\#\mathcal{G}}\int_{\mu^{-1}(\zeta)/\mathcal{G}} \frac{\tilde{\varpi}^r}{r!} \\ &= \frac{1}{\#\mathcal{G}} vol(\mathcal{N}_\zeta), \end{aligned} \tag{3.1.13}$$

where

$$\mathcal{N}_\zeta = \mu^{-1}(\zeta)/\mathcal{G}. \tag{3.1.14}$$

In the above we assumed that $\mathcal{G}$ acts freely on the locus $\mu^{-1}(\zeta) \subset X$. Thus we could simply integrate ϕ out, which gives rise to the delta function supported on $\mu^{-1}(\zeta)$. Then the quotient space $\mathcal{N}_\zeta$ is smooth. Our action functional S_ζ reduces to the Kähler form on the subspace $\mu^{-1}(\zeta)$. Since it is $\mathcal{G}$ invariant it becomes, after the parity change, the Kähler from $\tilde{\varpi}$ on the quotient space $\mathcal{N}_\zeta$. What we showed is the symplectic reduction theorem of Marsden and Weinstein [54].

We call our extended toy model the equivariant toy model. We note that the equivariant toy model makes perfect sense even if we start from a flat Kähler manifold X as our initial target space. We call the space $\mathcal{N}_\zeta$ *the effective target space*, which can be a very complicated non-linear space even if our initial target space X is flat.

Before examining further properties of our model, we turn to a review of the equivariant cohomology and momentum map. We refer for details on the equivariant cohomology and relation with momentum maps to a beautiful exposition of Atiyah and Bott [50]. The idea is to replace X by a bigger space $X \times E\mathcal{G}$ such that the extended space has a nice quotient

$$X_\mathcal{G} = (X \times_\mathcal{G} E\mathcal{G})$$

which is equivalent to the original quotient $M/\mathcal{G}$ when it has a nice quotient.[5] The $\mathcal{G}$-equivariant cohomology $H^*_{\mathcal{G}}(X)$ of X is defined as the ordinary cohomology $H^*(X_{\mathcal{G}})$ of $X_{\mathcal{G}}$. For instance the $\mathcal{G}$-equivariant cohomology of M is the ordinary cohomology of $X/\mathcal{G}$ if $\mathcal{G}$ acts freely on X.[6]

We will briefly review a convenient model of equivariant cohomology due to Cartan, of which variants will be used in this thesis. A crucial reference on the Cartan model for us is Witten's paper [55]. The path integral of the equivariant toy model reproduces a Kähler version of Witten's non-Abelian equivariant integration formula.

3.1.2 Equivariant Cohomology and Momentum Map

Consider a manifold X with $\mathcal{G}$ action. Let $Lie(\mathcal{G})$ be the Lie algebra of $\mathcal{G}$. We will always assume that we have a bi-invariant inner product $<\ ,\ >$ on $Lie(\mathcal{G})$ such that we can identify $Lie(\mathcal{G})$ with its dual $Lie(\mathcal{G})^*$.

Let $Fun^*(Lie(\mathcal{G}))$ denote the algebra of polynomial functions on $Lie(\mathcal{G})$ so that an m^{th} order homogeneous polynomial is considered to be of degree $2m$. The equivariant differential forms $\Omega^*_{\mathcal{G}}(X)$ on X are represented by

$$\Omega^*_{\mathcal{G}}(X) := (\Omega^*(X) \otimes Fun^*(Lie(\mathcal{G})))^{\mathcal{G}}\,, \tag{3.1.16}$$

where $^{\mathcal{G}}$ denote the $\mathcal{G}$-invariant part. The degree of such a form is the sum of degrees of $\Omega^*(X)$ and $Fun^*(Lie(G))$. One endows $\Omega^*_{\mathcal{G}}(X)$ with the equivariant differential operator $d_{\mathcal{G}}$

$$d_{\mathcal{G}} = d - i\phi^a j_a, \qquad d^2_{\mathcal{G}} = -i\phi^a \mathcal{L}_a, \tag{3.1.17}$$

where $j_a^2 = 0$ and $\phi = \phi^a T^a \in Lie(\mathcal{G})$. That is, $d^2_{\mathcal{G}} = 0$ modulo an infinitesimal gauge transformation generated by ϕ^a. Thus on the space $\Omega^*_{\mathcal{G}}(X)$ we have[7]

$$\widehat{d^2_{\mathcal{G}}} = 0.$$

The $\mathcal{G}$-equivariant de Rham cohomology on X is the cohomology of the complex $(\Omega^*_{\mathcal{G}}(X), d_{\mathcal{G}})$. The equivariant cohomology of X is the ordinary cohomology of the quotient space if the group acts freely, otherwise it is something else. For example $H^*_{\mathcal{G}}(pt)$ is $Fun^*(Lie(\mathcal{G}))$.

The Symplectic Case

Now we consider a symplectic manifold X with symplectic form ϖ. Assume that we have a $\mathcal{G}$ action on X. Under an infinitesimal $\mathcal{G}$ action $X^I \to X^I + \varepsilon^a V^I_a$ the

[5]The additional space $E\mathcal{G}$ is a fixed universal $\mathcal{G}$-bundle over the classifying space $B\mathcal{G}$. The homotopy quotient $X_{\mathcal{G}}$ forms a fiber bundle $\pi : X_{\mathcal{G}} \to B\mathcal{G}$ with fiber X. Then we have the following diagram

$$\begin{array}{ccccc} E\mathcal{G} & \leftarrow & E\mathcal{G}\times X & \to X \\ \downarrow & & \downarrow & \downarrow \\ B\mathcal{G} & \leftarrow & E\mathcal{G}\times_{\mathcal{G}} X & \to X/\mathcal{G} \end{array} \tag{3.1.15}$$

[6]Note, however, $\mathcal{G}$-equivariant cohomology of a point is $H^*(B\mathcal{G})$ which is highly non-trivial.

[7]An element in $\Omega^*_{\mathcal{G}}(X)$ is annihilated by $L_a = \mathcal{L}_a + f_{ab}{}^c\phi^b\frac{\partial}{\partial\phi^c}$, where $f_{ab}{}^c = -f_{ba}{}^c$ are the structure constants of $\mathcal{G}$. Then, it is also annihilated by $\phi^a\mathcal{L}_a$ since $\phi^a L_a = \phi^a \mathcal{L}_a$.

symplectic form transforms as $\varpi \to \varpi + \varepsilon^a \mathcal{L}_a \varpi$. Thus we have a vector field V_a^I which is an infinitesimal symplectic transformation whenever $\mathcal{L}_a \varpi = 0$. Since $d\varpi = 0$ we have $d(j_a \varpi) = 0$, thus at least locally we can write

$$j_a \varpi = d\mu_a. \tag{3.1.18}$$

The $\mu = \mu_a T^a : X \to Lie(\mathcal{G})^*$ is called the $\mathcal{G}$-momentum map.[8] The obstruction for global existence of μ_a is $H^1(X)$. The momentum map is a generalization of the familiar classical mechanical notion that X is a classical phase space and $\mathcal{G}$ is a group of rotation and μ is the angular momentum. The momentum map is equivariant if $\mu(g(x)) = (ad\ g)^*(\mu(x))$. Then $\mathcal{G}$ preserve the subspace $\mu^{-1}(\zeta)$ when ζ is a central element. Then the reduced phase space or the symplectic quotient is defined by

$$\mathcal{N}_\zeta = (X \cap \mu^{-1}(\zeta))/\mathcal{G} \tag{3.1.19}$$

The quotient space is a smooth symplectic manifold if ζ is a regular value. The symplectic form $\widetilde{\varpi}$ on N_ζ is obtained from ϖ by restriction and reduction [54].

The equivariant cohomology and the momentum map are closely related [50]. Note that the symplectic form ϖ is not equivariantly closed, $d_\mathcal{G} \varpi \neq 0$. We have a unique form, due to the degree, of equivariant extension $\varpi_\mathcal{G}$ of ϖ

$$\varpi_\mathcal{G} = \varpi + i(\phi, \mu) \tag{3.1.20}$$

The condition $d_\mathcal{G} \varpi_\mathcal{G} = 0$ reduces to using $d_\mathcal{G} \phi = 0$

$$< \phi, d\mu - j\varpi >= 0. \tag{3.1.21}$$

Thus $\varpi_\mathcal{G}$ is equivariantly closed iff μ is the momentum map (3.1.18). Note that $\varpi_\mathcal{G}$ is $\mathcal{G}$ invariant, $L_a \varpi_\mathcal{G} = 0$, iff the momentum map μ is equivariant.

The Kähler Case

Now we specialize to the case that X is a Kähler manifold with Kähler form ϖ and with $\mathcal{G}$ action, which preserve the complex structure and the Kähler form. The vector field V^I induced by the $\mathcal{G}$ action is decomposed into $V^I = V^i + V^{\bar{i}}$. Thus one can introduce interior derivatives ι_a and $\bar{\iota}_a$ by contracting with V_a^i and $V_a^{\bar{i}}$, respectively, such that $j_a = \iota_a + \bar{\iota}_a$;

$$\begin{aligned} \iota_a &: \Omega^{p,q}(X) \to \Omega^{p-1,q}(X), \\ \bar{\iota}_a &: \Omega^{p,q}(X) \to \Omega^{p,q-1}(X). \end{aligned} \tag{3.1.22}$$

From the relation $j_a^2 = 0$ we have

$$\iota_a^2 = 0, \qquad \{\iota_a, \bar{\iota}_a\} = 0, \qquad \bar{\iota}_a^2 = 0. \tag{3.1.23}$$

It follows that

$$\mathcal{L}_a = \partial \iota_a + \iota_a \partial + \bar{\partial} \bar{\iota}_a + \bar{\iota}_a \bar{\partial}. \tag{3.1.24}$$

[8] Note that we identified $Lie(\mathcal{G})$ with its dual $Lie(\mathcal{G})^*$.

We also decompose $Fun^*(Lie(\mathcal{G}))$ such that an m^{th} order homogeneous polynomial in $Fun(Lie(\mathcal{G}))$ is considered to be of degree (m, m). Then equivariant differential forms $\Omega_{\mathcal{G}}^{*,*}(X)$ on X are represented by

$$\Omega_{\mathcal{G}}^{*,*}(X) := (\Omega^{*,*}(X) \otimes Fun^*(Lie(\mathcal{G})))^{\mathcal{G}} \tag{3.1.25}$$

Similarly we decompose d_G into

$$d_{\mathcal{G}} = \partial_G + \overline{\partial}_G : \Omega_{\mathcal{G}}^0(M) = \Omega_{\mathcal{G}}^{1,0}(M) \oplus \Omega_{\mathcal{G}}^{0,1}(M). \tag{3.1.26}$$

where

$$\begin{aligned} \partial_{\mathcal{G}} &= \partial - i\phi^a \overline{\iota}_a, \\ \overline{\partial}_{\mathcal{G}} &= \overline{\partial} - i\phi^a \iota_a. \end{aligned} \tag{3.1.27}$$

Remark that ϕ is assigned to degree $(1, 1)$. The anti-commutation relations between $\partial_{\mathcal{G}}$ and $\overline{\partial}_{\mathcal{G}}$ are

$$\partial_{\mathcal{G}}^2 = 0, \qquad \{\partial_{\mathcal{G}}, \overline{\partial}_{\mathcal{G}}\} = -i\phi^a \mathcal{L}_a, \qquad \overline{\partial}_{\mathcal{G}}^2 = 0. \tag{3.1.28}$$

This defines equivariant Dolbeault cohomology on a Kähler manifold. Comparing with the anti-commutation relations (3.1.6) we can identify our supercharges s and $\overline{s}$ with $\partial_{\mathcal{G}}$ and $\overline{\partial}_{\mathcal{G}}$ after the parity change (2.1.16). Thus

$$\begin{aligned} s &= i\psi^i \frac{\partial}{\partial X^i} - \phi^a V_a^{\bar{i}} \frac{\partial}{\partial \psi^{\bar{i}}}, \\ \overline{s} &= i\psi^{\bar{i}} \frac{\partial}{\partial X^{\bar{i}}} - \phi^a V_a^i \frac{\partial}{\partial \psi^i}. \end{aligned} \tag{3.1.29}$$

Now we examine the relation between the momentum map and equivariant Dolbeault cohomology. For the Kähler case the relation (3.1.18) becomes, by matching form degrees

$$\iota_a \varpi = \overline{\partial} \mu_a. \tag{3.1.30}$$

Since the Kähler form ϖ can be written locally in terms of a Kähler potential f,

$$\varpi = i\partial\overline{\partial} f, \tag{3.1.31}$$

we have

$$i\iota_a(\partial\overline{\partial} f) = \overline{\partial}\mu_a. \tag{3.1.32}$$

Using the relations $\{\iota_a, \overline{\partial}\} = \{\partial, \overline{\partial}\} = 0$ we deduce that

$$\mu_a = i\iota_a(\partial f) \tag{3.1.33}$$

up to a constant. Combining all together we find an important identity

$$i\partial_{\mathcal{G}}\overline{\partial}_{\mathcal{G}} f = \varpi + i < \phi, \mu >, \tag{3.1.34}$$

which we obtained earlier in (3.1.9) and (3.1.10). Thus minus the action functional, $-S$, is a $\mathcal{G}$-equivariant Kähler form after the parity change. Note that the momentum map derived above is equivariant if the Kähler potential is $\mathcal{G}$ invariant. Thus we showed all the assertions made in Sect. 3.1.1.

3.1.3 Path Integrals and Non-Abelian Localization Theorem

We now return to the equivariant toy model. We return to the partition function Z (3.1.13) and ask what will happen as we vary the FI term ζ.

We have a classical theorem; the image of a proper momentum map of a compact group is a convex polytope divided by walls [56][57][58]. As we vary ζ the symplectic quotient $\mathcal{N}_\zeta$ may undergo birational transformations if the path of ζ crosses a wall, otherwise diffeomorphic. For the non-proper case the symplectic quotient does not exist. This does not imply that the partition function is empty. Recall that space of all bosonic fields is a copy of X and the space of all ϕ. The correct picture is that the path integral is localized to $\mathcal{N}_\zeta \subset X/\mathcal{G}$ for regular values of ζ. The full equation for the localization is

$$\begin{aligned} \mu - \zeta &= 0, \\ \phi^a \mathcal{L}_a(X^i) &= 0. \end{aligned} \tag{3.1.35}$$

We call the non-trivial solutions ϕ_0 of above equations the zero-modes of ϕ. It is clear that we have zero-modes of ϕ whenever the $\mathcal{G}$ action has fixed points in $\mu^{-1}(\zeta)$, thus when ζ lies on a non-regular value ζ_0. Clearly the path integral degenerates at such a value since the path integral measure contains zero-modes of ϕ. Let $\zeta_+ < \zeta_0 < \zeta_-$ we have

$$Z(\zeta_+) \neq Z(\zeta_-) \tag{3.1.36}$$

due to topology change. At ζ_0 the partition function should be singular.

It is clear how to resolve the singularity of the path integral. We have to regularize. We consider a more general action functional $S(\zeta,\varepsilon)$,

$$\begin{aligned} S(\zeta,\varepsilon) &= S(\zeta) + \frac{\varepsilon}{2} < \phi, \phi > \\ &= -i < \phi, \mu - \zeta > + \frac{\varepsilon}{2}(\phi,\phi) - i g_{i\bar{j}} \psi^i \psi^{\bar{j}}. \end{aligned} \tag{3.1.37}$$

Note that the additional term is invariant under $\mathcal{G}$ as well as all the supersymmetry. The additional ε dependent term changes the fixed point equations of the supersymmetry since the ϕ equation of motion is now

$$i(\mu - \zeta) = \varepsilon\phi. \tag{3.1.38}$$

Consequently, for $\varepsilon \neq 0$, the path integral is localized to the locus of the following equations

$$\left(\frac{\partial \mu_a}{\partial X}\right)(\mu^a - \zeta^a) = 0. \tag{3.1.39}$$

Now we also have contributions from *higher critical points.* Thus, we have two branches; (i) $\mu^a - \zeta^\alpha = 0$, (ii) $\frac{\partial}{\partial X}\mu = 0$. Clearly the quotient $\mathcal{N}_\zeta$ space develops singularities when branches (i) and (ii) intersect. In such a case the integrand of path integral contains the Gaussian measure

$$e^{-\varepsilon \sum_\ell |\phi_{0,\ell}|^2} \tag{3.1.40}$$

for the space of zero-modes $\phi_{0,\ell}$ of ϕ. Thus the path integral is non-singular. Consequently the politically correct version of the model is defined by the action functional $S(\zeta, \varepsilon)$.

Now we consider the correlation functions. A supersymmetric observable should be $\mathcal{G}$-invariant as well as invariant under $\boldsymbol{s}$ and $\overline{\boldsymbol{s}}$. Such an observable should be constructed from an equivariantly closed differential form. An equivariant differential form $\mathcal{O}^{p,q}$ of total degree (p,q) can be expanded as

$$\mathcal{O}^{p,q} = \alpha_0^{p,q} + \phi^a \alpha_a^{p-1,q-1} + \phi^a \phi^b \alpha_{ab}^{p-2,q-2} + \cdots, \tag{3.1.41}$$

where $\alpha_*^{p,q} \in \Omega^{p,q}(X)$. Let $\widehat{\mathcal{O}}^{p,q}$ be the parity change of $\mathcal{O}^{p,q}$, thus carrying ghost number (p,q). We have

$$\overline{\boldsymbol{s}}\widehat{\mathcal{O}}^{p,q} = \widehat{\overline{\partial}_{\mathcal{G}}\mathcal{O}}^{p,q}. \tag{3.1.42}$$

Thus $\widehat{\mathcal{O}}^{p,q}$ is an $\overline{\boldsymbol{s}}$-invariant observable if $\mathcal{O}^{p,q}$ is an $\overline{\partial}_{\mathcal{G}}$-closed equivariant differential form.

The correlation function of observables or the expectation value is defined by

$$\left\langle \prod_{m=1}^{r} \widehat{\mathcal{O}}^{p_m,q_m} \right\rangle = \int [\mathcal{D}\phi \mathcal{D}X \mathcal{D}\overline{X} \mathcal{D}\psi \mathcal{D}\overline{\psi}] \prod_{m=1}^{r} \widehat{\mathcal{O}}^{p_m,q_m} \cdot e^{-S} \tag{3.1.43}$$

For the present model one can show that

$$\begin{aligned} \left\langle \prod_{m=1}^{r} \widehat{\mathcal{O}}^{p_m,q_m} \right\rangle =& \frac{1}{vol(\mathcal{G})} \int_{\mathcal{G}} \frac{d\phi_1 d\phi_2 \ldots d\phi_s}{(2\pi)^s} \\ &\times \int_X \mathcal{O}^{p_1,q_2} \wedge \ldots \wedge \mathcal{O}^{p_r,q_r} \cdot \exp\left(\varpi + i(\phi, \mu - \zeta) - \frac{\varepsilon}{2}(\phi,\phi)\right), \end{aligned} \tag{3.1.44}$$

where $s = dim(\mathcal{G})$. Applying the fixed point theorem for the global supersymmetry we see that the above integral can be written as a sum of contribution of the critical points (3.1.39) of $I =< \mu, \mu >$. This is the non-Abelian localization theorem of Witten [55], generalizing the more familiar abelian Duistermaat-Heckman (DH) integration formula [56]. In the end our equivariant toy model turns out to be very non-trivial.

3.2 Equivariant $N_c = (2,2)$ Model

In this section we develop the equivariant generalization of the $N_c = (2,2)$ model in Sect. 2.2. We assume the same group $\mathcal{G}$ acting on X as in the previous section. This naturally extend to the tangent space TX. Recall that the partition function of our toy model is the symplectic volume of the target space, while the partition function of the equivariant toy model is the symplectic volume of the symplectic quotient $\mathcal{N}_\zeta$, for generic values of ζ, of X by $\mathcal{G}$. Similarly, the partition function the equivariant version of $N_c = (2,2)$ model, without holomorphic potential $\mathcal{W}$, will be the Euler characteristic $\chi(T\mathcal{N}_\zeta)$ of the symplectic quotient $\mathcal{N}_\zeta$, for generic value of ζ, of X by $\mathcal{G}$. After turning on $\mathcal{W}$, the path integral reduces to the symplectic quotient $\mathcal{M}_\zeta$ of the critical subset $X_{crit} \subset X$ of the potential $\mathcal{W}$ by $\mathcal{G}$.

We consider the same "type" of supercharges carrying the same ghost numbers (p, q);

$$\begin{aligned} s_+ &: (+1, 0), & \bar{s}_+ &: (0, +1), \\ s_- &: (-1, 0), & \bar{s}_- &: (0, -1). \end{aligned} \tag{3.2.1}$$

Now we postulate the supercharges to satisfy the following anti-commutation relations

$$\begin{aligned} &\{s_+, s_+\} = 0, & &\{s_+, \bar{s}_+\} = -i\phi^a_{++}\mathcal{L}_a, & &\{\bar{s}_+, \bar{s}_+\} = 0, \\ &\{s_+, s_-\} = 0, & &\{s_+, \bar{s}_-\} = -i\sigma^a\mathcal{L}_a, & &\{\bar{s}_+, \bar{s}_-\} = 0, \\ &\{s_-, s_-\} = 0, & &\{\bar{s}_+, s_-\} = -i\bar{\sigma}^a\mathcal{L}_a, & &\{\bar{s}_-, \bar{s}_-\} = 0, \\ & & &\{s_-, \bar{s}_-\} = -i\phi^a_{--}\mathcal{L}_a, & & \end{aligned} \tag{3.2.2}$$

which are equivariant generalizations of the commutation relations (2.2.2) and (2.2.3) for the $N_c = (2,2)$ model. For the $\mathcal{G}$-invariant subspace the equivariant supercharges are the same as the non-equivariant ones. Here, in total, we introduced four bosonic fields $\phi_{\pm\pm}$, σ and $\bar{\sigma}$ taking values in $Lie(\mathcal{G})$. They carry the following ghost numbers

$$\begin{aligned} \phi_{++} &: (+1, +1), & \sigma &: (+1, -1), \\ \phi_{--} &: (-1, -1), & \bar{\sigma} &: (-1, +1). \end{aligned} \tag{3.2.3}$$

The anti-commutation relations above define balanced $\mathcal{G}$-equivariant Dolbeault cohomology [59]. This is the Kähler version of the balanced equivariant cohomology [38].[9]

We should remark that the above algebra can be obtained by dimensional reduction of the $N = 1$ supersymmetry algebra of four-dimensional super-Yang-Mill theory and, equivalently, the algebra of $N_{ws} = (2,2)$ super-Yang-Mills theory in two-dimensions. Thus we may introduce other quantum numbers, as in two-dimensions, the left and right $U(1)$ $\mathcal{R}$-charges (J_L, J_R) as follows

$$\begin{aligned} s_+ &: (+1, 0), & \bar{s}_+ &: (-1, 0), \\ s_- &: (0, +1), & \bar{s}_- &: (0, -1). \end{aligned} \tag{3.2.4}$$

The analogy with the two-dimensional $N_{ws} = (2,2)$ space-time supersymmetric gauge theory, equivalently the linear gauged sigma-model [32][33] will be very useful. Indeed it is a trivial step to obtain a $N_{ws} = (2,2)$ model, and vice versa, just by replacing $\phi^a_{\pm\pm}\mathcal{L}_a$ by the left and right moving covariant derivatives $D_{\pm\pm}$ everywhere. Then the indices $\pm$ are identified with the left and right spinor indices in two-dimensions. For example requiring the ghost number symmetry is equivalent to requiring the two-dimensional Lorentz symmetry.

[9] In our approach a balanced cohomological field theory [60][7][38] is a $N_c = (1,1)$ supersymmetric sigma-model in (0+0) dimensions, whose target space can be a general Riemannian space.

3.2.1 Basic Structures

Now we examine the basic structure of the model.

$N_c = (2,2)$ **Multiplets**

- Chiral multiplets

  We have the same chiral multiplets introduced in the non-equivariant $N_c = (2,2)$ model,

$$\overline{s}_\pm X^i = 0. \tag{3.2.5}$$

  We have

$$\begin{array}{ccccc} \psi^i_- & \xleftarrow{s_-} & X^i & \xrightarrow{s_+} & \psi^i_+ \\ & {}_{s_+}\searrow & & \swarrow_{s_-} & \\ & & H^i & & \end{array} \tag{3.2.6}$$

  We denote their anti-chiral partners $(X^{\bar{i}}, \psi^{\bar{i}}_\pm, H^{\bar{i}})$, which are their Hermitian conjugates.

- Gauge multiplet

  The internal consistency of the anti-commutation relations (3.2.2) determines uniquely the following multiplet

$$\begin{array}{ccccc} \overline{\sigma} & \xrightarrow{s_+} & \eta_+ & \xleftarrow{s_-} & \phi_{++} \\ \Big\downarrow{\scriptstyle \overline{s}_-} & & \Big\downarrow{\scriptstyle \overline{s}_-} & & \Big\downarrow{\scriptstyle \overline{s}_-} \\ \overline{\eta}_- & \xrightarrow{s_+} & D & \xleftarrow{s_-} & \overline{\eta}_+ \\ \Big\uparrow{\scriptstyle \overline{s}_+} & & \Big\uparrow{\scriptstyle \overline{s}_+} & & \Big\uparrow{\scriptstyle \overline{s}_+} \\ \phi_{--} & \xrightarrow{s_+} & \eta_- & \xleftarrow{s_-} & \sigma \end{array} \tag{3.2.7}$$

  where D is real auxiliary field. All the fields above take values in $Lie(\mathcal{G})$. We call the above multiplet a $N_c = (2,2)$ *gauge multiplet* since it originated from the $\mathcal{G}$ action on X. Remark that σ is *twisted-chiral*; i.e.,

$$s_+\sigma = \overline{s}_-\sigma = 0. \tag{3.2.8}$$

- The ghost numbers

  The ghost numbers (p, q) of the fields in the gauge multiplet are determined from the assignments (3.2.1) and the commutation relations (3.2.2). We set the ghost number of X^i to $(0, 0)$. For the bosonic fields we have

$$\begin{array}{cccccccc} & \phi & \overline{\phi} & \sigma & \overline{\sigma} & D & X^i & H^i \\ p & +1 & -1 & +1 & -1 & 0 & 0 & 0 \\ q & +1 & -1 & -1 & +1 & 0 & 0 & 0 \end{array} \tag{3.2.9}$$

- The $\mathcal{R}$-charges

  The $\mathcal{R}$-charges (J_L, J_R) of the fields in the gauge multiplet are also determined from the assignments (3.2.4) and the commutation relations (3.2.2). We set the $\mathcal{R}$-charges of X^i to $(0,0)$. For the bosonic fields we have

$$\begin{array}{c|ccccccc} & \phi_{++} & \phi_{--} & \sigma & \overline{\sigma} & D & X^i & H^i \\ J_L & 0 & 0 & +1 & -1 & 0 & 0 & 1 \\ J_R & 0 & 0 & -1 & +1 & 0 & 0 & 1 \end{array} \tag{3.2.10}$$

The Supersymmetry Transformation Laws

The explicit transformation laws for the fields in the $N_c = (2,2)$ gauge multiplet are uniquely determined by the internal consistency

$$\begin{aligned}
\delta\phi &= i\overline{\epsilon}_+\eta_+ + i\epsilon_+\overline{\eta}_+, \\
\delta\overline{\phi} &= i\overline{\epsilon}_-\eta_- + i\epsilon_-\overline{\eta}_-, \\
\delta\sigma &= -i\overline{\epsilon}_+\eta_- - i\epsilon_-\overline{\eta}_+, \\
\delta\overline{\sigma} &= -i\overline{\epsilon}_-\eta_+ - i\epsilon_+\overline{\eta}_-, \\
\delta\eta_+ &= +i\epsilon_+D - \frac{1}{2}\epsilon_+[\sigma,\overline{\sigma}] - \frac{1}{2}\epsilon_+[\phi_{++},\phi_{--}] - \epsilon_-[\phi_{++},\overline{\sigma}], \\
\delta\overline{\eta}_+ &= -i\overline{\epsilon}_+D + \frac{1}{2}\overline{\epsilon}_+[\sigma,\overline{\sigma}] - \frac{1}{2}\overline{\epsilon}_+[\phi_{++},\phi_{--}] - \overline{\epsilon}_-[\phi_{++},\sigma], \\
\delta\eta_- &= +i\epsilon_-D + \frac{1}{2}\epsilon_-[\sigma,\overline{\sigma}] + \frac{1}{2}\epsilon_-[\phi_{++},\phi_{--}] - \epsilon_+[\phi_{--},\sigma], \\
\delta\overline{\eta}_- &= -i\overline{\epsilon}_-D - \frac{1}{2}\overline{\epsilon}_-[\sigma,\overline{\sigma}] + \frac{1}{2}\overline{\epsilon}_-[\phi_{++},\phi_{--}] - \overline{\epsilon}_+[\phi_{--},\overline{\sigma}], \\
\delta D &= +\frac{1}{2}\overline{\epsilon}_-[\phi_{++},\eta_-] + \frac{1}{2}\overline{\epsilon}_-[\sigma,\eta_+] + \frac{1}{2}\overline{\epsilon}_+[\phi_{--},\eta_+] + \frac{1}{2}\overline{\epsilon}_+[\overline{\sigma},\eta_-] \\
&\quad - \frac{1}{2}\epsilon_-[\phi_{++},\overline{\eta}_-] - \frac{1}{2}\epsilon_-[\overline{\sigma},\overline{\eta}_+] - \frac{1}{2}\epsilon_+[\phi_{--},\overline{\eta}_+] - \frac{1}{2}\epsilon_+[\sigma,\overline{\eta}_-],
\end{aligned} \tag{3.2.11}$$

where D is an auxiliary field and the commutators are for $Lie(\mathcal{G})$.

The transformation laws for chiral multiplets are also uniquely determined from the conditions $\overline{s}_\pm X^i = 0$.

$$\begin{aligned}
\delta X^i &= i\overline{\epsilon}_+\psi^i_- + i\overline{\epsilon}_-\psi^i_+, \\
\delta\psi^i_+ &= +\,\overline{\epsilon}_+H^i - \epsilon_-\phi^a_{++}\mathcal{L}_a(X^i) - \epsilon_+\sigma^a\mathcal{L}_a(X^i), \\
\delta\psi^i_- &= -\,\overline{\epsilon}_-H^i - \epsilon_+\phi^a_{--}\mathcal{L}_a(X^i) - \epsilon_-\overline{\sigma}^a\mathcal{L}_a(X^i), \\
\delta H^i &= +\,i\epsilon_-\phi^a_{++}\mathcal{L}_a(\psi^i_-) + i\epsilon_-\eta^a_+\mathcal{L}_a(X^i) - i\epsilon_-\overline{\sigma}^a\mathcal{L}_a(\psi^i_+) \\
&\quad - i\epsilon_+\phi^a_{--}\mathcal{L}_a(\psi^i_+) - i\epsilon_+\eta^a_-\mathcal{L}_a(X^i) + i\epsilon_+\sigma^a\mathcal{L}_a(\psi^i_-),
\end{aligned} \tag{3.2.12}$$

where H^i are auxiliary fields as in the non-equivariant $N_c = (2,2)$ model. The details of the transformation laws above depend on the ways the group $\mathcal{G}$ acts on X^i. Since this can be determined easily we will rarely write down the explicit forms and always refer to the above formulas. One may have several different chiral multiplets. Their transformation laws are also determined as above once the complex structure and the group action are given for the bosonic fields.

The Fixed Point Equations

One can never over emphasize the importance of the fixed point theorem of Witten. We have seen many times that the existence of global supersymmetry determine the theories almost uniquely. Such uniqueness becomes stronger as many global supercharges we have.

From the above supersymmetry transformation laws we see that the simultaneous fixed point equations for all the $N_c = (2, 2)$ are given by

$$\begin{aligned} H^i &= 0, \\ D &= 0, \\ \varphi_m^a \mathcal{L}_a(X^i) &= 0, \\ [\varphi_m, \varphi_n] &= 0, \end{aligned} \tag{3.2.13}$$

where φ_m, $m = 1, \ldots, 4$ denote the four independent real $Lie(\mathcal{G})$-valued scalar components of $\phi_{\pm\pm}$, σ and its Hermitian conjugate $\overline{\sigma}$. The action functional, in many respects, just gives the detailed form of the values of the auxiliary fields D and H^i. The path integral is localized to the solution space of the above set of equations modulo the $\mathcal{G}$-action. The third equation implies that φ_m are identically zero if $\mathcal{G}$ act freely on the subset $H^{-1}(0) \cap D^{-1}(0) \subset X$. In such a case the path integral reduces to an integral over the quotient space

$$\left(H^{-1}(0) \cap D^{-1}(0)\right)/\mathcal{G}. \tag{3.2.14}$$

We call this *the effective target space*. The $N_c = (2, 2)$ supersymmetry further implies, as we shall see shortly, that the above space is a Kähler manifold.

If one is interested in evaluating correlation functions of observables invariant only under the supersymmetry generated by s_+ and $\overline{s}_+$, the path integral is localized to the locus of the following equations

$$\begin{aligned} H^i &= 0, \\ D - \frac{i}{2}[\sigma, \overline{\sigma}] &= 0, \\ \overline{\sigma}^a \mathcal{L}_a(X^i) &= 0, \end{aligned} \tag{3.2.15}$$

and

$$\begin{aligned} \phi_{++}^a \mathcal{L}_a(X^i) &= 0, \\ [\phi_{++}, \phi_{--}] &= 0, \\ [\phi_{++}, \overline{\sigma}] &= 0. \end{aligned} \tag{3.2.16}$$

3.2.2 Action Functional and Partition Function

We define the general action functional S by demanding $N_c = (2, 2)$ supersymmetry, the $\mathcal{G}$-symmetry and the ghost number symmetry. We may, however, not require the $U(1)_{\mathcal{R}}$ symmetry in general. Then S should have the following form[10]

$$\begin{aligned} S =& s_+ s_- \bar{s}_+ \bar{s}_- \mathcal{K}(X^i, X^{\bar{i}}) + s_+ s_- \mathcal{W}(X^i) + \bar{s}_+ \bar{s}_- \overline{\mathcal{W}}(X^{\bar{i}}) \\ & - s_+ s_- \bar{s}_+ \bar{s}_- \langle \sigma, \bar{\sigma} \rangle + \bar{s}_+ s_- < t, \sigma > + s_+ \bar{s}_- \operatorname{Tr} < \bar{\sigma}, \bar{t} >, \end{aligned} \tag{3.2.17}$$

where all potentials $\mathcal{K}(X^i, X^{\bar{i}})$, $\mathcal{W}(X^i)$ and its Hermitian conjugate $\overline{\mathcal{W}}(X^{\bar{i}})$ are $\mathcal{G}$-invariant and[11]

$$t = \frac{\theta}{2\pi} - i\zeta \tag{3.2.18}$$

belongs to the center of $Lie(\mathcal{G})$. The first line of the action functional (3.2.17) has the same form as the non-equivariant $N_c = (2,2)$ action functional. We remark that the above action functional can be a quite strange object if $\mathcal{K}(X^i, X^{\bar{i}})$ is non-linear as well as if X^i are certain matrices.

Expanding the action functional above we have the following terms depending on the auxiliary fields

$$S = \langle D, D\rangle - i\langle D, \mu - \zeta\rangle + \langle g_{i\bar{j}} H^i, H^{\bar{j}}\rangle - i\langle H^i, \partial_i \mathcal{W}\rangle - i\langle H^{\bar{i}}, \partial_{\bar{i}} \overline{\mathcal{W}}\rangle + \ldots, \tag{3.2.19}$$

where μ is the $\mathcal{G}$-momentum map on the target space[12] X as defined earlier in (3.1.11), $g_{i\bar{j}} := \partial_i \bar{\partial}_j \mathcal{K}$ and $\partial_i \mathcal{W} = \partial \mathcal{W}/\partial X^i$. We integrate out the auxiliary fields D, H^i and $H^{\bar{i}}$ by imposing the following algebraic equations of motion

$$\begin{aligned} D &= \frac{i}{2}(\mu - \zeta), \\ H^i &= i g^{i\bar{j}} \frac{\partial \overline{\mathcal{W}}}{\partial X^{\bar{j}}}. \end{aligned} \tag{3.2.20}$$

From our general discussion earlier, we see that the path integral is localized to the space of solutions of the following equations

$$\begin{aligned} \mu - \zeta &= 0, \\ \frac{\partial \mathcal{W}}{\partial X^i} &= 0, \end{aligned} \tag{3.2.21}$$

modulo the $\mathcal{G}$-symmetry. In other words the effective target space (3.2.14) is the symplectic quotient at level ζ of the critical set $H_i^{-1}(0) \subset X$ of the holomorphic potential

$$\mathcal{M}_\zeta := \left(H_i^{-1}(0) \cap \mu^{-1}(\zeta)\right)/\mathcal{G}. \tag{3.2.22}$$

Equivalently $\mathcal{M}_\zeta$ is the restriction of $\mathcal{N}_\zeta$, the symplectic quotient of X by $\mathcal{G}$, to the critical subset. Those are compatible since H^i is $\mathcal{G}$-equivariant as $\mathcal{W}$ and S are $\mathcal{G}$-invariant. Thus $\mathcal{M}_\zeta$ is a Kähler manifold, provided that ζ is generic. Note

[10] The total "Kähler" potential $K(X^i, X^{\bar{i}}) - \langle \sigma, \bar{\sigma} \rangle$ can be generalized to an arbitrary $\mathcal{G}$-invariant real functional $\widetilde{\mathcal{K}}(X^i, X^{\bar{i}}; \sigma, \bar{\sigma})$. Then we may obtain a model whose effective target space is non-Kähler but has torsion and generally a dilaton.

[11] The theta term plays no roles in the $(0+0)$-dimension we are considering here.

[12] A better terminology is to regard X as the space of all X^i's.

that the space of all bosonic fields is much bigger than X due to the additional affine space of four real scalars φ^m, $m = 1, \ldots, 4$. The path integral is localized, in addition to (3.2.21), to the space of solutions of

$$[\varphi^m, \varphi^n] = 0, \qquad \varphi^a_m \mathcal{L}_a(X^i) = 0, \tag{3.2.23}$$

modulo the gauge symmetry. As the basic principle of the equivariant cohomology $\varphi_m = 0$ if $\mathcal{G}$ acts freely while, otherwise, there is something else.

Now we assume that $\mathcal{M}_\zeta$ is smooth. Then our model is equivalent to the non-equivariant $N_c = (2,2)$ model with target space $\mathcal{M}_\zeta$. Thus the partition function is the Euler characteristic of the effective target space;

$$Z = \chi(T\mathcal{M}_\zeta) = \chi(\mathcal{M}_\zeta). \tag{3.2.24}$$

A beautiful fact is that our initial target space X may be infinite dimensional with an infinite dimensional group $\mathcal{G}$ acting on it, while the final target space $\mathcal{M}_\zeta$ can be finite dimensional.

The Geometry of Effective Target Space

It is obvious that the group action preserves the condition $H_i = 0$ and the subvariety $H_i^{-1}(0) \subset X$ inherits the complex and Kähler structures by restriction. The quotient space $\mathcal{M}_\zeta$ inherits the Kähler structure from $H_i^{-1}(0)$ by the restrictions and the reduction.

If ζ takes on a generic value, the group $\mathcal{G}$ acts freely and $\mathcal{M}_\zeta$ is a smooth Kähler manifold. For such a case the model can be identified with the non-linear non-equivariant $N_c = (2,2)$ model in Sect. 2.2.2 with target space $\mathcal{M}_\zeta$. This property is equivalent to the property of equivariant cohomology that the equivariant cohomology is the ordinary cohomology of the quotient space if it is smooth.

For non-generic ζ the quotient space develops singularities or even may not exist at all. For such cases however one always has some extra degrees of freedom not described by the moduli space, due to the extension of $X/\mathcal{G}$ to $X_\mathcal{G}$. Those extra degrees of freedom are represented by the solutions of (3.2.23) modulo gauge symmetry. The first equation in (3.2.23) show that no such a solution exists if the $\mathcal{G}$ action act freely, without fixed points, on X. If there are solutions they span an affine space,[13] which looks like a symmetric products of $\mathbb{R}^4$.

The beautiful relation between the symplectic and geometrical invariant theory (GIT) quotients also is an important part of the story [61][62][58]. The essential point is that the condition $H_i = 0$ is preserved by the complexified group action $\mathcal{G}^\mathbb{C}$, while the condition $D = 0$ is only preserved by the real group action. Thus we may consider a complex quotient $H_i^{-1}(0)/\mathcal{G}^\mathbb{C}$ and try to compare with the real quotient $(H_i^{-1}(0) \cap D^{-1}(0))/\mathcal{G}$. In general there can be $\mathcal{G}^\mathbb{C}$-orbits in $H_i^{-1}(0)$ which contain several $\mathcal{G}$ orbits in $H_i^{-1}(0) \cap D^{-1}(0)$. Thus we need to consider a suitable subset in $H_i^{-1}(0)$ for which a $\mathcal{G}^\mathbb{C}$-orbit contains exactly one solution of the equation $D = 0$. Then the real equation $D = 0$ can be identified

[13] We will relate those degrees of freedom, in certain cases, with the degrees transverse to the D-brane world volume in the bulk.

with the gauge fixing condition of the complex gauge symmetry of the complex equations $H_i = 0$.

The complex gauge group in general does not act freely on the submanifold $H_i^{-1}(0)$, so that taking the quotient directly would lead to unwanted singularities. One first removes such obvious bad points B. However there are subsets in $(H_i^{-1}(0) - B)$ which can be arbitrarily close to B by $\mathcal{G}^{\mathbb{C}}$ action. One call a point in $H_i^{-1}(0)$ semi-stable if the closure of its $\mathcal{G}^{\mathbb{C}}$ orbit does not contain B. Let $H_i^{-1}(0)_{ss}$ be the semi-stable subset of $H_i^{-1}(0)$. Now the beautiful fact is that the complex quotient $H_i^{-1}(0)_{ss}/\mathcal{G}^{\mathbb{C}}$ contains the symplectic quotient $\mathcal{M}_0$ as open subset. A stable orbit is a semi-stable orbit if the points of the orbit have at most finite stabilizers under the real $\mathcal{G}$ action. Then the various symplectic quotients $\mathcal{M}_\zeta$ can be identified with the quotient space $H^{-1}(0)_s/\mathcal{G}^{\mathbb{C}}$ in dense open subset. Thus we have

$$H_i^{-1}(0)_{ss}/\mathcal{G}^{\mathbb{C}} \supset \mathcal{M}_\zeta \supset H^{-1}(0)_s/\mathcal{G}^{\mathbb{C}}. \tag{3.2.25}$$

The first relation implies that we have a natural compactification of $\mathcal{M}_\zeta$ by taking the closure in $H_i^{-1}(0)_{ss}/\mathcal{G}^{\mathbb{C}}$. The second relation implies that the various symplectic quotients $\mathcal{M}_\zeta$ are birational with each others.

Two Finite Dimensional Examples

We now consider two examples for finite dimensional target space X borrowed from Witten's papers on two-dimensional gauged linear sigma models [32][33].

Abelian Case

We consider the complex linear space $X = C^{n+1}$ with $U(1)$ action. Let X^i, $i = 1, \dots, n+1$ parameterize coordinates of a single instanton on C^d. Let $U(1)$ act on C^n such that X^i has charge Q_i. We have

$$\mathcal{K} = \sum_{i=1}^{n+1} |X^i|^2, \tag{3.2.26}$$

leading to the momentum map

$$\mu = \sum_i Q_i |X^i|^2. \tag{3.2.27}$$

To be more specific we consider n chiral fields s_i $i = 1, \dots, n$ with charge 1 and other chiral field p with charge $-n$. We pick holomorphic potential

$$\mathcal{W} = p \cdot G(s_i) \tag{3.2.28}$$

where G is a homogeneous polynomials of degree n. We also demand transversality, that $\partial G/\partial s_i = 0$ have no common root except at $s_i = 0$. Then the conditions in (3.2.21) and (3.2.23) become

$$\begin{aligned} \varphi_m p &= 0, \\ \varphi_m s_i &= 0, \\ G(s_i) &= 0, \\ p \cdot \frac{\partial G}{\partial s_i} &= 0, \\ \sum_{i=1}^{n} |s_i|^2 - n|p|^2 - r &= 0, \end{aligned} \tag{3.2.29}$$

Now we examine how the effective target space varies as we vary ζ.

- $r > 0$. The last equation requires that the s_i can not all vanish. Then the second to last equations together with the transversality implies that $p = 0$. The second equation implies that $\varphi_m = 0$. So we are left with

$$\begin{aligned} \sum_i |s_i|^2 - \zeta &= 0, \\ G(s_i) &= 0. \end{aligned} \tag{3.2.30}$$

  Thus the classical vacumn space is the hypersurface X in CP^{n-1} defined by $G(s_i) = 0$. It is a smooth (via transversality) Calabi-Yau space (via anomaly free $\mathcal{R}$-invariance).

- $r < 0$. The last equation requires that $p \neq 0$. The second equations with transversality implies that $s_i = 0$ for all $i = 1, \ldots, n$. The first equation implies that $\varphi_m = 0$. Thus

$$|p| = \sqrt{-r/n}. \tag{3.2.31}$$

- $r = 0$. The only solution for the last two equations is the origin of $\mathbb{C}^{n+1}$ which is fixed by the $U(1)$ action thus a singular point. The equations do not impose any restriction on φ_m thus they span $\mathbb{C}^2$.

A non-Abelian case

We consider the space X of all $N \times k$ hermitian matrices q In the space X we introduce a complex structure such that $\overline{s}_{\pm} q = 0$. We have natural $\mathcal{G} = U(N)$ action on X given by $q \to gq$ where $g \in \mathcal{G}$. The $U(N)$ action preserve the natural Hermitian structure f given by

$$\mathcal{K} = \operatorname{Tr} qq^*. \tag{3.2.32}$$

Then the conditions (3.2.21) and (3.2.23) become

$$\begin{aligned} \varphi_m q &= 0, \\ qq^* - \zeta I &= 0. \end{aligned} \tag{3.2.33}$$

Thus the model reduces to a zero-dimensional sigma model whose target space $\mathcal{M}_\zeta$, for $\zeta > 0$, is the Grassmannian $G(N, k)$ – the space of N complex planes in $\mathbb{C}^k$. For $\zeta = 0$ we have $q = 0$ while φ_m span $Sym^N(\mathbb{R}^4)$.

3.2.3 $N_c = (4,4)$ Model

In this subsection we briefly consider a special case of the $N_c = (2,2)$ model with $N_c = (4,4)$ supersymmetry. Historically the hyper-Kähler quotient was first discovered with the help of the $N_{ws} = (4,4)$ theory [27].

The initial target space X should have a hyper-Kähler structure. Then we have three independent complex structures J_1, J_2, J_3 satisfying $J_i J_j = -\delta_{ij} + \varepsilon_{ijk} I_k$. For each complex structure I_ℓ, $\ell = 1,2,3$, we have a Kähler form ϖ_ℓ. Now we assume that we have $\mathcal{G}$ action preserving all the Kähler forms, i.e., $\mathcal{L}_a \varpi_\ell = 0$. Then for each Kähler form ϖ_ℓ we have a momentum map μ_ℓ. The hyper-Kähler quotient is defined by [27]

$$\mathcal{M}_{\vec{\zeta}} := \bigcap_{\ell=1}^{3} \mu_\ell^{-1}(\zeta_\ell)/\mathcal{G}, \tag{3.2.34}$$

which inherits a hyper-Kähler structure from X by restrictions and reduction. Let d be the exterior derivative on X. For each complex structure J_ℓ we define a real operator $\widetilde{d}_\ell$ by

$$\widetilde{d}_\ell = J_\ell^{-1} d J_\ell^{-1}. \tag{3.2.35}$$

Then we have a decomposition $d = \partial_\ell + \overline{\partial}_\ell$ for each complex structure J_ℓ, where

$$\begin{aligned} \partial_\ell &= \frac{1}{2}(d + i\widetilde{d}_\ell), \\ \overline{\partial}_\ell &= \frac{1}{2}(d - i\widetilde{d}_\ell). \end{aligned} \tag{3.2.36}$$

Now it is obvious that we have supercharges $(\boldsymbol{s}^\ell_\pm, \overline{\boldsymbol{s}}^\ell_\pm)$ for each complex structure, defining balanced Dolbeault equivariant cohomology

$$\begin{aligned} &\{\boldsymbol{s}^\ell_+, \overline{\boldsymbol{s}}^\ell_+\} = -\mathcal{L}_{\phi_{++}}, \quad \{\boldsymbol{s}^\ell_+, \overline{\boldsymbol{s}}^\ell_-\} = -\mathcal{L}_{\sigma^\ell}, \quad \{\boldsymbol{s}^\ell_+, \boldsymbol{s}^\ell_-\} = 0, \quad \{\boldsymbol{s}^\ell_\pm, \boldsymbol{s}^\ell_\pm\} = 0, \\ &\{\boldsymbol{s}^\ell_-, \overline{\boldsymbol{s}}^\ell_-\} = -\mathcal{L}_{\phi_{--}}, \quad \{\overline{\boldsymbol{s}}^\ell_+, \boldsymbol{s}^\ell_-\} = -\mathcal{L}_{\overline{\sigma}^\ell}, \quad \{\overline{\boldsymbol{s}}^\ell_+, \overline{\boldsymbol{s}}^\ell_-\} = 0, \quad \{\overline{\boldsymbol{s}}^\ell_\pm, \overline{\boldsymbol{s}}^\ell_\pm\} = 0, \end{aligned} \tag{3.2.37}$$

where σ^j is a complex scalar obtained by a certain combination of the four real $Lie(\mathcal{G})$-valued scalars. Consequently we have $N_c = (4,4)$ supersymmetry.

It is convenient to pick a complex structure J, say, $J = J_1$ with the corresponding Kähler form $\varpi = \varpi_1$, real momentum map $\mu = \mu_1$ and FI term $\zeta = \zeta_1$ once and for all. Then $\varpi_2 + i\varpi_3$ is the holomorphic symplectic form $\varpi^{0,2}$. We define the complex momentum map $\mu_{\mathbb{C}} = \mu_2 + i\mu_3$ and FI term $\zeta_{\mathbb{C}} = \zeta_2 + i\zeta_3$. Now the internal consistency of the supersymmetry algebra leads to $N_c = (4,4)$ gauge multiplet consisting of the $N_c = (2,2)$ gauge multiplet (3.2.7) and a $N_c = (2,2)$ $Lie(\mathcal{G})$-valued chiral multiplet,

$$\begin{array}{ccccc} \lambda_- & \xleftarrow{\boldsymbol{s}_-} & \tau & \xrightarrow{\boldsymbol{s}_+} & \lambda_+ \\ & {\scriptstyle \boldsymbol{s}_+}\searrow & & \swarrow{\scriptstyle \boldsymbol{s}_-} & \\ & & D_{\mathbb{C}} & & \end{array}, \tag{3.2.38}$$

with the usual supersymmetry transformation laws. It turns out to be natural to modify the $N_c = (2,2)$ transformation laws (3.2.11) of the $N_c = (2,2)$ gauge multiplet by replacing $D \to D + \frac{i}{2}[\tau,\overline{\tau}]$.

Consequently we have in total the $Lie(\mathcal{G})$ valued complex scalars ϕ, σ, τ associated with $\mathcal{G}$-equivariant cohomology. The target space X must be $2n$ complex dimensional, thus we have $2n$ bi-holomorphic fields X^i, $i = 1, \ldots, 2n$.

Now the general action functional with $N_c = (4,4)$ supersymmetry is given by

$$
\begin{aligned}
S =& s_+ s_- \overline{s}_+ \overline{s}_- \mathcal{K}(X^i, X^{\bar{i}}) - s_+ s_- \overline{s}_+ \overline{s}_- \left(\langle \sigma, \overline{\sigma}\rangle - \langle \tau, \overline{\tau}\rangle\right) \\
&+ \frac{1}{\sqrt{2}} s_+ s_- \langle \tau, \mu_{\mathbb{C}}(X^i) - \zeta_{\mathbb{C}}\rangle + h.c. \\
&+ \overline{s}_+ s_- \langle \sigma, t\rangle + h.c.
\end{aligned}
\tag{3.2.39}
$$

where $\mathcal{K}(X^i, X^{\bar{i}})$ is a $\mathcal{G}$-invariant hermitian structure on X. We expand the action functional and determine the on-shell values of the auxiliary fields. We find that the path integral is localized to the space of solutions of the equations, modulo $\mathcal{G}$ symmetry

$$
\begin{aligned}
\mu_{\mathbb{C}}(X^i) - \zeta_{\mathbb{C}} &= 0, \\
\mu(X^i, X^{\bar{i}}) - \zeta &= 0,
\end{aligned}
\tag{3.2.40}
$$

and

$$
\begin{aligned}
\varphi^a_m \mathcal{L}_a(X^i) &= 0, \\
[\varphi^m, \varphi^n] &= 0,
\end{aligned}
\tag{3.2.41}
$$

where φ^m, $m = 1, \ldots, 6$ denote the six independent real components of $(\phi_\pm, \sigma, \overline{\sigma}, \tau, \overline{\tau})$. The equations in (3.2.40) say that the effective target space is the hyper-Kähler quotient $\mathcal{M}_{\vec{\zeta}}$. The first equation in (3.2.40) says that φ^m has non-zero solutions if $\mathcal{G}$ does not act freely. Thus there are no solutions for φ^m if the hyper-Kähler quotient is smooth. As usual, whenever the quotient contains singularity we have non-trivial solutions for φ^m spanning an affine space given by, due to the equation (3.2.41), a certain symmetric product of $\mathbb{R}^6$.

Two Finite Dimensional Examples

Here we give two examples relevant to D-brane physics [16][41].

We consider the space X of all complex $N \times N$ matrices X^i, $i = 1, 2$. We have a natural complex structure on X by demanding X^i are chiral, $\overline{s}_\pm X^i = 0$ while their Hermitian conjugates $(X^i)^* := X^{\bar{i}}$ are anti-chiral. We have a natural $\mathcal{G} = U(N)$ action on X given by $X^i \to g X^i g^{-1}$, preserving the Hermitian structure

$$
\mathcal{K} = \sum_{i=1}^{2} \operatorname{Tr} |X^i|^2. \tag{3.2.42}
$$

The space X has an affine hyper-Kähler structure with complex momentum map

$$
\mu_{\mathbb{C}} = [X^1, X^2]. \tag{3.2.43}
$$

Together with ϕ^ℓ, $\ell = 1,2,3$, where we complexified the six real $N \times N$ matrices φ^n, we have total five $N \times N$ Hermitian matrices as bosonic fields. The action functional actually has $N_c = (8,8)$ supersymmetry, and in fact is the world-volume theory of N D-instantons, or the IKKT matrix theory [43]. Denoting (X^i, ϕ^m) by Z^α, $\alpha = 1, \ldots, 5$, the equations (3.2.40) and (3.2.41) become

$$\begin{aligned}&[Z^\alpha, Z^\alpha] = 0,\\ &[Z^\alpha, Z^{\overline{\beta}}] = 0.\end{aligned} \tag{3.2.44}$$

Thus Z^i can be simultaneously diagonalized $Z^i = diag(z_1^i, \ldots, z_N^i)$ and the ℓ-th eigenvalues $\{z_\ell^i\}$ can be interpreted as the position of the ℓ-th D-instanton in $\mathbb{R}^{10}$ in complex coordinates.

Now we introduce additional chiral multiplets q and $\widetilde{q}$ which are $N \times k$ and $k \times N$ matrices, respectively. The $U(N)$-action is given by $(q, \widetilde{q}) \to (gq, \widetilde{q}g^{-1})$. In the total space of matrix quadruples $(X^1, X^2, q, \widetilde{q})$ we have an $U(N)$ invariant Hermitian structure

$$\mathcal{K} = \mathrm{Tr}\left(\sum_{i=1}^{2} \mathrm{Tr}\,|X^i|^2 + qq^* + \widetilde{q}^*\widetilde{q}\right). \tag{3.2.45}$$

The complex momentum map is

$$\mu_C = [X^1, X^2] + q\widetilde{q}. \tag{3.2.46}$$

The resulting model may be interpreted as the description of N $D0$-branes in the background of k parallel $D3$ branes, which breaks the $N_c = (8,8)$ symmetry down to $N_c = (4,4)$ symmetry. The three FI terms $(\zeta, \zeta_{\mathbb{C}})$ represent an anti-symmetric self-dual two-form being turned on in the $D3$-brane world-volume $\mathbb{R}^4$.

The equations (3.2.40) and (3.2.41) become

$$\begin{aligned}\phi_m q &= 0,\\ \widetilde{q}\phi_m &= 0,\\ [\phi_m, X^i] &= 0,\\ [X^1, X^2] + q\widetilde{q} - \zeta_{\mathbb{C}} I &= 0,\\ \sum_{i=1}^{2}[X^i, X^{\overline{i}}] + qq^* - \widetilde{q}^*\widetilde{q} - \zeta I &= 0.\end{aligned} \tag{3.2.47}$$

For $\zeta \neq 0$ q and $\widetilde{q}$ can not have common zeros. Then we have $\phi^m = 0$ and

$$\begin{aligned}&[X^1, X^2] + q\widetilde{q} - \zeta_{\mathbb{C}} I = 0,\\ &\sum_{i=1}^{2}[X^i, X^{\overline{i}}] + qq^* - \widetilde{q}^*\widetilde{q} - \zeta I = 0.\end{aligned} \tag{3.2.48}$$

Thus the target space[14] is the compactification and the subsequent resolution of singularities of moduli space of N $U(k)$ instantons on $\mathbb{R}^4$. This describes N

[14]The equations are the ADHM description [63][64] of torsion free sheaves on R^4 [65] or the ADHM description of instanton on non-commutative R^4 [66].

D-instantons bound to the world-volume of k coinciding $D3$-branes. Note that the equivariant degrees of freedom represented by ϕ_m are the degrees of freedom transverse to $D3$-branes in the bulk $\mathbb{R}^{10}$. With the FI terms being turned on those degrees of freedom decouple.

Now we turn off the FI terms. Then the $N \times k$ and $k \times N$ matrices q and $\widetilde{q}$ can degenerate to $(N-\ell) \times k$ and $k \times (N-\ell)$ matrices q' and $\widetilde{q}'$, respectively,

$$q = \begin{pmatrix} 0 \\ q' \end{pmatrix}, \qquad \widetilde{q} = (\,0 \quad \widetilde{q}'\,)\,. \tag{3.2.49}$$

Then the first two equations in (3.2.47) imply that the ϕ^m can be non-vanishing $\ell \times \ell$ matrices Z^m,

$$\phi^m = \begin{pmatrix} Z^m & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.2.50}$$

The last three equations in (3.2.47) imply that the X^i can be put into the following form,

$$X^i = \begin{pmatrix} Z^i & 0 \\ 0 & X'^i \end{pmatrix}. \tag{3.2.51}$$

where Z^i and X'^i are $\ell \times \ell$ and $(N-\ell) \times (N-\ell)$ matrices, respectively. Using (3.2.49), (3.2.50) and (3.2.51), the last two equations in (3.2.47) lead to

$$\begin{aligned} [Z^\alpha, Z^\beta] &= 0, \\ [Z^\alpha, Z^{\overline{\alpha}}] &= 0, \end{aligned} \tag{3.2.52}$$

and

$$\begin{aligned} [X'^1, X'^2] + q'\widetilde{q}' &= 0, \\ \sum_{i=1}^{2} [X'^i, X'^{\bar{i}}] + q'q'^* - \widetilde{q}'^* \widetilde{q}' &= 0, \end{aligned} \tag{3.2.53}$$

where we relabeled the $\ell \times \ell$ matrices, Z^i and Z^m as Z^α, $\alpha = 1, \ldots, 5$. The set of equations in (3.2.52) imply that Z^α can be diagonalized

$$Z^\alpha = diag(z_1^\alpha, \ldots, z_\ell^\alpha), \tag{3.2.54}$$

parameterizing positions (in complex coordinates) of ℓ point-like D-instantons on $\mathbb{R}^{10}$. The set of equations in (3.2.53) describe $(N-\ell)$ $U(k)$ instantons on $\mathbb{R}^4$. This bound state is not stable since some of the $(N-\ell)$ instantons can freely degenerate to point-like instantons and escape to the bulk to become D-instantons. Combining all together we see that the model without FI terms reduces to a zero-dimensional $N_c = (4,4)$ sigma-model with target space

$$\bigcup_{\ell=0}^{N} \mathcal{M}_{k,N-\ell} \times Sym^\ell(\mathbb{R}^{10}). \tag{3.2.55}$$

3.3 Generalization to Equivariant $N_c = (2,0)$ Model

Now we consider the equivariant extension of the $N_c = (2,0)$ model introduced in Sect. 2.3 or, equivalently, the generalization of the equivariant $N_{ws} = (2,2)$ model in the previous section. We consider the same group $\mathcal{G}$ acting on X as before but now we allow the $\mathcal{G}$ action to extend to a Hermitian holomorphic vector bundle $\mathbb{E} \to X$ preserving the Hermitian structure. We have two supercharges $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$, isomorphic to the differentials of $\mathcal{G}$-equivariant Dolbeault cohomology as in the equivariant toy model in Sect. 3.1;

$$\boldsymbol{s}_+^2 = 0, \qquad \{\boldsymbol{s}_+, \overline{\boldsymbol{s}}_+\} = -i\phi^a_{++}\mathcal{L}_a, \qquad \overline{\boldsymbol{s}}_+^2 = 0. \tag{3.3.1}$$

Comparing with the non-equivariant counterpart, the equivariant $N_c = (2,0)$ model has essentially one addition structure that the path integral is further localized to the vanishing locus $\mu^{-1}(\zeta)$ of $\mathcal{G}$-moment map. If $\mathcal{G}$ acts freely on $\mu^{-1}(\zeta)$ the model reduce to a standard $N_c = (2,0)$ model associated with the symplectic quotients. The observables of the model are given by $\mathcal{G}$-equivariant closed differential forms, after the parity changes, as our equivariant toy model. If $\mathcal{G}$ acts freely on $\mu^{-1}(\zeta)$ those observables become ordinary closed differential form on the symplectic quotient. Comparing with our equivariant toy model the additional structure is that the path integral is further localized to the locus of vanishing holomorphic sections on $\mathbb{E}$. We will use such property to define a more general hybrid $N_c = (2,0)$ model. Following the discussion in Sect. 3.2 the model is related with $N_{ws} = (2,0)$ world-sheet gauged sigma-model in $(1+1)$ dimensions by dimensional reduction [32].

3.3.1 Basic Structures

We may follow exactly the same route as we followed to arrive at the non-equivariant $N_c = (2,0)$ model from the non-equivariant $N_c = (2,2)$ models.

First we write the $N_c = (2,2)$ action functional S (3.2.17) in a form such that only the $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ are manifest - compare with (2.3.1)-

$$\begin{aligned} S(\zeta) = &- \boldsymbol{s}_+\overline{\boldsymbol{s}}_+\Big(\Big\langle \phi_{--}, \mu(X^i, X^{\bar{i}}) - \zeta\Big\rangle - \langle \eta_-, \overline{\eta}_-\rangle + \Big\langle g_{i\bar{j}}(X^k, X^{\bar{k}})\psi^i_-, \psi^{\bar{j}}_-\Big\rangle\Big) \\ &+ i\boldsymbol{s}_+\langle \psi^i_-, V_i(X^j)\rangle + i\overline{\boldsymbol{s}}_+\Big\langle \psi^{\bar{i}}_-, V_{\bar{i}}(X^{\bar{j}})\Big\rangle, \end{aligned} \tag{3.3.2}$$

where $V_i = \partial\mathcal{W}/\partial X^i$. Similarly we disconnect the diagram (3.2.6) by removing the link $\boldsymbol{s}_-$,

$$\begin{array}{ccc} \psi^i_- & X^i & \xrightarrow{\boldsymbol{s}_+} \psi^i_+ \\ & {}_{\boldsymbol{s}_+}\!\searrow & \\ & H^i & \end{array} \tag{3.3.3}$$

Now we regard the above as two independent sets of multiplets. Then we rename various fields as follows, exactly the same as earlier (2.3.3)

$$\begin{aligned} &\psi^i_- \to \chi^\alpha_-, && H^i \to H^\alpha, && V_i \to \mathfrak{S}_\alpha(X^j), \\ &\psi^{\bar{i}}_- \to \chi^{\bar{\alpha}}_-, && H^{\bar{i}} \to H^{\bar{\alpha}}, && V_{\bar{i}} \to \mathfrak{S}_{\bar{\alpha}}(X^{\bar{j}}), \end{aligned} \qquad g_{i\bar{j}} \to h_{\alpha\bar{\beta}}(X^i, X^{\bar{i}}), \tag{3.3.4}$$

where the new indices run as $\alpha, \beta = 1, \ldots, r$ and we maintain the Hermiticity of $h_{\alpha\bar{\beta}}$. The $N_c = (2,0)$ multiplets (X^i, ψ^i_+) are holomorphic, i.e., $\bar{s}_+ X^i = 0$. We call the multiplets $(\chi^\alpha_-, H^\alpha)$ Fermi multiplets. We also disconnect the diagram (3.2.7) for the $N_c = (2,2)$ gauge multiplet by removing the links s_- and $\bar{s}_-$,

$$\begin{array}{ccccc} \bar{\sigma} & \xrightarrow{s_+} & \eta_+ & & \phi_{++} \\ \\ \bar{\eta}_- & \xrightarrow{s_+} & D & & \bar{\eta}_+ \\ \uparrow \bar{s}_+ & & \uparrow \bar{s}_+ & & \uparrow \bar{s}_+ \\ \phi_{--} & \xrightarrow{s_+} & \eta_- & & \sigma \end{array} \tag{3.3.5}$$

Note that $\bar{\sigma}$ is holomorphic, i.e., $\bar{s}_+\bar{\sigma} = 0$. Thus the $N_c = (2,0)$ multiplet $(\bar{\sigma}, \eta_+)$ is another holomorphic multiplet, while their Hermitian conjugates $(\sigma, \bar{\eta}_+)$ form an anti-holomorphic multiplet. We may simply remove them, or keep them as they are still valued in $Lie(\mathcal{G})$, or just regard them as another holomorphic multiplet supplementing the multiplets (X^i, ψ^i_+).[15] We call the multiplet $(\phi_{--}, \eta_-, \bar{\eta}_-, D)$ $N_c = (2,0)$ gauge multiplet taking values in $Lie(\mathcal{G})$.

Now we consider the transformation laws for the s_+ and $\bar{s}_+$ supersymmetry. For the holomorphic multiplets (X^i, ψ^i_+), i.e., $\bar{s}_+ X^i = 0$, and their conjugates we have

$$\begin{aligned} &s_+ X^i = i\psi^i_+, && s_+\psi^i_+ = 0, && \\ &\bar{s}_+ X^i = 0, && \bar{s}_+\psi^i_+ = \phi^a_{++}\mathcal{L}_a X^i, && s_+\phi_{++} = 0, \\ &s_+ X^{\bar{i}} = 0, && s_+\psi^{\bar{i}}_+ = \phi^a_{++}\mathcal{L}_a X^{\bar{i}}, && \bar{s}_+\phi_{++} = 0, \\ &\bar{s}_+ X^{\bar{i}} = i\psi^{\bar{i}}_+, && \bar{s}_+\psi^{\bar{i}}_+ = 0, && \end{aligned} \tag{3.3.6}$$

which are, of course, the same as (3.1.7). The transformation laws for Fermi multiplets $(\chi^\alpha_-, H^\alpha)$ and their conjugates are given by

$$\begin{aligned} &s_+\chi^\alpha_- = -H^\alpha, && s_+ H^\alpha = 0, \\ &\bar{s}_+\chi^\alpha_- = \mathfrak{J}^\alpha(X^i), && \bar{s}_+ H^\alpha = -i\phi^a_{++}\mathcal{L}_a\chi^\alpha_- + i\psi^i_+\partial_i\mathfrak{J}^\alpha(X^j), \\ &s_+\chi^{\bar{\alpha}}_- = \mathfrak{J}^{\bar{\alpha}}(X^{\bar{i}}), && s_+ H^{\bar{\alpha}} = -i\phi^a_{++}\mathcal{L}_a\chi^{\bar{\alpha}}_- + i\psi^{\bar{i}}_+\partial_{\bar{i}}\mathfrak{J}^{\bar{\alpha}}(X^{\bar{j}}), \\ &\bar{s}_+\chi^{\bar{\alpha}}_- = -H^{\bar{\alpha}}, && \bar{s}_+ H^{\bar{\alpha}} = 0, \end{aligned} \tag{3.3.7}$$

where $\partial_{\bar{i}}\mathfrak{J}^\alpha(X^j) = 0$. Note that $\bar{s}_+\chi^\alpha_- \neq 0$ but rather equals $\mathfrak{J}^\alpha(X^i)$, while the above transformation laws are consistent, since $\bar{s}^2_+\chi^\alpha_- = \bar{s}_+\mathfrak{J}^\alpha(X^i) = 0$,

[15] It is our convention that all the holomorphic multiplets are collectively denoted as (X^i, ψ^i_+) where each multiplet may transform differently under $\mathcal{G}$ and other global symmetries. We also denote X as the space of all X^i's.

with the commutation relations (3.3.1). Finally the transformation laws for the $N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \overline{\eta}_-)$ are given by

$$\begin{aligned} & & s_+\eta_- &= 0, \\ s_+\phi_{--} &= i\eta_-, & \overline{s}_+\eta_- &= +iD + \frac{1}{2}[\phi_{++}, \phi_{--}], \\ \overline{s}_+\phi_{--} &= i\overline{\eta}_-, & s_+\overline{\eta}_- &= -iD + \frac{1}{2}[\phi_{++}, \phi_{--}], \\ & & \overline{s}_+\overline{\eta}_- &= 0. \end{aligned} \tag{3.3.8}$$

The general $N_c = (2,0)$ action functional, with the vanishing ghost number, is given by the following form[16]

$$\begin{aligned} S(\zeta) = &- s_+\overline{s}_+\Big(\langle \phi_{--}, \mu - \zeta\rangle - \langle \eta_-, \overline{\eta}_-\rangle + \left\langle h_{\alpha\overline{\beta}}\chi_-^\alpha, \chi_-^{\overline{\alpha}}\right\rangle\Big) \\ &+ is_+\langle \chi_-^\alpha, \mathfrak{S}^\alpha(X^i)\rangle + i\overline{s}_+\left\langle \chi_-^{\overline{\alpha}}, \mathfrak{S}^{\overline{\alpha}}(X^{\overline{i}})\right\rangle \end{aligned} \tag{3.3.9}$$

Here $h_{\alpha\overline{\beta}}(X^i, X^{\overline{i}})$ is a Hermitian structure on a Hermitian vector bundle $\mathbb{E}$ over X, $\mathfrak{S}^\alpha(X^i)$ a holomorphic section and $\mu(X^i, X^{\overline{i}})$ is the $\mathcal{G}$-momentum map on X. Note that $N_c = (2,0)$ symmetry of the above action functional is not obvious due to the second line in (3.3.9). For example the $\overline{s}_+$ supersymmetry of the term $s_+\langle \chi_-^\alpha, \mathfrak{S}_\alpha\rangle$ is not obvious if $\mathfrak{J}(X^i) \neq 0$ due to the transformation law $\overline{s}_+\chi_-^\alpha = \mathfrak{J}(X^i)$. The condition that the action functional $S(\zeta)$ has $N_c = (2,0)$ is

$$\overline{s}_+\langle \chi_-^\alpha, \mathfrak{S}^\alpha(X^i)\rangle = \langle \mathfrak{J}^\alpha(X^i), \mathfrak{S}^\alpha(X^i)\rangle = 0. \tag{3.3.10}$$

Let us summarize the basic structure of an equivariant $N_c = (2,0)$ model.

1. A complex Kähler target space X with a $\mathcal{G}$ symmetry as an isometry. These data determine holomorphic multiplets and gauge multiplets as well as their transformation laws and $\mathcal{G}$-equivariant momentum map $\mu : X \to Lie(\mathcal{G})^*$.
2. A Hermitian holomorphic vector bundle $\mathbb{E} \to X$ over the target space X with the $\mathcal{G}$ action preserving the Hermitian structure. We may have up to two $\mathcal{G}$-equivariant holomorphic sections $\mathfrak{S}$ and $\mathfrak{J}$ orthogonal with each others by a natural non-degenerated $\mathcal{G}$ invariant parings. Those sections determine Fermi multiplets and their transformation laws.

Given the data above, we have an unique family of equivariant $N_c = (2,0)$ models parameterized by the FI term ζ.

3.3.2 Path Integrals

Expanding the action functional S (3.3.9) we have the following terms depending on the auxiliary fields D, H^α and $H^{\overline{\alpha}}$,

$$S = \langle D, D\rangle - \langle D, \mu - \zeta\rangle + \langle h_{\alpha\overline{\beta}}H^\alpha, H^{\overline{\beta}}\rangle - i\langle H^\alpha, \mathfrak{S}^\alpha\rangle - i\langle H^{\overline{\alpha}}, \mathfrak{S}^{\overline{\alpha}}\rangle + \ldots \tag{3.3.11}$$

[16] The repeated indices are summed over unless otherwise stated.

We integrate the auxiliary fields out by imposing the following algebraic equations of motion,

$$
\begin{aligned}
D &= \frac{1}{2}(\mu - \zeta),\\
H_\alpha &= ih_{\alpha\overline{\beta}}\mathfrak{S}^{\overline{\beta}}.
\end{aligned} \tag{3.3.12}
$$

From our general discussion earlier, we see that the bosonic part of the path integral reduces to an integral over the space of solutions of the following equations,

$$
\begin{aligned}
\mathfrak{J}^\alpha(X^i) &= 0,\\
\mathfrak{S}_\alpha(X^i) &= 0,\\
\mu - \zeta &= 0,
\end{aligned} \tag{3.3.13}
$$

and

$$
\begin{aligned}
\phi^a_{++}\mathcal{L}_a X^i &= 0,\\
[\phi_{++}, \phi_{--}] &= 0,
\end{aligned} \tag{3.3.14}
$$

modulo $\mathcal{G}$-symmetry.

Now we examine the properties of the path integral in some detail by applying the fixed point theorem of Witten. For simplicity assume that the space X and the Hermitian holomorphic bundle E are flat. We also turn off the section $\mathfrak{J}^\alpha$, keeping $\mathfrak{S}$ only. Then the fixed point locus of the s_+ and $\overline{s}_+$ supersymmetry is the symplectic quotient $\mathcal{M}_\zeta$ of $\mathfrak{S}^{-1}_\alpha(0) \subset X$ by $\mathcal{G}$;

$$
\mathcal{M}_\zeta = \left(\mu^{-1}(\zeta) \cap \mathfrak{S}^{-1}_\alpha(0)\right)/\mathcal{G}. \tag{3.3.15}
$$

We have the same set of observables as in the equivariant toy model, given by s_+ and $\overline{s}_+$ closed $\mathcal{G}$-equivariant differential forms $\widehat{\mathcal{O}}^{r,s}$ with ghost numbers (r,s).

The explicit expression of the action functional is

$$
\begin{aligned}
S' =& D^2 + \sum |H_\alpha|^2 - \frac{1}{4}[\phi_{++}, \phi_{--}]^2 - i[\phi_{++}, \eta_-]_a \overline{\eta}^a_-\\
&- \overline{\eta}^a_- \partial_i \mu_a \psi^i_+ - \eta^a_- \partial_{\overline{i}} \mu_a \psi^{\overline{i}}_+ + \chi^\alpha_- \partial_i \mathfrak{S}_\alpha \psi^i_+ + \chi^{\overline{\alpha}}_- \partial_i \mathfrak{S}_{\overline{\alpha}} \psi^{\overline{i}}_+\\
&- ih_{\alpha\overline{\beta}}\phi^a_{++}\mathcal{L}_a \chi^\alpha_- \chi^{\overline{\beta}}_- + i\phi^a_{--}\left(\phi^b_{++}\partial_{\overline{i}}\mu_a V^{\overline{i}}_b + \partial_i\partial_{\overline{j}}\mu_a \psi^i_+ \psi^{\overline{j}}_+\right),
\end{aligned} \tag{3.3.16}
$$

where $V^{\overline{i}}_b = \mathcal{L}_a X^{\overline{i}}$. In doing the path integral one replaces all fields by their zero-modes. The zero-modes of the fermions are solutions of the following equations

$$
\begin{aligned}
\partial_{\overline{i}}\mu_a \psi^{\overline{i}}_+ &= 0, \qquad & \eta^a_- \partial_{\overline{i}}\mu_a &= 0,\\
\partial_{\overline{i}}\mathfrak{S}_{\overline{\alpha}} \psi^{\overline{i}}_+ &= 0, \qquad & \chi^{\overline{\alpha}} \partial_{\overline{i}}\mathfrak{S}_{\overline{\alpha}} &= 0.
\end{aligned} \tag{3.3.17}
$$

The above equations implies that the net ghost number violation Δ in the path integral measure due to fermionic zero-modes of $(\psi^{\overline{i}}_+, \chi^{\overline{\alpha}}_-, \eta_-)$ always equals

$$
\Delta = n - r - dim\,\mathcal{G}. \tag{3.3.18}
$$

We call $\triangle$ the virtual complex dimension of $\mathcal{M}_\zeta$. From the equations $\partial_{\bar{i}}\mu_a\psi_+^{\bar{i}} = 0$ we have the following integrability condition

$$\phi^b_{++}\partial_{\bar{i}}\mu_a V_b^{\bar{i}} + \partial_i\partial_{\bar{j}}\mu_a\psi_+^i\psi_+^{\bar{j}} = 0, \tag{3.3.19}$$

which is also the ϕ^a_{--} equation of motion. This implies that one can simply replace ϕ^a_{++} with the solutions of the above. Such an argument can not be justified if there are zero-modes of ϕ^a_{++}, which are given by the non-trivial solutions of (3.3.14), for instance $\phi^b_{++}V_b^{\bar{i}} = 0$.

Here we specialize to the case that $\mathcal{G}$ acts freely, thus there are no zero-modes of η^a_- and $\phi^a_{\pm\pm}$. Then the only non-trivial term in the action functional S' in the $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ invariant neighborhood $\mathcal{C}$ of the fixed point locus is

$$S'|_\mathcal{C} = -ih_{\alpha\overline{\beta}}\phi^a_{++}\mathcal{L}_a\chi^\alpha_-\chi^{\overline{\beta}}_-|_\mathcal{C}. \tag{3.3.20}$$

Using (3.3.19) we can solve ϕ^a_{++} in terms of the zero-modes $(u^{\bar{i}'}, \widetilde{\psi}_+^{\bar{i}'}, \widetilde{\chi}_-^{\overline{\alpha}'})$ of $(X^{\bar{i}}, \psi_+^{\bar{i}}, \chi_-^{\overline{\alpha}})$

$$< \phi^a_{++}(u^{i'}, u^{\bar{i}'}) >= -\left(\partial_{\bar{\ell}'}\mu_b V_a^{\bar{\ell}'}\right)^{-1}\partial_{i'}\partial_{\bar{j}'}\mu_b\widetilde{\psi}_+^{i'}\widetilde{\psi}_+^{\bar{j}'} \tag{3.3.21}$$

where the primed indices above are understood to label independent zero-modes - $\bar{i}' = 1, \ldots, n'$, $\overline{\alpha}' = 1, \ldots, r'$, with the condition

$$\triangle = n' - r' = n - r - dim\,\mathcal{G}. \tag{3.3.22}$$

Then we may write

$$S'|_\mathcal{C} = -\mathcal{F}(u^{\ell'}, u^{\bar{\ell}'})_{i'\bar{j}'\alpha'\overline{\beta}'}\widetilde{\psi}_+^{i'}\widetilde{\psi}_+^{\bar{j}'}\widetilde{\chi}_-^{\alpha'}\widetilde{\chi}_-^{\overline{\beta}'}, \tag{3.3.23}$$

where $\mathcal{F}_{i'\bar{j}'\alpha'\overline{\beta}'}\widetilde{\psi}_+^{i'}\widetilde{\psi}_+^{\bar{j}'}$ can be interpreted as the curvature two-form of the anti-ghost bundle $\mathbb{V}$ over $\mathcal{M}_\zeta$. Consequently the path integral reduces to

$$\begin{aligned}\left\langle\prod_{m=1}^k \widehat{\mathcal{O}}^{r_m,s_m}\right\rangle = \int_{\mathcal{M}_\zeta}\prod_{\gamma'=1}^{r'} d\widetilde{\chi}_-^{\gamma'}d\widetilde{\chi}_-^{\overline{\gamma}'}\prod_{\ell'=1}^{n'} du^{\ell'}du^{\bar{\ell}'}d\widetilde{\psi}_+^{\ell'}\widetilde{\psi}_+^{\bar{\ell}'} \\ \times \exp\left(\mathcal{F}_{i'\bar{j}'\alpha'\overline{\beta}'}\widetilde{\psi}_+^{i'}\widetilde{\psi}_+^{\bar{j}'}\widetilde{\chi}_-^{\alpha'}\widetilde{\chi}_-^{\overline{\beta}'}\right)\prod\widetilde{\mathcal{O}}^{r_m,s_m},\end{aligned} \tag{3.3.24}$$

where $\widetilde{\widehat{\mathcal{O}}}$ denote the expression of an observable $\widehat{\mathcal{O}}$ in terms of zero-modes and $< \phi_{++} >$. The necessary condition for a non-vanishing correlation function is

$$\sum_{m=1}^k (r_m, s_m) = (\triangle, \triangle). \tag{3.3.25}$$

Let us first assume that the section is generic and $\mathcal{G}$ acts freely on $\mathfrak{S}_\alpha^{-1}(0) \subset X$. Then $\mathcal{M}_\zeta$ is a smooth non-linear Kähler manifold with complex dimensions

$$dim_\mathbb{C}\mathcal{M}_\zeta = \triangle = n - r - dim\,\mathcal{G}. \tag{3.3.26}$$

The above counting goes as follows. Since $\mathfrak{S}_\alpha$, $\alpha = 1, \ldots, r$, are generic they are all independent and transverse. Thus the condition $\mathfrak{S}_\alpha = 0$ cuts out a complex $(n-r)$ smooth submanifold inside the complex n-dimensional ambient space X. On the subspace we further impose $dim\mathcal{G}$ real equations $\mu_a - \zeta = 0$ and take the quotient by the free $\mathcal{G}$ action. Now we do not have zero-modes of χ_- and the path integral becomes

$$\left\langle \prod_{m=1}^{k} \widehat{\mathcal{O}}^{r_m,s_m} \right\rangle = \int_{\mathcal{M}_\zeta} \prod_{\ell'=1}^{\triangle} du^{\ell'} du^{\bar{\ell}'} d\widetilde{\psi}_+^{\ell'} \widetilde{\bar{\psi}}_+^{\bar{\ell}'} \prod_m \widetilde{\widehat{\mathcal{O}}}^{r_m,s_m}$$
$$= \int_{\mathcal{M}_\zeta} \widetilde{\mathcal{O}}^{r_1,s_1} \wedge \ldots \wedge \widetilde{\mathcal{O}}^{r_k,s_k}. \tag{3.3.27}$$

A non-generic situation arises when $\mathfrak{S}_{\alpha'}$, $\alpha' = 1, \ldots, r'$, are linearly dependent to the remaining sections. Then the complex dimension of $\mathcal{M}_\zeta$ is given by $n' = \triangle + r'$. The resulting space is smooth if the linearly independent components of the section are transverse. We have r' χ_- zero-modes which span the anti-ghost bundle $\mathbb{V}$ over $\mathcal{M}_\zeta$. The path integral becomes

$$\left\langle \prod_{m=1}^{k} \widehat{\mathcal{O}}^{r_m,s_m} \right\rangle = \int_{\mathcal{M}_\zeta} e(\mathbb{V}) \wedge \widetilde{\mathcal{O}}^{r_1,s_1} \wedge \ldots \wedge \widetilde{\mathcal{O}}^{r_k,s_k}. \tag{3.3.28}$$

A beautiful fact about this is that $(X, \mathbb{E}, \mathcal{G})$ can be all infinite dimensional while the space $\mathcal{M}_\zeta$ can be a finite dimensional space. In particular X can be a certain function space defined by the space of all fields of a certain gauge field theory on a manifold M. Then the integral we are dealing with is a genuine path integral of a non-trivial quantum field theory on M, while the path integral eventually reduces to an ordinary integral on a smooth finite dimensional space $\mathcal{M}_\zeta$. The above is a key principle underlying cohomological field theory [1][24]. In principle the above path integral formalism is well-defined regardless of the properties the moduli space $\mathcal{M}_\zeta$.

Finally we remark that a proper mathematical interpretation of our formalism may be a certain equivariant version of Fulton and MacPherson's intersection theory [40].

3.3.3 Deformation to Holomorphic $N_c = (2,0)$ Model

In this subsection we introduce hybrid $\mathcal{N}_c = (2,0)$ model of the equivariant $N_c = (2,0)$ mode and the equivariant toy model in Sect. 3.2. The resulting hybrid model will have much better behavior than the original model when the effective target space $\mathcal{M}_\zeta$ has singularities. To motivate such a model we first compare the two models.

First of all both the models have the same supersymmetry generated by s_+ and $\bar{s}_+$, which are the differentials of equivariant Dolbeault cohomology after the parity change. Secondly both the models share the same holomorphic multiplets (X^i, ψ_+^i) and their Hermitian conjugates, which are anti-holomorphic multiplets $(X^{\bar{i}}, \psi_+^{\bar{i}})$. Thus they share the same observables, given by equivariantly closed differential forms on X, the space of all X^i, after the parity change.

A difference is that the equivariant $N_c = (2,0)$ model has the additional Fermi multiplets $(\chi_-^\alpha, H^\alpha)$ and their Hermitian conjugates $(\chi_-^{\overline{\alpha}}, H^{\overline{\alpha}})$. The roles of the Fermi multiplets are to restrict the (path) integral over X to the subspace defined by $\mathfrak{J}_\alpha^{-1}(0) \cap \mathfrak{S}_\alpha^{-1}(0) \subset X$. For convenience we denote this subspace by $X^{1,1} \subset X$. We saw that the path integral of the $N_c = (2,0)$ model is localized to the symplectic quotients $\mathcal{M}_\zeta = (X^{1,1} \cap \mu^{-1}(\zeta)$ of $X^{1,1}$ by $\mathcal{G}$. Now we consider an equivariant toy model whose initial target space is $X^{1,1}$. Then, its path integral is also localized to the same space $\mathcal{M}_\zeta$, provided that we set $\varepsilon = 0$ in the action functional $S(\zeta, \varepsilon)$ defined by (3.1.37). We also note that the partition function of the above equivariant toy model is the expectation value of $\exp(\widehat{\varpi}_\mathcal{G})$ evaluated by the $N_c = (2,0)$ model, where

$$\widehat{\varpi}_\mathcal{G} := S(\zeta, 0) = i\,\langle \phi_{++}, \mu - \zeta \rangle + i g_{i\overline{j}} \psi_+^i \psi_+^{\overline{i}}. \tag{3.3.29}$$

The first term above is irrelevant as the path integral of the $N_c = (2,0)$ is localized the to the locus $\mu - \zeta = 0$, while the second term above becomes the Kähler from $\widetilde{\varpi}$ on $\mathcal{M}_\zeta$. We note that it is the $N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \overline{\eta}_-, D)$, which is responsible for such a localization. One the other hand, the above is the action functional of the equivariant toy model on $X^{1,1}$ and the integration over ϕ localizes the path integral by a delta function supported on $\mathcal{M}_\zeta$ in $X^{1,1}$. Note that $\mathcal{M}_\zeta = \mathcal{N}_\zeta|_{X^{1,1}}$ is the restriction of $\mathcal{N}_\zeta$ - the symplectic quotient of X by $\mathcal{G}$ - to $X^{1,1}$.

The above discussion motivates us to define a new $N_c = (2,0)$ model with the following action functional $S_h(\zeta, 0)$, modifying the original $N_c = (2,0)$ action functional S in (3.3.9)

$$\begin{aligned} S_h(\zeta, 0) = & -i s_+ \overline{s}_+ \left\langle h_{\alpha\overline{\beta}} \chi_-^\alpha, \chi_-^{\overline{\alpha}} \right\rangle + i s_+ \langle \chi_-^\alpha, \mathfrak{S}_\alpha \rangle + i \overline{s}_+ \langle \chi_-^{\overline{\alpha}}, \mathfrak{S}_{\overline{\alpha}} \rangle \\ & - i\,\langle \phi_{++}, \mu - \zeta \rangle - i g_{i\overline{j}} \psi_+^i \psi_+^{\overline{i}}, \end{aligned} \tag{3.3.30}$$

where we removed the $N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \overline{\eta}_-, D)$ and added the action functional $S(\zeta, 0)$ of the equivariant toy model. According to the previous discussion we see that the partition function defined by the new action $S_h(\zeta, 0)$ is equivalent to the expectation value of $\exp(\widehat{\varpi}_\mathcal{G})$, evaluated by the original $N_c = (2,0)$ action functional S (3.3.9).

Now we define more general action functional $S_h(\zeta, \varepsilon)$ by

$$\begin{aligned} S_h(\zeta, \varepsilon) := & - s_+ \overline{s}_+ \left\langle h_{\alpha\overline{\beta}} \chi_-^\alpha, \chi_-^{\overline{\alpha}} \right\rangle + i s_+ \langle \chi_-^\alpha, \mathfrak{S}_\alpha \rangle + i \overline{s}_+ \langle \chi_-^{\overline{\alpha}}, \mathfrak{S}_{\overline{\alpha}} \rangle \\ & - i\,\langle \phi_{++}, \mu - \zeta \rangle - i g_{i\overline{j}} \psi_+^i \psi_+^{\overline{i}} + \frac{\varepsilon}{2} \langle \phi_{++}, \phi_{++} \rangle\,. \end{aligned} \tag{3.3.31}$$

We call the $N_c = (2,0)$ model with the above action functional $S_h(\zeta, \varepsilon)$ a holomorphic $N_c = (2,0)$ model, see [67] for the first example. Now we immediately see that the path integral of the holomorphic $N_c = (2,0)$ model is governed by Witten's non-Abelian localization principle [55]. The first line of the above action functional localizes the path integral to $X^{1,1}$. Then, following the discussions in Sect. 2.2.3, the path integral can be written as the sum of contributions of the critical points $I = \langle \mu - \zeta, \mu - \zeta \rangle$ in $X^{1,1}$. Also from the discussions in Sect. 2.2.3 the ε-dependent term regularizes the path integral when $\mathcal{M}_\zeta$ develops singularities.

The Mapping Between the Two Models

Now we will give more wider viewpoints which contain the original and holomorphic $N_c = (2,0)$ models as two special limits, following the original method of Witten [55]. Witten considered the case without the Fermi multiplets but for general manifolds. The Fermi multiplets will be purely spectators, and the specialization to a Kähler case will simplify the procedure.

Consider the following one-parameter family of $N_c = (2,0)$ supersymmetric action functional $S(\zeta)_\lambda$,

$$\begin{aligned} S(\zeta)_\lambda :=& S(\zeta) + \frac{\lambda}{2} s_+ \bar{s}_+ \langle \phi_{--}, \phi_{--} \rangle \\ =& - s_+ \bar{s}_+ \left\langle h_{\alpha\bar{\beta}} \chi^\alpha_-, \chi^{\bar{\alpha}}_- \right\rangle + i s_+ \langle \chi^\alpha_-, \mathfrak{S}_\alpha \rangle + i \bar{s}_+ \langle \chi^{\bar{\alpha}}_-, \mathfrak{S}_{\bar{\alpha}} \rangle \\ & - s_+ \bar{s}_+ \left(\langle \phi_{--}, \mu - \zeta - \frac{\lambda}{2} \phi_{--} \rangle - \langle \eta_-, \bar{\eta}_- \rangle \right). \end{aligned} \tag{3.3.32}$$

If we set $\lambda = 0$ we have the original $N_c = (2,0)$ model. For $\lambda \neq 0$ we can integrate out the $N_c = (2,0)$ gauge multiplet, and we are left with

$$\begin{aligned} S'(\zeta)_\lambda =& - s_+ \bar{s}_+ \left\langle h_{\alpha\bar{\beta}} \chi^\alpha_-, \chi^{\bar{\alpha}}_- \right\rangle + i s_+ \langle \chi^\alpha_-, \mathfrak{S}_\alpha \rangle + i \bar{s}_+ \langle \chi^{\bar{\alpha}}_-, \mathfrak{S}_{\bar{\alpha}} \rangle \\ & + \frac{1}{2\lambda} s_+ \bar{s}_+ \langle \mu - \zeta, \mu - \zeta \rangle + \mathcal{O}(1/\lambda^2). \end{aligned} \tag{3.3.33}$$

Since the additional λ-dependent term is closed by s_+ and $\bar{s}_+$, the path integral does not depend on λ as long as $\lambda \neq 0$. The models with $\lambda = 0$ and $\lambda \neq 0$ can be different since new fixed points can flow from the infinity $\lambda \to \infty$ in the field space [55].

If we take the limit $\lambda \to 0$, while $\lambda \neq 0$, we see that the dominant contributions to the path integral come from the critical points of $I = \langle \mu - \zeta, \mu - \zeta \rangle$. Now we add s_+ and $\bar{s}_+$-closed observables, $-\widehat{\varpi} + \frac{\varepsilon}{2} \langle \phi_{++}, \phi_{++} \rangle$, to the above action functional,

$$\begin{aligned} S'(\zeta, \varepsilon)_\lambda =& - s_+ \bar{s}_+ \left\langle h_{\alpha\bar{\beta}} \chi^\alpha_-, \chi^{\bar{\alpha}}_- \right\rangle + i s_+ \langle \chi^\alpha_-, \mathfrak{S}_\alpha \rangle + i \bar{s}_+ \langle \chi^{\bar{\alpha}}_-, \mathfrak{S}_{\bar{\alpha}} \rangle \\ & - i \langle \phi_{++}, \mu - \zeta \rangle - i \left\langle g_{i\bar{j}} \psi^i_+, \psi^{\bar{i}}_+ \right\rangle + \frac{\varepsilon}{2} \langle \phi_{++}, \phi_{++} \rangle \\ & + \frac{1}{2\lambda} s_+ \bar{s}_+ \langle \mu - \zeta, \mu - \zeta \rangle + \mathcal{O}(1/\lambda^2). \end{aligned} \tag{3.3.34}$$

In the above the path integral should be independent of $\lambda \neq 0$. Consequently we see that the partition function of the above action functional can still be written as a sum of contributions from the critical points of I. Finally we may take the limit $\lambda \to \infty$ to remove all the λ-dependent terms and obtain the action functional $S_h(\zeta, \varepsilon)$ (3.3.31) of the holomorphic $N_c = (2,0)$ model. Thus we showed that the partition function of the holomorphic $N_c = (2,0)$ model can be written as a sum of contributions from the critical points of $I = \langle \mu - \zeta, \mu - \zeta \rangle$.

A Historical Example

In this subsection we recall the first physical application of the non-Abelian localization principle to physical Yang-Mills theory on a Riemann surface Σ. Witten showed that physical Yang-Mills theory can be obtained by deforming the two dimensional version of Donaldson-Witten theory [55]. It is also an ideal example showing that a cohomological field theory in certain space-time manifold M is realized as a $(0+0)$-dimensional supersymmetric sigma model whose target space is the space of all fields. Furthermore the path integral over the infinite dimensional space of fields reduces to a nice integral over a finite dimensional space.

We consider the equivariant $N_c = (2,0)$ model whose target space $X = \mathcal{A}_\Sigma$ is the space $\mathcal{A}_\Sigma$ of all connections of a $SU(2)$ bundle E over a Riemann surface. To write down the model we need some data, namely the nature the $\mathcal{G}$ action, and the complex and Kähler structure on the target space. The group $\mathcal{G}$ is the group of all gauge transformations, i.e., $g \in \mathcal{G}$ where $g : \Sigma \to SU(2)$. The Lie algebra $Lie(\mathcal{G})$ of $\mathcal{G}$ is $\Omega^0(\Sigma, End(E))$ and we use integration over Σ to identity $Lie(\mathcal{G})^*$ with $\Omega^2(\Sigma, End(E))$. Thus the bi-invariant inner product on $Lie(\mathcal{G})$ is the integral over Σ combined with the trace of $SU(2)$;

$$< a, a >= -\int_\Sigma \operatorname{Tr}(a \wedge *a). \tag{3.3.35}$$

According to a complex structure on Σ the connection 1-form A is decomposed as

$$A = A^{1,0} + A^{0,1}. \tag{3.3.36}$$

We introduce a complex structure on $\mathcal{A}_\Sigma$ by declaring $\delta A^{0,1} \in \Omega^{0,1}(\Sigma, End(E))$ to be a holomorphic tangent vector in $T\mathcal{A}_\Sigma$. The Kähler form on $\mathcal{A}_\Sigma$ is defined by

$$\varpi = \frac{1}{4\pi^2}\int_M \operatorname{Tr}\delta A^{1,0} \wedge \delta A^{0,1}. \tag{3.3.37}$$

Finally the action of $\mathcal{G}$ preserves the above complex and Kähler structure.

Now we consider the corresponding $N_c = (2,0)$ model. According to the above complex structure we have holomorphic and anti-holomorphic multiplets, $(A^{0,1}, \psi_+^{0,1})$ and $(A^{1,0}, \overline{\psi}_+^{1,0})$, with the supersymmetry transformation laws

$$\begin{aligned}
&\boldsymbol{s}_+ A^{0,1} = i\psi_+^{0,1}, && \boldsymbol{s}_+\psi_+^{0,1} = 0, && \\
&\overline{\boldsymbol{s}}_+ A^{0,1} = 0, && \overline{\boldsymbol{s}}_+\psi_+^{0,1} = -\overline{\partial}_A\phi_{++}, && \boldsymbol{s}_+\phi_{++} = 0, \\
&\boldsymbol{s}_+ A^{1,0} = 0, && \boldsymbol{s}_+\overline{\psi}_+^{1,0} = -\partial_A\phi_{++}, && \overline{\boldsymbol{s}}_+\phi_{++} = 0, \\
&\overline{\boldsymbol{s}}_+ A^{1,0} = i\overline{\psi}_+^{1,0}, && \overline{\boldsymbol{s}}_+\overline{\psi}_+^{1,0} = 0. &&
\end{aligned} \tag{3.3.38}$$

where $\phi_{++} \in \Omega^0(M, End(E))$ and $\psi^{0,1} \in \Omega^{0,1}(M, End(E))$. Note that

$$\{\boldsymbol{s}, \overline{\boldsymbol{s}}\}A = -id_A\phi_{++}, \qquad \{\boldsymbol{s}, \overline{\boldsymbol{s}}\}\psi^{0,1} = i[\phi_{++}, \psi_{++}^{0,1}]. \tag{3.3.39}$$

which are the infinitesimal gauge transformations generated by ϕ_{++}. We have a $N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \overline{\eta}_-, D)$, which take values in $Lie(\mathcal{G}) =$

$\Omega^0(\Sigma, End(E))$. The supersymmetry transformations are given by (3.3.8). We do not consider any Fermi multiplets. We do not have a FI term. One finds that the $\mathcal{G}$-equivariant Kähler form $\widehat{\varpi}_{\mathcal{G}}$ on $\mathcal{A}_\Sigma$ is

$$\widehat{\varpi}_{\mathcal{G}} = \frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\left(i\phi_{++}F + \psi_+^{0,1}\wedge\overline{\psi}_+^{1,0}\right), \tag{3.3.40}$$

where $F \in \Omega^2(\Sigma, End(E)) = Lie(\mathcal{G})^*$ is the Yang-Mills curvature two-form. Thus the momentum map is

$$\mu = \frac{1}{4\pi^2}F. \tag{3.3.41}$$

Now the $N_c = (2,0)$ action functional (3.3.9) becomes

$$S = s_+\overline{s}_+\int_\Sigma \mathrm{Tr}\left(\frac{1}{4\pi^2}\phi_{--}F + \frac{1}{2\pi^2}\eta_-\overline{\eta}_-\omega\right), \tag{3.3.42}$$

where ω is the Kähler form on Σ. The above action functional defines Donaldson-Witten theory on the Riemann surface Σ.[17] The path integral is an integral over the space of all connections;

$$\left\langle\widehat{\mathcal{O}}\right\rangle = \frac{1}{vol(\mathcal{G})}\int[\mathcal{D}A\mathcal{D}\psi_+\mathcal{D}\overline{\psi}_+\mathcal{D}\phi_{++}\mathcal{D}\phi_{--}\mathcal{D}\eta_-\mathcal{D}\overline{\eta}_-]e^{-S}\cdot\widehat{\mathcal{O}}. \tag{3.3.43}$$

According to our discussion the path integral reduces to an integral over the moduli space $\mathcal{M}_\Sigma$ of flat connections on Σ, provided that $\widehat{\mathcal{O}}$ is an s_+ and $\overline{s}_+$-closed observable (thus an element of $\mathcal{G}$-equivariant cohomology). Thus the correlation functions of supersymmetric observables are intersection pairing on the moduli space $\mathcal{M}_\zeta$ of flat connections.

The action functional (3.3.31) of the holomorphic $N_c = (2,0)$ model is

$$S_h(\varepsilon) = -\frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\left(i\phi_{++}F + \psi_+^{0,1}\wedge\overline{\psi}_+^{1,0}\right) - \frac{\varepsilon}{8\pi^2}\int\omega\,\mathrm{Tr}\,\phi_{++}^2. \tag{3.3.44}$$

This action functional defines physical Yang-Mills theory on Σ. The relation between the two models are as described previously. Finally we note that the solutions of the above Donaldson-Witten theory are found by solving physical Yang-Mills theory [55].

3.4 Flows from $N_c = (2,2)$ to $N_c = (2,0)$ Models

In this section we try to complete the circle of ideas by relating a $N_c = (2,0)$ model with a $N_c = (2,2)$ model. This section is not for introducing new model but for introducing an useful method of computing the path integrals. We will utilize techniques developed here in the remaining chapters of this thesis concerning infinite dimensional examples.

[17] This cohomological field theory can be obtained by a twisting of $N_{ws} = (2,2)$ space-time supersymmetric Yang-Mills theory in two dimensions.

Consider an equivariant $N_c = (2,0)$ model as described in Sect. 3.2, with $\mathfrak{J}^\alpha = 0$. Such a model was classified by a $\mathcal{G}$-equivariant Hermitian holomorphic bundle $\mathbb{E} \to X$ with holomorphic section $\mathfrak{S}_A$. In a generic situation the model is equivalent to a non-linear $N_c = (2,0)$ model which target space $\mathcal{M}_\zeta$ is $\mathcal{M}_\zeta = (X \cap \mathfrak{S}^{-1}(0) \cap \mu^{-1}(\zeta))/\mathcal{G}$. In this section we define a canonical embedding of such a model to a $N_c = (2,2)$ model based on the tangent space $T\mathbb{E}$ of the total space of the bundle $\mathbb{E} \to X$. Then we study perturbation of the $N_c = (2,2)$ to a more general $N_c = (2,0)$ model. We will see that the above circle of ideas leads us to find a $N_c = (2,0)$ model which is "equivalent" to the original $N_c = (2,0)$ model. From the viewpoint of the original $N_c = (2,0)$ model there is no a priori reason of such an "equivalence" to a completely different model. For simplicity we restrict to linear models.

3.4.1 Embedding of a $N_c = (2,0)$ Model to a $N_c = (2,2)$ Model.

Recall that the $N_c = (2,0)$ model has a $Lie(\mathcal{G})$-valued gauge multiplet associated with the group action of $\mathcal{G}$. We add a $Lie(\mathcal{G})$-valued holomorphic multiplet $\overline{\sigma} \xrightarrow{s_+} \eta_+$, to form a $N_c = (2,2)$ gauge multiplet

$$\begin{array}{ccccc} \overline{\sigma} & \xrightarrow{s_+} & \eta_+ & \xleftarrow{s_-} & v_{++} \\ \downarrow{\scriptstyle \overline{s}_-} & & \downarrow{\scriptstyle \overline{s}_-} & & \downarrow{\scriptstyle \overline{s}_-} \\ \overline{\eta}_- & \xrightarrow{s_+} & D & \xleftarrow{s_-} & \overline{\eta}_+ \, . \\ \uparrow{\scriptstyle \overline{s}_+} & & \uparrow{\scriptstyle \overline{s}_+} & & \uparrow{\scriptstyle \overline{s}_+} \\ v_{--} & \xrightarrow{s_+} & \eta_- & \xleftarrow{s_-} & \sigma \end{array} \tag{3.4.1}$$

We had holomorphic multiplets $(X^i \xrightarrow{s_+} \psi^i_+)$, $i = 1, \ldots, n$, associated with the base space X of $\mathbb{E} \to X$. By adding new Fermi multiplets $(\psi^i_- \xrightarrow{s_+} H^i)$, we extend them to $N_c = (2,2)$ chiral multiplets;

$$\begin{array}{ccccc} \psi^i_- & \xleftarrow{s_-} & X^i & \xrightarrow{s_+} & \psi^i_+ \\ & {\scriptstyle s_+}\searrow & & \swarrow{\scriptstyle s_-} & \quad . \\ & & H^i & & \end{array} \tag{3.4.2}$$

We also had Fermi multiplets $(\chi^\alpha_- \xrightarrow{s_+} H^\alpha)$, $\alpha = 1, \ldots, r$, associated with the fiber of $\mathbb{E} \to X$. By adding new holomorphic multiplets $(B^\alpha \xrightarrow{s_+} \chi^\alpha_+)$, we extend them to $N_c = (2,2)$ chiral multiplets;

$$\begin{array}{ccccc} \chi^\alpha_- & \xleftarrow{s_-} & B^\alpha & \xrightarrow{s_+} & \chi^\alpha_+ \\ & {\scriptstyle s_+}\searrow & & \swarrow{\scriptstyle s_-} & \quad . \\ & & H^\alpha & & \end{array} \tag{3.4.3}$$

Now we consider the following $N_c = (2,2)$ supersymmetric action functional

$$\begin{aligned} S =& \boldsymbol{s}_+\overline{\boldsymbol{s}}_+\boldsymbol{s}_-\overline{\boldsymbol{s}}_-\left(\sum_{i=1}^n \langle X^i, X^{\bar{i}}\rangle + \sum_{\alpha=1}^r \langle B^\alpha, B^{\overline{\alpha}}\rangle - <\sigma, \overline{\sigma}>\right) \\ &+ \boldsymbol{s}_+\boldsymbol{s}_- \mathcal{W}\left(X^i, B^\alpha\right) + \overline{\boldsymbol{s}}_+\overline{\boldsymbol{s}}_- \overline{\mathcal{W}}\left(X^{\bar{i}}, B^{\overline{\alpha}}\right) \\ &+ \boldsymbol{s}_+\overline{\boldsymbol{s}}_- <t, \overline{\sigma}> + \overline{\boldsymbol{s}}_+\boldsymbol{s}_- <\sigma, \bar{t}>. \end{aligned} \tag{3.4.4}$$

To relate the above model with the initial $N_c = (2,0)$ model we assume the following conditions

$$\frac{\partial \mathcal{W}}{\partial B^\alpha} = \mathfrak{S}_\alpha(X^i), \tag{3.4.5}$$

where $\mathfrak{S}_\alpha$ is the holomorphic section of $\mathbb{E}$. This condition implies that $\mathcal{W}(X^i, B^\alpha)$ is linear in B^α. We will utilize this property later. It is useful to rewrite the action functional S (3.4.4) such that only the $N_c = (2,0)$ symmetry is manifest

$$\begin{aligned} S =& -i\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\left(\langle \phi_{--}, \mu_X + \mu_F - \zeta\rangle + \sum \langle \psi^i_-, \psi^{\bar{i}}_-\rangle + \sum \langle \chi^\alpha_-, \chi^{\overline{\alpha}}_-\rangle - \langle \eta_-, \overline{\eta}_-\rangle\right) \\ &+ i\boldsymbol{s}_+\left(\langle \psi^i_-, G_i\rangle + \langle \chi^\alpha_-, \mathfrak{S}_\alpha\rangle\right) + i\overline{\boldsymbol{s}}_+\left(\langle \psi^{\bar{i}}_-, G_{\bar{i}}\rangle + \langle \chi^{\overline{\alpha}}_-, \mathfrak{S}_{\overline{\alpha}}\rangle\right) \end{aligned} \tag{3.4.6}$$

where μ_X and μ_F are the momentum maps on X and the fiber of $\mathbb{E}$, respectively, while

$$G_i(X^j, B^\alpha) := \frac{\partial}{\partial X^i}\mathcal{W}(X^j, B^\alpha). \tag{3.4.7}$$

Note that G_i is linear in B^α since $\mathcal{W}$ is linear in B^α.

Applying the fixed point theorem we see that the path integral is localized to the solution space of the following equations, modulo the group action of $\mathcal{G}$

$$\begin{aligned} \mathfrak{S}_\alpha(X^i) &= 0, \\ G_i(X^j, B^\alpha) &= 0, \\ \mu_X(X^i, X^{\bar{i}}) + \mu_F(B^\alpha, B^{\overline{\alpha}}) - \zeta &= 0, \end{aligned} \tag{3.4.8}$$

and

$$\begin{aligned} \varphi^a_m \mathcal{L}_a(X^i) &= 0, \\ \varphi^a_m \mathcal{L}_a(B^\alpha) &= 0, \\ [\varphi_m, \varphi_n] &= 0, \\ [\varphi_m, \overline{\varphi}_m] &= 0. \end{aligned} \tag{3.4.9}$$

This model, for a generic value of ζ implying $\varphi_m = 0$ as usual, reduce to the non-linear $N_c = (2,2)$ model whose target space $\mathfrak{M}_\zeta$ is the space of all solutions of the equations (3.4.8) modulo $\mathcal{G}$-symmetry.

3.4.2 Perturbation to a $N_c = (2,0)$ Model

Now we want to perturb the $N_c = (2,2)$ model above to a $N_c = (2,0)$ model by breaking the $N_c = (0,2)$ supersymmetry generated by $\boldsymbol{s}_-$ and $\overline{\boldsymbol{s}}_-$. This can be done by giving bare "mass" to all the newly introduced multiplets given by

$$(\overline{\sigma}, \eta_+), \qquad (\psi^i_-, H^i), \qquad (B^\alpha, \chi^\alpha_+) \tag{3.4.10}$$

and their conjugates. Then the model flows to the original $N_c = (2,0)$ model if we take the bare "mass" to infinity. Such bare mass terms will have special geometrical meaning.

Note that there is a natural $U(1) = S^1$ group acting on B^α, while leaving fixed the X^i, such that the momentum map μ_F remains invariant. This S^1-action is given by

$$S^1 : (X^i, B^\alpha) \to (X^i, \xi B^\alpha), \tag{3.4.11}$$

where $\xi\overline{\xi} = 1$. Note that the above S^1-action does not change the first and the last equations of (3.4.8). The LHS of the second equation of (3.4.8) will be multiplied by ζ, which does not alter the solution space of the equation. Thus the S^1-action is a symmetry of the effective target space $\mathfrak{M}_\zeta$.

It is important to note that the above $U(1)$ needs *not* be a symmetry of our $N_{ws} = (2,2)$ model. To be such a symmetry, the S^1-action (3.4.11) should be extended to all the superpartners. That is, $\psi^i_\pm$ and H^i should be invariant under $U(1)$ while $\chi^\alpha_\pm$ and H^α should carry the $U(1)$-charge 1. We, however, demand that the above $U(1)$ is compatible with the $N_c = (2,0)$ supersymmetry generated by $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ supercharges. From the expression (3.4.6) of S with manifest $N_c = (2,0)$ symmetry we see that the ψ^i_- should carry $U(1)$ charge -1, since $G_i(X^j, B^\alpha)$ is linear in B^α. Then, by examining the supersymmetry transformation laws for the supercharges $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$, we see that the S^1-symmetry (3.4.11) should be extended to all the $N_c = (2,0)$ multiplets in (3.4.10) as follows

$$\begin{aligned} S^1 &: (B^\alpha, \chi^\alpha_+) \to \xi(B^\alpha, \chi^\alpha_+) \\ S^1 &: (\psi^i_-, H^i) \to \overline{\xi}(\psi^i_-, H^i), \\ S^1 &: (\overline{\sigma}, \eta_+) \to \overline{\xi}(\overline{\sigma}, \eta_+). \end{aligned} \tag{3.4.12}$$

That is, we give $U(1)$-charges to the fields in (3.4.10) while all the other fields remain neutral. Clearly this can't be done while maintaining the full $N_c = (2,2)$ supersymmetry.

Recall that the $N_c = (2,0)$ supercharges $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ satisfy now familiar anti-commutation relations

$$\boldsymbol{s}_+^2 = 0, \qquad \{\boldsymbol{s}_+, \overline{\boldsymbol{s}}_+\} = -i\phi^a_{++}\mathcal{L}_a, \qquad \overline{\boldsymbol{s}}_+^2 = 0, \tag{3.4.13}$$

defining the $\mathcal{G}$-equivariant Dolbeault cohomology. Since we have an additional S^1 acting on our system it is natural to extend the above to $\mathcal{G} \times S^1$-equivariant cohomology. Then the new supercharges, still to be denoted $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$, satisfy the following anti-commutation relations

$$\boldsymbol{s}_+^2 = 0, \qquad \{\boldsymbol{s}_+, \overline{\boldsymbol{s}}_+\} = -i\phi^a_{++}\mathcal{L}_a - im\mathcal{L}_{S^1}, \qquad \overline{\boldsymbol{s}}_+^2 = 0, \tag{3.4.14}$$

where we introduced a parameter m taking values in $Lie(S^1)$. The supersymmetry transformation laws should be modified accordingly.

Finally we define the following $N_c = (2,0)$ supersymmetric action functional

$$S(m,\overline{m}) = S' + \overline{m}\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\Big(\sum_{\alpha=1}^{r}\langle B^\alpha, B^{\overline{\alpha}}\rangle - \langle\sigma,\overline{\sigma}\rangle\Big), \tag{3.4.15}$$

where S' is defined by the same formula as the action functional in (3.4.6) but with the modified supersymmetry. The new action functional $S(m,\overline{m})$, compared to the $N_c = (2,2)$ symmetric action S, is

$$S(m,\overline{m}) = S + m\overline{m}\sum_\alpha\langle B^\alpha, B^{\overline{\alpha}}\rangle - i\overline{m}\sum_\alpha\langle\chi_+^\alpha,\chi_+^{\overline{\alpha}}\rangle + m\overline{m}\langle\sigma,\overline{\sigma}\rangle - i\overline{m}\langle\overline{\eta}_+,\eta_+\rangle$$
$$- im\langle\phi_{--},\mu_F - [\sigma,\overline{\sigma}]\rangle + i\overline{m}\langle\phi_{++},\mu_F - [\sigma,\overline{\sigma}]\rangle + im\sum_i\langle\psi_-^i,\psi_-^{\overline{i}}\rangle, \tag{3.4.16}$$

containing the desired mass terms. We note that the mass terms contain the *Hamiltonian* H_{S_1} of the S^1 symmetry on the space of all B^α and σ;

$$H_{S^1} = i\sum_\alpha\langle B^\alpha, B^{\overline{\alpha}}\rangle + i\langle\sigma,\overline{\sigma}\rangle. \tag{3.4.17}$$

This fact will play a crucial role later.

Now we examine the equation for fixed points. Since we only have $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ supersymmetry the path integral is localized to the fixed point locus of those symmetries, modulo the $\mathcal{G}$ symmetry. We have

$$\begin{aligned}
\overline{\sigma}^a\mathcal{L}_a(X^i) &= 0,\\
\overline{\sigma}^a\mathcal{L}_a(B^\alpha) &= 0,\\
\mathfrak{S}_\alpha(X^i) &= 0,\\
G_i(X^j,B^\alpha) &= 0,\\
\mu_X(X^i,X^{\overline{i}}) + \mu_F(B^\alpha,B^{\overline{\alpha}}) - [\sigma,\overline{\sigma}] - \zeta &= 0,
\end{aligned} \tag{3.4.18}$$

and

$$\begin{aligned}
[\phi,\overline{\phi}] &= 0,\\
\phi^a\mathcal{L}_a(X^i) &= 0,\\
\phi^a\mathcal{L}_a(B^\alpha) + mB^\alpha &= 0,\\
[\phi,\overline{\sigma}] - m\sigma &= 0.
\end{aligned} \tag{3.4.19}$$

The set of equations in (3.4.18) cut out a subspace of the space of all X^i, B^α and σ. After modding out the $\mathcal{G}$-symmetry we get the effective target space $\widetilde{\mathfrak{M}}_\zeta$ of our $N_c = (2,0)$ model. Following the previous general discussions we expect that $\widetilde{\mathfrak{M}}_\zeta$ is a Kähler manifold at least for the generic case. The set of equations in (3.4.19) represent gauge degrees of freedom. In particular those equations implies the path integral is localized to the fixed point of S^1-action on $\widetilde{\mathfrak{M}}_\zeta$.

We always have trivial fixed points, namely $B^\alpha = \sigma = 0$. We call such fixed points branch (i). In branch (i) the path integral is localized to the solution space of the following equations, modulo $\mathcal{G}$-symmetry,

$$\begin{aligned} \phi^a \mathcal{L}_a(X^i) &= 0, \\ \mu_X(X^i, X^{\bar{i}}) - \zeta &= 0. \end{aligned} \tag{3.4.20}$$

which are exactly the generic fixed point equations for the original $N_c = (2,0)$ model. There are other fixed points with $B^\alpha, \overline{\sigma} \neq 0$ when the S^1-action can be undone by the $\mathcal{G}$ action. The last two equations in (3.4.19) exactly stand for such property. We call such fixed points branch (ii).

The above localization principle can also be obtained from a different viewpoint. We consider a limit $|m| \to \infty$. Then the dominant contributions to the path integral come from the set of critical points of the Hamiltonian H_{S^1} defined by (3.4.17). It is well-known that the critical points of the Hamiltonian of a S^1-action are exactly the same as the fixed points of the S^1-action. One may evaluate the partition function in such a limit and set $|m| = 0$ afterwards, to get the partition function of the $N_c = (2,2)$ model.

Now we assume that everything is generic, so that we do not have any zero-modes of anti-ghosts, χ^α_-, ψ^i_-, as well as any zero-modes of the $N_c = (2,0)$ gauge multiplets. Then the partition function of the action functional $S(m, \overline{m})$ in (3.4.19) reduces to the following integral

$$Z = \int_{\widetilde{\mathfrak{M}}_\zeta} exp\left(imH_{S^1} + i\sum_\alpha \langle \chi^\alpha_+, \chi^{\overline{\alpha}}_+ \rangle + i\langle \overline{\eta}_+, \eta_+ \rangle \right), \tag{3.4.21}$$

where we regard m and $\overline{m}$ as independent numbers and scaled away the overall $\overline{m}$. The above resembles the DH integration formula on $\widetilde{\mathfrak{M}}_\zeta$. We see, however, that there is a missing term since the fermionic terms above correspond to the Kähler form only on the subspace of $\widetilde{\mathfrak{M}}_\zeta$ given by $X^i = 0$. We can provide the missing term by evaluating the correlation function of $\exp(i\langle \phi_{++}, \mu_X \rangle + i\sum \langle \psi^i_+, \psi^{\bar{i}}_+ \rangle)$, where the exponent is the $\mathcal{G}$-equivariant Kähler form $\widehat{\varpi}^X_\mathcal{G}$ on X. Note that it is an observable of the original $N_c = (2,0)$ model we started from. Assuming the same generic situation as above, the correlation function reduces to the following integral

$$\left\langle e^{\widehat{\varpi}^X_\mathcal{G}} \right\rangle = \int_{\widetilde{\mathfrak{M}}_\zeta} exp\,(imH_{S^1} + \widetilde{\varpi}), \tag{3.4.22}$$

where $\widetilde{\varpi}$ denote the Kähler form on $\widetilde{\mathfrak{M}}_\zeta$. Now we have exactly the DH integration formula [56]. The integral can be written as the sum of contributions from the fixed points of the S^1-action.

We saw that we have two branches. In branch (i) the fixed point locus is the effective target space $\mathcal{M}_\zeta$ of the original $N_c = (2,0)$ model. The Hamiltonian H_{S_1} in this branch is simply zero. Thus we are evaluating the symplectic volume of $\mathcal{M}_\zeta$. This is a correlation function of the original $N_c = (2,0)$ model. In branch (ii) the value $H^f_{S^1}$ of H_{S^1} at a fixed point is non-zero. So the integral for each fixed point is weighted by a phase factor $\exp(imH^f_{S^1})$. For both branches the integral is weighted by a one loop determinant coming from the transverse degrees of freedom. We note that such a determinant contains factors of m with certain weights depending on the particular fixed points. After evaluating the DH integral we can set $m = 0$. Then we may obtain many relations by imposing that the poles should be cancelled order by order between the two different branches, since the limit $m \to 0$ should be smooth in the path integral of the massive $N_c = (2,0)$ model. The partition function of the $N_c = (2,2)$ model is given by a sum of terms with order zero in m. One can also obtain the symplectic volume of $\mathcal{M}_\zeta$ in terms of a sum of contributions coming from branch (ii).

In the real situation life is more complicated since it is difficult to achieve the generic conditions and the space $\widetilde{\mathfrak{M}}_\zeta$ may be non-compact. Its is in principle possible to elaborate on the above procedure and perform the integral. Even if we can't do such an integral due to technicalities we can at least see that the essential information on the correlation function of the original $N_c = (2,0)$ model is contained in the fixed points which belongs to branch (ii).

Chapter 4

Cohomological Yang-Mills Theories on Kähler 2-Folds

In this chapter we apply the general constructions of the previous chapter to an important class of infinite dimensional target spaces with infinite dimensional group actions. We take as our target space the configuration space of Yang-Mills theory, namely the space $\mathcal{A}$ of all connections on a Hermitian vector bundle, on a complex d-dimensional compact Kähler manifold M. The infinite dimensional group $\mathcal{G}$ acting on our target space $\mathcal{A}$ is the group of all gauge transformations. Then we have a natural infinite dimensional holomorphic bundle over $\mathcal{A}$ with holomorphic section $\mathfrak{S} = F_A^{0,2}$. We will consider the two types of models; the $N_c = (2,0)$ model in Sect. 3.3 and its canonical embedding to the $N_c = (2,2)$ model in Sect. 3.4. Those models will be realized as (cohomological) field theories on the manifold M. For the $d = 2$ case, in particular, those models are precisely the Donaldson-Witten [1] and the Vafa-Witten [7] theories, respectively, specialized on Kähler manifolds [52][10][59]. The $d = 2$ case is very special as those models can be obtained by twisting $N = 2$ and $N = 4$ space-time supersymmetric Yang-Mills (SYM) theories in four dimensions [1][68].

The Donaldson-Witten theory in four dimensions is the first example of a cohomological field theory. It was introduced, more than a decade ago, by Witten as a quantum field theoretic approach to the four-dimensional differential topological invariant of Donaldson [8][69]. This approach opened up completely new horizon in mathematics [12] via the quantum properties of underlying physical theory uncovered by Seiberg and Witten [11]. The Vafa-Witten theory played a crucial role in physics by providing the first strong coupling test of the S-duality of $N = 4$ SYM theory, first conjectured by Montonen and Olive [70][71]. To this date, there are well-defined general procedures to determine all those differential-topological quantities based on the exact solutions of the underlying physical theories [11][72]. Here we will not follow those steps [73][74][75].

4.1 Donaldson-Witten Theory

In this section we consider the $N_c = (2,0)$ model whose target space is the infinite dimensional space $\mathcal{A}$ of all connections on a vector bundle over a complex d-dimensional compact Kähler manifold M [52][67]. We will see, following the discussions in Sect. 3.3, the $N_c = (2,0)$ supersymmetry uniquely leads us to construct a model whose effective target space is the moduli space of Einstein-Hermitian connections or, equivalently the moduli space of stable bundles. We will also see that the resulting model for $d \geq 3$ runs into a serious troubles. For $d = 2$ case the model is well-defined and gives rise to Donaldson-Witten theory specialized to a Kähler surface [52][10][53]. Still we work for general dimensions since the general model here will be used in the later chapters after small changes.

4.1.1 $N_c = (2,0)$ Model

To define an equivariant $N_c = (2,0)$ model we need to introduce complex and Kähler structure on our infinite target space $\mathcal{A}$ with the infinite dimensional $\mathcal{G}$ action given by local gauge transformations on the gauge fields - see [69][48] for general references on complex vector bundles. The above data determine $N_c = (2,0)$ holomorphic multiplets and gauge multiplets as well as their supersymmetry transformation laws and the $\mathcal{G}$-equivariant momentum map $\mu : \mathcal{A} \to Lie(\mathcal{G})^*$. Then the path integral of the resulting model will localized to the symplectic quotient $\mu^{-1}(\zeta)/\mathcal{G}$. For $d \geq 1$ the quotient space is still infinite dimensional. Thus one may consider certain infinite dimensional $\mathcal{G}$-equivariant Hermitian holomorphic vector bundle $\mathbb{E} \to \mathcal{A}$ over $\mathcal{A}$ with certain holomorphic sections, which determine anti-ghost multiplets accordingly. According to our general discussions in Sect. 3.3.1 we may pick two different orthogonal holomorphic sections $\mathfrak{S}$ and $\mathfrak{J}$ of $\mathbb{E}$. Then the path integral will be further localized to $(\mathfrak{J}^{-1}(0) \cap \mathfrak{S}^{-1}(0) \cap \mu^{-1}(\zeta))/\mathcal{G}$, which can be a finite dimensional Kähler manifold. To supply above data we need some preparation.

Description of Target Space $\mathcal{A}$

We consider a compact complex Kähler d-fold M with Kähler form ω. According to a complex structure on M the space $\Omega^r(M)$ of r-form on M has decompositions $\Omega^r(M) = \oplus_{p+q=r}\Omega^{p,q}(M)$. On M any two-form $\alpha \in \Omega^2(M)$ can be decomposed into $\alpha = \alpha^+ + \alpha^-$ such that

$$\begin{aligned} \alpha^+ &= \alpha^{2,0} + \alpha_0\omega + \alpha^{0,2}, \\ \alpha^- &= \alpha^{1,1}_\perp, \end{aligned} \tag{4.1.1}$$

where $\alpha_o \in \Omega^0(M)$ is a real scalar and $\alpha^{1,1}_\perp$ is $(1,1)$-form orthogonal to ω. For a complex Kähler 2-fold the above decomposition coincides with the self-dual and anti-self dual two-forms. We denote by $\Omega^p(M,E)$ the space of real p-forms on M taking values in E. Let E be a rank r vector bundle over M with a Hermitian metric on E. This fix the topological type for the connections on E.

We denote by $\mathcal{A}$ the space of all connections and by $\mathcal{G}$ the group of all gauge transformations, i.e., $g \in \mathcal{G}$ such that $g : M \to U(r)$. The group $\mathcal{G}$ is equivalent to the group of all unitary automorphisms of E. The Lie algebra $Lie(\mathcal{G})$ of $\mathcal{G}$ is $\Omega^0(M, End(E))$ and we use integration over M to identity $Lie(\mathcal{G})^*$ with $\Omega^{2d}(M, End(E))$. Thus the bi-invariant inner product on $Lie(\mathcal{G})$ is the integral over M combined with the trace of $U(r)$;

$$< a, a >= -\int_M \mathrm{Tr}(a \wedge *a). \tag{4.1.2}$$

We take $\mathcal{A}$ as our initial target space.

Let A denote a connection one-form, which is decomposed into $A = A^{1,0} + A^{0,1}$. We denote by $d_A = \partial_A + \overline{\partial}_A$ the covariant derivative,

$$d_A = \partial_A + \overline{\partial}_A : \Omega^0(M, E) \longrightarrow \Omega^{1,0}(M, E) \oplus \Omega^{0,1}(M, E). \tag{4.1.3}$$

The space $\mathcal{A}$ is an infinite dimensional affine space which tangent vector is represented by $\delta A \in \Omega^1(M, End(E))$.

$$A' - A \in \Omega^1(M, End(E)). \tag{4.1.4}$$

Note that there is no natural complex structure on $\mathcal{A}$. Any complex structure must be induced from the complex structure on M. One introduces a complex structure $\mathcal{A}$ by declaring $\delta A^{0,1} \in \Omega^{0,1}(M, End(E))$ as holomorphic tangent vector. Then $\mathcal{A}$ becomes an infinite dimensional flat Kähler manifold with Kähler form ϖ,

$$\varpi = \frac{1}{4(d)!\pi^2} \int_M \mathrm{Tr}(\delta A \wedge \delta A) \wedge \omega^{d-1}, \tag{4.1.5}$$

and $\mathcal{G}$ acts as isometry. The Kähler potential for the Kähler form ϖ of $\mathcal{A}$ is given by

$$\mathcal{K}(A^{1,0}, A^{0,1}) = \frac{1}{4(d)!\pi^2} \int_M \kappa\, \mathrm{Tr}(F \wedge F) \wedge \omega^{d-2}, \tag{4.1.6}$$

where κ is a Kähler potential for ω, i.e., $\omega = i\partial\overline{\partial}\kappa$.

Now we introduce our $N_c = (2,0)$ supercharges s_+ and $\overline{s}_-$ with the familiar commutation relations

$$s_+^2 = 0, \qquad \{s_+, \overline{s}_+\} = -i\phi^a_{++}\mathcal{L}_a, \qquad \overline{s}_+^2 = 0. \tag{4.1.7}$$

The supercharges are identified with the differentials of $\mathcal{G}$-equivariant cohomology of our target space $\mathcal{A}$. Thus $\phi^a_{++}\mathcal{L}_a$ is the infinitesimal gauge transformation generated by the adjoint scalar $\phi_{++} \in Lie(\mathcal{G}) = \Omega^0(M, End(E))$. We have $N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \overline{\eta}_-, D)$ taking values in $\Omega^0(M, End(E)$. Their transformation laws for are given by the general formula (3.3.8).

From the complex structure of $\mathcal{A}$ introduced above we have the holomorphic and anti-holomorphic multiplets $(A^{0,1}, \psi_+^{0,1})$ and $(A^{1,0}, \overline{\psi}_+^{1,0})$, respectively, with the following transformation laws,

$$\begin{aligned} &s_+A^{0,1} = i\psi_+^{0,1}, && s_+\psi_+^{0,1} = 0, && \\ &\bar{s}_+A^{0,1} = 0, && \bar{s}_+\psi_+^{0,1} = -\bar{\partial}_A\phi_{++}, && s_+\phi_{++} = 0, \\ &s_+A^{1,0} = 0, && s_+\bar{\psi}_+^{1,0} = -\partial_A\phi_{++}, && \bar{s}_+\phi_{++} = 0, \\ &\bar{s}_+A^{1,0} = i\bar{\psi}_+^{1,0}, && \bar{s}_+\bar{\psi}_+^{1,0} = 0. && \end{aligned} \tag{4.1.8}$$

where $\psi^{0,1} \in \Omega^{0,1}(M, End(E))$ represents a holomorphic tangent vector in $\mathcal{A}$. Note that

$$\{s,\bar{s}\}A = -id_A\phi, \qquad \{s,\bar{s}\}\psi^{0,1} = i[\phi,\psi^{0,1}], \tag{4.1.9}$$

which are the infinitesimal gauge transformations generated by ϕ. From the transformation laws we have the following equivariant Kähler form

$$\begin{aligned} \widehat{\varpi}^{\mathcal{G}} &= is_+\bar{s}_+\mathcal{K} \\ &= \frac{i}{2(d)!\pi^2}\int_M \mathrm{Tr}(\phi_{++}F)\wedge\omega^{d-1} + \frac{1}{2(d)!\pi^2}\int_M \mathrm{Tr}(\psi_+^{0,1}\wedge\bar{\psi}_+^{1,0})\wedge\omega^{d-1}, \end{aligned} \tag{4.1.10}$$

where we used the Bianchi identity $d_AF = 0 \to \bar{\partial}_AF^{0,2} = \partial_AF^{0,2} + \bar{\partial}_AF^{1,1} = 0$ and integration by parts. The second term in the above is the Kähler form ϖ and the first term is the $\mathcal{G}$-momentum map $\phi^a_{++}\mu_a$, $\mu : \mathcal{A} \to Lie(\mathcal{G})^* = \Omega^{2n}(M, End(E))$;

$$\mu(A) = \frac{1}{2(d)!\pi^2}F\wedge\omega^{d-1} = \frac{1}{2d(d)!\pi^2}(\Lambda F)_A\omega^d, \tag{4.1.11}$$

where Λ denotes the adjoint of wedge multiplication with ω.

Description of Holomorphic Section

The remaining task is to determine an infinite dimensional vector bundle over our target space $\mathcal{A}$ with an appropriate $\mathcal{G}$-equivariant holomorphic section $\mathfrak{S}(A^{0,1})$, i.e. $\bar{s}_+\mathfrak{S} = 0$. From our general discussion a choice of section $\mathfrak{S}$ should be compatible with the Kähler quotient such that the effective target space $\mathcal{M} = (\mathfrak{S}^{-1}(0)\cap\mu^{-1}(\zeta))/\mathcal{G}$ inherits a Kähler structure when $\mathcal{G}$ acts freely. We introduce a bundle $\mathbb{E}$ over our target space $\mathcal{A}$ which a holomorphic section $\mathfrak{S}(A^{0,1})$ is given by

$$\mathfrak{S} : A^{0,1} \to F_A^{0,2} \in \Omega^{0,2}(M, End(E)). \tag{4.1.12}$$

We note that the above is the only possible choice on *general* Kähler manifolds, since there are no other holomorphic "functions" of $A^{0,1}$ which are gauge covariant. Further obvious requirement is that the resulting action functional should be invariant under the Lorentz symmetry - the holonomy of a Kähler manifold M.[1] In general we do not have a room to introduce a second holomorphic section $\mathfrak{J}(A^{0,1})$ of $\mathbb{E} \to \mathcal{A}$. Thus we set $\mathfrak{J} = 0$. Then our effective target space will

[1]There are two special cases. On a Calabi-Yau 4-fold or an arbitrary hyper-Kähler manifold one can take certain projection of $F^{0,2}$ as the holomorphic section of $\mathbb{E} \to \mathcal{A}$. We will return to this later.

be the moduli space $\mathcal{M}_{EH}$ defined by

$$\mathcal{M}_{EH} = (\mathfrak{S}^{-1}(0) \cap \mu^{-1}(\zeta))/\mathcal{G}. \tag{4.1.13}$$

Since our section takes values in $\Omega^{0,2}(M, End(E))$ we have corresponding Fermi multiplets $(\chi_-^{2,0}, H^{2,0})$, taking values in $\Omega^{2,0}(M, End(E))$;

$$\begin{aligned} s_+\chi_-^{2,0} &= -H^{2,0}, & s_+H^{2,0} &= 0, \\ \overline{s}_+\chi_-^{2,0} &= 0, & \overline{s}_+H^{2,0} &= -i[\phi_{++}, \chi^{2,0}], \\ s_+\overline{\chi}_-^{0,2} &= 0, & s_+H^{0,2} &= -i[\phi_{++}, \overline{\chi}^{0,2}], \\ \overline{s}_+\overline{\chi}_-^{0,2} &= -H^{0,2}, & \overline{s}_+H^{0,2} &= 0. \end{aligned} \tag{4.1.14}$$

Comparing with the general transformation laws (3.3.7), we have $\mathfrak{J} = 0$.

Action Functional and Localization

Now we have all the ingredient to write down a $N_c = (2,0)$ model. Combining everything together we obtain the action functional of $N_c = (2,0)$ on a complex d-dimensional Kähler manifold, from the general formula (3.3.9),

$$\begin{aligned} S =& \frac{1}{2(d)!\pi^2} s_+\overline{s}_+ \int_M \mathrm{Tr}\left(\phi_{--}\left(F \wedge \omega^{d-1} + \frac{i\zeta}{d}\omega^d I_E\right)\right) \\ &+ \frac{1}{4\pi^2} s_+\overline{s}_+ \int_M \mathrm{Tr}\left(\chi_-^{2,0} \wedge *\overline{\chi}_-^{0,2}\right) + \frac{1}{2d\pi^2} s_+\overline{s}_+ \int_M \mathrm{Tr}\left(\eta_- * \overline{\eta}_-\right) \\ &+ \frac{i}{4\pi^2} s_+ \int_M \mathrm{Tr}\left(\chi_-^{2,0} \wedge *F^{0,2}\right) + \frac{i}{4\pi^2}\overline{s}_+ \int_M \mathrm{Tr}\left(\overline{\chi}_-^{0,2} \wedge *F^{2,0}\right). \end{aligned} \tag{4.1.15}$$

Expanding above we have the following terms depending on the auxiliary fields

$$\begin{aligned} S =& \frac{1}{2d\pi^2} \int \mathrm{Tr}\left(D * D + D * (\Lambda F + i\zeta I_E)\right) \\ &- \frac{1}{4\pi^2} \int \mathrm{Tr}\left(H^{2,0} \wedge *H^{0,2} + iH^{2,0} \wedge *F^{0,2} + iH^{2,0} \wedge *F^{0,2}\right) + \ldots \end{aligned} \tag{4.1.16}$$

We integrate out auxiliary fields by imposing their algebraic equations of motion

$$\begin{aligned} H^{0,2} &= -iF^{0,2}, \\ D &= -\frac{1}{2}\left(\Lambda F + i\zeta I_E\right). \end{aligned} \tag{4.1.17}$$

The explicit form of the action functional is

$$\begin{aligned} S' =& \frac{1}{2\pi^2} \int \mathrm{Tr}\Big(-\frac{1}{4}F^+ \wedge *F^+ - \frac{1}{d}\phi_{--} * d_A^* d_A \phi_{++} + \frac{1}{4d}[\phi_{++}, \phi_{--}]^2 \\ &+ \frac{1}{d}\phi_{--} * \Lambda[\psi_+^{0,1}, \overline{\psi}_+^{1,0}] + \frac{i}{2}\chi_-^{2,0} \wedge *[\phi_{++}, \overline{\chi}_-^{0,2}] + \frac{i}{d}[\phi_{++}, \eta_-] * \overline{\eta}_- \\ &- \frac{i}{d}\overline{\eta}_- * \overline{\partial}_A^* \psi_+^{0,1} - \frac{i}{d}\eta_- * \partial_A \overline{\psi}_+^{1,0} - \frac{1}{2}\chi_-^{2,0} \wedge *\overline{\partial}_A \psi_+^{0,1} - \frac{1}{2}\overline{\chi}_-^{0,2} \wedge *\partial_A \overline{\psi}_+^{1,0}\Big), \end{aligned} \tag{4.1.18}$$

where set $\zeta = 0$ for simplicity and used the Kähler identities

$$\overline{\partial}_A^* = -i[\Lambda, \partial_A], \quad \partial_A^* = i[\Lambda, \overline{\partial}_A]. \tag{4.1.19}$$

We also note that

$$\begin{aligned} -\int_M \mathrm{Tr}\left(F^+ \wedge *F^+\right) &= -\int_M \mathrm{Tr}\left(2F^{2,0} \wedge *F^{0,2} + f\omega * f\omega\right) \\ &= -\int_M \mathrm{Tr}\left(2F^{2,0} \wedge *F^{0,2} + \frac{1}{d}\Lambda F * \Lambda F\right), \end{aligned} \tag{4.1.20}$$

where $f = \frac{1}{d}\Lambda F$.

A Hermitian connection is called Einstein-Hermitian (EH) with factor ζ if[2]

$$\begin{aligned} F^{0,2} &= 0, \\ i\Lambda F - \zeta I_E &= 0. \end{aligned} \tag{4.1.21}$$

We denote by $\mathcal{M}_{EH}$ the moduli space of EH connections. From the general discussions in Sect. 3.3.2 we see that the path integral of this model is localized to the moduli space $\mathcal{M}_{EH}$ of EH connections.

Now consider the subspace $\mathfrak{S}^{-1}(0) := \mathcal{A}^{1,1} \subset \mathcal{A}$ consisting of unitary connections satisfying $F^{0,2} \equiv \overline{\partial}_A^2 = 0$.[3] That is, the partial connection $\overline{\partial}_A$ is integrable. The space $\mathcal{A}^{1,1}$ is preserved by $\mathcal{G}$ and inherits a complex and Kähler structure from $\mathcal{A}$. Thus we have a symplectic quotient of $\mathcal{A}^{1,1}$ by $\mathcal{G}$, which is equivalent to $\mathcal{M}_{EH}$;

$$\mathcal{M}_{EH} = \left(\mathcal{A}^{1,1} \cap \mu^{-1}(\zeta)\right)/\mathcal{G}. \tag{4.1.22}$$

Thus the moduli space $\mathcal{M}_{EH}$ is a Kähler manifold if it is smooth. The equivariant differential form $\varpi^{\mathcal{G}}$, then, may be identified with the Kähler form $\tilde{\varpi}$, after the restriction and reduction, on $\mathcal{M}_{EH}$. An EH connection can be reducible. We denote by $\mathcal{M}^*_{EH}$ the moduli space of irreducible EH connections. A connection $A \in \mathcal{A}^{1,1}$ endows E with the structure of holomorphic vector bundle $\mathcal{E}_A$. The moduli space $\mathcal{M}_{hol}$ of holomorphic vector bundles is the space $\mathcal{A}^{1,1}$ modulo bundle isomorphisms generated by the complexification $\mathcal{G}^{\mathbb{C}}$ of $\mathcal{G}$, i.e.,

$$\mathcal{M}_{hol} := \mathcal{A}^{1,1}/\mathcal{G}^{\mathbb{C}}. \tag{4.1.23}$$

With a choice of polarization (typically an ample line bundle on M whose curvature two-form is Kähler form ω on M), one can define the notion of semi-stability. The GIT quotient is defined by taking a quotient by $\mathcal{G}^{\mathbb{C}}$ restricted to the semi-stable orbits. The moduli space $\mathcal{M}^{ss}_{hol}$ of semi-stable holomorphic bundles is then

$$\mathcal{M}^{ss}_{hol} := \mathcal{A}^{1,1}//\mathcal{G}^{\mathbb{C}} \tag{4.1.24}$$

It is shown by Kobayashi that every EH connection induces a semi-stable bundle and every irreducible EH connection induces a stable bundle. The inverse is also true and is the Donaldson-Uhlenbeck-Yau theorem [76][77]. Thus we have isomorphisms

$$\mathcal{M}_{EH} \simeq \mathcal{M}^{ss}_{hol}, \qquad \mathcal{M}^*_{EH} \simeq \mathcal{M}^s_{hol}. \tag{4.1.25}$$

[2] Note that $\zeta = \left(\int_M c_1(E) \wedge \omega^{d-1}\right)/\left(\frac{r}{2d\pi}\int_M \omega^n\right)$. Thus the FI term ζ depends only on the cohomology class of $c_1(E)$ and ω.

[3] For general complex vector bundle $F^{0,2} = 0$ does not imply $F^{2,0} = 0$.

Observables and Correlation Functions

Now we consider observables which are $\mathcal{G}$-equivariant closed differential forms, after the parity change, on the space $\mathcal{A}$ of all connections. Those observables generate cohomology rings of the moduli space of EH connections via restriction and reduction. From $s_+\phi_{++} = \overline{s}_+\phi_{++} = 0$ we see that an arbitrary G-invariant polynomial $\mathcal{Q}(\phi_{++})$ of ϕ_{++} with degree r is an observable. It corresponds to an equivariant (r,r)-form. The other observables can be obtained by the usual descent procedure. Equivalently we may use the universal bundle to construct those observables [78].

From the Bianchi identity $d_A F = 0$ and the transformation laws in (4.1.8), we have the following generalized Bianchi identity

$$\mathcal{D}\mathcal{F} = 0, \tag{4.1.26}$$

where

$$\begin{aligned} \mathcal{D} &= s + \overline{s} + \partial_A + \overline{\partial}_A, \\ \mathcal{F} &= \phi_{++} + i\overline{\psi}_+^{1,0} + i\psi_+^{0,1} + F^{2,0} + F^{1,1} + F^{0,2} \end{aligned} \tag{4.1.27}$$

We define a generalized Chern class c_n by

$$c_n = \frac{i^n}{(2\pi)^n n!}\,\mathrm{Tr}\,\mathcal{F}^n. \tag{4.1.28}$$

We expand the generalized Chern class as

$$c_n = \sum_{p+q+r+s=2n} \mathcal{V}^{r,s}_{p,q}, \tag{4.1.29}$$

where the upper indices denote the form degree on M while the lower indices denote the degree of the ghost number. Now it follows from the Bianchi identity (4.1.26) that we have the following descent equations

$$(s_+ + \overline{s}_+ + \partial + \overline{\partial})c_n = 0, \tag{4.1.30}$$

leading to

$$\overline{s}_+\mathcal{V}^{r,s}_{p,q} + s_+\mathcal{V}^{r,s}_{p-1,q+1} + \overline{\partial}\mathcal{V}^{r,s-1}_{p,q+1} + \partial\mathcal{V}^{r-1,s}_{p,q+1} = 0. \tag{4.1.31}$$

We define

$$\widehat{\mathcal{O}}^{(r,s)}_{p,q} = \int_M \alpha^{d-r,d-s} \wedge \mathcal{V}^{r,s}_{p,q}, \tag{4.1.32}$$

where $\alpha^{d-r,d-s} \in H^{d-r,d-s}(M)$, $0 \le r,s \le d$ and $0 \le p,q$. Then we have

$$s_+\widehat{\mathcal{O}}^{(r,s)}_{p-1,q} + \overline{s}_+\widehat{\mathcal{O}}^{(r,s)}_{p,q-1} = 0. \tag{4.1.33}$$

The above relation implies that not every candidates $\widehat{\mathcal{O}}^{(r,s)}_{p,q}$ are both s_+ and $\overline{s}_+$ closed.[4] If we impose the equations of motions all those candidates are both

[4] The relation (4.1.33) implies that $Q_+\widehat{\mathcal{O}}^{(\ell)}_m = 0$ where $Q_+ = s_+ + \overline{s}_+$ and $\widehat{\mathcal{O}}^{(\ell)}_m = \sum_{m=p+q}^{\ell=r+s} \widehat{\mathcal{O}}^{(r,s)}_{p,q}$. The relation (4.1.31) implies that the Q_+ cohomology depends only on the de Rham cohomology on M.

s and $\bar{s}$ closed. In general, however, one should not use a quantity which is invariant only on-shell as an observable. To define a cohomological theory it is sufficient to have one global supercharges. Thus we may maintain only the $\bar{s}_+$ symmetry and use the $\bar{s}_+$-closed equivariant differential forms as observables. A s_+-closed, but $\bar{s}_+$-closed only on-shell, equivariant differential form can be added to the original action functional. Then one can always change the $\bar{s}_+$ transformation laws such that the new action functional has $\bar{s}_+$ symmetry. Such a perturbation, for the $d = 2$ case, was studied by Witten and led him to determine Donaldson invariants of a Kähler surface almost completely [10]. The similar perturbation was considered earlier in the topological sigma B-model, and led to the notion extended moduli space of complex structure on a Calabi-Yau [3].[5]

One may consider correlation functions of other observables $\widehat{\mathcal{O}}^{r,s}$ with the ghost number (r, s) given by s_+ and $\bar{s}_+$ closed $\mathcal{G}$ equivariant differential forms $\mathcal{O}^{r,s}$ - see Sect. 3.3.2. We have

$$\left\langle \prod_{i=1}^{\ell} \widehat{\mathcal{O}}^{r_i,s_i} \right\rangle = \int_{\mathcal{M}_{EH}} e(\mathbb{V}) \wedge \widetilde{\mathcal{O}}^{r_1,s_1} \wedge \ldots \wedge \widetilde{\mathcal{O}}^{r_\ell,s_\ell} \tag{4.1.34}$$

where $\widetilde{\mathcal{O}}^{r,s}$ denote the equivariant differential form $\mathcal{O}^{r,s}$ after the restriction and reduction to $\mathcal{M}_{EH}$ and $e(\mathbb{V})$ denotes the Euler class of the anti-ghost bundle $\mathbb{V}$ over $\mathcal{M}_{EH}$. The above correlation function can be non-vanishing for

$$\sum_{i=1}^{\ell} (r_i, s_i) = (\triangle, \triangle), \tag{4.1.35}$$

where $(\triangle, \triangle)$ denotes the net ghost number anomaly in the path integral measure due to fermionic zero-modes.

From the action functional S' (4.1.18) we obtain the following equations for fermionic zero-modes,

$$\bar{\partial}_A \bar{\eta}_- = 0, \qquad \begin{array}{c} \bar{\partial}^*_A \psi^{0,1}_+ = 0, \\ \bar{\partial}_A \psi^{0,1}_+ = 0, \end{array} \qquad \bar{\partial}^*_A \bar{\chi}^{0,2}_- = 0. \tag{4.1.36}$$

For $d = 2$ we also have $\bar{\partial}_A \bar{\chi}^{0,2}_- = 0$ by dimensional reasons. Thus a $\bar{\chi}^{0,2}_-$ zero-mode is an adjoint-valued harmonic $(0, 2)$-form -recall that $\bar{\partial}^2_A = 0$. The net ghost number carried by the above fermionic zero-modes is precisely the complex formal dimension of the moduli space $\mathcal{M}_{EH}$, which is equivalent to the moduli space of anti-self-dual connections. For $d \geq 3$, the zero-modes of the anti-ghost $\bar{\chi}^{0,2}_-$ are no longer constrained to be harmonic. Then we run into a serious problem that we may have too many zero-modes of the anti-ghost $\bar{\chi}^{0,2}_-$. This implies that the moduli space $\mathcal{M}_{EH}$ and anti-ghost bundle $\mathbb{V}$ over it may have components which are too high in dimensions. It is also doubtful if the Euler class $e(\mathbb{V})$ is well-defined. We note that the zero-modes of the anti-ghost

[5] This may be regarded as the starting point of the homological mirror conjecture [79].

are related with the choice of vector bundle $\mathbb{E} \to \mathcal{A}$ with holomorphic section $\mathfrak{S}(A^{0,1}) = F^{0,2}$. Thus one may try to extend the target space $\mathcal{A}$ or use different holomorphic section to have well-defined anti-ghost bundle. We will return to this problem in the later chapters.

In the remaining part of this chapter we restrict our attention exclusively to the $d = 2$ case.

4.1.2 Donaldson-Witten Theory

Our model above specialized to a complex $d = 2$ dimensional Kähler manifold is Donaldson-Witten theory [1]. The correlation functions (4.1.34) of supersymmetric observables are the path integral representation of Donaldson's invariants. For a manifold with $b_2^+ \geq 3$ Donaldson showed that one can avoid zero-modes of $\overline{\eta}_-$ and $\overline{\chi}_-^{0,2}$. Thus the correlation functions (4.1.34) can be interpreted as intersection pairings of homology cycles on the moduli space $\mathcal{M}_{EH}$ of anti-self-dual connections.

We may compare our model with global $N = 2$ supersymmetric Yang-Mills theory on $\mathbb{R}^4 \cong \mathbb{C}^2$. We first recall the field contents of our model in $d = 2$. For Bosons we have a gauge field $A^{0,1}$ and a complex scalar ϕ.[6] In real coordinates, we have (A^I, ϕ). For Fermions we have an anti-commuting vector $\psi_+^{0,1}$ an anti-commuting scalar $\overline{\eta}_-$ and a $(0,2)$-form $\overline{\chi}_-^{0,2}$. The latter two, the real coordinates can be recombined into a scalar and self-dual two-form. Then the field contents is

$$\begin{array}{ccc} & A^I & \\ \psi_+^{\dot{\alpha}\beta} & & \psi_-^{\alpha\beta} \\ & \phi & \end{array} \tag{4.1.37}$$

where α and $\dot{\alpha}$ denote the undotted and dotted spinor indices from the decomposition of the Lorentz group $SO(4) \cong SU(2)_L \times SU(2)_R$. The expression $\psi_-^{\alpha\beta}$ contains both symmetric and anti-symmetric parts corresponding to a self-dual two-form and a scalar. Since the holonomy of $\mathbb{R}^4$ is contained in $SU(2)$ a physical observation does not see either $SU(2)_L$ or $SU(2)_R$. Let's pick $SU(2)_R$ and replace the index β with another index $i = 1, 2$ of a certain $SU(2)_I$, while we keep the index α in $\psi_-^{\alpha\beta}$ as the index for $SU(2)_R$. Then we have

$$\begin{array}{ccc} & A^I & \\ \psi_{i+}^{\dot{\alpha}} & & \psi_{i-}^{\alpha} \\ & \phi & \end{array} \tag{4.1.38}$$

Our action functional S' in (4.1.18) still remains invariant under the new global symmetry $SU(2)_L \times SU(2)_R \times SU(2)_I \times U(1)$ where $U(1)$ denotes the classical ghost number symmetry. The above is exactly the field contents of $N = 2$ supersymmetric Yang-Mills theory with correct global symmetry, which is called a $N = 2$ vector multiplet. With the above field redefinition the action functional S' in (4.1.18) becomes that of $N = 2$ super-Yang-Mill theory. The ghost number anomaly of the path integral measure becomes the well-known chiral anomaly due to the instantons.

[6] We regard ϕ_{++} as an adjoint valued complex scalar ϕ and ϕ_{--} as its conjugates.

A $N = 2$ super-Yang-Mills theory has global supercharges $Q^{\dot{\alpha}}_{i-}$ and Q^{α}_{i+} which transform under $SU(2)_L \times SU(2)_R \times SU(2)_I \times U(1)$ as indicated by the various indices. Now we go in the reverse direction of the above discussion by taking the diagonal subgroup of $SU(2)_R \times SU(2)_I$. Thus we get $Q^{\dot{\alpha}\beta}_{-}$ and $Q^{\alpha\beta}_{i+}$. The anti-symmetric part of $Q^{\alpha\beta}_{i+}$ now transforms as a scalar Q_+ and the symmetric part transform as a self-dual two-form. On a Kähler surface a self-dual two-form is isomorphic to a scalar $\widetilde{Q}_+$ and a holomorphic two-form. Thus we have two scalar supercharges. They corresponds to our $\boldsymbol{s}_+$ and $\bar{\boldsymbol{s}}_+$ as

$$\begin{aligned} \boldsymbol{s}_+ &= \frac{1}{2}(Q_+ + i\widetilde{Q}_+), \\ \bar{\boldsymbol{s}}_+ &= \frac{1}{2}(Q_+ - i\widetilde{Q}_+), \end{aligned} \tag{4.1.39}$$

which are identified with the differentials of $\mathcal{G}$-equivariant Dolbeault cohomology on the space $\mathcal{A}$ of all gauge fields. On a hyper-Kähler surface a self-dual two-form is equivalent to three scalars $\widetilde{Q}^{\ell}_{+}$, $\ell = 1, 2, 3$, due to the three independent Kähler forms ω_ℓ. Then we have

$$\begin{aligned} \boldsymbol{s}^{\ell}_+ &= \frac{1}{2}(Q_+ + i\widetilde{Q}^{\ell}_+), \\ \bar{\boldsymbol{s}}^{\ell}_+ &= \frac{1}{2}(Q_+ - i\widetilde{Q}^{\ell}_+), \end{aligned} \tag{4.1.40}$$

which are identified with the differentials of hyper-Kähler $\mathcal{G}$-equivariant Dolbeault cohomology on the space $\mathcal{A}$ of all gauge fields - see (3.2.36).

The above procedure is called twisting and is originally due to Witten [1]. It leads for any global supersymmetric theory with a suitable internal symmetry to a cohomological field theory. On a manifold $\mathbb{R}^4$, $K3$, T^4 or ALE, for which the holonomy is contained in $SU(2)$, twisting does nothing. Thus the physical supersymmetric Yang-Mills theory and the cohomological Yang-Mills theory are indistinguishable. We should emphasis that a "cohomological" field theory is cohomological only as far as it computes correlation functions of observables annihilated by the global supercharges of the theory. Otherwise the theory is not "cohomological" at all. In the above respect the only benefit of cohomological field theory compared with a possible equivalent globally space-time supersymmetric theory is that it can be defined on a general manifold, which usually does not admit any constant spinor, or not even spinors. Then a cohomological field theory assigns a certain set of cohomological quantities of differential-topological nature of the manifold in terms of the correlation functions of supersymmetric observables. For the present model these are Donaldson's polynomial invariants, which depend only on the diffeomorphism class of the four manifold.

According to our approach a global supersymmetric field theory on a Kähler manifold M, including flat Euclidean space, is nothing but an equivariant $N_c = (2,0)$ supersymmetric sigma-model in $(0+0)$ dimensions, whose target space is the function space of all fields in M. For example $N = 2$ supersymmetric Yang-Mills theory on $\mathbb{R}^4$ is equivalent to a $N_c = (2,0)$ sigma model with target space given by the space of all gauge fields. There is literally nothing wrong in the above identification. However such an aptitude is overlooking an open "secret" of quantum field theory.

One of the most beautiful properties of quantum field theory is the dependence on energy scale known as asymptotic freedom. One may define a so called microscopic theory with certain associated geometric structures like vector bundles etc. At a certain energy scale massive degrees of freedom above the scale are not essential and can be integrated out. Then we have a (Wilsonian) effective theory in terms of light degrees of freedom. A wonderful thing is that certain completely new massless degrees of freedom can appear at certain scales, and the above prescription breaks down. This leads to certain "singularities" which can be mended by including these new massless degrees of freedom. By lowering the energy scale arbitrarily one may have an effective theory described by the new massless degrees of freedom only. Now one is interested in differential-topological quantities of a compact manifold M represented by a twisted version of the above quantum field theory. Such a quantity may be independent to arbitrary scaling of the metric on M such that everywhere M looks like flat Euclidean space. Then the essential information of such a differential-topological quantity on M should be contained in the effective field theory of new massless degrees of freedom. Such an effective field theory should be much simpler since all the irrelevant degrees are already decoupled. The above considerations then lead to an equivalence between completely different mathematical entities. This is exactly what happened in the so called Seiberg-Witten revolution in differential topology of four manifolds [11][12].

For a certain class of $N_c = (2,0)$ models we saw that there is a canonical embedding to a $N_c = (2,2)$ model, which is connected with the original model by a massive perturbation – see Sect. 3.4. Then one may use an analogy with the physical system such that the original (non-Abelian) $N_c = (2,0)$ model may be equivalent to a new (Abelian) $N_c = (2,0)$ model which can be discovered by the massive perturbation of the $N_c = (2,2)$ model. In the next section we consider such an embedding of our $N_c = (2,0)$ model ($N = 2$ SYM) to a $N_c = (2,2)$ model. The resulting model turns out to be a twisted version of $N = 4$ super-Yang-Mills theory [68][7]. The massive perturbation corresponds to giving bare mass to the $N = 2$ hypermultiplet of $N = 4$ SYM theory. The new $N_c = (2,0)$ model corresponds to the cohomological field theory computing Seiberg-Witten invariants. A beautiful property of $N = 4$ theory is scale independence as well as higher symmetry known as S-duality.

It seems to be a good analogy to compare any well-defined $N_c = (2,0)$ model and its extension to a $N_c = (2,2)$ model with $N = 2$ and $N = 4$ supersymmetric Yang-Mills theory. For example on may, roughly, regard a well-defined $N_c = (2,0)$ model as an asymptotically free global supersymmetric theory while its $N_c = (2,2)$ extension can then be viewed as a scale independent theory. One may also interpret the massive perturbation as using a S^1 symmetry as a certain renormalization group flow, such that the original and the deformed $N_c = (2,0)$ models lie in different fixed points. Then the equivalence between the two models may be interpreted as the two models not entirely forgetting their origin.

4.2 Vafa-Witten Theory

In the paper [7] Vafa and Witten presented strong evidence for S-duality of $N = 4$ super-Yang-Mills theory. They used a topological, twisted version of the $N = 4$ theory [68] and were able to determine the partition function of $N = 4$ super-Yang-Mills theory on certain Kähler manifolds. In particular they identified the partition function with the Euler characteristic of the moduli space of instantons, provided certain vanishing theorems hold.

This section, based on [59], is an elaboration and generalization of the work of Vafa and Witten. We want to determine the partition function for a general compact Kähler surface M with $b_2^+ \geq 3$. Our computation of the partition function involves a a series of perturbations which break the supersymmetry down to $N = 2$ and $N = 1$ (topological) supersymmetry. The perturbation down to $N = 2$ is achieved by adding a bare mass term for the $N = 2$ adjoint hypermultiplet. Geometrically, this term may be viewed as the equivariant momentum map of a $\mathcal{G} \times S^1$-action on the hypermultiplet. As a result of its inclusion in the action, the path integral is localized on the fixed point set of the $\mathcal{G} \times S^1$-action, which consists of two branches: (i) the moduli space of anti-self-dual connections, (ii) the moduli space of a certain class of Seiberg-Witten monopoles. Perturbing further down to $N = 1$ leads to the factorization of the Seiberg-Witten classes contributing to branch (ii). Specializing to gauge groups $SU(2)$ and $SO(3)$ we propose a formula for the branch (ii) contribution on a general Kähler manifold with $b_2^+ \geq 3$. Then S-duality of $N = 4$ super-Yang-Mills theory enables us to determine the entire partition function. As a corollary we obtain a formula for the Euler characteristic of the moduli space of instantons (branch (i)). Finally we consider the pure $N = 2$ limit and obtain the essential part of Witten's formula for Donaldson invariants [12].

Our construction sketched above is an example of the construction in Sect. 3.4. The twisted $N = 4$ super-Yang-Mills theory on Kähler surface is an example of our $N_c = (2,2)$ model. The massive perturbation to twisted $N = 2$ super-Yang-Mills theory corresponds to the perturbation to $N_c = (2,0)$ mode described in Sect. 3.4.2. Finally massive perturbation to $N = 1$ super-Yang-Mills theory corresponds to perturbation of $N_c = (2,0)$ mode down to a $N_c = (1,0)$ model.

4.2.1 Embedding to $N_c = (2,2)$ Model

In this section we apply the construction in Sect. 3.4. to embed the previous $N_c = (2,0)$ model on a Kähler surface M to a $N_c = (2,2)$ model.

As usual we have the $N_c = (2,2)$ gauge multiplet

$$\begin{array}{ccccc}
\bar{\sigma} & \xrightarrow{s_+} & \eta_+ & \xleftarrow{s_-} & \phi_{++} \\
\big\downarrow{\scriptstyle \bar{s}_-} & & \big\downarrow{\scriptstyle \bar{s}_-} & & \big\downarrow{\scriptstyle \bar{s}_-} \\
\bar{\eta}_- & \xrightarrow{s_+} & D & \xleftarrow{s_-} & \bar{\eta}_+ \\
\big\uparrow{\scriptstyle \bar{s}_+} & & \big\uparrow{\scriptstyle \bar{s}_+} & & \big\uparrow{\scriptstyle \bar{s}_+} \\
\phi_{--} & \xrightarrow{s_+} & \eta_- & \xleftarrow{s_-} & \sigma
\end{array} \quad , \tag{4.2.1}$$

which consists of adjoint valued scalars on M. We will denote $\phi_{++} = \phi$ and $\phi_{--} = \overline{\phi}$ in certain occasions. From the holomorphic multiplets $(A^{0,1}, \psi_+^{0,1})$ we build up the following chiral multiplets,

$$\begin{array}{ccccc} \psi_-^{0,1} & \stackrel{s_-}{\longleftarrow} & A^{0,1} & \stackrel{s_+}{\longrightarrow} & \psi_+^{0,1} \\ & {}_{s_+}\searrow & & \swarrow_{s_-} & . \\ & & H^{0,1} & & \end{array} \tag{4.2.2}$$

From the Fermi multiplets $(\chi_-^{2,0}, H^{2,0})$ we build up another set of chiral multiplets,

$$\begin{array}{ccccc} \chi_-^{2,0} & \stackrel{s_-}{\longleftarrow} & B^{2,0} & \stackrel{s_+}{\longrightarrow} & \chi_+^{2,0} \\ & {}_{s_+}\searrow & & \swarrow_{s_-} & . \\ & & H^{2,0} & & \end{array} \tag{4.2.3}$$

Following the discussion in Sect. 3.2 and 3.4 we have the following manifestly $N_c = (2, 2)$ invariant functional

$$\begin{aligned} S = & s_+\overline{s}_+ s_-\overline{s}_- \left(\mathcal{K}(A^{1,0}, A^{0,1}) + \mathcal{K}(B^{2,0}, B^{0,2}) - \int_M \operatorname{Tr}(\sigma * \overline{\sigma}) \right) \\ & + s_+ s_- \mathcal{W}\left(A^{0,1}, B^{2,0}\right) + \overline{s}_+\overline{s}_- \overline{\mathcal{W}}\left(A^{1,0}, B^{0,2}\right). \end{aligned} \tag{4.2.4}$$

It is obvious that the Hermitian structure $\mathcal{K}(B^{2,0}, B^{0,2})$ of the space

$$\Omega^{2,0}(M, End(E)) \oplus \Omega^{0,2}(M, End(E))$$

is given by

$$\mathcal{K}(B^{2,0}, B^{0,2}) = -\frac{1}{4\pi^2} \int_M \operatorname{Tr}\left(B^{2,0} \wedge *B^{0,2}\right). \tag{4.2.5}$$

The holomorphic potential $\mathcal{W}$ is also uniquely determined as follows,

$$\mathcal{W}\left(A^{0,1}, B^{2,0}\right) = \frac{1}{4\pi^2} \int_M \operatorname{Tr}\left(B^{2,0} \wedge F^{0,2}\right). \tag{4.2.6}$$

Now, from the discussions in Sect. 3.4, we see that the path integral is localized to the solution space of the following equations, modulo the gauge symmetry,

$$\begin{aligned} F^{0,2} &= 0, \\ \partial_A B^{0,2} &= 0, \\ iF \wedge \omega + [B^{2,0}, B^{0,2}] &= 0, \\ [\overline{\sigma}, B^{0,2}] &= 0, \\ [\sigma, B^{0,2}] &= 0, \\ [\sigma, \overline{\sigma}] &= 0, \\ d_A \overline{\sigma} &= 0, \end{aligned} \tag{4.2.7}$$

and

$$\begin{aligned}[\phi_{\pm\pm}, B^{0,2}] &= 0,\\ [\phi_{\pm\pm}, \overline{\sigma}] &= 0,\\ [\phi_{++}, \phi_{--}] &= 0,\\ d_A \phi_{\pm\pm} &= 0.\end{aligned} \tag{4.2.8}$$

The set of equations in (4.2.7) is the Vafa-Witten equation on a Kähler surface.

For a completeness we recall the original form of the Vafa-Witten equation on a Riemann 4-manifold

$$\begin{aligned} iF^+_{mn} + \frac{1}{4}[B_{mp}, B^p{}_n]^+ &= [C, B_{mn}] = 0,\\ D^n B_{mn} &= D_m C = 0,\end{aligned} \tag{4.2.9}$$

where B_{mn}, $m,n = 1,\ldots,4$, is an adjoint-valued self-dual two-form and C is an adjoint valued real scalar. On a Kähler surface, using (4.1.1), (C, B_{mn}) are equivalent to a complex scalar σ and a $(0,2)$-form $B^{0,2}$. The equations (4.2.9) become (4.2.7) on a Kähler surface.

4.2.2 Perturbation to $N_c = (2,0)$ Model

Now we want to perturb the above $N_c = (2,2)$ model to a $N_c = (2,0)$ model by maintaining the $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ symmetry only. Following the discussions in Sect. 3.4 we enlarge the gauge symmetry group $\mathcal{G}$ to $\mathcal{G} \times S^1$. Under the S^1 only fields newly introduced for the $N_c = (2,2)$ model are charged . We have

$$\begin{aligned} S^1 &: (B^{2,0}, \chi_+^{2,0}) \to \xi(B^{2,0}, \chi_+^{2,0}),\\ S^1 &: (\psi_-^{0,1}, H^{0,1}) \to \overline{\xi}(\psi_-^{0,1}, H^{0,1}),\\ S^1 &: (\overline{\sigma}, \eta_+) \to \overline{\xi}(\overline{\sigma}, \eta_+),\end{aligned} \tag{4.2.10}$$

and the opposite charges to the conjugate fields. Following the procedure in Sect. 3.4, the perturbed action functional $S(m, \overline{m})$ is given by

$$\begin{aligned} &S(m,\overline{m})\\ &= S - m\overline{m} \int_M \mathrm{Tr}\Big(B^{2,0} \wedge * B^{0,2} + \sigma * \overline{\sigma}\Big)\\ &\quad + im \int_M \mathrm{Tr}\Big(\phi_{--}([B^{2,0}, B^{0,2}] - [\sigma, *\overline{\sigma}]) + \psi_-^{0,1} \wedge * \overline{\psi}_-^{1,0}\Big)\\ &\quad - i\overline{m} \int_M \mathrm{Tr}\Big(\mathrm{Tr}\, \phi_{++}([B^{2,0}, B^{0,2}] - [\sigma, *\overline{\sigma}]) + \chi_+^{2,0} \wedge \overline{\chi}_+^{0,2} + \overline{\eta}_+ * \eta_+\Big).\end{aligned} \tag{4.2.11}$$

The $|m|$ dependent terms are exactly the physical mass terms for the $N = 2$ adjoint hypermultiplet after the twisting.

After the above perturbation the path integral is localized to the space of solutions of the following set of equations, modulo the gauge symmetry;

$$\begin{aligned} F^{0,2} &= [\overline{\sigma}, B^{0,2}] = 0,\\ iF \wedge \omega + [B^{2,0}, B^{0,2}] - \frac{1}{2}[\sigma, \overline{\sigma}]\omega \wedge \omega &= 0, \qquad \partial_A B^{0,2} = \overline{\partial}_A \overline{\sigma} = 0,\end{aligned} \tag{4.2.12}$$

and

$$[\phi, \overline{\phi}] = 0, \qquad d_A \phi = 0, \tag{4.2.13}$$

and

$$\begin{aligned} [\phi, B^{0,2}] - mB^{0,2} = 0, \qquad [\phi, \overline{\sigma}] - m\overline{\sigma} = 0, \\ [\phi, B^{2,0}] + mB^{2,0} = 0, \qquad [\phi, \sigma] + m\sigma = 0. \end{aligned} \tag{4.2.14}$$

In studying solutions of these fixed point equations we specialize to the gauge group $SU(2)$. We also restrict to Kähler surfaces with $b_+^2(M) \geq 3$. Then there are no reducible instantons for generic choice of the metric.

First of all, (4.2.13) implies that ϕ should be diagonalized at the fixed points. Thus we have two branches

- Branch (i): $\phi = 0$. The gauge symmetry is unbroken. Then (4.2.14) implies that $B^{2,0}, B^{0,2}, \sigma$, and $\overline{\sigma}$ vanish. So the fixed point equation (4.2.12) reduces to the anti-self-duality equation for the connection A: $F_A^+ = 0$.
- Branch (ii): $\phi \neq 0$. The gauge symmetry is broken down to $U(1)$. Thus the bundle E splits into line bundles $E = L \oplus L^{-1}$ with $L \cdot L = -k$, where k is the instanton number. And ϕ takes the form $\phi = diag(a, -a)$. Then the only non-trivial solutions of (4.2.14) are, with $m - a = 0$:

$$\begin{aligned} B^{0,2} = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \qquad \overline{\sigma} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \\ B^{2,0} = \begin{pmatrix} 0 & 0 \\ \overline{\beta} & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 & 0 \\ \overline{\alpha} & 0 \end{pmatrix}. \end{aligned} \tag{4.2.15}$$

Then (4.2.12) reduces to

$$\begin{aligned} F_{L^2}^{0,2} = \alpha\beta = 0, \\ iF_{L^2} \wedge \omega = \beta \wedge \overline{\beta} - \alpha\overline{\alpha}\omega^2, \end{aligned} \qquad \partial_{L^2}\beta = \overline{\partial}_{L^2}\alpha = 0. \tag{4.2.16}$$

Here α is a section of L^{-2} and β is a section of $K^{-1} \otimes L^2$, with K denoting the canonical line bundle. To make progress it it is useful to regard the above equation as a perturbation of another equation. To achieve this note that

$$F_{K^{-1/2} \otimes L^2} = \frac{1}{2} F_{K^{-1}} + F_{L^2} \quad \to F_{L^2} = F_{K^{-1/2} \otimes L^2} - \frac{1}{2} F_{K^{-1}}, \tag{4.2.17}$$

so that we can write

$$\begin{aligned} F_{L^2}^{0,2} = \alpha\beta = 0, \\ \frac{i}{2} F_\zeta \wedge \omega = \beta \wedge \overline{\beta} - \alpha\overline{\alpha}\omega^2 + \frac{1}{2} F_{K^{-1}} \wedge \omega, \end{aligned} \qquad \partial_{L^2}\beta = \overline{\partial}_{L^2}\alpha = 0. \tag{4.2.18}$$

This is a perturbation of the Seiberg-Witten equation [12] for a *spin*^{c} structure $\zeta = K^{-1} \otimes L^4$; this fact will be crucial in the next section. For later use we also note that $c_1(\zeta) = w_2(M)$ modulo 2 since $c_1(K) = w_2(M)$ mod 2.

4.2.3 Perturbation to $N_c = (1,0)$ Theory

We can further break the remaining $N_c = (2,0)$ symmetry down to $N_c = (1,0)$ by introducing a bare mass for the twisted $N = 1$ matter-multiplet. We will do this by preserving the $\overline{\boldsymbol{s}}_+$-symmetry only. Note that, among the twisted $N = 2$ vector multiplet given by (3.2.7), the twisted $N = 1$ matter multiplet consists of $(\psi_+^{0,1}, \phi, \overline{\phi}, \overline{\eta}_-, \chi_-^{2,0})$.

The required mass term involves a holomorphic two-form $\omega^{2,0} \in H^{2,0}(M)$ and has the form

$$I_{\omega^{2,0}} = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(\psi_+^{0,1} \wedge \psi_+^{0,1}) \wedge \omega^{2,0}. \tag{4.2.19}$$

This term is invariant under $\boldsymbol{s}_+$-symmetry, but not invariant under the $\overline{\boldsymbol{s}}_+$-symmetry;

$$\overline{\boldsymbol{s}}_+ I_{\omega^{2,0}} = -\frac{1}{4\pi^2}\int_M \mathrm{Tr}\,\phi\overline{\partial}_A\psi_+^{0,1} \wedge \omega^{2,0}. \tag{4.2.20}$$

However, the imposition $\chi_-^{2,0}$ equation of motion leads to invariance. The relevant term in the action $S(m,\overline{m})$ is $-\int_M M\,\mathrm{Tr}\,\chi_-^{2,0} \wedge \overline{\partial}_A\psi_+^{0,1}$. If we add (4.2.19) to the action $S(m,\overline{m})$ of (4.2.11) and at the same time change the $\overline{\boldsymbol{s}}_+$-transformation of $\chi_-^{2,0}$ according to

$$\overline{\boldsymbol{s}}_+\chi_-^{2,0} = [\overline{\sigma}, B^{2,0}] \longrightarrow \overline{\boldsymbol{s}}_+\chi_-^{2,0} = [\overline{\sigma}, B^{2,0}] - \phi\omega^{2,0}, \tag{4.2.21}$$

the new action $S(m,\overline{m})' + I_{\omega^{2,0}}$ enjoys $\overline{\boldsymbol{s}}_+$-symmetry. Here $S(m,\overline{m})'$ is given by

$$S(m,\overline{m})' = S_2(m) - \frac{1}{4\pi^2}\int_M \mathrm{Tr}\,\phi[\sigma, B^{0,2}] \wedge \omega^{2,0}, \tag{4.2.22}$$

where the additional term is due to the modification (4.2.21). Since $\overline{\boldsymbol{s}}_+\phi = 0$, we still have the property $\overline{\boldsymbol{s}}_+^2 = 0$. In this way the one component $\psi_+^{0,1}$ of the $N = 1$ chiral superfield has obtained a mass. To give mass to the remaining components in the $N = 1$ matter multiplet we add the following $\overline{\boldsymbol{s}}_+$-exact terms to the action

$$\begin{aligned} I_{\phi\overline{\phi}} =& -\frac{\overline{\boldsymbol{s}}_+}{4\pi^2}\int_M \mathrm{Tr}\left(\overline{\phi}\chi_-^{2,0}\right) \wedge \omega^{0,2} \\ =& -\frac{1}{4\pi^2}\int_M \mathrm{Tr}\left(\overline{\phi}[\overline{\sigma}, B^{2,0}]\right) \wedge \omega^{0,2} + \frac{1}{4\pi^2}\int_M \mathrm{Tr}\left(\phi\overline{\phi}\right)\omega^{2,0} \wedge \omega^{0,2} \\ & -\frac{1}{4\pi^2}\int_M \mathrm{Tr}\left(\overline{\eta}_-\chi_-^{2,0}\right) \wedge \omega^{0,2}. \end{aligned} \tag{4.2.23}$$

A similar prescription for breaking pure $N = 2$ theory down to $N = 1$ was given by Witten in [10].

To sum up, the total action

$$S(m,\overline{m},\omega^{0,2}) = S(m,\overline{m})' + I_{\omega^{0,2}} + I_{\phi\overline{\phi}}, \tag{4.2.24}$$

has only $\overline{\boldsymbol{s}}_+$ supersymmetry and all the matter multiplets have a bare mass.

Now the fixed point equations (4.2.12) undergo an important change due to the modification of the $\overline{s}_+$ transformation law of $\chi_-^{0,2}$ given by (4.2.21). The new fixed point equations are

$$\begin{aligned} F^{0,2} = [\overline{\sigma}, B^{0,2}] - \phi\omega^{0,2} = 0, \qquad & \\ iF \wedge \omega + [B^{2,0}, B^{0,2}] - \frac{1}{2}[\sigma, \overline{\sigma}]\omega \wedge \omega = 0, \qquad & \overline{\partial}_A^* B^{0,2} = \partial_A \overline{\sigma} = 0, \end{aligned} \tag{4.2.25}$$

while (4.2.13) and (4.2.14) remain unchanged. Thus there are again two branches

- Branch (i): This branch is unchanged.
- Branch (ii): We have

$$\begin{aligned} F_\zeta^{0,2} &= \alpha\beta - m\omega^{0,2} = 0, \\ \frac{i}{2}F_\zeta \wedge \omega &= \beta \wedge \overline{\beta} - \alpha\overline{\alpha}\omega \wedge \omega - \frac{1}{2}F_{K^{-1}}, \end{aligned} \tag{4.2.26}$$

  where $\zeta = K^{-1} \otimes L^4$. This is a perturbed version of the Seiberg-Witten equation, containing the perturbation introduced by Witten in [12]. The condition

$$\alpha\beta = m\omega^{0,2}, \tag{4.2.27}$$

  gives the crucial factorization condition of the Seiberg-Witten basic classes.

Analysis of Branch (ii) Fixed Points

Our analysis of branch (ii) exploits the relation of the the defining equations with the the Seiberg-Witten equation.

As a first step we need to classify which Seiberg-Witten classes contribute to branch (ii). For an arbitrary *spin*^{c} structure x, which can always be written in terms of an arbitrary integral line bundle ξ as

$$x = K^{-1} \otimes \xi^2, \tag{4.2.28}$$

we have an associated Seiberg-Witten equation. If the square root $\xi^{1/2} = L$ of ξ exists, the Seiberg-Witten is identical to the fixed point equation (4.2.18) of branch (ii). The inclusion of the perturbation in (4.2.26) further implies that we also have to satisfy the factorization condition $\alpha\beta = \omega^{0,2}$, where we have scaled $m = 1$ in (4.2.27).

Let the canonical divisor K be given by $K = \sum_i r_i C_i$, where the C_i are irreducible components. The factorization means that

$$K^{1/2} \otimes x^{1/2} = \xi = \sum_i s_i C_i, \tag{4.2.29}$$

where s_i are integers with $0 \le s_i \le r_i$. Thus, the question of which Seiberg-Witten classes contribute to branch (ii) reduces to finding line bundles L satisfying $L \cdot L = -k$ and

$$2L = \sum_{i=1}^n s_i C_i, \qquad 0 \le s_i \le r_i. \tag{4.2.30}$$

Now let x be a Seiberg-Witten basic class. If $(x^{1/4} \otimes K^{1/4})$ exists as a line bundle, then the associated SW invariant n_x contributes to the path integral in branch (ii). Note that $(x^{1/2} \otimes K^{1/2})$ always exist as a line bundle. The question is thus whether the square root of $(x^{1/2} \otimes K^{1/2})$ exists, which is the case iff

$$\frac{1}{2}[x+K]=0, \tag{4.2.31}$$

or, equivalently

$$\frac{1}{2}[x+w_2(M)]=0, \tag{4.2.32}$$

where [...] means the mod 2 reduction. Here $w_2(M)$ is the second Stiefel-Whitney class of our Kähler manifold M. In the $SU(2)$ case such a square root may not exist. However, if we repeat the analysis for an $SO(3)$ bundle E, the factorization condition can be met provided the second Stiefel-Whitney class $w_2(E)$ of E satisfies

$$\frac{1}{2}[x+w_2(M)] \equiv w_2(E). \tag{4.2.33}$$

With the abbreviations $z_0 = w_2(M)$, $z = w_2(E)$ and $2x' = x+K$ the branch (ii) contribution has the form

$$\sum_x n_x \delta_{z,[x']} \times (\ldots), \tag{4.2.34}$$

where the summation is over all Seiberg-Witten basic classes x and n_x denotes the Seiberg-Witten invariant defined by x. This general form applies to both the $SU(2)$ and the $SO(3)$ case. In principle one could proceed to compute the branch (ii) contribution directly using localization techniques. However, in practice this requires that one starts with a suitable compactification of the moduli space of the Vafa-Witten equations in order to make the integration over the normal bundle of branch (ii) well-defined – this will fill the unwritten part $(\ldots)$ in (4.2.34). Here we are not able to follow that path. Though we have technical limitation we demonstrated that the general principle advocated in Sect. 2.5 works fine, since we saw a glimpse of purely non-perturbative quantum properties by a simple classical analysis.

In the remaining chapter we determine the branch (ii) contribution to the partition function by a generalization of the results of Vafa and Witten in [7]. Then we apply the S-duality to determine the full partition function.

4.2.4 Partition Function

Vafa and Witten make a precise statement about the expected behavior of the partition function of $N = 4$ super-Yang-Mills theory under S-duality [7]. According to [7] the partition function of $N = 4$ theory is a modular form invariant under the $\Gamma_0(4)$ subgroup of $SL(2, \mathbb{Z})$. If this is true the total partition function can be determined from the contribution of branch (ii) alone, as we shall now show.

Consider the case $K = \sum_i^n [C_i]$ where the $[C_i]$ are irreducible and disjoint. Vafa and Witten made a prediction for what we call the contribution to the partition function from branch (ii) (eq. 5.50 in [7]): for the $SU(2)$ case the answer is

$$\left(\frac{G(q^2)}{4}\right)^{\nu/2}\left(\frac{\theta_0}{\eta^2}\right)^{\sum_{i=1}^n(1-g_i)}\sum_{\vec{\varepsilon}}\delta_{0,w_2(\vec{\varepsilon})}\left(\prod_{i=1}^n t_i^{\varepsilon_i}\right)\left(\frac{\theta_1}{\theta_0}\right)^{\sum\varepsilon_i(1-g_i)}, \quad (4.2.35)$$

where $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ and $\varepsilon_i = 0$ or 1 chosen independently. Here $\nu = (\chi + \sigma)/4$ and χ and σ denote the Euler number and signature of the manifold, respectively. The expression $G(q) = 1/\eta^{24}$ is defined by the Dedekind eta-function.

From our perspective the sum and the delta function $\delta_{0,w_2(\vec{\varepsilon})}$ can be understood as follows. What is called $w_2(\vec{\varepsilon})$ in [7] is a special form of $[x']$, so that the sum in (4.2.35) is over the same range as the sum (4.2.34). In our notation, the factorization condition has the form

$$x' = 2L = \sum_{i=1}^n \varepsilon_i[C_i], \qquad 0 \le s_i \le 1 \quad (4.2.36)$$

From

$$k = -L\cdot L = -\frac{1}{4}x'\cdot x' = -\frac{1}{4}\sum_i s_i^2(g_i - 1) \quad (4.2.37)$$

and since $s_i^2 = s_i$ for $s_i = 0$ or 1, we recover the formula

$$k = -\frac{1}{4}\sum_i \varepsilon_i(g_i - 1), \qquad 0 \le \varepsilon_i \le 1. \quad (4.2.38)$$

given in [7]. Note also that $\sum_{i=1}^n (g_i - 1) = K\cdot K = 2\chi + 3\sigma$.

We now propose a formula for the branch (ii) contributions on general Kähler manifolds with $b_2^+ \ge 3$. We replace

$$\sum_{\vec{\varepsilon}}\delta_{0,w_2(\vec{\varepsilon})}\left(\prod_{i=1}^n t_i^{\varepsilon_i}\right)\left(\frac{\theta_1}{\theta_0}\right)^{\sum\varepsilon_i(1-g_i)} \longrightarrow (-1)^\nu \sum_x \delta_{0,[x']}n_x\left(\frac{\theta_1}{\theta_0}\right)^{-x'\cdot x'}, \quad (4.2.39)$$

where the summation is over all Seiberg-Witten basic classes x with the Seiberg-Witten invariants n_x. Then, (4.2.35) can be written as

$$(-1)^\nu\left(\frac{G(q^2)}{4}\right)^{\nu/2}\left(\frac{\theta_0}{\eta^2}\right)^{-2\chi-3\sigma}\sum_x \delta_{0,x'}n_x\left(\frac{\theta_1}{\theta_0}\right)^{-x'\cdot x'}. \quad (4.2.40)$$

In the $SO(3)$ case, for a fixed $z = w_2(E)$, we immediately get

$$(-1)^\nu\left(\frac{G(q^2)}{4}\right)^{\nu/2}\left(\frac{\theta_0}{\eta^2}\right)^{-2\chi-3\sigma}\sum_x \delta_{z,x'}n_x\left(\frac{\theta_1}{\theta_0}\right)^{-x'\cdot x'}. \quad (4.2.41)$$

The basic idea is to examine the terms generated by applying the S-duality transformations corresponding to $\tau \to -1/\tau$ and $\tau \to \tau + 1$ to (4.2.41). Combining the resulting terms in a convenient fashion one gets

$$\begin{aligned} Z_z =&(-1)^\nu \left(\frac{G(q^2)}{4}\right)^{\nu/2} \left(\frac{\theta_0}{\eta^2}\right)^{-2\chi-3\sigma} \sum_x \delta_{z,[x']} n_x \left(\frac{\theta_1}{\theta_0}\right)^{-x'\cdot x'} \\ &+ 2^{1-b_1} \left(\frac{G(q^{1/2})}{4}\right)^{\nu/2} \left(\frac{\theta_0+\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \sum_x (-1)^{[x']\cdot z} n_x \left(\frac{\theta_0-\theta_1}{\theta_0+\theta_1}\right)^{-x'\cdot x'} \\ &+ 2^{1-b_1} i^{-z^2} \left(\frac{G(-q^{1/2})}{4}\right)^{\nu/2} \left(\frac{\theta_0 - i\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \\ &\times s \sum_x (-1)^{[x']\cdot z} n_x \left(\frac{\theta_0+i\theta_1}{\theta_0-i\theta_1}\right)^{-x'\cdot x'}, \end{aligned} \tag{4.2.42}$$

where the sum $\sum_x$ is over all Seiberg-Witten basic classes, as before. In principle there could be a contribution to the partition function which can not be obtained by performing modular transformations of the contribution of branch (ii). However, such a contribution should vanish for a manifold with $b_2^+ > 1$.

According to [7] the required transformation behavior under S-duality is:

$$Z_y(-1/\tau) = 2^{-b_2/2} (-1)^\nu \left(\frac{\tau}{i}\right)^{-\chi/2} \sum_z (-1)^{z\cdot y} Z_z(\tau). \tag{4.2.43}$$

We can check that our proposed expression (4.2.42) transforms correctly as follows. First we insert (4.2.42) into the RHS of (4.2.43) and obtain

$$\begin{aligned} RHS =& \left(\frac{\tau}{i}\right)^{-\chi/2} \Bigg[2^{-b_2/2} \left(\frac{G(q^2)}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0}{\eta^2}\right)^{-2\chi-3\sigma} \sum_x (-1)^{z\cdot[x']} n_x \left(\frac{\theta_1}{\theta_0}\right)^{-x'\cdot x'} \\ &+ 2^{1-b_1+b_2/2} (-1)^\nu \left(\frac{G(q^{\frac12})}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0+\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \sum_x \delta_{z,[x']} n_x \left(\frac{\theta_0-\theta_1}{\theta_0+\theta_1}\right)^{-x'\cdot x'} \\ &+ 2^{1-b_1} (-1)^\nu i^{z^2-\frac{\sigma}{2}} \left(\frac{G(-q^{\frac12})}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0-i\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \\ &\times \sum_x (-1)^{[x']\cdot z} i^{x'\cdot x'} n_x \left(\frac{\theta_0+i\theta_1}{\theta_0-i\theta_1}\right)^{-x'\cdot x'} \Bigg], \end{aligned} \tag{4.2.44}$$

where we used

$$\begin{aligned} &\sum_z (-1)^{z\cdot y} \delta_{z,x'} = (-1)^{y\cdot x'}, \\ &\sum_z (-1)^{z\cdot y + z\cdot x'} = 2^{b_2} \delta_{y,x'}, \\ &\sum_z (-1)^{z\cdot y} i^{-z^2} (-1)^{z\cdot x'} = 2^{b_2/2} i^{+y^2-\sigma/2+x'\cdot x'} (-1)^{x'\cdot y}. \end{aligned} \tag{4.2.45}$$

Carefully taking into account differences in notation these formulae follow from those noted as eq. (5.40) in [7]. In comparing (4.2.44) with the expression (4.2.42) evaluated at $-1/\tau$ one finds that the first line and second line in (4.2.44) equal, respectively, the second and first line in (4.2.42) evaluated at $-1/\tau$. The third line of (4.2.44) should thus be compared with the third line in (4.2.42) at $-1/\tau$. The equality here may require some explanation. Performing $\tau \to -1/\tau$ in the third line on (4.2.42) one finds, with some rearrangements, that

$$s2^{1-b_1}(-1)^{\nu} i^{z^2-\sigma/2}\left(\frac{\tau}{i}\right)^{-\chi/2}\left(\frac{G(-q^{1/2})}{4}\right)^{\nu/2}\left(\frac{\theta_0-i\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma}$$
$$\times\sum_x(-1)^{-z^2+[x']\cdot z} i^{-x'\cdot x'} i^{\sigma}(-1)^{\nu} n_x\left(\frac{\theta_0-i\theta_1}{\theta_0+i\theta_1}\right)^{-x'\cdot x'+2\chi+3\sigma}. \tag{4.2.46}$$

We want to show that the above is identical to the third term in (4.2.44). A crucial property is that $-x$ is a Seiberg-Witten basic class if x is. Note also that $x'=\frac{1}{2}x+\frac{1}{2}K$. Writing $\overline{x}'=-\frac{1}{2}x+\frac{1}{2}K$ we have $-x'\cdot x'+2\chi+3\sigma=\overline{x}'\cdot\overline{x}'$, since a Seiberg-Witten basic class x satisfies $x\cdot x=2\chi+3\sigma$. Now the second line of (4.2.46) can be rewritten as

$$\sum_{-x}(-1)^{-z^2+x^2}(-1)^{[\overline{x}']\cdot z} i^{\overline{x}'\cdot\overline{x}'} n_{-x}\left(\frac{\theta_0+i\theta_1}{\theta_0-i\theta_1}\right)^{-\overline{x}'\cdot\overline{x}'}, \tag{4.2.47}$$

where we used $n_{-x}=(-1)^{\nu}n_x$ and the fact that $\nu=(\chi+\sigma)/4$ is an integer. The Wu formula implies $(-1)^{-z^2+x^2}=1$, and we replace the dummy variable $-x,\overline{x}'$ with x,x' to complete the proof.

A Relation with Strings

Taubes proved that Seiberg-Witten invariants (SW) are equivalent to Gromov-Witten invariants (Gr) for a symplectic 4-manifold of simple type [80]. Here we only consider a Kähler surface. Let ξ be a non-trivial, complex line bundle over M and use ξ to define a *spin*^{c} structure $x=K^{-1}\otimes\xi^2$. Then $SW(K^{-1}\otimes\xi^2)=Gr(\xi)$. Consider a line bundle ξ such that $SW(K^{-1}\otimes\xi^2)\neq 0$, then the Poincaré dual of $c_1(\xi)$ is represented by the fundamental class of an embedded, holomorphic submanifold with, say, n irreducible components. Then each component H_i satisfies the adjunction formula $g(H_i)=1+H_i\cdot H_i$, where $g(H_i)$ is the genus of H_i. We can define the integer multiplicities a_i by writing $\xi=\sum_{i=1}^n a_iH_i$.

Let the canonical divisor K be given by a union of irreducible components C_i with multiplicities r_i, i.e $K=\sum_i r_iC_i$. The factorization of a Seiberg-Witten basic class x means that

$$K^{1/2}\otimes x^{1/2}=\xi=\sum_i s_iC_i \tag{4.2.48}$$

where the s_i are integers with $0\le s_i\le r_i$. Consequently, Taubes' result leads to the identifications

$$a_i=s_i,\qquad C_i=H_i. \tag{4.2.49}$$

Physically speaking this means that the world sheets of the superconducting cosmic strings discussed by Witten in [10] are embedded holomorphic curves.

Recall that for a fixed instanton number k the Seiberg-Witten classes $x = -K + 2x'$, with $x' \cdot x' = -4k$ and $z = [x']$, contribute to the partition function of $N = 4$ theory in branch (ii). From the above discussion we identify x' with holomorphic curve $x' = \sum_{i=1}^{n} s_i H_i$ and $1 - g(x') = -x' \cdot x'$. So the branch (ii) contribution can be written as the sum of contributions of all holomorphic curves Σ with $[\Sigma] = z$. The summation over the (space-time) instanton numbers is replaced with the summation over the genus of the holomorphic curves (the world-sheet instantons). So our formula (4.2.42) for the partition function of $N = 4$ theory can also be viewed as a genus expansion:

$$
\begin{aligned}
Z_z &= (-1)^{\nu} \left(\frac{G(q^2)}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0}{\eta^2}\right)^{-2\chi-3\sigma} \sum_{\Sigma} \delta_{z,[\Sigma]} Gr(\Sigma) \left(\frac{\theta_1}{\theta_0}\right)^{1-g(\Sigma)} \\
&+ 2^{1-b_1} \left(\frac{G\left(q^{1/2}\right)}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0+\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \sum_{\Sigma} (-1)^{[\Sigma]\cdot z} Gr(\Sigma) \left(\frac{\theta_0-\theta_1}{\theta_0+\theta_1}\right)^{1-g(\Sigma)} \\
&+ 2^{1-b_1} i^{-z^2} \left(\frac{G\left(-q^{1/2}\right)}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0 - i\theta_1}{2\eta^2}\right)^{-2\chi-3\sigma} \\
&\times \sum_{\Sigma} (-1)^{[\Sigma]\cdot z} Gr(\Sigma) \left(\frac{\theta_0+i\theta_1}{\theta_0-i\theta_1}\right)^{1-g(\Sigma)} .
\end{aligned}
\tag{4.2.50}
$$

The $N = 2$ Limit and Donaldson-Witten Invariants

It is instructive to rewrite the formula (4.2.42) as follows:

$$
\begin{aligned}
Z_z &= (-1)^{\nu} \left(\frac{G(q^2)}{4}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0}{\eta^2}\right)^{-2\chi-3\sigma} \sum_{x} \delta_{z,[x']} n_x \left(\frac{\theta_1}{\theta_0}\right)^{-x'\cdot x'} \\
&+ 2^{1-b_1+\frac{1}{4}(7\chi+11\sigma)} G\left(q^{1/2}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0+\theta_1}{\eta^2}\right)^{-2\chi-3\sigma} \sum_{x} (-1)^{[x']\cdot z} n_x \left(\frac{\theta_0-\theta_1}{\theta_0+\theta_1}\right)^{-x'\cdot x'} \\
&+ 2^{1-b_1+\frac{1}{4}(7\chi+11\sigma)} i^{-z^2} G\left(-q^{1/2}\right)^{\frac{\nu}{2}} \left(\frac{\theta_0 - i\theta_1}{\eta^2}\right)^{-2\chi-3\sigma} \\
&\times \sum_{x} (-1)^{[x']\cdot z} n_x \left(\frac{\theta_0+i\theta_1}{\theta_0-i\theta_1}\right)^{-x'\cdot x'} .
\end{aligned}
\tag{4.2.51}
$$

The fact that this formula can naturally be grouped into three terms whereas we classically think of contributions from two branches can be understood physically as follows. The first term is the contribution from branch (ii) and stems from the singularity in the u-plane due to the massless adjoint hypermultiplet. The remaining two terms are the contribution from branch (i), which classically corresponds to the singularity at the origin of the u-plane. Geometrically, the branch (i) contribution is the Euler characteristic of the moduli space of instantons [7]. The fact that this contribution is made up from two terms is due to a quantum effect: the classical singularity at the origin of the u-plane bifurcates quantum mechanically [72].

From the above formula we can recover the Donaldson invariants for gauge groups $SU(2)$ and $SO(3)$ as follows. For a simply connected simple type manifold, Witten's formula for the generating functional of Donaldson's invariants is [10]

$$\begin{aligned}\left\langle e^{\widehat{v+\lambda u}}\right\rangle_z =& 2^{1+\frac{1}{4}(7\chi+11\sigma)}\exp(v^2/2+2\lambda)\sum_x(-1)^{[x']\cdot z}n_x e^{v\cdot x} \\ &+2^{1+\frac{1}{4}(7\chi+11\sigma)}i^{\nu-z^2}\exp(-v^2/2-2\lambda)\sum_x(-1)^{[x']\cdot z}n_x e^{-iv\cdot x}.\end{aligned} \tag{4.2.52}$$

Here $\widehat{v}$ is the observable $\widehat{\mathcal{O}}_2^{(2)}$ associated with a two-dimensional cycle v and $u=\widehat{\mathcal{O}}_4^o$, see (4.1.32). The expectation value is computed using (twisted) $N=2$ super-Yang-Mills theory.

To obtain the above formula from the $N=4$ theory we could turn on the observables (4.1.32)after breaking the supersymmetry down to $N=2$ and, following [72][11], take the double scaling limit $m\to\infty$ and $q\to 0$ with $\Lambda^2=2q^{1/2}m^2$ being fixed. In this limit the singularity coming from the massless adjoint hypermultiplet (branch (ii)) moves to infinity in the u-plane and no longer contributes to the path integral. On the other hand the two other singularities remain at the points $u=\pm\Lambda^2$ (in Donaldson theory Λ^2 is normalized to 2). Here we are not able to consider general expectation values of observables. However, we can compute the $N=2$ limit of the partition function (4.2.51) (the $q\to 0$ limit since (4.2.51) does not depend on m). The leading terms only come from the second and the third lines and are given by

$$2^{1-b_1+\frac{1}{4}(7\chi+11\sigma)}\left(\sum_x(-1)^{[x']\cdot z}n_x+i^{\nu-z^2}\sum_x(-1)^{[x']\cdot z}n_x\right)\left(q^{3\nu/4}+\dots\right). \tag{4.2.53}$$

Note that this partition function vanishes unless the dimension of the moduli space of instantons is zero. Since $dim_{\mathbb{C}}\mathcal{M}_k=4k-3\nu$, this occurs when $k=3\nu/4$, thus explaining the leading term $q^{3\nu/4}$ in (4.2.53). In fact, the expression (4.2.53) contains all the non-trivial information about Donaldson's invariants.

The relation between the S^1-action and the mass term of the hypermultiplet described here were summarized and used in the paper [81]. There the same sequence of perturbations $N=4,2,1$ was used to relate the zero-dimensional reduction of the Vafa-Witten equation ($N=4$) to the ADHM description of instantons ($N=2$) [63][64] and of torsion-free-sheaves ($N=1$) [65]. It established the first concrete relation between D-instantons and Yang-Mills instantons [41]. The similar S^1-action and its application to the mass perturbation of the $N=4$ theory was also considered in [82].

Recently some progress was reported on the entire generating functional of the $N=2$ theory with a massive adjoint hypermultiplet with more general group [82][83] and on other four-manifolds [22].

4.2.5 Stringy Donaldson-Witten Theory ?

In this chapter we formulated a twisted $N=4$ SYM theory on a Kähler surface as a $N_c=(2,2)$ supersymmetric sigma model in zero-dimensions. As we remarked in the first part of Sect. 2.3 our model can be generalized to $N_{ws}=(2,2)$ space-time supersymmetric gauged linear sigma model in two-dimensions.

The two equivariant differentials s_+ and $\bar{s}_+$ can be identified with the two left-moving supercharges on a Riemann surface Σ. The other two differentials s_- and $\bar{s}_-$ are identified with the two right-moving supercharges on Σ. With the above extension $\phi_{\pm\pm}$ correspond to the components of two-dimensional vector in the light coordinate (or in the complex coordinate). Now we identify $\bar{\epsilon}_-$ and ϵ_- with sections of $K_\Sigma^{-1/2}$ and $\bar{\epsilon}_+$ and ϵ_+ with sections of $\overline{K_\Sigma}^{-1/2}$, where K_Σ denotes the canonical line bundle on Σ. The gauge multiplet (4.2.1) becomes the $N_{ws}=(2,2)$ vector multiplet, while the two types of bi-holomorphic multiplets (4.2.2) and (4.2.3) correspond to two types of the $N_{ws}=(2,2)$ chiral matter multiplets.

The resulting theory can be viewed as an infinite dimensional $N_{ws}=(2,2)$ supersymmetric gauged linear-sigma model [32][33]. Then one may twist the theory to obtain A model [49][3]. After the A model twisting we have s_+ and $\bar{s}_-$ (or s_- and $\bar{s}_+$) transform as scalars on Σ. Then they are identified with the differentials of $\mathcal{G}$-equivariant cohomology of the target space - the space $\mathcal{A}$ of all connections A and $B^{2,0}$;

$$\mathcal{A}\oplus\Omega^{2,0}(M,End(E)\oplus\Omega^{0,2}(M,End(E)). \tag{4.2.54}$$

Denoting $s_+:=s$ and $\bar{s}_-=\bar{s}$ they satisfy the familiar anti-commutation relations

$$s^2=0,\qquad \{s,\bar{s}\}=-i\sigma^a\mathcal{L}_a,\qquad \bar{s}^2=0. \tag{4.2.55}$$

In certain cases[7] one can show that all the degrees of freedom due to the $N_{ws}=(2,2)$ vector multiplet as well as to σ, $\bar{\sigma}$ and its fermionic partner decouple in the infrared limit in Σ. Then the twisted model flows to the non-linear topological sigma model with target space given by the space of solutions (of the Vafa-Witten equations)

$$\begin{aligned} F^{0,2}&=0,\\ \partial_A B^{0,2}&=0,\\ iF\wedge\omega+[B^{2,0},B^{0,2}]&=0, \end{aligned} \tag{4.2.56}$$

modulo the gauge symmetry. The observables of the model are given by $\mathcal{G}$-equivariant closed differential forms which flows in the infrared limit to the usual observables (the closed differential forms on the above moduli space). Thus the correlation functions of the resulting model are the quantum cohomology rings of the moduli space of Vafa and Witten.

[7] A example is the case with the $SU(2)$ group and a Kähler surface M with $b_2^+\geq 3$. The required properties are analyzed in details in [7].

We may use the S^1 symmetry acting on $B^{0,2}$ to by modifying the anti-commutation relations as usual

$$s^2 = 0, \qquad \{s, \overline{s}\} = -i\sigma^a \mathcal{L}_a - im\mathcal{L}_{S^1}, \qquad \overline{s}^2 = 0. \tag{4.2.57}$$

Then the path integral can be written as the sum over contributions from the moduli space of $SU(2)$ instantons (branch (i)) and the moduli space of the following abelian (branch (ii)) Seiberg-Vafa-Witten equations first considered in [7]

$$\begin{aligned} F^{0,2}_{L^2} &= 0, \\ \partial_{L^2}\beta &= 0, \\ iF_{L^2} \wedge \omega &= \beta \wedge \overline{\beta}, \end{aligned} \tag{4.2.58}$$

which is a special case $deg(L) > 0$ of the equations (4.2.16). In this way, we may obtain the quantum cohomology rings of the moduli space of $SU(2)$ instantons on a Kähler surface M with $b_2^+ \geq 3$. The question is if such a stringy generalization of Donaldson-Witten theory would lead to more subtle four manifold invariants than the Seiberg-Witten's. The author do not know the answer but suspect that it is no at least for the Kähler case. The above argument based localization by the S^1-action strongly suggest that the quantum cohomology rings can be, in principle, determined in terms of of the quantum cohomology rings of the moduli space of Seiberg-Witten monopoles. We expect to have only the constant map since the later moduli space is a collection of non-degenerated points [12]. This implies that quantum Donaldson-Witten invariant may not contain any information beyond the Seiberg-Witten's at least for the Kähler case. There seems to be still some hope that one may get non-trivial result by considering the general almost complex surface. Then the moduli space of Seiberg-Witten monopoles can be non-zero dimensional almost complex manifold.

We may use the S^1-equivariant integration formula with the modified commutation relations as usual

$$s^2 = 0, \quad \{s, \bar{s}\} = -i\delta_{\phi}, \quad \bar{s}^2 = 0.$$

Then the path integral can be written as the sum over contributions from the moduli space of $SU(2)$ instantons (through E) and the moduli space of the following abelian (through $U(1)$) Seiberg-Witten equations first considered in [7],

$$F^{0,2}_{A} = 0,$$
$$\bar{\partial}_A \alpha = 0,$$
$$iF_A \wedge \omega = \frac{1}{2}|\alpha|^2,$$

which is a special case $\beta = 0$ of the equations (4.2.10). In this way we may obtain the quantum cohomology rings of the moduli space of $SU(2)$ instantons on a Kähler surface M with $b_2^+ \geq 3$. The question is if such a stringy generalization of Donaldson-Witten theory could lead to more subtle four manifold invariants than the Seiberg-Witten's. The author do not know the answer but suspect that it is no at least for the Kähler case. The above argument based localization by the S^1-action strongly suggest that the quantum cohomology rings can be, in principle, determined in terms of the quantum cohomology rings of the moduli space of Seiberg-Witten monopoles. We expect to have only the constant map since the later moduli space is a collection of non-degenerated points [12]. This implies that quantum Donaldson-Witten invariant may not contain any information beyond the Seiberg-Witten's at least for the Kähler case. There seems to be still some hope that one may get non-trivial result by considering the general almost complex surface. Then the moduli space of Seiberg-Witten monopoles can be non-zero dimensional almost complex manifold.

Chapter 5

Cohomological Yang-Mills Theories On 3-Folds

5.1 Introduction

The discovery of D-branes has greatly enriched our understanding of non-perturbative strings [16]. The configurations of D-branes may be viewed as a stringy description of vector bundles (more generally sheaves) and the dynamics is effectively described by supersymmetric gauge theory [41]. The celebrated M(atrix) conjecture, then, provides a microscopic definition of M theory by the later theory [42]. The matrix string theory is the M(atrix) theory compactification on the circle [84] in the eleventh direction and is described by the maximal $N_{ws} = (8,8)$ supersymmetric gauge theory in $(1+1)$ dimensions [29]. The matrix string theory compactified on a non-trivial manifold should involve degrees of freedom of branes wrapped around non-trivial homology cycles. Such a theory may be viewed with some assumptions as a $(1+1)$-dimensional gauged linear sigma model with the space of all bundles on compactified space as target space [30]. Due to the brane configuration not all the supersymmetry will be preserved. For example the $K3$ and CY_3 compactification have $N_{ws} = (4,4)$ and $N_{ws} = (2,2)$ supersymmetry.

Formally the infrared limit from the string world-sheet viewpoint corresponds to the limit where the bulk string coupling constant becomes zero. Then the theory flows to a superconformal non-linear sigma model whose target space is the moduli space of semi-stable bundles together with a linear space spanned by the zero-modes of adjoint scalars. Those zero-modes represent bulk degrees of freedom transverse to the compactified space (and branes in it). When the brane configuration is a BPS state the semi-stable bundles are actually stable and there are no zero-modes of adjoint scalars. Thus the stable bundles represent the degrees of freedom completely decoupled from the bulk. Those phenomena are directly related with the equivariant nature of world-sheet supercharges of gauged linear sigma-models. When the target space given by a symplectic quotient is smooth the equivariant cohomology is the ordinary cohomology of the quotient space. Otherwise there is always something more. Recall that the total extended space where the equivariant cohomology is defined is always smooth. Similarly the total physical system is always smooth if we include all the degrees of freedom.

Consequently the infrared superconformal non-linear sigma model whose target space is the moduli space of stable bundle describes decoupled matrix string theory from the bulk. As a superconformal $(1+1)$-dimensional non-linear sigma model the chiral rings can be described by topological sigma-models [3]. In general $N_{ws} = (2,2)$ superconformal theory has two types of chiral rings, the (c,c) and (a,c) ring [4]. The (c,c) ring is described by the A model corresponding to the quantum cohomology ring of the moduli space of stable bundles. Thus the Donaldson-Witten type polynomials associated with stable bundles are the A model correlation functions without worldsheet instanton corrections. The Donaldson-Witten type invariants can also be viewed as the correlation functions of chiral primaries of superconformal field theory obtained by M(atrix) theory compactification. The other (a,c) chiral ring is described by the B model. For the case of a Calabi-Yau 3-fold we argue that the B model is equivalent to the holomorphic Chern-Simons theory [85].

The above discussions based on [30] motivates us to look up cohomological field theory which computes the classical cohomology rings of the moduli space of stable bundles on a Calabi-Yau 3-fold. In the previous chapter we already considered, among others, a natural generalization of the Donaldson-Witten theory on a complex Kähler surface to a $d > 2$ dimensional Kähler manifold M. The path integral of the resulting model is localized to the moduli space of the Einstein-Hermitian connections, equivalently the moduli space of stable bundles. However we had a serious problem due to the uncontrollable abundance of anti-ghost zero-modes. In this chapter we find resolution of this problem by starting off we have failed. A simple observation is that one has to introduce additional degrees of freedom to control the anti-ghost zero-modes. This inductive procedure leads us to a natural extension of the moduli space of Einstein-Hermitian connections or, equivalently, stable bundles. It turns out that we have a well-defined model only for the $d = 3$ case.

Obviously the $d = 3$ case is the one most relevant to string theory [86]. We also note that the notion of stable bundles, on a Calabi-Yau 3-fold, appears naturally in the non-perturbative string theory in terms of BPS states [87][88][89]. It is important to note, although usually not being emphasized, that what one actually has in string theory is the extension of stable bundles. In terms of D-branes, a rank r stable bundle on a CY 3-fold M is a BPS configuration of r D_6-branes wrapped around M, while the topological type of the bundle is determined by D_4, D_2, and D_0 wrapped around non-trivial cycles in M. Such a D_6 brane also has two complex transverse degree of freedom in the bulk. All together one has the extended stable bundles.

The moduli space of stable bundles on a CY 3-fold is also crucial for homological mirror symmetry [79][90], which is essentially mirror symmetry with D-branes [91][92]. The mirror partner of a stable bundle on M is represented by a special Lagrangian submanifold $C \subset \widetilde{M}$ with flat line bundle L in the mirror $\widetilde{M}$ Calabi-Yau. The extended mirror conjecture is that the moduli space of stable bundles on M should be isomorphic to the moduli space of (C, L) on $\widetilde{M}$. However this can't be literally true since, for example, the deformation of special Lagrangian submanifold is not obstructed [93] but the deformation of stable bundles is. The situation is analogous, in the old mirror symmetry, to the obvious differences between the moduli space of complex structures on M and the moduli space of Kähler structure on $\widetilde{M}$. The natural resolution was extending both the moduli spaces [3].[1] A natural resolution may be that both moduli spaces of stable bundles and Lagrangian submanifolds should be extended [94][95].

We begin with constructing a well-defined $N_c = (2,0)$ model on a Kähler 3-fold. This model gives a concrete formula for Donaldson-Witten type polynomials which is valid regardless of the properties the extended moduli space has. We also argue that, using a S^1 symmetry and the DH integration formula, that Donaldson-Witten type invariants may be equivalent to Seiberg-Witten type invariants on Kähler 3-folds. The dimensional reduction of the model gives rise to the $N_c = (2,2)$ Vafa-Witten model on a Kähler 2-fold. Then we specialize to the Calabi-Yau case. On a Calabi-Yau 3-fold the $N_c = (2,0)$ supersymmetry is automatically enhanced to $N_c = (2,2)$ supersymmetry. The partition function of this model gives a concrete and well-defined formula for holomorphic Casson invariants defined by Thomas [96][97]. On Calabi-Yau 3-folds the actual dimension of the moduli space of stable bundle can never be equal to the formal dimension, which is zero. This property causes one of the main difficulties in defining holomorphic Casson invariants. We give a concrete prescription of dealing with the above problem, different from that of Thomas. We also give a concrete prescription of resolving the problem caused by reducible connections by combining certain deformations and perturbations of the initial model.

The $N_c = (2,2)$ model can be obtained by dimensional reduction of the $N_{ws} = (2,2)$ gauged linear sigma model in $(1+1)$ dimensions introduced in [30]. A quantum field theoretic approach to Donaldson-Witten type invariants on a general Kähler manifold based on the moduli space of stable bundles is first studied in [52][67]. We note other related papers; [98][99] for Kähler case and [100][101] for Calabi-Yau case.

5.2 Motivating the Extended Moduli Space of Stable Bundles

In this section we motivate the notion of extended moduli space of stable bundles [95] in the context of resolving the problems of anti-ghost zero-modes discussed at the end of Sect. 4.1.1.

First we set up our notation. Consider a d complex dimensional compact Kähler manifolds (M, ω) with Kähler form ω. We consider a rank r Hermitian vector bundle $E \to M$. On M any two-form $\alpha \in \Omega^2(M)$ can be decomposed as

[1] We should also mention that the homological mirror conjecture was derived from a deeper study of the extend moduli space of complex structures.

follows

$$\begin{aligned} \alpha &= \alpha^+ + \alpha^-, \\ \alpha^+ &= \alpha^{2,0} + \alpha_0\omega + \alpha^{0,2}, \\ \alpha^- &= \alpha^{1,1}_{\perp}, \end{aligned} \tag{5.2.1}$$

where $\alpha_o = \frac{1}{d}\Lambda\alpha$ is a scalar and $\alpha^{1,1}_{\perp}$ is (1,1)-form orthogonal to ω. One defines the following projections

$$P^{\pm} : \Omega^2_M \to \Omega^{2\pm}(M), \quad P^{0,2} : \Omega^2_M \to \Omega^{0,2}(M). \tag{5.2.2}$$

The curvature two-form decomposes as $F = F^+ + F^-$ according to (5.2.1). A connection on E is called Einstein-Hermitian (EH) with factor ζ if

$$\begin{aligned} F^{0,2} &= 0, \\ i\Lambda F &- \zeta I_E. \end{aligned} \tag{5.2.3}$$

Now we return to the problem of the anti-ghost zero-modes Let A be an EH connection. We consider a nearby connection $A + \delta A$, $\delta A \in \Omega^1(M, End(E))$, which also is EH. After linearization we have $P^+ d_A \delta A := d_A^+ \delta A = 0$. Supplying the Coulomb gauge condition $d_A^* \delta A = 0$, a local deformation δA around a point A in $\mathcal{M}_{EH}$ represented by the kernel of an operator $d_A^+ \oplus d_A^*$ in $\Omega^1(M, End(E))$. From the above one introduces the associated elliptic complex of Atiyah-Hitchin-Singer [102];

$$0 \to \Omega^0(M, End(E)) \xrightarrow{d_A} \Omega^1(M, End(E)) \xrightarrow{d_A^+} \Omega^{2+}(M, End(E)). \tag{5.2.4}$$

We compare the above with the fermionic zero-modes of $(\overline{\eta}_-, \psi_+^{0,1}, \overline{\chi}_-^{0,2})$ governed by the equations (4.1.36);

$$\overline{\partial}_A \overline{\eta}_- = 0, \qquad \begin{aligned} \overline{\partial}_A^* \psi_+^{0,1} &= 0, \\ \overline{\partial}_A \psi_+^{0,1} &= 0, \end{aligned} \qquad \overline{\partial}_A^* \overline{\chi}_-^{0,2} = 0. \tag{5.2.5}$$

After decomposing $\overline{\eta}_- = \boldsymbol{\eta} + i\chi^0_-$ into real and imaginary parts, we can form real fermions $(\boldsymbol{\eta}_-, \boldsymbol{\psi}_+, \boldsymbol{\chi}_-)$

$$\boldsymbol{\psi}_+ = \overline{\psi}_+^{1,0} + \psi_+^{0,1}, \qquad \boldsymbol{\chi}_- = \overline{\chi}_-^{2,0} + \chi^0_-\omega + \chi_-^{0,2}, \tag{5.2.6}$$

where $\boldsymbol{\eta}_- \in \Omega^0(M, End(E))$, $\boldsymbol{\psi}_+ \in \Omega^1(M, End(E))$ and $\boldsymbol{\chi}_- \in \Omega^{2+}(M, End(E))$. The equations for zero-modes (5.2.5) are translated into the following

$$d_A \boldsymbol{\eta}_- = 0, \qquad \begin{aligned} d_A^* \boldsymbol{\psi}_+ &= 0, \\ d_A^+ \boldsymbol{\psi}_+ &= 0, \end{aligned} \qquad d_A^{+*} \boldsymbol{\chi}_- = 0. \tag{5.2.7}$$

Thus the zero-modes of fermions $(\boldsymbol{\eta}_-, \boldsymbol{\psi}_+, \boldsymbol{\chi}_-)$ are elements of the AHS complex (5.2.4). The above correspondence is one of the crucial ingredients of Witten's approach to Donaldson theory in four real dimensions [1]. The path integral measure contains such fermionic zero-modes and the net ghost number anomaly precisely the index of the above complex, which is the formal dimension of the moduli space of instantons on a four manifold.

Now we undo the combination (5.2.6) and return to the initial equations (5.2.5) for the complex fermions $(\overline{\eta}_-, \psi_+^{0,1}, \overline{\chi}_-^{0,2})$. The equations (5.2.5) imply that the fermionic zero-modes are one to one correspondence with the following Dolbeault complex [103]

$$0 \to \Omega^{0,0}(M, End(E)) \xrightarrow{\overline{\partial}_A} \Omega^{0,1}(M, End(E)) \xrightarrow{\overline{\partial}_A} \Omega^{0,2}(M, End(E)). \quad (5.2.8)$$

Note that $\overline{\partial}_A^2 = 0$ at the fixed point locus. Our problem for $d \geq 3$ is that a fermionic zero-mode of $\overline{\chi}_-^{0,2}$ only needs to satisfy the condition $\overline{\partial}_A^* \overline{\chi}_-^{0,2} = 0$ so that we have too many of them. As a result we always have an infinite dimensional anti-ghost bundle. Therefore the path integral would hardly make any sense. But this is exactly what the EH condition gives us via local deformation. For $d = 2$ the desired condition $\overline{\partial}_A \overline{\chi}_-^{0,2} = 0$ is automatic due to the dimensional reason. For $d \geq 3$ the only way of imposing the desired condition $\overline{\partial}_A \overline{\chi}_-^{0,2} = 0$ is to introduce another fermionic field $\lambda_+^{3,0}$ with ghost numbers $(1, 0)$ such that the action functional contains the following term

$$S \sim \int_M \mathrm{Tr}(\lambda_+^{3,0} \wedge *\overline{\partial}_A \overline{\chi}_-^{0,2}) + \ldots \quad (5.2.9)$$

Then we obtain in addition to (5.2.5)

$$\overline{\partial}_A \overline{\chi}_-^{0,2} = 0, \qquad \partial_A^* \lambda_+^{3,0} = 0. \quad (5.2.10)$$

Thus we have to generalize the $N_c = (2, 0)$ model in Sect. 4.1.1 by introducing a new holomorphic multiplet $(C^{3,0}, \lambda_+^{3,0}) \in \Omega^{3,0}(M, End(E))$. For $d = 3$ the above additional conditions are sufficient. For $d = 4$ we should supply yet another additional condition $\partial_A \lambda_+^{3,0} = 0$, otherwise we have too many zero-modes of $\lambda_+^{3,0}$. Thus we should introduce another fermionic fields $\overline{\xi}_-^{0,4}$ with ghost numbers $(-1, 0)$ such that now the action contains

$$S \sim \int_M \mathrm{Tr}(\lambda_+^{3,0} \wedge *\overline{\partial}_A \overline{\chi}_-^{0,2} + \partial_A \lambda_+^{3,0} \wedge *\overline{\xi}_-^{0,4}) + \ldots, \quad (5.2.11)$$

and so on.

Thus a natural resolution of our problem is to extend the complex (5.2.8) all the way to the end

$$0 \longrightarrow C^{0,0} \xrightarrow{\overline{\partial}_A} C^{0,1} \xrightarrow{\overline{\partial}_A} C^{0,2} \xrightarrow{\overline{\partial}_A} C^{0,3} \xrightarrow{\overline{\partial}_A} \ldots \xrightarrow{\overline{\partial}_A} C^{0,d} \to 0, \quad (5.2.12)$$

where $C^{0,\ell} := \Omega^{0,\ell}(M, End(E))$. To give any meaning to the above Dolbeault complex, we have to introduce the following set of fermionic fields

$$\overline{\eta}_-^{0,0}, \psi_+^{0,1}, \overline{\chi}_-^{0,2}, \overline{\lambda}_+^{0,odd}, \overline{\xi}_-^{0,even}, \quad (5.2.13)$$

where $2 < odd, even \leq d$. It is obvious, from the basic structure of our $N_c = (2,0)$ model, that $\overline{\lambda}_+^{0,odd}$ are superpartners of anti-holomorphic bosonic fields $C^{0,odd}$ to form holomorphic multiplets;

$$(C^{0,odd} \xrightarrow{\overline{s}_+} \overline{\lambda}_+^{0,odd}). \tag{5.2.14}$$

It is also obvious that $\overline{\xi}_-^{0,even}$ should be parts of Fermi multiplets

$$(\overline{\xi}_-^{0,even} \xrightarrow{s_+} H^{0,even}), \tag{5.2.15}$$

where $H^{0,even}$ are auxiliary fields. Then we may try to design an action functional which gives the following equations, in addition to (5.2.5), for fermionic zero-modes

$$\begin{aligned} \overline{\partial}_A \overline{\lambda}_+^{0,odd} &= 0, \quad & \overline{\partial}_A \overline{\xi}_-^{0,even} &= 0, \\ \overline{\partial}_A^* \overline{\lambda}_+^{0,odd} &= 0, \quad & \overline{\partial}_A^* \overline{\xi}_-^{0,even} &= 0. \end{aligned} \tag{5.2.16}$$

Thus the $(0,q)$-form fermionic zero-modes become the elements of the q-th cohomology group $\boldsymbol{H}^{0,q} := H^{0,q}_{\overline{\partial}_A}(M, End(E))$ of the complex (5.2.12). Then the net ghost number violation due to the fermionic zero-modes is precisely the index $\sum_{q=0}^{d}(-1)^{q+1} dim_{\mathbb{C}} \boldsymbol{H}^{0,q}$ of the complex (5.2.12). Now we are in the same situation as the Donaldson-Witten theory in the $d = 2$ case.

Finally let's consider how the above extension fits into the framework EH connections. Kim introduced the following complex, generalizing the Atiyah-Hitchin-Singer-Itoh complex [104][48]

$$0 \longrightarrow \boldsymbol{B}^0 \xrightarrow{d_A} \boldsymbol{B}^1 \xrightarrow{d_A^+} \boldsymbol{B}^{2+} \xrightarrow{d_A^{0,2}} \boldsymbol{B}^{0,3} \xrightarrow{\overline{\partial}_A} \ldots \xrightarrow{\overline{\partial}_A} \boldsymbol{B}^{0,d} \longrightarrow 0, \tag{5.2.17}$$

where $d^{0,2} = \overline{\partial}_A \circ P^{0,2}$, $\boldsymbol{B}^p = \Omega^p(M, End(E))$ and $\boldsymbol{B}^{p,q} = \Omega^{p,q}(M, End(E))$. It is shown that the above is a complex if the connection A is EH and elliptic. We denote the associated q-th cohomology group by $\boldsymbol{H}^q$. It is not difficult to show that

$$\sum_{q=0}^{d}(-1)^{q+1} dim_{\mathbb{R}} \boldsymbol{H}^q = 2\sum_{q=0}^{d}(-1)^{q+1} dim_{\mathbb{C}} \boldsymbol{H}^{0,q}. \tag{5.2.18}$$

It is also obvious that the two extended complexes (5.2.17) and (5.2.12) are related in the same way as the unextended complexes (5.2.4) and (5.2.8).

We remark that Kim's complex is not the genuine deformation complex of EH connections, but rather a natural extension of it. As in the $d = 2$ case we require that the index is the complex formal dimension of a certain extended moduli space of stable bundles. We define the extended moduli space $\mathfrak{M}$ of EH connections or of stable bundles by extending the EH condition as follows

$$\begin{aligned} \overline{\mathfrak{D}} \circ \overline{\mathfrak{D}} &= 0, \\ \exp(\omega) \cdot (\mathfrak{D} \circ \overline{\mathfrak{D}} + \overline{\mathfrak{D}} \circ \mathfrak{D})\,|_{\text{top form}} + id\zeta\omega^d I_E &= 0, \end{aligned} \tag{5.2.19}$$

where $\overline{\mathfrak{D}}$ is the extended holomorphic connections

$$\mathfrak{D} = \overline{\partial}_A + \sum_{2<odd\leq d} C^{0,odd} \tag{5.2.20}$$

The versal deformation complex of the above equation is then precisely Kim's complex (5.2.17). This can be check by using the Kähler identities (4.1.19). In the above scheme the infinitesimal deformations of the extended moduli space always lie in $\boldsymbol{H}^{0,odd}$, while the obstructions, by Kuranishi's method, lie in $\boldsymbol{H}^{0,even}$. Thus the local model of the extended moduli space is $f^{-1}(0)$ [94], where[2]

$$f : \boldsymbol{H}^{0,odd} \otimes \boldsymbol{H}^{0,odd} \to \boldsymbol{H}^{0,even}. \tag{5.2.21}$$

The complex formal dimension of the extended moduli space $\mathfrak{M}$ can be computed using the Riemann-Roch formula

$$\sum_{q=0}^{d}(-1)^{q+1}dim_{\mathbb{C}}\boldsymbol{H}^{0,q}. = -\int_M td(M)\cdot ch(E)\cdot ch(E^*), \tag{5.2.22}$$

where $td(M)$ denotes the Todd class of M and $ch(E)$ denotes the Chern character of E.

Now we have all the ingredients to construct a well-defined $N_c = (2,0)$ model. Unfortunately it turns out to be impossible to implant the above ideas except for the case of three complex dimensions. It is not possible to maintain $N_c = (2,0)$ supersymmetry and impose the desired equations (5.2.16) for all fermions unless $d = 3$.

5.3 On Kähler 3-Folds

We consider the $N_c = (2,0)$ model studied in Sect. 4.1 specializing to the case when M is a Kähler 3-fold. According to the discussion in the previous section we introduce one more bosonic field $C^{0,3} \in \Omega^{0,3}(M, End(E))$ and its Hermitian conjugate $C^{3,0}$. Our goal is to construct a $\mathcal{G}$-equivariant $N_c = (2,0)$ model whose target space is the space $\mathcal{A}$ of all connections together with the space of all $C^{0,3}$ fields. Furthermore the fermionic zero-modes should be elements of the Dolbeault cohomology of the complex (5.2.12). It turns out there is only one way of achieving this goal.

5.3.1 Basic Properties

The $N_c = (2,0)$ model here will be an example of the construction in Sect. 3.3 with $\mathfrak{J} \neq 0$ (3.3.7). We first recall that the path integral of a general $N_c = (2,0)$ model is localized to the space of the following equations, modulo $\mathcal{G}$ symmetry,

$$\begin{aligned}\mathfrak{J}^\alpha(X^i) &= 0,\\ \mathfrak{S}_a(X^i) &= 0,\\ \mu(X^i, X^{\bar{i}}) - \zeta &= 0.\end{aligned} \tag{5.3.1}$$

[2]One may view the condition for a good deformation as a Maurer-Cartan equation of a differential graded algebra without derivation. The authors [95] studied the conditions for a good extended Lagrangian deformation given by a master equation. The homological mirror symmetry seems to imply that there should be A^∞ or L^∞ morphisms between them. We examined if the extended moduli space always gives a well-defined cohomological field theory. So far we are unsuccessful except for the $d = 3$ case.

The momentum map μ is determined from the Kähler potential on the space of all X^i and from the $\mathcal{G}$ action on it. The sections $\mathfrak{J}^\alpha$ and $\mathfrak{S}_\alpha$ above should satisfy the following equations to have $N_c = (2,0)$ supersymmetry,

$$\begin{aligned} \overline{s}_+ \mathfrak{J}^a &= 0, \\ \overline{s}_+ \mathfrak{S}_\alpha &= 0, \\ \langle \mathfrak{J}^\alpha, \mathfrak{S}_\alpha \rangle &= 0. \end{aligned} \tag{5.3.2}$$

In the present case our infinite dimensional target space is

$$\mathcal{A} \oplus (\Omega^{3,0}(M, End(E)) \oplus \Omega^{0,3}(M, End(E))), \tag{5.3.3}$$

and the infinite dimensional group $\mathcal{G}$ acts on the above space as the group of all local gauge transformation on M, i.e., $g \in \mathcal{G}$ for $g : M \to G$. The Lie algebra $Lie(\mathcal{G})$ of $\mathcal{G}$ is $\Omega^0(M, End(E))$ and the bi-invariant inner product on $Lie(\mathcal{G})$ is $\langle a, a \rangle = -\int_M \mathrm{Tr}(a \wedge *a)$. We already gave a complex structure on $\mathcal{A}$ in Sect. 4.1.1 by demanding that $A^{0,1}$ is a holomorphic field, i.e., $\overline{s}_+ A^{0,1} = 0$. We also have a unique holomorphic section $F^{0,2}$ from the subspace $\mathcal{A}$ and the corresponding Fermi multiplet $(\overline{\chi}_-^{0,2}, H^{0,2}) \in \Omega^{0,2}(M, End(E))$ with the following transformation laws

$$\begin{aligned} s_+ \overline{\chi}_-^{0,2} &= -\overline{\mathfrak{J}}, \\ \overline{s}_+ \overline{\chi}_-^{0,2} &= -H^{0,2}. \end{aligned} \tag{5.3.4}$$

Then we only have two possibilities to fit the additional bosonic fields; either $\mathfrak{S} = F^{0,2} - \overline{\partial}_A^* C^{0,3}$ or $\overline{\mathfrak{J}} = \overline{\partial}_A^* C^{0,3}$. The first choice is not possible since $\overline{\partial}_A^* = -* \partial_A *$, thus the additional term is not holomorphic, i.e., $\overline{s}_+ \mathfrak{S} \neq 0$ since $\overline{s}_+ A^{1,0} \neq 0$. For the second choice we see $s_+ \overline{\mathfrak{J}} = 0$, thus $\overline{s}_+ \mathfrak{J} = 0$, if we demand $s_+ C^{0,3} = 0$. Thus the additional holomorphic multiplet is $(C^{3,0}, \lambda_+^{3,0})$. We conclude

$$\begin{aligned} \mathfrak{J} &= \partial_A^* C^{3,0}, \\ \mathfrak{S} &= F^{0,2}. \end{aligned} \tag{5.3.5}$$

Finally we check the last condition in (5.3.2) as follows

$$\int_M \mathrm{Tr}(\partial_A^* C^{3,0} \wedge *F^{0,2}) = \int_M \mathrm{Tr}(C^{3,0} \wedge *\overline{\partial}_A F^{0,2}) = 0, \tag{5.3.6}$$

by the Bianchi identity $d_A F = 0 \to \overline{\partial}_A F^{0,2} = 0$.

The above considerations determine, following Sect. 3.3, an equivariant $N_c = (2,0)$ model.

Fields and Their Transformation Laws

Here we recall again the fields and their supersymmetry transformation laws, just to refresh our memory. Associated with the $\mathcal{G}$ symmetry we have the

$N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \overline{\eta}_-, D)$, all transforming as adjoint valued scalars on M. The transformation laws are given by (3.3.8),

$$
\begin{aligned}
& & s_+\eta_- &= 0, & & \\
s_+\phi_{--} &= i\eta_-, & \overline{s}_+\eta_- &= +iD + \frac{1}{2}[\phi_{++}, \phi_{--}], & s_+\phi_{++} &= 0, \\
\overline{s}_+\phi_{--} &= i\overline{\eta}_-, & s_+\overline{\eta}_- &= -iD + \frac{1}{2}[\phi_{++}, \phi_{--}], & \overline{s}_+\phi_{++} &= 0. \\
& & \overline{s}_+\overline{\eta}_- &= 0, & &
\end{aligned}
\tag{5.3.7}
$$

We have two sets of holomorphic multiplets and their anti-holomorphic partners. One set of holomorphic multiplets is $(A^{0,1}, \psi_+^{0,1})$ with anti-holomorphic partners $(A^{1,0}, \overline{\psi}_+^{1,0})$

$$
\begin{aligned}
s_+A^{0,1} &= i\psi_+^{0,1}, & s_+\psi_+^{0,1} &= 0, \\
\overline{s}_+A^{0,1} &= 0, & \overline{s}_+\psi_+^{0,1} &= -\overline{\partial}_A\phi_{++}, \\
s_+A^{1,0} &= 0, & s_+\overline{\psi}_+^{1,0} &= -\partial_A\phi_{++}, \\
\overline{s}_+A^{1,0} &= i\overline{\psi}_+^{1,0}, & \overline{s}_+\overline{\psi}_+^{1,0} &= 0.
\end{aligned}
\tag{5.3.8}
$$

The other holomorphic multiplet is $(C^{3,0}, \lambda_+^{3,0})$ with anti-holomorphic partner $(C^{0,3}, \overline{\lambda}_+^{0,3})$,

$$
\begin{aligned}
s_+C^{3,0} &= i\lambda_+^{3,0}, & s_+\lambda_+^{3,0} &= 0, \\
\overline{s}_+C^{3,0} &= 0, & \overline{s}_+\lambda_+^{3,0} &= -i[\phi_{++}, C^{3,0}], \\
s_+C^{0,3} &= 0, & s_+\overline{\lambda}_+^{0,3} &= -i[\phi_{++}, C^{0,3}], \\
\overline{s}_+C^{0,3} &= i\overline{\lambda}_+^{0,3}, & \overline{s}_+\overline{\lambda}_+^{0,3} &= 0.
\end{aligned}
\tag{5.3.9}
$$

Finally we have Fermi multiplets $(\chi_-^{2,0}, H^{2,0})$ and anti-Fermi multiplets $(\overline{\chi}_-^{0,2}, H^{0,2})$,

$$
\begin{aligned}
s_+\chi_-^{2,0} &= -H^{2,0}, & s_+H^{2,0} &= 0, \\
\overline{s}_+\chi_-^{2,0} &= -\partial_A^* C^{3,0}, & \overline{s}_+H^{2,0} &= -i[\phi_{++}, \chi^{2,0}] + i[*\psi_+^{0,1}*, C^{3,0}] + i\partial_A^*\lambda_+^{3,0}, \\
s_+\overline{\chi}_-^{0,2} &= -\overline{\partial}_A^* C^{0,3}, & s_+H^{0,2} &= -i[\phi_{++}, \overline{\chi}^{0,2}] + i[*\overline{\psi}_+^{1,0}*, C^{0,3}] + i\overline{\partial}_A^*\overline{\lambda}_+^{0,3}, \\
\overline{s}_+\overline{\chi}_-^{0,2} &= -H^{0,2}, & \overline{s}_+H^{0,2} &= 0.
\end{aligned}
\tag{5.3.10}
$$

The above transformation laws imply that the resulting $N_c = (2,0)$ model, in general, can not be embedded into a $N_c = (2,2)$ theory since $s_+\overline{\chi}_-^{0,2} \neq 0$. Such an embedding is only possible if M is a Calabi-Yau 3-fold, where our $N_c = (2,0)$ supersymmetry will automatically enhance to $N_c = (2,2)$ even without adding additional field. For a later purpose we summarize the field contents by the following diagrams;

$$
\begin{array}{ccccccc}
C^{3,0} & \xrightarrow{s_+} & \lambda_+^{3,0} & \phi_{++} & & & \\
 & & & & \overline{\chi}_-^{0,2} & A^{0,1} \xrightarrow{s_+} \psi_+^{0,1} & \\
\overline{\eta}_- & \xrightarrow{s_+} & D & \overline{\lambda}_+^{0,3} & , & \overline{s}_+\searrow & . \\
\uparrow \overline{s}_+ & & \uparrow \overline{s}_+ & \uparrow \overline{s}_+ & & H^{0,2} & \\
\phi_{--} & \xrightarrow{s_+} & \eta_- & C^{0,3} & & &
\end{array}
\tag{5.3.11}
$$

The Action Functional

The final ingredient for the action functional is the $\mathcal{G}$-momentum map on the total space (5.3.3). The total space has a natural $\mathcal{G}$-invariant Kähler potential

$$\mathcal{K}_T = \frac{1}{24\pi^2}\int_M \left(\kappa \operatorname{Tr}(F\wedge F)\wedge\omega^2 - i\operatorname{Tr}\left(C^{3,0}\wedge C^{0,3}\right)\right). \tag{5.3.12}$$

From the transformation laws (5.3.8)(5.3.9) we have the following equivariant Kähler form,

$$
\begin{aligned}
\widehat{\varpi}_T^{\mathcal{G}} :=& i s_+ \overline{s}_+ \mathcal{K}_T \\
=& \frac{1}{12\pi^2}\int_M \operatorname{Tr}(i\phi_{++}\left(F\wedge\omega^2 + \frac{1}{2}[C^{3,0}, C^{0,3}]\right) \\
&+ \frac{1}{12\pi^2}\int_M \operatorname{Tr}(\psi_+^{0,1}\wedge\overline{\psi}_+^{1,0}\wedge\omega^2 - \frac{i}{2}\lambda_+^{3,0}\wedge\overline{\lambda}_+^{0,3}).
\end{aligned}
\tag{5.3.13}
$$

The term in the third line is the Kähler form $\widehat{\varpi}_T$, after the parity change, and the term in the second line is proportional to the $\mathcal{G}$-momentum map m_T on the total space (5.3.3)

$$\mu_T = \frac{1}{12\pi^2}\left(F\wedge\omega^3 + \frac{1}{2}[C^{3,0}, C^{0,3}]\right). \tag{5.3.14}$$

Thus the $N_c = (2,0)$ action functional is given by, see (3.3.9) and (4.1.15)

$$
\begin{aligned}
S =& \frac{s_+\overline{s}_+}{12\pi^2}\int_M \operatorname{Tr}\left(\phi_{--}\left(F\wedge\omega^2 + \frac{1}{2}[C^{3,0}, C^{0,3}] + \frac{i}{3}\zeta\omega^3 I_E\right)\right) \\
&+ \frac{s_+\overline{s}_+}{4\pi^2}\int_M \operatorname{Tr}\left(\chi_-^{2,0}\wedge *\overline{\chi}_-^{0,2}\right) + \frac{s_+\overline{s}_+}{6\pi^2}\int_M \operatorname{Tr}\left(\eta_- * \overline{\eta}_-\right) \\
&+ \frac{i s_+}{4\pi^2}\int_M \operatorname{Tr}\left(\chi_-^{2,0}\wedge *F^{0,2}\right) + \frac{i\overline{s}_+}{4\pi^2}\int_M \operatorname{Tr}\left(\overline{\chi}_-^{0,2}\wedge *F^{2,0}\right).
\end{aligned}
\tag{5.3.15}
$$

Now we examine if the above action functional gives the desired equations for the fermionic zero-modes. After expanding the action functional S we have the following terms relevant for fermionic zero-modes,

$$S = -\frac{1}{6\pi^2}\int_M \mathrm{Tr}\Big(i\overline{\eta}_- * \overline{\partial}_A^* \psi_+^{0,1} + i\eta_- * \partial_A^* \overline{\psi}_+^{1,0} + \frac{3}{2}\chi_-^{2,0} \wedge *\overline{\partial}_A \psi_+^{0,1} + \frac{3}{2}\overline{\chi}_-^{0,2} \wedge *\partial_A \overline{\psi}_+^{1,0} + \frac{3i}{2}\chi_-^{2,0} \wedge *\overline{\partial}_A^* \overline{\lambda}_+^{0,3} + \frac{3i}{2}\overline{\chi}_-^{0,2} \wedge *\partial_A^* \lambda_+^{3,0} \Big) + \dots \tag{5.3.16}$$

From the above we obtain the following fermionic equations of motion,

$$\begin{aligned} \overline{\partial}_A^* \psi_+^{0,1} &= 0, \\ i\overline{\partial}_A \eta_- + \frac{3}{2}\overline{\partial}_A^* \overline{\chi}_-^{0,2} &= 0, \\ \overline{\partial}_A \psi_+^{0,1} + i\overline{\partial}_A^* \overline{\lambda}_+^{0,3} &= 0, \\ \overline{\partial}_A \overline{\chi}_-^{0,2} &= 0. \end{aligned} \tag{5.3.17}$$

We will see below that these give rise to exactly the required equations (5.2.5) and (5.2.10).

5.3.2 Path Integrals

The path integral of our model is localized to the locus of the following equations, modulo $\mathcal{G}$ symmetry, see (3.3.13) and (3.3.14),

$$\begin{aligned} \overline{\partial}_A^* C^{0,3} &= 0, \\ F^{0,2} &= 0, \\ iF\wedge\omega\wedge\omega + \frac{i}{2}[C^{3,0}, C^{0,3}] - \frac{\zeta}{3}\omega^3 I_E &= 0, \end{aligned} \tag{5.3.18}$$

and

$$\begin{aligned} d_A \phi_{++} &= 0, \\ [\phi_{++}, C^{0,3}] &= 0, \\ [\phi_{++}, \phi_{--}] &= 0. \end{aligned} \tag{5.3.19}$$

We call the moduli space defined by the eq. (5.3.18) the *extended* moduli space $\mathfrak{M}$ of EH connections (with factor ζ) or stable bundles.

Since the path integral is localized to integrable connections $\overline{\partial}_A^2 = 0$, the fermionic equations of motion in (5.3.17) become

$$\overline{\partial}_A \eta_- = 0, \quad \begin{aligned} \overline{\partial}_A^* \psi_+^{0,1} &= 0, \\ \overline{\partial}_A \psi_+^{0,1} &= 0, \end{aligned} \quad \begin{aligned} \overline{\partial}_A^* \overline{\chi}_-^{0,2} &= 0, \\ \overline{\partial}_A \overline{\chi}_-^{0,2} &= 0, \end{aligned} \quad \overline{\partial}_A^* \overline{\lambda}_+^{0,3} = 0. \tag{5.3.20}$$

Thus the zero-modes of fermions

$$\eta_-, \psi_+^{0,1}, \overline{\chi}_-^{0,2}, \overline{\lambda}_+^{0,3} \tag{5.3.21}$$

are elements of the cohomology group $\boldsymbol{H}^{0,p}$ of the following Dolbeault complex (5.2.12),

$$0 \longrightarrow \boldsymbol{C}^{0,0} \xrightarrow{\overline{\partial}_A} \boldsymbol{C}^{0,1} \xrightarrow{\overline{\partial}_A} \boldsymbol{C}^{0,2} \xrightarrow{\overline{\partial}_A} \boldsymbol{C}^{0,3} \to 0, \tag{5.3.22}$$

where $\boldsymbol{C}^{0,\ell} := \Omega^{0,\ell}(M, End(E))$. It is also easy to check that the above is isomorphic to the versal deformation complex of the extended moduli space $\mathfrak{M}$ of stable bundles. Thus minus the index of the above Dolbeault cohomology group correspond to the net ghost number violations in the path integral measure due to the zero-modes of fermions in (5.3.21). We have

$$\begin{aligned}\Delta &= -\#(\overline{\eta}_-)_0 + \#(\psi_+^{0,1})_0 - \#(\overline{\chi}_-^{0,2})_0 + \#(\overline{\lambda}_+^{0,3})_0 \\ &= \sum_{q=0}^{3}(-1)^{q+1} dim \boldsymbol{H}^{0,q}.\end{aligned} \tag{5.3.23}$$

The net ghost number violation of the path integral due to all the fermions – the fermions in (5.3.21) and their conjugates – is (Δ, Δ). The above index can be computed by applying the standard Riemann-Roch formula. We have

$$\Delta := \int_M c_1(M) \wedge \left(rc_2(E) - \frac{r-1}{2}c_1(E)^2\right) - r^2(1 - h^{0,1} + h^{0,2} - h^{0,3}), \tag{5.3.24}$$

where $h^{p,q}$ denote the Hodge numbers of M. We also note that a Hermitian vector bundle E admits an EH connection if

$$\int_M \omega \wedge \left(rc_2(E) - \frac{r-1}{2}c_1(E)^2\right) \geq 0 \tag{5.3.25}$$

and the equality holds if and only if E is projectively flat.

Now we take a closer look at the path integral. We note that the zero-modes of $\psi_+^{0,1}$ and $\overline{\lambda}_+^{0,3}$, thus $\boldsymbol{H}^{0,1}$ and $\boldsymbol{H}^{0,3}$, correspond to local deformations of the extended moduli space $\mathfrak{M}$. The other fermionic zero-modes $\overline{\eta}_- \in \boldsymbol{H}^{0,0}$ and $\overline{\chi}_-^{0,2}$ will cause some trouble. Note that we have a decomposition into trace and trace-free parts

$$\begin{aligned}\boldsymbol{H}^{0,0} &= H^{0,0}(M) + \widetilde{\boldsymbol{H}}^{0,0}, \\ \boldsymbol{H}^{0,2} &= H^{0,2}(M) + \widetilde{\boldsymbol{H}}^{0,2}.\end{aligned}$$

We call $\Delta - 1 - h^{0,2}$ the complex formal dimension of $\mathfrak{M}$. If we assume a situation that $\mathcal{G}$ acts freely on the locus of solutions of (5.3.18), i.e., the connection is irreducible, the extend moduli space $\mathfrak{M}$ is an analytic space with the Kähler structure induced from the $\mathcal{G}$-equivariant Kähler form (5.3.13). The moduli space will not have the right complex dimension $\Delta - 1 - h^{0,2}$ unless $\widetilde{\boldsymbol{H}}^{0,2} = 0$ as well. In the ideal situation $\widetilde{\boldsymbol{H}}^{0,0} = \widetilde{\boldsymbol{H}}^{0,2} = 0$, the extended moduli space $\mathfrak{M}$ is smooth and the zero-modes of $\psi_+^{0,1}, \lambda_+^{3,0}$ span the holomorphic tangent space.[3] Thus the formal complex dimension is the actual dimension.

However the assumption made above, in particular $\widetilde{\boldsymbol{H}}^{0,2} = 0$, is too naive. We note that the obstruction to deformation of the extended moduli space $\mathfrak{M}$ lies in $\widetilde{\boldsymbol{H}}^{0,2}$. In two complex dimensions Donaldson proved that one can

[3]We will establish this later. We remark that the case with $\widetilde{\boldsymbol{H}}^{0,3} \neq 0$ has no problem which is associated with deformation of $\mathfrak{M} \subset \mathcal{M}_{EH}$ along the direction of $C^{0,3}$. It would be a problem if we work with $\mathcal{M}_{EH}$.

always achieve $\widetilde{\boldsymbol{H}}^{0,2} = 0$ after suitable perturbation of the metric. In three complex dimensions one can hardly expect such a result to continue to hold. The assumption $\widetilde{\boldsymbol{H}}^{0,0} = 0$ is valid for a bundle E with degree and rank coprime.

Now we examine how the path integral deals with the above problems. We assume, for simplicity, that our gauge group is $SU(r)$, so that $End(E)$ is always trace-free. Then the formal complex dimension Δ is given by

$$\Delta := r\int_M \Big(c_1(M)\wedge c_2(E)\Big) - (r^2-1)(1-h^{0,1}+h^{0,2}-h^{0,3}). \tag{5.3.26}$$

A typical observable of the theory is the total $\mathcal{G}$-equivariant Kähler form, after parity change, $\widehat{\varpi}_T^{\mathcal{G}}$ given by (5.3.13). First we consider an idealistic case that $\boldsymbol{H}^{0,0} = \boldsymbol{H}^{0,2} = 0$. Then the correlation function $< \exp\widehat{\varpi}_T^{\mathcal{G}} >$ can be identified with the symplectic volume of $\mathfrak{M}$,

$$< \exp\widehat{\varpi}_{\mathcal{G}}^{T} >= \int_{\mathfrak{M}} \exp\widetilde{\varpi}_T = vol(\mathfrak{M}). \tag{5.3.27}$$

If we have the anti-ghost $\overline{\chi}_-^{0,2}$ zero-modes, i.e., $\boldsymbol{H}^{0,2} \neq 0$, the above correlation function becomes

$$< \exp\widehat{\varpi}_{\mathcal{G}}^{T} >= \int_{\mathfrak{M}} e(\mathbb{V})\wedge\exp\widetilde{\varpi}_T, \tag{5.3.28}$$

where $e(\mathbb{V})$ denotes the Euler class of the anti-ghost bundle $\mathbb{V}$. One may consider correlation functions of other observables $\widehat{\mathcal{O}}^{r,s}$ with ghost numbers (r,s) given by $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ closed $\mathcal{G}$ equivariant differential forms $\mathcal{O}^{r,s}$ – see Sect. 3.1.2. We have

$$\left\langle \prod_{i=1}^{\ell}\widehat{\mathcal{O}}^{r_i,s_i}\right\rangle = \int_{\mathfrak{M}} e(\mathbb{V})\wedge\widetilde{\mathcal{O}}^{r_1,s_1}\wedge\ldots\wedge\widetilde{\mathcal{O}}^{r_\ell,s_\ell} \tag{5.3.29}$$

where $\widetilde{\mathcal{O}}^{r,s}$ denotes the equivariant differential form $\mathcal{O}^{r,s}$ after the restriction and reduction to $\mathfrak{M}$. The above correlation function can be non-vanishing if

$$\sum_{i=1}^{\ell}(r_i,s_i) = (\Delta,\Delta), \tag{5.3.30}$$

due to the ghost number anomaly. Almost all properties are essentially the same as for the detailed discussions for the $N_c = (2,0)$ models in Sect. 2.3.2 and Sect. 3.3.2. Repeating the same analysis here will be unnecessary. What is remarkable is that the path integral is well-defined even if the moduli space $\mathfrak{M}$ does not satisfy nice conditions like $\boldsymbol{H}^{0,2} = 0$.

For that purpose let's look up some details about how the Euler class of the anti-ghost bundle emerges. The action function S (5.3.15) contains the following Yukawa coupling involving the anti-ghost,

$$S = \frac{i}{4\pi^2}\int \mathrm{Tr}\left(\chi_-^{2,0}\wedge *[\phi_{++},\overline{\chi}_-^{0,2}]\right) + \ldots \tag{5.3.31}$$

It also contains the following terms, soley from the first line of the expression (5.3.15), depending on ϕ_{--},

$$S = -\frac{1}{6\pi^2}\int \mathrm{Tr}\Big[\phi_{--}\Big(*d_A^* d_A\phi_{++} - \frac{1}{4}[C^{3,0},[\phi_{++},C^{0,3}]] - *\Lambda[\psi_+^{0,1},\overline{\psi}^{1,0}] - \frac{1}{4}[\lambda_+^{3,0},\overline{\lambda}_+^{0,3}]\Big)\Big] + \dots \tag{5.3.32}$$

Assuming, for simplicity, that there are no-zero modes of $\overline{\eta}_-$ ($\boldsymbol{H}^{0,0}=0$) one can evaluate the expectation value $<\phi_{++}>$ of ϕ_{++} by solving the ϕ_{--} equations of motion and replacing all the other fields to their zero-modes. Then the only non-vanishing term in the action functional S in the $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ invariant neiborhood $\mathcal{C}$ of the fixed point locus comes from the expression (5.3.31), which can be written as

$$S|_{\mathcal{C}} = -\mathcal{F}_{\alpha\overline{\beta}i\overline{j}}\widetilde{\psi}_+^i\widetilde{\overline{\psi}}_+^{\overline{j}}\widetilde{\chi}_-^{\alpha}\widetilde{\chi}_-^{\overline{\beta}}, \tag{5.3.33}$$

where $\widetilde{\psi}_+^i$ and $\widetilde{\chi}_-^{\alpha}$ denote the zero-modes of $(\psi_+^{0,1},\lambda_+^{3,0})$ and $\chi_-^{2,0}$, respectively. In the above the indices i and α run over $i=1,\dots,\boldsymbol{h}^{0,1}+\boldsymbol{h}^{0,3}$ and $\alpha=1,\dots,\boldsymbol{h}^{0,2}$, where $\boldsymbol{h}^{0,*}=dim_{\mathbb{C}}\boldsymbol{H}^{0,*}$. The expression $\mathcal{F}_{\alpha\overline{\beta}i\overline{j}}\widetilde{\psi}_+^i\widetilde{\overline{\psi}}_+^{\overline{j}}$ denotes the curvature two form of the anti-ghost bundle $\mathbb{V}$ over $\mathfrak{M}$ – the space of the zero-modes a^i of $A^{0,1}$ and $C^{3,0}$ modulo $\mathcal{G}$. Consequently the expectation value, for example, $<\exp\widehat{\varpi}_T^{\mathcal{G}}>$, becomes

$$<\exp\widehat{\varpi}_T^{\mathcal{G}}> = \int_{\mathfrak{M}}\prod_{\ell=1}^{\triangle+\boldsymbol{h}^{0,2}} da^{\ell}da^{\overline{\ell}}d\widetilde{\psi}_+^{\ell}d\widetilde{\overline{\psi}}^{\overline{\ell}}\prod_{\gamma=1}^{\boldsymbol{h}^{0,2}} d\widetilde{\chi}_-^{\gamma}d\widetilde{\chi}_-^{\overline{\gamma}} \times\exp\Big(\mathcal{F}_{\alpha\overline{\beta}i\overline{j}}(a^{\ell},a^{\overline{\ell}})\widetilde{\psi}_+^i\widetilde{\overline{\psi}}_+^{\overline{j}}\widetilde{\chi}_-^{\alpha}\widetilde{\chi}_-^{\overline{\beta}} + \widetilde{\omega}_{i\overline{j}}(a^{\ell},a^{\overline{\ell}})\widetilde{\psi}_+^i\widetilde{\overline{\psi}}^{\overline{j}}\Big), \tag{5.3.34}$$

which leads exactly to (5.3.28).

Some Properties of $\mathfrak{M}$

This is a mathematical digression to establish a property of the extended moduli space. First we recall a theorem [48][104] on the moduli space $\mathcal{M}_{EH}$ of EH connections - if $\widetilde{\boldsymbol{H}}^{0,0}=0$ the moduli space $\mathcal{M}_{EH}$ is a complex analytic space. It is nonsingular at a neighborhood of a connection if $\widetilde{\boldsymbol{H}}^{0,2}=0$ and its tangent space is naturally isomorphic to the space of $\boldsymbol{H}^{0,1}$. Here $\widetilde{\boldsymbol{H}}^{0,*}$ denotes the cohomology group defined by tracefree endomorphisms.

Now we state an analogous theorem about the extended moduli space $\mathfrak{M}$ of EH connections on a complex Kähler 3-fold – if $\widetilde{\boldsymbol{H}}^{0,0}=0$ the moduli space $\mathfrak{M}$ is a complex analytic space. It is nonsingular at a neighborhood of an extended connection if $\widetilde{\boldsymbol{H}}^{0,2}=0$ and its tangent space is naturally isomorphic to the space $\boldsymbol{H}^{0,1}\oplus\boldsymbol{H}^{3,0}$. The extended moduli space $\mathfrak{M}$ is a smooth Kähler manifold with the formal dimension equal to the actual dimension if $\widetilde{\boldsymbol{H}}^{0,0}=\widetilde{\boldsymbol{H}}^{0,2}=0$.

The proof of the above theorem is similar to that of the Einstein-Hermitian case [48]. Given an extended EH connection $\overline{\mathfrak{D}}$, a nearby deformation $\overline{\partial}_A+\alpha$,

$C^{3,0} + \beta$ is governed by the equations

$$\begin{aligned} \overline{\partial}_A\alpha + \alpha \wedge \alpha &= 0, \\ \overline{\partial}_A^*\alpha &= 0, \\ \Lambda(\overline{\partial}_A\beta + \alpha \wedge \beta) &= 0. \end{aligned} \tag{5.3.35}$$

We only need to consider the last equation since the theorem quoted above already dealt with the first two equations. The last equation has the following orthogonal decomposition

$$\overline{\partial}_A\beta + \alpha \wedge \beta = 0 \leftrightarrow \begin{cases} \overline{\partial}_A\left(\beta + \overline{\partial}_A^* \circ G(\alpha \wedge \beta)\right) = 0, \\ \overline{\partial}_A^* \circ \overline{\partial}_A \circ G(\alpha \wedge \beta) = 0, \\ H(\alpha \wedge \beta) = 0, \end{cases} \tag{5.3.36}$$

where G is Green's operator and H is the harmonic projection. We define Kuranishi map k'

$$k' : \boldsymbol{C}^{3,0} \to \boldsymbol{C}^{3,0}, \qquad k'(\beta) = \beta + \overline{\partial}_A^* \circ G(\alpha \wedge \beta). \tag{5.3.37}$$

Then, from the first equation on the right of (5.3.36) we have $\overline{\partial}_A(k'(\beta)) = 0$, while $\overline{\partial}_A^*(k'(\beta)) = 0$ by the dimensional reason. Thus we obtain $\Lambda\overline{\partial}_A(k'(\beta)) = 0 \to \overline{\partial}_A^*(k'(\beta)) = 0$. Consequently we have

$$k'(\beta) \in \boldsymbol{H}^{3,0}. \tag{5.3.38}$$

Now we examine if the Kuranishi map is invertible for a given $\rho \in \boldsymbol{H}^{3,0}$, i.e., $\beta = k'^{-1}(\rho)$ and $\Lambda(\overline{\partial}_A\beta + \alpha \wedge \beta) = 0$. Taking the orthogonal decomposition of $\alpha \wedge \beta$ one finds that

$$\Lambda(\overline{\partial}_A\beta + \alpha \wedge \beta) = \Lambda\overline{\partial}_A^* \circ \overline{\partial}_A \circ G(\alpha \wedge \beta) + \Lambda(H(\alpha \wedge \beta)). \tag{5.3.39}$$

Note that $\Lambda(H(\alpha\wedge\beta))$ is in $\widetilde{\boldsymbol{H}}^{2,0}$, which is isomorphic to $\widetilde{\boldsymbol{H}}^{0,2}$. By our assumption we have $H(\alpha \wedge \beta)) = 0$. Denoting $\gamma = \overline{\partial}_A\alpha + \alpha \wedge \alpha$ and $\delta = \overline{\partial}_A\beta + \alpha \wedge \beta$ we have

$$\begin{aligned} \delta &= \overline{\partial}_A^* \circ G(\overline{\partial}_A\alpha \wedge \beta - \alpha \wedge \overline{\partial}_A\beta) \\ &= \overline{\partial}_A^* \circ G(\gamma \wedge \beta - \alpha \wedge \delta) \\ &= -\overline{\partial}_A^* \circ G(\alpha \wedge \delta), \end{aligned} \tag{5.3.40}$$

where we used the fact that $\gamma = 0$ for $\widetilde{\boldsymbol{H}}^{0,2} = 0$. Applying the following estimate

$$||\overline{\partial}_A * \circ Gv||_{2k+1} \le c||v||_{2,k}, \tag{5.3.41}$$

we have

$$\begin{aligned} ||\delta||_{2,k} \le ||\delta||_{2,k+1} &= ||\overline{\partial}_A^* \circ G(\alpha \wedge \delta)||_{2,k+1} \\ &\le c||\delta||_{2,k} \cdot ||\alpha||_{2,k}. \end{aligned} \tag{5.3.42}$$

Taking α sufficiently close to 0 so that $||\alpha||_{2,k} < 1/c$, we conclude $\delta = 0$. Thus the Kuranishi map k' is invertible if $\widetilde{\boldsymbol{H}}^{0,2} = 0$. Consequently the local model of the extended moduli space $\mathfrak{M}$ is given by $f^{-1}(0)$ where

$$\begin{aligned} f : \boldsymbol{H}^{0,1} \oplus \boldsymbol{H}^{3,0} &\to \widetilde{\boldsymbol{H}}^{0,2}, \\ (\alpha, \beta) &\to H(\alpha \wedge \beta), \Lambda(H(\alpha^* \wedge \beta^*)). \end{aligned} \tag{5.3.43}$$

5.3.3 A Use of S^1 Symmetry

The extended equations (5.3.18) we have may be very useful. On the extended moduli space $\mathfrak{M}$ of EH connections we have the natural S^1-action

$$S^1 : C^{0,3} \to e^{i\theta} C^{0,3}, \tag{5.3.44}$$

which preserves the complex and the Kähler structure. Thus any cohomological computations can be further localized to the fixed point locus of the S^1-action. For the $SU(2)$ case we are concentrating on it is easy to determine the fixed point. We have two branches.

- Branch (i)

  $\phi_{++} = 0$ and the $SU(2)$ symmetry is unbroken. Then we have a trivial fixed point where simply $C^{0,3} = 0$ where we have EH connections.

- Branch (ii)

  ϕ_{++} is a constant diagonal trace-free matrix. The non-trivial fixed point occur if the gauge symmetry can undo the S^1-action. For this the $SU(2)$ symmetry should be broken to $U(1)$, i.e., $\mathcal{E}_A = L \oplus L^{-1}$ where $A \in \mathcal{A}^{1,1}$. While $C^{0,3}$ and $C^{3,0}$ become

$$C^{0,3} = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}, \qquad C^{3,0} = \begin{pmatrix} 0 & 0 \\ \overline{\gamma} & 0 \end{pmatrix}, \tag{5.3.45}$$

  where γ is a section of $K^{-1} \otimes L^2$, with K denoting the canonical line bundle of our Kähler 3-fold. Then we have the following fixed point equations

$$\begin{aligned} F_L^{0,2} &= 0, \\ iF_L \wedge \omega \wedge \omega - \frac{1}{2}\gamma \wedge \overline{\gamma} &= 0. \end{aligned} \qquad \overline{\partial}_L^* \gamma = 0, \tag{5.3.46}$$

  where F_L denotes the curvature of the line bundle L. Obviously we have a non-trivial solution if $deg(L) > 0$. If $\gamma = 0$ we can have abelian EH connections, and also if $deg(L) = 0$.

The above equation (5.3.46) is analogous to the abelian Seiberg-Vafa-Witten equations [12][7], perhaps equally powerful. Thus we expect that the above equations may contains all the non-trivial information about the Donaldson-Witten type theory on Kähler 3-folds. It should be possible to establish our conjecture quite rigorously. Here we will sketch the idea.

As a first step we map the $N_c = (2,0)$ model defined by the action functional S (5.3.15) to its deformed version, following the discussions in Sect. 3.4. The action functional is then defined by, see (3.3.31),

$$\begin{aligned} S_h(\varepsilon) =& \frac{1}{4\pi^2} s_+ \overline{s}_+ \int_M \mathrm{Tr}\left(\chi_-^{2,0} \wedge * \overline{\chi}_-^{0,2}\right) \\ &+ \frac{i}{4\pi^2} s_+ \int_M \mathrm{Tr}\left(\chi_-^{2,0} \wedge * F^{0,2}\right) + \frac{i}{4\pi^2} \overline{s}_+ \int_M \mathrm{Tr}\left(\overline{\chi}_-^{0,2} \wedge * F^{2,0}\right) \\ &- i s_+ \overline{s}_+ \mathcal{K}_T + \frac{\varepsilon}{4\pi^2} \int_M \frac{\omega^3}{3!} \mathrm{Tr}(\phi_{++}^2), \end{aligned} \tag{5.3.47}$$

where $\mathcal{K}_T$ is given by (5.3.12). As we established earlier the partition function of the above for $\varepsilon = 0$ is the correlation function (5.3.28) with the same conditions. If the reducible connections are unavoidable we turn on ε to regularize and utilize the non-abelian localization.

Now we examine the supersymmetry transformation laws (5.3.9) and (5.3.10) to find that the S^1-action (5.3.44) should be extended to as follows

$$\begin{aligned} S^1 &: (C^{0,3}, \overline{\lambda}_+^{0,3}, \overline{\chi}_-^{0,2}, H^{0,2}) \to \xi (C^{0,3}, \overline{\lambda}_+^{0,3}, \overline{\chi}_-^{0,2}, H^{0,2}), \\ S^1 &: (C^{3,0}, \lambda_+^{3,0}, \chi_-^{2,0}, H^{2,0}) \to \overline{\xi} (C^{3,0}, \lambda_+^{3,0}, \chi_-^{2,0}, H^{2,0}), \end{aligned} \tag{5.3.48}$$

where $\xi\overline{\xi} = 1$. Thus the above fields are now charged under S^1. A problem might be that the above S^1-action is not a symmetry of the action functional.[4] However the S^1-action preserves the supersymmetry transformation laws as well as the localization equations. Thus we can use it anyway. Now we modify the transformation laws of the charged fields under the S^1 by extending the $\mathcal{G}$-equivariant cohomology to $\mathcal{G} \times S^1$;

$$s_+^2 = 0, \qquad \{s_+, \overline{s}_+\} = -i\phi_{++}^a \mathcal{L}_a - im\mathcal{L}_{S^1}, \qquad \overline{s}_+^2 = 0. \tag{5.3.49}$$

We use the same form of the deformed action functional (5.3.47) but with the new transformation laws for supercharges according to (5.3.49). We obtain a new $N_c = (2,0)$ supersymmetric action functional[5]

$$\begin{aligned} S_h(m,\varepsilon) =& \frac{1}{4\pi^2} s_+ \overline{s}_+ \int_M \mathrm{Tr}\left(\chi_-^{2,0} \wedge * \overline{\chi}_-^{0,2}\right) \\ &+ \frac{i}{4\pi^2} s_+ \int_M \mathrm{Tr}\left(\chi_-^{2,0} \wedge * F^{0,2}\right) + \frac{i}{4\pi^2} \overline{s}_+ \int_M \mathrm{Tr}\left(\overline{\chi}_-^{0,2} \wedge * F^{2,0}\right) \\ &- \widehat{\varpi}_T^{\mathcal{G}} - imH_{S^1}, \end{aligned} \tag{5.3.50}$$

where H_{S^1} is the Hamiltonian of the S^1-action,

$$H_{S^1} = \frac{1}{24\pi^2} \int_M \mathrm{Tr}\left(C^{3,0} \wedge C^{0,3}\right), \tag{5.3.51}$$

[4] This is due to a term like $\mathrm{Tr}(\chi_-^{2,0} \wedge * \overline{\partial}_A \overline{\psi}_+^{0,1})$.

[5] We turn off ε. We can turn on ε whenever necessary.

The first and second lines in the action functional localize the path integral to the locus $\overline{\partial}_A^* C^{0,3} = F^{0,2} = 0$. The first term in the third line further localize the path integral to the locus $\mu_T = 0$. For simplicity we assume that there are no zero-modes of $\chi_-^{2,0}$. Then the partition function of the model reduces to[6]

$$Z = \int_{\mathfrak{M}} e^{imH_{S^1} + \widetilde{\varpi}_T}, \tag{5.3.53}$$

where $\widetilde{\varpi}_T$ is the Kähler form of $\mathfrak{M}$, obtained by the restriction and reduction from our equivariant Kähler form $\widehat{\varpi}_T^{\mathcal{G}}$ (5.3.13). Thus the partition function is given by the familiar DH integral formula over a finite dimensional Kähler manifold $\mathfrak{M}$ [56][105]. It is therefore an integral over the set of critical points of H_{S^1}, which is the same as the fixed point locus of the S^1-action on $\mathfrak{M}$. Thus we have the same two branches.

The following is a formal argument since I do not understand the compactification of $\mathfrak{M}$. However it will be sufficient to serve our purpose. We will just apply the exactness of the stationary phase integral. By setting $m \to \infty$ we may have

- Branch (i)

  Note that the value of the Hamiltonian H_{S^1} is zero at Branch (i). So its contribution to the integral is simply the volume of $\mathcal{M}_{EH}$ weighted by the one loop determinant of due to the normal bundle $N(\mathcal{M}_{EH})$ in $\mathfrak{M}$. Note that such one loop determinant contains weight m^{-s} where s denotes codimension of $\mathcal{M}_{EH}$) in $\mathfrak{M}$. Thus

  $$Z(i) \sim \frac{1}{m^s} vol(\mathcal{M}_{EH}) \times \ldots \tag{5.3.54}$$

  The unwritten part is due to contribution from the normal bundle $N(\mathcal{M}_{EH})$, while we extracted its dependence on m.

- Branch (ii)

  Note that the value of the Hamiltonian at Branch (ii) is $H_{S^1} = \frac{1}{12\pi} deg(L) := \frac{1}{24\pi^2} \int c_1(L) \wedge \omega \wedge \omega$, where L is a line bundle defined in (5.3.46). Thus

  $$Z(ii) \sim \sum_L \frac{1}{m^{s'}} \int_{\mathcal{F}(L)} \exp\left(-\frac{im}{12\pi} deg(L) + \widetilde{\omega}\,|_{\mathcal{F}(L)} \right) \times .. \tag{5.3.55}$$

  where $\mathcal{F}(L)$ denotes the fixed point locus, s' denotes its codimension and $\widetilde{\omega}|_{\mathcal{F}(L)}$ denote the Kähler form on $\mathcal{F}(L)$. The unwritten part is due to

[6] We remark that the action functional contains the mass term for the anti-ghosts $\chi_-^{2,0}$ and $\overline{\chi}_+^{0,2}$. If there are no-zero modes for anti-ghost such the term plays no roles. If there are zero-modes of anti-ghosts we have to include contribution from the anti-ghost bundles and the mass term. Then the partition function Z become

$$Z = \int_{\mathfrak{M}} det(\mathcal{F}_{\alpha\overline{\beta}} - imh_{\alpha\overline{\beta}}) \exp\left(imH_{S^1} + \widetilde{\varpi}_T\right), \tag{5.3.52}$$

where $\mathcal{F}_{\alpha\overline{\beta}} - imh_{\alpha\overline{\beta}}$ is S^1-equivariant curvature two form of the anti-ghost bundle $\mathbb{V}$ over $\mathfrak{M}$.

contributions from the normal bundle over the fixed point locus, while we extracted its dependence on m.

We assume that $s < s'$, otherwise the above formal formula does not make sense. Then one can take $m = 0$. Since the original formula was smooth in the limit of the reduction to the symplectic volume of $\mathfrak{M}$ the poles in $Z(i)$ and $Z(ii)$ should cancel order by order. Thus we have

$$\begin{aligned} vol(\mathcal{M}_{EH}) \sim \sum_L \frac{1}{(s'-s)!}\left(\frac{im}{12\pi}deg(L)\right)^{s'-s} \\ \times \int_{\mathcal{F}(L)} \exp\left(-\frac{im}{12\pi}deg(L) + \widetilde{\varpi}|_{\mathcal{F}(L)}\right) \times \ldots \end{aligned} \tag{5.3.56}$$

and

$$\begin{aligned} vol(\mathfrak{M}) \sim \sum_L \frac{1}{s'!}\left(\frac{im}{12\pi}deg(L)\right)^{s'} \\ \times \int_{\mathcal{F}(L)} \exp\left(-\frac{im}{12\pi}deg(L) + \widetilde{\varpi}|_{\mathcal{F}(L)}\right) \times \ldots \end{aligned} \tag{5.3.57}$$

We conclude that the above formal evaluation justifies, at least, our conjecture that Seiberg-Vafa-Witten type invariants defined by the equation (5.3.46) should be equivalent to the Donaldson-Witten type invariants on a Kähler 3-fold. It is possible to perform a similar analysis for the case with anti-ghost zero-modes, which makes life more complicated but does not alter the essential points advocated above.

5.3.4 Reduction To a Kähler Surface

In this subsection we perform a dimensional reduction of our models on a Kähler 3-fold M to a complex Kähler surface M_2. We first assume that M is a product manifold $M_3 = M_2 \times \mathbb{C}$ and, then, remove dependence of our fields on $\mathbb{C}$. We have the following correspondence

$$\begin{aligned} &A^{0,1} \to A^{1,0}, \overline{\sigma}, \\ &\psi_+^{0,1} \to \psi_+^{1,0}, \eta_+, \\ &\chi_-^{2,0} \to \psi_-^{1,0}, \chi_-^{2,0}, \\ &H^{0,2} \to H^{0,1}, H^{0,2}, \\ &C^{0,3} \to B^{0,2}, \end{aligned} \tag{5.3.58}$$

as well as the similar decomposition for their Hermitian conjugates. The other fields $(\phi_{\pm\pm}, \eta_-, \overline{\eta}_-, D)$ remain as they were. Thus we obtain the $N_c = (2,2)$ model in Sect. 3.2. Similarly the equation (5.3.18) for the extended EH connection reduces to the Vafa-Witten equations. Furthermore our equation (5.3.46) for branch (ii) fixed point become the Abelian Seiberg-Witten equations. Thus our conjecture on Donaldson-Witten type invariants on a Kähler 3-fold becomes a fact [12].

Now instead of the above trivial reduction we consider a product manifold $M = M_2 \times \Sigma$, where Σ is a compact Riemann surface. Then we can follow the same steps with the same sort of assumption as [106] to conclude that the models discussed in the previous subsection are equivalent to topological sigma model discussed in Sect. 4.2.5. Thus the stringy Donaldson-Witten invariants on a Kähler surface may be obtained from formulas like (5.3.56) and (5.3.57) on the product 3-fold. This support an earlier suspicion in Sect. 4.2.5 since the Seiberg-Vafa-Witten type invariants on a manifold $M_2 \times \Sigma$ most likely are just the Seiberg-Witten invariants on M_2.

5.4 On Calabi-Yau 3-Folds

In this section we specialize to the case that the Kähler 3-fold M is Calabi-Yau with holomorphic 3-form $\omega^{0,3}$. For the Calabi-Yau case a very special thing happens that our $N_c = (2,0)$ supersymmetry enhances to $N_c = (2,2)$ supersymmetry. Following the discussions in Sect. 2.1.3 we shall see that the partition function of our model is the path integral representation of the holomorphic Casson invariants defined by Thomas [96][97]. We will also discuss various issues related with string theory. We argued that our model is the world-volume theory of parallel type IIB (Euclidean) $D5$-branes wrapped on the CY_3. We show that the $\mathcal{G}$-equivariant degrees of freedom correspond to the bulk degrees of freedom transverse to the (Euclidean) $D5$-branes. We use such a correspondence as supporting evidence that our path integral should be well-defined in any situation.

5.4.1 Enhanced Supersymmetry

We consider the $N_c = (2,0)$ theory with superchages $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ defined in the previous section specializing to a Calabi-Yau 3-fold M with a holomorphic 3-form $\omega^{0,3}$. Using the non-degeneracy of $\omega^{0,3}$ we may redefine the fields $(\overline{\chi}_-^{0,2}, H^{0,2}, \lambda_+^{0,3}, C^{0,3})$ as

$$\psi_-^{0,1}, H^{0,1}, \eta_+, \sigma, \tag{5.4.1}$$

where[7]

$$\begin{aligned} \overline{\chi}_-^{0,2} &= *(\overline{\omega^{3,0} \wedge \psi_-^{0,1}}), & \lambda_+^{0,3} &= \eta_+ \omega^{0,3}, \\ H^{0,2} &= *(\overline{\omega^{3,0} \wedge H^{0,1}}), & C^{0,3} &= \sigma \omega^{0,3}. \end{aligned} \tag{5.4.2}$$

Now it is not difficult to show that the action functional S has additional global supersymmetries generated by $\boldsymbol{s}_-$ and $\overline{\boldsymbol{s}}_-$. We have the following dia-

[7] We recall that the Hodge star operator $*$ acting on a (p,q)-form on a complex d-folds gives a $(d-q, d-p)$,

$$* : \Omega^{p,q}(M) \to \Omega^{d-q,d-p}(M).$$

grams to be compared with (5.3.11);

$$
\begin{array}{ccccc}
\overline{\sigma} & \xrightarrow{s_+} & \eta_+ & \xleftarrow{s_-} & \phi_{++} \\
\Big\downarrow{\scriptstyle \overline{s}_-} & & \Big\downarrow{\scriptstyle \overline{s}_-} & & \Big\downarrow{\scriptstyle \overline{s}_-} \\
\overline{\eta}_- & \xrightarrow{s_+} & D & \xleftarrow{s_-} & \overline{\eta}_+ \\
\Big\uparrow{\scriptstyle \overline{s}_+} & & \Big\uparrow{\scriptstyle \overline{s}_+} & & \Big\uparrow{\scriptstyle \overline{s}_+} \\
\phi_{--} & \xrightarrow{s_+} & \eta_- & \xleftarrow{s_-} & \sigma
\end{array}
\quad , \qquad
\begin{array}{ccccc}
\psi_-^{0,1} & \xleftarrow{s_-} & A^{0,1} & \xrightarrow{s_+} & \psi_+^{0,1} \\
 & {\scriptstyle s_+}\searrow & & \swarrow{\scriptstyle s_-} & \\
 & & H^{0,1} & &
\end{array}
\quad . \tag{5.4.3}
$$

The four supercharges satisfy the following anti-commutation relations, see (3.2.1) and (3.2.2),

$$
\begin{aligned}
& & &\{s_+, \overline{s}_+\} = -i\phi^a_{++}\mathcal{L}_a, & & \\
&\{s_\pm, s_\pm\} = 0, & &\{s_+, \overline{s}_-\} = -i\sigma^a\mathcal{L}_a, & &\{s_+, s_-\} = 0, \\
&\{\overline{s}_\pm, \overline{s}_\pm\} = 0, & &\{s_-, \overline{s}_+\} = -i\overline{\sigma}^a\mathcal{L}_a, & &\{\overline{s}_+, \overline{s}_-\} = 0. \\
& & &\{s_-, \overline{s}_-\} = -i\phi^a_{--}\mathcal{L}_a, & &
\end{aligned} \tag{5.4.4}
$$

The above anti-commutation relations define balanced $\mathcal{G}$-equivariant Dolbeault cohomology on the space $\mathcal{A}$ of all connections [59]. Thus our model becomes a $N_c = (2,2)$ model.

For convenience we write down the explicit supersymmetry transformation laws. The $N_c = (2,0)$ gauge multiplet in (5.3.7) together with holomorphic and anti-holomorphic multiplets in (5.3.9) become the $N_c = (2,2)$ gauge multiplet with the transformation laws given by (3.2.11). The holomorphic multiplets in (5.3.8) and Fermi multiplets in (5.3.10) form $N_c = (2,2)$ chiral multiplets. For the chiral multiplets $(A^{0,1}, \psi^{0,1}_\pm, H^{0,1})$ we have

$$
\begin{aligned}
\delta A^{0,1} &= i\overline{\epsilon}_+\psi_-^{0,1} + i\overline{\epsilon}_-\psi_+^{0,1}, \\
\delta\psi_+^{0,1} &= +\overline{\epsilon}_+ H^{0,1} - \epsilon_-\overline{\partial}_A\phi_{++} - \epsilon_+\overline{\partial}_A\sigma, \\
\delta\psi_-^{0,1} &= -\overline{\epsilon}_- H^{0,1} - \epsilon_+\overline{\partial}_A\phi_{--} - \epsilon_-\overline{\partial}_A\overline{\sigma}, \\
\delta H^{0,1} &= -i\epsilon_-[\phi_{++}, \psi_-^{0,1}] + i\epsilon_-\overline{\partial}_A\eta_+ + i\epsilon_-[\overline{\sigma}, \psi_+^{0,1}] \\
&\quad + i\epsilon_+[\phi_{--}, \psi_+^{0,1}] - i\epsilon_+\overline{\partial}_A\eta_- - i\epsilon_+[\sigma, \psi_-^{0,1}],
\end{aligned} \tag{5.4.5}
$$

while for their conjugate multiplets $(A^{1,0}, \overline{\psi}^{1,0}_\pm, H^{1,0})$ we find

$$
\begin{aligned}
\delta A^{1,0} &= i\epsilon_+\overline{\psi}_-^{1,0} + i\epsilon_-\overline{\psi}_+^{1,0}, \\
\delta\overline{\psi}_+^{1,0} &= +\epsilon_+ H^{1,0} - \overline{\epsilon}_-\partial_A\phi_{++} - \overline{\epsilon}_+\partial_A\overline{\sigma}, \\
\delta\overline{\psi}_-^{1,0} &= -\epsilon_- H^{1,0} - \overline{\epsilon}_+\partial_A\phi_{--} - \overline{\epsilon}_-\partial_A\sigma, \\
\delta H^{1,0} &= -i\overline{\epsilon}_-[\phi_{++}, \overline{\psi}_-^{1,0}] + i\overline{\epsilon}_-\partial_A\overline{\eta}_+ + i\overline{\epsilon}_-[\sigma, \overline{\psi}_+^{1,0}] \\
&\quad - i\overline{\epsilon}_+[\phi_{--}, \overline{\psi}_+^{1,0}] - i\overline{\epsilon}_+\partial_A\overline{\eta}_- - i\overline{\epsilon}_+[\overline{\sigma}, \overline{\psi}_-^{1,0}],
\end{aligned} \tag{5.4.6}
$$

where $\delta = \overline{\epsilon}_- s_+ + \overline{\epsilon}_+ s_- + \epsilon_+\overline{s}_- + \epsilon_+\overline{s}_+$.

Now the action functional S in (5.3.15) can be rewritten in the form with manifest $N_c = (2,2)$ symmetry, see (3.2.17),

$$S = s_+\overline{s}_+s_-\overline{s}_-\left(\mathcal{K} - \frac{1}{6\pi^2}\int_M \mathrm{Tr}(\sigma * \overline{\sigma})\right) + s_+s_-\mathcal{W}(A^{0,1}) + \overline{s}_+\overline{s}_-\overline{\mathcal{W}}(A^{1,0}), \tag{5.4.7}$$

where $\mathcal{K}$ is the Kähler potential on the space $\mathcal{A}$ of all connections,

$$\mathcal{K} = \frac{1}{24\pi^2}\int_M \kappa\,\mathrm{Tr}\,(F\wedge F)\wedge\omega, \tag{5.4.8}$$

and $\mathcal{W}(A^{0,1})$ is the holomorphic Chern-Simons form

$$\mathcal{W}(A^{0,1}) = \frac{1}{8\pi^2}\int_M \omega^{3,0}\wedge \mathrm{Tr}\left(A\wedge\overline{\partial}A + \frac{2}{3}A\wedge A\wedge A\right). \tag{5.4.9}$$

We remark that the above action functional can be obtained by the dimensional reduction of the $(1+1)$-dimensional $N_{ws} = (2,2)$ spacetime supersymmetric linear gauged sigma model in two real dimensions [30], whose target space is the space $\mathcal{A}$ of all connections on a Calabi-Yau 3-fold M. In [30] we interpreted the model as the matrix string theory [29] compactified on a Calabi-Yau by regarding $\mathcal{A}$ as the configuration space of all D-branes wrapped on the Calabi-Yau. We will return to the related topics later.

5.4.2 Path Integral and Holomorphic Casson Invariants

Now we examine the partition function of our model. For simplicity we consider the $SU(r)$ case so that we only have trace-free parts. Examining the simultaneous fixed point locus of all the supercharges we see that the path integral is localized to the moduli space $\mathcal{M}_{EH}$ of EH connections,

$$\begin{aligned} F^{0,2} &= 0,\\ F\wedge\omega\wedge\omega &= 0, \end{aligned} \tag{5.4.10}$$

together with the solutions space, modulo $\mathcal{G}$, of

$$\begin{aligned} d_A\varphi_m &= 0,\\ [\varphi_m,\varphi_n] &= 0,\\ [\varphi_m,\varphi^{\overline{n}}] &= 0, \end{aligned} \tag{5.4.11}$$

where φ_m, $m = 1,2$, denotes the two adjoint valued complex scalars build from σ and $\phi_{\pm\pm}$. If there are no reducible EH connections all the adjoint scalars should be zero and the path integral reduces to an integration over the moduli space $\mathcal{M}^*_{EH}$ of stable holomorphic bundles.

Serre duality implies $\boldsymbol{H}^{0,i} \simeq \boldsymbol{H}^{0,3-i}$. Thus the formal complex dimension $\triangle$ is always zero. We can convert the fermionic fields (5.3.21) to

$$\eta_-, \psi_+^{0,1}, \psi_-^{0,1}, \eta_+. \tag{5.4.12}$$

The zero-modes of fermions are elements of the cohomology group of the following complex,

$$0 \longrightarrow \boldsymbol{C}^{0,0} \oplus \boldsymbol{C}^{0,0} \xrightarrow{\overline{\partial}_A} \boldsymbol{C}^{0,1} \oplus \boldsymbol{C}^{0,1} \xrightarrow{\overline{\partial}_A} 0. \tag{5.4.13}$$

Now we always have a non-zero number of anti-ghost $\psi_-^{0,1}$ unless $\boldsymbol{H}^{0,1} = 0$. At the end of the day, the fermionic path integral measure will be reduced to

$$\prod_{\rho=1}^{h^{0,1}} d\widetilde{\psi}_+^\rho d\widetilde{\psi_+^{\overline{\rho}}} d\widetilde{\psi}_-^\rho d\widetilde{\psi_-^{\overline{\rho}}} \prod_{b=1}^{h^{0,0}} d\widetilde{\eta}_+^b d\widetilde{\eta}_+^{\overline{b}} d\widetilde{\eta}_-^b d\widetilde{\eta}_-^{\overline{b}}. \tag{5.4.14}$$

The net ghost number violation of the measure is zero as the formal dimension.

For simplicity we assume that there are no fermionic zero-modes of $\eta_\pm$ and $\overline{\eta}_\pm$, i.e., $\boldsymbol{H}^{0,0} = 0$. Even if there are no reducible connections the *formal* dimension is not equal to the *actual* dimension unless it is zero. Whenever we have zero-modes $\widetilde{\psi}_+^\ell$ of $\psi_+^{0,1}$ we have corresponding zero-modes $\widetilde{\psi}_-^\ell$ of $\psi_-^{0,1}$. Thus we have two cases.

(i) There are no fermionic zero-modes. Then the formal dimension $\triangle = 0$ is the actual dimensions and the moduli space consists of a collection of non-degenerate points. The partion function then simply counts the number of solutions. But we hardly expect such a situation to arise.

(ii) There are fermionic zero-modes. Then the formal dimension $\triangle = 0$ is not the actual dimensions and the moduli space $\mathcal{M}_{EH}$ contains components with positive dimension. Then repeating the same anaysis as in Sect. 5.2.2, see eqs. (5.3.31) - (5.3.33), the partition function becomes

$$Z = \int_{\mathcal{M}_{EH}} \prod_{\rho=1}^{h^{0,1}} da^\rho da^{\overline{\rho}} d\widetilde{\psi}_+^\rho d\widetilde{\psi^{\overline{\rho}}} d\widetilde{\psi}_-^\rho d\widetilde{\psi_-^{\overline{\rho}}} \exp\left(\mathcal{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}(a^\rho, a^{\overline{\rho}}) \widetilde{\psi}_+^\alpha \widetilde{\psi_+^{\overline{\beta}}} \widetilde{\psi}_-^\gamma \widetilde{\psi_-^{\overline{\delta}}}\right), \tag{5.4.15}$$

where the curvature two-form $\mathcal{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}(a^\rho, a^{\overline{\rho}}) \widetilde{\psi}_+^\alpha \widetilde{\psi_+^{\overline{\beta}}}$ of the anti-ghost bundle $\mathbb{V}$ is now interpreted as the curvature two-form of the tangent bundle $T\mathcal{M}_{EH}$. Then the partition function is the Euler characteristic of $\mathcal{M}_{EH}$.

Thus our partition function can be identified with the holomorphic Casson invariant defined by Thomas [96][97]. It is interesting to compare our quantum field theoretic approach to the holomorphic Casson invariant with the definition of Thomas. Mathematically speaking we are using an infinite dimensional $\mathcal{G}$ equivariant version of Fulton-MacPherson's intersection theory starting all the way from infinite dimensional holomorphic bundles with a holomorphic section $F^{0,2}(A^{0,1})$ over the infinite dimensional space $\mathcal{A}$ of all gauge fields. Thomas used, very roughly speaking, infinitesimal data about the moduli space of stable sheaves to construct a virtual moduli cycle, whose role may be identified with that of the anti-ghost bundle.

The assumption that $\boldsymbol{H}^{0,0} = 0$ is reasonable since it is indeed true when certain conditions on the topological type of E are met. One may imagine various mathematical difficulties without such a condition. However, we can deal with such a situation as well, as we will now discuss.

5.4.3 Taking Care of Reducible Connections

Now we remove the assumption that there are no zero-modes of $\eta_\pm$ and $\overline{\eta}_\pm$, i.e., $\boldsymbol{H}^{0,0} \neq 0$. Then the above analysis is no longer valid. In such a case we perturb the model to a $N_c = (2,0)$ theory by giving bare mass to $(\sigma, \overline{\sigma}, \eta_+, \overline{\eta}_+, \psi_-^{0,1}, \overline{\psi}_-^{1,0})$ and further deform the model to a hybrid version, similar to holomorphic Yang-Mills theory [67]. This procedure can be sketched as follows

1. Write down the action functional S in (5.4.7) such that only the $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ symmetry are manifest,

$$S = \frac{1}{12\pi^2}\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\int_M \mathrm{Tr}\Big(\phi_{--}\left(F\wedge\omega^2 + 2[\sigma, *\overline{\sigma}]\right) - i\psi_-^{0,1}\wedge\overline{\psi}_-^{1,0}\wedge\omega^2 + 2\eta_- * \overline{\eta}_-\Big)$$
$$+\frac{i}{4\pi^2}\int_M \omega^{3,0}\wedge\mathrm{Tr}\left(\psi_-^{0,1}\wedge F^{0,2}\right) + \frac{i}{4\pi^2}\int_M \omega^{0,3}\wedge\mathrm{Tr}\left(\overline{\psi}_-^{1,0}\wedge F^{2,0}\right). \tag{5.4.16}$$

2. Modify transformation the laws of the $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ supersymmetries as (5.3.49) by extending $\mathcal{G}$ to $\mathcal{G}\times S^1$ according to

$$S^1 : (\sigma, \overline{\eta}_+, \psi_-^{0,1}, H^{0,1}) \longrightarrow \xi(\sigma, \overline{\eta}_+, \psi_-^{0,1}, H^{0,1}) \tag{5.4.17}$$

where $\xi\overline{\xi} = 1$. The above S^1-action is compatible with the $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ supersymmetry. The S^1-action is a symmetry of our original model if the holomorphic 3-form $\omega^{3,0}$ is rotated by $\overline{\xi}$ at the same time. There is no inconsistency since picking a holomorphic 3-form always has an ambiguity up to a $\mathbb{C}^*$ action. We denote the actional functional given by the same form as (5.4.16) but with the modified transformation laws by S_m. Then we consider the following more general action functional,

$$\begin{aligned} S(m,\overline{m})_0 =& S_m + \frac{\overline{m}}{3\pi^2}\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\int_M \mathrm{Tr}\Big(\sigma * \overline{\sigma}\Big) \\ =& S + \frac{1}{3\pi^2}\int_M \mathrm{Tr}\Big(-im\overline{m}\sigma * \overline{\sigma} - i\overline{m}\phi_{++}[\sigma, *\overline{\sigma}] + m\overline{\eta}_+ * \eta_+\Big) \\ & - \frac{im}{3\pi^2}\int_M \mathrm{Tr}\left(\phi_{--}[\sigma, *\overline{\sigma}]\right) + \frac{im}{12\pi^2}\int_M \mathrm{Tr}\left(\psi_-^{0,1}\wedge\overline{\psi}_-^{1,0}\right)\wedge\omega^2. \end{aligned} \tag{5.4.18}$$

The new action functional has only $N_c = (2,0)$ supersymmetry, while its partition function is the *same* as for the original $N_c = (2,2)$ theory.

3. Now we deform the model by adding a $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ exact term

$$\begin{aligned} S(m,\overline{m})_1 &= S(m,\overline{m})_0 + \frac{m}{6\pi^2}\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\int_M \mathrm{Tr}(\phi_{--}^2) \\ &= S(m,\overline{m})_0 - \frac{m}{3\pi^2}\int_M \mathrm{Tr}(\eta_- * \overline{\eta}_- - \phi_{--} * D). \end{aligned} \tag{5.4.19}$$

We can eliminate the auxiliary field D by setting

$$D = -\frac{1}{2}\Lambda F - \phi_{--}. \tag{5.4.20}$$

Then we have

$$
\begin{aligned}
S(m,\overline{m})_1'
&= S' + \frac{1}{3\pi^2}\int_M \mathrm{Tr}\Big(-i\overline{m}\phi_{++}[\sigma,*\overline{\sigma}] + \overline{m}\eta_+ * \eta_+ - im\phi_{--}[\sigma,*\overline{\sigma}] - m\eta_- * \overline{\eta}_-\Big)\\
&- \frac{im\overline{m}}{3\pi^2}\int_M \mathrm{Tr}\,(\sigma * \overline{\sigma}) - \frac{m}{12\pi^2}\int_M \mathrm{Tr}\left(\phi_{--}F - i\psi_-^{0,1}\wedge\overline{\psi}_-^{1,0}\right)\wedge\omega^2\\
&- \frac{m^2}{3\pi^2}\int_M \frac{\omega^2}{3!}\mathrm{Tr}\,(\phi_{--}^2),
\end{aligned}
\tag{5.4.21}
$$

where S' is the original $N_c = (2,2)$ action functional S after integrating out D. Now we see that the above deformed $N_c = (2,0)$ model also receives contributions from the higher critical points $d_A f = 0$ in additions to the original fixed points $f = \frac{1}{3}\Lambda F = 0$.

4. Finally we define a new action functional $I(m,\overline{m})$ by adding $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ closed observables $\widehat{u}$ and $\widehat{v}$,

$$
I(m,\overline{m}) := S'(m,\overline{m}) - \overline{m}\widehat{v} - \overline{m}^2\widehat{u},
\tag{5.4.22}
$$

where

$$
\begin{aligned}
\widehat{v} &= \frac{1}{12\pi^2}\int_M \mathrm{Tr}\left(\phi_{++}F - i\psi_+^{0,1}\wedge\overline{\psi}_+^{1,0}\right)\wedge\omega^2,\\
\widehat{u} &= \frac{1}{3\pi^2}\int_M \frac{\omega^3}{3!}\mathrm{Tr}(\phi_{++}^2).
\end{aligned}
\tag{5.4.23}
$$

We have

$$
\begin{aligned}
I =& S' - \frac{im\overline{m}}{3\pi^2}\int_M \frac{\omega^3}{3!}\mathrm{Tr}\,(\sigma\overline{\sigma}) + \frac{m^2}{3\pi^2}\int_M \frac{\omega^3}{3!}\mathrm{Tr}(\phi_{--})^2 + \frac{\overline{m}^2}{3\pi^2}\int_M \frac{\omega^3}{3!}\mathrm{Tr}(\phi_{++})^2\\
&- \frac{m}{12\pi^2}\int_M \mathrm{Tr}\Big(\phi_{--}\left(F\wedge\omega^2 + 4i[\sigma,*\overline{\sigma}]\right) - i\overline{\psi}_-^{1,0}\wedge\psi_-^{0,1}\wedge\omega^2 + 4\eta_- * \overline{\eta}_-\Big)\\
&- \frac{\overline{m}}{12\pi^2}\int_M \mathrm{Tr}\Big(\phi_{++}\left(F\wedge\omega^2 + 4i[\sigma,*\overline{\sigma}]\right) - i\overline{\psi}_+^{1,0}\wedge\psi_+^{0,1}\wedge\omega^2 + 4\eta_+ * \overline{\eta}_+\Big).
\end{aligned}
\tag{5.4.24}
$$

Now we examine the properties of the new action functional $I(m,\overline{m})$. First of all we can regard m and $\overline{m}$ as independent real numbers. We have chosen the notation to make the action functional look symmetric under $+$ and $-$ indices. Nevertheless it is interesting to observe that the action functional $I(m,\overline{m})$ almost retains the symmetric structure of the original model, despite the asymmetric perturbation and deformation. This property may be interpreted as a quantum background independence related with the holomorphic anomaly [107][108]. We will not elaborate on this issue here. For $m = \overline{m} = 0$ we recover the original $N_c = (2,2)$ model. For $m = 0$ and $\overline{m} \neq 0$ we are turning on $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ closed observables in the original $N_c = (2,2)$ model. Similarly for $\overline{m} = 0$ and $m \neq 0$ we are turning on $\boldsymbol{s}_-$ and $\overline{\boldsymbol{s}}_-$ closed observables. Hower both the processes do not change the partition function since the asymmetry of the added terms between fermions with positive and negative ghost numbers, while we do not have any ghost number anomaly in the path integral measure. The new model defined by $I(m,\overline{m})$ is different from the original $N_c = (2,2)$ model for $m,\overline{m} \neq 0$. The original model has a problem when we have reducible connections, i.e., $\boldsymbol{H}^{0,0} \neq 0$. Then we have fermionic zero-modes of $\eta_\pm$ and $\overline{\eta}_\pm$ as well as bosonic zero-modes of $\phi_{\pm\pm}$ and $\sigma,\overline{\sigma}$. The moduli space $\mathcal{M}_{EH}$ becomes singular and the path integral may be not well-defined. Furthermore we have new obvious affine non-compact directions spanned by arbitrary linear combinations of bosonic zero-modes. The salient feature of the new action functional $I(m,\overline{m})$ is that its value restricted to the fixed point locus of the original $N_c = (2,2)$ model is non-zero if and only if the group $\mathcal{G}$ does not act freely, i.e., $\boldsymbol{H}^{0,0} \neq 0$. In such a case the quadratic terms of adjoint scalars $\phi_{\pm\pm}$ and $\sigma,\overline{\sigma}$ regularize the singularities. By turning on m and $\overline{m}$ we also see new fixed points flowing from infinity $m,\overline{m} \to \infty$. Thus we are essentially doing a non-abelian equivariant integral of Witten [55]. The partion function of the new action functional $I(m,\overline{m})$ can be written as the sum of contributions of higher critical points $d_A f = 0$ of $\int_M \mathrm{Tr}(f * f)$ where $f = \frac{1}{4}\Lambda F \in \Omega^0(M, End(E)$. According to a general estimate of Witten one can always, in principle, extract precise information of the contribution coming from the original fixed point $f = 0$, the moduli space $\mathcal{M}_{EH}$, by taking $m,\overline{m} \to 0$.

As a summary we have a quantum field theoretic formalism of the holomorphic Casson invariant, which is well-defined regardless to whatever the properties of the moduli space $\mathcal{M}_{EH}$.

5.4.4 Relations With String Theory

Stable bundles appear very naturally in non-perturbative string theory. They correspond to stable BPS configurations of type IIB branes wrapped around non-trivial cycles in the compactified part M of the bulk space time Z. Consider a Calabi-Yau 3-fold M with Kähler form ω and holomorphic 3-form $\omega^{3,0}$. We fix a rank N C^∞ bundle E over M, endowed with a Hermitian structure. We fix the topological type of the bundle, by specifying its Chern character $\mathrm{ch}(E)$, or rather the Mukai vector $\mathrm{ch}(E)\sqrt{\widehat{A}(M)}$. For a Calabi-Yau 3-fold, the Mukai vector is given by

$$Q = \left(\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E) - \frac{p_1(M)}{48}\mathrm{ch}_0(E), \mathrm{ch}_3(E) - \frac{p_1(M)}{48}\mathrm{ch}_1(E)\right), \tag{5.4.25}$$

where $p_1(M)$ is the first Pontryagin class of the Calabi-Yau manifold. We may sum over different topological types later. The bundles may be seen as describing D-branes wrapped around the Calabi-Yau manifold M. The D-brane charges are precisely given by the components of the Mukai vector [89][88]. Since we are dealing with Euclidean branes we call a type IIB $D5$-brane, for example, a $D6$-brane. For example, the rank $r = \mathrm{ch}_0(E)$ corresponds to the number of $D6$-branes wrapped around M and more generally the charges $Q_{3-n}(E) \sim \mathrm{ch}_{3-n}(E)$ correspond to $D2n$-branes wrapped around cycles in M [92].

They are associated to extremal blackhole solutions of the low energy effective supergravity. The suitable counting of the number of stable orbits corresponds to counting the microscopic degrees of freedom leading to the black-hole entropy. The semistable bundles, which are not stable, correspond to marginally stable configurations. Namely they correspond to branes wrapped around vanishing cycles. Physically these represent new massless (or tensionless) states [11][109]. In the Wilsonian effective theory those degrees attribute to the singularities in the effective theory.[8] Mathematically they also correspond to the singularities in the moduli space $\mathcal{M}_{EH}$ of EH connections. However this does not mean that the physics breaks down at such a point. It is only the effective theory which failed at that point. The singularities can be mended by including to new massless (tensionless) degrees of freedom in the effective description. We also emphasis that ***not only the new semi-stable orbits but also related new bulk (transverse to the compactified space M) degrees of freedom are created in such a case.*** Altogether, the total system (string or M theory), nothing singular has happened.

Now we apply the above discussions to our model. The r $D6$-branes wrapped on the Calabi-Yau 3-fold M induces a rank r Hermitian vector bundle E over M whose topological type is determined by other lower dimensional D-branes wrapping homology cycles. On the $D6$-branes world-volume we have $U(r)$ gauge field A. The degrees of freedom transverse to M in the bulk, $Z = M \times \mathbb{C}^2$ in our case, are represented by two $End(E)$ valued complex scalars φ_m, $m = 1, 2$, on M. In sum we have exactly the bosonic field content of our model. Among the 16 space-time supercharges of the effective supersymmetric Yang-Mills theory of $D6$-branes on $\mathbb{C}^5$ we have 4 unbroken supercharges (covariant constant spinors) since the holonomy of a Calabi-Yau is $SU(3)$. Since we do not have any propagating gravitons on a D-brane world-volume the covariantly constant spinors should be twisted to become scalar supercharges [17]. These 4 supercharges can be identified with $s_\pm$ and $\bar{s}_\pm$. Consequently we may interpret our model as the effective world-volume theory of $D6$-branes wrapped on M.[9]

To be more concrete consider a holomorphic vector bundle $\mathcal{E}_A$, $A \in \mathcal{A}^{1,1}$. If the holomorphic connection A is reducible we have the reduction $\mathcal{E}_A = L \oplus L^{-1}$, where L is a line bundle whose Chern class is determined by the topological type of $\mathcal{E}_A$, thus by E. A reducible holomorphic connection is EH iff the degree $deg(L)$ of the associated line bundle L vanishes;

$$deg(L) = \int_M c_1(L) \wedge \omega^2 = \int_{D_4} \omega^2 = 0, \qquad (5.4.26)$$

where D_4 denotes the 4-cycle (a positive divisor) Poincaré dual to $c_1(L)$. Physically the above situation corresponds to a $D4$-brane wrapped around the vanishing cycle D_4. The degree is related with the mass of the wrapped degrees of

[8] The Wilsonian effective theory is defined in terms of massless degree of freedoms. In the beginning one has to specify what is the massless degrees of freedom. Such criterion in the present situation corresponds to, as we shall see shortly, a choice of polarization for stability.

[9] We should emphasis here that the world-volume theory is not entirely a "cohomological" field theory. Any global (space-time or not space-time) supersymmetric theory is a "cohomological" theory and vice versa if we compute the path integral of observables invariant under some of the global supercharges. Otherwise the fixed point theorem of Witten says nothing and we do not have such a drastic localization of the path integral to a finite dimensional moduli space.

freedom.[10] Thus the reducible EH connections correspond to massless states, represented by the zero-modes of φ_m. Once massless the state associated with (5.4.26) can freely propagate into (or escape to) the bulk.[11] The corresponding configuration space is obviously non-compact. Our moral is that one should include such configurations as well to have a well-defined total system. This sounds contrary to the usual belief. However we are stating the conventional wisdom of an equivariant approach.

Actually Witten's reformulation of Donaldson theory is based on such wisdom. As if well-known the equivariant cohomology is always something much more than the usual cohomology if the quotient space is singular. The extended space, where one defines equivariant cohomology, is bigger than the space of all bundles on M (the brane configuration on the compactified part M in the bulk Z). Those additional parts correspond to degrees transverse to M in Z.

5.4.5 Open String Field Theory, Homological Mirror Symmetry and D-branes

This subsection is for a brief history, with some risks due to my own prejudice, on related subjects. Our purpose here is to place the previous model in a larger prospective and also to motivate the remaining part of this thesis.

The holomorphic Chern-Simons term was first introduced by Witten as the action functional of the space-time field theory of the topological B model of open string field theory [85]. In the paper [85] Witten showed that topological sigma models on a Riemann surface Σ with boundaries can interpreted as string theory backgrounds, where the usual decoupling of ghost and matter does not hold. There are two types of topological sigma models called A and B models [3]. For the A model the path integral is localized to the moduli space of holomorphic maps from a Riemann surface to a Calabi-Yau 3-fold M. The correlation functions compute cohomology rings of the moduli space of holomorphic maps. The moduli space contains M as its zero-instanton sector (the constant maps) so that the classical part of the correlation functions is just the cohomology ring of M itself. Summing up higher instanton contributions the correlation functions are named quantum cohomology rings. For the B model there are no such instanton corrections and the path integral is localized to an integral over M. The correlation functions of the model compute the variation of Hodge structures. Now Witten considered those models on Riemann surface Σ with boundaries $(\partial\Sigma)_i$. For the A model the boundary condition is that each

[10] One may also consider the case of $D2$-branes wrapped on a 2-cycle Poincaré dual to $rC_2(E)-\frac{r-1}{2}c_1(E)^2$. If the cycle shrinks to zero-size we only have projectively flat connections as BPS states, see (5.3.25). One may also imagine that the area of the 2-cycle becomes "negative" - a flop type topology change, we do not have any EH connections left. However the partition function of our model is still non-empty.

[11] Mathematically the situation can be viewed as follows; one is interested in the problem associated with the moduli space of EH connections with a suitable polarization such that semi-stability implies stability, i.e, the mathematical term of the Wilsonian effective theory. That is, every reducible holomorphic connection is a non-EH connection. By changing the polarization, however, certain reducible holomorphic connections induce semi-stable holomorphic bundles. These are new degrees of freedom coming from non-EH holomorphic connections.

component $(\partial\Sigma)_i$ is mapped to a Lagrangian submanifold $\mathcal{L}_i \subset M$. For the B model he picked "free" boundary condition. Then he studied the space-time field theory of the open string field theories of the resulting string background. For the A model we have Chern-Simons theory with instanton corrections,

$$S_A = \frac{1}{2}\int_{\mathcal{L}} \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) + \text{instanton corrections.} \qquad (5.4.27)$$

For the B model he obtained the holomorphic Chern-Simons form as the action functional.

The A model above is closely related with the symplectic Floer theory – Floer homology of Lagrangian intersections involving pseudo holomorphic curves. Fukaya discovered a certain A^∞ category on the Floer homology which is roughly given by the genus zero correlation functions of topological open string theory – the A model above. Kontsevich extended Fukaya's category by supplying a flat line bundle L_i for each Lagrangian submanifolds $\mathcal{L}_i$. Then he conjectured so called homological mirror symmetry, that is, the derived category constructed from the Fukaya-Kontsevich category on a Calabi-Yau is equivalent to the derived category of coherent sheaves on a mirror Calabi-Yau. Note that Kontsevich' extension of the Fukaya category naturally fits in the topological open string A model (due to the Chan-Paton degrees of freedom). The coherent sheaves on a Calabi-Yau also naturally appear as holomorphic vector bundles due to the Chan-Paton degrees of freedom in the topological open string B model. The homological mirror conjecture is essentially the physical equivalence of the A and B models on mirror Calabi-Yau manifolds. It is interesting to note that the underlying structure of of the open string fied theory of Gaberdiel and Zwiebach [110], the refined version of Witten's construction [111], is a (cyclic) A^∞ algebra. The homological mirror symmetry could be a physical equivalence between two "different" backgrounds in the open string field theory.

In the modern language of physics the A model above corresponds to (topological) $D3$-branes wrapped on a CY_3, while the B model corresponds to $D6$-branes wrapped on a CY_3. It is shown that the BPS states of type IIA strings are represented by special Lagrangian submanifolds with flat line bundles on them, while for type IIB the BPS states are represented by stable holomorphic bundles [87]. Subsequently Strominger-Yau-Zaslow (SYZ) suggested a full quantum equivalence between IIA and IIB models [91]. Based on the simplest case, SYZ concluded that the mirror symmetry is a T-duality along a special Lagrangian T^3 fibration on the CY_3. Vafa went further for the more general case and argued, essentially, the homological mirror symmetry can be explained in terms of T-duality of D-branes wrapped on the Calabi-Yau [92]. Vafa also conjectured an equivalence between certain variations of holomorphic bundles and the counting of holomorphic disks with Lagrangian boundary conditions.

5.4.6 A Program

Here we consider a program towards an understanding of generalized mirror symmetry. Unfortunately this program is still speculative and not yet conclusive if it will lead to a concrete understanding of mirror symmetry. It gives also some motivations for the last chapter of this thesis. The idea is, roughly, to try to understand the mirror symmetry as something like the equivalence between Donaldson-Witten and Seiberg-Witten theories. Our source of inspiration are the facinating properties of Hitchin's moduli space [112] as well as of quiver varieties of Nakajima [113].

In Chapter 4 we showed that such an equivalence can be understood from S-duality of $N = 4$ supersymmetric Yang-Mills theory. The picture was that both Donaldson-Witten and Seiberg-Witten theories are obtained as different fixed points of the same renormalization group flows from $N = 4$ supersymmetric Yang-Mills theory. Then the equivalence between two theories was understood as remnants of the S-duality of the original $N = 4$ theory surviving the renormalization group flows. Such renormalization group flows were generated by the bare masses of certain fields, which in turn the Hamiltonian of a natural S^1 action on the function space of fields. From the viewpoint of the function space the renormalization group flows are just the gradient flows generated by the Hamiltonian vector fields.

For mirror symmetry the natural analogue of the above function space is the space all all maps $\Sigma \to M$ from a Riemann surface to a Calabi-Yau 3-fold M. For a pair $(M, \widetilde{M})$ of such manifolds one associates two different topological sigma $A(M)$ and $B(\widetilde{M})$ models obtained by twisting the worldsheet $N_{ws} = (2,2)$ supersymmetric sigma model in $(1 + 1)$ dimension [3]. The original mirror symmetry states that the two models are physically equivalent if $(M, \widetilde{M})$ is a mirror pair. A natural step towards understanding mirror symmetry could be the following. We pick a Calabi-Yau 3-fold M and consider the total space of the cotangent bundle T^*M as the target space of a supersymmetric sigma model in $(1 + 1)$ dimensions. Let us denote the corresponding A and B model by $\boldsymbol{A}$ and $\boldsymbol{B}$. Since the total space T^*M is a hyper-Kähler manifold we may expect it is self-mirror, i.e., the $\boldsymbol{A}$ and $\boldsymbol{B}$ model are physically equivalent. Assuming this, our goal will be to recover physical equivalence between $A(M)$ and $B(\widetilde{M})$ as a remnant of the equivalence $\boldsymbol{A} \simeq \boldsymbol{B}$ after a suitable renormalization group flow.

An important property of T^*M is that there is always an S^1 symmetry acting on the fiber.[12] This S^1-action also naturally extends to the function space of all maps $\Sigma \to T^*M$. We denote the function space by $\mathfrak{N} \simeq T^*\mathfrak{M}$ where $\mathfrak{M}$ denotes the space of all maps $\Sigma \to M$. The upshot is that the Hamiltonian $\boldsymbol{h}$ of the vector field generating the S^1-action (the momentum map of S^1 on the function space $\mathfrak{N}$ all maps $\Sigma \to T^*M$) corresponds to the physical bare mass m of the fields representing maps to the fiber space of T^*M. Thus we consider a family of supersymmetric sigma models parametrized by m,

$$S(m) = S + m\overline{m}\boldsymbol{h} + \dots,$$

where S denote the action functional of the original sigma model with target

[12] A compact hyper-Kähler is certainly self-mirror. A problem would be the fact that T^*M is a non-compact space. Perhaps one has to consider a natural compactifcation of T^*M compatible with the S^1 symmetry.

space T^*M and the dots above correspond to the supersymmetric completion. Now we take limit $|m| \to \infty$. The dominant contributions to the path integral are from arbitrary close neighborhoods of the locus of the critical points of h. It is well-known that the critical points of a Hamiltonian of a S^1-action coincide with the fixed points of the S^1-action. Let us denote the fixed point set of the S^1-action by $\mathfrak{F}$. The space generally decomposes into disjoint sets

$$\mathfrak{F} = \bigcup_\alpha \mathfrak{F}_\alpha, \tag{5.4.28}$$

with corresponding Morse index m_α – the number of negative eigenvalues of the Hessian of $\boldsymbol{h}$ at $\mathfrak{F}_\alpha$. Let us denote the value of $\boldsymbol{h}$ at a fixed points locus $\mathfrak{F}_\alpha$ by $\boldsymbol{h}_\alpha$. A key point is that the function space of all maps $\Sigma \to M$ to the (base) Calabi-Yau 3-fold M always corresponds to the trivial fixed point $\mathfrak{F}_0$ with zero Morse index $m_0 = 0$ and vanishing Morse function $h_0 = 0$, since the S^1 acts only on the fiber of T^*M. Thus we recover the supersymmetric sigma model with target space M, the $A(M)$ or $B(M)$ depending on twisting, in the trivial fixed point locus $\mathfrak{F}_0$ of the S^1-action. We also find the Calabi-Yau 3-fold M among the infinite dimensional space $\mathfrak{F}_0$ as a subspace consists of constant maps. Now an immediate question is: where is $\widetilde{M}$ and how can we recover $B(\widetilde{M})$? To state our proposed answer we need to give some more details of our set up.

To begin we consider our hyper-Kähler target space T^*M. Let (I, J, K) and $(\omega_1, \omega_2, \omega_3)$ be the hyper-Kähler structures on T^*M, where (ω_1, I) were extended from the complex structure and Kähler form on M. In total we have a S^2 worth of complex structures since $aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$ is also a complex structure. Now we consider the twistor space $T^*M \times S^2$ such that the fiber over a point (a, b, c) in S^2 is a copy of T^*M with complex structure $aI + bJ + cK$. We replace the real coordinates on S^2 by complex affine coordinate $\xi \in \mathbb{CP}^1$. Then

$$\omega^{2,0} = (\omega_2 + i\omega_3) - 2\omega_1\xi - (\omega_2 - i\omega_3)\xi^2, \tag{5.4.29}$$

is the holomorphic 2-form on the fiber over ξ. If V is the vector field generated by our S^1-action on the fiber of T^*M we have

$$\mathcal{L}_V\omega_1 = 0, \qquad \mathcal{L}_V\omega_2 = \omega_3, \qquad \mathcal{L}_V\omega_3 = -\omega_2. \tag{5.4.30}$$

This S^1-action can be extended to a holomorphic (with respect to the complex structure I) $\mathbb{C}^*$ action. This $\mathbb{C}^*$ action than covers the $\mathbb{C}^*$ action on $\mathbb{CP}^1$ in the twistor space $T^*M \times \mathbb{CP}^1$. The $\lambda \in \mathbb{C}^*$ action has two limit points $\zeta = 0, \infty$, corresponding to $\pm I$.

We can extend all the above structures to the function space $\mathfrak{N}$ of all maps $\Sigma \to T^*M$. We will maintain the same notations. Now we examine the role of the $\mathbb{C}^*$ action. For $\lambda \to 0$ any point on the space $\mathfrak{N}$ flows to some fixed point of the S^1-action. This leads to a cell decomposition of $\mathfrak{N}$ by attracting sets. For inverse flow $\lambda \to \infty$ the limit points are again the fixed points of $\mathbb{C}^*$ action but some flows from fixed points $\mathfrak{F}_\alpha$ of the S^1 action may stay in compact sets;

$$\mathfrak{L} = \bigcup_\alpha \mathfrak{L}_\alpha \supset \mathfrak{F}_\alpha, \tag{5.4.31}$$

and each component $\mathfrak{L}_\alpha$ is Lagrangian with respect to the holomorphic 2-form [113]. Equivalently we can work with the gradient flow associated with the Hamiltonian h of the S^1-action and examine paths of the flow (steepest descent) for the past $t \to -\infty$ and the future $t \to \infty$. We have the correspondence that $|\lambda| = e^t$.

The upshot is that we can naturally identify the generator λ of the above $\mathbb{C}^*$ action with the inverse of the physical bare mass m of our sigma model with target space T^*M. Thus we regard the sigma model with target space M as an end point of the renormalization group flow generated by the mass. Certainly we do not expect that all the physical properties of the model in the past $m = 0$ will be preserved after such a flow. However we may also expect that the chiral rings remember the past where the self-mirror property is recovered. Thus it is natural to find hints by trying to reverse the flow. What we find are the Lagrangian subvarieties $\mathfrak{L}_\alpha$. We also know that $\mathfrak{L}_0 = \mathfrak{F}_0 = \mathfrak{M}$, which is the function space of all maps $\Sigma \to M$. Now a natural question is; can we associate sigma models with other Lagrangian subvarieties $\mathfrak{L}_\alpha$, $\alpha \neq 0$, such that their function space of all maps $\Sigma \to M_\alpha$ can be identified with $\mathfrak{L}_\alpha$. Then our discussions so far seem to imply that one may find mirror partner among (?) those models. One may repeat the same procedure by considering only constant maps. It will produce a set of Lagrangian subspaces $M_\alpha \subset T^*M$ where $M_0 = M$. We recall that the B model depends only on the complex structure. It is possible that other Lagrangian subspaces M_α, $\alpha \neq 0$, are birational to each others and we might have $A(M) = B(M_\alpha)$.[13]

If the above speculation turns out to be true one may consider arbitrary dimensional Calabi-Yau space and examine Lagrangian subvarieties of its cotangent bundle. Unfortunately we do not even know how to establish the basic necessary condition like $h^{p,q}(M) = h^{p,3-q}(M_\alpha)$. It is quite possible that our consideration may not shed light on the mirror symmetry itself. But still we may use the above set up as a useful way of computing quantum cohomology rings of M in terms of sum of contributions from the (other) fixed points $\mathfrak{F}_\alpha$, $\alpha \neq 0$.

Now we can consider a natural generalization of the above setting. A sigma-model with the target space M is equivalent to the sigma-model whose target space is the configuration space of a single $D0$-brane on M. Thus we can replace the target space M with the space of all D-brane configurations on the Calabi-Yau 3-fold M. Such a configuration space can be identified with the space of all (Chan-Paton) sheaves on M. Similarly the sigma-model with the target space T^*M can be generalized to a sigma-model whose target space is the total space of the cotangent bundle of the space of all sheaves on M.

Our initial goal was to define $N_{ws} = (4,4)$ model whose target space is the cotangent bundle of the space of all connections on a Calabi-Yau 3-folds, and to study some of its properties along the line of the above ideas. Some

[13] The $\mathbb{C}^*$ action acts transitively on $\mathbb{CP}^1\backslash\{0,\infty\}$ hence will carry T^*M with any complex structure to any other within $\mathbb{CP}^1\backslash\{0,\infty\}$. Thus all the complex structures of the hyperKählerian family other that $\pm I$ are equivalent [112]. The base space M is equip with the complex structure I.

of ideas above can be applied to $N_{ws} = (4,0)$ theory. We may also consider $(0+0)$-dimensional sigma model, thus a cohomological field theory, instead of $(1+1)$-dimensional model.

[illegible]
[illegible]
Hamiltonian h of the S^1-action and examine paths of [illegible] for the past $t \to -\infty$ and the future $t \to \infty$. We have the correspondence that [illegible].

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Chapter 6

Cohomological Yang-Mills Theories on Calabi-Yau 4-Folds

6.1 Introduction

In the previous chapter we used the holomorphic Chern-Simons form as the holomorphic potential of a $N_c = (2, 2)$ supersymmetric model, which led to the holomorphic Casson invariants of a Calabi-Yau 3-fold. We also recall the other use of the same form as the action functional of the space-time field theory of topological open string field theory of the B model. In this chapter we discuss yet another use of the holomorphic Chern-Simons form, namely as a holomorphic Morse function.

In the paper [34], Witten constructed a complex spanned by the critical points of a Morse function whose boundary operator is given in terms of the gradient flows connecting critical points. In terms of physics, the critical points are supersymmetric ground states, while the gradients flows are instantons tunneling between different ground states. Floer constructed the Morse-Witten complex on the space of all connections on a real 3-manifold Y using the real Chern-Simons form as the Morse functional [114]. In the Floer case the critical points are flat connections on a real 3-manifold Y and the gradient flow lines between them are Yang-Mills instanton (anti-self-dual connections) on the real 4-manifold $Y \times R$. Taubes proved that the Euler characteristic of the Floer homology is (twice) the Casson invariant [115]. Atiyah interpreted the Floer theory as a non-relativistic quantum field theory and conjectured that its relativistic generalization is Donaldson theory [8] on a real 4-manifold [116].

Witten studied such a relativistic generalization and obtained the path integral representation of Donaldson theory [1]. The resulting Donaldson-Witten theory is the first example of a cohomological field theory and is related with physical $N = 2$ space-time supersymmetric Yang-Mills theory on a real 4-

manifold by twisting. The Hamiltonian formalism of the resulting theory (on $Y \times R$) gives rise to the Floer homology. Further dimensional reduction of the theory to Y leads to a cohomological field theory, whose partition function computes the Casson invariant [117][35].

Donaldson and Thomas observed that Casson and Floer theories have natural "holomorphic" or "complex" counterparts on Calabi-Yau 3- and 4-folds [118][96]. The idea is, very roughly speaking, to replace real coordinates of real 3- and 4-manifolds with complex coordinates of Calabi-Yau 3- and 4-folds. One may replace the Chern-Simons form on Y by the holomorphic Chern-Simons form on a Calabi-Yau 3-fold M, whose critical points are holomorphic bundles. Between the critical points one has a complex version of gradient flow lines, which are the holomorphic analogue of anti-self-dual connections on $M \times S^1 \times R$.[1] According to the program of Donaldson and Thomas most of the themes in the real case can be played in Calabi-Yau 3- and 4-folds with some variations.

On the other hand, in the paper [30], we proposed that the matrix string theory compactified on a Calabi-Yau 3-fold should be the $N_{ws} = (2,2)$ supersymmetric gauged-linear-sigma model in $\Sigma = R \times S^1$ whose target space is the space of all bundles on M.[2] In the infrared limit of Σ the model flows to $N_{ws} = (2,2)$ superconformal sigma model whose target space is the moduli space of stable bundles on Calabi-Yau 3-fold. We studied chiral rings of the resulting superconformal theory by twisting the sigma model. We argued that the resulting B model is equivalent to the holomorphic Chern-Simons theory. We also studied A model which involves holomorphic maps (the worldsheet instantons) from Σ to the moduli space of stable bundles. It turn out that one can identify the states in the Hilbert space of the topological string with the "holomorphic" version of Floer homology.[3] Furthermore the A model can be viewed as the $N_c = (2,0)$ cohomological field theory for the "holomorphic" Donaldson-Witten theory on $M \times S^1 \times R$.[4]

Here we follow the historical footsteps reviewed in the beginning to consider a quantum field theoretic approach to "holomorphic" or "complex" versions of Floer and Donaldson-Witten theories on Calabi-Yau 3- and 4-folds. We will start from motivating holomorphic Floer theory adopting Atiyah's approach. Then we propose, adopting Witten's approach, holomorphic Donaldson-Witten theory (a $N_c = (2,0)$ model) which can be used to give a quantum field theoretic definition of holomorphic Floer homology based on a "Hamiltonian" analysis. It seems that holomorphic Floer homology still lacks a mathematical definition beyond the original idea in [118][96]. A detailed study of this topic will appear elsewhere [119].

A quantum field theoretic approach to Donaldson-Witten type theories on general Kähler manifolds based on the moduli space of stable bundles was first

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[1] One should not read the above sentence literally.

[2] The dimensional reduction of the model along Σ is the $N_c = (2,2)$ model in Sect. 5.4, which partition function gives the holomorphic Casson invariant.

[3] We didn't realize this relation at the time of writing the paper [30].

[4] The prefix "holomorphic" can cause some confusions. We will use it since the names *holomorphic Chern-Simons theory* and *holomorphic Casson invariants* are already well-established.

considered in [52][67]. A quantum field theoretic approach to Donaldson-Witten type theories on Calabi-Yau 4-folds based on the moduli space of the holomorphic analogue of anti-self-dual connections was first considered by Baulieu et. al. [100]. Our model for holomorphic Donaldson-Witten theory is different from the model in [100], though it may morally be equivalent to it. Our model will be essentially a special example of considerations in [52] after some modifications and shares the same kind of observables as Donaldson-Witten theory. Our model is also exactly the (Euclidean) supersymmetric Yang-Mills theory on Calabi-Yau 4-folds.

6.2 Holomorphic Donaldson-Witten Theory

In this section we develop the holomorphic version of Donaldson-Witten theory on a Calabi-Yau 4-fold M_4 with holomorphic 4-form $\omega^{4,0}$ and Kähler form ω.

It will be useful to give a quick sketch of our model along the lines of the general approach of this thesis. Our model will be an example of an equivariant $N_c = (2,0)$ model, whose target space is the space of all connections $\mathcal{A}$ on a Hermitian vector bundle E over M_4, or equivalently the space of all unitary gauge fields on M_4. Thus the resulting model can not be much different from the model for $d = 4$ in Sect. 4.1.1. For the given data above the only freedom we have in a $N_c = (2,0)$ model is the choice of a suitable infinite dimensional holomorphic Hermitian vector bundle $\mathbb{E} \to \mathcal{A}$ over $\mathcal{A}$ with $\mathcal{G}$-equivariant holomorphic section $\mathfrak{S}$. Instead of the former ill-fated choice $\mathfrak{S} = F^{0,2}$ in (4.1.12), we take only the "holomorphic self-dual" part $F^{0,2+}$ of $F^{0,2}$, i.e., $F^{0,2+} = \frac{1}{2}(F_{\overline{\alpha}\overline{\beta}} + \frac{1}{2}\varepsilon_{\overline{\alpha}\overline{\beta}\overline{\gamma}\overline{\delta}}(F^*)^{\overline{\gamma}\overline{\delta}})dz^{\overline{\alpha}} \wedge dz^{\overline{\beta}}$. The above considerations already determines the model uniquely. However we will take an interesting detour by imitating Atiyah's approach to Floer theory and Witten's relativistic generalization.

6.2.1 Imitating Atiyah-Floer Theory

We begin by a brief review of the description of the original Floer theory by Atiyah-Witten [1]. We consider a gauge field $A^a_I(x)$ on a real 3-fold Y, where a denotes the index of the Lie algebra and $I = 1, 2, 3$ is the vector index on Y. Let $\mathcal{A}$ be the space of all gauge fields on Y. Consider the exterior derivative of $\mathcal{A}$, after parity change

$$Q = \int d^3x \psi^a_I(x) \frac{\delta}{\delta A^a_I(x)}, \tag{6.2.1}$$

and its adjoint

$$Q^* = \int d^3x \chi^a_I(x) \frac{\delta}{\delta A^a_I(x)}, \tag{6.2.2}$$

where $\psi_I^a(x)$ and $\chi_I^a(x)$ represent a one-form and its dual vector field, after the parity change, on $\mathcal{A}$. They satisfy the following anti-commutation relations

$$\begin{aligned} \{\psi_I^a(x), \psi_J^b(y)\} &= 0, \\ \{\psi_I^a(x), \chi_J^b(y)\} &= g_{IJ}\delta^{ab}\delta^3(x-y), \\ \{\chi_I^a(x), \chi_J^b(y)\} &= 0, \end{aligned} \tag{6.2.3}$$

where g_{IJ} denotes the Riemann metric tensor of Y. Using the real Chern-Simons form W one defines $Q_t = e^{-tW}Qe^{tW}$ and $Q_t^* = e^{-tW}Q^*e^{tW}$, where t is a real number. One finds

$$Q_t^2 = 0, \qquad \{Q_t, Q_t^*\} = 2H, \qquad Q_t^{*2} = 0, \tag{6.2.4}$$

where H is the Hamiltonian of the non-relativistic theory. The ground states of H form the Floer homology group, which is equivalent to the Q-cohomology group. The Floer homology group is graded by the ghost number U, which is such that $[U, Q] = +Q$ and $[U, Q^*] = -Q^*$. This ghost number is conserved modulo certain integer d due to instanton corrections.

We now consider a Calabi-Yau 3-fold M and the space $\mathcal{A}$ of all connections on a Hermitian vector bundle E. We follow all the other settings in this chapter. We introduce fermionic fields $\psi_{\bar{i}}^a$ and χ^{ai}, representing (0,1)-forms and (1,0)-vectors on $\mathcal{A}$ respectively. We consider the $\bar{\partial}$-operator and its "adjoint" on $\mathcal{A}$

$$\begin{aligned} \boldsymbol{s} &= \int d^6x\, \psi_{\bar{i}}^a(x) \frac{\delta}{\delta A_{\bar{i}}^a(x)}, \\ \widetilde{\boldsymbol{s}} &= -\int d^6x\, g_{i\bar{j}}\chi^{ai}(x) \frac{\delta}{\delta A_{\bar{j}}^a(x)}, \end{aligned} \tag{6.2.5}$$

where $i = 1, 2, 3$ runs over the complex coordinates on M. They satisfy the following anti-commutation relations

$$\begin{aligned} \{\psi_{\bar{i}}^a(x), \psi^{b\bar{j}}(y)\} &= 0, \\ \{\psi_{\bar{i}}^a(x), \chi^{b\bar{j}}(y)\} &= \delta_{\bar{i}}^{\bar{j}}\delta^{ab}\delta^6(x-y), \\ \{\chi_{\bar{i}}^a(x), \chi^{b\bar{j}}(y)\} &= 0. \end{aligned} \tag{6.2.6}$$

Using the holomorphic Chern-Simons form $\mathcal{W}(A^{\bar{i}})$

$$\mathcal{W} = \frac{1}{8\pi^2}\int_M \omega^{3,0} \wedge \mathrm{Tr}\left(A \wedge \bar{\partial}A + \frac{2}{3}A \wedge A \wedge A\right), \tag{6.2.7}$$

one defines $\boldsymbol{s}_{\bar{t}} = e^{-\bar{t}\mathcal{W}}\boldsymbol{s}e^{\bar{t}\mathcal{W}}$ and $\widetilde{\boldsymbol{s}}_{\bar{t}} = e^{-\bar{t}\mathcal{W}}\widetilde{\boldsymbol{s}}e^{\bar{t}\mathcal{W}}$, where t is a complex number. One finds

$$\boldsymbol{s}_{\bar{t}}^2 = 0, \qquad \{\boldsymbol{s}_{\bar{t}}, \widetilde{\boldsymbol{s}}_{\bar{t}}\} = H, \qquad \widetilde{\boldsymbol{s}}_{\bar{t}}^2 = 0, \tag{6.2.8}$$

where the holomorphic Hamiltonian H is given by

$$H = \frac{1}{4\pi^2}\sum_{\bar{i},a}\int d^6x \left[-\left(\frac{\delta}{\delta A_{\bar{i}}^a(x)}\right)^2 + \bar{t}^2\left(B^{a\bar{i}}\right)^2\right] + \frac{\bar{t}}{12\pi^2}\int d^6x\, \varepsilon^{\overline{ijk}}\,\mathrm{Tr}\left(\psi_{\bar{i}}D_{\bar{j}}\chi_{\bar{k}}\right) \tag{6.2.9}$$

where $B^{\bar{i}} = \frac{1}{2}\varepsilon^{\overline{ijk}}F_{\overline{jk}}$. The first two terms above give the Hamiltonian of bosonic Yang-Mills theory in "four" dimensions whose real 4-vector indices are replaced by holomorphic indices. The critical points of the Morse function are holomorphic vector bundles. The complex gradient flow lines between them are governed by the following equation

$$\frac{\partial A^{\bar{i}}}{\partial \bar{z}} = -\frac{1}{2}\varepsilon^{\overline{ijk}}F_{\overline{jk}}. \tag{6.2.10}$$

However one should not accept the above literally. We naturally want to have stable holomorphic vector bundles as critical points. One may supply such a stability by hand and then take the quotient by $\mathcal{G}^{\mathbb{C}}$. However the above equation is not invariant under $\mathcal{G}^{\mathbb{C}}$. Thus we should fix a Hermitian metric on the complex vector bundle and supply an Einstein-Hermitian condition, thus imposing the momentum map equation, and take a quotient by $\mathcal{G}$. Another problem is that holomorphic Morse theory is different from the real one. We refer to those problems and resolutions to the papers [118][96].

Thus our strategy is to first consider a relativistic generalization such that the supersymmetric ground states of the model on $M_3 \times \mathbb{C}/\mathbb{Z}$ are given by stable holomorphic bundles on M_3. Then we can just define holomorphic Floer homology as a suitable BRST (co)-homology of supersymmetric ground states. By taking such a definition we do not need to worry about the precise mathematical definition of coboundary operators and the holomorphic Morse complex etc.[5]

6.2.2 Covariant Generalization

Now we consider the covariant generalization of the holomorphic Floer theory. The resulting model will be an equivariant $N_c = (2,0)$ model based on the space of all connections on a Calabi-Yau 4-fold M_4 with holomorphic 4-form $\omega^{4,0}$. It is sufficient to retain $SU(4)$ invariance, which is the holonomy group of M_4. First we promote our fields $A_{\bar{i}}$ $\psi_{\bar{i}}$, and $\chi^{a}_{\overline{ij}}$ into "Lorentz" multiplets. Obviously we should introduce $(A_{\bar{\alpha}}, \psi_{\bar{\alpha}})$, where $\bar{\alpha}, \bar{\beta} = 1,2,3,4$ runs for anti-holomorphic tangent vector indices on M_4. As for $\chi_{\overline{ij}}$ we can take a holomorphic self-dual two form $\chi_{\bar{\alpha}\bar{\beta}}$;

$$\chi_{\bar{\alpha}\bar{\beta}} = -\chi_{\bar{\beta}\bar{\alpha}} = \frac{1}{2}\varepsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}(\chi^*)^{\bar{\gamma}\bar{\delta}}, \tag{6.2.11}$$

which has three complex components, the same as $\chi_{\overline{ij}}$. The star in the above denotes complex conjugation. This holomorphic analogue of anti-self-duality was introduced by Donaldson and Thomas [118].

The above "relativistic" extension uniquely fixes the associated $N_c = (2,0)$ model. The multiplets $(A_{\bar{\alpha}}, \psi_{\bar{\alpha}})$ form $N_c = (2,0)$ holomorphic multiplets, i.e., $\bar{\boldsymbol{s}}A_{\bar{\alpha}} = 0$. The anti-ghosts $\chi_{\bar{\alpha}\bar{\beta}}$, together with corresponding auxiliary fields, form the anti-ghost multiplets $(\chi_{\bar{\alpha}\bar{\beta}}, H_{\bar{\alpha}\bar{\beta}})$. Finally we have the usual $N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \bar{\eta}_-, D)$. The two supercharges, $\boldsymbol{s}_+$ and its conjugate

[5] these are, according to [118], related with the Picard-Lefschetz theory of the Lefschetz fibration over $\mathbb{CP}^1$.

$\overline{s}_+$, are differentials of $\mathcal{G}$-equivariant cohomology on the space $\mathcal{A}$ of all connections on a Hermitian vector bundle E on M_4. The property of the anti-ghosts $\chi_{\overline{\alpha}\overline{\beta}}$ fixes the infinite dimensional bundle $\mathbb{E}$ over $\mathcal{A}$ where other holomorphic section $\mathfrak{S}$ of an to the holomorphic self-dual part of the type $(0,2)$ curvature tensor $F^+_{\overline{\alpha}\overline{\beta}}(A_{\overline{\gamma}})$. Thus we have all the ingredients to define the model.

It is convenient to work with differential forms. For any two-form $\alpha \in \Omega^2(M_4)$ we have the familiar decomposition

$$\begin{aligned}\alpha^+ &= \alpha^{0,2} + \alpha_0\omega + \alpha^{0,2},\\ \alpha^- &= \alpha^{1,1}_\perp,\end{aligned} \tag{6.2.12}$$

where $\alpha_\perp$ denotes the $(1,1)$-form orthogonal to the Kähler form ω. On a Calabi-Yau 4-fold the $(0,2)$-form $\alpha^{0,2}$ can be further decomposed into

$$\alpha^{0,2} = \alpha^{0,2+} + \alpha^{0,2-}, \tag{6.2.13}$$

where $\alpha^{0,2\pm}$ denotes the eigenstates of holomorphic Hodge star operator $\star$ defined by [96]

$$\star\alpha^{0,2\pm} = *\overline{(\alpha^{0,2\pm}\wedge\omega^{4,0})} = \pm\alpha^{0,2\pm}. \tag{6.2.14}$$

We denote the corresponding eigenspace decomposition of $\Omega^{0,2}(M)$ as

$$\Omega^{0,2}(M) = \Omega^{0,2+}(M) \oplus \Omega^{0,2-}(M). \tag{6.2.15}$$

Thus $\chi^{0,2}_- := \frac{1}{2}\chi_{\overline{\alpha}\overline{\beta}}dz^{\overline{\alpha}}dz^{\overline{\beta}}$ is an element of $\Omega^{0,2+}(M, End(E))$. There is a bilinear form

$$\Omega^{0,2}(M)\times\Omega^{0,2}(M)\to\mathbb{C},\qquad (\alpha^{0,2},\beta^{0,2})\to\int_M \alpha^{0,2}\wedge\beta^{0,2}\wedge\omega^{4,0}. \tag{6.2.16}$$

Then

$$\int_{M_4}\alpha^{0,2}\wedge\alpha^{0,2}\wedge\omega^{4,0} \tag{6.2.17}$$

is positive definite on $\Omega^{0,2+}(M)$ and negative definite on $\Omega^{0,2-}(M)$. We also note that for any $\alpha^{0,2+}\in\Omega^{0,2+}(M)$ we have

$$\int \mathrm{Tr}\left(\beta^{0,2+}\wedge\alpha^{0,2+}\right)\wedge\omega^{4,0} = \int \mathrm{Tr}\left(\beta^{0,2+}\wedge *\alpha^{2+,0}\right) \tag{6.2.18}$$

since $*\alpha^{2+,0} = **\left(\alpha^{0,2+}\wedge\omega^{4,0}\right) = \alpha^{0,2+}\wedge\omega^{4,0}$.

Though obvious we give the explicit transformation laws. We have two sets of holomorphic multiplets and their anti-holomorphic partners. One set of holomorphic multiplets is $(A^{0,1},\psi^{0,1}_+)$ and its anti-holomorphic partner $(A^{1,0},\overline{\psi}^{1,0}_+)$,

$$\begin{aligned} s_+A^{0,1} &= i\psi^{0,1}_+, & s_+\psi^{0,1}_+ &= 0,\\ \overline{s}_+A^{0,1} &= 0, & \overline{s}_+\psi^{0,1}_+ &= -\overline{\partial}_A\phi_{++},\\ s_+A^{1,0} &= 0, & s_+\overline{\psi}^{1,0}_+ &= -\partial_A\phi_{++},\\ \overline{s}_+A^{1,0} &= i\overline{\psi}^{1,0}_+, & \overline{s}_+\overline{\psi}^{1,0}_+ &= 0.\end{aligned} \tag{6.2.19}$$

We have Fermi multiplet $(\chi_-^{2,0}, H^{2,0}) \in \Omega^{2+,0}(M, End(E))$ and anti-Fermi multiplet $(\overline{\chi}_-^{0,2}, H^{0,2}) \in \Omega^{0,2+}(M, End(E))$

$$\begin{aligned}
&\boldsymbol{s}_+\chi_-^{2,0} = -H^{2,0}, && \boldsymbol{s}_+ H^{2,0} = 0,\\
&\overline{\boldsymbol{s}}_+\chi_-^{2,0} = 0, && \overline{\boldsymbol{s}}_+ H^{2,0} = -i[\phi_{++}, \chi_-^{2,0}],\\
&\boldsymbol{s}_+\overline{\chi}_-^{0,2} = 0, && \boldsymbol{s}_+ H^{0,2} = -i[\phi_{++}, \overline{\chi}_-^{0,2}],\\
&\overline{\boldsymbol{s}}_+\overline{\chi}_-^{0,2} = -H^{0,2}, && \overline{\boldsymbol{s}}_+ H^{0,2} = 0.
\end{aligned} \tag{6.2.20}$$

Finally we have the usual $N_c = (2,0)$ gauge multiplet with transformation laws

$$\begin{aligned}
& && \boldsymbol{s}_+\eta_- = 0, && \\
&\boldsymbol{s}_+\phi_{--} = i\eta_-, && \overline{\boldsymbol{s}}_+\eta_- = +iD + \frac{1}{2}[\phi_{++}, \phi_{--}], && \boldsymbol{s}_+\phi_{++} = 0,\\
&\overline{\boldsymbol{s}}_+\phi_{--} = i\overline{\eta}_-, && \boldsymbol{s}_+\overline{\eta}_- = -iD + \frac{1}{2}[\phi_{++}, \phi_{--}], && \overline{\boldsymbol{s}}_+\phi_{++} = 0.\\
& && \overline{\boldsymbol{s}}_+\overline{\eta}_- = 0, &&
\end{aligned} \tag{6.2.21}$$

The $\mathcal{G}$-equivariant Kähler form on $\mathcal{A}$ is defined by (4.1.10), which we rewrite here for convenience;

$$\begin{aligned}
\widehat{\varpi}^{\mathcal{G}} &= i\boldsymbol{s}_+\overline{\boldsymbol{s}}_+\mathcal{K}\\
&= \frac{i}{2(4)!\pi^2}\int_M \mathrm{Tr}(\phi_{++}F)\wedge\omega^3 + \frac{1}{2(4)!\pi^2}\int_M \mathrm{Tr}(\psi_+^{0,1}\wedge\overline{\psi}_+^{1,0})\wedge\omega^3.
\end{aligned} \tag{6.2.22}$$

The action functional is given by the following familiar form

$$\begin{aligned}
S =& \frac{\boldsymbol{s}_+\overline{\boldsymbol{s}}_+}{2\cdot 4!\pi^2}\int_M \mathrm{Tr}\left(\phi_{--}\left(F\wedge\omega^3 + \frac{i}{4}\zeta\omega^4 I_E\right)\right)\\
&+ \frac{\boldsymbol{s}_+\overline{\boldsymbol{s}}_+}{4\pi^2}\int_M \mathrm{Tr}\left(\chi_-^{0,2}\wedge *\overline{\chi}_-^{2,0}\right) + \frac{\boldsymbol{s}_+\overline{\boldsymbol{s}}_+}{8\pi^2}\int_M \mathrm{Tr}\left(\eta_- * \overline{\eta}_-\right)\\
&+ \frac{i\boldsymbol{s}_+}{4\pi^2}\int_M \mathrm{Tr}\left(\chi_-^{2,0}\wedge *F^{0,2+}\right) + \frac{i\overline{\boldsymbol{s}}_+}{4\pi^2}\int_M \mathrm{Tr}\left(\overline{\chi}_-^{0,2}\wedge *F^{2+,0}\right).
\end{aligned} \tag{6.2.23}$$

Comparing with the action functional (4.1.15) the only difference is the last line above, involving a different choice of anti-ghost and holomorphic section over $\mathcal{A}$, i.e., $F^{0,2} \to F^{0,2+}$. After expanding the above we integrate out auxiliary fields D, $H^{2,0}$ and $H^{0,2}$ by imposing their algebraic equations of motion,

$$\begin{aligned}
H^{0,2} &= -iF^{0,2+},\\
D &= -\frac{1}{2}(\Lambda F + i\zeta I_E).
\end{aligned} \tag{6.2.24}$$

From the above and the general structure of an equivariant $N_c = (2,0)$ model we see that the path integral is localized to the moduli space defined by the following equations,

$$\begin{aligned}
F^{0,2+} &= 0,\\
i\Lambda F - \zeta I_E &= 0.
\end{aligned} \tag{6.2.25}$$

An integrable $F^{0,2} = \overline{\partial}_A^2 = 0$ connection is called Einstein-Hermitian if it further satisfies the second equation in (6.2.25). We may call the above moduli space the moduli space of "half-integrable" Einstein-Hermitian connections, since it imposes only half of the integrability. The equations in (6.2.25) are due to Lewis as cited in [118]. Note that

$$\begin{aligned}\int_M \operatorname{Tr}\left(F^{0,2} \wedge \star F^{0,2}\right) \wedge \omega^{4,0} =& 2\int_M \operatorname{Tr}\left(F^{0,2+} \wedge F^{0,2+}\right) \wedge \omega^{4,0} \\ &+4\pi^2 \int_M p_1(E) \wedge \omega^{4,0}.\end{aligned} \tag{6.2.26}$$

If $p_1(E)$ is of type $(2,2)$ we see from this that a half-integrable connection is actually integrable, $\overline{\partial}_A^2 = 0$. We denote the moduli space by $\mathfrak{M} \supset \mathcal{M}_{EH}$. We also have another familiar localization equation,

$$\begin{aligned} d_A\phi_{++} &= 0, \\ [\phi_{++}, \phi_{--}] &= 0. \end{aligned} \tag{6.2.27}$$

If the "half-integrable" EH bundle is irreducible $\phi_{\pm\pm} = 0$ and $\mathcal{G}$ acts freely on the locus of (6.2.25).

6.2.3 Path Integral

The explicit form of the action functional is given by

$$\begin{aligned} S' =& \frac{1}{8\pi^2}\int_M \operatorname{Tr}\Big(-2F^{0,2+}\wedge * F^{2+,0} - \frac{1}{4}\Lambda F * \Lambda F - d_A\phi_{++}\wedge * d_A\phi_{--} \\ &+ \frac{1}{4}[\phi_{++},\phi_{--}]^2 + \phi_{--} * \Lambda[\psi_+^{0,1}, \overline{\psi}_+^{1,0}] + 2i\chi_-^{2,0}\wedge *[\phi_{++}, \overline{\chi}^{0,2}] \\ &+ i[\phi_{++},\eta_-] * \overline{\eta}_- - i\overline{\eta}_- * \overline{\partial}_A^* \psi_+^{0,1} - i\eta_- * \overline{\partial}_A^* \overline{\psi}_+^{1,0} \\ &- 2\chi_-^{2,0}\wedge * \overline{\partial}_A^+ \psi_+^{0,1} - 2\overline{\chi}_-^{0,2}\wedge \partial_A^+ \overline{\psi}_+^{1,0}\Big), \end{aligned} \tag{6.2.28}$$

where

$$\partial_A^+ = \frac{1}{2}(1+\star)\partial_A, \qquad \overline{\partial}_A^+ = \frac{1}{2}(1+\star)\overline{\partial}_A. \tag{6.2.29}$$

Now we consider the fermionic zero-modes. Using $\overline{\partial}_A^+\overline{\partial}_A = F^{0,2+} = 0$ we obtain the following equations for fermionic zero-modes,

$$\overline{\partial}_A\eta_- = 0, \qquad \begin{aligned}\overline{\partial}_A^*\psi_+^{0,1} &= 0,\\ \overline{\partial}_A^+\psi_+^{0,1} &= 0,\end{aligned} \qquad \overline{\partial}_A^{+*}\overline{\chi}_-^{0,2} = 0. \tag{6.2.30}$$

Note that the equation $\overline{\partial}_A^+\overline{\chi}^{0,2} = 0$ is automatic here. Consequently the anti-ghost bundle over $\mathcal{M}$ will be finite dimensional. So we see that the fermionic zero-modes,

$$\eta_-, \psi_+^{0,1}, \overline{\chi}_-^{0,2}, \tag{6.2.31}$$

represent the cohomology of the following complex ,

$$0 \to \Omega^{0,0}(M, End(E)) \xrightarrow{\bar{\partial}_A} \Omega^{0,1}(M, End(E)) \xrightarrow{\bar{\partial}_A^+} \Omega^{0,2+}(M, End(E) \to 0. \tag{6.2.32}$$

The above is the holomorphic analogue of the deformation complex (5.2.4) of anti-self-dual connections in four real dimensions. The above is the deformation complex for the moduli space $\mathfrak{M}$ [96]. Thus the formal complex dimension of $\mathfrak{M}$ equals minus the index of the above complex or, equivalently, the net ghost number violation in the path integral measure due to the zero-modes of fermions,

$$\triangle = -\#(\eta_-)_0 + \#(\psi_+^{0,1})_0 - \#(\overline{\chi}_-^{0,2})_0 = -dim\boldsymbol{H}^{0,1} + dim\boldsymbol{H}^{0,2} - dim\boldsymbol{H}^{0,2+}. \tag{6.2.33}$$

The net ghost number violation of the path integral due to zero-modes of all the fermions – the fermions in (6.2.31) and their conjugates $(\overline{\eta}_-, \overline{\psi}_+^{1,0}, \chi_-^{2,0})$ – is $(\triangle, \triangle)$.

An observable of the theory can be constructed from any closed $\mathcal{G}$-equivariant differential form on $\mathcal{A}$ associated with any cohomology class on M, as defined in Sect. 4.1.1. A typical observable of the theory is the total $\mathcal{G}$-equivariant Kähler form, after the parity change, $\widehat{\varpi}^{\mathcal{G}}$ given by (6.2.22). Thus there are essentially no differences with Donaldson-Witten theory on a Kähler surface. Here the moduli space of integrable Einstein-Hermitian (anti-self-dual) connections on a Kähler surface is replaced by the moduli space of half-integrable Einstein-Hermitian connections on a Calabi-Yau 4-folds.

If we assume a situation that $\mathcal{G}$ acts freely on the locus of the solutions of (5.3.18), i.e. the connection is irreducible, the moduli space $\mathfrak{M}$ is an analytic space with a Kähler structure induced from the $\mathcal{G}$-equivariant Kähler form. The moduli space will not have the right complex dimension $\triangle$ unless $\boldsymbol{H}^{0,2+} = 0$ as well. However, in general, one can hardly expect to have such condition. In any case the correlation function $< \exp \widehat{\varpi}^{\mathcal{G}} >$ becomes – following Sect. 3.2.2 and 5.2.2 –

$$< \exp \widehat{\varpi}_{\mathcal{G}} > = \int_{\mathfrak{M}} e(\mathbb{V}) \wedge \exp \widetilde{\varpi}, \tag{6.2.34}$$

where $e(\mathbb{V})$ denotes the Euler class of the anti-ghost bundle. One may consider correlation functions of other observables $\widehat{\mathcal{O}}^{r,s}$ with ghost numbers (r, s) given by the degrees of s_+ and $\overline{s}_+$ closed $\mathcal{G}$ equivariant differential forms $\mathcal{O}^{r,s}$ – see Sect. 4.1.1. We have – see Sect. 3.2.2 –

$$\left\langle \prod_{i=1}^{\ell} \widehat{\mathcal{O}}^{r_i,s_i} \right\rangle = \int_{\mathfrak{M}} e(\mathbb{V}) \wedge \widetilde{\mathcal{O}}^{r_1,s_1} \wedge \ldots \wedge \widetilde{\mathcal{O}}^{r_\ell,s_\ell}, \tag{6.2.35}$$

where $\widetilde{\mathcal{O}}^{r,s}$ denotes the equivariant differential form $\mathcal{O}^{r,s}$ after the restriction and reduction to $\mathfrak{M}$. The above correlation function can be non-vanishing if

$$\sum_{i=1}^{\ell} (r_i, s_i) = (\triangle, \triangle), \tag{6.2.36}$$

due to the ghost number anomaly.

6.2.4 Relation to Super-Yang-Mills Theory

We now shortly discuss the relation of holomorphic Donaldson-Witten theory with physical $N = 2$ supersymmetric Yang-Mills theory in 8 real dimensions.

First note that because we defined the model for a Calabi-Yau 4-fold, the holonomy of the manifold is reduced from $SO(8)$ to $SU(4)$. Let us see how the various representations of these groups reduce under this restriction, see [98] for a similar analysis. First note that the group $SO(8)$ has three inequivalent 8-dimensional representations, which are called $\mathbf{8_v}$ (the vector representation), $\mathbf{8_s}$ and $\mathbf{8_c}$ (the two chiral spinors). These representations reduce as follows to $SU(4)$ representations

$$\mathbf{8_v}, \mathbf{8_s} \to \mathbf{4} \oplus \overline{\mathbf{4}}, \qquad \mathbf{8_c} \to \mathbf{6} \oplus 2 \times \mathbf{1}. \tag{6.2.37}$$

Note that the representations $\mathbf{4}$ and $\overline{\mathbf{4}}$ are pseudo-real and symplectic real. Therefore in fact the complexified $SO(8)$ representations $\mathbf{8_v}$ and $\mathbf{8_s}$ reduce in this way. Let us now look at the field content of physical Yang-Mills. It consist of a vector in $\mathbf{8_v}$ (the gauge field), two real (or one complex) scalars and two complex spinors of both chiralities, so $\mathbf{8_s} \oplus \mathbf{8_c}$. On a Calabi-Yau 4-fold, we see from (6.2.37) that the gauge field reduces to a $\mathbf{4}$ of $\overline{\mathbf{4}}$ of $SU(4)$ (the difference is just a matter of convention). Now let us see what happens to the spinors. The spinor transforming in the $\mathbf{8_s}$ goes to $\mathbf{4} \oplus \overline{\mathbf{4}}$. The other one reduces to $\mathbf{6} \oplus 2 \times \mathbf{1}$.

Now we can interpret this neatly in term of the field content of holomorphic Donaldson-Witten theory. Indeed, the gauge field is in the $\mathbf{4}$ of $SU(4)$. Furthermore, we have two real scalars $\phi_{\pm\pm}$ (or one complex). For the fermions in our model, the fermions the $\psi_+^{0,1}$ and $\overline{\psi}_+^{1,0}$ exactly transform according to $\mathbf{4} \oplus \overline{\mathbf{4}}$, while the holomorphic self-dual spinor $\chi_-^{0,2}$ has six real components, and therefore should transform in the $\mathbf{6}$ of $SU(4)$; and the remaining two real spinors are η_- and $\overline{\eta}_-$, having one complex of two real components. The supersymmetry charges of Yang-Mills transform in the the $SO(8)$ representation $\mathbf{8_c} \oplus \mathbf{8_s}$. Using (6.2.37) we readily see that we get two supercharges which transform as scalars under the holonomy $SU(4)$. These should therefore be identified with the global supercharges on a general Calabi-Yau 4-fold. Furthermore, these supercharges both originate from the same charge of the Yang-Mills theory, and therefore carry the same ghost number. So we should get $N_c = (2,0)$ supersymmetry. This is exactly the global supersymmetry of our model.

Therefore we see that both the field content and the global supersymmetry of holomorphic Donaldson-Witten theory is completely equivalent to that of physical Yang-Mills theory on a Calabi-Yau 4-fold. Note that the twisting in this situation does nothing. But we can even say more. It can be shown that the action functional (6.2.28) is exactly the action functional, up to a topological term, of $N = 2$ supersymmetric Yang-Mills in eight dimensions. This shows equivalence between $N = 2$ super-Yang-Mills theory and our holomorphic $N_c = (2,0)$ Donaldson-Witten model on a Calabi-Yau 4-fold.

Chapter 7

Cohomological Yang-Mills-Higgs Theory

7.1 Introduction

In this last chapter we introduce a new four manifold invariant which seems to have a good chance of carrying new information beyond Donaldson-Witten or Seiberg-Witten invariants. We define two models with $N_c = (2,0)$ and $N_c = (2,2)$ supersymmetry, respectively, which are generalizations of Donaldson-Witten and Vafa-Witten theories on Kähler 2-folds. Then we consider a general $N_c = (2,0)$ model which have the various interesting limits. Thus we are returning back to the subjects in Chapter 4. The similar generalizations of the theories in Chapter 5 and 6 are also possible which will appear elsewhere. Our models in this chapter combine the various general structures and ideas which we discuss before.

To motivate this chapter, it is useful to recall the models in the previous chapters. In Sect. 3.3 we studied a general equivariant $N_c = (2,0)$ model. Such a model is classified by a Kähler target space $\mathcal{A}$ with a group $\mathcal{G}$ acting as an isometry, which determines a $\mathcal{G}$-equivariant momentum map $\mu : \mathcal{A} \to Lie(\mathcal{G})^*$. We further have a Hermitian holomorphic vector bundle $\mathbb{E} \to \mathcal{A}$ with $\mathcal{G}$-equivariant holomorphic section $\mathfrak{S}$. Then the bosonic part of the path integral reduces to an integration over $\mathcal{M} := \mathfrak{S}^{-1} \cap \mu^{-1}(\zeta)/\mathcal{G}$; the solution space of the following equations, modulo $\mathcal{G}$,

$$\begin{aligned} \mathfrak{S} &= 0, \\ \mu - \zeta &= 0, \end{aligned} \tag{7.1.1}$$

provided that we are evaluating correlation functions for supersymmetric observables. Those observables correspond to elements of $\mathcal{G}$-equivariant cohomology of $\mathcal{A}$. The correlation functions of such observables are identified with intersection numbers of homology cycles, represented by the observables, in $[e(\mathbb{V})]$, where $[e(\mathbb{V})]$ denotes the cycle in $\mathcal{M}$ Poincaré dual to the Euler class $e(\mathbb{V})$ of the anti-ghost bundle $\mathbb{V}$ over $\mathcal{M}$. If the model has actually $N_c = (2,2)$ supersymmetry the anti-ghost bundle $\mathbb{V}$ can be identified with the tangent bundle $T\mathcal{M}$ and the partition function is the Euler characteristic of $\mathcal{M}$. The moral underlying cohomological field theory is that the triple $(\mathcal{A}, \mathcal{G}, \mathbb{E})$ can be all infinite dimensional but certain path integral can still be reduced to an integral over finite dimensional space $\mathcal{M}$.

In Sect. 4.1 we studied such an example of equivariant $N_c = (2,0)$ model where $\mathcal{A}$ is the space of all connections (gauge fields) on a Hermitian vector bundle $E \to M$ over a complex d-dimensional Kähler manifold M and $\mathcal{G}$ is the group of all gauge transformations. This determines a localization equation from the momentum map μ;

$$i\Lambda F - \zeta I_E = 0, \tag{7.1.2}$$

where ζ is the Fayet-Illiopolous term. The solution space of this equation modulo $\mathcal{G}$ is infinite dimensional except for $d = 1$. For $d \geq 2$ we consider an infinite dimensional bundle $\mathbb{E} \to \mathcal{A}$ with $\mathcal{G}$-equivariant holomorphic section $\mathfrak{S}$. We saw that there is an unique choice $\mathfrak{S} = F^{0,2}$ on a general Kähler manifold, leading to another localization equation,

$$F^{0,2} = 0. \tag{7.1.3}$$

An integrable connection $F^{0,2} = \bar{\partial}_A^2 = 0$ is called Einstein-Hermitian or Hermitian-Yang-Mills if it further satisfies (7.1.2). Thus the path integral is localized to the moduli space $\mathcal{M}_{EH}$ of Hermitian-Yang-Mills connections, or equivalently the moduli space of semi-stable holomorphic bundles. For $d = 1$ and $d = 2$ the Einstein-Hermitian condition is the same as the flatness and anti-self-duality of the gauge fields, respectively.

In Sect. 3.4 we showed that a class of equivariant $N_c = (2,0)$ model can be extended to a $N_c = (2,2)$ model. The essential point of such a construction is introducing additional bosonic fields corresponding to the local frame fields on the image of the section $\mathfrak{S} : \mathcal{A} \to \mathbb{E}$. In Sect. 4.2 we applied the method to Donaldson-Witten theory and obtained Vafa-Witten theory. Then we defined a family of models interpolating between the two theories and obtained useful information about both Donaldson-Witten and Vafa-Witten theories.

In this chapter we generalize Donaldson-Witten ($N = 2$ SYM) and Vafa-Witten ($N = 4$ SYM) theories on Kähler surfaces. The similar generalization would be possible for models in higher dimensions. The basic idea is to extend our target space $\mathcal{A}$ to the total space $T^*\mathcal{A}$ of its cotangent bundle. Since $\mathcal{A}$ is a flat affine Kähler manifold a cotangent vector is represented as an element of $\Omega^1(M, End(E))$. Thus we introduce additional bosonic fields φ given by an adjoint valued 1-form $\varphi \in \Omega^1(M, End(E))$. Then, by decomposing $\varphi = \varphi^{1,0} + \varphi^{0,1}$, we have to determine which component form holomorphic multiplets. We have to declare $\varphi^{1,0}$ to represent holomorphic coordinates on the fiber space of $T^*\mathcal{A}$, since we already fixed a complex structure of $\mathcal{A}$ by declaring that the $A^{0,1}$ component of a connection 1-form represents holomorphic coordinates. Thus we have $\bar{\mathbf{s}}_+ A^{0,1} = \bar{\mathbf{s}}_+ \varphi^{1,0} = 0$. Now one may proceed to construct a $N_c = (2,0)$ model with target space $T^*\mathcal{A}$.

A beautiful fact for any cotangent bundle of a Kähler manifold is that it always has canonical hyper-Kähler structure [120]. Thus it is natural to consider hyper-Kähler quotients. Then the real momentum map equation (7.1.2) for $\mathcal{A}$ is generalized to the hyper-Kähler momentum map equations for $T^*\mathcal{A}$,

$$\begin{aligned} \partial_A^* \varphi^{1,0} &= 0, \\ i\Lambda\left(F + [\varphi^{1,0}, \varphi^{0,1}]\right) - \zeta I &= 0. \end{aligned} \tag{7.1.4}$$

However the resulting hyper-Kähler quotient of $T^*\mathcal{A}$ by $\mathcal{G}$ will be infinite dimensional except for the $d = 1$ case, where the quotient space is Hitchin's moduli space [112]. To obtain a finite dimensional space we extend the bundle $\mathbb{E} \to \mathcal{A}$ to $\widetilde{\mathbb{E}} \to T^*\mathcal{A}$ and try to cut out the hyper-Kähler quotient space by the vanishing locus of suitable $\mathcal{G}$-equivariant holomorphic sections.

A natural choice on a Kähler surface is

$$\begin{aligned} F^{0,2} &= 0, \\ \overline{\partial}_A \varphi^{1,0} &= 0, \\ \varphi^{1,0} \wedge \varphi^{1,0} &= 0, \end{aligned} \tag{7.1.5}$$

which defines Higgs bundles of Simpson [121][122]. The above equation can be viewed as a generalization of the integrability $\overline{\partial}_A^2 = 0$ of the connection $\overline{\partial}_A$ to the integrability of the extended connection $\boldsymbol{D}'' = \overline{\partial}_A + \varphi^{1,0}$. Our model based on (7.1.5) is a generalization of Donaldson-Witten theory. We will also study the similar generalization of Vafa-Witten theory.

Another beautiful fact for any cotangent bundle of a Kähler manifold is that it always has the equivariant S^1-action acting on the fiber. Such a S^1-action on $T^*\mathcal{A}$ descends to the moduli spaces above. We will use the S^1 symmetry to define a family of models, which have many interesting limits.

7.1.1 Preliminaries

We consider a rank r Hermitian vector bundle $E \to M$ over a complex d-dimensional Kähler manifold M with Kähler form ω. Consider the space $\mathcal{A}$ of all connections of E and the cotangent bundle $T^*\mathcal{A}$. First we determine the fields representing the cotangent space $T^*\mathcal{A}$. For the base space $\mathcal{A}$ of $T^*\mathcal{A}$ we have connection 1-form $A = A^{1,0} + A^{0,1}$ with the usual gauge transformation law. We introduce a complex structure I on $\mathcal{A}$ using the complex structure of M by declaring $A^{0,1}$ to represent holomorphic coordinates. Since $\mathcal{A}$ is a flat affine Kähler manifold a cotangent vector is represented as an element of $\Omega^1(M, End(E))$. We introduce an adjoint valued bosonic 1-form $\varphi \in \Omega^1(M, End(E))$, which may be regarded as an element of the cotangent space of $\mathcal{A}$. According to the complex structure of M we have a decomposition $\varphi = \varphi^{1,0} + \varphi^{0,1}$. Then it is natural to fix the complex structure of the fiber space of $T^*\mathcal{A}$ by declaring $\varphi^{1,0}$ to be a holomorphic coordinate. Thus the (holomorphic) tangent space of $T^*\mathcal{A}$ is given by

$$\Omega^{0,1}(M, End(E)) \oplus \Omega^{1,0}(M, End(E)). \tag{7.1.6}$$

We denote the above complex structure also by I and call it the preferred complex structure, which has been induced from the complex structure of M. The total Kähler potential $\mathcal{K}(A,\varphi)$ of the total space $T^*\mathcal{A}$ is given by

$$\mathcal{K}(A,\varphi) = \mathcal{K}(A) - \frac{i}{2(d)!\pi^2}\int_M \mathrm{Tr}(\varphi^{1,0}\wedge\varphi^{0,1})\wedge\omega^{d-1} \tag{7.1.7}$$

where the Käher potential $\mathcal{K}(A)$ of $\mathcal{A}$ is

$$\mathcal{K}(A) = \frac{1}{4(d)!\pi^2}\int_M \kappa\,\mathrm{Tr}(F\wedge F)\wedge\omega^{d-2}. \tag{7.1.8}$$

and the added term is a Kähler potential in the space $\mathcal{B}$. On the total space $T^*\mathcal{A}$ we have a obvious action of the infinite dimensional group $\mathcal{G}$ of all gauge transformations, preserving the Kähler potential $\mathcal{K}(A,\varphi)$.

Now we introduce our $N_c = (2,0)$ supercharges s_+ and $\overline{s}_-$ with the familiar commutation relations

$$s_+^2 = 0, \qquad \{s_+, \overline{s}_+\} = -i\phi^a_{++}\mathcal{L}_a, \qquad \overline{s}_+^2 = 0. \tag{7.1.9}$$

The supercharges are identified with the differentials of $\mathcal{G}$-equivariant cohomology of our target space $T^*\mathcal{A}$. Thus $\phi^a_{++}\mathcal{L}_a$ is the infinitesimal gauge transformation generated by the adjoint scalar $\phi_{++}\in Lie(\mathcal{G}) = \Omega^0(M, End(E))$. From the complex structure of $T^*\mathcal{A}$ introduced above we have two sets of holomorphic multiplets $(A^{0,1},\psi_+^{0,1})$ and $(\varphi^{1,0},\lambda_+^{1,0})$ and their anti-holomorphic partners. The supersymmetry transformation laws are given by

$$\begin{aligned} s_+A^{0,1} &= i\psi_+^{0,1}, & s_+\psi_+^{0,1} &= 0,\\ \overline{s}_+A^{0,1} &= 0, & \overline{s}_+\psi_+^{0,1} &= -\bar{\partial}_A\phi_{++},\\ s_+A^{1,0} &= 0, & s_+\overline{\psi}_+^{1,0} &= -\partial_A\phi_{++},\\ \overline{s}_+A^{1,0} &= i\overline{\psi}_+^{1,0}, & \overline{s}_+\overline{\psi}_+^{1,0} &= 0, \end{aligned} \tag{7.1.10}$$

and

$$\begin{aligned} s_+\varphi^{1,0} &= i\lambda_+^{1,0}, & s_+\lambda_+^{1,0} &= 0,\\ \overline{s}_+\varphi^{1,0} &= 0, & \overline{s}_+\lambda_+^{1,0} &= [\phi_{++},\varphi^{1,0}],\\ s_+\varphi^{0,1} &= 0, & s_+\overline{\lambda}_+^{0,1} &= [\phi_{++},\varphi^{0,1}],\\ \overline{s}_+\varphi^{0,1} &= i\overline{\lambda}_+^{0,1}, & \overline{s}_+\overline{\lambda}_+^{0,1} &= 0. \end{aligned} \tag{7.1.11}$$

From the transformation laws we have the following total $\mathcal{G}$-equivariant Kähler form on $T^*\mathcal{A}$,

$$\begin{aligned} \widehat{\varpi}^T_{\mathcal{G}} =& is_+\overline{s}_+\mathcal{K}(A,\varphi)\\ =& \frac{i}{2(d)!\pi^2}\int_M \mathrm{Tr}\left(\phi_{++}\left(F + [\varphi^{1,0},\varphi^{0,1}]\right)\right)\wedge\omega^{d-1}\\ &+ \frac{1}{2(d)!\pi^2}\int_M \mathrm{Tr}\left(\psi_+^{0,1}\wedge\overline{\psi}_+^{1,0} + \lambda_+^{1,0}\wedge\overline{\lambda}_+^{0,1}\right)\wedge\omega^{d-1}. \end{aligned} \tag{7.1.12}$$

The second term in the above is the Kähler form ϖ^T of $T^*\mathcal{A}$ and the first term is the real $\mathcal{G}$-momentum map $\phi^a_{++}\mu_a$, $\mu^T_{\mathbb{R}}: T^*\mathcal{A} \to Lie(\mathcal{G})^* = \Omega^{2n}(M, End(E))$;

$$\begin{aligned}\mu^T_{\mathbb{R}} &= \frac{1}{2(d)!\pi^2}\left(F + [\varphi^{1,0}, \varphi^{0,1}]\right) \wedge \omega^{d-1} \\ &= \frac{1}{2d(d)!\pi^2}\Lambda\left(F + [\varphi^{1,0}, \varphi^{0,1}]\right)\omega^d,\end{aligned} \tag{7.1.13}$$

where Λ denote the adjoint of wedge multiplication with ω.

Following Hitchin [112] we have a natural hyper-Kähler structure I, J and K on $T^*\mathcal{A}$. Note that the additional complex structures J and K have no relation with the complex structure on the manifold M. Then we define the holomorphic symplectic form $\varpi_{\mathbb{C}}$ on $T^*\mathcal{A}$ by

$$\begin{aligned}&\varpi_{\mathbb{C}}((\delta_1 A^{0,1}, \delta_1\varphi^{1,0}), (\delta_2 A^{0,1}, \delta_2\varphi^{1,0})) \\ &\qquad = \frac{1}{2(d)!\pi^2}\int_M \mathrm{Tr}\left(\delta_2\varphi^{1,0} \wedge *\delta_1 A^{0,1} - \delta_1\varphi^{1,0} \wedge *\delta_2 A^{0,1}\right).\end{aligned} \tag{7.1.14}$$

The corresponding complex momentum map $\mu_{\mathbb{C}}$ on $T^*\mathcal{A}$ is given by

$$\mu_{\mathbb{C}} = \frac{1}{2(d)!\pi^2}\overline{\partial}_A\varphi^{1,0} \wedge \omega^{d-1} = \frac{1}{2d\cdot(d)!\pi^2}(\Lambda\overline{\partial}_A\varphi^{1,0}) \wedge \omega^d. \tag{7.1.15}$$

Using the Kähler identities

$$\overline{\partial}^*_A = i[\partial_A, \Lambda], \qquad \partial^*_A = -i[\overline{\partial}_A, \Lambda], \tag{7.1.16}$$

we see that the zeros of the complex momentum map is given by

$$\Lambda\overline{\partial}_A\varphi^{1,0} \to \partial^*_A\varphi^{1,0} = 0. \tag{7.1.17}$$

7.2 Generalized Donaldson-Witten Theory

From now on we consider a rank r Hermitian vector bundle $E \to M$ over a complex 2-dimensional Kähler manifold M with Kähler form ω. Consider the space $\mathcal{A}$ of all connections on E and the cotangent bundle $T^*\mathcal{A}$. We have the same holomorphic coordinates fields $A^{0,1}$ and $\varphi^{1,0} \in \Omega^{1,0}(M, End(E))$ of $T^*\mathcal{A}$, with the supersymmetry transformation laws in (7.1.10) and (7.1.11). We also have the usual $N_c = (2,0)$ gauge multiplet.

Now consider an infinite dimensional $\mathcal{G}$-equivariant holomorphic Hermitian vector bundle $\widetilde{\mathbb{E}} \to T^*\mathcal{A}$ over $T^*\mathcal{A}$ with a suitable $\mathcal{G}$-equivariant holomorphic section $\widetilde{\mathfrak{S}}(A^{0,1}, \varphi^{1,0})$, i.e., $\overline{\mathfrak{s}}_+\widetilde{\mathfrak{S}}(A^{0,1}, \varphi^{1,0}) = 0$. We only have the following possibility for this;

$$\widetilde{\mathfrak{S}}(A^{0,1}, \varphi^{1,0}) = F^{0,2} \oplus \overline{\partial}_A\varphi^{1,0} \oplus (\varphi^{1,0} \wedge \varphi^{1,0}). \tag{7.2.1}$$

We choose this most general form as our holomorphic section. We have a natural paring of the holomorphic section with corresponding anti-ghost fields Υ_- given by $\int_M \mathrm{Tr}(\Upsilon_- \wedge *\widetilde{\mathfrak{S}})$. Thus the anti-ghost for the $F^{0,2}$ bit of section belongs to $\Omega^{2,0}(M, End(E))$, the anti-ghost for the mixed part belongs to $\Omega^{1,1}(M, End(E))$ and the anti-ghost for $(\varphi^{1,0} \wedge \varphi^{1,0})$ belongs to $\Omega^{0,2}(M, End(E))$.

Associated with the holomorphic section $F^{0,2}$ over the base space $\mathcal{A}$ of $T^*\mathcal{A}$ we have Fermi multiplet $(\chi_-^{2,0}, H^{2,0}) \in \Omega^{2,0}(M, End(E))$ and anti-Fermi multiplet $(\overline{\chi}_-^{0,2}, H^{0,2})$,

$$\begin{aligned} s_+\chi_-^{2,0} &= -H^{2,0}, & s_+H^{2,0} &= 0,\\ \overline{s}_+\chi_-^{2,0} &= 0, & \overline{s}_+H^{2,0} &= -i[\phi_{++}, \chi_-^{2,0}],\\ s_+\overline{\chi}_-^{0,2} &= 0, & s_+H^{0,2} &= -i[\phi_{++}, \overline{\chi}_-^{0,2}],\\ \overline{s}_+\chi_-^{0,2} &= -H^{0,2}, & \overline{s}_+H^{0,2} &= 0. \end{aligned} \tag{7.2.2}$$

Associated with the mixed component of holomorphic section $\overline{\partial}_A\varphi^{1,0}$ over $T^*\mathcal{A}$ we have Fermi multiplets $(\chi_-^{1,1}, H^{1,1}) \in \Omega^{1,1}(M, End(E))$ and their anti-Fermi partners $(\overline{\chi}_-^{1,1}, \overline{H}^{1,1})$,

$$\begin{aligned} s_+\chi_-^{1,1} &= -H^{1,1}, & s_+H^{1,1} &= 0,\\ \overline{s}_+\chi_-^{1,1} &= 0, & \overline{s}_+H^{1,1} &= -i[\phi_{++}, \chi_-^{1,1}],\\ s_+\overline{\chi}_-^{1,1} &= 0, & s_+\overline{H}^{1,1} &= -i[\phi_{++}, \overline{\chi}_-^{1,1}],\\ \overline{s}_+\overline{\chi}_-^{1,1} &= -\overline{H}^{1,1}, & \overline{s}_+\overline{H}^{1,1} &= 0. \end{aligned} \tag{7.2.3}$$

Associated with the holomorphic section $\varphi^{1,0} \wedge \varphi^{1,0}$ over the fiber space of $T^*\mathcal{A}$ we have Fermi multiplet $(\eta_-^{0,2}, K^{0,2}) \in \Omega^{2,0}(M, End(E))$ and their anti-Fermi partner $(\overline{\eta}^{2,0}, H^{2,0})$

$$\begin{aligned} s_+\eta_-^{0,2} &= -K^{2,0}, & s_+K^{2,0} &= 0,\\ \overline{s}_+\eta_-^{0,2} &= 0, & \overline{s}_+K^{2,0} &= -i[\phi_{++}, \eta_-^{0,2}],\\ s_+\overline{\eta}_-^{2,0} &= 0, & \overline{s}_+K^{0,2} &= -i[\phi_{++}, \overline{\eta}_-^{2,0}],\\ \overline{s}_+\overline{\eta}_-^{2,0} &= -K^{0,2}, & s_+K^{0,2} &= 0. \end{aligned} \tag{7.2.4}$$

Now we consider the following $N_c = (2,0)$ supersymmetric action functional

$$\begin{aligned} S =& \frac{s_+\overline{s}_+}{4\pi^2}\int_M \mathrm{Tr}\Big(\phi_{--}\Big(F\wedge\omega + [\varphi^{1,0}, \varphi^{0,1}]\wedge\omega + \frac{i\zeta}{2}\omega^2 I_E\Big) + \eta_- * \overline{\eta}_-\Big)\\ &+ \frac{s_+\overline{s}_+}{4\pi^2}\int_M \mathrm{Tr}\left(\chi_-^{2,0}\wedge *\overline{\chi}_-^{0,2} + \chi_-^{1,1}\wedge *\overline{\chi}_-^{1,1} + \eta_-^{0,2}\wedge *\overline{\eta}_-^{2,0}\right)\\ &+ \frac{is_+}{4\pi^2}\int_M \mathrm{Tr}\Big(\chi_-^{2,0}\wedge *F^{0,2} + \chi_-^{1,1}\wedge *\overline{\partial}_A\varphi^{1,0} + \eta_-^{0,2}\wedge *(\varphi^{1,0}\wedge\varphi^{1,0})\Big)\\ &+ \frac{i\overline{s}_+}{4\pi^2}\int_M \mathrm{Tr}\Big(\overline{\chi}_-^{0,2}\wedge *F^{2,0} + \overline{\chi}_-^{1,1}\wedge *\partial_A\varphi^{0,1} + \overline{\eta}_-^{2,0}\wedge *(\varphi^{0,1}\wedge\varphi^{0,1})\Big), \end{aligned} \tag{7.2.5}$$

We set $\zeta = 0$ for simplicity by restricting to the case with $c_1(E) = 0$. By expanding the above and integrating out the auxiliary fields we see that the path integral is localized to the moduli space defined by the following equations

$$\begin{aligned} F^{0,2} &= 0, \\ \varphi^{1,0} \wedge \varphi^{1,0} &= 0, \\ \overline{\partial}_A \varphi^{1,0} &= 0, \\ i\Lambda(F + [\varphi^{1,0}, \varphi^{0,1}]) - \zeta I_E &= 0. \end{aligned} \tag{7.2.6}$$

The first three equations above are from $\widetilde{\mathfrak{S}} = 0$ and the last equation is from the total momentum map $\mu_{\mathbb{R}}$ (7.1.13). The Higgs bundle $(\overline{\partial}_A, \varphi^{1,0})$ of Simpson [121][122] is defined by the first three equations in (7.2.6), which can be regarded as integrability $(\boldsymbol{D}'')^2 = 0$ of the extended half "connection" $\boldsymbol{D}'' = \overline{\partial}_A + \varphi^{1,0}$. There is notion of semi-stable Higgs bundle and a theorem analogous to Donaldson-Uhlenbeck-Yau such that every semi-stable Higgs bundle $(E, \varphi^{1,0})$ has an Einstein-Hermitian metric;

$$i\Lambda(F + [\varphi^{1,0}, \varphi^{0,1}]) - \zeta I_E = 0. \tag{7.2.7}$$

Furthermore the extended connection is flat, i.e., $\boldsymbol{D}' \circ \boldsymbol{D}'' + \boldsymbol{D}'' \circ \boldsymbol{D}' = 0$, if and only if $c_1(E, \varphi^{1,0}) = c_2(E, \varphi^{1,0}) = 0$. Thus the path integral of our model is localized to the moduli space of semi-stable Higgs bundles. We also have other bosonic localization equations, as usual

$$\begin{aligned} d_A \phi_{++} &= 0, \\ [\phi_{++}, B] &= 0, \\ [\phi_{++}, \phi_{--}] &= 0. \end{aligned} \tag{7.2.8}$$

If the connections are irreducible we have $\phi_{\pm\pm} = 0$ and $\mathcal{G}$ acts freely on the solution space of (7.2.6). The resulting moduli space is then isomorphic to the moduli space of stable Higgs bundles. We denote the moduli space of semi-stable Higgs bundle by $\mathcal{N}$. Note that the moduli space $\mathcal{N}$ contains the moduli space $\mathcal{M}$ of semi-stable bundles, equivalently the moduli space of EH or anti-self-dual connections on a Kähler surface M.

From now on we set $\zeta = 0$ for simplicity.

7.2.1 Comparison with Donaldson-Witten Theory

At this point it is useful to compare with Donaldson-Witten theory. The path integral of Donaldson-Witten theory is localized to the moduli space $\mathcal{M}$ of anti-self-dual connections defined by

$$\begin{aligned} (\overline{\partial}_A)^2 &= 0, \\ \Lambda(\partial_A \circ \overline{\partial}_A + \overline{\partial}_A \circ \partial_A) &= 0. \end{aligned} \tag{7.2.9}$$

Define $\boldsymbol{D}'' = \overline{\partial}_A + \varphi^{1,0}$ and $\boldsymbol{D}' = \partial_A + \varphi^{0,1}$. Our localization equations (7.2.6) can be written as

$$\begin{aligned} (\boldsymbol{D}'')^2 &= 0, \\ \Lambda(\boldsymbol{D}' \circ \boldsymbol{D}'' + \boldsymbol{D}'' \circ \boldsymbol{D}') &= 0. \end{aligned} \tag{7.2.10}$$

Similarly we can combine the superpartners of $A^{0,1}$ and $\varphi^{1,0}$, and the anti-ghosts $(\chi_-^{2,0}, \chi_-^{1,1}, \eta_-^{0,2})$. To see this let us define extended fields

$$\begin{aligned}
\boldsymbol{A}^{(0,1)} &:= A^{0,1} + \varphi^{1,0}, & \boldsymbol{A}^{(1,0)} &:= A^{1,0} + \varphi^{0,1},\\
\Psi_+^{(0,1)} &:= \psi_+^{0,1} + \lambda_+^{1,0}, & \overline{\Psi}_+^{(1,0)} &:= \overline{\psi}_+^{1,0} + \overline{\lambda}_+^{0,1},\\
\Upsilon_-^{(2,0)} &:= \chi_-^{2,0} + \chi_-^{1,1} + \eta_-^{0,2}, & \overline{\Upsilon}_-^{(0,2)} &:= \overline{\chi}_-^{0,2} + \overline{\chi}_-^{1,1} + \overline{\eta}_-^{2,0},\\
\boldsymbol{H}^{(2,0)} &:= H^{2,0} + H^{1,1} + K^{0,2}, & \boldsymbol{H}^{(0,2)} &:= H^{0,2} + \overline{H}^{1,1} + K^{2,0},
\end{aligned} \tag{7.2.11}$$

where the superscript of the extended fields represent a graded form degree on M. That is we exchange holomorphic and anti-holomorphic differential form degree on M of fields associated with $\varphi^{1,0}$ and $\varphi^{0,1}$. For example the extended anti-ghost $\Upsilon_-^{(2,0)}$ is associated with the total holomorphic section $\widetilde{\mathfrak{S}} := \boldsymbol{F}^{(0,2)} := F^{0,2} + \overline{\partial}_A\varphi^{1,0} + \varphi^{1,0}\wedge\varphi^{1,0}$ of $\widetilde{\mathbb{E}} \to T^*\mathcal{A}$ by the pairing $\int_M \mathrm{Tr}\left(\Upsilon_-^{(2,0)} \wedge *\boldsymbol{F}^{(0,2)}\right)$. Note that the combinations (7.2.11) preserve the ghost numbers

$$\begin{aligned}
\Psi_+^{(0,1)} &: (+1,0), & \overline{\Psi}_+^{(1,0)} &: (0,+1),\\
\Upsilon_-^{(2,0)} &: (-1,0), & \overline{\Upsilon}_-^{(0,2)} &: (0,-1).
\end{aligned} \tag{7.2.12}$$

The supersymmetry transformation laws for the coordinate fields of $T^*\mathcal{A}$ are, combining (7.1.10) and (7.1.11),

$$\begin{aligned}
\boldsymbol{s}_+\boldsymbol{A}^{(0,1)} &= i\Psi_+^{(0,1)}, & \boldsymbol{s}_+\Psi_+^{(0,1)} &= 0,\\
\overline{\boldsymbol{s}}_+\boldsymbol{A}^{(0,1)} &= 0, & \overline{\boldsymbol{s}}_+\Psi_+^{(0,1)} &= -\boldsymbol{D}''\phi_{++},\\
\boldsymbol{s}_+\boldsymbol{A}^{(1,0)} &= 0, & \boldsymbol{s}_+\overline{\Psi}_+^{(1,0)} &= -\boldsymbol{D}'\phi_{++},\\
\overline{\boldsymbol{s}}_+\boldsymbol{A}^{(1,0)} &= i\overline{\Psi}_+^{(1,0)}, & \overline{\boldsymbol{s}}_+\overline{\Psi}_+^{(1,0)} &= 0.
\end{aligned} \tag{7.2.13}$$

The supersymmetry transformation laws for the Fermi multiplet $(\Upsilon_-^{(2,0)}, \boldsymbol{H}^{(2,0)})$ are, by combining (7.2.2), (7.2.3), and (7.2.4) together,

$$\begin{aligned}
\boldsymbol{s}_+\Upsilon_-^{(2,0)} &= -\boldsymbol{H}^{(2,0)}, & \boldsymbol{s}_+\boldsymbol{H}^{(2,0)} &= 0,\\
\overline{\boldsymbol{s}}_+\Upsilon_-^{(2,0)} &= 0, & \overline{\boldsymbol{s}}_+\boldsymbol{H}^{(2,0)} &= -i[\phi_{++}, \Upsilon_-^{(2,0)}],\\
\boldsymbol{s}_+\overline{\Upsilon}_-^{(0,2)} &= 0, & \boldsymbol{s}_+\boldsymbol{H}^{(0,2)} &= -i[\phi_{++}, \overline{\Upsilon}_-^{(0,2)}],\\
\overline{\boldsymbol{s}}_+\overline{\Upsilon}_-^{(0,2)} &= -\boldsymbol{H}^{(0,2)}, & \overline{\boldsymbol{s}}_+\boldsymbol{H}^{(0,2)} &= 0.
\end{aligned} \tag{7.2.14}$$

We have the usual $N_c = (2,0)$ gauge multiplet associated with the unitary gauge transformation. For convenience we rewrite down supersymmetry the transformation laws

$$\begin{aligned}
& & \boldsymbol{s}_+\eta_- &= 0, & &\\
\boldsymbol{s}_+\phi_{--} &= i\eta_-, & \overline{\boldsymbol{s}}_+\eta_- &= +iH_0 + \frac{1}{2}[\phi_{++}, \phi_{--}], & \boldsymbol{s}_+\phi_{++} &= 0,\\
\overline{\boldsymbol{s}}_+\phi_{--} &= i\overline{\eta}_-, & \boldsymbol{s}_+\overline{\eta}_- &= -iH_0 + \frac{1}{2}[\phi_{++}, \phi_{--}], & \overline{\boldsymbol{s}}_+\phi_{++} &= 0.\\
& & \overline{\boldsymbol{s}}_+\overline{\eta}_- &= 0, & &
\end{aligned} \tag{7.2.15}$$

Now the action functional S in (7.2.5) can be written as

$$\begin{aligned} S =& \frac{\boldsymbol{s}_+\overline{\boldsymbol{s}}_+}{4\pi^2}\int_M \mathrm{Tr}\Big(\phi_{--}\boldsymbol{F}^{(1,1)}\Big)\wedge\omega \\ &+ \frac{\boldsymbol{s}_+\overline{\boldsymbol{s}}_+}{4\pi^2}\int_M \mathrm{Tr}\Big(\eta_-\wedge *\overline{\eta}_- + \Upsilon_-^{(2,0)}\wedge *\overline{\Upsilon}_-^{(0,2)}\Big) \\ &+ \frac{i\boldsymbol{s}_+}{4\pi^2}\int_M \mathrm{Tr}\Big(\Upsilon_-^{(2,0)}\wedge *\boldsymbol{F}^{(0,2)}\Big) + \frac{i\overline{\boldsymbol{s}}_+}{4\pi^2}\int_M \mathrm{Tr}\Big(\overline{\Upsilon}_-^{(0,2)}\wedge *\boldsymbol{F}^{(2,0)}\Big), \end{aligned} \tag{7.2.16}$$

where

$$\mu^T_{\mathbb{R}} = \frac{1}{4\pi^2}\boldsymbol{F}^{(1,1)}\wedge\omega. \tag{7.2.17}$$

The above action functional has exactly same form as Donaldson-Witten theory on Kähler 2-folds, see (4.1.15). We remark that the Kähler identities (4.1.19) are important technical tools in analyzing Donaldson-Witten theory on Kähler manifolds. Simpson showed that one also has the Kähler identities for Higgs bundles,

$$(\boldsymbol{D}')^* = i[\Lambda, \boldsymbol{D}''], \qquad (\boldsymbol{D}'')^* = -i[\Lambda, \boldsymbol{D}'], \tag{7.2.18}$$

We will work with the above shorthand notations.

7.2.2 Path Integrals

The explicit form of the total action functional S' after integrating out all the auxiliary fields from S is given by

$$\begin{aligned} S' =& \frac{1}{4\pi^2}\int_M \mathrm{Tr}\Big(-\frac{1}{2}\boldsymbol{F}^+\wedge *\boldsymbol{F}^+ - \boldsymbol{D}\phi_{++} * \boldsymbol{D}\phi_{--} + \frac{1}{4}[\phi_{++},\phi_{--}] * [\phi_{++},\phi_{--}] \\ &+ \phi_{--} * \Lambda[\Psi_+^{(0,1)}, \overline{\Psi}_+^{(1,0)}] + i\Upsilon_-^{2,0}\wedge *[\phi_{++}, \overline{\Upsilon}_-^{(0,2)}] + i[\phi_{++},\eta_-] * \overline{\eta}_- \\ &- i\boldsymbol{D}'\overline{\eta}_-\wedge *\Psi_+^{(0,1)} - i\boldsymbol{D}''\eta_-\wedge *\overline{\Psi}_+^{(1,0)} - \Upsilon_-^{2,0}\wedge *\boldsymbol{D}''\Psi_+^{(0,1)} \\ &- \overline{\Upsilon}_-^{(0,2)}\wedge *\boldsymbol{D}'\overline{\Psi}_+^{(1,0)}\Big), \end{aligned} \tag{7.2.19}$$

where $\boldsymbol{D} = \boldsymbol{D}' + \boldsymbol{D}''$ and we we used the extended Kähler identities (7.2.18). We also used notation $\boldsymbol{F}^+$, which is given by

$$\boldsymbol{F}^+ = \boldsymbol{F}^{(2,0)} + \frac{1}{2}(\Lambda\boldsymbol{F}^{(1,1)})\omega + \boldsymbol{F}^{(0,2)}, \tag{7.2.20}$$

so that $\boldsymbol{F}^+|_{\varphi^{1,0}=\varphi^{0,1}=0} = F^+$, where F^+ denotes the self-dual part of the odinary curvature two-form F. Note that $\boldsymbol{F}^+$ also contains anti-self-dual two-form part as well.

Now we examine the equations for fermionic zero-modes. The equation of motions for fermions are, modulo infinitesimal gauge transformations, are

$$\begin{aligned} i\boldsymbol{D}''\overline{\eta}_- + (\boldsymbol{D}'')^*\overline{\Upsilon}_-^{(0,2)} &= 0, \\ (\boldsymbol{D}'')^*\Psi_+^{(0,1)} &= 0, \\ \boldsymbol{D}''\Psi_+^{(0,1)} &= 0. \end{aligned} \quad (7.2.21)$$

Using one of the bosonic localization equation $(\boldsymbol{D}'')^2 = 0$, we find that the fermionic zero-modes are governed by the following equations

$$\boldsymbol{D}''\overline{\eta}_- = 0, \qquad \begin{aligned} (\boldsymbol{D}'')^*\Psi_+^{(0,1)} &= 0, \\ \boldsymbol{D}''\Psi_+^{(0,1)} &= 0, \end{aligned} \qquad (\boldsymbol{D}'')^*\overline{\Upsilon}_-^{(0,2)} = 0. \quad (7.2.22)$$

Thus the fermionic zero-modes are elements of cohomology group of the following extended Dolbeault complex

$$0 \to \boldsymbol{S}^{(0,0)} \xrightarrow{\boldsymbol{D}''} \boldsymbol{S}^{(0,1)} \xrightarrow{\boldsymbol{D}''} \boldsymbol{S}^{(0,2)} \to 0, \quad (7.2.23)$$

where

$$\boldsymbol{S}^{(0,p)} = \bigoplus_{r+s=p} \Omega^{0,r}(M, \wedge^s(T_M^{*1,0}) \otimes End(E)). \quad (7.2.24)$$

The net ghost number violation in the path integral measure due to fermionic zero-modes is $(\widetilde{\Delta}, \widetilde{\Delta})$ where $\widetilde{\Delta}$ is the negative of the index of the above complex. Almost all of the standard procedure in Donaldson-Witten theory can be repeated here. For example observables are $\mathcal{G}$-equivariant closed differential forms, after the parity change, on the space $T^*\mathcal{A}$. As for a canonical observable we have the $\mathcal{G}$-equivariant Kähler form, after the parity change, on $T^*\mathcal{A}$;

$$\widehat{\varpi}_T^{\mathcal{G}} = \frac{i}{4\pi^2}\int_M \mathrm{Tr}\Big(\phi_{++}\boldsymbol{F}^{(1,1)}\Big)\wedge\omega + \frac{1}{4\pi^2}\int_M \mathrm{Tr}\Big(\Psi_+^{(0,1)}\wedge\overline{\Psi}_+^{(1,0)}\Big)\wedge\omega, \quad (7.2.25)$$

The correlation functions of supersymmetric observables are the path integral representations of a generalized Donaldson-Witten invariant.

We note that the fundamental group of four-manifold does not seem to play any essential roles in the original Donaldson-Witten theory. On the other hand the most crucial application of Simpson's Higgs bundle is on the non-Abelian Hodge theory associated with the representation variety $\pi_1(M) \to GL(r, \mathbb{C})$ of the fundamental group. For this purpose let us consider the case the $c_1(E) = c_2(E) = 0$.[1] It is known that there is a one-to-one correspondence between irreducible representations of $\pi_1(M)$ and stable Higgs bundles with vanishing Chern classes, see [122]. In this situation Donaldson-Witten invariants concern only the unitary irreducible representation variety. An important property of the moduli space of stable Higgs bundles is the existence of a $\mathbb{C}^*$ action

[1] It is not obvious if the moduli space of stable Higgs bundles has a hyper-Kähler structure. For the flat case the existence of hyper-Kähler structure has proved by Fujiki [123].

$(E, \varphi^{1,0}) \to (E, t\varphi^{1,0})$. Simpson showed that the fixed points of $\mathbb{C}^*$ action correspond to complex variations of Hodge structures. It also implies that any other representation of $\pi_1(M)$ can be deformed to a complex variation of Hodge structures. Among the fixed points the trivial complex variation of Hodge structures corresponds to unitary irreducible representations. A useful viewpoint of the $\mathbb{C}^*$ action is to regard it as Hodge decomposition of non-Abelian cohomology. Then a unitary representation is some kind of zero-form. The above results also imply that the path integral of our model for $c_1(E) = c_2(E) = 0$ can be written as the sum of contributions from every complex variation of Hodge structures. Thus it is natural to hope that our new invariants may have information beyond the Donaldson-Witten and Seiberg-Witten invariants for non-simply connected Kähler 2-folds. Of course we do not need to restrict our attention to the flat case.

The moduli space of stable Higgs bundles have many beautiful properties and applications. One of the properties is that the rank r stable Higgs sheaves on M can be identified with stable sheaves on the cotangent bundle T^*M which are supported on Lagrangian subvarieties of T^*M which are finite degree r branched coverings of M [124][125]. The above property may be relevant to generalized mirror symmetry on Calabi-Yau 4-folds [92]. If we consider the complex 2-torus, T^4, its cotangent bundle may be regarded as local model for T^4-fibered Calabi-Yau 4-folds. Then the moduli space of stable rank r Higgs sheaves may be viewed as parameterizing r $D4$-branes wrapped on Lagrangian cycles of Calabi-Yau 4-folds. Of course the above picture is too naive but somewhat suggestive. Here we will not be able to penetrate many of the applications and properties of Higgs bundles. We will use its S^1 symmetry to have an anatomy of our invariants.

7.2.3 Flows to Donaldson-Witten theory

In the lay men's terms Donaldson-Witten invariant is simply the symplectic volume of the moduli space $\mathcal{M}$ of stable bundles on M. Similarly, the invariants defined by the correlation function $\langle \exp(\widehat{\omega}_T^{\mathcal{G}}) \rangle$ is the symplectic volume of the moduli space $\mathcal{N}$ of stable Higgs bundles. One of most important properties of the moduli space $\mathcal{N}$ is that it has a symmetry under a S^1-action, which can be extended to a $\mathbb{C}^*$-action. The beautiful fact is that the $\mathbb{C}^*$ action is a very special one, related with a certain variation of Hodge structures[2]

First we note that our localization equations in (7.2.6) are more than the equations (7.2.10). We may replace $\boldsymbol{D}''$ by a family of extended derivatives by introducing a spectral parameter t,

$$\begin{aligned} \boldsymbol{D}'' &= \bar{\partial}_A + t\varphi^{1,0}, \\ \boldsymbol{D}' &= \partial_A + \bar{t}\varphi^{0,1}. \end{aligned} \tag{7.2.26}$$

[2]This notion will be relevant to the case when the Higgs bundle is flat. Then $\boldsymbol{D} = \boldsymbol{D}' + \boldsymbol{D}''$ can be identified with the Gauss-Manin connections of the associated local system. Then our localization equations are familiar tt^*-equations in special geometry [126]. In fact for any complex, not necessarily integral, variation of Hodge structures there is a corresponding flat Higgs bundle.

Then our localization equations in (7.2.6) imply that

$$\begin{aligned} (D'')^2 &= 0, \\ \Lambda(D' \circ D'' + D'' \circ D') &= 0, \end{aligned} \tag{7.2.27}$$

for any t with $t\bar{t} = 1$. Similarly we replace the extended fields defined in (7.2.11) as follows

$$\begin{aligned} A^{(0,1)} &:= A^{0,1} + t\varphi^{1,0}, & A^{(1,0)} &:= A^{1,0} + \bar{t}\varphi^{0,1}, \\ \Psi_+^{(0,1)} &:= \psi_+^{0,1} + t\lambda_+^{1,0}, & \overline{\Psi}_+^{(1,0)} &:= \overline{\psi}_+^{1,0} + \bar{t}\overline{\lambda}_+^{0,1}, \\ \Upsilon_-^{(2,0)} &:= \chi_+^{2,0} + \bar{t}\chi_-^{1,1} + \bar{t}^2\eta_-^{0,2}, & \overline{\Upsilon}_-^{(0,2)} &:= \overline{\chi}_-^{0,2} + t\overline{\chi}_-^{1,1} + t^2\overline{\eta}_-^{2,0}, \\ H^{(2,0)} &:= H^{2,0} + \bar{t}H^{1,1} + \bar{t}^2 K^{0,2}, & H^{(0,2)} &:= H^{0,2} + t\overline{H}^{1,1} + t^2 K^{2,0}. \end{aligned} \tag{7.2.28}$$

Then our action functional S in (7.2.5) or (7.2.16) is invariant for any t with $t\bar{t} = 1$.

We will show shortly that the S^1 action can be extended to a $\mathbb{C}^*$ action by "gauging" the $U(1) = S^1$ symmetry and scaling the unit $U(1)$ charge. Such a procedure is equivalent to giving physical bare mass m to the $U(1)$ charged fields. Thus one can consider an imaginary $\mathbb{CP}^1$ where the $\mathbb{C}^*$ action covers the natural $\mathbb{C}^*$ action on $\mathbb{CP}^1$ with limit points $(t = 0, t = \infty)$. Now we can identify the two limit points in $\mathbb{CP}^1$ with $(m = \infty, m = 0)$. Thus we can interpret the absolute flow generated by the $\mathbb{C}^*$ action as a renormalization group flow from the past or unbroken phase $m = 0$ to the future (present) or broken phase $m \to \infty$. This is not just a mere fantasy since we indeed have a twistor space constructed from the function space of fields namely the total space $T^*\mathcal{A}$ of the cotangent bundle over the space of all gauge fields. Our field space has a hyper-Kähler structure preserved by the $\mathcal{G}$ as well as by the S^1 symmetry acting on the fiber of $T^*\mathcal{A}$. Such a S^1 action can be extended to a $\mathbb{C}^*$ action and then cover the $\mathbb{C}^*$ action of $\mathbb{CP}^1$ in the twistor space $T^*\mathcal{A} \times \mathbb{CP}^1$. Furthermore the Hamiltonian of the S^1-action on the field space is precisely the physical bare mass of the bosonic fields, whose field space are the fiber of $T^*\mathcal{A}$ on space-time M. Now by taking the $m \to \infty$ limit the dominant contributions to path integral come from the critical points of the Hamiltonian, equivalently from the fixed points of S^1-action. Similarly in the $t \to 0$ limit any point in the field space flows to a certain fixed point of the S^1-action. In the trivial fixed point $\varphi^{1,0} = 0$ we recover original Donaldson-Witten theory. As a global supersymmetric field theory on M certain path integral of our model will be localized to a finite dimensional subspace $\mathcal{N}$ of the hyper-Kähler quotient of $T^*\mathcal{A}$ by $\mathcal{G}$. The above argument is valid regardless whether $\mathcal{N}$ preserves the hyper-Kähler structure or not.

We may ask an interesting physical question. Donaldson-Witten theory is the twisted $N = 2$ supersymmetric Yang-Mills theory. On a manifold with trivial canonical line bundle twisting does nothing and we have space-time supersymmetric Yang-Mills theory. Then where shall we place our model? Our proposal is that it may describe a certain unbroken phase of bigger symmetry which is connected to the physical super-Yang-Mills theory by renormalization group flows, and the physical theory lives in one of the fixed points.

Now we perturb our model by "gauging" the $U(1)$ symmetry. For this we modify the supersymmetry transformation laws according to the following anti-commutations relations

$$s_+^2 = 0, \qquad \{s_+, \bar{s}_+\} = -i\phi_{++}^a \mathcal{L}_a - im\mathcal{L}_{S^1}, \qquad \bar{s}_+^2 = 0. \tag{7.2.29}$$

We define a new action functional $S(m)$ by the same formula as S in (7.2.16) but with the modified transformation laws. Then we define a family of $N_c = (2,0)$ models parameterized by m and $\overline{m}$ with the following action functional

$$S(m, \overline{m}) = S(m) + i\overline{m} s_+ \bar{s}_+ \mathcal{K}(\boldsymbol{D}), \tag{7.2.30}$$

where $\mathcal{K}(\boldsymbol{D})$ is the Kähler potential of $T^*\mathcal{A}$ given by (7.1.7). Then the action functional contains bare mass terms for all the charged fields under the $U(1)$, except for auxiliary fields. The relevant terms in the action functional looks like

$$\begin{aligned} S(m,\overline{m}) = S & \\ & - \frac{\overline{m}}{4\pi^2}\int_M \mathrm{Tr}\left(i\phi_{++}(F + [\varphi^{1,0}, \varphi^{0,1}]) + \psi_+^{0,1} \wedge \overline{\psi}_+^{-1,0} + \lambda_+^{1,0} \wedge \overline{\lambda}_+^{-0,1}\right) \wedge \omega \\ & + \frac{im\overline{m}}{4\pi^2}\int_M \mathrm{Tr}\left(\varphi^{1,0} \wedge \varphi^{0,1}\right) \wedge \omega + \ldots . \end{aligned} \tag{7.2.31}$$

In the above the $m\overline{m}$ dependent term is the Hamiltonian of the S^1-action on $T^*\mathcal{A}$. The term in the second line is the equivariant Kähler form $\widehat{\varpi}_T^{\mathcal{G}}$ of $T^*\mathcal{A}$. Thus $\widehat{\varpi}^{\mathcal{G}} := \widetilde{\varpi}_T^{\mathcal{G}}|_{\mathcal{A}}$ is an observable of Donaldson-Witten theory which will descend to the Kähler form of moduli space $\mathcal{M}$ of anti-self-dual connections.

Now by taking the $m \to \infty$ limit we see that the dominant contributions to the path integral come from the critical points of the Hamiltonian of the S^1-action. Such critical points are identical to the fixed points of the S^1-action. As usual we always have trivial fixed points given by $\varphi^{1,0} = 0$ and the fixed point locus is the moduli space $\mathcal{M}$ of anti-self-dual connections. Thus the contribution from the trivial fixed points to the partition function of the model with the action functional $S(m, \overline{m})$ is given by a generating functional $\langle \exp(\widehat{\omega}^{\mathcal{G}}) \rangle_{DW}$ of Donaldson-Witten theory weighted by one loop contributions from the degrees of freedom normal to $\mathcal{M}$ in $\mathcal{N}$. We also note that the value of the Hamiltonian of the S^1-action at the trivial fixed point is zero. There are other non-trivial fixed points $\varphi^{1,0} \neq 0$ if the S^1-action can be undone by the gauge transformations,

$$g\varphi^{1,0}g^{-1} = t\varphi^{1,0}, \tag{7.2.32}$$

where $g \in \mathcal{G}$ and $t \in U(1)$.

7.3 Generalized Vafa-Witten Theory

In this section we apply the construction in Sect. 3.4 to embed the previous $N_c = (2,0)$ model on a Kähler surface M to a $N_c = (2,2)$ model. The resulting model generalizes Vafa-Witten theory and compute Euler characteristic of the moduli space of stable Higgs bundles together with extra (!) contributions. The construction of the model will be straightforward exactly as in Sect. 4.2 for Vafa-Witten theory on Kähler surface. Then we define $\mathbb{C}^*$ family of $N_c = (2,2)$ models which has various interesting limit.

We recall the basic setting for the previous $N_c = (2,0)$ model. We considered the total space $T^*\mathcal{A}$ of the cotangent bundle of the space of all connections of a rank r Hermitian vector bundle $E \to M$ over a Kähler surface M. As for the holomorphic coordinate fields on $T^*\mathcal{A}$ we have the extended connection $\boldsymbol{A}^{0,1}$ with superpartner $\Psi_+^{0,1}$. We also considered an infinite dimensional $\mathcal{G}$-equivariant holomorphic vector bundle $\widetilde{\mathbb{E}} \to T^*\mathcal{A}$ with holomorphic section $\widetilde{\mathfrak{S}}(\boldsymbol{D}'') = (\boldsymbol{D}'')^2 := \boldsymbol{F}^{(0,2)}$ and associated anti-ghost multiplet $(\Upsilon_-^{(2,0)}, \boldsymbol{H}^{(2,0)})$.

The basic idea behind the extension to a $N_c = (2,2)$ model is that one can regard the total space of the holomorphic bundle $\widetilde{\mathbb{E}} \to T^*\mathcal{A}$ as the target space of a $N_c = (2,2)$ model. Then we have to supply local holomorphic coordinate fields for fiber space of $\widetilde{\mathbb{E}} \to T^*\mathcal{A}$. Thus we introduce adjoint-valued bosonic spectral fields $\boldsymbol{B}^{(2,0)}$ and its superpartner $\Upsilon_+^{(2,0)}$. Now the former holomorphic section $\widetilde{\mathfrak{S}} = \boldsymbol{F}^{(0,2)}(\boldsymbol{D}'')$ of the bundle $\widetilde{\mathbb{E}} \to T^*\mathcal{A}$ corresponds to a holomorphic vector field on the target space $\widetilde{\mathbb{E}}$ but being supported only on $T^*\mathcal{A}$. Thus the $\mathcal{G}$-equivariant holomorphic vector $\mathfrak{S}(\boldsymbol{D}'')$ should be extended over the whole space $\widetilde{\mathbb{E}}$. Furthermore $N_c = (2,2)$ supersymmetry demands that a such holomorphic vector should be the gradient vector of a non-degenerated $\mathcal{G}$-invariant holomorphic function $\mathcal{W}$, i.e, $\overline{\boldsymbol{s}}_+\mathcal{W} = 0$, on the target space $\widetilde{\mathbb{E}}$.

Now demanding $N_c = (2,2)$ supersymmetry will take care of everything. From the $N_c = (2,0)$ holomorphic multiplets $(\boldsymbol{A}^{(0,1)}, \Psi_+^{(0,1)})$ we build up the following chiral multiplets, i.e., $\overline{\boldsymbol{s}}_\pm \boldsymbol{A}^{(0,1)} = 0$

$$\begin{array}{ccccc} \Psi_-^{(0,1)} & \xleftarrow{s_-} & \boldsymbol{A}^{(0,1)} & \xrightarrow{s_+} & \Psi_+^{(0,1)} \\ & {\scriptstyle s_+}\searrow & & \swarrow{\scriptstyle s_-} & \\ & & \boldsymbol{H}^{(0,1)} & & \end{array} \quad . \tag{7.3.1}$$

From the $N_c = (2,0)$ Fermi multiplets $(\Upsilon_-^{(2,0)}, \boldsymbol{H}^{(2,0)})$ we build up another set of chiral multiplets, i.e., $\overline{\boldsymbol{s}}_\pm \boldsymbol{B}^{(2,0)} = 0$

$$\begin{array}{ccccc} \Upsilon_-^{(2,0)} & \xleftarrow{s_-} & \boldsymbol{B}^{(2,0)} & \xrightarrow{s_+} & \Upsilon_+^{(2,0)} \\ & {\scriptstyle s_+}\searrow & & \swarrow{\scriptstyle s_-} & \\ & & \boldsymbol{H}^{(2,0)} & & \end{array} \quad . \tag{7.3.2}$$

Form the $N_c = (2,0)$ gauge multiplet $(\phi_{--}, \eta_-, \overline{\eta}_-, H_0, \phi_{++})$ we build up a

$N_c = (2,2)$ gauge multiplet

$$\begin{array}{ccccc} \overline{\sigma} & \xrightarrow{s_+} & \eta_+ & \xleftarrow{s_-} & \phi_{++} \\ \big\downarrow{\scriptstyle \overline{s}_-} & & \big\downarrow{\scriptstyle \overline{s}_-} & & \big\downarrow{\scriptstyle \overline{s}_-} \\ \overline{\eta}_- & \xrightarrow{s_+} & H_0 & \xleftarrow{s_-} & \overline{\eta}_+ \, , \\ \big\uparrow{\scriptstyle \overline{s}_+} & & \big\uparrow{\scriptstyle \overline{s}_+} & & \big\uparrow{\scriptstyle \overline{s}_+} \\ \phi_{--} & \xrightarrow{s_+} & \eta_- & \xleftarrow{s_-} & \sigma \end{array} \tag{7.3.3}$$

which are adjoint valued scalars on M.

To keep track of all the fields we write down the explicit spectral form of the extended fields. We have

$$\begin{aligned} \boldsymbol{A}^{(0,1)} &:= A^{0,1} + t\varphi^{1,0}, & \boldsymbol{A}^{(1,0)} &:= A^{1,0} + \bar{t}\varphi^{0,1}, \\ \Psi_\pm^{(0,1)} &:= \psi_\pm^{0,1} + t\lambda_\pm^{1,0}, & \overline{\Psi}_\pm^{(1,0)} &:= \overline{\psi}_\pm^{1,0} + \bar{t}\overline{\lambda}_\pm^{0,1}, \\ \boldsymbol{H}^{(1,0)} &:= H^{1,0} + tL^{1,0}, & \boldsymbol{H}^{(0,1)} &:= H^{0,1} + \bar{t}L^{0,1}, \end{aligned} \tag{7.3.4}$$

and

$$\begin{aligned} \boldsymbol{B}^{(2,0)} &:= B^{2,0} + \bar{t}B^{1,1} + \bar{t}^2 C^{0,2}, & \boldsymbol{B}^{(0,2)} &:= B^{0,2} + t\overline{B}^{1,1} + t^2 C^{2,0}, \\ \Upsilon_\pm^{(2,0)} &:= \chi_\pm^{2,0} + \bar{t}\chi_\pm^{1,1} + \bar{t}^2\eta_\pm^{0,2}, & \overline{\Upsilon}_\pm^{(0,2)} &:= \overline{\chi}_\pm^{0,2} + t\overline{\chi}_\pm^{1,1} + t^2\overline{\eta}_\pm^{2,0}, \\ \boldsymbol{H}^{(2,0)} &:= H^{2,0} + \bar{t}H^{1,1} + \bar{t}^2 K^{0,2}, & \boldsymbol{H}^{(0,2)} &:= H^{0,2} + t\overline{H}^{1,1} + t^2 K^{2,0}. \end{aligned} \tag{7.3.5}$$

Now we have the standard $N_c = (2,2)$ invariant functional

$$\begin{aligned} S = & s_+\overline{s}_+ s_-\overline{s}_- \left(\mathcal{K}(\boldsymbol{D}) + \mathcal{K}(\boldsymbol{B}^{(2,0)}, \boldsymbol{B}^{(0,2)}) - \int_M \mathrm{Tr}(\sigma * \overline{\sigma}) \right) \\ & + s_+ s_- \mathcal{W}\left(\boldsymbol{A}^{(0,1)}, \boldsymbol{B}^{(2,0)}\right) + \overline{s}_+\overline{s}_- \overline{\mathcal{W}}\left(\boldsymbol{A}^{(1,0)}, \boldsymbol{B}^{(0,2)}\right), \end{aligned} \tag{7.3.6}$$

where

$$\mathcal{K}(\boldsymbol{B}^{(2,0)}, \boldsymbol{B}^{(0,2)}) = -\frac{1}{4\pi^2}\int_M \mathrm{Tr}\left(\boldsymbol{B}^{(2,0)} \wedge *\boldsymbol{B}^{(0,2)}\right). \tag{7.3.7}$$

The holomorphic potential $\mathcal{W}$, i.e., $\overline{s}_\pm \mathcal{W} = 0$, is given as follows

$$\mathcal{W}\left(\boldsymbol{A}^{(0,1)}, \boldsymbol{B}^{(2,0)}\right) = \frac{1}{4\pi^2}\int_M \mathrm{Tr}\left(\boldsymbol{B}^{(2,0)} \wedge *\boldsymbol{F}^{(0,2)}\right). \tag{7.3.8}$$

We note that the above action functional remains invariant for any t in (7.3.4) and (7.3.5) with $t\bar{t} = 1$. We will use this S^1 symmetry to define a $\mathbb{C}^*$ family of the $N_c = (2,2)$ model.

Now, from the discussions in Sect. 3.4, we see that the path integral is localized to the zeros of the momentum map $\mu_{\mathbb{R}}$ and the critical points of the holomorphic potential $\mathcal{W}$, modulo the gauge symmetry,

$$\begin{aligned} \boldsymbol{F}^{(0,2)} &= 0, \\ \boldsymbol{D}'' * \boldsymbol{B}^{(2,0)} &= 0, \\ i\boldsymbol{F} \wedge \omega + [\boldsymbol{B}^{(2,0)}, *\boldsymbol{B}^{(0,2)}] &= 0. \end{aligned} \tag{7.3.9}$$

We also have other default localization equations

$$\begin{aligned} [\overline{\sigma}, \boldsymbol{B}^{(0,2)}] &= 0, & [\phi_{\pm\pm}, \boldsymbol{B}^{(0,2)}] &= 0, \\ [\sigma, \boldsymbol{B}^{(0,2)}] &= 0, & [\phi_{\pm\pm}, \overline{\sigma}] &= 0, \\ [\sigma, \overline{\sigma}] &= 0, & [\phi_{++}, \phi_{--}] &= 0, \\ \boldsymbol{D}\overline{\sigma} &= 0, & \boldsymbol{D}\phi_{\pm\pm} &= 0. \end{aligned} \tag{7.3.10}$$

When there are no reducible orbits in (7.3.9) we have $\sigma = \phi_{\pm\pm} = 0$, and the path integral is localized to the moduli space defined by (7.3.9). The equation (7.3.9) is a generalization of Vafa-Witten equation. We note that the equations in (7.3.9), as well as in (7.3.10), remain the same for any t in (7.3.4) and (7.3.5) with $t\bar{t} = 1$, which is a symmetry of the action functional.

The equations in (7.3.9) have another S^1 symmetry given by

$$(\boldsymbol{D}'', \boldsymbol{B}^{(2,0)}) \to (\boldsymbol{D}'', \xi \boldsymbol{B}^{(2,0)}) \tag{7.3.11}$$

with $\xi\bar{\xi} = 1$. However the above is not a symmetry of the action functional due to the holomorphic potential term (7.3.8);

$$S = \frac{\boldsymbol{s}_+\boldsymbol{s}_-}{4\pi^2}\int_M \mathrm{Tr}\left(\boldsymbol{B}^{(2,0)} \wedge *\boldsymbol{F}^{(0,2)}\right) + \ldots \tag{7.3.12}$$

The above situation is exactly same as for Vafa-Witten theory on a Kähler surface. We can use the S^1 symmetry (7.3.11) to break $N_c = (2,2)$ supersymmetry down to $N_c = (2,0)$ supersymmetry by breaking the supersymmetries generated by $\boldsymbol{s}_-$ and $\overline{\boldsymbol{s}}_-$. We expand (7.3.12) by one step to get

$$S = \frac{\boldsymbol{s}_+}{8\pi^2}\int_M \mathrm{Tr}\left(i\Upsilon_-^{(2,0)} \wedge *\boldsymbol{F}^{(0,2)} + \boldsymbol{B}^{(2,0)} \wedge *\boldsymbol{D}''\Psi_-^{(0,1)}\right) + \ldots \tag{7.3.13}$$

Then we see that

$$(\boldsymbol{D}'', \Upsilon_-^{2,0}, \boldsymbol{B}^{(2,0)}, \Psi_-^{(0,1)}) \to (\boldsymbol{D}'', \Upsilon_-^{2,0}, \xi\boldsymbol{B}^{(2,0)}, \overline{\xi}\Psi_-^{(0,1)}) \tag{7.3.14}$$

for $\xi\bar{\xi} = 1$ preserves the action functional. On the one hand the above rotation is not compatible with the supersymmetry generated by $\boldsymbol{s}_-$ since $\boldsymbol{s}_-\boldsymbol{A}^{(0,1)} = i\Psi_-^{(0,1)}$. On the other hand we can make it compatible with the $\boldsymbol{s}_+$ supersymmetry by assigning the same $U(1)$ charge to the pair $(\boldsymbol{B}^{(2,0)}, \Upsilon_+^{(2,0)})$ related by the $\boldsymbol{s}_+$ supersymmetry, etc.

In the next section we will use the above $S_t^1 \times S_\xi^1$ symmetry to define a $\mathbb{C}^* \times \mathbb{C}^*$ family of $N_c = (2,0)$ theories. The idea is that all the theories, both the original and the generalized Donaldson-Witten and Vafa-Witten theories, we discussed so far should be viewed as different semi-classical limits governed by different massless degrees of freedom of the same underlying theory.

7.3.1 A Family of $N_c = (2,2)$ Models

We begin with generalizing our $N_c = (2,2)$ model to a $\mathbb{C}^*$ family of $N_c = (2,2)$ models using the S_t^1 symmetry, whose action is given as (7.3.4) and (7.3.5). For that purpose we extend our $N_c = (2,2)$ supersymmetry by "gauging" the S^1_t symmetry;

$$\begin{aligned}
&&\{s_+, \bar{s}_+\} = -i\phi^a_{++}\mathcal{L}_a, &&\\
\{s_\pm, s_\pm\} = 0, &&\{s_+, \bar{s}_-\} = -i\sigma^a\mathcal{L}_a - im\mathcal{L}_{S^1_t}, &&\{s_+, s_-\} = 0,\\
\{\bar{s}_\pm, \bar{s}_\pm\} = 0, &&\{s_-, \bar{s}_+\} = -i\bar{\sigma}^a\mathcal{L}_a - i\overline{m}\mathcal{L}_{S^1_t}, &&\{\bar{s}_+, \bar{s}_-\} = 0,\\
&&\{s_-, \bar{s}_-\} = -i\phi^a_{--}\mathcal{L}_a, &&
\end{aligned} \tag{7.3.15}$$

where $\mathcal{L}_{S^1_t}$ denotes the Lie derivative defined by the vector field generating the S^1_t symmetry. Equivalently it is an infinitesimal $U(1)$ gauge transformation for fields with non-vanishing $U(1)$ charge as defined by (7.3.4) and (7.3.5). For convenience we write down explicit transformations for bosonic fields

$$\begin{aligned}
\boldsymbol{A}^{(0,1)} &:= A^{0,1} + t\varphi^{1,0}, & \boldsymbol{A}^{(1,0)} &:= A^{1,0} + t^{-1}\varphi^{0,1},\\
\boldsymbol{B}^{(2,0)} &:= B^{2,0} + t^{-1}B^{1,1} + t^{-2}C^{0,2}, & \boldsymbol{B}^{(0,2)} &:= B^{0,2} + t\overline{B}^{1,1} + t^2 C^{2,0}.
\end{aligned} \tag{7.3.16}$$

The new action functional $S(m,\overline{m})$ is defined by the same formula as given by (7.3.6) but with modified supersymmetry transformation laws for charged fields under the S^1_t arcodingly to (7.3.15). We write down the relevant terms depending on the bare mass

$$S(m,\overline{m}) = S + \frac{m\overline{m}}{4\pi^2}\int_M \mathrm{Tr}\left(\varphi^{1,0}\wedge *\varphi^{0,1} + B^{1,1}\wedge *\overline{B}^{1,1} + 4C^{0,2}\wedge *C^{2,0}\right) + \ldots \tag{7.3.17}$$

where the unwritten terms are supersymmetric completions including the bare mass terms of $N_c = (2,2)$ superpartners of bosonic fields charged under S^1_t. We remark that the above action functional preserves all the symmetry of the original model. We note that the bare mass terms written above are exactly the Hamiltonian of the S^1_t action on the space of all bosonic fields. There are two ways of examining the above action functional. One may take the $|m| \to \infty$ limit. Then the dominant contributions to the path integral come from the critical points of the Hamiltonian of the S^1_t action. Such critical points are identical to the fixed points of S^1_t action, equivalently the $\mathbb{C}^*$ action. However this viewpoint is rather limited, as it mainly concerns the moduli space defined by the equations in (7.3.9). We should not forget that such a moduli space is only a subsystem, and usually does not form a closed system.

A better viewpoint is to rely on the Higgs mechanism. We again take the limit that the bare mass is arbitrarily large. Then we can integrate out everything except for massless degrees of freedom. Here the adjoint scalar fields (Higgs fields) σ and $\bar{\sigma}$ play a crucial role since the effective mass of a field is the sum of the bare mass and the contribution from the expectation values of Higgs scalars. This phenomena can be most directly seen from the anti-commutation relations of supercharges (7.3.15). Since we have global supersymmetry the expectation values of supersymmetric observables, $< 1 >= Z$ in our case, are localized to an integral over the fixed point locus of unbroken global supersymmetry. Consequently the path integral is localized to the kernel of the right hand sides of (7.3.15) acting on the fields. Then we immediately get the following set of relevant equations for $\boldsymbol{A}^{0,1}$ and $\boldsymbol{B}^{2,0}$,

$$\begin{aligned} [\sigma, \varphi^{1,0}] + m\varphi^{1,0} &= 0, \\ [\sigma, B^{1,1}] - mB^{1,1} &= 0, \\ [\sigma, C^{0,2}] - 2mC^{0,2} &= 0, \end{aligned} \tag{7.3.18}$$

and

$$\begin{aligned} \overline{\partial}_A \sigma &= 0, \\ [\sigma, \overline{\sigma}] &= 0, \\ [\sigma, B^{2,0}] &= 0, \end{aligned} \tag{7.3.19}$$

We will now study several limits of these equations.

Three Different Limits

We consider an $SU(2)$ bundle $E \to M$ for simplicity. The set of equations in (7.3.18) are the conditions for masslessness of the fields charged under S^1_t. The second equation in (7.3.19) implies that σ and $\overline{\sigma}$ can be diagonalized, say $\sigma = \frac{1}{2} diag(a, -a)$. Since $\mathrm{Tr}\,\sigma^2$ is the gauge invariant object we will consider $a \geq 0$.

We see that there are three (semi-classical) limits governed by different massless degree of freedom while preserving $N_c = (2,2)$ supersymmetry.

1. Vafa-Witten or a twisted $N = 4$ super-Yang-Mills theory. (i) the gauge symmetry is unbroken $a = 0$. (ii) the gauge symmetry is broken to $U(1)$ $a > 0$ and $a \neq m, 2m$ Then $\varphi^{1,0} = B^{1,1} = C^{0,2} = 0$ is the only solution of (7.3.18). Equivalently those fields and their $N_c = (2,2)$ superpartners are all infinitely massive.

2. The gauge symmetry is broken to $U(1)$ and $a = m$ and we have the reduction $E = L \oplus L^{-1}$ Then

$$\begin{aligned} \overline{\partial}_A &= \begin{pmatrix} \overline{\partial}_L & 0 \\ 0 & -\overline{\partial}_L \end{pmatrix}, & B^{2,0} &= \begin{pmatrix} \beta^{2,0} & 0 \\ 0 & -\beta^{2,0} \end{pmatrix}, \\ \varphi^{1,0} &= \begin{pmatrix} 0 & 0 \\ \vartheta^{1,0} & 0 \end{pmatrix}, & B^{1,1} &= \begin{pmatrix} 0 & \beta^{1,1} \\ 0 & 0 \end{pmatrix}, \\ & & C^{0,2} &= 0. \end{aligned} \tag{7.3.20}$$

We have

$$\begin{aligned} F^{0,2} &= 0, \\ i(F - \vartheta^{1,0} \wedge \vartheta^{0,1}) \wedge \omega + \beta^{1,1} \wedge *\overline{\beta}^{1,1} &= 0, \\ \overline{\partial}_L \vartheta^{1,0} &= 0, \\ \overline{\partial}_L \beta^{2,0} + \vartheta^{1,0} \wedge *\beta^{1,1} &= 0, \\ \overline{\partial}_L * \beta^{1,1} &= 0. \end{aligned} \tag{7.3.21}$$

3. The gauge symmetry is broken to $U(1)$ and $a = 2m$ and we have the reduction $E = L \oplus L^{-1}$ Then

$$\overline{\partial}_A = \begin{pmatrix} \overline{\partial}_L & 0 \\ 0 & -\overline{\partial}_L \end{pmatrix}, \qquad B^{2,0} = \begin{pmatrix} \beta^{2,0} & 0 \\ 0 & -\beta^{2,0} \end{pmatrix}, \qquad B^{1,1} = 0, \qquad \varphi^{1,0} = 0, \qquad C^{0,2} = \begin{pmatrix} 0 & \gamma^{0,2} \\ 0 & 0 \end{pmatrix}. \tag{7.3.22}$$

We have

$$\begin{aligned} F^{0,2} &= 0, \\ \overline{\partial}_L \beta^{2,0} &= 0, \\ iF \wedge \omega + \gamma^{2,0} \wedge \gamma^{0,2} &= 0. \end{aligned} \tag{7.3.23}$$

7.3.2 Families of $N_c = (2,0)$ Models

Following the discussions in Sect. 3.4 and Sect. 4.2 we break the $N_c = (2,2)$ symmetry down to $N_c = (2,0)$ supersymmetry generated by $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$. The S^1_ξ-action (7.3.11) can be extended to all those additional fields introduced for the $N_c = (2,2)$ model, compared with the original $N_c = (2,0)$. The S^1_ξ action is given by

$$\begin{aligned} S^1_\xi &: \left(\boldsymbol{B}^{(2,0)}, \Upsilon_+^{(2,0)}\right) \to \xi\left(\boldsymbol{B}^{(2,0)}, \Upsilon_+^{(2,0)}\right), \\ S^1_\xi &: \left(\Psi_-^{(0,1)}, \boldsymbol{H}^{(0,1)}\right) \to \overline{\xi}\left(\Psi_-^{(0,1)}, \boldsymbol{H}^{(0,1)}\right), \\ S^1_\xi &: (\overline{\sigma}, \eta_+) \to \overline{\xi}\,(\overline{\sigma}, \eta_+), \end{aligned} \tag{7.3.24}$$

and the conjugate fields have the opposite $U(1)_\xi$-charges. Here we can just follow the procedure in Sect. 4.2 to obtain the general $N_c = (2,0)$ supersymmetric action functional $S(m, \overline{m}, m_{++}, m_{--})$ is given by

$$S(m, \overline{m}, m_{\pm\pm}) = S(m, \overline{m}) + m_{++}m_{--}\int_M \mathrm{Tr}\Big(\boldsymbol{B}^{(2,0)} \wedge *\boldsymbol{B}^{(0,2)} + \sigma * \overline{\sigma}\Big) + \ldots, \tag{7.3.25}$$

whose new mass terms contain the Hamiltonian of the S^1_ξ symmetry. The $N_c = (2,0)$ supercharges $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$ satisfy the following modified anti-commutation relations

$$\boldsymbol{s}_+^2 = 0, \qquad \{\boldsymbol{s}_+, \overline{\boldsymbol{s}}_+\} = -i\phi^a_{++}\mathcal{L}_a - im_{++}\mathcal{L}_{S^1_\xi}, \qquad \overline{\boldsymbol{s}}_+^2 = 0. \tag{7.3.26}$$

Now, in total, we have a $\mathbb{C}^* \times \mathbb{C}^*$ family of $N_c = (2,0)$ models. From the previous discussions all we need to do is collect all fixed point equations of the supercharges $\boldsymbol{s}_+$ and $\overline{\boldsymbol{s}}_+$. Then the localization equations (7.3.9) and (7.3.10) are changed by the following equations

$$\begin{aligned}
\boldsymbol{F}^{(0,2)} &= 0,\\
\boldsymbol{D}'' * \boldsymbol{B}^{(2,0)} &= 0,\\
\boldsymbol{F}\wedge\omega + [\boldsymbol{B}^{(2,0)}, *\boldsymbol{B}^{(0,2)}] - \frac{1}{2}[\sigma,\overline{\sigma}]\omega\wedge\omega &= 0,\\
\boldsymbol{D}'\sigma + m\varphi^{0,1} &= 0,\\
[\sigma, \boldsymbol{B}^{(2,0)}] - mB^{1,1} - 2mC^{0,2} &= 0,
\end{aligned} \tag{7.3.27}$$

and

$$\begin{aligned}
[\phi_{++}, \boldsymbol{B}^{(2,0)}] + m_{++}\boldsymbol{B}^{2,0} &= 0,\\
[\phi_{++}, \sigma] + m_{++}\sigma &= 0.\\
[\phi_{++}, \overline{\phi}_{--}] &= 0,\\
d_A\phi_{++} &= 0.
\end{aligned} \tag{7.3.28}$$

By sending all the bare masses to infinity we have various semi-classical limits governed by different massless degrees of freedom.

For our purpose it is suffice to examine a limit $m_{\pm\pm}\to\infty$ by setting $m = \overline{m} = 0$. For simplicity we consider the $SU(2)$ case. Then we can follow the discussions in Sect. 4.2.2 and see that the path integral can be written as the sum of contributions from two branches;

- branch (i): On a generic point on the vacuum moduli space we have the trivial fixed point $\boldsymbol{B}^{(0,2)} = 0$ and the fixed point locus is the moduli space $\mathcal{N}$ of stable Higgs bundles,

$$\begin{aligned}
\boldsymbol{F}^{(0,2)} &= 0,\\
\boldsymbol{F}\wedge\omega &= 0.
\end{aligned} \tag{7.3.29}$$

  Hence we recover the generalization Donaldson-Witten theory in Sect. 7.2.
- branch (ii): The $SU(2)$ symmetry is broken down to $U(1)$. We have $E = L\oplus L^{-1}$ and

$$\boldsymbol{D}'' = \begin{pmatrix} \boldsymbol{d}''_L & 0\\ 0 & \boldsymbol{d}'_L\end{pmatrix}, \quad \boldsymbol{B}^{(2,0)} = \begin{pmatrix} 0 & \boldsymbol{b}^{(2,0)}\\ 0 & 0\end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & \alpha\\ 0 & 0\end{pmatrix}, \tag{7.3.30}$$

  where $\boldsymbol{d}''_L = \overline{\partial}_L + \vartheta^{1,0}$ and $\boldsymbol{b}^{(2,0)} = \beta^{2,0} + \beta^{1,1} + \gamma^{0,2}$ takes values in L^{-2}. The fixed point equation are

$$\begin{aligned}
\boldsymbol{F}^{(0,2)} &= 0,\\
\boldsymbol{d}'_L\alpha &= 0,\\
\boldsymbol{d}''_L * \boldsymbol{b}^{(2,0)} &= 0,\\
i\boldsymbol{F}_L\wedge\omega - \boldsymbol{b}^{(2,0)}\wedge *\boldsymbol{b}^{(0,2)} + \alpha\overline{\alpha}\omega^2 &= 0.
\end{aligned} \tag{7.3.31}$$

  The above set of equations is a spectral generalization of Abelian Seiberg-Vafa-Witten equation.

It is a well-established fact that Donaldson-Witten (DW) theory is equivalent to Seiberg-Witten (SW) theory [12]. One of the strong evidences, or vice versa, for such equivalence is the S-duality of Vafa-Witten (VW) theory, which has both DW and SW theories as two different semi-classical limits after the massive perturbation. The S-duality, for $SU(2)$ and $SO(3)$, implies that one can recover the entire partition function from one of such semi-classical limits. We expect similar relations between the generalized versions. It remains to be seen if our generalized Seiberg-Witten theory contains new information beyond Seiberg-Witten invariants.

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$$[\varphi, B^{(2,0)}] - m B^{2,0} - 2m C^{2,0} = 0,$$

and

$$\begin{aligned} [\phi_{++}, B^{(2,0)}] + m_{++} B^{2,0} &= 0, \\ [\phi_{++}, \sigma] + m_{++} \sigma &= 0, \\ [\phi_{++}, \bar\phi_{--}] &= 0, \\ d_A \phi_{++} &= 0. \end{aligned} \tag{7.3.28}$$

By sending all the bare masses to infinity we have various semi-classical limits governed by different massless degrees of freedom.

For our purpose it is suffice to consider a limit $m_{++} \to \infty$ by setting $m = \bar m = 0$. For simplicity we consider the $SU(2)$ case. Then we can follow the discussions in Sect. 4.3.3 and see that the path integral can be written as the sum of contributions from two branches:

- branch (i): On a generic point on the vacuum moduli space we have the trivial fixed point $B^{(0,2)} = 0$ and the fixed point locus is the moduli space N of stable Higgs bundles,

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$$D'' = \begin{pmatrix} d_L'' & 0 \\ 0 & d_{L^{-1}}'' \end{pmatrix}, \quad B^{(2,0)} = \begin{pmatrix} 0 & b^{(2,0)} \\ 0 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \tag{7.3.30}$$

  where $d_L'' = \bar\partial_L + \theta^{1,0}$ and $b^{(2,0)} = \beta^{2,0} + \beta^{1,1} + \gamma^{0,2}$ takes values in L^{-2}. The fixed point equations are

$$\begin{aligned} F^{(0,2)} &= 0, \\ d_L'' \alpha &= 0, \\ d_L'' * b^{(2,0)} &= 0, \\ iF_L \wedge \omega - b^{(2,0)} \wedge \bar b^{(0,2)} + \alpha\bar\alpha \omega^2 &= 0. \end{aligned} \tag{7.3.31}$$

  The above set of equations is a spectral generalization of Abelian Seiberg-Vafa-Witten equations.

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Dankwoord

Op deze laatste pagina's wil ik een aantal mensen bedanken voor hun bijdrage aan de totstandkoming van dit proefschrift. Allereerst wil ik mijn promotor Herman Verlinde bedanken.

Nu in Engels. It is my great pleasure to express sincere gratitude of me and my family to my adviser and promoter Herman Verlinde. I admire him both as a physicist, since my Korean days, and a teacher as well as a person. I always have enjoyed the numerous discussions with him and learned a lot from his insightful comments and suggestions. I have tried to imitate his unique and often surprising approach to any problem. It is largely due to his influence that I explored, though with some criss-crossings, many new different horizons both in subjects and viewpoints for me in physics. Those interactions with him have helped me and will continuously do so to converge to a proper physicist. I am also grateful to him for his openness and generosity. I have a vivid memory of great joy of me and my wife Bok-Sun, subsequently, when I first heard from him. It is due to him that my family had a reunion and I have had a very special and wonderful PhD experience in many respects.[3] I am grateful to him for many suggestions and encouragements when I was writing this thesis. He directed me in a subtle and effective way to take to writing this thesis seriously. Otherwise, this thesis would have been a lot poorer than the present form. I am also grateful to him for giving me opportunities to experience this part of the planet as well as the exciting period for advances in string theory.

I am grateful to Bernd Schroers for many things he has done for me personally as well as for many delightful discussions and conversations. I first met him in Swansea where he happened to share an office with me during his short visit. At the time I had no idea, as usual, what to do after my controversial MSc. He greatly encouraged me and eventually directed me to Amsterdam as well as introduce me to Herman. Since then he became my wonderful office mate and friend. I often recall my encounters with him as a mysterious thing in my life.

I am grateful to Robbert Dijkgraaf for many delightful discussions as well as for lucid comments and suggestions. I always admire him as one of my teachers since my Korean days. I am also grateful to him for kindly introducing me into the first research project in Amsterdam, which became the actual back-bone of this thesis. I remember his exciting lecture in the I. Newton Institute, which greatly inspired me. It was the first time it occurred to me that I should try a

[3] I am sure that only the students of Herman would understand what these really mean.

place like Amsterdam, though I had been regarding DVV as my champions of the "new generation" of physicists.

I am grateful to Feike Hacquebord for many things he has done for me personally as well as for many delightful discussions and conversations. We together have been jolly fellow students of Herman during the last four years and developed a wonderful friendship. It is impossible to recall and thank for all those nuisance things that he and Bernd did for me. Otherwise those would have been my day-mares instead of their night-mares. Most recently, for example, he helped me out in this complicated PhD business.

I am grateful to Christiaan Hofman for our exciting scientific Journey together, numerous discussions and friendship. Together we learned and explored many different areas in physics and mathematics. I regard all of our collaborations as wonderful and wild adventures, though often they turned out to be unsuccessful after consuming lots of time, paper and coffee as well as telephone bills. I am also grateful to him for his patience in listening to all those nonsense of mine and for his sharp criticisms afterwards. This is just one of his virtues as a physicist and a person and I learned a lot from him. He also corrected several errors as well as almost literally translated my thesis into English, i.e., Ik Christiaan bedanken.

I am grateful to Robbert Dijkraaf, Christiaan Hofman, Lubos Motl, Bernd Schroers and Herman Verlinde for enjoyable collaborations and discussions. I am grateful to Sander Bais for useful discussions and for influence through his enthousiastic spirits. I am grateful to all the member of the Amsterdam/Utrecht string Journal clubs as well as the Department of Physics, the University of Utrecht, and the Spinoza Institute for their hospitality. I thank Erik Verlinde for offering me an office in the Spinoza Institute. I am also grateful to Gijsbert Zwart for discussions and nice experiences together in the various parts of the globe. I thank KIAS (Korea), where some parts of this thesis were written, for their hospitality. I thank all the staff and members of the Instituut voor Theoretische Fysica te Amsterdam for the exciting atmosphere. I am grateful to Yocklang Chung for helping me out of many complicated paper work.

I thank my dear friend Jorn Verway for his help and friendship. He is the first Dutch person that I ever met. He gave me nice impressions of Holland and Dutch people. He greatly helped me to settle down in Swansea and, afterwards, in Amsterdam. Jorn and his wonderful partner Mayoo also became dear friends to my children Yoomin and Yooha. Jorn taught Yoomin to speak Dutch, etc. I am grateful to Jost and his parents for providing me a room during the first week in Holland.

I am grateful to the wonderful families of Mr. Soonnku Kang, Mr.Youngshin Pak and, especially, Mr. Kihang Lee for their invaluable help and freindship. They helped my family in various ways and introduce us to Korean community. I am grateful to Mrs. Han for her caring for my wife after giving birth to Yooha.

Now, back to Korea, I thank all of my dear friends for their longstanding friendship and moral support. On top of this I am grateful to Seungjoon Hyun for his invaluable contribution to my understanding of physics as well as for collaborations in the related subjects.

I am grateful to my Mother and all of my big family. I would like to express my sincere thanks and respects to uncle Chungwon and uncle Jiwon for their devotion and support to me as well as my big family. It is due to them that I could have concentrate on my own problems, physics etc.

I am grateful to my beautiful and precious wife Bok-Sun for her great Love. I am also grateful to her parents and Sister for their support.