# Disclaimer

# Parameterization of the Multiple Coulomb Scattering Error in High Energy Physics Detectors

DØ NOTE 1184 A
Revised

UCR/DØ92-001
Fermilab Pub 92/46
A.Klatchko
University of California Riverside CA 92521
28-AUG-1992

**Abstract**

A closed form for the parameterization of the error matrix that arises due to multiple Coulomb scattering is described. The errors depend on only one angle, rather than the two quasi-independent projected angles which are commonly used.

## Introduction

Multiple Coulomb Scattering (MS) introduces small deviations into the track parameters compared with those of an unscattered track (i.e a particle traversing the vacuum). The effect is usually described by an angle, $\Theta^{MS}$ [1] and a corresponding lateral shift in the position, $\epsilon$ [2]. It is usually assumed that the error on the physical process of measurement (the resolution) and the MS errors are independent. Also note that the MS process can be decoupled from energy losses and thus, does not affect the momentum.

MS is a stochastic process, namely, the probability for a scattering event (denoted by the state $X(t)$ in the phase space) to take place at time $t_o$ (site $k_i$) depends only on the physical condition in the immediate past at time $t < t_o$ (site $k_{i-1}$). The stochastic nature of MS is described as a convolution of local probability density functions satisfying the Chapman - Kolmogorov identity,

$$w_2(X,t|Y,s) = \int w_2(X,t|\xi,u) w_2(\xi,u|Y,s) d\xi,$$

where $w_2(X,t|Y,s)dX$ is the probability that the event $X < X(t) \leq X + dX$ occurs at time $t$, given that $X(s) = Y$ for $t > s$. The subscript "2" emphasizes the fact that only the state in the immediate past matters. Traversing a material of thickness $L$ the particle undergoes successive small-angle deflections *symmetrically* distributed about the incident direction. Recall that the scattering process goes predominantly in the forward direction leading to small deflection angles. Applying the central limit theorem of statistics to a large number of independent scattering events the distribution of the deflection angle can be approximated by a Gaussian distribution of the scattering angle. The mean squared MS angle is defined as, $\overline{\Theta^2} = n\overline{\theta^2}$, where $n$ is the number of collisions ($n$ is proportional to the number of atoms in the material) and $\overline{\theta^2}$ is the mean squared angle of a single scattering event defined as [3]:

$$\overline{\theta^2} = \frac{\int \theta^2 \frac{d\sigma}{d\Omega} d\Omega}{\int \frac{d\sigma}{d\Omega} d\Omega}$$

with $\frac{d\sigma}{d\Omega}$ the differential (Rutherford) cross section for a single scattering. On the scale of position measurements as they are carried out in high energy physics (HEP) detectors, the accumulative scattering angle $\Theta^{MS}$, estimated by the *RMS* of the distribution, $\sqrt{\overline{\Theta^2}}$, can be viewed as a 'local' parameter. Locality is extended here to a step length long enough to consist of $n$ (large enough) collisions for which the Gaussian approximation is still valid ; and yet the particle hasn't lost too much of its energy passing through the material. Alternatively one can think of shrinking the material between successive planes of measurements to a 'single scatterer' with the above statistical characteristics. We thus picture the particle traversing the detector from one plane of measurement to the successive plane such that with each plane is associated a scattering angle distributed symmetrically about the incident direction. Following this concept of locality, it is most suitable to treat MS errors by convoluting their probability density functions along the particles trajectory (in the spirit of the Chapman - Kolmogorov identity).

This article deals with the estimation of the errors on track parameters due to MS, in the milieu of track reconstruction for HEP detectors. The article is organized in two sections. In the first section we describe the concept of the local tracking method [4] which is lately used in some of the largest HEP experiments, such as DELPHI and ZEUS. In the second section we describe a parameterization of the MS error and outline construction of the error matrix.

# 1 The concept of local tracking

A track is locally defined (at a fixed plane of measurement), by five parameters; two coordinates (i.e. two measurements at that fixed plane), two direction cosines (or angles), and the radius of curvature (when there is a magnetic field), which is proportional to the momentum. In Cartesian coordinates, one has a five dimensional vector, $\bar{V} = (x, \hat{x}, y, \hat{y}, \frac{1}{p})$. Note however, that the parameterization of the track and the errors of its parameters in one system can always be transformed to another system appropriate to the detector geometry. It is thus sufficient to evaluate the errors for one set of parameters, for example in the Cartesian parameterization.

The track model, $F(\bar{V})$, is a function of the track parameters, and describes the trajectory of the particle in the detector. In the case of nonlinearities (the presence of a magnetic field), the analytical function $F(\bar{V})$ can be linearized at a given point in space $a_\circ$. The linearized track model, $f(\bar{V})$, is expressed as

$$f(\bar{V}) = F(\bar{V}|_{a_\circ}) + \frac{\partial F(\bar{V})}{\partial V_i}|_{a_\circ}(\bar{V}|_a - \bar{V}|_{a_\circ}).$$

Propagation of the track from a measured location $a$ to the next measured location $b$ can be described by a matrix derived from the track model, $\Phi_{ij}^{ab} = \frac{\partial V_i|_a}{\partial V_j|_b}$. In the case of nonlinearities, it may be approximated by the linearized track model, $f(\bar{V})$.

In a similar fashion, errors that occur locally in location $a$ are propagated to location $b$ using the same propagation matrix, $\Phi_{ij}^{ab}$ [5]. The propagation of errors is done using the following relation:

$$\Sigma_P^b = \Phi_{ij}^{ab}\Sigma^a\Phi_{ji}^{ab} \tag{1}$$

where $\Phi_{ji}^{ab}$ is the transpose of $\Phi_{ij}^{ab}$. The resulting matrix, $\Sigma_P^b$, contains the error variance and covariance, estimated at location $a$, as they are propagated to location $b$.

In order to estimate ('locally') the MS effect on the track parameters, it is necessary to evaluate the error variance and covariance in a given plane of measurement (location $b$) due to the traversing of a scattering material with a given thickness, $L$, and radiation length $L_o$ that is located between the $b$ and $a$ planes of measurement. The propagation of these variance and covariance can be done using equation 1. For a complete estimation of the error matrix, the MS error variance and covariance are added to the measurement error matrix and the resulting matrix is propagated across the measurement planes.

This concept is best realized in the Kalman filter approach to track reconstruction in HEP detectors [6]. In the Kalman filter framework, one estimates the track parameters and their errors locally, adds to them the MS error matrix and then propagates both the track parameters and the resulting error matrix to the next plane of the detector. The track parameters are then updated by a fit procedure resulting in a new set of parameters for that plane. In this way, one optimally follows the particle trajectory in the detector and the errors associated with it.

# 2 Parameterization of the MS error

Next we derive a parameterization of the errors of the track parameters due to MS, in terms of the scattering angle $\Theta^{MS}$, and outline the calculation of the error variance and covariance of the track parameters.

Let us break the trajectory of the particle traversing the material in the detector into a series of quasi straight lines, (each with an infinite radius of curvature), such that the trajectory that associates the two locations $a$ and $b$ can be described to first order by the direction cosines at $a$, $\hat{x}_i{}^a$, where $i$ runs from 1 to 3.

The effect of MS is to scatter the track such that instead of reaching location b, the particle is most probably found in a cone of a solid angle $\Omega$ ($\approx \pi {\Theta^{MS}}^2$) around the original line, <u>ab</u>. Although the direction of the particle is described by two independent variables, since the length of the cone is fixed and due to azimuthal symmetry we can further reduce the problem to a *one parameter problem* - the cone opening angle. Our task is thus to propagate the angular error due to the MS process to the direction cosines of the particle trajectory.

To first order, the errors of the direction cosines are $\delta\hat{x}_i{}^a$, such that the scattered line is now defined by the new (scattered) direction cosines:

$$\hat{x}_i{}^{a'} = \hat{x}_i{}^a + \delta\hat{x}_i{}^a \tag{2}$$

Let us point out again that although the variables ,$\hat{x}_i{}^a$, $\delta\hat{x}_i{}^a$ are 'local' on the detector scale, in fact they are the result of an $n$ (large number) of collisions and should thus be considered as average quantities exactly as $\Theta^{MS}$ is viewed as an average over single scattering angles. The intersection of the two lines in a plane defines an angle $\theta_s$ with a variance which is equal to the projected scattering angle [3]:

$$\theta_s = \frac{\Theta^{MS}}{\sqrt{2}} \tag{3}$$

The cosine of the angle of intersection $\theta_s$ is given by:

$$cos(\theta_s) = \sum_i \hat{x}_i{}^a \cdot \hat{x}_i{}^{a'} = \sum_i \hat{x}_i{}^a \cdot (\hat{x}_i{}^a + \delta\hat{x}_i{}^a) = 1 + \sum_i \hat{x}_i{}^a \cdot \delta\hat{x}_i{}^a \quad (4)$$

For $\theta_s$ small enough, the cosine can be expressed as

$$cos(\theta_s) = 1 - \frac{\theta_s^2}{2} \quad (5)$$

Parameterizing the new direction cosines as a Taylor expansion of the original ones we have

$$\hat{x}_i{}^{a'} = \hat{x}_i{}^a - \delta\theta_i\sqrt{1 - \hat{x}_i{}^{a^2}} \quad (6)$$

where the parameters $\delta\theta_i$ are small angular deviations of the direction angles. Recall that after $n$ collisions these small deviations are averaged out such that in an isotropic material it is legitimate to assume that the $\delta\theta_i$ are equal i.e. $\delta\theta_i = \delta\theta_j = \delta\theta_k = \delta\theta$ (compare to the Kinetic Theory of Gases) [7]. Despite the fact that in general, one needs 3 Euler angles to rotate a vector in space (we may neglect the translation parameters) the statistical nature of the process averages out these angles to a single parameter, which in turn, is proportional to the cone opening angle. The treatment of the problem using two quasi-independent variables [8] seems to us not necessary and leads to a wrong covariance. We would like to point out again that indeed the direction of the particle is characterized by two independent variables, say the polar angles, $\vartheta$ and $\varphi$, which after MS will suffer from the corresponding errors, $\delta\vartheta, \delta\varphi$. However, a formulation of the problem with the three direction angles (and the *symmetrical* errors associated with them), while constraining the direction cosines to be orthonormal, is effectively reducing the problem back to two independent degrees of freedom. Taking the advantage of the statistical characteristics of the process we further reduce the problem to a one parameter problem. We thus require the new direction cosines to satisfy orthonormality:

$$\sum_i \hat{x}_i{}^{a'} \cdot \hat{x}_i{}^{a'} = \sum_i [\hat{x}_i{}^a \cdot \hat{x}_i{}^a - 2\delta\theta\hat{x}_i{}^a\sqrt{1 - \hat{x}_i{}^{a^2}} + \delta\theta^2(1 - \hat{x}_i{}^{a^2})] = 1 \quad (7)$$

Solving equation 7 for $\delta\theta$ we obtain:

$$\delta\theta = \sum_i \hat{x}_i{}^a\sqrt{1 - \hat{x}_i{}^{a^2}} \quad (8)$$

Substitution of equation 8 in equation 4 with $\delta\hat{x}_i{}^a = -\delta\theta\sqrt{1 - \hat{x}_i{}^{a^2}}$, and using the Taylor expansion of equation 5 yields a parameterization of $\delta\theta$ in terms of the projected scattering angle $\theta_s$:

$$\delta\theta = \frac{\theta_s}{\sqrt{2}} \quad (9)$$

Hence, we identify the errors on the direction cosines as a function of the projected MS scattering angle:

$$\delta\hat{x}_i{}^a = -\frac{\theta_s}{\sqrt{2}}\sqrt{1 - \hat{x}_i{}^{a^2}} \quad (10)$$

Using this expression the error variance is:

$$\sigma_{\hat{x}_i} = (\frac{\partial \delta \hat{x}_i}{\partial \delta \theta} \delta \theta)^2 = \frac{\theta_s^2}{2}(1 - \hat{x}_i{}^2) \tag{11}$$

The angular direction errors $\delta\theta_{i,j,k}$ are independent and therefore there are *no correlations* between orthogonal planes. This feature manifests the stochastic nature of the MS process, where after a large number of independent scattering events any such nontrivial correlations are practically washed out. This conjecture is associated with the fact that the MS process is effectively a rotation of the coordinate system. To manifest that the correlations between the direction errors indeed vanish, we use the polar angle parameterization. Let us describe the direction cosines after MS as:

$$\hat{x}' = sin(\vartheta + \delta\vartheta)\ cos(\varphi + \delta\varphi)$$

$$\hat{y}' = sin(\vartheta + \delta\vartheta)\ sin(\varphi + \delta\varphi)$$

$$\hat{z}' = cos(\vartheta + \delta\vartheta)$$

We Taylor expand the above formulae to first order and obtain the expressions for the direction errors:

$$\delta\hat{x} = \delta\vartheta\ cos(\vartheta)\ cos(\varphi) - \delta\varphi\ sin(\vartheta)\ sin(\varphi)$$

$$\delta\hat{y} = \delta\vartheta\ cos(\vartheta)\ sin(\varphi) + \delta\varphi\ sin(\vartheta)\ cos(\varphi)$$

$$\delta\hat{z} = -\ \delta\vartheta\ sin(\vartheta)$$

Calculating the correlations we have:

$$\sigma_{\hat{x},\hat{y}} = \langle \frac{\partial \delta\hat{x}}{\partial \delta\vartheta}\delta\vartheta \frac{\partial \delta\hat{y}}{\partial \delta\varphi}\delta\varphi + \frac{\partial \delta\hat{x}}{\partial \delta\varphi}\delta\varphi \frac{\partial \delta\hat{y}}{\partial \delta\vartheta}\delta\vartheta \rangle_\varphi = \delta\vartheta\delta\varphi\ sin(\vartheta)cos(\vartheta)\langle cos^2(\varphi) - sin^2(\varphi)\rangle_\varphi,$$

Averaging over the azimuthal angle, $\varphi$, in the range $[0,2\pi]$ leads to:

$$\sigma_{\hat{x},\hat{y}} = 0 \tag{12}$$

This feature can be demonstrated without any loss of generality when the direction vector is fixed as (0,0,1) .i.e. $\vartheta, \varphi = 0$. In this case the correlations between orthogonal directions are of the third order in the angular error $(\delta\vartheta, \delta\varphi)$ and can thus be ignored:

$$\frac{\partial \delta\hat{x}}{\partial \delta\vartheta}\delta\vartheta \frac{\partial \delta\hat{y}}{\partial \delta\varphi}\delta\varphi = \delta\vartheta^2\delta\varphi \quad etc.$$

However, in an isotropic material the coordinate system can always be rotated such that any trajectory can be described by these fixed polar angles.

The errors of the parameters $x_i$ (the particle coordinates) are expressed as the distance between the two measured points as it is projected onto this particular direction with a $\delta\hat{x}_i$ projection error:

$$\delta x_i = \delta\hat{x}_i \bigtriangleup x_{ab} = -\frac{\theta_s}{\sqrt{2}} \bigtriangleup x_{ab}\sqrt{1-\hat{x}_i{}^2} \tag{13}$$

with, $\bigtriangleup x_{ab}$, the distance between the two successive measurements

$$\bigtriangleup x_{ab} = \frac{z^b - z^a}{\sqrt{1-\hat{x}^2-\hat{y}^2}}.$$

Using the above equations one can calculate the remaining error variance:

$$\sigma_{x_i} = \left(\frac{\partial \delta x_i}{\partial \delta\theta}\delta\theta\right)^2 = \frac{\theta_s^2}{2}(1-\hat{x}_i^2) \bigtriangleup x_{ab}^2 \tag{14}$$

and the covariance of the position and the direction errors at a given direction:

$$\sigma_{x_i\hat{x}_i} = \frac{\partial \delta x_i}{\partial \delta\theta}\delta\theta\frac{\partial \delta\hat{x}_i}{\partial \delta\theta}\delta\theta = -\frac{\theta_s^2}{2} \bigtriangleup x_{ab}(1-\hat{x}_i^2) \tag{15}$$

In view of equation (12) the covariance of orthogonal planes vanishes.

Let us emphasize again that the full covariance matrix, $V_{ij}^{ms}$, can be transformed to another set of parameters rather than the Cartesian coordinates and the direction cosines. Using the "propagation error formula" [9]

$$V_{nm}(\bar{f}) \approx \sum_{i,j}\frac{\partial f_n}{\partial x_i}\frac{\partial f_m}{\partial x_j}V_{ij}(\bar{x}) \tag{16}$$

one can express the errors on any other parameterization, $\bar{f}$, of the particle trajectory.

The propagation of the 'local' error matrix, $V_{ij}^{ms}$, is straight forward for the linear case. A linear track model at a given plane $z = z_k$ can be propagated to the following plane, $z = z_{k+1}$, with the following transfer matrix:

$$\Phi(\bigtriangleup z, \hat{x}, \hat{y}) = \begin{pmatrix} 1 & \frac{\bigtriangleup z}{\sqrt{1-\hat{x}^2-\hat{y}^2}} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{\bigtriangleup z}{\sqrt{1-\hat{x}^2-\hat{y}^2}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

where $\bigtriangleup z = z_{k+1} - z_k$ is the distance between the $k, k+1$ planes. Assuming that the direction cosines do not change drastically along the particle trajectory from $z_1$ to $z_n$, equation 1 is applied $N$ times along the path, $L = N \bigtriangleup z = z_n - z_1$, such that

$$\Sigma_1^n = \Phi_{ij}^N V_{ij}^{ms} \Phi_{ji}^N \tag{18}$$

Note that due to the block diagonal form of the $\Phi$ matrix the off diagonal terms are given by:

$$\Phi_{1;12,34}^n = \frac{N \bigtriangleup z}{\sqrt{1-\hat{x}^2-\hat{y}^2}} = \frac{z_n - z_1}{\sqrt{1-\hat{x}^2-\hat{y}^2}} \tag{19}$$

Even when a magnetic field is present one can still use the above scheme for short enough paths i.e. cords that approximate the arc, for which the average change in the direction cosines is tolerable (small compared to the MS errors). One then propagates the error matrices of a given cord at the break point to the next cord, using the direction cosines of this cord etc.

To summarize, we have shown how the MS error may be parameterized using only one variable, the scattering angle $\Theta^{MS}$, which is evaluated in the theory of MS. Using this parameterization, a full error matrix can be constructed locally and propagated across the detector. Unlike the calculation found in [8], we find that there are no correlations between orthogonal planes. This approach is a natural consequence of the stochastic character of the MS process.

## Acknowledgements

I would like to thank P. Kasper, R. Früwirth and C. Murphy for helpful remarks.

Work supported by DOE contract # DEAM03765F00010.

## References

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[7] The average over the deflection angles is zero due to the stochastic nature of the process; however we approximate the angular deflection errors $\delta\theta_i$ by the *RMS* of the squared deflection angle, $\delta\theta = \sqrt{\overline{\delta\theta_i^2}}$.

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Note that the Eichinger & Regler parameterization does not fulfill orthonormality i.e. the new direction cosines in Regler notation: $\underline{r_1} = \underline{r_\circ} + \theta_1 \underline{l_1} + \theta_2 \underline{l_2}$, with $\underline{l_{1,2}, r_\circ}$ orthogonal to each other, are off by $\Theta^{MS}$ from unity which is of the order of the cone opening angle between the 'old' and the 'new' set of direction cosines and is thus a poor approximation. Moreover, instead of the expected relation, $\underline{r_1} \cdot \underline{r_1} = 1$ we find in the Eichinger & Regler parameterization the following, $\underline{r_1} \cdot \underline{r_\circ} = 1$ which means that the scattered particle is pointing in its original direction, namely, the particle is not scattered at all. Also note that for a track that is fully pointing in one of the directions, say $\hat{z}$, the covariance matrix obtained in this parameterization yields vanishing off diagonal terms compared to other directions. This is an unacceptable violation of isotropy, because any track can be rotated into a coordinate system such that it points in that particular direction and thereby will 'loose' these artificial correlations.

[9] Particle Properties Data Booklet Particle Data Group, M. Aguilar - Benitez et al. April 1990 pp. 148.