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# Exploring the Percolation Phenomena in Quantum Networks

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**Abstract:** Quantum entanglement as a non-local correlation between particles is critical to the transmission of quantum information in quantum networks (QNs); the key challenge lies in establishing long-distance entanglement transmission between distant targets. This issue aligns with percolation theory, and as a result, an entanglement distribution scheme called "Classical Entanglement Percolation" (CEP) has been proposed. While this scheme provides an effective framework, "Quantum Entanglement Percolation" (QEP) indicates a lower percolation threshold through quantum preprocessing strategies, which will modify the network topology. Meanwhile, an emerging statistical theory known as "Concurrence Percolation" reveals the unique advantages of quantum networks, enabling entanglement transmission under lower conditions. It fundamentally belongs to a different universality class from classical percolation. Although these studies have made significant theoretical advancements, most are based on an idealized pure state network model. In practical applications, quantum states are often affected by thermal noise, resulting in mixed states. When these mixed states meet specific conditions, they can be transformed into pure states through quantum operations and further converted into singlets with a certain probability, thereby facilitating entanglement percolation in mixed state networks. This finding greatly broadens the application prospects of quantum networks. This review offers a comprehensive overview of the fundamental theories of quantum percolation and the latest cutting-edge research developments.

**Keywords:** entanglement transmission; quantum communication; complex quantum network; entangled state; percolation

**MSC:** 81P01



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## 1. Introduction

With the rapid development of quantum information, quantum communication has become a cutting-edge technology in the field of information transmission [1]. Compared to classical communication methods, quantum communication offers a significantly more reliable means of information transmission, primarily due to its superior confidentiality and security features. In the context of increasing demands for information security and privacy protection, quantum communication, with its unique properties of quantum entanglement and superposition, is emerging as an effective way to solve the problem of information transmission.

In quantum communication, qubits function as the fundamental units of information transmission, with their entanglement playing a crucial role. Quantum entanglement represents a unique non-classical correlation wherein qubits, regardless of the distance between them, remain interconnected. Thus, a change in the state of one qubit will instantly affect the state of another qubit. Qubits also play a significant role in quantum communication as they can provide better channel capacity and stronger noise resilience, which are key to achieving long-distance quantum communication [2–4]. In recent years, the non-local entanglement properties of qubits have been verified, opening up new possibilities for constructing more efficient quantum communication systems [5–8]. This phenomenon underpins the theoretical framework for quantum teleportation, facilitating non-local transmission of information between qubits through entangled states [9]. Given this characteristic, quantum communication demonstrates significant potential for secure information transmission, particularly in technologies such as quantum key distribution, superdense coding, where the application of quantum entanglement substantially enhances the confidentiality of the communication process.

However, despite quantum entanglement laying a solid theoretical foundation for quantum communication, there are still many challenges in practical applications. The fragility of quantum entanglement makes it prone to degradation when exposed to thermal noise, and other quantum disturbances, causing the entangled states to lose their correlation. The degradation effect becomes more pronounced as the transmission distance increases, significantly undermining the reliability and stability of long-distance quantum communication. Therefore, in this field, the question of how to maintain and improve the quality of entangled states during transmission has emerged as a critical research topic. To address these challenges, researchers have proposed various optimization strategies, one of which is the introduction of quantum repeaters. This method divides entangled transmission into several shorter links, where entanglement can be "purified" and "swapped", thereby effectively enhancing the fidelity of entangled states [10,11]. This scheme extends the transmission distance of quantum communication while significantly reducing the degradation of entangled states caused by environmental interference, making it an effective tool for entanglement distribution.

From the perspective of network science [12–14], researching how to optimize the entanglement distribution within a complex network to achieve long-distance quantum transmission efficiently and reliably has become a challenging, yet promising, research field. The phenomenon of quantum percolation, as an important research direction in quantum networks, is closely related to percolation theory. The entanglement distribution exhibits similar behavior to percolation theory: when a sufficient number of entangled links are formed within the network, an entanglement connection can be established between the source node and the target node. This provides new theoretical support for quantum communication and has sparked further research on how to achieve efficient entanglement distribution.

This article primarily explores the percolation phenomena in quantum networks, organized as follows: In Section 2, we briefly introduce percolation theory in complex networks. In Section 3, we analyze the basic model of a quantum network. In Section 4, we review Classical Entanglement Percolation and Quantum Entanglement Percolation. In Section 5, we delve into exploring Concurrence Percolation and study its critical phenomenon. In Section 6, we discuss entanglement percolation in a mixed state network. These methods not only provide a theoretical foundation for the design of quantum communication but also offer new ideas and directions for the development of quantum networks.

## 2. Percolation Theory

Percolation theory was initially developed to study the flow behavior and characteristics of fluids in porous media [15]. However, as research has progressed, its applications have gradually extended beyond fluid mechanics to various fields such as statistical physics and complex networks. Today, percolation theory has become an important tool for studying randomness, phase transitions, and complex systems, particularly in analyzing changes in network structure and connectivity [16].

One of the most classic percolation models is a square lattice, where percolation employs probabilistic methods to investigate how the overall connectivity of a system changes when random nodes or edges are removed. In this model, each edge in the lattice is retained with a certain probability $p$ and removed with a probability of $1 - p$. As the retention probability increases, when $p$ reaches a critical value, a sufficiently large connected cluster begins to form, and the system exhibits a percolation phenomenon, meaning that one or more paths connect the two ends of the lattice (as shown in Figure 1).

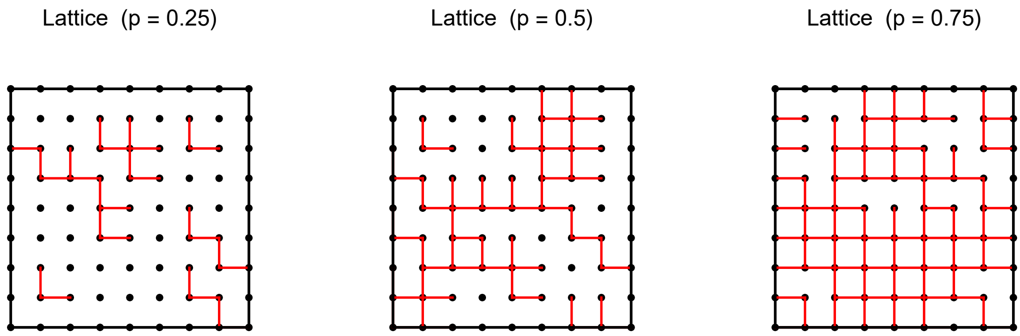


**Figure 1.** The three graphs illustrate the connectivity of a square lattice network at different probabilities $p$. When $p = 0.25$, there are only scattered clusters; at $p = 0.5$, a clear connected path spans the entire lattice; while at $p = 0.75$, large-scale connected paths become even more apparent, with almost all nodes being connected.

In the subcritical region, when the percolation probability $p < p_c$, the nodes in the lattice form many small and dispersed clusters, with almost no connections between each cluster, making them isolated from one another. This low connectivity means that it is almost impossible to find a complete path from one side of the lattice to the other, leading the entire system to exhibit disordered behavior. The number of clusters $n_s$ decreases rapidly as the cluster size increases, which can be described by an exponential decay function [17],

$$n_s \sim s^{-\tau} e^{-s/s_\xi}, \tag{1}$$

where $s$ is the size of the cluster, and $s_\xi$ is the characteristic size.

As the percolation probability approaches the critical point $p \approx p_c$, the system begins to display specific critical behaviors, showing characteristics of transitioning from a locally disconnected state to a globally connected state. Near this tipping point, some properties follow power-law distributions [16,18,19], such as

The size of the giant cluster $S$,

$$S \sim |p - p_c|^{\beta}; \tag{2}$$

The correlation length $\xi$,

$$\xi \sim |p - p_c|^{-v}; \tag{3}$$

The mean cluster size $\langle s \rangle$,

$$\langle s \rangle \sim |p - p_c|^{-\gamma}; \tag{4}$$

The characteristic size $s_\xi$,

$$s_\xi \sim |p - p_c|^{-1/\sigma}; \tag{5}$$

with the following relationships:

$$\beta = \tfrac{\tau-2}{\sigma}, \tag{6}$$

$$\gamma = \tfrac{3-\tau}{\sigma}. \tag{7}$$

Here, the exponents $\beta, v, \gamma, \sigma$, and $\tau$ are known as critical exponents [16–18]. This power-law relationship reflects the universal behavior of the system near the critical point, indicating that the statistical properties of the percolation system are independent of the specific shape of the lattice but are determined by some universal parameters.

In the supercritical region, where $p > p_c$, a massive connected cluster begins to form. As $p$ continues to increase, the average size of connected clusters also grows, and the vast majority of nodes join the largest cluster, forming the dominant structure of the system. At this point, the entire system exhibits high global connectivity, allowing for connection between distant nodes.

Percolation theory provides an effective method for studying the transition of complex systems from local random behaviors to global structural changes. It has demonstrated significant theoretical and practical value in interdisciplinary research areas, such as the spread of infectious diseases and network robustness, and offers a fresh perspective for the development of quantum communication.

## 3. Quantum Network

A qubit is the fundamental unit for transmitting quantum information, and quantum entanglement between qubits is a unique phenomenon in quantum information processing that allows two or more particles to be closely correlated at the quantum level. This entanglement enables instant information sharing between particles, even when they are far apart, playing a crucial role in quantum communication [20].

The Bell state is a maximally entangled two-qubit state. In this state, the correlation between two qubits is maximized, allowing perfect information transmission in quantum communication. Different quantum states exhibit varying degrees of entanglement, which can affect transmission. Therefore, we need specific measurement tools to describe their degree of entanglement [21].

For a bipartite pure state, it can be expressed using Dirac notation as

$$|\psi_i(\theta)\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle, \tag{8}$$

where $0 \leq \theta \leq \pi/4$. In such a pure state, the entanglement degree can be described using the Concurrence , defined as

$$C(|\psi\rangle) = \sqrt{2[1 - \mathrm{Tr}(\rho_A^2)]}, \tag{9}$$

where $\rho_A$ is the reduced density matrix [22,23]. The density matrix of a pure state $\rho_\psi$ can be written as

$$\rho_\psi = |\psi(\theta)\rangle\langle\psi(\theta)| = (\cos\theta|00\rangle + \sin\theta|11\rangle)(\cos\theta\langle 00| + \sin\theta\langle 11|). \tag{10}$$

Expanding this expression gives

$$\rho_\psi = \cos^2\theta|00\rangle\langle 00| + \cos\theta\sin\theta|00\rangle\langle 11| + \cos\theta\sin\theta|11\rangle\langle 00| + \sin^2\theta|11\rangle\langle 11|. \tag{11}$$

Taking the partial trace of $\rho_\psi$ over one subsystem yields

$$\rho_A = \mathrm{Tr}_B(\rho_\psi) = \cos^2\theta|0\rangle\langle 0| + \sin^2\theta|1\rangle\langle 1|, \tag{12}$$

which can be written as

$$\rho_A = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}. \tag{13}$$

Thus, $C(|\psi(\theta)\rangle) = \sin 2\theta$, with a range of $[0,1]$. When $\theta = \pi/2$, $c = 1$, representing the maximally entangled pure state, i.e., the singlet. If $\theta = 0$, it clearly indicates no entanglement [23].

In quantum communication, quantum entanglement is highly sensitive to the external environment and is prone to degradation due to noise, fiber loss, and other external factors, leading to a gradual decrease in transmission efficiency over long distances. Quantum repeaters use "entanglement swapping" technology to connect multiple segments of entangled states, allowing the quantum state to be "recovered" after a certain distance, enabling long-distance distribution of entangled states. Therefore, similar to a network model, quantum networks also have a graph-like structure (as shown in Figure 2). However, in this case, the nodes consist of multiple qubits, and each edge (or connection) in the network represents a pure entangled state $|\psi_i(\theta)\rangle$ established between two qubits, with $\theta$ determining the degree of entanglement in this pure state.

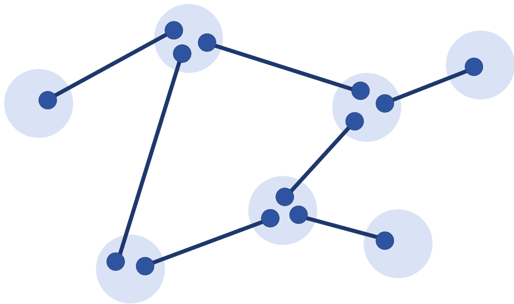

**Figure 2.** Small dots represent individual qubits, while large dots represent nodes in the network, with each node typically consisting of multiple qubits. The edges represent the entangled pure state formed between two qubits.

## 4. Entanglement Percolation in Quantum Networks

### *4.1. Classical Entanglement Percolation*

In quantum networks, Classical Entanglement Percolation (CEP) offers a new perspective for analyzing the entanglement transmission of quantum states. This theory is similar to traditional percolation theory, but its core concept is leveraging the properties of quantum entanglement to establish connections between nodes in the network [24].

In this framework, each edge represents the same entangled pure state. Using the Procrustean method [25], each partially entangled pure state can be transformed into a singlet with a certain probability $p = 2\min(\sin^2\theta, \cos^2\theta) = 2\sin^2\theta$, known as the Singlet Conversion Probability (SCP). This means that for each edge, there is a probability $p$ that it will be retained, and a probability $1 - p$ that it will break. As the conversion probability $p$ increases, when it reaches a critical value $p_{\text{th}}$, a giant cluster forms within the quantum network. Based on this, by using entanglement swapping, two connected singlet edges can be transformed into a singlet that connects their endpoints, thus enabling any two nodes in the cluster to be linked through a singlet. When $p$ is large enough, it is possible to find a connected path between the boundaries, thus enabling entanglement transmission.

It becomes clear that CEP is essentially a problem of path connectivity, which can be described using the "sponge-crossing" probability (Psc). This represents the probability of finding a connected path between distant boundaries. For irregular networks that lack obvious boundaries, the boundaries are defined as two nodes *Source* and *Target*, with the shortest distance between them being the diameter of the network.

According to classical percolation theory, for sufficiently large networks and for $p \subset [0,1]$, there always exists a critical probability $p$, at which a giant cluster appears in the network that can effectively connect the two boundaries. In other words, $p_{th}$ is the minimum value that makes $Psc > 0$, representing the percolation threshold

$$p_{\text{th}} = \inf\Big\{p \subset [0,1] \mid \lim_{n\to\infty} P_{\text{SC}} > 0\Big\}. \tag{14}$$

The percolation threshold represents the critical point in a network for transitioning from local to global access. When the entangled resources, connectivity density, or transmission capacity in the network reach this threshold, the quantum network can enable global quantum communication or entanglement distribution. So, how do we calculate the percolation threshold $p_{\text{th}}$ for a quantum network? Specifically, $Psc$ is actually a function of the variable $p$. In a series-parallel network, the series-parallel rules [23],

$$\begin{cases} \text{Series:} & p = p_1 p_2 \cdots; \\ \text{Parallel:} & 1 - p = (1 - p_1)(1 - p_2) \cdots, \end{cases} \tag{15}$$

can be used to calculate $Psc$, similar to the way resistances are calculated in electrical networks, thereby finding $p_{\text{th}}$. For example, in CEP, the percolation thresholds vary across different lattice structures. For common lattice types, such as a square lattice, honeycomb lattice, and triangular lattice, their percolation thresholds are 0.670, 0.777, and 0.545, respectively. These values are expressed in units of $(\pi/4)^{-1}\theta$ [23].

However, real quantum networks are often complex in structure and may contain loops and multiple connections, requiring higher-order calculation rules to handle these situations.

Fortunately, the properties of Classical Entanglement Percolation allow us to focus on finding the probability of a maximally entanglement path between the two boundaries, making Monte Carlo simulations an effective tool. In simulations, we randomly assign conversion probabilities $p$ to each edge and calculate whether a maximally entanglement path between the source and target can form at a given $p$. As the number of simulations increases, we can gather statistics on the connectivity probability for each $p$ value, and thus plot the relationship between $Psc$ and $p$. When we observe that $Psc$ suddenly jumps from zero to a positive value, we can determine the critical probability.

### 4.2. Quantum Entanglement Percolation

In fact, it has been shown that entanglement transmission does not necessarily require reaching the above threshold, meaning that CEP is not the optimal solution. Through certain preprocessing operations and adjustments to the network topology, a lower percolation threshold may be achieved, a method known as Quantum Entanglement Percolation (QEP) [26–28].

For two entangled states, $|\psi_i(\theta_1)\rangle$ and $|\psi_i(\theta_2)\rangle$, each containing one qubit located at the same node, by performing an entanglement swapping operation, these two entangled states can be transformed into a new pure state and then converted into a singlet state. In this case, the total success probability is given by

$$p = \min\left\{2\sin^2\theta_1, 2\sin^2\theta_2\right\}, \tag{16}$$

which is equivalent to the probability of transforming the state with the least entanglement in $|\psi_i(\theta_1)\rangle$ and $|\psi_i(\theta_2)\rangle$ into a singlet using the Procrustean method, and this is optimal [29].

Furthermore, if two nodes are connected by two pure states, applying majorization [30] can further optimize this transmission process, resulting in the highest conversion probability

$$p = \min\left\{1, 2(1 - \cos^2\theta_1\cos^2\theta_2)\right\}. \tag{17}$$

For certain specific network structures, the QEP can reduce thresholds through optimization operations, thereby improving the entanglement transmission efficiency. However, this does not imply that QEP is the optimal solution in all cases. In fact, the effectiveness of QEP depends on multiple factors, including the topology of the network, the nature of the entangled states between nodes, and the preprocessing operations employed. Compared to classical situations, quantum networks exhibit their unique "advantages", achieving more efficient transmission through these quantized operations.

## 5. Concurrence Percolation in Quantum Networks

### 5.1. Concurrence Percolation Theory

In fact, although QEP can lower the threshold of the CEP, it is still not the optimal result. This suggests that compared to classical theory, quantum networks have their own uniqueness, and QEP cannot fully explain the percolation phenomenon.

The core of both CEP and QEP is to find a singlet path between the source and target. To reduce this necessary condition, an alternative scheme called Concurrent Percolation Theory (ConPT) has been proposed [23,31]. This scheme focuses on the concurrence between quantum states, which ranges between 0 and 1, rather than the probability $p$ of singlet conversion. Similar to CEP, the concurrence between the source and target is described by $C_{SC}$, the "sponge-crossing" concurrence, and $c_{th}$ is defined as its corresponding threshold. In the thermodynamic limit, $C_{SC}$ tends toward 0 in the subcritical region and 1 in the supercritical region. The threshold is defined as

$$c_{\text{th}} = \inf\left\{C \subset [0,1] \mid \lim_{n\to\infty} C_{\text{SC}} > 0\right\}. \quad (18)$$

Moreover, for the same $\theta$, $c_{\text{th}}$ is always smaller than $p_{\text{th}}$ [17].

Interestingly, concurrent entanglement percolation also follows certain series-parallel rules [23]:

$$\begin{cases} \text{Series:} & c = c_1 c_2 \cdots; \\ \text{Parallel:} & \frac{1+\sqrt{1-c^2}}{2} = \max\left\{\frac{1}{2}, \frac{1+\sqrt{1-c_1^2}}{2} \frac{1+\sqrt{1-c_2^2}}{2} \cdots\right\}. \end{cases} \quad (19)$$

In a series-parallel network, the above formula can be directly applied to calculate the corresponding $C_{\text{SC}}$. However, for networks containing loops, other calculation methods are required. Based on the idea of equivalent resistance, a method called the Star-Mesh transform has been proposed to replace unknown high-order rules, allowing for the equivalent calculation of $C_{\text{SC}}$ (as shown in Figure 3). Under ConPT, the thresholds for a square lattice, honeycomb lattice, and triangular lattice are 0.42, 0.51, and 0.32, respectively, which are clearly lower than those obtained under CEP [23].

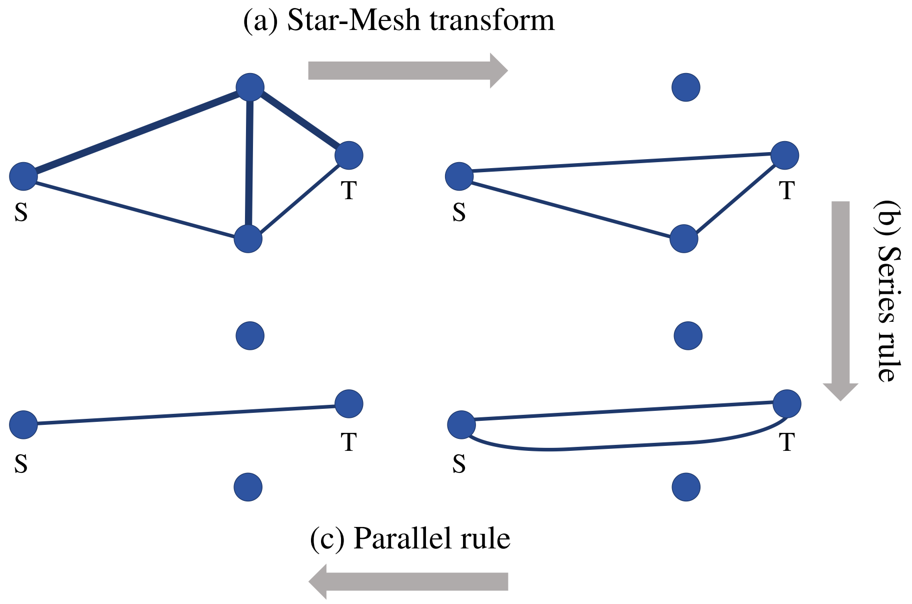


**Figure 3.** (**a**) Star-Mesh transform compresses a star graph with loops (n nodes) into a complete graph with fewer nodes (n-1 nodes), thereby simplifying the topology. During this process, the connectivity equivalence between two specified nodes S and T is preserved. (**b**,**c**) Series rule and parallel rule: When multiple quantum channels (or connections) are connected in series or parallel, their overall equivalent transmission characteristics can be calculated using the series rule or parallel rule.

However, the calculation of the Star-Mesh transform is a double-recursive process, and as the network size increases, the complexity grows exponentially, making it impractical to compute for larger network structures. To solve this issue, researchers proposed a fast approximation method combining the Sm approximation and parallel approximation [32].

Since entanglement percolation is essentially a problem of path connectivity [33], paths of different lengths have varying impacts on the total concurrence. As the path length increases, the influence of a single path gradually weakens, so considering only the shortest $m$ paths can basically reflect the percolation of the entire network. Meanwhile, the parallel approximation ignores overlapping edges between different paths and treats them as being in parallel. It has been verified that the $C'_{\text{SC}}$ obtained from the parallel approximation is always an upper bound of the actual value $C_{\text{SC}}$. By combining these two approximation methods, the selected $m$ paths can be fully regarded as being in parallel, while ignoring their internal shared edges. After calculation, an approximate threshold will also emerge.

In the thermodynamic limit, this approximate threshold will gradually approach the true $c_{\text{th}}$. Simulations have verified that the results obtained by the approximation method are very close to the theoretical values, which greatly accelerates the computation process and provides a feasible solution for solving the percolation threshold of large scales, greatly advancing the study of ConPT.

### *5.2. Critical Phenomena in Concurrence Percolation*

Percolation theory is closely related to critical exponents. In typical lattice networks (such as Bethe lattices, square lattices, honeycomb lattices, etc.), Concurrence Percolation also exhibits critical phenomena, with these critical exponents being independent of the specific details of the network [17,23].

To investigate the critical phenomena of Concurrence Percolation in quantum networks, a type of hierarchical scale-free network called $(U, V)$ flowers was chosen as the research model (as shown in Figure 4). This type of network effectively simulates the topological features of many real-world networks, such as social networks and the internet [34]. The $(U, V)$ flower network is a hierarchical scale-free network with self-similar properties, and it achieves this by iteratively replacing each edge with the same structure [35,36]. Nodes are connected through two parallel paths with lengths $U$ and $V$, satisfying $U \leq V$. This repeated structure generation ensures that the network maintains similar topological features across different levels.

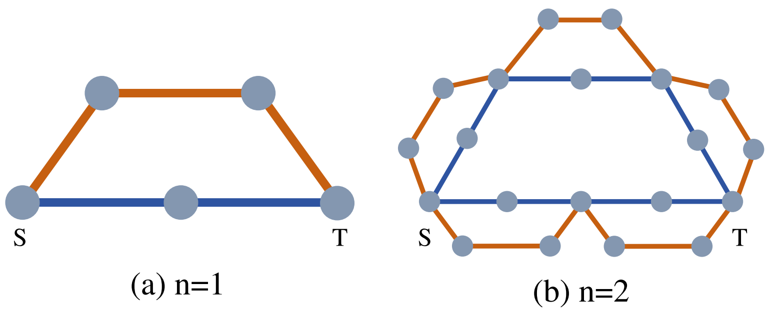


**Figure 4.** $(U, V)$ flower: The $n_{th}$ generation $(U, V)$ flower is shown in the figure, with (**a**) $n = 1$ and (**b**) $n = 2$, where the blue paths represent the shorter path $U = 2$, and the orange paths represent the longer path $V = 3$.

Through numerical simulations and theoretical derivations, it was found that the relationship between longer and shorter paths has markedly different effects on the two types of percolation.

For classical percolation, the critical probability can be expressed as

$$p_{\text{th}} \simeq 1 - \left(\ln \frac{U}{U-1}\right) V^{-1} + O\left(V^{-2}\right). \tag{20}$$

For Concurrence Percolation, the critical value is

$$c_{\text{th}} \simeq 1 - \left(\frac{1}{4} \ln V\right) V^{-1} + O\left(V^{-1} \ln \ln V\right). \tag{21}$$

This indicates that, in classical percolation, shorter percolation paths play a crucial role near the critical point, while in Concurrence Percolation, longer paths have a more significant impact on the percolation properties [37].

The asymptotic behavior of critical exponents shows significant differences between classical percolation and Concurrence Percolation. In classical percolation, the indices $\nu$, $d - d_f$, and $\beta$ are primarily determined by the shorter path length $U$, indicating that shorter paths dominate the percolation process. In contrast, for Concurrence Percolation, the main factors affecting $\nu$ and $d_f$ are related to the longer path $V$, with slow corrections appearing, which leads to cancellations in the calculation of $\beta$, resulting in a constant $\beta = 1$ that is independent of the values of $U$ or $V$.

From theoretical and statistical physics perspectives, Concurrence Percolation and classical percolation belong to different universality classes. This difference in universality classes arises from how the two percolation problems handle the increase of length scale $V$, which controls $U$. As $V$ approaches infinity, the critical exponents of classical percolation decouple from $V$ and depend only on the shorter length scale $U$. In contrast, the critical exponents of Concurrence Percolation depend on both $U$ and $V$. This distinction extends to the behavior of critical thresholds: as $V$ approaches infinity, both $p_{th}$ and $c_{th}$ converge, but the convergence of the concurrence threshold $c_{th}$ is slower, indicating that concurrence has greater resilience as $V$ increases [38,39].

These findings emphasize the role of longer paths in quantum networks: although entanglement decays exponentially along longer paths, if the paths are abundant, longer paths still significantly contribute to the overall connectivity of the quantum network.

## 6. Entanglement Percolation in Mixed State Networks

The entanglement percolation of mixed states is a theoretical model that describes the behavior of entanglement transmission in mixed state quantum networks. Previous research has primarily focused on the theory of Concurrence Percolation in pure state quantum networks, which is somewhat similar to classical percolation theory. However, the assumption of pure state networks is overly idealized, as in the real world, due to factors such as thermodynamic noise, quantum states are often mixed states. Therefore, to better understand the entanglement transmission behavior in actual quantum networks, it is necessary to extend the existing theoretical framework to handle mixed states [40,41].

For mixed states, they cannot be described by a single state vector but are instead represented by a density matrix, usually written as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{22}$$

satisfying the normalization condition $\sum_i p_i = 1$.

In conjunction with CEP, the problem of entanglement transmission in a mixed state network also involves finding a path that can eventually generate a singlet. To generate a singlet between any two nodes in a mixed state quantum network, these two nodes need to be connected by mixed states of a particular form [42]. Specifically, the form of this mixed state should be expressed as

$$\rho(\alpha, \gamma, \lambda) = \lambda|\alpha, \gamma\rangle\langle\alpha, \gamma| + (1-\lambda)|01\rangle\langle 01|, \tag{23}$$

where $|\alpha, \gamma\rangle$ represents a pure state that can be written as

$$|\alpha, \gamma\rangle = \sqrt{\alpha}|00\rangle + \sqrt{1-\alpha-\gamma}|11\rangle + \sqrt{\gamma}|01\rangle, \tag{24}$$

with parameters $0 \le \lambda \le 1$.

In this framework, if two nodes are connected by two quantum states satisfying the form $\rho(\alpha_1, \gamma_1, \lambda_1)$ and $\rho(\alpha_2, \gamma_2, \lambda_2)$, through pure state conversion measurement (PCM), there is a probability $p_c = \lambda_1\lambda_2(\alpha_1(1-\alpha_2-\gamma_2) + \alpha_2(1-\alpha_1-\gamma_1))$ that a pure state $|\alpha', \gamma = 0\rangle = \sqrt{\alpha'}|00\rangle + \sqrt{1-\alpha'}|11\rangle$ can be obtained [41], where

$$\alpha' = \frac{\min(\alpha_1(1-\alpha_2-\gamma_2), \alpha_2(1-\alpha_1-\gamma_1))}{\alpha_1(1-\alpha_2-\gamma_2) + \alpha_2(1-\alpha_1-\gamma_1)}. \tag{25}$$

Next, by using the Procrustean method, a singlet can be obtained with a probability $p = 2\min(1-\alpha', \alpha')$. Thus, the final SCP can be expressed as

$$p_{\text{scp}} = 2\lambda_1\lambda_2 \min[\alpha_1(1-\alpha_2-\gamma_2), \alpha_2(1-\alpha_1-\gamma_1)]. \tag{26}$$

For more complex network structures, where a path may contain multiple quantum states, entanglement can be transmitted through an entanglement swapping operation. Specifically, this is achieved by performing standard Bell basis measurements between the two quantum states connected along the path. There are four possible outcomes of the measurement, respectively, $|\Psi^{\pm}\rangle$ and $|\Phi^{\pm}\rangle$. When the result is $|\Psi^{\pm}\rangle$, the new state becomes

$$\rho\left(\frac{\alpha_1\alpha_2}{h_\pm}, \frac{(\sqrt{\alpha_1\gamma_2} \pm \sqrt{\gamma_1(1-\alpha_2-\gamma_2)})^2}{h_\pm}, \frac{\lambda_1\lambda_2 h_\pm}{2p(\Psi_\pm)}\right), \tag{27}$$

which still satisfies the specific mixed-state form mentioned above, and thus entanglement swapping can continue. However, if the result is $|\Phi^{\pm}\rangle$, the form of the mixed state cannot be preserved, and the process cannot proceed further. Therefore, as the number of entanglement swapping increases, the fidelity of the final result rapidly declines, making long-distance entanglement transmission very difficult [10].

To improve the transmission efficiency of mixed state entanglement percolation, one can consider setting multiple edges on each bond to meet the conditions for generating a singlet. This can transform the problem of entanglement percolation in a mixed state network into CEP. Similar to QEP, some preprocessing operations, such as the swapping procedure or the square protocol, can be used to lower the percolation threshold, thereby achieving more efficient entanglement transmission.

Through these methods, the efficiency of entanglement percolation in mixed state quantum networks can be improved, reducing the loss that occurs during long-distance transmission and providing a more practical theoretical foundation for future quantum communication and quantum information processing.

## 7. Discussion

Percolation theory provides a novel framework for investigating connectivity and phase transition phenomena in quantum networks. Due to the unique features of quantum mechanics, such as the non-locality of quantum entanglement, percolation in quantum networks presents greater complexity than in classical networks.

A central issue in the study of percolation phenomena in quantum networks is determining the percolation threshold. In complex quantum networks, the parallel approximation method may introduce considerable errors, as it fails to adequately account for the local complexities of the network. These errors may be non-negligible, particularly in practical applications where imprecise threshold estimates can result in inefficiencies in quantum communication or the misallocation of network resources. Consequently, reducing errors in the parallel approximation method or developing more accurate and efficient algorithms has become a critical research challenge [43,44].

Additionally, under specific conditions, the percolation threshold in quantum networks can be reduced through the optimization of quantum resource allocation. This observation suggests that, with the same level of resources, an optimized network structure can facilitate more efficient global quantum communication. This leads to a fundamental question: in the limiting case, is it possible to determine the minimum percolation threshold? If so, how can this exact limit be calculated, and what is its physical significance?

Although Concurrence Percolation offers a new theoretical framework for quantum network design, it remains limited to pure state networks. Extending the concept of Concurrence Percolation to mixed state quantum networks has become another key challenge. Since the degree of entanglement in mixed states is typically lower than in pure states, achieving global connectivity may require more entanglement resources and paths. Therefore, does a well-defined percolation threshold exist in mixed state quantum networks as well? How does it compare to the threshold in pure state?

Addressing these questions could lead to further optimization in the design of quantum networks and provide valuable theoretical insights for the practical implementation of quantum information processing in the future.

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