Homological methods for Gauge Theories with singularities

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THESE DE DOCTORAT DE
L'UNIVERSITE CLAUDE BERNARD LYON1

École doctorale (ED 512)
InfoMaths
Discipline : Mathématiques

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Soutenue publiquement le 05/10/2023 par
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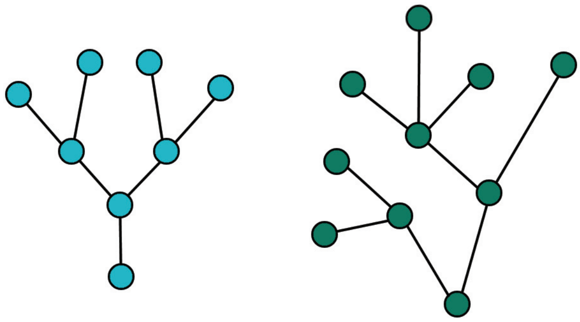

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Acknowledgements

I would like to express my gratitude to my PhD advisor Thomas Strobl, who guided me professionally in the last four years. I admire and aspire his creativity as well as his deep look into details. His mentorship did not end at professional matters, he was there to support me in different times. I am grateful to Camille Laurent-Gengoux, whose help was invaluable to me in the last two years. I was fortunate to work with and to learn from him. Camille also serves as an exceptional example of fortitude, humility and endurance. I also want to thank Camille for his friendship and hospitality.

I thank the members of the jury, Eveline, Marc, Oleksii, Sophie and Vladimir who agreed to participate in my Ph. D. defence. An additional thank you to reviewers, Pavel and Vladimir who managed to read the text of my thesis.

I would like to personally thank Leonid Ryvkin, whose friendship and support I cherish. Many thanks to my colleagues at Institut Camille Jordan and not only there - Martin, Ri, Yanni, Vladimir, Oscar, Eduard for many laughs and a great time. An additional thank you to my friends and my parents, who played a part to bring me on the path I am on today.

Contents

Introduction

Frequently physical systems are described by differential equations on a manifold M usually assumed to be symplectic. This means that M is even dimensional and there exists a 2-form ω which is closed, $d\omega = 0$ and it induces an isomorphism $\omega : TM \longrightarrow T^*M$. A typical example of a symplectic manifold is the cotangent bundle $M = T^*Q$, where the canonical symplectic form can be written in coordinates as $\omega = \sum_{i=1}^{\dim Q} dq_i \wedge dp_i$. From the algebraic point of view we may think of a ring of smooth functions $C^\infty(M)$ equipped with a Poisson bracket $\{\cdot\,,\,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$:

$$\{f, g\} = \omega(X_f, X_g),$$

where the vector field X_f is uniquely determined through the equation $df = \omega(X_f, \cdot)$. This bracket is Lie bracket, i.e. it is anticommutative and satisfies the Jacobi identity. The usual multiplication on $C^\infty(M)$ turns $C^\infty(M)$ into a commutative ring, which relates to the Poisson bracket via the Leibniz rule:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g.$$

This means that for any $f \in C^\infty(M)$, $\{f, \}$ is a derivation of a commutative ring $C^\infty(M)$. A typical physical system consists of a fixed function H, called the Hamiltonian, and a set of functions $\{\phi_\alpha : M \longrightarrow \mathbb{K}, \alpha \in J\}$, for some index set J, called the constraints. The dynamics is defined on the constraint surface V, which is a zero locus of $\{\phi_\alpha\}$. The ring of functions on V can be identified with the $C^\infty(M)/I$, where I is the ideal of functions vanishing on V. If $\mathcal{I}$ is coisotropic, that is preserved under Poisson bracket $\{\mathcal{I}, \mathcal{I}\} \subset \mathcal{I}$, then the vector fields X_{ϕ_α} are tangent to V (if V is smooth). Thus V is foliated and the space of orbits $V^{\mathcal{I}}$ inherits a Poisson structure. A simple example is given by the Marsden-Weinstein reduction [MW74]:

i. Suppose M is equipped with a Lie group action $\Phi : G \times M \longrightarrow M$. The infinitesimal action is a Lie algebra homomorphism that sends elemenents of Lie algebra $\mathfrak{g}$ of G to vector fields on M, $\phi : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$. Assume that this action is hamiltonian, i.e. $\forall a \in \mathfrak{g}$ $\phi_a = \{h_a, \cdot\}$, where $h : \mathfrak{g} \longrightarrow C^\infty(M)$ is a Lie algebra homomorphism, $h_{[a,b]} = \{h_a, h_b\}$.

ii. The map h defines a momentum map $\mu : M \longrightarrow g^*$ via $\mu(p)(a) = h_a(p)$. Then the zero locus $h_a(p) = 0$ for all $a \in \mathfrak{g}$ can be identified with the $\mu^{-1}(0)$. If in addition G acts freely on $\mu^{-1}(0)$, $\mu^{-1}(0)/G$ is a symplectic manifold with symplectic structure induced by ω.

A more general treatment that incorporates Poisson manifolds can be found in [MR86, OR14]. An alternative approach is a homological construction, known as BFV(also known as the Hamiltonian-BRST) construction. The essence of the BFV construction is captured by a differential graded Poisson algebra, which encodes a resolution of the zero locus $\phi_\alpha = 0$ and a complex resembling the Chevalley-Eilenberg complex whose cohomology upon the restriction to the constraint surface is given by $N(\mathcal{I})/\mathcal{I}$, where $N(\mathcal{I}) = \{f \in \mathcal{O} \,|\, \{f, \mathcal{I}\} \in \mathcal{I}\}$ is the normalizer of $\mathcal{I}$. In good cases this cohomology can be identified with $V^{\mathcal{I}}$. We discuss this construction in more details:

Given a commutative Poisson algebra $(\mathcal{O}, \{\cdot,\cdot\}_{\mathcal{O}})$ and a Poisson ideal $\mathcal{I}$, $\{\mathcal{I}, \mathcal{I}\}_{\mathcal{O}} \subset \mathcal{I}$, the BFV construction can be obtained as follows:

i. The first building block is a Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$, first introduced by Tate [Tat57]. If a generating set $\varphi_1, \ldots, \varphi_k$ of $\mathcal{I}$ (which does not neccesarily coincide with $\{\phi_\alpha\}$) forms a regular sequence, then the Koszul-Tate resolution is isomorphic to the Koszul complex, hence the name. The Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$ is a graded symmetric algebra $(S_{\mathcal{O}}(\mathcal{E}), \delta)$ built on a free (or projective) graded $\mathcal{O}$-module $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_n \oplus \ldots$. Any element a of $\mathcal{E}_i$ is said to have a degree i, also known in the physics literature as the antighost number $\mathrm{agh}(\cdot)$, $\mathrm{agh}(a) = i$. By construction this graded symmetric algebra is quasi-isomorphic to $\mathcal{O}/\mathcal{I}$.

   In contrast to free (projective) resolutions, the Koszul-Tate resolutions are highly infinite, in particular if the generating set $\varphi_1, \ldots, \varphi_k$ of $\mathcal{I}$ is not a regular sequence, then the length of the Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$ is infinite.

ii. To any $\mathcal{O}$-submodule $\mathcal{E}_i$ we introduce its dual $\mathcal{E}_i^* := \mathrm{Hom}(\mathcal{E}_i, \mathcal{O})$ for all $i \geq 1$. Any element $a \in \mathcal{E}_i^*$ is said to have a ghost degree $\mathrm{gh}(a) = i$. Let us write $\mathcal{E}^* = \bigoplus_{i=1}^{\infty} \mathcal{E}_i^*$. Then $S_{\mathcal{O}}(\mathcal{E} \oplus \mathcal{E}^*)$ is equipped with a Poisson bracket $[\cdot,\cdot]$ which coinsides with $\{\cdot,\cdot\}_{\mathcal{O}}$ whenever the entries are in $\mathcal{O}$. On the basis $\{e_{m;i}\}$, $\{e^*_{m;i}\}$ of $\mathcal{E}_i, \mathcal{E}_i^*$ respectively the bracket is given by:

$$[e^*_{m;i}, e_{n;j}] = \delta_{mn}\delta_{ij},$$

   with other combinations vanishing. The bracket is graded antisymmetric: $[e^*_{m;i}, e_{n;j}] = -(-1)^{mn}[e_{n;j}, e^*_{m;i}]$. The lift of the bracket to $S_{\mathcal{O}}(\mathcal{E} \oplus \mathcal{E}^*)$ is then given by a graded Leibniz rule. On $\mathcal{E} \oplus \mathcal{E}^*$ one introduces a total degree, known in physics also as the total ghost number $\mathrm{tgh}(\cdot)$. By definition one assigns $\mathrm{tgh}(a) = -i$ for all $a \in \mathcal{E}_i$ $\forall i$, $\mathrm{tgh}(b) = j$ for all $b \in \mathcal{E}_j^*$ $\forall j$, and $\mathrm{tgh}(F) = 0$ for all $F \in \mathcal{O}$. Then the Poisson bracket $[\cdot,\cdot]$ has a total degree 0.

iii. The BFV construction is then the differential graded Poisson algebra $(S_{\mathcal{O}}(\mathcal{E} \oplus \mathcal{E}^*), [\cdot,\cdot], \mathcal{Q})$ equipped with a homological vector field $\mathcal{Q} = [Q, \cdot]$ which we outline below. The existence of such Q is due to the following well-known statement (see e.g. [HT92]).

   Theorem 0.1. *Let $\mathcal{O}$ be a Poisson algebra and $\mathcal{I} \subsetneq \mathcal{O}$ be a Poisson ideal. Let $(S(\mathcal{E}), \delta)$ be the Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$ and $S(\mathcal{E} \oplus \mathcal{E}^*)$ the graded Poisson algebra described above. Then there exists a vector field $\mathcal{Q}$ of total degree 1 on $S(\mathcal{E} \oplus \mathcal{E}^*)$ such that*

   1. *$\mathcal{Q} = [Q, \cdot] = [\overline{\delta} + \ldots, \cdot]$, where $\overline{\delta}$ is the differential δ viewed as an element of $S_{\mathcal{O}}(\mathcal{E}) \otimes \mathcal{E}^*$, $\ldots$ correspond to terms at least quadratic in $\mathcal{E}^*$.*
   2. *$\mathcal{Q} \circ \mathcal{Q} = 0$*

   The proof of Theorem 0.1 is essentially an application of the Homological Perturbation Lemma, see [HT92, Sta97]. Let us list the properties that such $\mathcal{Q}$ satisfies:

   1. Cohomology of $\mathcal{Q}$ at negative total degree is equal to 0.
   2. $H_0(\mathcal{Q}, S_{\mathcal{O}}(\mathcal{E} \oplus \mathcal{E}^*)) = N(\mathcal{I})/\mathcal{I}$.
   3. With respect to a filtration by antighost degree, $\mathcal{Q} = \delta + \sum_{i=1}^{k} e_i^* X_{\varphi_i} + \ldots$, where $\ldots$ stands for terms of agh > 1 and agh $= 0$ distinct from $\sum_{i=1}^{k} e_i^* X_{\varphi_i}$. By abusing the notations, we denote by δ here the Koszul-Tate differential trivially extended to $\mathcal{E}^*$.

A Hamiltonian of a system with constraints is an element H of $N(\mathcal{I})$. Having found the BFV construction $(S_{\mathcal{O}}(\mathcal{E}\oplus\mathcal{E}^*), [\cdot,\cdot], \mathcal{Q})$, it is possible to construct an extension $H_{\mathcal{Q}} \in S_{\mathcal{O}}(\mathcal{E}\oplus\mathcal{E}^*)$ of H such that its total degree is 0, $H_{\mathcal{Q}} = H + \dots$, where $\dots$ stands for terms at least linear in $\mathcal{E}^*$, and $[Q, H_{\mathcal{Q}}] = 0$. In such a case $H_{\mathcal{Q}}$ is referred as BFV Hamiltonian.

Remark 0.2. The BFV construction intrinsically depends on choices. In particular, the Koszul-Tate resolutions depends on the choice of generators of $\mathcal{I}$, as well as choices within the Tate algorithm [Tat57]. The Perturbation Lemma ensures the vector field $\mathcal{Q}$ exists, but it is not canonically given. However, as it was shown in a broad generality in [FKS14, Mue17], the cohomology of $\mathcal{Q}$ does not depend on such choices. See [HT92] for earlier discussions in the physical context.

Historical remark. From a physics perspective, one of the pioneering works that deal with studying gauge theories was written by Faddeev and Popov [FP67], where the authors presented a new method to calculate one-loop diagrams in a path integral approach by introducing auxiliary fields. Further developments are known now as a BRST quantization [BRS76, Tyu75] for Hamiltonian systems and BV quantization [BV83a, BV83b, BV84] for the Lagrangian counterpart. The cohomological interpreration of BRST method is due to Henneaux [Hen85] and collaborators [FHST89], where the authors employed the homological algebra tools to show the existence of BFV construction. See also [KS87, Sta97] for related works from the point of view of Poisson reduction. An application to field theory is given in [BBH00], where the higher cohomologies of $\mathcal{Q}$ are studied and its interpretation is given as well as the application to Yang-Mills theory. Another related result is the celebrated AKSZ construction [ASZK97], which canonically associates a BV-type construction to a classical action functional and the Hamiltonian counterpart, first appeared in [GD00].

Example 0.3 (BFV construction and equivariant moment maps)**.** As an entry data, let us have a manifold M with a Lie group G action on it. Furthermore, let the momentum map $\mu : M \longrightarrow \mathfrak{g}^*$ be G-equivariant, which is tantamount to $h : \mathfrak{g} \longrightarrow C^\infty(M)$ being a Lie algebra homomorphism. The relation between h and μ is $h_a(p) = \mu(p)(a)$, for all $p \in M, a \in \mathfrak{g}$. Denote by $\mathcal{I}$ the ideal of functions vanishing on the zero locus of $\{h_a\}$. The first step is the Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$. For simplicity, we assume that the generators of $\mathcal{I}$, $h_1, \dots h_{\dim(\mathfrak{g})}$ form a regular sequence, i.e. h_1 is not a zero divisor on $C^\infty(M)$, and h_j is not a zero divisor on $C^\infty(M)/(h_1, \dots, h_{j-1})$ for all $j \in 2, \dots, \dim(\mathfrak{g})$. Then the Koszul-Tate resolution is isomorphic to the Koszul complex, and is given by $(S_{\mathcal{O}}(\mathcal{E}_1), \delta)$, where $\mathcal{E}_1$ is a free $\mathcal{O}$-module of rank $\dim(\mathfrak{g})$ with a basis parametrized by the basis of $\mathfrak{g}$. $\delta : \mathcal{E}_1 \longrightarrow \mathcal{O}$ is then $\delta(e_k) = h_k$ and it is extended to the graded symmetric product $S_{\mathcal{O}}(\mathcal{E}_1)$ as a graded derivation. We may view δ as an element in $S_{\mathcal{O}}(\mathcal{E}_1 \otimes \mathcal{E}_1^*)$ as

$$\overline{\delta} = \sum_{k=1}^{\dim(\mathfrak{g})} e_k^* h_k.$$

A simple check shows that $[\overline{\delta}, \overline{\delta}] \neq 0$. If C_{ij}^k are structure constants of the Lie algebra $\mathfrak{g}$, then

$$[\overline{\delta} - \sum_{i,j,k=1}^{\dim \mathfrak{g}} \tfrac{1}{2} C_{ij}^k e_i^* e_j^* e_k,\ \overline{\delta} - \sum_{i,j,k=1}^{\dim \mathfrak{g}} \tfrac{1}{2} C_{ij}^k e_i^* e_j^* e_k] = 0,$$

so the nilpotent vector field is given by

$$\mathcal{Q} = \delta + \sum_{k=1}^{\dim(\mathfrak{g})} e_k^* X_{h_k} - \sum_{i,j,k=1}^{\dim \mathfrak{g}} \tfrac{1}{2} C_{ij}^k e_i^* e_j^* \frac{\partial}{\partial e_k^*} - \sum_{i,j,k=1}^{\dim \mathfrak{g}} C_{ij}^k e_j^* e_k \frac{\partial}{\partial e_i}.$$

Let us write $\mathrm{p}_{\mathcal{E}}$ and $\mathrm{p}_{\mathcal{E}^*}$ for projections $S_{\mathcal{O}}(\mathcal{E}\oplus\mathcal{E}^*) \longrightarrow S_{\mathcal{O}}(\mathcal{E})$ and $S_{\mathcal{O}}(\mathcal{E}\oplus\mathcal{E}^*) \longrightarrow S_{\mathcal{O}}(\mathcal{E}^*)$ respectively. Then $\mathrm{p}_{\mathcal{E}} \circ \mathcal{Q} \circ \mathrm{p}_{\mathcal{E}}$ is the Koszul-Tate differential and $\mathrm{p}_{\mathcal{E}^*} \circ \mathcal{Q} \circ \mathrm{p}_{\mathcal{E}^*}$ is the Chevalley-Eilenberg differential.

In practice whenever the contraints do not form a regular sequence, construction of the BFV data is problematic. Despite that the general algorthm is known [FHST89], it may not terminate at the finite number of steps. The problem arises already at the level of Koszul-Tate resolutions, where the known resolutions are essentially the one isomorphic to Koszul complex.

This thesis is devoted to studying Koszul-Tate resolutions. There are several instances when Koszul-Tate resolutions appear: it is not restricted to only BV/BFV techniques, but it is also a building block in rational homotopy theory [Sul77] and the theory of PDE [Ver01], see [GK19] for a development in the direction of gauge PDE. The Koszul-Tate resolutions are interesting by themselves as well, since they provide an example of infinite minimal free resolutions [MP15], a topic largely undeveloped. In particular, let $\mathcal{O}$ be a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$, $\varphi_1, \ldots, \varphi_k$, where $\varphi_j = a_{j,1}x_1 + \cdots + a_{j,n}x_n$, be a regular R sequence. Then, the minimal free resolution of $\mathbb{K}$ in free $R/\langle\varphi_1, \ldots, \varphi_k\rangle$- modules is a Koszul-Tate resolution $(S(\mathcal{E}_1 \oplus \mathcal{E}_2), \delta)$, where $\mathcal{E}_1, \mathcal{E}_2$ are free $R/\langle\varphi_1, \ldots, \varphi_k\rangle$- modules of ranks n and k respectively. The differential δ is defined as:

$$\begin{aligned} &\text{In homological degree 1} \quad \delta(e_i) = x_i. \\ &\text{In homological degree 2} \quad \delta(y_j) = a_{j,1}e_1 + \cdots + a_{j,n}e_n. \end{aligned}$$

and then extended by the graded Leibniz rule to $S(\mathcal{E} \oplus \mathcal{E}^*)$. However, up to a tensor product with a graded symmetric algebra generated by a trivial complex $\mathbf{T}$, this is the only known example of the Koszul-Tate resolution different from the Koszul complex. In this thesis we introduce a new class of examples we call arborescent Koszul-Tate resolutions. The name is due to $\mathcal{E}$ being a graded module of decorated rooted trees. In contrast to the original Tate algorithm, the arborescent resolution is constructed in a finite number of steps for a large class of examples, precisely those admitting as modules finite length free or projective resolutions $(\mathfrak{M}_\bullet, d)$. This covers in particular the resolutions of finitely generated R-modules, with R being a polynomial (or local) ring. The arborescent resolution by construction gives a deformation retract on $(\mathfrak{M}_\bullet, d)$ thus inducing an A_∞-algebra on $(\mathfrak{M}_\bullet, d)$. We give particular examples of such resolutions and the induced A_∞-algebras. Moreover, whenever one has a free or projective resolution $(\mathfrak{M}_\bullet, d)$ which has a differential graded algebra structure, the arborescent Koszul-Tate resolution is canonically defined.

We find a notion of a minimal Koszul-Tate resolution whenever $\mathcal{O}$ is a Noetherian local ring or a graded polynomial ring. We prove that in this case any Koszul-Tate resolution contains a minimal one, which can be recovered. Lastly, in case $\mathcal{O}$ is a polynomial ring and $\mathcal{I} \subsetneq$ a monomial ideal of $\mathcal{O}$ whose generators do not form a regular $\mathcal{O}$-sequence, we show that the Koszul-Tate resolutions are highly infinite: the graded module $\mathcal{E}$ entering the resolution has non-zero components of arbitrary high homological degree. In physical terms, this means that the number of generations of "ghosts-for-ghosts" is infinite.

The thesis is organized as follows: In the first Chapter we review the material on free resolutions. We study in details the notion of minimality and we specialize to the case of graded resolutions of polynomial rings, where a lot of development took place in the recent years, see e.g. [Pee10].

In the second Chapter we study and develop the *arborescent* Koszul-Tate resolutions. This is based on a joint work with Camille Laurent-Gengoux and Thomas Strobl.

The third Chapter is a reprint of a joint paper with Thomas Strobl [HS23], where we studied the connection between the universal Lie-∞ algebroid [LLS20] and the BFV construction on the example of Lie-2 algebroids.

The objective of this chapter is to establish the foundation for the next chapter. We offer the reader an introduction to certain aspects of commutative algebra. Specifically, we study free $\mathcal{O}$-resolutions of an $\mathcal{O}$-module M, with $\mathcal{O}$ being a commutative ring. In Section 1, we focus primarily to the case when $\mathcal{O}$ is local. We investigate the minimality of such resolutions and provide various examples. We devote special attention to the study of the Koszul complex. When $\mathcal{I}$ is generated by a regular $\mathcal{O}$-sequence, the Koszul complex of an $\mathcal{O}$-module $\mathcal{O}/\mathcal{I}$ serves as an example of an $\mathcal{O}$-resolution equipped with an exterior (or graded symmetric algebra) that aligns with the resolution's differential. The Koszul complex, whenever it is acyclic in non-zero homological degrees, becomes relevant for the next Chapter on algebra resolutions as well, since it is the sole explicit example of the Koszul-Tate resolution. The content in Section 1 is based on references [Eis95, Mat70, MR89]. Meanwhile, the material in Section 2 focuses on polynomial rings and modules graded with respect to the polynomial degree. Concentrating on this specific case yields fruitful results; we demonstrate that if $\mathcal{I}$ is a monomial ideal, then a particular canonical simplicial construction known as the Taylor complex (or the algebraic Scarf complex in specific cases) serves as a resolution of $\mathcal{O}/\mathcal{I}$. Section 2 draws from [Pee10, MS04]. Finally, in Section 3, we compile some fundamental concepts of homological algebra, alongside some technical lemmas and propositions utilized within the chapter.

1 Free resolutions

Let $\mathcal{O}$ be a commutative ring and $\mathcal{I} \subset \mathcal{O}$ a proper ideal. We call a complex

$$\cdots \xrightarrow{d_{k+1}} \mathfrak{M}_k \xrightarrow{d_k} \cdots \xrightarrow{d_2} \mathfrak{M}_1 \xrightarrow{d_1} \mathcal{O} \xrightarrow{0} 0 \tag{1}$$

a *free $\mathcal{O}$-resolution of* $\mathcal{O}/\mathcal{I}$ if it satisfies

i. each $\mathfrak{M}_k$ is a free $\mathcal{O}$-module and each map d_k is $\mathcal{O}$-linear.

ii. The complex has no homology in positive degrees and the degree 0 homology is $\mathcal{O}/\mathcal{I}$.

For further convenience, we define the sum of all $\mathfrak{M}_i$ as $\mathfrak{M}_\bullet$, i.e, $\mathfrak{M}_\bullet = \oplus_{i\in\mathbb{N}}\mathfrak{M}_i$. We adopt the convention that $\mathfrak{M}_0$ is equal to $\mathcal{O}$ and d_0 is equal to 0. To encode a collection of maps d_i, we introduce a map $d : \mathfrak{M}_\bullet \longrightarrow \mathfrak{M}_\bullet$ such that its restriction to $\mathfrak{M}_i$ reproduces d_i, denoted as $d|_{\mathfrak{M}_i} = d_i$.

We call $(\mathfrak{M}_\bullet, d)$ the complex defined in Equation (1). We denote the homological degree of the complex by $\deg(\cdot)$. According to this definition, for any $a \in \mathfrak{M}_k$ and $F \in \mathcal{O}$, we have $\deg(a) = k$ and $\deg(F) = 0$.

Remark 1.1. In a broader context, for any $\mathcal{O}$-module M, an $\mathcal{O}$-resolution of M refers to a chain complex

$$\cdots \xrightarrow{d_{k+1}} \mathfrak{M}_k \xrightarrow{d_k} \cdots \xrightarrow{d_2} \mathfrak{M}_1 \xrightarrow{d_1} \mathfrak{M}_0 \xrightarrow{0} 0 \tag{2}$$

made of free $\mathcal{O}$-modules. This complex is required non-zero homology only at degree 0, denoted as $H_0(\mathfrak{M}_\bullet, d)$, which is equal to $\mathfrak{M}_0/\mathrm{Im}(d_1)$ and is isomorphic to M. Furthermore, each d_i in the complex is $\mathcal{O}$-linear.

A classical result in homological algebra states that it is possible to construct a free resolution. This can be demonstrated easily using the following algorithm:

Construction 1.2. *Free resolution of $\mathcal{O}/\mathcal{I}$*

1. Take a set $\mathcal{C}_0 = \{\phi_1, \dots, \phi_k, \dots\}$, possibly infinite, of generators of $\mathcal{I}$. Form a free $\mathcal{O}$-module $\mathfrak{M}_1$, with a basis $\{e_\phi,\ \phi \in \mathcal{C}_0\}$ parametrized by the generators of $\mathcal{I}$. The differential d_1 is an $\mathcal{O}$-linear map $\mathfrak{M}_1 \longrightarrow \mathcal{O}$:
$$d_1(e_\phi) = \phi.$$
2. Here we describe a pair $(\mathfrak{M}_{i+1}, d_{i+1})$ by recursion. Assume that a collection $\{(\mathfrak{M}_j, d_j) \,|\, j \in [\![1, i]\!]\}$ of free $\mathcal{O}$-modules and $\mathcal{O}$-linear maps has been constructed. Consider a kernel of d_i. Clearly, $Ker(d_i)$ is an $\mathcal{O}$-submodule of $\mathfrak{M}_i$. Denote its set of generators as $\mathcal{C}_i = \{\chi_1, \dots, \chi_k, \dots\}$. Then $\mathfrak{M}_{i+1}$ is a free $\mathcal{O}$-module whose basis $\{f_\chi,\ \chi \in \mathcal{C}_i\}$ is parametrized by $\mathcal{C}_i$. The differential $d_{i+1} : \mathfrak{M}_{i+1} \longrightarrow \mathfrak{M}_i$ is defined as
$$d_{i+1}(f_\chi) = \chi.$$
3. The algorithm continues on unless at a step N we find $Ker(d_N) = 0$. In that case we set $\mathfrak{M}_{N+1} = 0$ and the algrorithm terminates.

Definition 1.3. *The length of a resolution $(\mathfrak{M}_\bullet, d)$ is $max\{i \,|\, \mathfrak{M}_i \neq 0\}$. $(\mathfrak{M}_\bullet, d)$ is a finite resolution if its length is finite, otherwise we call it an infinite resolution.*

Remark 1.4. If $\mathcal{O}$ is a polynomial ring of N variables, then by the Hilbert Syzygy Theorem (see Section 1.6), for any $\mathcal{O}$-module M there exists a resolution $(\mathfrak{M}_\bullet, d)$ of length $\leq N$.

Remark 1.5. For $\mathcal{O}$ being a Noetherian ring(see details in Section 3) Construction 1.2 implies that every $\mathfrak{M}_i$ can be chosen to be finitely generated. Indeed, at the step 1 we can choose a finite number of generators of $\mathcal{I}$, and therefore a Noetherian $\mathcal{O}$-module $\mathfrak{M}_1$ is finitely generated. At degree 2, $Ker(d_1)$ is a submodule of $\mathfrak{M}_1$ and therefore finitely generated by the Noetherian property of $\mathfrak{M}_1$. This argument extends to all homological degrees.

1.1 Minimal resolutions and first examples

We start this subsection by giving a proper definition for a minimal set of generators of an $\mathcal{O}$-module M.

Definition 1.6. *Let $\Phi = \{\phi_1, \dots, \phi_k\}$ be a set of generators of an $\mathcal{O}$-module M, i.e. $M = \mathcal{O}\phi_1 + \dots + \mathcal{O}\phi_k$. Φ is said to be minimal if any proper subset of Φ does not generate M.*

While constructing a resolution of $\mathcal{O}/\mathcal{I}$ one can pose a natural question: can we find a "minimal" free resolution? The property of being minimal can be addressed at each step of the Construction 1.2: use a minimal set of generators of $\mathcal{I}$ to construct $\mathfrak{M}_1$. Then at a homological degree i choose a minimal set of generators of kernel d_i and use it for defining $\mathfrak{M}_{i+1}$. However, using a minimal set of generators turns out to be a not well-thought idea in general. As a simple example shows, these sets are not unique and even the minimal number of generators is not defined:

Example 1.7. Let $\mathcal{I}$ be an ideal generated by 7 in $\mathbb{Z}$. Then both $G_1 = \{7\}$ and $G_2 = \{28, 35\}$ generate $\mathcal{I}$. These two generating sets are minimal in the sense of Definition 1.6.

However, in two distinct situations, when $\mathcal{O}$ is a local or a graded polynomial ring, due to Nakayama's lemma the concept of minimality persists, a minimal free resolution exists and is unique.

Let us briefly introduce the neccessary concepts:

Definition 1.8. *A local ring R is a unital commutative ring with exactly one maximal ideal $\mathcal{J}$.*

To emphasize the maximal ideal $\mathcal{J}$ we denote sometimes a local ring as a pair $(R, \mathcal{J})$. A typical example of local rings is a ring of formal power series $\mathbb{C}[[x_1, \ldots, x_n]]$. The maximal ideal $\mathcal{J}$ comprises of all elements whose constant term is zero. A non-example is a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ whose maximal ideals $\mathcal{J}_{\mathbf{a}}$ are parametrized by $\mathbf{a} \in \mathbb{C}^n$. Each $\mathcal{J}_{\mathbf{a}}$ is the collection of polynomials vanishing at $\mathbf{a}$.

Lemma 1.9. *The following statements are equivalent:*

- *R is a local ring.*
- *R has a maximal ideal $\mathcal{M}$ and any non-zero element of $R \backslash \mathcal{M}$ is a unit.*

Proof. Proof by contraposition. Let $\mathcal{M}$ be a maximal ideal in R.

If $x \in R \backslash \mathcal{M}$ and is not a unit (i.e. non-invertible), then it is contained in a maximal ideal $\mathcal{M}'$. Therefore, R has at least two distinct maximal ideals.

If $\mathcal{M}'$ is another maximal ideal of R, choose $x \in \mathcal{M}'$, $x \notin \mathcal{M}$. Then x is not a unit. □

Since every element of $R \backslash \mathcal{M}$ is a unit, $\mathbb{K} := R/\mathcal{M}$ is a field called a *residue field*. More generally, for any R-module N, $N/\mathcal{M}N$ is a $\mathbb{K}$-vector space.

Example 1.10. *A residue field of $\mathbb{C}[[x]]$ is just $\mathbb{C}$. Any element of the form $1 + f$, where f being any formal power series with zero constant coefficient, has an inverse $\frac{1}{1+f}$ understood as a formal Taylor expansion around 0.*

Lemma 1.11 (Nakayama's Lemma)**.**

Let M be a finitely generated R-module and $\mathcal{I} \subset R$ a proper ideal. If $M = \mathcal{I}M$ then there exists $a \in R$ such that $aM = 0$ and $a = 1 \mod \mathcal{I}$.

Proof. Let $y_1, \ldots y_s$ be generators of M. We prove the statement by induction on the number of generators of M. Take $M' = M/Ry_s$, which satisfies by induction assumption $(1 + x)M' = 0$ for some $x \in \mathcal{I}$. Note if $s = 1$ this is trivially satisfied.

Since $M = \mathcal{I}M$, $(1 + x)M \subset Ry_s \subset \mathcal{I}y_s$. In particular $(1 + x)y_s = zy_s$ for some $z \in \mathcal{I}$. Therefore $(1 + x - z)(1 + x)M = 0$. The last equality can be expressed as $(1 + f)M = 0$ for some $f \in \mathcal{I}$. □

Corollary 1.12. *In the setting of Lemma 1.11 if R is a local ring then $M = \mathcal{J}M$ implies $M = 0$. Moreover, if $M = W + JM$ for some R-module W, then $M = W$.*

Proof. The first statement follows from the fact that $1 + f$ is an invertible element in R. The second is a corollary to the first one for the module M/W. □

Nakayama's lemma proves to be extremely useful when dealing with minimal sets of generators. The main properties of such sets can be summarized in the following proposition:

Proposition 1.13. *Let $(R, \mathcal{J})$ be a local ring with a residue field $\mathbb{K}$ and M be a finitely generated $\mathcal{O}$-module. Let us denote $\overline{M} = M/\mathcal{J}M$ a $\mathbb{K}$-vector space, whose $\dim_{\mathbb{K}}(\overline{M})$ we denote n. Then*

(1) *A preimage in M of a basis $\{\overline{e}_1, \ldots, \overline{e}_n\}$ of $\overline{M}$ is a minimal set of generators of M.*

(2) *Every minimal set of generators is obtained as in (1), therefore it has n elements.*

(3) *If* $\{e_1,\dots,e_n\}$ *and* $\{f_1,\dots,f_n\}$ *are minimal sets of generators of* M *and* $e_i = \sum_{j=1}^n A_{ij} f_j$. *Then the determinant of the matrix* (A_{ij}) *is a unit, therefore* (A_{ij}) *is invertible.*

Proof. (1): Let us denote a preimage of $\{\overline{e}_1,\dots,\overline{e}_n\}$ in M by $\{e_1,\dots,e_n\}$. We can write $M = \sum_{i=1}^n Re_i, +\mathcal{J}M$. Since M is finitely generated, $M/\sum_{i=1}^n Re_i$ is finitely generated as well, and $M/\sum_{i=1}^n Re_i = \mathcal{J}(M/\sum_{i=1}^n Re_i)$. By Nakayama's lemma we deduce $M = \sum_{i=1}^n Re_i$. If $\{e_1,\dots,e_n\}$ is not a minimal set of generators, then one of its proper subsets $\{e_{j_1},\dots,e_{j_k}\}$, $k < n$ is. By quotienting out $\mathcal{J}M$ we obtain that $\{\overline{e}_{j_1},\dots,\overline{e}_{j_k}\}$ is a basis of $\overline{M}$. Contradiction.

(2): Let $e = \{e_1,\dots,e_n\}$ be a minimal set of generators of M. Then its image $\{\overline{e}_1,\dots,\overline{e}_n\}$ under the quotient by $\mathcal{J}M$ is a basis of $\overline{M}$. Indeed, if its any proper subset $\{\overline{e}_{j_1},\dots,\overline{e}_{j_k}\}$ is a basis of $\overline{M}$, then by (1) we get that $\{e_{j_1},\dots,e_{j_k}\}$ is a generating set of M and is a proper subset of e. Contradiction.

(3): Take a quotient by $\mathcal{J}$ of $e_i = \sum_{j=1}^n A_{ij} f_j$. The corresponding matrix $(\overline{A}_{ij})$ has a non-zero determinant valued in $\mathbb{K}$. Its preimage, $detA \notin \mathcal{J}$, therefore it is invertible, i.e. a unit. The inverse matrix is a multiple of Cramer adjoint, $A^{-1} = (detA)^{-1} \cdot A^{\vee}$

□

Now we are finally able to give a definition of a minimal resolution:

Definition 1.14. *Let* $(R,\mathcal{J})$ *be a local ring and* $\mathcal{I} \subset R$ *a proper ideal. A resolution of* $R/\mathcal{I}$

$$\cdots \xrightarrow{d_{k+1}} \mathfrak{M}_k \xrightarrow{d_k} \cdots \xrightarrow{d_2} \mathfrak{M}_1 \xrightarrow{d_1} \mathcal{O} \xrightarrow{0} 0$$

is called minimal if $d_{i+1}(\mathfrak{M}_{i+1}) \subseteq \mathcal{J}\mathfrak{M}_i$.

This, at the first sight strange, definition gives exactly the naive notion of "minimality" of a free resolution, i.e. such resolutions can be obtained by choosing a minimal set of generators in Construction 1.2 as illustrated in the following proposition:

Proposition 1.15. *Let* $(R,\mathcal{J})$ *be a Noetherian local ring with a residual field* $\mathbb{K}$ *and* $\mathcal{I} \subset R$ *a proper ideal. The resolution described in Construction 1.2 is minimal if and only if at each degree* i *we choose* $\mathcal{C}_i$ *to be a minimal set of generators of* $Ker(d_i)$.

Proof. Assume that the resolution $(\mathfrak{M}_\bullet, d)$ is not minimal. Therefore $\exists i \geq 0$ such that $d_{i+2}(\mathfrak{M}_{i+2}) \nsubseteq \mathcal{J}\mathfrak{M}_{i+1}$ (in case $i = -1$ the image of d_{i+2} is automatically in $\mathcal{I} \subset \mathcal{J}\mathfrak{M}_0$). This means that there is an element a in $\mathrm{Im}(d_{i+1}) = \mathrm{Ker}(d_{i+1})$ such that

$$a = \sum_{\phi\in\mathcal{C}_i} a_\phi f_\phi \tag{3}$$

with at least one of a_j being a unit. Recall that as in Construction 1.2 we denote $\{f_\phi \,|\, \phi \in \mathcal{C}_i\}$ a basis of $\mathfrak{M}_{i+1}$. After multiplying (3) by a_j^{-1} and renumbering elements of $\mathcal{C}_i$ we obtain

$$a = f_{\phi_1} + \sum_{\phi\in\mathcal{C}_i/\phi_1} a_\phi f_\phi \tag{4}$$

Applying d to (4) we obtain $\phi_1 + \sum_{\phi\in\mathcal{C}_i/\phi_1} a_\phi \,\phi = 0$, therefore the set of generators $\mathcal{C}_i$ is not minimal.

Now, let us assume that at a homological degree $i \geq 0$ we chose a non-minimal system of generators of kernel of d_i. Renumbering the generators, we can write

$$\phi_1 + \sum_{\phi\in\mathcal{C}_i/\phi_1} a_\phi \,\phi = 0. \tag{5}$$

For some $a_\phi \in R$. (5) is tantamount to a generator ϕ_1 being redundant. Due to each ϕ being in the kernel of d, at degree $i+1$ the folowing combination is exact:

$$f_{\phi_1} + \sum_{\phi \in \mathcal{C}_i/\phi_1} a_\phi f_\phi \tag{6}$$

and therefore, belongs to $\mathrm{Im}(d_{i+2})$. Thus $d_{i+2}(\mathfrak{M}_{i+2}) \nsubseteq \mathcal{J}\mathfrak{M}_{i+1}$. □

Remark 1.16. In Proposition 1.15 we asked additionally for $(R, \mathcal{J})$ to be Noetherian. This is done in order to ensure that $\mathrm{Ker}(d_i)$ is finitely generated, thus the properties of minimal sets of generators as in Proposition 1.13 hold.

Example 1.17. Let $R = \mathbb{C}[[x, y, z]]$ and $\mathcal{I}$ is generated by $\mathcal{C}_0 = \{x^3, y^2, yz\}$. The minimal free resolution of $R/\mathcal{I}$ is

$$0 \xrightarrow{0} R \xrightarrow{d_3} R^3 \xrightarrow{d_2} R^3 \xrightarrow{d_1} R \xrightarrow{0} 0$$

The ranks of $\mathfrak{M}_\bullet$ are $1, 3, 3, 1$ in degree $3, 2, 1, 0$ respectively. The differential d is given by:

In homological degree 1 $\quad d_1e_1 = x^3, \quad d_1e_2 = y^2, \quad d_1e_3 = yz.$

In homological degree 2 $\quad d_2f_1 = ze_2 - ye_3, \quad d_2f_2 = yze_1 - x^3e_3, \quad d_2f_3 = y^2e_1 - x^3e_2$

In homological degree 3 $\quad d_3g = x^3f_1 - yf_2 + zf_3$

1.2 Betti numbers

Given an $\mathcal{O}$-module N and a free resolution $(\mathfrak{M}_\bullet, d)$ of an $\mathcal{O}$-module M a natural question is to study a complex $(\mathfrak{M}_\bullet \otimes_R N, d \otimes \mathrm{id})$. As we will see in a short while, this complex in general is not acyclic. The homology of this complex at homological degree i is denoted by $\mathrm{Tor}_i^R(M, N)$.

Example 1.18. *Let R be $\mathbb{R}[x, y]$, $\mathcal{I}$ be an ideal generated by x and N be a quotient R/x^2. A resolution of $\mathcal{O}/\mathcal{I}$ has a form* $0 \xrightarrow{0} R \xrightarrow{x} R \xrightarrow{0} 0$. *Then a tensor product with N gives rise to a complex* $0 \xrightarrow{0} R \otimes_R N \xrightarrow{x} R \otimes_R N \xrightarrow{0} 0$. *A straightforward calculation shows that $Tor_1^R(\mathcal{O}/\mathcal{I}, N)$ is generated by $1 \otimes_R x$ as an $\mathcal{O}$-module and $Tor_0^R(\mathcal{O}/\mathcal{I}, N) = \mathcal{O}/\mathcal{I} \otimes_R N$.*

Proposition 1.19. List of properties of $\mathrm{Tor}_\bullet^R(M, N)$

i. $\mathrm{Tor}_0^R(M, N) = M \otimes_R N$.

ii. If N is free, then $\mathrm{Tor}_i^R(M, N) = 0$ for $i \geq 1$.

iii. It is symmetric in M and N, i.e. $\mathrm{Tor}_\bullet^R(M, N) = \mathrm{Tor}_\bullet^R(N, M)$

iv. $\mathrm{Tor}_\bullet^R(M, N)$ is unique up to an isomorphism, i.e. it does not depend on a choice of $(\mathfrak{M}_\bullet, d)$.

The proof of this Proposition can be found in a textbook of commutative algebra, see e.g. [Eis95].

Definition 1.20. *Let $(R, \mathcal{J})$ be a local ring with a residue field $\mathbb{K}$, let $\mathcal{I}$ be a proper ideal of R and $(\mathfrak{M}_\bullet, d)$ is a resolution of $R/\mathcal{I}$. Then the Betti numbers b_i are defined as the $\mathbb{K}$-dimension of i-th homology group of a complex $(\mathfrak{M}_\bullet \otimes_R \mathbb{K}, d \otimes \mathrm{id})$, i.e. $b_i = \dim_{\mathbb{K}}(\mathrm{Tor}_i^R(R/\mathcal{I}, \mathbb{K}))$.*

Remark 1.21. *The Betti numbers b_i do not depend on a choice of resolution of $\mathcal{O}/\mathcal{I}$ since $\mathrm{Tor}_i^R(R/\mathcal{I}, \mathbb{K})$ does not. If $(\mathfrak{M}_\bullet, d)$ is a minimal resolution, then the rank of $\mathfrak{M}_i$ coincides with the Betti number b_i. In Example 1.17 they are equal to $1, 3, 3, 1$ respectively.*

1.3 Uniqueness of a minimal resolution

In this subsection R is a Noetherian local ring with a maximal ideal $\mathcal{J}$ and a residue field $\mathbb{K}$.

Definition 1.22. *$(\mathcal{A}_\bullet, d_\mathcal{A})$ and $(\mathcal{B}_\bullet, d_\mathcal{B})$ be two chain complexes. A direct sum of $(\mathcal{A}_\bullet, d_\mathcal{A})$ and $(\mathcal{B}_\bullet, d_\mathcal{B})$ is a chain complex $(\mathcal{C}_\bullet, d_\mathcal{C})$ such that*

i. $\mathcal{C}_i = \mathcal{A}_i \oplus \mathcal{B}_i$ *as R-modules.*

ii. $(d_\mathcal{C})_i = (d_\mathcal{A})_i \oplus (d_\mathcal{B})_i$

Homology of a direct sum of complexes is a direct sum of homologies of its component complexes.

Definition 1.23. *A trivial complex is a direct sum of complexes of the form*

$$0 \xrightarrow{0} R \xrightarrow{1} R \xrightarrow{0} 0. \tag{7}$$

Example 1.24. *We can form a trivial complex*

$$0 \xrightarrow{0} R \longrightarrow R^2 \longrightarrow R^2 \longrightarrow R \longrightarrow 0.$$

by taking several copies of (7) with shifted degrees, e.g.:

$$\begin{array}{ccccccccccc}
0 & \xrightarrow{0} & R & \xrightarrow{1} & R & \xrightarrow{0} & 0 & & & & \\
 & & \oplus & & \oplus & & \oplus & & & & \\
 & & 0 & \xrightarrow{0} & R & \xrightarrow{1} & R & \xrightarrow{0} & 0 & & \\
 & & & & \oplus & & \oplus & & \oplus & & \\
 & & & & 0 & \xrightarrow{0} & R & \xrightarrow{1} & R & \xrightarrow{0} & 0
\end{array}$$

Trivial complexes have no homology, so if $(\mathfrak{M}_\bullet, d)$ is a resolution of M then so is $(\mathfrak{M}_\bullet \oplus \mathbf{T}_\bullet, d \oplus d_\mathbf{T})$ for a trivial complex $(\mathbf{T}_\bullet, d_\mathbf{T})$. It turns out that over a local ring (also for a graded polynomial ring, see Section 2) this is the only source of a nonuniqueness of free resolutions.

Lemma 1.25. *If*

$$(\mathfrak{M}_\bullet, d): \cdots \xrightarrow{d_{k+1}} \mathfrak{M}_k \xrightarrow{d_k} \cdots \xrightarrow{d_2} \mathfrak{M}_1 \xrightarrow{d_1} \mathfrak{M}_0 \xrightarrow{0} 0$$

is a complex of finitely generated at each degree free modules with trivial homology, i.e. a free resolution of 0, then $(\mathfrak{M}_\bullet, d)$ is a trivial complex.

Proof. Since $\mathfrak{M}_0$ is free, by Lemma 3.4 $\mathfrak{M}_1$ splits into $\mathfrak{M}_1 \cong \mathfrak{M}_0 \oplus \mathfrak{M}_1'$. Note that $\operatorname{Im}(d_2) = \operatorname{Ker}(d_1) = \mathfrak{M}_1'$. Thus the complex $(\mathfrak{M}_\bullet, d)$ is a direct sum of complexes

$$\mathcal{A}_0: \ 0 \longrightarrow \mathfrak{M}_0 \xrightarrow{1} \mathfrak{M}_0 \xrightarrow{0} 0$$

and

$$(\mathfrak{M}^1_\bullet, d): \cdots \xrightarrow{d_{k+1}} \mathfrak{M}_k \xrightarrow{d_k} \cdots \xrightarrow{d_3} \mathfrak{M}_2 \xrightarrow{d_2} \mathfrak{M}_1' \xrightarrow{0} 0$$

$\mathcal{A}_0$ is a trivial complex. $(\mathfrak{M}^1_\bullet, d)$ is a complex with trivial homology which consists of free modules with a possible exception for $\mathfrak{M}_1'$, which is at least projective. Due to Lemma 3.6 $\mathfrak{M}_1'$ is actually free. Applying the same argument we find that $(\mathfrak{M}^1_\bullet, d) = \mathcal{A}_1 \oplus (\mathfrak{M}^2_\bullet, d)$ where $\mathcal{A}_1$ is a trivial complex and $(\mathfrak{M}^2_\bullet, d)$ is a complex of free modules with trivial homology that starts with terms at least of degree 2. Repeating the argument, we find $(\mathfrak{M}_\bullet, d) = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_k \oplus \ldots$ where each summand is a trivial complex. □

Theorem 1.26. *Let R be a Noetherian local ring and let M be a finitely generated R-module. If $(\mathfrak{M}_\bullet, d)$ is a minimal free resolution of M, then any free resolution of M is isomorphic to the direct sum of $(\mathfrak{M}_\bullet, d)$ and a trivial complex.*

Proof. Let $(\mathcal{G}_\bullet, \delta)$ be a free resolution of M and $(\mathfrak{M}_\bullet, d)$ be a minimal resolution of M. By Corollary 3.13 they are homotopy equivalent, that is there exists chain maps $\alpha : (\mathfrak{M}_\bullet, d) \longrightarrow (\mathcal{G}_\bullet, \delta)$ and $\beta : (\mathcal{G}_\bullet, \delta) \longrightarrow (\mathfrak{M}_\bullet, d)$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are homotopic to identity. In particular this means $\beta \circ \alpha - 1 = d \circ h + h \circ d$ for some chain homotopy h. Since $d(\mathfrak{M}_\bullet) \subseteq \mathcal{J}\mathfrak{M}_\bullet$, $\beta \circ \alpha = 1 \operatorname{mod} \mathcal{J}$. In particular for each homological degree i $\det(\beta_i \circ \alpha_i)$ is invertible. Therefore, there exists an inverse of $\beta \circ \alpha$ that we denote by γ. For each i consider the following sequence of morphisms:

$$\mathfrak{M}_i \xrightarrow{\alpha_i} \mathcal{G}_i \xrightarrow{\gamma_i \circ \beta_i} \mathfrak{M}_i$$

This sequence is splitting, and by Lemma 3.3 $\mathcal{G}_i = \alpha(\mathfrak{M}_i) \oplus \operatorname{Ker}(\gamma_i \circ \beta_i)$. Denote R-module $\operatorname{Ker}(\gamma_i \circ \beta_i)$ by T_i, which is at least projective. By Lemma 3.6 T_i is free. Let us check that δ preserves the direct sum decomposition, i.e. $\delta(\alpha(\mathfrak{M}_i)) \subset \alpha(\mathfrak{M}_{i-1})$ and $\delta(T_i) \subset T_{i-1}$. The first inclusion follows directly from α being a chain map. To verify the second one, take an arbitrary $x \in T_i$. Then

$$\gamma_{i-1} \circ \beta_{i-1} \circ \delta(x) = \gamma_{i-1} \circ d \circ \beta_i(x) = \gamma_{i-1} \circ d \circ \gamma_i^{-1} \circ \gamma_i \circ \beta_i(x) = 0.$$

Thus $\delta(T_i) \subset T_{i-1}$. Therefore $\mathcal{G}_\bullet = T_\bullet \oplus \mathfrak{M}_\bullet$ as chain complexes. Since homology of $\mathcal{G}_\bullet$ is a direct sum of homologies of $T_\bullet$ and $\mathfrak{M}_\bullet$ and, on the other hand is equal to homology of $\mathfrak{M}_\bullet$ since both are resolutions of M we deduce that $T_\bullet$ is acyclic. Since $T_\bullet$ is acyclic and consists of free modules T_i by Lemma 1.25 we obtain that $T_\bullet$ is a trivial complex. □

Remark 1.27. In practice, Theorem 1.26 is behind the algorithm of elimination of "contractible pairs": suppose we are given a resolution $(\mathcal{G}_\bullet, d)$ such that it satisfies $d(\mathcal{G}_j) \subset \mathcal{J}\mathcal{G}_{j-1}$ for $j < i+1$. Then we can find a direct sum decomposition $\mathcal{G}_k = T_k \oplus \mathfrak{M}_k$ for $k = i, i+1$ such that $d_{i+1} : \mathcal{G}_{i+1} \longrightarrow \mathcal{G}_i$ preserves the decomposition and its restriction to T_{i+1} is an isomorphism. By Lemma 3.6 $\mathfrak{M}_{i+1}$, $\mathfrak{M}_i$ and T_{i+1}, T_i are free, therefore a complex

$$0 \xrightarrow{0} T_{i+1} \xrightarrow{d} T_i \xrightarrow{0} 0.$$

is a trivial complex. Thus a "smaller" complex $(\mathcal{G}'_\bullet, d')$ obtained by replacing $\mathcal{G}_j$ by $\mathfrak{M}_j$ for $j = i, i+1$ is still a free resolution of M. Note that $(\mathcal{G}'_\bullet, d')$ is well defined, i.e. $d(\mathcal{G}_{i+2}) \cap T_{i+1} = 0$, otherwise it contradicts the decomposition and the action of d on it. The procedure can be repeated for degrees $i+2, i+1$ and its outcome is always an elimination of pairs T_{i+2}, T_{i+1}, hence the name.

1.4 Koszul complex

In this section we assume R is a Noetherian ring and M is a finitely generated R-module. An element $x \in R$ is a non-zero divisor on M if $x \cdot m \neq 0$ for all non-zero $m \in M$. It is also said to be an *M-regular* element. We abbreviate by $(y_1, \ldots, y_n)$ an ideal $\mathcal{I} \subset R$ generated by $y_1, \ldots, y_n$.

Definition 1.28. *A sequence $f = f_1, \ldots, f_k$ of elements in R is M-regular if*

i. $(f_1, \ldots, f_k)M \neq M$.

ii. f_1 is a non-zero divisor on M and for all $i > 1$ f_i is a non-zero divisor on $M/(f_1, \ldots, f_{i-1})M$.

Construction 1.29. Let $f = f_1, \dots, f_k$ be a sequence of elements in R. Let V be a free R-module $\underbrace{R \oplus R \oplus \cdots \oplus R}_{k \text{ times}}$ with a basis $e = \{e_1, \dots, e_k\}$. Form an exterior R-algebra $E = \wedge^{\bullet} V$, i.e. take a quotient of a tensor algebra $T(V) = R \oplus V \oplus V \otimes_R V \oplus \cdots$ by an ideal I generated by $\{e_i^2, e_i e_j + e_j e_i \,|\, i, j \in [\![1, \dots, k]\!]\}$. Multiplication on E is denoted by $\wedge$, which is associative. and is anticommutative on e: $e_i \wedge e_j = -e_j \wedge e_i$. E is a graded free R-module, its degree i component E_i is generated by elements $\{e_{j_1} \wedge \cdots \wedge e_{j_i} \,|\, 1 \le j_1 < \cdots < j_i \le k\}$, thus E_i has a rank $\binom{k}{i}$ The multiplication $\wedge$ is compatible with the grading, i.e $E_i \wedge E_j \subseteq E_{i+j}$, thus E is a graded algebra.

Definition 1.30. *A Koszul complex $\mathcal{K}(f)$ on a sequence $f = f_1, \dots, f_k$ is the graded algebra E from Construction 1.29 equipped with a differential*

$$d(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{l=1}^{i} (-1)^{l+1} f_{j_l} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_l} \wedge \cdots \wedge e_{j_i}$$

where ˆ means that the corresponding term is omitted from the product.

Remark 1.31. The following computation ensures that the differential d squares to zero on $\mathcal{K}(f)$:

Evaluate d^2 on $e_{j_1} \wedge \cdots \wedge e_{j_i}$:

$$d^2(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{1 \le l < p \le i}^{i} A_{lp} f_{j_l} f_{j_p} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_l} \wedge \cdots \wedge \hat{e}_{j_p} \wedge \cdots \wedge e_{j_i}$$

for some coefficient A_{lp}. To find A_{lp} we should consider two terms obtained from the action of d^2 on $(e_{j_1} \wedge \cdots \wedge e_{j_i})$:

1. Remove e_{j_l} and then e_{j_p}.
2. Remove e_{j_p} and then e_{j_l}.

In 1. we obtain $f_{j_l} f_{j_p} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_l} \wedge \cdots \wedge \hat{e}_{j_p} \wedge \cdots \wedge e_{j_i}$ with a sign prefactor $(-1)^{l+1+p}$. In 2. the prefactor is $(-1)^{l+p}$. A_{lp} is a sum of the prefactors, thus $A_{lp} = 0$.

Remark 1.32. $\mathcal{K}(f)$ is a complex of a finite length k:

$$\mathcal{K}(f):\ 0 \xrightarrow{d} \mathcal{K}_k \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{K}_1 \xrightarrow{d} \mathcal{K}_0 \xrightarrow{0} 0$$

with $\mathrm{rk}(\mathcal{K}_i) = \binom{k}{i}$. If f is empty, then $\mathcal{K}(f) = \mathcal{K}_0 = R$.

We write $\mathcal{K}(f; M)$ for $\mathcal{K}(f) \otimes_R M$. Clearly $\mathcal{K}(f; R) = \mathcal{K}(f)$. The homology of $\mathcal{K}(f; M)$ is called the *Koszul homology* of M.

Remark 1.33. $H_0(\mathcal{K}(f; M)) = M/(f_1, \dots, f_k)M$. This follows directly from observing that $d(\mathcal{K}_1(f; M)) = (f_1, \dots, f_k)M$ and $H_0(\mathcal{K}(f; M)) = M/d(\mathcal{K}_1(f; M))$.

Example 1.34. Let us give an example of the Koszul complex. We use the data of Example 1.17: $R = C[[x, y, z]]$ and f is given by x^3, y^2, yz. The Koszul complex is

$$0 \xrightarrow{0} R \xrightarrow{d_3} R^3 \xrightarrow{d_2} R^3 \xrightarrow{d_1} R \xrightarrow{0} 0$$

The ranks of $\mathcal{K}_i$ are $1, 3, 3, 1$ in degree $3, 2, 1, 0$ respectively. The differential d is given by:

In homological degree 1	$d_1 e_1 = x^3, \quad d_1 e_2 = y^2, \quad d_1 e_3 = yz.$
In homological degree 2	$d_2(e_3 \wedge e_2) = yze_2 - y^2 e_3, \quad d_2(e_3 \wedge e_1) = yze_1 - x^3 e_3$ $d_2(e_2 \wedge e_1) = y^2 e_1 - x^3 e_2.$
In homological degree 3	$d_3 g = x^3 f_1 - y^2 f_2 + yz f_3.$

In contrast to Example 1.17, the Koszul complex has a homology in degrees $i > 0$: $H_1(\mathcal{K}(f))$ is generated by $ze_2 - ye_3$.

One of the main properties of Koszul complex is that whenever f is M-regular, then $\mathcal{K}(f)$ is a resolution of $M/(f_1, \dots, f_k)M$. To prove this result, we consider first the following lemmas:

Lemma 1.35. *Let $f = f_1, \dots, f_k$ be a sequence of elements in R, and let $\overline{f} = f \backslash f_k$. Then there is a short exact sequence of complexes*

$$0 \longrightarrow \mathcal{K}(\overline{f}; M) \longrightarrow \mathcal{K}(f; M) \longrightarrow \mathcal{K}(\overline{f}; M)[-1] \longrightarrow 0$$

and a connecting morphism τ of the corresponding long exact sequence

$$\cdots \xrightarrow{\tau_{i+1}} H_i(\mathcal{K}(\overline{f}; M)) \longrightarrow H_i(\mathcal{K}(f; M)) \longrightarrow H_{i-1}(\mathcal{K}(\overline{f}; M)) \xrightarrow{\tau_i} H_{i-1}(\mathcal{K}(\overline{f}; M)) \longrightarrow \cdots$$

is given by $\tau_i = (-1)^{i-1} f_k$, i.e. a multiplication by $(-1)^{i-1} f_k$.

Proof. First note that we can decompose $\mathcal{K}(f; M)$ as a direct sum of graded modules $\mathcal{K}(f; M) = \mathcal{K}(\overline{f}; M) \oplus \mathcal{K}(\overline{f}; M) \wedge e_k$, which reads at degree i as $\mathcal{K}(f; M)_i = \mathcal{K}(\overline{f}; M)_i \oplus \mathcal{K}(\overline{f}; M)_{i-1} \wedge e_k$. Inclusion of a first factor $\mathcal{K}(\overline{f}; M)$ into $\mathcal{K}(f; M)$ is obviously a chain map. Projection of $\mathcal{K}(f; M)$ onto $\mathcal{K}(\overline{f}; M)$ of the second factor is a chain map as well. Therefore we have the short exact sequence.

Let us compute explicitly the connecting homomorphism. Let $x \in H_i(\mathcal{K}(\overline{f}; M)[-1]) = H_{i-1}(\mathcal{K}(\overline{f}; M)$ Take its reperesentative x', a closed element in $\mathcal{K}(\overline{f}; M)[-1]_i = \mathcal{K}(\overline{f}; M)_{i-1}$. Choose its preimage in $\mathcal{K}(f; M)_i$ as $x' \wedge e_k$. Then $d(x' \wedge e_k) = (-1)^{i-1} f_k \cdot x'$ which belongs to $\mathcal{K}(\overline{f}; M)_{i-1}$. Therefore on a homology level a connecting homomorphism τ_i is a multiplication by $(-1)^{i-1} f_k$. □

Lemma 1.36. *Let $f = f_1, \dots, f_k$ a sequence of elements in R. Then $(f_1, \dots, f_k) H_i(\mathcal{K}(f; M)) = 0$ for all $i \geq 0$.*

Proof. Let $a \in H_i(\mathcal{K}(f; M))$. Choose its reperesentative a' in $\mathcal{K}(f; M)_i$. Then, for any f_j, $j \in [\![1, \dots, k]\!]$ we have $f_j a' = d(e_j a') - e_j d(a') = d(e_j a')$. Therefore $f_j a'$ is exact and its homology class is zero. □

Now we are ready to prove the acyclicity of Koszul complex when f is M-regular.

Proposition 1.37. *If $f = f_1, \dots, f_k$ is M-regular, then $H_i(\mathcal{K}(f; M)) = 0$ for $i > 0$.*

Proof. We use induction on a number k of generators to prove that $H_i(\mathcal{K}(f; M)) = 0$ for $i > 0$:

For $k = 1$ the Koszul complex reads as

$$0 \xrightarrow{0} M \xrightarrow{f_1} M \xrightarrow{0} 0.$$

With $H_1(\mathcal{K})(f; M)) = \{x \in M \mid f_1 \cdot x = 0\}$, which is 0 since f_1 is a non-zero divisor.

Now, assume for $f^{q-1} = f_1, \dots, f_{q-1}$, $H_i(\mathcal{K}(f^{q-1}; M)) = 0$ for $i > 0$. We show that for $f^q = f_1, \dots, f_q$, $H_i(\mathcal{K}(f^q; M)) = 0$:

From the long exact sequence of Lemma 1.35 we obtain:

$$0 \longrightarrow H_i(\mathcal{K}(f^q; M)) \longrightarrow 0$$

for $i > 1$ therefore $H_j(\mathcal{K}(f^q; M)) = 0$ for $j > 1$. If $i = 1$, then

$$0 \longrightarrow H_1(\mathcal{K}(f^q; M)) \longrightarrow H_0(\mathcal{K}(f^{q-1}; M)) \xrightarrow{\tau_1} H_0(\mathcal{K}(f^{q-1}; M))$$

Since τ_1 is a multiplication by f_q, and f_q is a non-zero divisor on $H_0(\mathcal{K}(f^{q-1}; M)) = M/(f_1, \dots, f_q)M$, $\mathrm{Ker}(\tau_1) = 0$, thus $H_1(\mathcal{K}(f^q; M)) = 0$. The claim of the proposition follows. □

Furthermore, if R is local, then the converse holds true: if we are given an acyclic Koszul complex $\mathcal{K}(f;M)$, then f is M-regular. Initially, verifying the acyclicity of a Koszul complex may seem like a tedious task, as one needs to identify the kernel of d within modules containing $\binom{k}{j}$ generators. However, simply requiring the vanishing of the first homology $H_1(\mathcal{K}(f;M)) = 0$ is sufficient to conclude that f is M-regular. This proposition succinctly captures this statement:

Proposition 1.38. *If R is Noetherian local ring, and $f = f_1, \dots, f_k$ be sequence of elements in R generating a proper ideal. Then $H_1(\mathcal{K}(f;M)) = 0$ implies that f is M-regular.*

Proof. As in the previous proposition, we denote a sequence $f_1, \dots, f_q$ by f^q (therefore $f = f^k$). The maximal ideal of R is $\mathcal{J}$.

In order to prove the statement we first prove an auxiliary one: $H_1(\mathcal{K}(f;M)) = 0$ for $i > 0$ implies $H_1(\mathcal{K}(f^q;M)) = 0$ for all $q \in [\![1,k]\!]$. This is done by recursion on q: Consider a part of the long exact sequence

$$H_2\mathcal{K}(f^q;M)) \longrightarrow H_1(\mathcal{K}(f^{q-1};M)) \xrightarrow{\tau_2} H_1(\mathcal{K}(f^{q-1};M)) \longrightarrow H_1(\mathcal{K}(f^q;M)) \qquad (8)$$

For $q = k$ we have zero entries on the rightmost position, therefore $H_1(\mathcal{K}(f^{k-1};M)) = \tau_2 H_1(\mathcal{K}(f^{k-1};M))$. Since τ_2 is a multiplication by f_q up to a sign, by Nakayama's lemma we obtain $H_1(\mathcal{K}(f^{k-1};M)) = 0$. Assume now that $H_1(\mathcal{K}(f^q;M))$ is zero. Using the same argument we obtain $H_1(\mathcal{K}(f^{q-1};M)) = 0$, thus the auxiliary statement holds.

Now, to prove the proposition, consider the following part of the long exact sequence: for any $q > 0$

$$0 \longrightarrow H_1(\mathcal{K}(f^q;M)) \longrightarrow H_0(\mathcal{K}(f^{q-1};M)) \xrightarrow{\tau_1} H_0(\mathcal{K}(f^{q-1};M))$$

Since $H_1(\mathcal{K}(f^q;M)) = 0$, multiplication of $H_0(\mathcal{K}(f^{q-1};M)) = M/(f_1, \dots, f_q)M$ by f_q has no kernel. Therefore f_q, $q > 1$ is an M-regular element in $M/(f_1, \dots, f_{q-1})M$. For $q = 1$, f_q is a regular element on M. □

Corollary 1.39. *Let R be a Noetherian local ring. If $f = f_1, \dots, f_k$ is M-regular then a sequence $\sigma(f) = f_{\sigma_1} \dots f_{\sigma_k}$ obtained by permuting elements of f is M-regular.*

Proof. The Koszul complex $\mathcal{K}(\sigma(f);M)$ is $\mathcal{K}(f;M)$ up to a change of basis, therefore it is exact in positive degrees. By Proposition 1.38 $\sigma(f)$ is regular. □

Lastly, we give an example of an acyclic Koszul complex:

Example 1.40. Let $R = \mathbb{K}[x_1, \dots, x_n]$ and $f = x_1, \dots, x_n$. The Koszul complex $\mathcal{K}(f)$ is a minimal free resolution of $\mathbb{K} = R/(x_1, \dots, x_n)$ in R-modules.

1.5 Polynomial rings and graded resolutions

We now proceed with studying resolutions of ideals of polynomial rings. In this chapter $\mathcal{O}$ denotes a polynomial ring $\mathbb{K}[x_1, \dots, x_n]$. We use the *standard grading* of the polynomial ring $\mathcal{O}$, i.e we set for each x_i $\operatorname{pol}(x_i) = 1$. We define the degree of multiplication of two monomials u, v to be $\operatorname{pol}(uv) = \operatorname{pol}(u) + \operatorname{pol}(v)$. In particular, a monomial $x_1^{a_1} x_2^{a_2} \cdot \dots \cdot x_k^{a_k}$ has a degree $a_1 + \dots + a_k$. The ideal $\mathcal{J} = \langle x_1, \dots, x_n \rangle_{\mathcal{O}}$ is called the (irrelevant) maximal ideal of $\mathcal{O}$.

Denote the $\mathbb{K}$-vector space of all monomials of degree i by $\mathcal{O}_i$. From the definition of deg we deduce $\mathcal{O}_i \cdot \mathcal{O}_j \subseteq \mathcal{O}_{i+j}$. A polynomial f is *homogeneous* if $f \in \mathcal{O}_i$. In this case we say f has a degree i. Any polynomial can be written as a finite sum of homogeneous components. A proper ideal $\mathcal{I} \subset \mathcal{O}$ is called *homogeneous* or *graded* if for any f in $\mathcal{I}$ its homogeneous components are in $\mathcal{I}$, or equivalently $\mathcal{I} = \oplus_{j \in \mathbb{N}} \mathcal{I}_j$, $\mathcal{I}_j = \mathcal{O}_j \cap \mathcal{I}$. The $\mathbb{K}$-vector spaces $\mathcal{I}_j$ are caled *homogeneous components* of $\mathcal{I}$. More generally,

Definition 1.41. *An $\mathcal{O}$-module M is graded if $M = \oplus_{j\in\mathbb{Z}} M_i$ and $\mathcal{O}_j \cdot M_i \subseteq M_{i+j}$.*

An immediate consequence of this definition is the following Lemma:

Lemma 1.42. *Any graded $\mathcal{O}$-module has a system of homogeneous generators.*

Indeed, generators of M can be written as a sum of its homogeneous components, which are generators themselves. We use the following convention: the graded free $\mathcal{O}$-module $\mathcal{O}[-p]$ has a non-zero component only in degree p which is isomorphic to $\mathcal{O}$.

Remark 1.43. Any free finitely generated graded $\mathcal{O}$-module M can be written as a direct sum $M = \oplus_{j\in\mathbb{Z}} \mathcal{O}[-c_j]$, where only finitely many components are non-zero.

The property of being graded is naturally extended to chain complexes:

Definition 1.44. *A chain complex $(\mathfrak{M}_\bullet, d)$ over $\mathcal{O}$ is called graded, if each $\mathfrak{M}_i$ is a graded $\mathcal{O}$-module and each differential d_i has a degree 0 with respect to the standard grading.*

Similarly,

Definition 1.45. *Let $\mathcal{I}$ be a graded proper ideal of $\mathcal{O}$. A graded $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ is a graded chain complex over $\mathcal{O}$ which is a resolution of $\mathcal{O}/\mathcal{I}$.*

The existence of a graded $\mathcal{O}$ resolution of a graded finitely generated module M can be shown along the lines of Construction 1.2:

Construction 1.46. *A refinement of Construction 1.2 for the graded case:*

1. At the step 1, the set $C_0 = \{\phi_1, \dots, \phi_k, \dots\}$ is chosen to be homogeneous with respect to the standard grading. Then the corresponding basis $\{e_\phi,\ \phi \in \mathcal{C}_0\}$ inherits the grading. Thus $d_1 \colon \mathfrak{M}_1 \longrightarrow \mathcal{O}$ satisfies $\mathrm{pol}(d_1) = 0$.

2. At the step $i+1$ we assume that a family $\{(\mathfrak{M}_j, d_j) \,|\, j \in [\![1, i]\!]\}$ of graded $\mathcal{O}$-modules and $\mathcal{O}$-linear differentials with $\mathrm{pol}(d_i) = 0$ has been constructed. Since d_i preserves the polynomial grading, for any $x \in \mathrm{Ker}(d_i)$, the homogeneous components of x also belong to $\mathrm{Ker}(d_i)$. Thus $\mathrm{Ker}(d_i)$ is a graded $\mathcal{O}$-submodule of $\mathfrak{M}_i$. Therefore we can choose a family $\mathcal{C}_i = \{\chi_1, \dots, \chi_k, \dots\}$ which contains homogeneous elements and is finitely generated. Now, as in the Construction 1.2, we obtain a graded $\mathfrak{M}_{i+1}$ whose basis is parametrized by $\mathcal{C}_i$ and $\mathrm{pol}(d_{i+1}) = 0$.

Example 1.47. Consider an ideal $\mathcal{I} \subset \mathcal{O} \coloneqq \mathbb{R}[x, y, z]$ generated by monomials x^2, xy^3, yz^2. We apply the Construction 1.46 in order to find a graded free resolution of $\mathcal{O}/\mathcal{I}$.

First, we construct a graded free $\mathcal{O}$-module $\mathfrak{M}_1$ whose basis is parametrized by x^2, xy^3, yz^2 and has a polynomial degree $2, 4, 3$ respectively. Thus, $\mathfrak{M}_1 = \mathcal{O}[-2] \oplus \mathcal{O}[-4] \oplus \mathcal{O}[-3]$ and the differential d_1 is given by:

$$d_1 e_1 = x^2, \quad d_1 e_2 = xy^3, \quad d_1 e_3 = yz^2.$$

At the second step, we have to find the kernel of d_1. This requires to solve $d_1(\alpha e_1 + \beta e_2 + \gamma e_3) = 0$, which translates into $\alpha x^2 + \beta xy^3 + \gamma yz^2 = 0$. From the equality $\gamma yz^2 = -x(\alpha x + \beta y^3)$ it follows that x divides γ. And from $\alpha x^2 = -y(\beta xy^2 + \gamma z^2)$ we deduce that y divides α. Write $\alpha = y \cdot \tilde{\alpha}$ and $\gamma = x \cdot \tilde{\gamma}$. Then we obtain

$$\tilde{\alpha} x + \beta y^2 + \tilde{\gamma} z^2 = 0 \tag{9}$$

We can study this equation further and learn its space of solutions. However, we can use the following trick: since x, y^2, z^2 is a regular sequence, the solutions of (9) come out of relations derived from Koszul complex, namely

$$-y^2 \cdot x + x \cdot y^2 = 0, \quad -z^2 \cdot x + x \cdot z^2 = 0, \quad -z^2 \cdot y^2 + y^2 \cdot z^2 = 0.$$

Thus admissible $(\tilde{\alpha}, \beta, \tilde{\gamma})$ are described by $\mathcal{O}$-linear combination of the following 3-tuples:

$$(\tilde{\alpha}, \beta, \tilde{\gamma}) = \langle\, (-y^2, x, 0),\ (-z^2, 0, x),\ (0, -z^2, y^2)\,\rangle_{\mathcal{O}}$$

And (α, β, γ) is parametrized by:

$$(\alpha, \beta, \gamma) = \langle\, (-y^3, x, 0),\ (-yz^2, 0, x^2),\ (0, -z^2, xy^2)\,\rangle_{\mathcal{O}}$$

Thus we set $\mathfrak{M}_2$ to be a free module of rank 3 with a differential d_2 defined as

$$d_2 f_1 = xe_2 - y^3 e_1, \quad d_2 f_2 = xy^2 e_3 - z^2 e_2, \quad d_2 f_3 = x^2 e_3 - yz^2 e_1.$$

With $\mathrm{pol}(f_1) = 5$, $\mathrm{pol}(f_2) = 6$ and $\mathrm{pol}(f_3) = 5$. Therefore $\mathfrak{M}_2 = \mathcal{O}[-5] \oplus \mathcal{O}[-6] \oplus \mathcal{O}[-5]$. Now we have to construct $\mathfrak{M}_3$, this is done by studying the kernel of d_2. Taking a general ansatz $\alpha f_1 + \beta f_2 + \gamma f_3$, we obtain the following equation on the paramaters (α, β, γ):

$$\alpha(xe_2 - y^3 e_1) + \beta(xy^2 e_3 - z^2 e_2) + \gamma(x^2 e_3 - yz^2 e_1) = 0. \tag{10}$$

which can be rewritten as:

$$\alpha y^3 + \gamma yz^2 = 0, \quad \alpha x - z^2 \beta = 0, \quad \beta xy^2 + \gamma x^2 = 0.$$

The solution of this set of equations can be easily found and is parametrized by

$$(\alpha, \beta, \gamma) = \langle\, (z^2, x, -y^2)\,\rangle_{\mathcal{O}}.$$

Thus $\mathfrak{M}_3$ is a free module of rank 1. A simple count of polynomial degree shows that in fact $\mathfrak{M}_3 = \mathcal{O}[-7]$. The differential d_3 is given by:

$$d_3 g = xf_2 - y^2 f_3 + z^2 f_1.$$

Any element αg of $\mathrm{Ker}(d_3)$ implies that $\alpha(xf_2 - y^2 f_3 + z^2 f_1) = 0$, which in turn indicates that $\alpha = 0$. Thus the Construction 1.46 terminates at a step 3. The graded $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ we found is:

$$0 \xrightarrow{0} \mathcal{O}[-7] \xrightarrow{d_3} \mathcal{O}[-6] \oplus \mathcal{O}[-5]^2 \xrightarrow{d_2} \mathcal{O}[-3] \oplus \mathcal{O}[-4] \oplus \mathcal{O}[-2] \xrightarrow{d_1} \mathcal{O} \xrightarrow{0} 0$$

As in the case of local rings, Nakayama's lemma holds for graded modules over a graded polynomial ring:

Lemma 1.48 (Nakayama's lemma, graded version)**.** *Let $\mathcal{I}$ be a proper graded ideal in $\mathcal{O}$ and let M be a finitely generated graded $\mathcal{O}$-module.*

i. If $M = \mathcal{I}M$, then $M = 0$.

ii. If N is a graded $\mathcal{O}$-submodule of W such that $W = N + \mathcal{I}W$, then $W = N$.

We give a proof of this lemma since its logic is different compared to the local case.

Proof. Let us write $M = \mathcal{O}m_1 + \cdots + \mathcal{O}m_k$ for $m_1, \ldots m_k$ being homogeneous generators of M. Without any loss of generality we may assume $\mathrm{pol}(m_1) \leq \mathrm{pol}(m_2) \leq \cdots \leq \mathrm{pol}(m_k)$. Since $\mathcal{I}$ has a strictly positive degree, from the $M = \mathcal{I}M$ we obtain $m_1 = 0$. Then the statement i. follows by recursion.

ii. follows from i. if we choose $M = W/N$. □

Moreover, the notion of minimal generating sets does make sense for polynomial graded rings as well. Proposition 1.13 holds true, with a small refinement that each set of generators is homogeneous and the matrix $(A_i j)$ has homogeneous entries. For more details, see [Pee10]. The notion of a minimal resolution exists as well for graded polynomial rings:

Definition 1.49. *Let $\mathcal{O}$ be a polynomial ring with $\mathcal{J}$ being the irrelevant maximal ideal and let $\mathcal{I} \subsetneq \mathcal{O}$. A graded resolution $(\mathfrak{M}_\bullet, d)$ of $\mathcal{O}/\mathcal{I}$ is minimal if $d_{i+1}(\mathfrak{M}_{i+1}) \subseteq \mathcal{J}\mathfrak{M}_i$ for all i.*

Proposition 1.15 Theorem 1.26 hold true in the graded case as well with small corrections. In particular, a graded resolution obtained in Construction 1.46 is minimal if and only if at each step $i+1$ one chooses a minimal set $\mathcal{C}_i$ of homogeneous generators of $mathrmKer(d_i)$. Any graded free resolution can be written as a direct sum of a graded trivial complex and a graded minimal resolution. The proof mimics one of the local case. The detais are to be found in [Pee10].

1.6 Hilbert Syzygy Theorem

We continue studying resolutions of polynomial rings. In this section $\mathcal{O}$ is a polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$.

Definition 1.50. *Let M be a graded, finitely generated $\mathcal{O}$-module. A projective dimension of M is the minimum length of all projective resolutions of M. It is denoted by* $\mathrm{pd}(M)$*.*

Theorem 1.51 (Hilbert Syzygy Theorem)**.** *The projective dimension of a finitely generated graded $\mathcal{O}$-module M is at most n.*

Proof. Let $(\mathfrak{M}_\bullet, d)$ be a minimal resolution of M. Then, then rank of each $\mathfrak{M}_i$ coincides with the dimension of i-th Tor group: $\mathrm{Tor}_i(\mathcal{O}/\mathcal{I}, \mathbb{K}) = \dim_{\mathbb{K}}(\mathfrak{M}_i \otimes \mathcal{O}/\mathcal{J})$. Since the resolution of $\mathbb{K}$ is given by a Koszul complex which has a length n, and due to the symmetric property of Tor, we conclude $\mathfrak{M}_i = 0$ for $i > n$. □

There is an even more refined result, called Auslander-Buchsbaum formula concerning the projective dimension of M. It relates $\mathrm{pd}(M)$ to the so-called depth of M, which we introduce below.

Definition 1.52. *The depth of a graded $\mathcal{O}$-module M is the maximal length of a regular sequence on M, denoted by* $\mathrm{depth}(M)$*.*

Example 1.53. *If we view $\mathbb{K} = \mathcal{O}/\mathcal{J}$ as an $\mathcal{O}$-module, then for any $x \in \mathcal{J}$ $x\mathbb{K} = 0$. Therefore* $\mathrm{depth}(\mathbb{K}) = 0$.

Theorem 1.54 (Auslander-Buchsbaum formula)**.** *Let M be a graded, finitely generated $\mathcal{O}$-module. Then*

$$\mathrm{pd}(M) = n - \mathrm{depth}(M)$$

The proof of this theorem requires some results on theory of associated primes of an $\mathcal{O}$-module M, which we do not use anywhere else in the text. A systematical study can be found in [Eis95].

2 Simplicial constructions and applications to the polynomial rings

The construction of free resolutions of $\mathcal{O}/\mathcal{I}$ can be a daunting task, and one often needs to consult a software [EGSS02] to find a manifest description. However, in a particular case where $\mathcal{O}$ is a ring of polynomials and $\mathcal{I}$ is an ideal generated by monomials, there are important results and explicit constructions of free resolutions. In particular, the Taylor resolution[Tay66], and even the Koszul complex underline a particular reduced simplicial complex. The refinement of the Taylor resolution known as Scarf complex [BPS98] under certain condition provides an explicit minimal resolution of $\mathcal{O}/\mathcal{I}$. A canonical construction of a minimal resolution of monomial ideals is found in [EMO19]. In this Chapter we study and employ the aforementioned techniques for several important examples. The material is based on [MS04] and [Pee10]. Throughout this section $\mathcal{O} = \mathbb{K}[x_1, \ldots, x_n]$. and $\mathcal{I}$ is a graded proper ideal of $\mathcal{O}$.

2.1 A (reduced) simplicial complex

We begin this section with a brief overview of some particular simplicial constructions. The aim is to quickly obtain the neccessary knowledge to specialize to the setting of polynomial rings and obtain a description of resolutions of monomial ideals, as well as an alternative description of Koszul complex.

Definition 2.1. *A simplicial complex* Δ *on a set of vertices* $\mathcal{V} = \{v_1, \ldots, v_n\}$ *is a collection of subsets of* $\mathcal{V}$*, called faces or simplices such that if* $\tau \subset \sigma \in \Delta$*, then* $\tau \in \Delta$*.*

A dimension of a simplex $\sigma \in \Delta$ is defined as a $\mathrm{card}(\sigma) - 1$. In particular, σ containing i elements has $\dim(\sigma) = i - 1$, and an empty set $\varnothing$ has a dimension -1. The dimension of a simplicial complex $\dim(\Delta)$ is defined as a a maximum of the dimensions of its faces. By historical conventions, the dimension of the void simplicial complex $\{\}$ is $-\infty$ and this complex is not to be confused with the *irrelevant* simplicial complex $\{\varnothing\}$, whose dimension is -1.

Example 2.2. In this example, a simplicial complex on $\{A, B, C, D, E, F\}$ consists of all subsets of the sets $\{A, B, D\}$, $\{A, C\}$, $\{C, D\}$, $\{E, F\}$

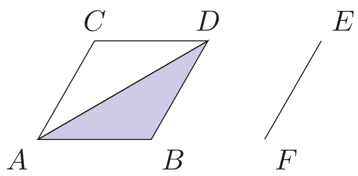


Figure 1: An example of a simplicial complex

A simplicial complex is completely determined by its maximal faces, called *facets*. In the Example 2.2 those are $\{A, B, D\}$, $\{A, C\}$, $\{C, D\}$, $\{E, F\}$.

Let $F_i(\Delta)$ be a set of faces of dimension i and let $\mathbb{K}^{F_i(\Delta)}$ be a $\mathbb{K}$-vector space with a basis $e_\tau, \tau \in F_i(\Delta)$ parametrized by i-dimensional faces. Let us introduce a total order on the set of vertices (e.g. a lexicographic order in Example 2.2). The order on $\mathcal{V}$ determines the order on the vertices of any simplex $\sigma \in \Delta$.

Let Δ be a k-dimensional simplicial complex.

Definition 2.3. *A reduced simplicial complex is a complex*

$$C(\Delta, \mathbb{K})\colon\ 0 \longrightarrow \mathbb{K}^{F_k(\Delta)} \xrightarrow{d_k} \cdots \xrightarrow{d_1} \mathbb{K}^{F_0(\Delta)} \xrightarrow{d_0} \mathbb{K}^{F_{-1}(\Delta)} \xrightarrow{0} 0$$

of $\mathbb{K}$*-vector spaces, where*

$$d_i e_\tau = \sum_{j \in \tau} \mathrm{sign}(j, \tau) e_{\tau \setminus j} \tag{11}$$

In (11) the sum goes over all vertices containing in τ. $\tau \setminus j$ is a $i - 1$ dimensional simplex obtained from τ by removing the vertex i. The $\mathrm{sign}(j, \tau)$ is given by $(-1)^{j_\tau - 1}$ with j_τ being the position of j in τ. It is a straightforward observation that $d_{i-i} \circ d_i = 0$: crossing out two vertices j and k from e_τ can be done in two different ways; the corresponding summands $\mathrm{sign}(k, \tau \setminus j)\mathrm{sign}(j, \tau) e_{\tau \setminus \{j,k\}}$ and $\mathrm{sign}(j, \tau \setminus k)\mathrm{sign}(k, \tau) e_{\tau \setminus \{j,k\}}$ have a relative sign "$-$", so they cancel each other out.

Example 2.4. Let us work this differential out for the Example 2.2:

$$C(\Delta, \mathbb{K})\colon\ 0 \longrightarrow \mathbb{K}^{F_2(\Delta)} \xrightarrow{d_2} \mathbb{K}^{F_1(\Delta)} \xrightarrow{d_1} \mathbb{K}^{F_0(\Delta)} \xrightarrow{d_0} \mathbb{K}^{F_{-1}(\Delta)} \xrightarrow{0} 0$$

The sets F_i are: $F_2(\Delta) = \{A, B, C\}$, $F_1(\Delta) = \{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, D\}$, $\{C, D\}$, $\{E, F\}$, $F_0(\Delta) = \{A\}$, $\{B\}$, $\{C\}$, $\{D\}$, $\{E\}$, $\{F\}$ and $F_{-1}(\Delta) = \{\varnothing\}$. The differential d acts on the corresponding basis of $\mathbb{K}^{F_i(\Delta)}$ in the following way:

In degree -1: $d(e_\varnothing) = 0.$

In degree 0: $d(e_A) = d(e_B) = d(e_C) = d(e_D) = d(e_E) = d(e_F) = e_\varnothing.$

In degree 1:
$$\begin{array}{ccc} d(e_{AB}) = e_B - e_A, & d(e_{AC}) = e_C - e_A, & d(e_{AD}) = e_D - e_A, \\ d(e_{BD}) = e_D - e_B, & d(e_{CD}) = e_D - e_C, & d(e_{EF}) = e_F - e_E. \end{array}$$

In degree 2: $d(e_{ABD}) = e_{BD} - e_{AD} + e_{AB}.$

While studying complexes we naturally arrive at a notion of homology, which in the case of reduced simplicial complexes has the following defintion:

Definition 2.5. *The $\mathbb{K}$ vector space $H_i(\Delta, \mathbb{K}) = Ker(d_i)/Im(d_{i+1})$ of homological degree i, $i \in [\![0, k-1]\!]$ is the i-th reduced homologogy of $C(\Delta, \mathbb{K})$. For $i = k$ the homology is defined as $H_k(\Delta, \mathbb{K}) = Ker(d_k)$. In degree -1: $H_{-1}(\Delta, \mathbb{K}) = \mathbb{K}/Im(d_0)$*

Whenever Δ is not $\{\varnothing\}$ the homology $H_{-1}(\Delta, \mathbb{K}) = H_k(\Delta, \mathbb{K}) = 0$. For $\Delta = \{\varnothing\}$ the homology $H_{-1}(\Delta, \mathbb{K}) = \mathbb{K}$. In the Example 2.2 the non-zero homology is $H_0(\Delta, \mathbb{K}) = \mathbb{K}$ (the dimension of the homology in degree 0 is the number of connected components of Δ minus one), while the rest vanishes.

Lemma 2.6. *If Δ is a k-dimensional symplex, then $H_i(\Delta, \mathbb{K}) = 0$ $\forall i \in [\![-1, k]\!]$.*

Proof. This is a standard lemma in many textbooks, see [Hat02]. Let us give a simple and manifest proof that $C(\Delta, \mathbb{K})$ is contractible. Define the homotopy $h : C_i(\Delta, \mathbb{K}) \longrightarrow C_{i+1}(\Delta, \mathbb{K}), i < k$:

$$h(e_\sigma) = \frac{1}{k+1} \sum_{j \notin \sigma} \text{sign}(j, \sigma) e_{\sigma \cup j}$$

while for $i > k-1$ and $i < -1$:

$$h = 0$$

The element $\sigma \cup j$ is an $i+1$-dimensional face obtained by inserting a vertex j into σ. The $\text{sign}(j, \sigma)$ is $(-1)^{j_\sigma - 1}$ with j_σ being the position of j in $\sigma \cup j$. For Δ being a symplex $\{A, B, D\}$ of Example 2.2, we have

$$\begin{array}{ccc} & h(e_\varnothing) = e_A + e_B + e_D, & \\ h(e_A) = -e_{AB} - e_{AD}, & h(e_B) = e_{AB} - e_{BD}, & h(e_D) = e_{AD} + e_{BD}, \\ h(e_{AB}) = e_{ABD}, & h(e_{AD}) = -e_{ABD}, & h(e_{BD}) = e_{ABD} \\ & h(e_{ABD}) = 0. & \end{array}$$

Now, take any element e_σ of degree $i \in [\![0, k-1]\!]$ and evaluate $(h \circ d_i + d_{i+1} \circ h)(e_\sigma)$:

$$(h \circ d_i + d_{i+1} \circ h)(e_\sigma) = \frac{1}{k+1} \sum_{j \notin \{\sigma/m\}} \sum_{m \in \sigma} \text{sign}(j, \sigma/m)\text{sign}(m, \sigma) e_{\{(\sigma/m) \cup j\}} + \frac{1}{k+1} \sum_{m \in \{\sigma \cup j\}} \sum_{j \notin \sigma} \text{sign}(m, \sigma \cup j)\text{sign}(j, \sigma) e_{\{\sigma \cup j\}/m}.$$

The expressions for $i = -1$ and k are obtained in a similar way. It is straightforward to check that the terms with $m \neq j$ cancel each other out. The summands with $m = j$ reproduce e_σ:

$$\frac{1}{k+1}\sum_{m\in\sigma} e_\sigma + \frac{1}{k+1}\sum_{m\notin\sigma} e_\sigma = e_\sigma$$

So we have established $C(\Delta, \mathbb{K})$ is contractible, i.e. $(h \circ d_i + d_{i+1} \circ h - \mathrm{id}) = 0$, so that the homology vanishes. □

We have already introduced the Koszul complex in Section 1.4. As we have seen, if $\mathcal{O}$ is Noetherian and a proper ideal $\mathcal{I} \subset \mathcal{O}$ is generated by a regular sequence $\varphi = \varphi_1, \ldots, \varphi_n$, then the Koszul complex $\mathcal{K}(\varphi)$ is acyclic. This required some work on our part. It turns out that the Koszul complex admits a description via a *labeled* reduced symplicial complex. Later in the text we give a simple proof of its acyclicity when φ is a regular sequence of monomials.

Definition 2.7. *Let $\varphi = \varphi_1, \ldots, \varphi_n$ be a sequence of elements of $\mathcal{O}$. The Koszul complex $\mathcal{K}(\varphi)$ is a labeled reduced simplicial complex in the following sense:*

i. The set of vertices $\mathcal{V} = \{1, \ldots, n\}$ corresponds to a system of generators $\varphi_1, \ldots, \varphi_n$ of $\mathcal{I}$. The i-th vertex is labeled by the i-th generator φ_i of $\mathcal{I}$. Δ is an $(n-1)$-simplex.

ii. The homological degree i component of Koszul complex is a free $\mathcal{O}$-module $M_i = \mathcal{O}^{\times F_{i-1}(\Delta)}$ with a basis $\{e_\sigma,\ \sigma \in F_{i-1}(\Delta)\}$

iii. The $\mathcal{O}$-linear differential d is defined as

$$d(e_\sigma) = \sum_{j\in\sigma} \mathrm{sign}(j,\sigma)\varphi_j \cdot e_{\sigma/j}$$

It is clear that this complex is isomorphic to the previously defined Koszul complex (see Definition 1.30): the ranks of free $\mathcal{O}$-modules of degree k are $\binom{n}{k}$, just as for the wedge product $V \wedge \cdots \wedge V$ in Construction 1.29, and once a total order is chosen on the basis $e_1, \ldots, e_n$ of V. It is clear that the formulas for d coincide.

Remark 2.8. If $\mathcal{I}$ is a graded ideal in a polynomial ring $\mathcal{O}$ and $\varphi_1, \ldots, \varphi_n$ are homogeneous generators, then Koszul complex is a graded chain complex, with

$$\mathfrak{M}_i = \bigoplus_{\sigma\in F_{i-1}(\Delta)} \mathcal{O}[-\sum_{j\in\sigma}\mathrm{pol}(\varphi_j)].$$

In a particular case when $\mathcal{I}$ is the irrelevant maximal ideal $\langle x_1, \ldots, x_n\rangle_{\mathcal{O}}$ in a polynomial ring $\mathcal{O}$ we can very quickly verify the exactness of the Koszul complex relying on the exactness of the reduced simplicial complex. For this we introduce a notion of a *multidegree*:

Definition 2.9. *A multidegree of monomial $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ is an n-tuple $(a_1, \ldots, a_n) \in \mathbb{N}^n$. We denote $\mathbf{pol}(\mathbf{x}^{\mathbf{a}}) = \mathbf{a}$.*

The polynomial ring $\mathcal{O}$ is then a direct sum of its $\mathbb{N}^n$-homogeneous components $\mathcal{O} = \oplus_{\mathbf{a}\in\mathbb{N}^n}\mathcal{O}_{\mathbf{a}}$, and $\mathcal{O}_{\mathbf{a}} \cdot \mathcal{O}_{\mathbf{b}} = \mathcal{O}_{\mathbf{a}+\mathbf{b}}$. The notions of a graded ideal, graded $\mathcal{O}$-module, graded chain complex and graded resolutions generalize directly to the $\mathbb{N}^n$-setting. In particular, a multigraded chain complex is a graded chain complex $\{(\mathfrak{M}_i, d_i),\ i \in \mathbb{N}\}$ such that each $\mathfrak{M}_i$ is a free $\mathcal{O}$-module graded with respect to the multidegree and $\mathbf{pol}(d_i) = 0$.

Observe that Koszul complex preserves the multidegree if $\mathcal{I}$ is generated by monomials, therefore it is a multigraded chain complex. In particular for any e_σ, $\sigma \in F_\bullet(\Delta)$, $\mathbf{pol}(e_\sigma) = \sum_{j\in\sigma}\mathbf{pol}(\varphi_j)$. For further analysis we introduce a notion of support of an n-tuple $\mathbf{a}$:

Definition 2.10. *For n-tuple $\mathbf{a}$, a* support *of $\mathbf{a}$ is an n-tuple with entries $\in \{0, 1\}$ such that the entry at i-th place is 1 if $a_i \neq 0$ and zero otherwise.*

We denote the support of $\mathbf{a}$ by $\mathrm{supp}(\mathbf{a})$. Viewed as a set, we identify $\mathrm{supp}(\mathbf{a})$ with a subset of $\mathcal{V}$: $\{i \in \mathcal{V} \,|\, a_i \neq 0\}$.

Proposition 2.11. *The Koszul complex is a minimal (multi)graded free resolution of $\mathcal{O}/\mathcal{I}$ for $\mathcal{I} = \langle x_1, \ldots, x_n \rangle_{\mathcal{O}}$.*

Proof. Denote $\mathcal{K}_{\mathbf{b}}$ - a restriction of Koszul complex to elements of multidegree $\mathbf{b}$. This is a collection of $\mathbb{K}$-vector spaces. Since d preserves the multidegree, clearly $(\mathcal{K}_{\mathbf{b}}, d)$ is a subcomplex. If $\mathbf{b} = 0$ this subcomplex is a reduced simplicial complex $\mathcal{C}(\{\varnothing\}, \mathbb{K})$ shifted by one, therefore $H_i(\mathcal{K}_{\mathbf{b}}, d) = \mathbb{K} = \mathcal{O}/\langle x_1, \ldots, x_n \rangle$ for $i = 0$ and is 0 otherwise. In a general case $\mathbf{b} \neq 0$ support of $\mathbf{b}$ viewed as a set is a face of Δ. An element $\mathbf{x}^{\mathbf{a}} e_\tau$ of multidegree $\mathbf{b}$ belongs to $\mathcal{K}_{\mathbf{b}}$ if and only if $\tau \in \mathrm{supp}(\mathbf{b})$. Therefore, $\mathcal{K}_{\mathbf{b}}$ is a reduced simplicial complex with Δ being $\mathrm{supp}(\mathbf{b})$, thus acyclic. This concludes that $\mathcal{K}$ is a resolution of $\mathcal{O}/\mathcal{I}$. The minimality follows directly from $d(\mathcal{K}) \subset \mathcal{I}\mathcal{K}$. □

2.2 General monomial constructions and Taylor resolutions

We have seen in the previous section that there exist a simplicial complex underlying the Koszul complex, and based on the acyclicity of that complex we showed that the Koszul complex $\mathcal{K}(\varphi)$ is exact in a particular case when $\varphi = x_1, \ldots, x_n$. Here we provide a more general construction, which produces out of a complex of $\mathbb{K}$-vector spaces a particular chain complex of free $\mathcal{O}$-modules which encodes some information of a set $\varphi = \varphi_1, \ldots, \varphi_k$ of generators of $\mathcal{I} \subsetneq \mathcal{O}$, see Construction 2.15. In the case when the complex of vector spaces is exact, we prove that the complex obtained in Construction 2.15 is in fact a resolution of $\mathcal{O}/(\varphi_1, \ldots \varphi_k)$. In this section we additionally assume that $\mathcal{I}$ is generated by monomials.

We start by giving a definition to a so-called k-frame V, which is a generalization of the reduced simplicial complex.

Definition 2.12 (k-frame V)**.** *A k-frame V is a chain complex of $\mathbb{K}$-vector spaces $\{(V_i, d_i),\, i \in \mathbb{N}\}$ such that*

- *$V_0 = \mathbb{K}$.*
- *V_1 has a fixed basis $e_1, \ldots, e_k$ such that $d(e_i) = 1$ for $i \in [\![1, k]\!]$.*

Remark 2.13. Note that the reduced simplicial complex on a set of vertices $\mathcal{V} = \{1, \ldots, k+1\}$, see Definition 2.3, is a $(k+1)$-frame $\mathbb{K}^{F_\bullet(\Delta)}$ upon a shift by 1 in the homological degree.

A more general version of a labeled reduced simplicial complex is called a φ-complex and is defined as:

Definition 2.14. *Let $\varphi = \varphi_1 \ldots, \varphi_k$ be a set of homogeneous generators of $\mathcal{I} \subsetneq \mathcal{O}$. A φ-complex is a multigraded chain complex $\{(\mathfrak{M}_i, d_i),\, i \in \mathbb{N}\}$ of free $\mathcal{O}$-modules satisfying the following conditions:*

- *$\mathfrak{M}_0 = \mathcal{O}$.*
- *$\mathfrak{M}_1$ is a graded free module of rank k, $\mathfrak{M}_1 = \mathcal{O}[-\mathbf{pol}(\varphi_1)] \oplus \cdots \oplus \mathcal{O}[-\mathbf{pol}(\varphi_k)]$. The corresponding fixed basis is $e_\varphi = e_{\varphi_1}, \ldots, e_{\varphi_k}$. The diffential d defined on e_φ as:*

$$d(e_{\varphi_i}) = \varphi_i.$$

In particular, the Koszul complex $\mathcal{K}(\varphi)$ (see Definition 2.7) is a φ-complex.

Construction 2.15 (φ-homogenization of a k-frame V)**.** Given a k-frame $V = \{(V_i, \bar{d}_i),\ i \in \mathbb{N}\}$ and a set of homogeneous generators $\varphi = \varphi_1, \dots, \varphi_k$ of $\mathcal{I} \subsetneq \mathcal{O}$ we construct a φ-complex $M(\varphi)$ as follows:

1. We set $M_1(\varphi) = \mathcal{O}[-\mathbf{pol}(\varphi_1)] \oplus \cdots \oplus \mathcal{O}[-\mathbf{pol}(\varphi_k)]$. We use the basis of the k-frame V $e_1, \dots, e_k$ to form the basis $e_{\varphi_1}, \dots, e_{\varphi_k}$ of multidegree $\mathbf{pol}(\varphi_1), \dots, \mathbf{pol}(\varphi_k)$ respectively. The diffential d_1 is defined then as:

$$d_1(e_{\varphi_i}) = \varphi_i.$$

   so that d preserves the multidegree.

2. Assume that up to a homological degree j we have constructed a collection $\{(M_i(\varphi), d_i),\ i \in [\![1, j]\!]\}$ of multigraded $\mathcal{O}$-modules and $\mathcal{O}$-linear differentials. Denote the basis of V_j, V_{j+1} is $\bar{f} = \bar{f}_1, \dots, \bar{f}_{a_j}$ and $\bar{g} = \bar{g}_1, \dots, \bar{g}_{b_{j+1}}$ respectively. The basis of $M_j(\varphi)$ is then $f = f_1, \dots, f_{a_j}$ and it has a multidegree $\mathbf{pol}(f_1), \dots, \mathbf{pol}(f_{a_j})$. Let us write explicitly the action of $\bar{d}_{j+1}$:

$$\bar{d}_{j+1}(\bar{g}_l) = \sum_{m=1}^{a_j} \alpha_{lm} \bar{f}_m. \tag{12}$$

   Let us denote a subset of f that corresponds to those $\bar{f}_m$ who enters (12): $A_l = \{f_m \mid \alpha_{lm} \neq 0\}$. We write a least common multiple of $\{\mathbf{x}^{\mathbf{pol}(a)} \mid a \in A_l\}$ as $\operatorname{lcm}(A_l)$.

   Then the basis $g_1, \dots, g_{b_{j+1}}$ of $C_{j+1}(\varphi)$ has a multidegree

$$\mathbf{pol}(g_l) = \mathbf{pol}(\operatorname{lcm}(A_l))$$

   So that $M_{j+1}(\varphi) = \mathcal{O}[-\mathbf{pol}(\operatorname{lcm}(A_1))] \oplus \cdots \oplus \mathcal{O}[-\mathbf{pol}(\operatorname{lcm}(A_l))]$. The differential d_{j+1} is given by:

$$d_{j+1}(g_l) = \sum_{m=1}^{a_j} \frac{\operatorname{lcm}(A_l)}{\mathbf{x}^{\mathbf{pol}(f_m)}} \alpha_{lm} f_m = \sum_{m=1}^{a_j} \frac{\mathbf{x}^{\mathbf{pol}(g_l)}}{\mathbf{x}^{\mathbf{pol}(f_m)}} \alpha_{lm} f_m$$

   According to this definition, $\mathbf{pol}(d_{j+1}) = 0$.

Proposition 2.16. *φ-homogenization of a k-frame V is a complex.*

Proof. Let us check that the differential d described in Construction 2.15 does square to zero. In the notation of Construction 2.15 let us write for the basis $\bar{f}$ of V_j, $\bar{d}\bar{f}_m = \sum_{q=1}^{c_{j-1}} \beta_{mq} \bar{h}_q$, where $\bar{h} = \bar{h}_1, \dots \bar{h}_{c_{j-1}}$ is a basis of V_{j-1}. Since $\bar{d}^2 = 0$, we obtain

$$\sum_{m=1}^{a_j} \alpha_{lm} \beta_{mq} = 0$$

for all possible l, q. The basis of $M_{j-1}(\varphi)$ we denote by $h = h_1, \dots, h_{c_{j-1}}$. According to the construction,

$$d_j(f_m) = \sum_{q=1}^{c_{j-1}} \frac{\mathbf{x}^{\mathbf{pol}(f_m)}}{\mathbf{x}^{\mathbf{pol}(h_q)}} \beta_{mq} h_q.$$

Then,

$$d_{j+1} \circ d_j(g_l) = \sum_{m=1}^{a_j} \sum_{q=1}^{c_{j-1}} \frac{\mathbf{x}^{\mathbf{pol}(g_l)}}{\mathbf{x}^{\mathbf{pol}(f_m)}} \alpha_{lm} \frac{\mathbf{x}^{\mathbf{pol}(f_m)}}{\mathbf{x}^{\mathbf{pol}(h_q)}} \beta_{mq} h_q = \sum_{m=1}^{a_j} \sum_{q=1}^{c_{j-1}} \frac{\mathbf{x}^{\mathbf{pol}(g_l)}}{\mathbf{x}^{\mathbf{pol}(h_q)}} \alpha_{lm} \beta_{mq} h_q = 0$$

□

Remark 2.17. *Note that* $H_0(M(\varphi)) = \mathcal{O}/\mathcal{I}$.

Example 2.18. Let us explain the φ-homogenization $M(\varphi)$ on Example 2.4. First, we shift the degrees of all generators by 1, so the generators start from degree 0. Then this reduced symplicial complex is a 6-frame. Let us choose a set of generators that corresponds to the set of vertices A, B, C, D, E, F as $\varphi = x^2, xy, z^2, xy^3z, z, xz$. Then $M_1(\varphi) = \mathcal{O}[-(2,0,0)] \oplus \mathcal{O}[-(1,1,0)] \oplus \mathcal{O}[-(0,0,2)] \oplus \mathcal{O}[-(1,3,1)] \oplus \mathcal{O}[-(0,0,1)] \oplus \mathcal{O}[-(0,1,1)]$. We abuse the notation and denote the corresponding basis of $M_1(\varphi)$ as $e_A, e_B, e_C, e_D, e_E, e_F$. Then

$$d \text{ on generators of degree 1:} \quad \begin{array}{ccc} d(e_A) = x^2, & d(e_B) = xy, & d(e_C) = z^2, \\ d(e_D) = xy^3z, & d(e_E) = z, & d(e_F) = xz. \end{array}$$

According to Construction 2.15 we set the multidegree of generators of degree 2 as:

$$\begin{array}{ccc} \mathbf{pol}(e_{AB}) = (2,1,0), & \mathbf{pol}(e_{AC}) = (2,0,2), & \mathbf{pol}(e_{AD}) = (2,3,1), \\ \mathbf{pol}(e_{BD}) = (1,3,1), & \mathbf{pol}(e_{CD}) = (1,3,2), & \mathbf{pol}(e_{EF}) = (1,0,1). \end{array}$$

The differential d on these generators is given by:

$$\begin{array}{ccc} d(e_{AB}) = xe_B - ye_A, & d(e_{AC}) = x^2e_C - z^2e_A, & d(e_{AD}) = xe_D - y^3ze_A, \\ d(e_{BD}) = e_D - y^2ze_B, & d(e_{CD}) = ze_D - xy^3e_C, & d(e_{EF}) = e_F - xe_E. \end{array}$$

The degree 3 generator e_{ABD} has a multidegree $(2,3,1)$ and

$$d(e_{ABD}) = xe_{BD} - e_{AD} + y^2ze_{AB}.$$

We introduce a dehomogenization of a φ-complex as well.

Definition 2.19. *For a φ-complex $M(\varphi)$, a dehomogenization is a quotient $M(\varphi)/(x_1 - 1, \ldots, x_n - 1)$.*

Remark 2.20. It follows directly from the definition of $M(\varphi)$ that its dehomogenization is some k-frame V. We call such V *a frame of* $M(\varphi)$. The dehomogenization of the φ-homogenization of a k-frame V is the same k-frame V.

For a multigraded complex M let us write $M_{\leq \mathbf{a}}$ for a subcomplex of M whose elements has a multidegree $\leq \mathbf{a}$, and $M_{\mathbf{a}}$ for a subcomplex of multidegreee $\mathbf{a}$.

Proposition 2.21. *Let M be a $\varphi-$complex. The subcomplex $M_{\mathbf{a}}$ is isomorphic to the dehomogenization of $M_{\leq \mathbf{a}}$*

Proof. We can chooses a basis of $M_{\mathbf{a}}$ as follows:

$$\{\frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{x}^{\mathbf{pol}(g)}}g, \mid g \text{ is a basis element of } M \text{ whenever } \mathbf{x}^{\mathbf{pol}(g)} \text{ divides } \mathbf{x}^{\mathbf{a}}\}$$

. Clearly the set of all such g is a basis of $M_{\leq \mathbf{a}}$, with the underlying basis $\bar{g}$ of a frame of $M_{\leq \mathbf{a}}$. The isomorphism is then given by:

$$\frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{x}^{\mathbf{pol}(g)}}g \mapsto \bar{g}$$

□

Proposition 2.22. *Let $\varphi = \varphi_1 \ldots, \varphi_k$ be a set of homogeneous generators of $\mathcal{I} \subsetneq \mathcal{O}$, and let M be a φ-complex. M is a multigraded resolution of $\mathcal{O}/\mathcal{I}$ if and only if for any $\mathbf{a}$ such that $\mathbf{x}^{\mathbf{a}} \in \mathcal{I}$ the frame of $M_{\leq \mathbf{a}}$ is acyclic.*

Proof. If M is a multigraded resolution, then it follows that for any $\mathbf{a}$, $\mathbf{x}^{\mathbf{a}} \in \mathcal{I}$ $M_{\mathbf{a}}$ is acyclic thus by the isomorphism of Proposition 2.21 the frame of $M_{\leq \mathbf{a}}$. If $\mathbf{x}^{\mathbf{a}} \notin \mathcal{I}$, then $M_{\mathbf{a}}$ is concetrated in homological degree 0 and belongs to $\mathcal{O}/\mathcal{I}$. The same isomorphism shows the converse as well, if the frame of $M_{\leq \mathbf{a}}$ is acyclic, then so is $M_{\mathbf{a}}$. The statement follows. □

Definition 2.23. *Let $\varphi = \varphi_1 \dots, \varphi_k$ be a set of homogeneous generators of $\mathcal{I} \subsetneq \mathcal{O}$. A Taylor complex $C(\varphi)$ is the φ-homogenization of the reduced symplicial complex $C(\Delta, \mathbb{K})$, where Δ is a $(k-1)$-simplex on a set of vertices $\mathcal{V} = \varphi$.*

Proposition 2.24 (Manifest description of the Taylor complex)**.** *The basis of the Taylor complex is parametrized by faces σ of Δ:*

$$\{e_\sigma \,|\, \sigma = \varphi_{i_1}, \dots, \varphi_{i_m}, 1 \leq i_1 < \dots < i_m \leq k\}, \quad \mathbf{pol}(e_\sigma) = \mathbf{pol}(\mathrm{lcm}\{\sigma\}).$$

The differential d acts on the basis elements as

$$d(e_\sigma) = \sum_{\phi \in \sigma} \mathrm{sign}(\phi, \sigma) \frac{\mathbf{x}^{\mathbf{pol}(e_\sigma)}}{\mathbf{x}^{\mathbf{pol}(e_{\sigma/\phi})}} e_{\sigma/\phi}$$

Proof. The statement that basis e_σ is parametrized by faces of Δ follows from the definition of φ-homogenization and the one of reduced simplicial complex. The multidegree of e_σ can be easily found by recursion; if σ is a 0-simplex φ_{i_m}, then indeed $\mathbf{pol}(e_\sigma) = \mathbf{pol}(\varphi_{i_m})$. If σ is $(m-1)$ simplex $\varphi_{i_1}, \dots, \varphi_{i_m}$, then, from Construction 2.15, $\mathbf{pol}(e_\sigma) = \mathbf{pol}(\mathrm{lcm}\{\sigma/\phi \,|\, \phi \in \sigma\}) = \mathbf{pol}(\mathrm{lcm}\{\sigma\})$. The formula for the differential follows directly from the construction of φ-homogenization. □

Proposition 2.25. *The Taylor complex $C(\varphi)$ is exact.*

Proof. By Proposition 2.22 we have to show that dehomogenization of $C(\varphi)_{\leq \mathbf{a}}$ is acyclic if $\mathbf{x}^{\mathbf{a}} \in \mathcal{I}$. Write $\varphi_{\mathbf{a}}$ for a set $\{\phi \in \varphi = \varphi_1 \dots, \varphi_k, \,|\, \phi \text{ divides } \mathbf{x}^{\mathbf{a}}\}$. Then $\mathrm{lcm}\{\varphi_{\mathbf{a}}\}$ divides $\mathbf{x}^{\mathbf{a}}$ as well, thus the dehomogenization of $C(\varphi)_{\leq \mathbf{a}}$ is a reduced simplical simplex on $\varphi_{\mathbf{a}}$, which is exact. □

Since Taylor complex is exact, we call it the *Taylor resolution*. The standard example is given by the Koszul complex 2.7 whenever φ is a regular sequence. The regularity in case of monomials translates into $\mathrm{lcm}\{\varphi_i, \varphi_j\} = \varphi_i \cdot \varphi_j$ if $i \neq j$.

Example 2.26. Let us find the Taylor resolution $C(\varphi)$ of the Example 1.47. The ideal $\mathcal{I} \subset \mathcal{O} := \mathbb{R}[x, y, z]$ generated by x^2, xy^3, yz^2. Then the multigraded $\mathcal{O}$-modules $C_i(\varphi)$, $i = 1, 2, 3$ entering the Taylor resolution have ranks $3, 3, 1$ respectively, and d is given by:

In hom. degree 1: $d(e_{x^2}) = x^2, \quad d(e_{xy^3}) = xy^3, \quad d(e_{yz^2}) = yz^2.$

In hom. degree 2: $d(e_{x^2,xy^3}) = xe_{xy^3} - y^3 e_{x^2}, \quad d(e_{x^2,yz^2}) = x^2 e_{yz^2} - yz^2 e_{x^2}$ $d(e_{xy^3,yz^2}) = xy^2 e_{yz^2} - z^2 e_{xy^3}.$

In hom. degree 3: $d(e_{x^2,xy^3,yz^2}) = xe_{xy^3,yz^2} - y^2 e_{x^2,yz^2} + z^2 e_{x^2,xy^3}.$

In this example, the Taylor resolution coincides with the one of Example 1.47 and is in fact minimal: $d(e_\bullet) \in \mathcal{J}C_\bullet(\varphi)$.

Let us give an example of Taylor resolution which is not minimal:

Example 2.27 (A non-minimal Taylor resolution)**.** Let $\mathcal{I}$ be an ideal $\mathcal{I} \subset \mathcal{O} := \mathbb{R}[x, y, z]$ generated by xy, xz, z^2. The ranks of $C_i(\varphi)$ are $3, 3, 1$ for $i = 1, 2, 3$. The differential d acts as follows:

$$\text{In hom. degree 1:} \quad d(e_{xy}) = xy, \quad d(e_{xz}) = xz, \quad d(e_{z^2}) = z^2.$$

$$\text{In hom. degree 2:} \quad \begin{array}{l} d(e_{xy,xz}) = ye_{xz} - ze_{xy}, \quad d(e_{xy,z^2}) = xye_{z^2} - z^2 e_{xy} \\ d(e_{xz,z^2}) = xe_{z^2} - ze_{xz}. \end{array}$$

$$\text{In hom. degree 3:} \quad d(e_{xy,xz,z^2}) = ye_{xz,z^2} - e_{xy,z^2} + ze_{xy,xz}.$$

The coefficient in from of the second term in (e_{xy,xz,z^2}) is constant, thus the resolution is not minimal.

Remark 2.28. The fact that the Taylor resolution is not minimal is not surprising: the length of the Taylor resolution on $\varphi = \varphi_1, \dots, \varphi_n$ is always n. If it is minimal, then it follows that $\operatorname{depth}(\mathcal{O}/\mathcal{I}) = 0$ by Auslander-Buchsbaum formula. This is not the case already when $\varphi = x_2^2, x_2x_3, x_3^2, x_4, x_5, \dots, x_n$. Then x_1 is not a zero divisor of $\mathcal{O}/\mathcal{I}$, therefore $\operatorname{depth}(\mathcal{O}/\mathcal{I}) \geq 1$ (and is 1 in fact).

A "smaller" complex can be obtained if one chooses a "smaller" r-frame than a simplex. One of the notable examples is given by an algebraic Scarf complex, introduced below. This complex is a resolution of $\mathcal{O}/\mathcal{I}$ if $\mathcal{I}$ is a *generic* monomial ideal, i.e. if x_i enters two distinct generators φ_l, φ_m as a multiple $x_i^{a_l}$ and $x_i^{a_m}$, then $a_l \neq a_m$.

Definition 2.29. *Let $\varphi = \varphi_1, \dots, \varphi_k$ be a minimal set of generators of $\mathcal{I} \subsetneq \mathcal{O}$. A Scarf complex Δ_φ is a collection of subsets of φ such that whenever* $\operatorname{lcm}\{\sigma\} = \operatorname{lcm}\{\tau\}$ *we have $\sigma = \tau$.*

Proposition 2.30. *The Scarf complex Δ_φ is a simplical complex.*

Proof. Let σ be an element of Δ_φ, and $\phi \in \sigma$. We have to show that Δ_φ is closed under taking subsets, i.e. $\sigma \in \Delta_\varphi \implies \sigma \backslash \phi \in \Delta_\varphi$. Suppose there exists another $\tau \in \Delta_\varphi$ such that $\operatorname{lcm}\{\tau\} = \operatorname{lcm}\{\sigma \backslash \phi\}$. If $\operatorname{lcm}\{\tau\} = \operatorname{lcm}\{\sigma\}$ then $\sigma = \tau = \sigma \backslash \phi$. Otherwise, $\operatorname{lcm}\{\tau, \phi\} = \operatorname{lcm}\{\sigma\}$, therefore $\tau \cup \phi = \sigma$. Thus $\tau = \sigma \backslash \phi$. □

Definition 2.31. *Let $\varphi = \varphi_1, \dots, \varphi_k$ be a set of generators of $\mathcal{I} \subsetneq \mathcal{O}$. The algebraic Scarf complex $C_S(\varphi)$ is a φ-homogenization of the Scarf complex Δ_φ.*

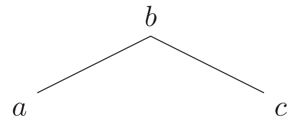
Example 2.32. Let $\mathcal{I} = \langle xy, xz, yz \rangle_{\mathcal{O}}$ with $\mathcal{O} = \mathbb{K}[x, y, z]$. The Scarf complex can be found as a subcomplex of a 2-simplex via the Taylor resolution of $\mathcal{O}/\mathcal{I}$. The set of generators of the Taylor resolution parametrized by faces of the 2-simplex is

In hom. degree 1: e_{xy}, e_{xz}, e_{yz}, the multidegrees are: $(1, 1, 0), (1, 0, 1), (0, 1, 1)$.

In hom. degree 2: $e_{xy,xz}, e_{xy,yz}, e_{xz,yz}$, the multidegrees are: $(1, 1, 1), (1, 1, 1), (1, 1, 1)$.

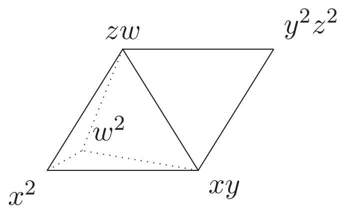
In hom. degree 3: $e_{xy,xz,yz}$, the multidegree is: $(1, 1, 1)$.

Here we crossed all the terms having the same multidegree. The remaining generators correspond to a simplicial complex Δ_φ consisting of the points $\{xy\}$, $\{xz\}$, $\{yz\}$. It is clear that the homogenization of Δ_φ is not a minimal graded resolution. In this example, the minimal resolution corresponds to the homogenization of the following simplicial complex:



where a, b, c are arbitrary assigned between xy, xz, yz.

Example 2.33 ([Kat19])**.** $\mathcal{O}$ is $\mathbb{K}[x, y, z, w]$ and $\mathcal{I} \subset \mathcal{O}$ is generated by x^2, xy, y^2z^2, zw, w^2. The Scarf complex is given by a tetrahedron on vertices $\{x^2, xy, zw, w^2\}$ and a triangle $\{xy, y^2z^2, zw\}$:



Homogenization of this complex by x^2, xy, y^2z^2, zw, w^2, gives in fact a minimal graded resolution. Some algebraic properties of this example will be studied further in the next Chapter.

Both examples show that the algebraic Scarf complex is not a resolution in general, in contrast to the Taylor complex. However, the algebraic Scarf complex is always contained in any graded resolution, as the following proposition shows:

Proposition 2.34. *Let φ be a minimal set of generators of $\mathcal{I}$. Any resolution of $\mathcal{O}/\mathcal{I}$ contains $C_S(\varphi)$.*

Proof. By contstruction it is clear that the algebraic Scarf complex is a subcomplex of the Taylor resolution. For any (multi)graded resolution of $\mathcal{O}/\mathcal{I}$, the graded version of Theorem 1.26 states that any resolution of $\mathcal{O}/\mathcal{I}$ is a direct sum of a graded minimal resolution and a *graded* trivial complex, see [Pee10]. The graded trivial complex is a sum of acyclic complexes of the form

$$0 \xrightarrow{0} \mathcal{O}[-\mathbf{a}] \xrightarrow{\text{id}} \mathcal{O}[-\mathbf{a}] \xrightarrow{0} 0.$$

Thus the algorithm of obtaining a graded minimal resolution consists in elimination of (at least some) pairs of generators in subsequent homological degrees $j, j+1$ having the same multidegree. Therefore the algebraic Scarf complex must be contained in the minimal multigraded resolution, thus in any multigraded resolution. □

If $\mathcal{I}$ is generic, then

Theorem 2.35 ([BPS98])**.** *If $\mathcal{I}$ is a generic monomial ideal minimally generated by $\varphi = \varphi_1, \ldots, \varphi_k$, then the algebraic Scarf complex $C_S(\varphi)$ is a minimal multigraded resolution of $\mathcal{O}/\mathcal{I}$.*

3 Basic definitions and elements of homological algebra

Let A, B, C are $\mathcal{O}$-modules and $\alpha : A \longrightarrow B$, $\beta : B \longrightarrow C$ are homomorphisms of $\mathcal{O}$-modules, then a pair of homomorphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is exact if $\text{Im}(\alpha) = \text{Ker}(\beta)$. In general, given a sequence of maps between modules

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \longrightarrow \cdots$$

a sequence of maps is exact if every two consecutive maps are exact.

Definition 3.1. *A short exact sequence is a sequence of exact maps*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

i.e α *is an injection,* β *is an injection,* $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$.

A short exact sequence is called *split* if there is a homomorphism $\iota : C \longrightarrow B$ such that a composition of β and ι is an identity on C, $\beta \circ \iota = \mathrm{id}$.

Definition 3.2. *A sequence of homomorphisms of* $\mathcal{O}$*-modules*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} A$$

is called splitting, if $\beta \circ \alpha = \mathrm{id}$.

Lemma 3.3. *If* $A \xrightarrow{\alpha} B \xrightarrow{\beta} A$ *is splitting, then* $B = \mathrm{Ker}(\beta) \oplus \mathrm{Im}(\alpha)$

Proof. Since $\beta \circ \alpha = \mathrm{id}$, clearly $\mathrm{Ker}(\beta) \cap \mathrm{Im}(\alpha) = 0$. By definitions of α and β we obviously have $\mathrm{Ker}(\beta) \oplus \mathrm{Im}(\alpha) \subseteq B$. Now, for any $x \in B$, $x - \alpha \circ \beta(x) \in \mathrm{Ker}(\beta)$. Therefore, $x \in \mathrm{Im}(\alpha) \oplus \mathrm{Ker}(\beta)$. This concludes the proof. □

Lemma 3.4. *If* $A \xrightarrow{\alpha} B$ *is surjective and* B *is free, then* $A \cong Ker(\alpha) \oplus B$.

Proof. Choose a basis $b = \{b_1, \ldots, b_k, \ldots\}$ of B and choose a preimage of b in A denoted by $a = \{a_1, \ldots, a_k, \ldots\}$. Define a map $\beta : B \longrightarrow A$, $\beta(b_i) = a_i$. Then $\alpha \circ \beta = \mathrm{id}$, therefore the pair α, β is splitting and $A = \mathrm{Ker}(\alpha) \oplus \mathrm{Im}(\beta) \cong \mathrm{Ker}(\alpha) \oplus B$. □

Definition 3.5. *A module* M *is called projective if it is a direct summand of a free module,* $F = Q \oplus M$.

Lemma 3.6. *Any finitely generated projective module* M *over a local ring* $(R, \mathcal{J})$ *is free.*

Proof. Let us denote a minimal set of generators of M by $\{m_1, \ldots, m_n\}$. Form a free R-module F by taking a direct sum of n copies of M. Denote its basis $\{f_1, \ldots, f_n\}$. Define an R-linear map $\beta : F \longrightarrow M$ by $\beta(f_i) = m_i$ for all $i \in [\![1, n]\!]$. Clearly β is surjective. Let us study the kernel of this map, if $\sum_{i=1}^{n} a_i m_i = \beta(\sum_{i=1}^{n} a_i f_i) = 0$, then each $a_i \in \mathcal{J}$, otherwise $\{m_1, \ldots, m_n\}$ is not a minimal set of generators. Therefore $\mathrm{Ker}(\beta) \subseteq \mathcal{J}F$.

Now we make a use of M being projective, that is there exists a free R-module G such that $G = Q \oplus M$ for some R-module Q. Denote a projection $G \longrightarrow M$ by π. Since both π and α are surjective, for any $g \in G$ there exists $f \in F$ such that $\pi(g) = \beta(f)$. Define a map γ such that the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{\gamma} & F \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \beta} \\ M & \xrightarrow[\mathrm{id}]{} & M \end{array}$$

And its restriction to M by α. Clearly, $\beta \circ \alpha = \mathrm{id}$ on M, therefore α and β are splitting. By lemma 3.3 $F = \mathrm{Ker}\beta \oplus \mathrm{Im}\alpha$ The inclusion $\mathrm{Ker}(\beta) \subseteq \mathcal{J}F$ implies $F = \mathcal{J}F + \alpha(M)$ By Corollary 1.12 to Nakayama's lemma we get $F = \alpha(M)$. Therefore $\mathrm{Ker}(\beta) = 0$ and the surjective morphism β is an isomorphism, therefore M is free. □

Definition 3.7. *Let $(A_\bullet, d)$ and $(B_\bullet, \delta)$ be two chain complexes. A chain map $f : (A_\bullet, d) \longrightarrow (B_\bullet, \delta)$ is a collection of morphisms $\{f_i : A_i \longrightarrow B_i \,|\, i \in \mathbb{Z}\}$ such that the following diagram commutes:*

$$\begin{array}{ccc} A_{i+1} & \xrightarrow{d_{i+1}} & A_i \\ {\scriptstyle f_{i+1}}\downarrow & & \downarrow{\scriptstyle f_i} \\ B_{i+1} & \xrightarrow[\delta_{i+1}]{} & B_i \end{array}$$

In other words, $f_i \circ d_{i+1} = \delta_i \circ f_{i+1}$

Definition 3.8. *Let $(A_\bullet, d)$ and $(B_\bullet, \delta)$ be two chain complexes. A chain homotopy h between two chain maps $f, g : (A_\bullet, d) \longrightarrow (B_\bullet, \delta)$ is a collection of morphisms $\{h_i : A_i \longrightarrow B_{i+1} \,|\, i \in \mathbb{Z}\}$ such that $f_i - g_i = \delta_{i+1} \circ h_i + h_{i-1} \circ d_i$ for all $i \in \mathbb{Z}$. We say f is homotopic to g if a chain homotopy exists and denote it by $f \sim g$.*

Remark 3.9. $\sim$ is an equivalence relation on the space of chain maps from $(A_\bullet, d)$ to $(B_\bullet, \delta)$.

Definition 3.10. *Two chain complexes $(A_\bullet, d)$ and $(B_\bullet, \delta)$ are said to be homotopy equivalent if there exists two chain maps $f : (A_\bullet, d) \longrightarrow (B_\bullet, \delta)$ and $g : (B_\bullet, \delta) \longrightarrow (B_\bullet, d)$ such that $f \circ g \sim \mathrm{id}_B$ and $g \circ f \sim id_A$.*

Lemma 3.11. *If two chain complexes are homotopy equivalent, then their homologies are isomorphic.*

Proof. Let us denote H_i the i-th homology of $(A_\bullet, d)$ and H'_i the i-th homology of $(B_\bullet, \delta)$. Chain maps f, g induce maps $\overline{f} : H_i \longrightarrow H'_i$ and $\overline{g} : H'_i \longrightarrow H_i$. This is easily verified by observing that a chain map maps closed elements to closed elements and exact elements to exact ones. Since $f \circ g \sim \mathrm{id}_B$, it follows that $\overline{f}_i \circ \overline{g}_i = \mathrm{id}_{H'_i}$ and $\overline{g}_i \circ \overline{f}_i = \mathrm{id}_{H_i}$. Therefore, $g \circ f$ is a bijection, thus H'_i and H_i are isomorphic. ☐

Proposition 3.12. *Let $(A_\bullet, d)$ and $(B_\bullet, \delta)$ be two free resolutions of an R-module M. Then*

i. There exists a lift of identity on M to a chain map f between $(A_\bullet, d)$ and $(B_\bullet, \delta)$, i.e. $f_0 : M \longrightarrow M$ is equal to id.

ii. Any two such lifts f and g are homotopic in the sense that there exists a collection of R-morphisms $\{h_i : A_i \longrightarrow B_{i+1} \,|\, i \in \mathbb{N}\}$ such that

$$f_i - g_i = \delta_{i+1} \circ h_i + h_{i-1} \circ d_i.$$

Proof. Claim i. can be showed by recursion. Since for any element $x \in A_1$ $\mathrm{id} \circ d_1(x) \in \mathrm{Im}(\delta_1)$, we can built f_1 as follows: choose a basis $e = \{e_1, \dots, e_k, \dots\}$ of A_1 and define $f_1(e_j)$ to be any δ_1-preimage of $d_1(e_j)$. Then f_1 is extended to A_1 as an $\mathcal{O}$-linear map.

Now, assume that a collection of R-module morphisms$\{f_1, \dots, f_i\}$ is constructed such that the diagram

$$\begin{array}{ccc} A_{j+1} & \xrightarrow{d_{j+1}} & A_j \\ {\scriptstyle f_{j+1}}\downarrow & & \downarrow{\scriptstyle f_j} \\ B_{j+1} & \xrightarrow[\delta_{j+1}]{} & B_j \end{array}$$

commutes for all $j < i$. Then for any $x \in A_{i+1}$, $\delta_i \circ f_i \circ d_{i+1}(x) = f_{i-1} \circ d_i \circ d_{i+1}(x) = 0$, thus $f_i \circ d_{i+1}(x)$ is closed, therefore exact. Let $r = \{r_1, \dots, r_k, \dots\}$ be a basis of A_{i+1}. Form an $\mathcal{O}$-linear map f_{i+1} by choosing a preimage of $f_i \circ d_{i+1}(r)$. Constructed in this way f_{i+1} satisfies $\delta_{i+1} \circ f_{i+1} = f_i \circ d_{i+1}$.

Claim *ii.* can be proved by a similar diagram chasing:

$$
\begin{array}{ccccccccccc}
A_{i+1} & \xrightarrow{d_{i+1}} & A_i & \longrightarrow & \cdots & \longrightarrow & A_2 & \xrightarrow{d_2} & A_1 & \xrightarrow{d_1} & M \\
\downarrow{\scriptstyle f_{i+1}-g_{i+1}} & \swarrow{\scriptstyle h_i} & \downarrow{\scriptstyle f_i-g_i} & & & & \downarrow{\scriptstyle f_2-g_2} & \swarrow{\scriptstyle h_1} & \downarrow{\scriptstyle f_1-g_1} & & \downarrow{\scriptstyle 0} \\
B_{i+1} & \xrightarrow[\delta_{i+1}]{} & B_i & \longrightarrow & \cdots & \longrightarrow & B_2 & \xrightarrow{\delta_2} & B_1 & \xrightarrow{\delta_1} & M
\end{array}
$$

Define $h_0 = 0$. At degree 1, take a element $x \in A_1$. Then $(f_1 - g_1)(x)$ is δ_1-closed, therefore δ_2-exact. Form an R-linear map h_1 by choosing a preimage of a basis e of A_1 in B_2. Thus we obtain $f_1 - g_1 = \delta_2 \circ h_1 + h_0 \circ d_1$. Now, suppose that up to degree $i - 1.i > 1$ we coonstructed a collection of R-linear maps $\{h_j \mid j \in [\![0, i-1]\!]\}$ satisfying $f_j - g_j = \delta_{j+1} \circ h_j + h_{j-1} \circ d_j$. In order to ensure the existence of h_{i+1} we must check that $(f_i - g_i - h_{i-1} \circ d_i)(x)$ is closed for all $x \in A_i$:

$$\delta_i \circ (f_i - g_i - h_{i-1} \circ d_i)(x) = (f_i - g_i) \circ d_i(x) - \delta_i \circ h_{i-1} \circ d_i(x) = 0.$$

Therefore, a R-linear map h_{i+1} with desired properties exists. □

Corollary 3.13. *Any two free resolutions $(A_\bullet, d)$ and $(B_\bullet, \delta)$ of M are homotopy equivalent.*

Proof. Let $f : (A_\bullet, d) \longrightarrow (B_\bullet, \delta)$ and $g : (B_\bullet, \delta) \longrightarrow (A_\bullet, d)$ be chain maps covering the identity on M. Then $f \circ g : (B_\bullet, \delta) \longrightarrow (B_\bullet, \delta)$ and $g \circ f : (A_\bullet, d) \longrightarrow (A_\bullet, d)$ are chain maps covering id on M. Since id_A and id_B are chain maps as well, $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_B$ by Proposition 3.12 are homotopic. □

Koszul-Tate resolutions

4 Overview of algebra resolutions

Given an associative (and not neccesarily commutative) algebra $\mathcal{A}$, one may want to study its homological properties. The generic approach to such questions is to consider a "simpler" object than $\mathcal{A}$ itself, whose homology coincides with $\mathcal{A}$. There are several answers to this question in the literature. One of the answers we have already explored in the previous Chapter, i.e. a Construction 1.2, which can be easily adapted to the case of non-commutative algebras. The main drawback of such resolutions is that they are not canonical, i.e. they depend on choices made in Construction 1.2. The standard result in homological algebra is the bar-cobar resolution [HMS74] for augmented associative algebras. The bar-cobar resolution is done in two steps. The first one called the bar construction [EL53], gives as an output a cofree coalgebra together with a coderivation built on the multiplication of A. The second step, which is the cobar construction [Ada56], uses the coalgebra data to arrive at a free differential graded algebra, which is a resolution of A. We study the bar-cobar resolution in details in the Subsection 4.1. In the Subsection 4.2 we review a nice result by Chuang and King [CK12], which provides a free algebra resolution in the following setting: $\mathcal{A}$ is an associative algebra (not neccesarily unital) over a ring R, which is given as a quotient algebra $T^R(V)/\mathcal{I}$ of a tensor algebra on R-bimodule V by some ideal $\mathcal{I}$. The result of Chuang and King is essentially a cobar construction of a particular ∞-coalgebra of rooted trees or "dots and brackets" $B^\bullet_\infty(V;\mathcal{I})$, which depends on V and $\mathcal{I}$. Both constructions are canonical, but they are "too big" for practical purposes. Nevertheless, the ∞-coalgebra of dots and brackets turns out to be a good starting idea for studying "smaller" algebra resolutions of Koszul-Tate type, which we introduce and develop in the subsequent sections. This overview section is based on [Gin05, LV12, AJ13, CK12, FHT92].

4.1 Bar complex and bar-cobar resolution

For a short background on differential graded algebras and coalgebras we refer the reader to Appendix 10. Let A be an augmented differential graded associative algebra over $\mathbb{K}$, $A = \mathbb{K}1_A \oplus \bar{A}$, with the $\mathbb{K}$-linear differential d and the product μ.

Construction 4.1. The *bar construction* is a differential graded coassociative coalgebra $(\overline{T}_c(s\bar{A}), D)$ built as follows:

i. $T_c(s\bar{A})$ is a graded tensor coalgebra with s being the suspension map (see Appendix 10). $T_c(s\bar{A})$ is coaugmented, i.e. it can written as a direct sum of $\mathbb{K}$ and the reduced tensor coalgebra $\overline{T}_c(s\bar{A})$, $\overline{T}_c(s\bar{A}) = s\bar{A} \oplus s\bar{A} \otimes s\bar{A} \oplus s\bar{A} \otimes s\bar{A} \otimes s\bar{A} \oplus \ldots$. An element $(sa_1)(sa_2)\ldots(sa_n)$ in $(s\bar{A})^{\otimes n}$ has degree $n + \deg(a_1) + \cdots + \deg(a_n)$ and arity n, the coproduct is given by the deconcatenation $\bar{\Delta}$ and is of degree 0.

ii. The differential D can be written as a sum of two differential $D_1 + D_2$ defined in the following way:

- On $s\bar{A}$ D_1 reproduces the diffential of $\bar{A}$, $D_1 = s \circ d \circ s^{-1}$. D_2 is identically zero on $s\bar{A}$.
- On $s\bar{A} \otimes s\bar{A}$ D_2 is given by

$$D_2 = -s \circ \mu \circ (s^{-1} \otimes s^{-1})$$

  In other words, D_2 is a linear map of degree -1, $D_2 \colon s\bar{A} \otimes s\bar{A} \longrightarrow s\bar{A}$ such that the diagram

$$\begin{array}{ccc} sA \otimes sA & \xrightarrow{-D_2} & sA \\ {\scriptstyle s^{-1}\otimes s^{-1}}\downarrow & & \uparrow{\scriptstyle s} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

  commutes. In particular, $D_2(sa \otimes sb) = s\mu(a \otimes b)$, for all $a, b \in \bar{A}$.
- According to Lemma 10.4 we can extend D_1 and D_2 as coderivations $\overline{T}_c(s\bar{A}) \longrightarrow \overline{T}_c(s\bar{A})$. In particular on $(s\bar{A})^{\otimes n}$, $D_1 = \sum_{j=1}^{n} \mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \underbrace{s \circ d \circ s^{-1}}_{j\text{-th position}} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}$.
  For $n = 3$, $D_2(sa \otimes sb \otimes sc) = s\mu(a \otimes b) \otimes sc - sa \otimes s\mu(b \otimes s)$. In general,

$$D_2(sa_1 \otimes sa_2 \otimes \cdots \otimes sa_n) = \sum_{i=1}^{n-1} (-1)^{i-1} (sa_1 \otimes sa_2 \otimes s\mu(a_i \otimes a_{i+1}) \otimes \cdots \otimes sa_n)$$

  The signs follow from the application of the Koszul sign rule to (37).

Proposition 4.2. *The coderivation D of Construction 4.1 is a differential, $D^2 = 0$.*

Proof. One has to check that $D^2 = 0$. In fact one can show that $D_1^2 = 0, D_1 \circ D_2 + D_2 \circ D_1 = 0$ and $D_2^2 = 0$. The first equality follows from $d^2 = 0$ of the underlying differential graded algebra $\bar{A}$. The second equality comes from the compatibility of d and μ, and the third one follows from the associativity of μ. We show here explicitly the third equality $D_2^2 = 0$, the first two are treated analogously. For $n \leq 2$ it follows from the definition that $D_2^2 = 0$. For $n \geq 3$ let us apply D_2^2 to $sa_1 \otimes sa_2 \otimes \cdots \otimes sa_n$. We obtain summonds of the following form:

i. Those where D_2 acts on two different pairs $(a_\bullet, a_{\bullet+1})$:
$sa_1 \otimes sa_2 \otimes s\mu(a_i \otimes a_{i+1}) \otimes \cdots \otimes s\mu(a_j \otimes a_{j+1}) \otimes \cdots \otimes sa_n$.

ii. Summonds where D_2 hits the same pair twice:
$sa_1 \otimes sa_2 \otimes s(\mu(\mu(a_i \otimes a_{i+1}) \otimes a_{i+2}) \otimes \cdots \otimes sa_n$ and
$sa_1 \otimes sa_2 \otimes s(\mu(a_{i-1} \otimes \mu(a_i \otimes a_{i+1})) \otimes \cdots \otimes sa_n$.

Summands of type i. can be obtained in two different ways, first one acts by D_2 on a pair starting with a_j and then on a pair starting with a_i, or vice versa. It is easy to check that one gets a relative $-$ sign in both cases, thus the sum of such summands vanishes. Using the same argument as well as the associativity of μ helps to establish that summands of type ii. cancel each other out as well. □

Since $D^2 = 0$ the bar construction is actually a chain complex, called the *bar complex*.

Construction 4.3. Let C be a coaugmented coassociative differential graded coalgebra with a comultiplication Δ, and $\overline{C}$ be its reduced component, $C = 1_C \oplus \overline{C}$ with a reduced coproduct $\bar{\Delta}$. The differential d is a codifferential with respect to $\bar{\Delta}$. The *cobar construction* is a differential graded algebra $(T(s^{-1}\overline{C}), D)$ such that

i. $T(s^{-1}\overline{C}) = \mathbb{K} \oplus s^{-1}\overline{C} \oplus s^{-1}\overline{C} \boxtimes s^{-1}\overline{C} \oplus s^{-1}\overline{C} \boxtimes s^{-1}\overline{C} \boxtimes s^{-1}\overline{C} \oplus \dots$

ii. The differential D is defined as follows:

- D is zero on $\mathbb{K}$.
- D on $s^{-1}\overline{C}$ incorporates d of the coalgebra $\overline{C}$ and the comultiplication $\bar{\Delta}$:

$$D|_{s^{-1}\overline{C}} = s^{-1} \circ d \circ s + (s^{-1} \otimes s^{-1}) \circ \Delta \circ s$$

To illustrate the action of D, take $a \in \overline{C}$ such that $\overline{\Delta}a = \sum_{(i,j)\in I} a_i \boxtimes a_j$ for some index set I, a_i, a_j are elements of $\overline{C}$. Then

$$D(s^{-1}a) = s^{-1}(da) + \sum_{(i,j)\in I} (-1)^{\deg(a_i)} s^{-1}a_i \boxtimes s^{-1}a_j$$

The sign in the second summand appears due to the Koszul sign rule, s^{-1} has to "jump" over a_i in order to be applied to a_j

- D is extended as a derivation to $T(s^{-1}\overline{C})$.

Proposition 4.4. *D of the cobar construction is a differential, $D^2 = 0$*

The proof of this proposition is along the same lines as the proof of Proposition 4.2

Proposition 4.5. *Let $A = \mathbb{K}1_A \oplus \bar{A}$ be an augmented associative algebra. The cobar construction of the bar construction of A, $T(s^{-1}(T_c(s\bar{A})))$ is a free differential graded algebra quasi-isomorphic to A.*

For the proof we refer to [LV12]. Let us check that the degree homology $H_0(T(s^{-1}(T_c(s\bar{A}))))$ does coincide with A. Note that degree zero subspace of $T(s^{-1}(T_c(s\bar{A})))$ is given by $T(\bar{A}) = \mathbb{K}\oplus\bar{A}\oplus\bar{A}\boxtimes\bar{A}\oplus\dots$. The image of D in degree zero is generated by $I = \{a\boxtimes b - \mu(a\otimes b) \mid a, b \in \bar{A}\}$ viewed as a bimodule over A. Then the zero homology of D is given by the quotient $T(\bar{A})/I \simeq A$.

The resolution of Proposition 4.5 is called the *bar-cobar resolution*. Despite being canonical, this resolution is known to be "too big" to analyze A.

4.2 Coalgebra of rooted trees

We may generalize the viewpoint of the previous section. Instead of starting with an associative algebra over $\mathbb{K}$, one can start from a tensor algebra $T(V)$, with V being an R-bimodule for some ring R, and a set of relations I of $T(V)$ i.e. a two-sided ideal of $T(V)$. In the setting before the set of relations $\mathcal{I}_{\text{bar}}$ is generated by $\{a\boxtimes b - \mu(a\otimes b) \mid a, b \in V\}$. The bar-cobar resolution is then a free differential graded algebra which is quasi-isomorphic to $T(V)/\mathcal{I}_{\text{bar}}$. Chuang and King in [CK12] generalized this construction to an arbitrary set of relations. The main tool used by the authors is the A_∞ coalgebra of rooted trees $B^\infty(V; I)$ which is described below. The differential graded algebra quasi-isomorphic to $T(V)/I$ is then given by the cobar construction of $B^\infty(V; I)$. We discuss this construction in more details.

Definition 4.6. *Let $C = \{C_i \mid i \in \mathbb{Z}\}$ be a graded R-bimodule. An A_∞ coalgebra structure on C is a collection of R-morphisms $\{\Delta_i : C_n \longrightarrow C_{a_1}\boxtimes C_{a_2}\boxtimes\cdots\boxtimes C_{a_i} \mid n \in \mathbb{Z},\ a_1+\cdots+a_i = n-2+i\}$ satisfying*

$$\sum_{r+s+t=n} (-1)^{r+st}(\mathrm{id}^{\boxtimes r} \boxtimes \Delta_s \boxtimes \mathrm{id}^{\boxtimes t}) \circ \Delta_{1+r+t} = 0$$

for all $n \geq 1$.

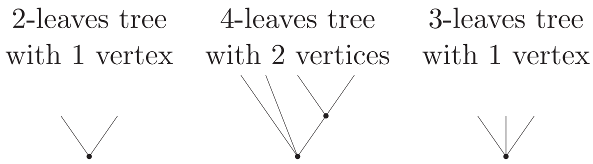
Example 4.7. If the collection consists of only Δ_1, Δ_2, then one recovers the relations of coassociative differential coalgebra:

$$\begin{aligned}
&\text{For } n = 1: \quad \Delta_1 \circ \Delta_1 = 0,\\
&\text{For } n = 2: \quad \Delta_2 \circ \Delta_1 - (\mathrm{id} \boxtimes \Delta_1) \circ \Delta_2 - (\Delta_1 \boxtimes \Delta_1) \circ \Delta_2 = 0\\
&\text{For } n = 3: \quad (\Delta_2 \boxtimes \mathrm{id}) \circ \Delta_2 - (\mathrm{id} \boxtimes \Delta_2) \circ \Delta_2 = 0.
\end{aligned}$$

First, let us describe the coalgebra $B_\infty(V)$.

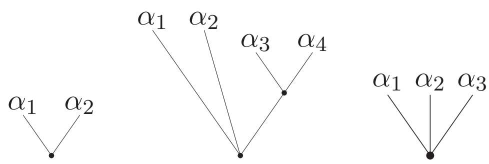
Construction 4.8 (A_∞ coalgebra of rooted trees)**.** Let V be an R-bimodule. The coalgebra $B^\infty(V)$ is constructed as follows:

i. Let Tree_n^k be a set of all planar rooted trees with k vertices and n leaves. Typical examples of planar rooted trees:



The degree $k+1$ component $B_\infty^k(V)$ is generated by pairs $\{(t_n, V^{\otimes_R n}) \mid t_n \in \mathrm{Tree}_n^k, n \in \mathbb{N}\}$ as an R-bimodule. The component of degree 1 is defined to be $T(V)$. We invite to think that $B_\infty^k(V)$ is given by trees with k vertices and leaves decorated by words in V.

Denote by $B_n^k(V)$ the set of pairs of degree k whose correspoding trees have n leaves. Note that any tree $t \in \mathrm{Tree}_n^k$ can be viewed equivalently as a word of "dots and brackets". The trees with 2, 4, 3 leaves as above can be viewed as $[\bullet\bullet], [\bullet\bullet[\bullet\bullet]]$ and $[\bullet\bullet\bullet]$. We view any generating element $\alpha \in B_n^k(V)$ as a word in brackets. For example,



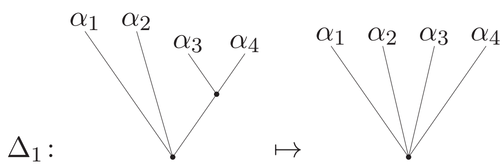
can be rewritten as $[\alpha_1\alpha_2]$, $[\alpha_1\alpha_2[\alpha_3\alpha_4]]$ and $[\alpha_1\alpha_2\alpha_3]$.

ii. Let us write any generating element $\alpha \in B_n^k(V)$ that underlines a tree t as $[\beta_1 \ldots \beta_m]$, where β_j is the j-th child of the root of t. A collection of R-morphisms $\{\Delta_i\}$ is given by:

1. Δ_1 acts by removing the *internal* brackets:

$$\Delta_1([\beta_1 \ldots \beta_m]) = \sum_{j=1}^{k-1} (-1)^{j-1} [\beta_1 \ldots \beta_m]_j$$

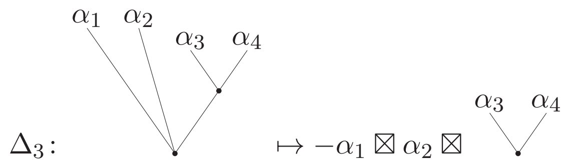
where $[\beta_1 \ldots \beta_m]_j$ is obtained from $[\beta_1 \ldots \beta_m]$ by removing the j-th internal bracket, counted from the left. In the language of trees, Δ_1 acts as



2. The non-zero components of Δ_m, $m > 1$ are given on the trees with m children of the root $[\beta_1 \dots \beta_m]$:

$$\Delta_m([\beta_1 \dots \beta_m]) = (-1)^{\sum_{i=1}^{m}(m-i)\deg(\beta_i)} \beta_1 \boxtimes \dots \boxtimes \beta_m.$$

where degree of each child β_i which has d_i pairs of brackets is $d_i + 1$. In particular for a decorated tree $\alpha = [\alpha_1 \alpha_2 [\alpha_3 \alpha_4]]$ Δ_3 is given by



A straightforward computation shows that this collection $\{\Delta_i\}$ satisfies the relations of A_∞ coalgebra. Let us describe now the coalgebra $B_\infty(V; I)$. We do so in the case when $I \subset T^{\geq 2}(V) := V \otimes_R V \oplus V^{\otimes_R 3} \oplus \dots$. The general case can be found in [CK12]. Let us distinguish the *innermost* brackets - those that contains inside only words in V and no brackets. Then $B_\infty(V; I)$ as a graded R-module is obtained from $B_\infty(V)$ by putting words in I instead of V in every innermost bracket. This means if $\alpha = [[\alpha_1 \alpha_2]\alpha_3]$, then $\alpha_1 \alpha_2 \in \mathcal{I}$ and $\alpha_3 \in V$. The coalgebra structure is given by the same set of operations as for $B_\infty(V)$.

Proposition 4.9 ([CK12]). *The cobar construction of $B_\infty(V; I)$ is quasi-isomorphic to $T(V)/I$.*

This resolution of $T(V)/I$ is quite big: the degree 1 generators are generators of R-module I, and there are infinitely many of them. This is in sharp contrast with free resolutions examined in the previous Chapter. However, the idea of using decorated rooted trees proves to be useful in studying the algebra resolutions of commutative rings, which we study and develop in the next sections.

5 Koszul-Tate resolutions, questions and motivation

Let $\mathcal{O}$ be a commutative algebra and $\mathcal{I} \subset \mathcal{O}$ an ideal. Then the quotient $\mathcal{O}/\mathcal{I}$ is:

i. an $\mathcal{O}$-module (in fact, a *cyclic module*, i.e. a module with 1 generator),

ii. and also a commutative algebra.

To both previous contexts correspond two different meanings of "a resolution of $\mathcal{O}/\mathcal{I}$".

i. If we forget the algebra structure and only keep in mind the $\mathcal{O}$-module structure, then, as for any $\mathcal{O}$-module, it is natural to look for a resolution $(\mathfrak{M}, d)$ of $\mathcal{O}/\mathcal{I}$ by free or projective $\mathcal{O}$-modules, starting with $\mathcal{O}$ itself:

$$\cdots \xrightarrow{d} \mathfrak{M}_k \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{M}_1 \xrightarrow{d} \mathcal{O} \longrightarrow 0 \tag{13}$$

More precisely, we call *projective (or free) $\mathcal{O}$-resolutions of the quotient $\mathcal{O}/\mathcal{I}$* complexes as (13) such that

(a) each $\mathfrak{M}_k$ is a projective (or free) $\mathcal{O}$-module and each map d is $\mathcal{O}$-linear,

(b) the complex has no homology in positive degrees and $H_0(\mathfrak{M}_\bullet, d) = \mathcal{O}/\mathcal{I}$.

We draw the attention of the reader to the fact that $\mathcal{O} \oplus \mathfrak{M}_\bullet := \mathcal{O} \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots$ is simply an $\mathcal{O}$-module, but not a ring. However, later on in this article, we will equip $\mathcal{O} \oplus \mathfrak{M}_\bullet$ with the structure of an A_∞-algebra.

ii. If we do not forget the algebra structure on $\mathcal{O}/\mathcal{I}$, then one may look for a projective resolution as in (13) such that $\mathcal{O} \oplus \mathfrak{M}_\bullet$ admits a graded commutative product making it a *DGA-$\mathcal{O}$-resolution*. It is known (See [Avr75]-Appendix 7, answering negatively a conjecture in [BE77], or [Avr79]. More conter-examples are given in [Kat19]) that such a commutative product may not exist on a given $\mathcal{O}$-resolution. The alternative is to look for a *Koszul-Tate resolution* [Tat57] of $\mathcal{O}/\mathcal{I}$. By definition this is a pair $(\mathcal{A}, \delta)$ such that

(a) $\mathcal{A}$ is a graded commutative $\mathcal{O}$-algebra isomorphic to the symmetric algebra generated by some projective $\mathcal{O}$-module $(\mathcal{E}_i)_{i\geq 1}$

(b) and δ is a degree -1 derivation, squaring to zero, whose homology is zero in all degrees except for in degree 0, where it is $\mathcal{O}/\mathcal{I}$.

Let us give some explanations on these two conditions. Condition (a) means that:

$$\mathcal{A} \quad := \quad S(\oplus_{i=1}^{\infty}\mathcal{E}_i)$$

where the symbol S stands for the graded symmetric algebra generated by a graded $\mathcal{O}$-module $\mathcal{E}_\bullet$ and we denote by $\odot$ its product. Let us write the first terms:

$$\mathcal{A} \quad = \quad \underbrace{\mathcal{O}}_{=\mathcal{A}_0} \oplus \underbrace{\mathcal{E}_1}_{=\mathcal{A}_1} \oplus \underbrace{\mathcal{E}_2 \oplus \mathcal{E}_1 \odot \mathcal{E}_1}_{=\mathcal{A}_2} \oplus \underbrace{\mathcal{E}_3 \oplus \mathcal{E}_1 \odot \mathcal{E}_2 \oplus \mathcal{E}_1 \odot \mathcal{E}_1 \odot \mathcal{E}_1}_{=\mathcal{A}_3} \oplus \cdots$$

In this article, the tensor product $\otimes$ of $\mathcal{O}$-modules and its graded symmetrization $\odot$ are by default taken over $\mathcal{O}$. When considering the tensor product over a field $\mathbb{K}$, the notations $\otimes_{\mathbb{K}}$ will be used. Condition (b) means that the complex

$$\cdots \xrightarrow{\delta} \mathcal{A}_3 \xrightarrow{\delta} \mathcal{A}_2 \xrightarrow{\delta} \mathcal{A}_1 \xrightarrow{\delta} \mathcal{O} \xrightarrow{0} 0$$

admits the following homology

$$H_i(\mathcal{A}, \delta) = 0 \text{ for } i \geq 1 \text{ and } H_0(\mathcal{A}, \delta) = \mathcal{O}/\mathcal{I}.$$

The last condition means that $\delta(\mathcal{A}_1) = \mathcal{I}$.

Unlike DG-$\mathcal{O}$-resolutions on a given projective $\mathcal{O}$-resolution, Koszul-Tate resolutions always exist [Tat57]. Notice that a Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$ is also a projective $\mathcal{O}$-resolution of $\mathcal{O}$. Since any two projective $\mathcal{O}$-resolutions of the same $\mathcal{O}$-module are homotopy equivalent, any Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$ is homotopy equivalent to any projective $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$.

It is remarkable that the two previously described "resolutions" exist for any commutative algebra $\mathcal{O}$ and any ideal $\mathcal{I}$.

i. The existence of free resolutions of any $\mathcal{O}$-module is a classical result of commutative algebra.

Construction 5.1. For the quotient algebra $\mathcal{O}/\mathcal{I}$, the construction goes as follows:

(a) We choose generators $(E_i)_{i\in I}$ of the $\mathcal{O}$-module $\mathcal{I}$.

(b) We define $\mathfrak{M}_1$ to be the free $\mathcal{O}$-module generated by elements in I, i.e. sums of the form

$$\sum_{i\in I} f_i[i]$$

where only finitely many of the elements $f_i \in \mathcal{O}$ are not equal to zero. We define $d \colon \mathfrak{M}_1 \to \mathcal{O}$ to be the $\mathcal{O}$-linear map

$$d: \sum_{i\in I} f_i[i] \mapsto \sum_{i\in I} f_i E_i. \tag{14}$$

(c) We choose generators $(F_j)_{j\in J}$ of the kernel of $d\colon \mathfrak{M}_1 \to \mathcal{O}$, then define $\mathfrak{M}_2$ to be the free $\mathcal{O}$-module generated by J, and we define $d\colon \mathfrak{M}_2 \to \mathfrak{M}_1$ by

$$d: \sum_{j\in J} g_j[j] \mapsto \sum_{j\in I} g_j F_j.$$

The procedure continues by recursion. This completes the proof.

Moreover, there are wide classes of algebras $\mathcal{O}$ for which this construction can be chosen to be "small". For instance, if $\mathcal{O}$ is a Noetherian algebra, then each one of the free $\mathcal{O}$-module $\mathfrak{M}_k$ can be chosen to be finitely generated. Moreover some Noetherian algebras $\mathcal{O}$ have the property that for any ideal $\mathcal{I}$ the quotient $\mathcal{O}/\mathcal{I}$ admit a projective $\mathcal{O}$-resolution $(\mathfrak{M}, d)$ of finite length, i.e. satisfy that there exists an integer N such that $\mathfrak{M}_k = 0$ for $k \geq N+1$. A classical example of such an algebra is the algebra $\mathbb{C}[X_1, \ldots, X_n]$ of polynomials in n variables. An another example is germs at 0 of real analytic functions on $\mathbb{R}^n$, or holomorphic functions on $\mathbb{C}^n$. In any of these cases, one can even choose $N \leq n+1$. For practical purposes one may find concrete realization of module resolutions using Macaulay software [EGSS02].

ii. Koszul-Tate resolutions can be constructed by a similar algorithm, called the *Tate algorithm.*

Construction 5.2. *Tate algorithm*

(a) The modules $\mathcal{E}_1$ can be constructed exactly as the module $\mathfrak{M}_1$: it is a free module equipped with an $\mathcal{O}$-linear map $d_1 : \mathfrak{M}_1 \longrightarrow \mathcal{O}$ such that $d_1(\mathfrak{M}_1) = \mathcal{I}$.

(c) The differential $d_1\colon \mathcal{E}_1 \to \mathcal{O}$ extends to $S^\bullet(\mathcal{E}_1)$ as a graded derivation of degree -1 squaring to zero that we denote by δ_1. By construction, $H_0(S^\bullet(\mathcal{E}_1), \delta_1) = \mathcal{O}/\mathcal{I}$. Of course, $H_1(S^\bullet(\mathcal{E}_1), \delta_1) \neq 0$ in general, but there exists a free $\mathcal{O}$-module $\mathcal{E}_2$ equipped with $\mathcal{O}$-linear map $d_2\colon \mathcal{E}_2 \to S(\mathcal{E}_1)_1$ whose image generates $H_1(S(\mathcal{E}_1), \delta_1)$. The maps δ_1, d_2 then extend to a degree -1 derivation δ_2 of $S(\mathcal{E}_1 \oplus \mathcal{E}_2)$ that squares to zero. By construction,

$$H_0(S(\mathcal{E}_1 \oplus \mathcal{E}_2), \delta_2) = \mathcal{O}/\mathcal{I} \text{ and } H_1(S(\mathcal{E}_1 \oplus \mathcal{E}_2), \delta_2) = 0$$

(c) The procedure is repeated in order to construct a sequence $(\mathcal{E}_k)_{k\geq 1}$ of free $\mathcal{O}$-modules and a sequence $d_k\colon \mathcal{E}_k \to S^\bullet(\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{k-1})_{k-1}$ of $\mathcal{O}$-linear maps such that for every $k \geq 1$ the image of d_k generates $H_{k-1}(S(\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{k-1}), \delta_k)$ where δ_k stands for the degree -1-derivation whose restriction to $\mathcal{E}_i$ is d_i for all $i \leq k-1$. It is easily proven by recursion that for all $k \geq 1$, we have $\delta_k^2 = 0$ and

$$H_0\left(\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{k-1}, \delta_k\right) = \mathcal{O}/\mathcal{I} \text{ and } H_i\left(\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{k-1}, \delta_k\right) = 0 \text{ for all } i \leq k$$

This implies that the pair

$$(S(\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots), \delta)$$

with δ being the degree -1 derivation whose restriction to $\mathcal{E}_i$ is d_i for all $i \geq 1$, is a Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$.

Although both algorithms seem to be similar at first look, the Tate algorithm is in general more complicated than the algorithm that permits to compute Koszul-Tate resolution.

In general, even when we start from an ideal $\mathcal{I}$ in a polynomial ring $\mathcal{O}$, there might exist infinitely many $\mathcal{E}_k$ appearing in the Tate algorithm. This means that one has to compute infinitely many homologies, but also that the construction may depend on infinitely many choices.

As a result, there are quite a few ideals $\mathcal{I}$ in algebras $\mathcal{O}$ for which the Koszul-Tate $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ is not explicitly known although it is possible to find a free or projective $\mathcal{O}$-resolution.

The purpose of the present article is mainly to answer the following natural question:

Question 5.3. *Is there a way to construct a Koszul-Tate resolution by doing only finitely many homology computations and having only finitely many choices?*

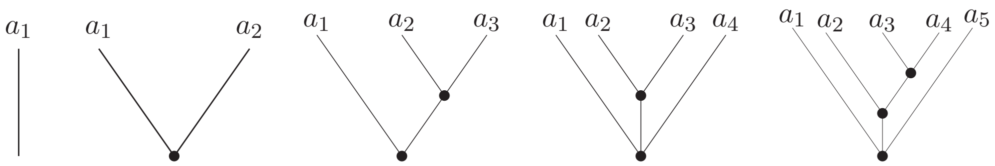
Alternatively, one can rephrase the question as: Is there a manner to construct a Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$ out of the data of a free/projective $\mathcal{O}$-resolution $(\mathfrak{M}_\bullet, d)$ of $\mathcal{O}/\mathcal{I}$? We claim to have a convincing answer to the latter question, that in particular answers Question 5.3 positively when $(\mathfrak{M}, d)$ has finite length. More precisely, with our construction, not only there are finitely many computations and choices, but we also know when we can stop the construction: at the n-th step if n is the length of $(\mathfrak{M}, d)$.
Let us now explain how we answer Question 5.3. First, let us show why a very natural answer turns to be wrong.

Remark 5.4. There is a natural and tempting answer to Question 5.3, which unfortunately fails in general: given $(\mathfrak{M}, d)$ as in (13), it is tempting to consider $S(\mathfrak{M})$, equipped with the degree -1 graded derivation δ extending d. Although it is true that $\delta^2 = 0$, and that $H^0(S(\mathfrak{M}_\bullet), \delta) = \frac{\mathcal{O}}{d\mathfrak{M}_1} = \frac{\mathcal{O}}{\mathcal{I}}$, it is not true even in the case of a complete intersection that $H_i(S(\mathfrak{M}_\bullet), \delta) = 0$ for $i \geq 1$. It is easily verified in the following example: take $\mathcal{O}$ to be a ring of smooth functions on $\mathbb{R}^2$, and $\mathcal{I}$ is an ideal generated by $< x, y >_{\mathcal{O}}$. The module $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ has a length two: $\mathfrak{M}_1$ has two generators π^x, π^y, $\mathfrak{M}_2$ has one generator π_2 and

$$\begin{array}{ll} \text{Generators in degree 1:} & d\pi_1^{xy} = xy, \qquad d\pi_1^z = z, \\ \text{Generator in degree 2:} & d\pi_2 = z\pi_1^{xy} - xy\pi_1^z. \end{array}$$

In the graded commutative algebra $(S(\mathfrak{M}_\bullet), \delta)$ (with δ the extension of d to a graded derivation), $\pi_2 + \pi_1^x \pi^y$ is δ-closed, but it cannot be exact, for π_2 is "out of range" for such a differential δ.

To answer Question 5.3, given a projective $\mathcal{O}$-resolution $(\mathfrak{M}, d)$ as in (13), our strategy is to consider the free $\mathcal{O}$-module $\mathcal{T}ree[\mathfrak{M}_\bullet]$ of rooted trees with leaves decorated by elements in $\mathfrak{M}_i$, i.e. elements of the form:



(15)

with $a_i \in \mathfrak{M}$ for $i = 1, \ldots, 5$. Such trees have a natural $\mathcal{O}$-module structure. The main point of the present article is to equip the symmetric algebra of $\mathcal{T}ree[\mathfrak{M}_\bullet]$ (that can be thought as a forest of such trees) with a differential δ that makes it a Koszul-Tate $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$. We call it the *arborescent Koszul-Tate $\mathcal{O}$-resolution of* $\mathcal{O}/\mathcal{I}$ (attached to the free or projective $\mathcal{O}$-resolution $(\mathfrak{M}, d)$). This answers positively Question 5.3.

Moreover, this differential has several components, which are of the following types:

1. One component consists in acting by d on the elements of $\mathfrak{M}$ attached to the leaves.

2. Several components are natural and mechanical operations on trees, e.g. removing a root to create a forest out of the tree, removing an inner edge, applying the previously constructed δ to each one of the possible sub-trees, or binding a forest into a tree by adding a root vertex.

3. The last and the most intriguing components can be characterized by a family indexed by rooted trees associating to any tree t with n leaves and k inner vertices a n-linear endomorphism ψ_t of degree k on $\mathcal{O} \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots$. We will call these maps the *arborescent operations.* The construction of these arborescent operations is the only one which is not mechanical and requires a choice of a coboundary. For degree reasons, if the resolution $(\mathfrak{M}, d)$ has length n, there exists finitely many non-trivial arborescent operations.

We call *arborescent Koszul-Tate resolution of* $(\mathfrak{M}, d)$ the henceforth obtained structure.
It is a natural question to ask what structure these mysterious coboundary operations encode on the resolution $(\mathfrak{M}_\bullet, \delta)$. A complete answer probably involves operads, but we have a partial answer. Since any Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$ is in particular a projective $\mathcal{O}$-resolution of $\mathcal{O}\mathcal{I}$, and since any two projective $\mathcal{O}$-resolutions of $\mathcal{O}/\mathcal{I}$ are homotopy equivalent, $(\mathfrak{M}, d)$ is homotopy equivalent to its arborescent Koszul-Tate resolution. In fact, we will show that it is even a homotopy retract. As a consequence, homotopy transfer theorem applies and yields an A_∞-algebra structure on the complex $\cdots \xrightarrow{d} \mathfrak{M}_k \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{M}_1 \xrightarrow{d} \mathcal{O} \longrightarrow 0$ We will give explicit formulas for the A_∞-algebra products in terms of the arborescent operations.

Last, we answer the following question:

Question 5.5. *Is there a minimal Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$?*

We provide a positive answer in the case when $\mathcal{O}$ is a local Noetherian ring or a polynomial ring, where $\mathcal{I}$ is graded with respect to polynomial degree. Moreover, we show that any Koszul-Tate resolution encodes a minimal one and is in fact isomorphic to $S(\mathcal{E}) \odot S(T)$ as a graded algebra, where T is a trivial complex.
Section 6 is devoted to the construction of the arborescent Koszul-Tate resolution, with a particular emphasize on the arborescent products. Examples shall be given in Section 7. In Section 9 we explore consequences and applications of this explicit construction of a Koszul-Tate resolution and we study minimal Koszul-Tate resolutions.

We conclude the introduction by a few conventions:

Convention 5.6. *Given an $\mathcal{O}$-resolution $(\mathfrak{M}_\bullet, d)$ of $\mathcal{O}/\mathcal{I}$ as in (13), $(\mathfrak{M}, d)$ shall denote the complex truncated at* 0*:*

$$\cdots \xrightarrow{d} \mathfrak{M}_k \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{M}_1$$

Last, $\mathfrak{M}_\bullet$ and $\mathcal{O} \oplus \mathfrak{M}_\bullet$ denote the graded vector spaces $\mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots$ and $\mathcal{O} \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots$, without any reference to the differential d.

Also, we recall our conventions for tensor products.

Convention 5.7. *Given $\mathcal{O}$-modules $\mathcal{E}$ and $\mathcal{F}$, the notations $\mathcal{E} \otimes \mathcal{F}$ and $\mathcal{E} \odot \mathcal{F}$ stand for the tensor and symmetric products over the algebra $\mathcal{O}$. The tensor and symmetric products over the base field $\mathbb{K}$ are denoted by $\otimes_{\mathbb{K}}$ and $\odot_{\mathbb{K}}$ respectively.*

For $\mathcal{E}$ an $\mathcal{O}$-module, $S(\mathcal{E})$ stands by default for the symmetric algebra over $\mathcal{O}$. Also,

$$S^k(\mathcal{E}) := \underbrace{\mathcal{E} \odot \cdots \odot \mathcal{E}}_{k-times} \quad \text{and} \quad S^{\geq 2}(\mathcal{E}) := \oplus_{k \geq 2} S^k(\mathcal{E}).$$

6 Construction of the arborescent Koszul-Tate resolution.

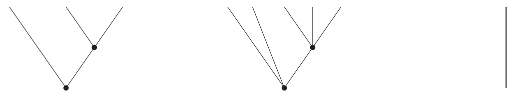
In this section, we present an explicit and constructive way to obtain a Koszul-Tate $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ out of an $\mathcal{O}$-module resolution $(\mathfrak{M}_\bullet, d)$ of $\mathcal{O}/\mathcal{I}$ as in (13).

We proceed as follows. In Section 6.1, we describe the free graded commutative algebra $S(\mathcal{T}ree[\mathfrak{M}_\bullet])$ generated by the $\mathcal{O}$-module $\mathcal{T}ree[\mathfrak{M}_\bullet]$ of trees with leaves decorated by elements of $\mathfrak{M}_\bullet$. In Section 6.2, we explain how to equip the graded commutative algebra $S(\mathcal{T}ree[\mathfrak{M}_\bullet])$ with a differential δ_ψ, that depends on some recursively constructed multi-linear maps on $\mathfrak{M}_\bullet$ that we call arborescent operations. Theorem 6.19 will show that the homology of δ_ψ is zero, hence proving it is a Koszul-Tate resolution, that we will call arborescent Koszul-Tate $\mathcal{O}$-resolutions of $\mathcal{O}/\mathcal{I}$.

6.1 Arborescent Koszul-Tate resolutions: the graded algebra of decorated trees

Throughout this section, $\mathfrak{M}_\bullet = (\mathfrak{M}_i)_{i\geq 1}$ stands for an arbitrary sequence of $\mathcal{O}$-modules. We define the $\mathcal{O}$-module $\mathcal{T}ree[\mathfrak{M}_\bullet]$ of decorated trees:

(i) First, let us consider the set of *planar rooted trees*, i.e. the set of rooted trees with ordered leaves satisfying the property that each vertex has at least two descendants. The set is enlarged by adding a trivial tree. Typical examples are listed below:



We denote by $Tree$ the free $\mathbb{K}$-vector space generated by rooted planar trees. The free $\mathbb{K}$-vector spaces generated by trees with n leaves (resp k vertices[1] shall be denoted by $Tree^n$ (resp. $Tree_k$). Also $Tree^n_k := Tree^n \cap Tree_k$.

For instance, $Tree^1_0$ is the vector space generated by the trivial tree, it is therefore isomorphic to $\mathbb{K}$.

(ii) The free $\mathcal{O}$-module of *ordered decorated trees* is defined then as

$$Tree[\mathfrak{M}_\bullet] = \bigoplus_{n=1}^{\infty} Tree^n \otimes_{\mathbb{K}} \underbrace{\mathfrak{M}_\bullet \otimes \ldots \otimes \mathfrak{M}_\bullet}_{n \text{ times}} \tag{16}$$

The space of ordered decorated trees with n leaves (resp. k vertices) is denoted by $Tree^n[\mathfrak{M}_\bullet]$ (resp. $Tree_k[\mathfrak{M}_\bullet]$). Also, $Tree^n_k[\mathfrak{M}_\bullet] = Tree_k[\mathfrak{M}_\bullet] \cap Tree^n[\mathfrak{M}_\bullet]$. Elements in $Tree^n[\mathfrak{M}_\bullet]$ shall be denoted as:

$$t\,[a_1, \ldots, a_n]$$

with $t \in Tree^n$ and $a_1, \ldots, a_n \in \mathfrak{M}_\bullet$. We invite the reader to think that $a_1, \ldots, a_n$ are attached to the leaves of t, as in (15). By construction the $\mathcal{O}$-module structure is given for all $F \in \mathcal{O}$ by

$$F \cdot t\,[a_1, \ldots, a_n] = t\,[Fa_1, \ldots, a_n] = \cdots = t\,[a_1, \ldots, Fa_n].$$

Remark 6.1. In particular, elements in $Tree^1_0 \otimes_{\mathbb{K}} \mathfrak{M}_\bullet$ shall be denoted as $|\,[a]$ with $a \in \mathfrak{M}_\bullet$. The vertical bar $|$ refers to the trivial tree.

(iii) Let us introduce a grading on ordered trees by declaring the degree of an element in $Tree^n_k$ to be $k = \#\,\{\text{Vertices of } t\}$. With this convention, the free $\mathcal{O}$-module of ordered decorated trees has an induced degree defined by:

$$deg(t\,[a_1 \otimes \cdots \otimes a_n]) = |a_1| + \cdots + |a_n| + \#\,\{\text{Vertices of } t\} \tag{17}$$

We denote the space of elements of degree i by $Tree[\mathfrak{M}_\bullet]_i$, so that $Tree[\mathfrak{M}_\bullet] = \oplus_{i=1}^{\infty} Tree[\mathfrak{M}_\bullet]_i$.

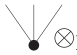
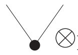
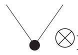
For instance:

[1] Our convention for trees is that by a vertex, we mean either an inner vertex or a root, i.e. leaves are not vertices: for instance, the rightmost tree of (15) is has 3 vertices. By convention also, the trivial tree has 0 vertex.

$i=3$ We have $Tree[\mathfrak{M}_\bullet]_3 :=$

 $\otimes_{\mathbb{K}} (\mathfrak{M}_3) \bigoplus$  $\otimes_{\mathbb{K}} (\mathfrak{M}_1 \otimes \mathfrak{M}_1)$

$i=4$ We have $Tree[\mathfrak{M}_\bullet]_4 :=$

 $\otimes_{\mathbb{K}}(\mathfrak{M}_4) \bigoplus$  $\otimes_{\mathbb{K}}(\mathfrak{M}_1\otimes\mathfrak{M}_1\otimes\mathfrak{M}_1) \bigoplus$  $\otimes_{\mathbb{K}}(\mathfrak{M}_2\otimes\mathfrak{M}_1) \bigoplus$  $\otimes_{\mathbb{K}}(\mathfrak{M}_1\otimes\mathfrak{M}_2)$

In addition to this degree decomposition, one will also need to consider the graded sub-$\mathcal{O}$-module $T^{\geq 2}[\mathfrak{M}_\bullet]$ generated by trees which have at least two leaves (= trees whose root has at least two children). For every fixed degree i:

$$Tree[\mathfrak{M}_\bullet]_i = Tree_0^1 \otimes_{\mathbb{K}} \mathfrak{M}_i \;\oplus\; T^{\geq 2}[\mathfrak{M}_\bullet]_i\,, \tag{18}$$

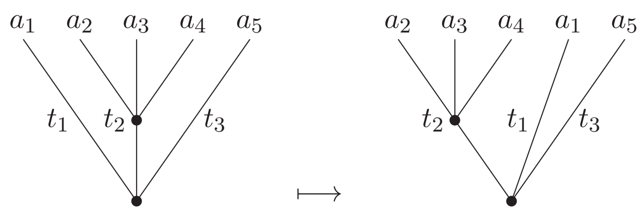
(iv) Let us a define a natural quotient of the $\mathcal{O}$-module $Tree[\mathfrak{M}_\bullet]$ of ordered decorated trees, exploiting permutations of the leaves of a tree.

Consider an ordered decorated tree with n leaves and k inner vertices $t[a_1,\ldots,a_n] \in Tree_k^n[\mathfrak{M}_\bullet]$. Consider an inner vertex V of t which has v children, which are themselves ordered decorated trees

$$t_1[a_1,\ldots,a_{|t_1|}]\,,\;\ldots\;,\; t_v[a_{|t_1|+\cdots|t_{v-1}|+1},\ldots,a_m]$$

of degrees $\theta_1,\ldots,\theta_v$, $|t_i|$ being the number of leaves of the corresponding tree t_i, and m being the cardinal of teh leaves that descent from V. A permutation $\sigma \in S_v$ acts by exchanging the position of the children, keeping the rest of the tree untouched. The outcome is a tree that we denote by $t_{V,\sigma}[a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}]$. Also, we denote by $\epsilon(t_{V,\sigma})$ the Koszul sign of the permutation σ with respect to the degrees $\theta_1,\ldots,\theta_v$.

An example of such a permutation is provided below for an ordered decorated tree $t[a_1 \otimes \cdots \otimes a_5]$. As a vertex we choose the root, which has three children of degrees $|a_1|, |a_2| + |a_3|+|a_4|+1$ and $|a_5|$, and we exchange the two first ones, leaving the third one untouched:



The Koszul sign of this permutation is $(-1)^{|a_1|(|a_2|+|a_3|+|a_4|+1)}$.

Let us identify, for every $t[a_1 \otimes \cdots \otimes a_n] \in Tree_n[\mathfrak{M}_\bullet](k)$ and every permutation σ as above, the following elements:

$$t[a_1 \otimes \cdots \otimes a_n] \;\sim\; \epsilon(t_{A,\sigma})\, t_{A,\sigma}[a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(1)}], \tag{19}$$

where σ is any permutation of the children of any vertex A of t. Equivalently, let us introduce the following quotient:

Definition 6.2. *We call $\mathcal{O}$-module of decorated trees the quotient $\mathcal{O}$-module $\mathcal{T}ree[\mathfrak{M}_\bullet] := Tree[\mathfrak{M}_\bullet]/\sim$ with $\sim$ as in (19).*

Let us give some properties of this quotient:

Proposition 6.3. *The $\mathcal{O}$-module $\mathcal{T}ree[\mathfrak{M}_\bullet]$ of decorated trees is a positively graded $\mathcal{O}$-module, which is free if the $\mathcal{O}$-module $\mathfrak{M}$ is free.*

Proof. The only difficulty is to show that the quotient is still a free $\mathcal{O}$-module. It suffices to prove this result for trees with a fixed number n of leaves.

We shall use the following fact: the quotient of a free $\mathcal{O}$-module $W \otimes_{\mathbb{K}} \mathcal{O}$, with W a $\mathbb{K}$-vector space by the $\mathcal{O}$-module $W' \otimes_{\mathbb{K}} \mathcal{O}$, with $W' \subset W$ a sub-$\mathbb{K}$-vector space, is the free $\mathcal{O}$-module $W/W' \otimes_{\mathbb{K}} \mathcal{O}$. We apply this point to the $\mathbb{K}$-vector space $W = T_n \otimes_{\mathbb{K}} M_\bullet^{\otimes n}$, with $M_\bullet$ a graded $\mathbb{K}$-vector space such that $\mathfrak{M}_\bullet = M_\bullet \otimes_{\mathbb{K}} \mathcal{O}$. Since $\sim$ consists in dividing by linear relations whose coefficients are in $\mathbb{K}$, the equivalence relation can be seen as the quotient by a free $\mathcal{O}$-module generated by a sub-$\mathbb{K}$-vector space W' of $W = Tree_n \otimes_{\mathbb{K}} M_\bullet^{\otimes n}$. The result follows. □

The $\mathcal{O}$-module $\mathcal{T}ree[\mathfrak{M}_\bullet]$ of decorated trees, which has been proven to be free in Proposition 6.3, is the one on which a Koszul-Tate resolution will be constructed. More precisely, we will consider the graded commutative free algebra $S^\bullet(\mathcal{T}ree[\mathfrak{M}_\bullet])$ generated (over $\mathcal{O}$) by $\mathcal{T}ree[\mathfrak{M}_\bullet]$ and equip it with some differential δ_ψ. In order to have a short and consistent definition of a yet to be constructed differential, we need to adopt a new convention, and allow leaves of a decorated tree to be elements in $\mathcal{O}$.

Convention 6.4. *For t a tree with $n > 1$ leaves, and $a_1, \ldots, a_n$ elements with are either in $\mathfrak{M}$ or in $\mathcal{O}$, we let $t[a_1, \ldots, a_n]$ to be the element in $\mathcal{T}ree[\mathfrak{M}_\bullet]$ defined following the following rule. Assume $a_i = F \in \mathcal{O}$:*

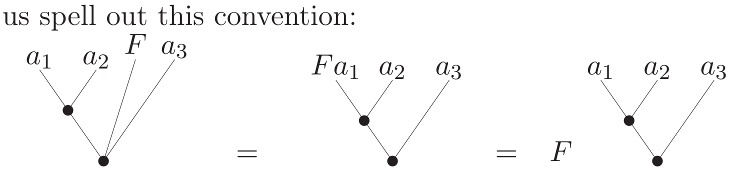
1. *If the parent of the i-th leaf is a vertex with ≥ 3 children, then we consider the tree t_i obtained by erasing the i-th leaf, and we set:*

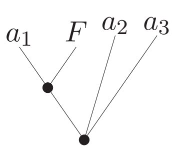
$$t[a_1, \ldots, \underbrace{F}_{i^{th}}, \ldots a_n] = F\, t_i[a_1, \ldots, \widehat{F}, \ldots, a_n].$$

2. *If the parent of the i-th leaf is a vertex with 2 children, then we set*

$$t[a_1, \ldots, \underbrace{F}_{i^{th}}, \ldots, a_n] = 0.$$

Let us spell out this convention:

1. 

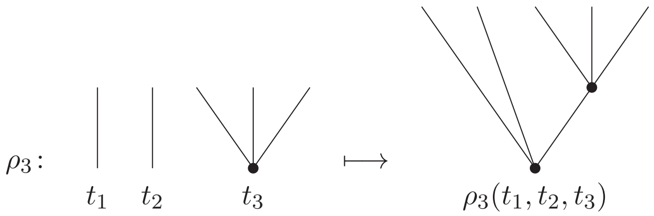
2. 

$= 0.$

Remark 6.5. Elements in $S^\bullet(\mathcal{T}ree[\mathfrak{M}_\bullet])$ may be thought of as linear combinations of forests of elements in $\mathcal{T}ree[\mathfrak{M}_\bullet]$, which some sign appearing when trees in the forest are permuted.

We need two more structures on $S^\bullet(\mathcal{T}ree[\mathfrak{M}_\bullet])$, namely the root map and its inverse, the unroot map.

Consider the $\mathbb{K}$-vector space $Tree$. For every $n \geq 2$, there is a natural n-linear operation ρ_n of degree 1 which associates to n rooted trees $(t_1, \ldots, t_n)$ the tree obtained by adding a root and linking it to the roots of $(t_1, \ldots, t_n)$. For $n = 3$, for instance



Altogether, the maps $(\rho_n)_{n\geq 2}$ define a map ρ from the tensor algebra

$$\otimes^{\geq 2} Tree = \otimes^2 Tree \oplus \otimes^3 Tree \oplus \cdots$$

to $Tree^{\geq 2}$. The map ρ above is injective, and is surjective on elements of $Tree$ coming from trees whose root has ≥ 2 children:

Proposition 6.6. *The map ρ is an isomorphism of $\mathbb{K}$-vector space*

$$\rho\colon\ \otimes^{\geq 2}(Tree) \xrightarrow{\sim} Tree^{\,\geq 2}.$$

The map ρ extends to ordered rooted trees with decorated leaves and this extension $\otimes^{\geq 2} Tree[\mathfrak{M}_\bullet] \longrightarrow T^{\,\geq 2}[\mathfrak{M}_\bullet]$ is $\mathcal{O}$-linear of degree $+1$. The map above is injective, and is surjective on elements of $Tree[\mathfrak{M}_\bullet]$ coming from trees whose root has ≥ 2 children. Morover, it goes to the quotient (see item (iv)) to induce a linear endomorphism $\otimes^{\geq 2}\mathcal{T}ree[\mathfrak{M}_\bullet] \to \mathcal{T}^{\,\geq 2}[\mathfrak{M}_\bullet]$. By definition of the quotient operation, this quotient map is graded symmetric. One can therefore describe it as a map:

$$\oplus_{k\geq 2} S^k(\mathcal{T}ree[\mathfrak{M}_\bullet]) = \oplus_{k\geq 2} \underbrace{\mathcal{T}ree[\mathfrak{M}_\bullet] \odot \cdots \odot \mathcal{T}ree[\mathfrak{M}_\bullet]}_{k-terms} \longrightarrow \mathcal{T}ree[\mathfrak{M}_\bullet].$$

By abuse of notations, we still denote it by ρ this quotient map and call it the *root map*. It remains true that for all $k \geq 2$,

$$\rho_k := S^k(\mathcal{T}ree[\mathfrak{M}_\bullet]) \to \mathcal{T}ree[\mathfrak{M}_\bullet]$$

is injective. It remains also true that it is surjective on the $\mathcal{O}$-module of decorated rooted trees with k leaves. As a consequence: ρ satisfies the following generalization of Proposition 6.6:

Proposition 6.7. *The root map ρ is an isomorphism of $\mathcal{O}$-modules*

$$\rho\colon \qquad S^{\,\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet]) \longrightarrow \mathcal{T}^{\,\geq 2}[\mathfrak{M}_\bullet] \tag{20}$$

This isomorphism maps $S^k(\mathcal{T}ree[\mathfrak{M}_\bullet])$, $k \geq 2$ to decorated trees whose root has k children.

Remark 6.8. The isomorphism ρ does not extend well to $S^1(\mathcal{T}ree[\mathfrak{M}_\bullet])$ because we do not allow the root to have only one descendant.

Remark 6.9. The constructions presented in this section are easily checked to be compatible with Convention 6.4.

Remark 6.10. The constructions and result of the present section extend easily to several contexts of interest. For $\mathcal{O} = C^\infty(M)$ the algebra of smooth function on a manifold, we could have chosen $\mathfrak{M}_i$ to be for all $i \geq 1$ sections of a vector bundle $V_i \to M$. Or we could have used sheaves: if one replaces the algebra $\mathcal{O}$ by the sheaf of holomorphic functions over a complex variety M (or regular functions over a scheme), and $\mathfrak{M}_i$ by a sequence of projective finitely generated $\mathcal{O}$-modules, (i.e. sections of some vector bundle $V_i \to M$), then Proposition 6.3 extends to state that $\mathcal{T}ree[\mathfrak{M}_\bullet]$ is made of sections of a graded vector bundle, which is of finite rank in every degree. Last, in all these contexts, Proposition 6.7 stands verbatim, upon replacing free $\mathcal{O}$-modules by the relevant sheaf of sections.

6.2 Arborescent operations and the differential δ_ψ

We will in the sequel associate to any non-trivial tree t with n leaves and k inner vertices a n-linear endomorphism of $\mathfrak{M}$ of degree $k-1$:

$$\psi_t \colon \mathrm{Hom}_{\mathcal{O}}^{k-1}(S^n(\mathcal{O}\oplus\mathfrak{M}_\bullet),\mathcal{O}\oplus\mathfrak{M}_\bullet).$$

We intend to do so in a way which is compatible with the operations defining the quotient considered in (19): For every tree t with n leaves and every $a_1,\ldots,a_n\in\mathfrak{M}$, we require $\psi_t\,(a_1,\ldots,a_n)=\epsilon(t_{A,\sigma})\psi_{\sigma(t_{A,\sigma})}\left(a_{\sigma(1)},\ldots,a_{\sigma(n)}\right)$ where σ is any permutation of the children of an inner vertex A as in (19). We also intend to make it compatible Convention 6.4.

Definition 6.11. *Let $(\mathfrak{M}_k)_{k\geq 1}$ be a sequence of projective $\mathcal{O}$-modules. We call* arborescent operation *a linear map:*

$$\psi\colon\ \mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet]\ \longrightarrow\ \mathfrak{M}. \tag{21}$$

Moreover, for any non-trivial tree $t\in Tree$ with $n\geq 2$ leaves and k inner vertices, we call arborescent operation associated to t the n-linear map of degree $k-1$:

$$\psi_t(a_1,\ldots,a_n)\mapsto\psi\left(t[a_1,\ldots,a_n]\right)\ \textit{for all } a_1,\ldots,a_n\in\mathfrak{M}.$$

We still denote by ψ_t and call arborescent operation associated to t the natural extension of ψ_t to a degree $k-1$ n-linear map $\psi\colon\otimes^n(\mathcal{O}\oplus\mathfrak{M}_\bullet)\longrightarrow\mathfrak{M}_\bullet$ obtained by allowing $a_1,\ldots,a_n$ to be in $\mathcal{O}$ and using Convention 6.4.

Let us introduce some notations. The component of the projections with respect to the direct sum:

$$S^\bullet(\mathcal{T}ree[\mathfrak{M}_\bullet])=\mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet]\oplus|[\mathfrak{M}_\bullet]\oplus S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet])$$

(recall that $|[\mathfrak{M}_\bullet]=Tree_0^1\otimes\mathfrak{M}_\bullet$) shall be denoted by:

$$\begin{aligned}\mathrm{p}^1_\vee : S(\mathcal{T}ree[\mathfrak{M}_\bullet])&\longrightarrow\mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet]\\ \mathrm{p}^1_| : S(\mathcal{T}ree[\mathfrak{M}_\bullet])&\longrightarrow|[\mathfrak{M}_\bullet]\\ \mathrm{p}^{\geq 2}\colon S(\mathcal{T}ree[\mathfrak{M}_\bullet])&\longrightarrow S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet]).\end{aligned} \tag{22}$$

The upper index of p refers to the polynomial degree in $S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet])$, while the symbols $|$ and $\vee$ indicates whether one projects on trivial or on non-trivial trees.

Description of the differential δ_ψ

From now on, we choose $(\mathfrak{M}_\bullet,d)$ a $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ as in (14). In the present section 6.2, we choose an arbitrary arborescent operation $\psi\colon t\mapsto\psi_t$ and we construct by recursion a degree -1 derivation δ_ψ of $S^\bullet(\mathcal{T}ree[\mathfrak{M}_\bullet])$, which at the moment does not square to zero yet.
In order to define a degree -1 derivation δ_ψ of $S^\bullet(\mathcal{T}ree[\mathfrak{M}_\bullet])$, it suffices to give its restriction to the $\mathcal{O}$-module $\mathcal{T}ree[\mathfrak{M}_\bullet]$. This restriction is obtained by recursion on the degree, a procedure that we now describe.
In degree 0**.** For degree reason, δ_ψ has to be identically zero on the subspace of elements of degree 0. Notice that it implies that δ_ψ will be an $\mathcal{O}$-linear derivation.
In degree 1**.** Elements of degree 1 in $S(\mathcal{T}ree[\mathfrak{M}_\bullet])$ are elements in $Tree_0^1\otimes_{\mathbb{K}}\mathfrak{M}_1$, i.e. elements of the form $|\otimes a$ for some $a\in\mathfrak{M}_1$. We chose the differential δ_ψ to map $|\,[a]$ to $d(a)\in\mathcal{O}$.
In degree ≥ 2**.** Let us assume that δ_ψ is defined on generators of $\mathcal{T}ree[\mathfrak{M}_\bullet]$ of degree less or equal to n. In order to describe its extension to elements in $\mathcal{T}ree[\mathfrak{M}_\bullet]_{n+1}$, we write

$$\delta_\psi=\tau^{\geq 2}+\tau^1_\vee+\tau^1_|$$

where

$$\tau^{\geq 2}:=\mathrm{p}^{\geq 2}\circ\delta_\psi,\quad \tau^1_\vee:=\mathrm{p}^1_\vee\circ\delta_\psi,\quad \tau^1_|:=\mathrm{p}^1_|\circ\delta_\psi. \tag{23}$$

We define $\tau^{\geq 2}, \tau_{|}^{1}, \tau_{\vee}^{1}$ on both components of the decomposition

$$\mathcal{T}ree[\mathfrak{M}_\bullet]_{n+1} = \mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet]_{n+1} \oplus |[\mathfrak{M}_{n+1}].$$

On the second summand, we impose $\tau^{\geq 2} = \tau_{\vee}^{1} = 0$ and $\tau_{|}^{1} := \mathrm{id} \otimes d$. Equivalently,

$$\delta_\psi\left(|\,[a]\right) := |\,[d(a)] \text{ for all } a \in \mathfrak{M}_{n+1}.$$

On the first summand $\mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet]_{n+1}$, we now define $\tau^{\geq 2}, \tau_{\vee}^{1}, \tau_{|}^{1}$ as follows:

- The map $\tau_{|}^{1}$ is the opposite of the chosen arborescent operation $\psi \colon \mathcal{T}ree^{\geq 2}[\mathfrak{M}_\bullet]_{n+1} \longrightarrow \mathfrak{M}_n$ composed with the map $a \mapsto |[a]$ from $\mathfrak{M}_n$ to $| \otimes \mathfrak{M}_n$. Explicitly

$$\tau_{|}^{1} \colon t[a_1, \ldots, a_m] \mapsto -|\,[\psi_t(a_1, \ldots, a_m)]$$

  for any tree t with m leaves and k inner vertices and any element $a_1 \otimes \cdots \otimes a_m \in \otimes^m \mathfrak{M}$ of degree $n + 1 - k$.

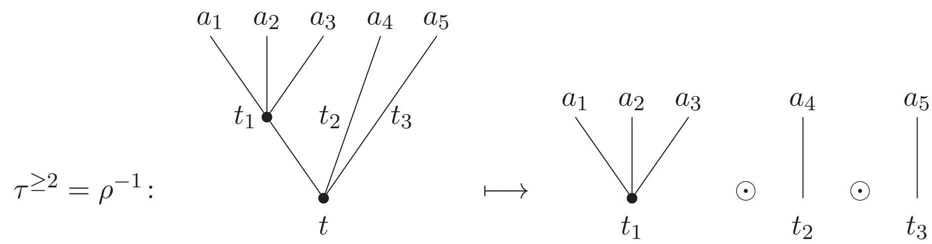
- $\tau^{\geq 2}$ is the *unroot map*, i.e. the inverse of the root map $\rho \colon S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet])_n \to \mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet]_{n+1}$. Explicitly, on an element $t[a_1, \ldots, a_{|t|}]$ such that the root of t has k children $t_1, \ldots, t_k$, it is defined by:

$$\rho^{-1} \colon t[a_1, \ldots, a_{|t|}] \mapsto t_1[a_1, \ldots, a_{|t_1|}] \odot t_2[a_{|t_1|+1}, \ldots, a_{|t_1|+|t_2|}] \odot \cdots \odot t_k[a_{|t_1|+\cdots+|t_{k-1}|+1}, \ldots, a_{|t|}].$$

  Here we denote by $|t|$ the number of leaves of t. The previous equation may be written in the following more practical form:

$$\rho^{-1}\left(t[a_1, \ldots, a_n]\right) = t_{A_1}[a_{A_1}] \odot \cdots \odot t_{A_k}[a_{A_k}], \tag{24}$$

  with the understanding that $t_{A_i}[A_i]$ stands for the subtree of all descendents of A_i and that $A_1, \ldots, A_k$ are the vertices children of the root. For instance:



- In contrast of the previous maps, $\tau_{\vee}^{1}$ is defined recursively. Assume δ_ψ is defined on elements of degree less or equal to n. Set

$$\tau_{\vee}^{1} := -\rho \circ \mathrm{p}^{\geq 2} \circ \delta_\psi \circ \rho^{-1}.$$

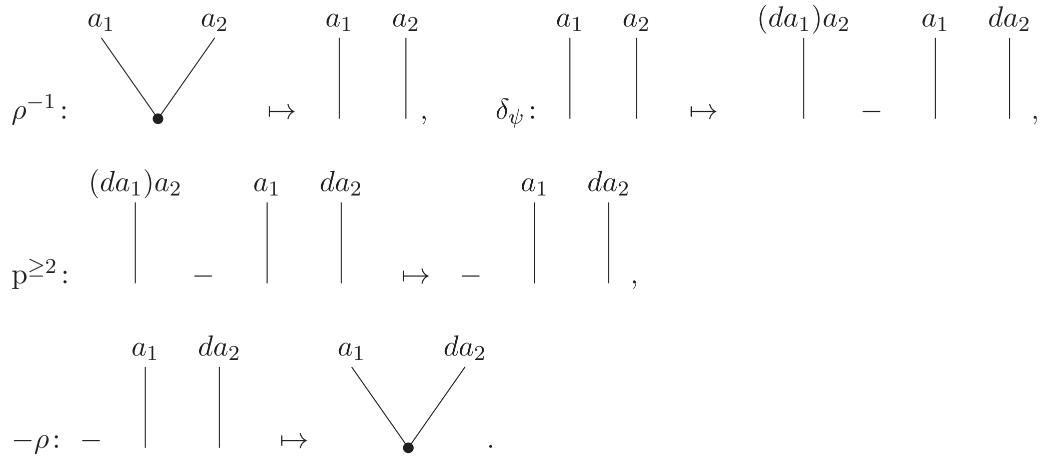
  In words, $\tau_{\vee}^{1}$ consists of doing the following operations:

  1. applying $\tau^{\geq 2}$, i.e. the inverse ρ^{-1} of the root map, $\rho \colon \mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet]_{n+1} \to S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet])_n$,
  2. applying derivation δ_ψ already constructed: $\delta_\psi \colon S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet])_n \to S^\bullet(\mathcal{T}ree[\mathfrak{M}_\bullet])_{n-1}$,
  3. applying the projection $\mathrm{p}^{\geq 2}$, i.e. suppressing elements in $\mathcal{T}ree[\mathfrak{M}_\bullet]_{n-1}$,
  4. Binding the resulting forest into one tree by means of the root map ρ and multiplying by -1.

In equation:

$$\mathcal{T}ree[\mathfrak{M}_\bullet]_{n+1} \xrightarrow{\rho^{-1}} S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet])_n \xrightarrow{\delta_\psi} S(\mathcal{T}ree[\mathfrak{M}_\bullet])_{n-1} \xrightarrow{p^{\geq 2}} S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet])_{n-1} \xrightarrow{-\rho} \mathcal{T}ree[\mathfrak{M}_\bullet]_n$$

To illustrate the operations let us consider the tree with 2 leaves decorated by elements a_1, a_2 with $|a_1| = 1$ and $|a_2| \geq 2$:



The derivation δ_ψ described in Section 6.2 has an explicit (i.e. non-recursive) description in terms of the arborescent operation ψ and several elementary operations on trees.

We first equip the $\mathcal{O}$-module of ordered decorated trees $T[\mathfrak{M}_\bullet]$ of all decorated rooted trees with a natural differential ∂ as follows.

1. For $t[a_1, \ldots, a_n]$ a decorated rooted tree, to any inner vertex $A \in t$ (that maybe the root or a leaf), we attach a weight $W_A \in \mathbb{Z}$ which is the degree of the tree obtained by summing the following two contributions:
   (a) the number of the vertices along the path γ from the root to A,
   (b) the sum of degrees of all subtrees located to the left of the path γ.
2. For any rooted tree t and inner vertex A (different from the root), let $\partial_A t$ stand for the tree obtained by merging vertex A with its parent vertex. If A is the root, we define $\partial_A t$ to be zero. Since ∂_A does not modify the cardinal of leaves, and is compatible with the symmetry relations, ∂_A can be defined on a decorated rooted tree as follows: $t[a_1, \ldots, a_n] \mapsto (\partial_A t)[a_1, \ldots, a_n]$. We still denote by ∂_A this induced map.

We then consider the degree -1 map

$$\begin{array}{rccc} \partial\colon & Tree[\mathfrak{M}_\bullet] & \longrightarrow & Tree[\mathfrak{M}_{\bullet-1}] \\ & t[a_1, \ldots, a_n] & \mapsto & \sum_{A \in Vertex(t)} (-1)^{W_A} \partial_A t[a_1, \ldots, a_n]. \end{array}$$

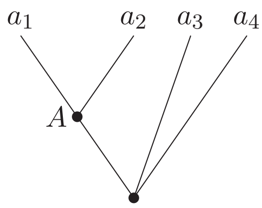
Proposition 6.12. *The degree -1 map ∂ squares to 0 and descends to define a differential on the module of decorated trees $\mathcal{T}ree[\mathfrak{M}_\bullet]$.*

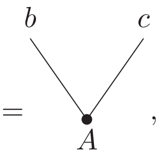
Proof. This is a recursion on a number of inner vertices and is left to the reader. □

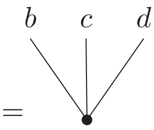
We can now give an alternative description of δ_ψ. We start by a definition:

Definition 6.13. *For any inner vertex A of a tree t, we denote by $t_{\downarrow A}$ the trees obtained by replacing that vertex by a leaf , and $t_{\uparrow A}$ the tree with a root A obtained by keeping only the descendants of this vertex.*

Example 6.14. Let us explain Definition 6.13.

For tree $t =$ 

, one has $t_{\uparrow A}[b,c] =$ 

, and $t_{\downarrow A}[b,c,d] =$ 

We now can describe δ_ψ in a direct manner.

Proposition 6.15. *Consider a tree t with $n \geq 2$ leaves. Let $Vertex(t)$ be the set of inner vertices of tree t. Then for all $a_1, \dots, a_n \in \mathfrak{M}$:*

$$\delta_\psi(t[a_1,\dots,a_n]) = \rho^{-1}\left(t[a_1,\dots,a_n]\right) + \partial\left(t[a_1,\dots,a_n]\right)$$
$$- \sum_{A\in Vertex(t)} (-1)^{W_A}\left(t_{\downarrow A}[a_1,\dots,\psi_{t_{\uparrow A}}(a_A),\dots a_n]\right) + \sum_{i=1}^{n}(-1)^{W_i} t[a_1,\dots,da_i,\dots,a_n], \tag{25}$$

where:

- *∂ is the differential on trees defined in Proposition 6.12,*
- *ρ^{-1} is the unroot map, see Equation (24),*
- *a_A denotes the leaves that descent from the vertex A*
- *In the fourth term of the r.h.s. of Equation 25, Convention 6.4 must be applied when $a_i \in \mathfrak{M}_1$.*

Proof. To prove the statement, we unify the last lines in Equation (25) by introducing a sum that runs on all vertices, leaves included. This requires to set, for A the i-th leaf, $t_{\uparrow A}[a_A] = |\otimes a_i$, and $\psi_|(a_i) = -d(a_i)$. With these conventions, Equation (25) reads

$$\delta_\psi(t[a_1,\dots,a_n]) = \rho^{-1}\left(t[a_1,\dots,a_n]\right)$$
$$+ \sum_{A\in Vertex+Leaves(t)} (-1)^{W_A}\left(-t_{\downarrow A}[a_1,\dots,\psi_{t_{\uparrow A}}(a_A),\dots a_n] + (\partial_A t)[a_1,\dots,a_n]\right),$$

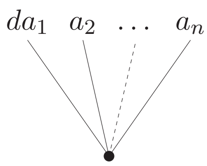
The first line in the equation is the unroot map $\rho^{-1} = \tau^{\geq 2}$. For A the root R of the tree t, we have $\partial_R = 0$, $a_R = a_1,\dots,a_n$ and $t_R = t$ so that the non-vanishing term in the second line is:

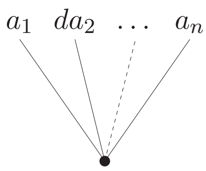
$$-t_{\downarrow R}[a_1,\dots,\psi_{t_{\uparrow R}}(a_R),\dots a_n] = -|\,[\psi_{t_{\uparrow A}}(a_1,\dots,a_n)].$$

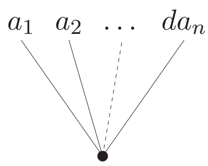
One recognizes $\tau_|^1(t(a_1,\dots,a_n))$. The expression (25) can be read as

$$\delta_\psi(t[a_1,\dots,a_n]) = \tau^{\,\geq 2}(t[a_1,\dots,a_n]) + \tau_|^1(t[a_1,\dots,a_n]) + \text{rest}$$

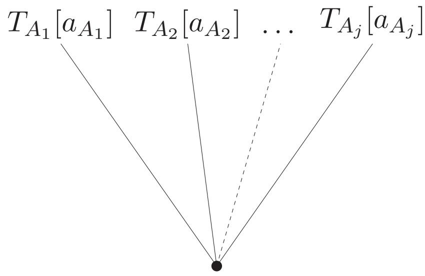
Where "rest" is a collection of terms valued in single trees with ≥ 2 leaves. It remains to check that the "rest" corresponds to $\tau_\vee^1 = -\rho \circ \mathrm{p}^{\,\geq 2} \circ \delta_\psi \circ \rho^{-1}$. This is done by a recursion on the number of inner vertices N. For $N = 0$, these residual terms are read as:

$-$ 

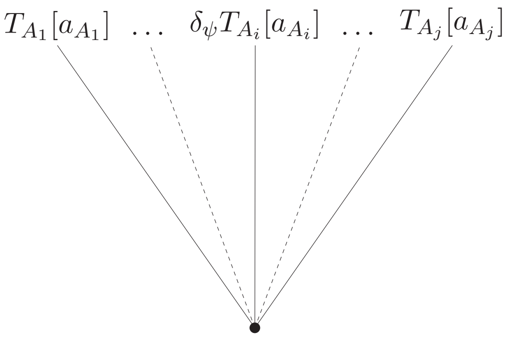
$+ (-1)^{|a_1|+1}$ 

$+ \cdots + (-1)^{|a_1|+\cdots+|a_{n-1}|+1}$ 

With the Conventions 6.4 in mind, it is straightforward to check that the summands above constitute $\tau^1_\vee$. For $N = k+1$ inner vertices the tree $t[a_1, \dots, a_n]$ we view as a rooting of j trees $T_{A_1}[a_{A_1}], \dots, T_{A_j}[a_{A_j}]$:

$t[a_1, \dots, a_n] =$


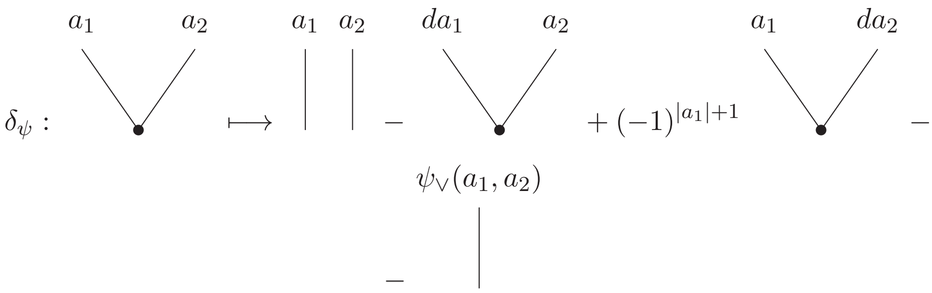
Then the terms that constitute the "rest" can be rewritten using the recursion assumption as:

$$\sum_{i=1}^{j} (-1)^{1+|T_{A_1}[a_{A_1}]|+\cdots+|T_{A_{i-1}}[a_{A_{i-1}}]|}$$


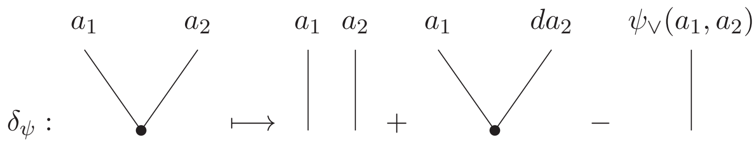
which are identifed then with the expression $\tau^1_\vee = -\rho \circ \mathrm{p}^{\geq 2} \circ \delta_\psi \circ \rho^{-1}(t[a_1, \dots, a_n])$. The statement follows. □

We illustrate the content of Proposition 6.15 by giving explicitly the differential δ_ψ on the following examples:

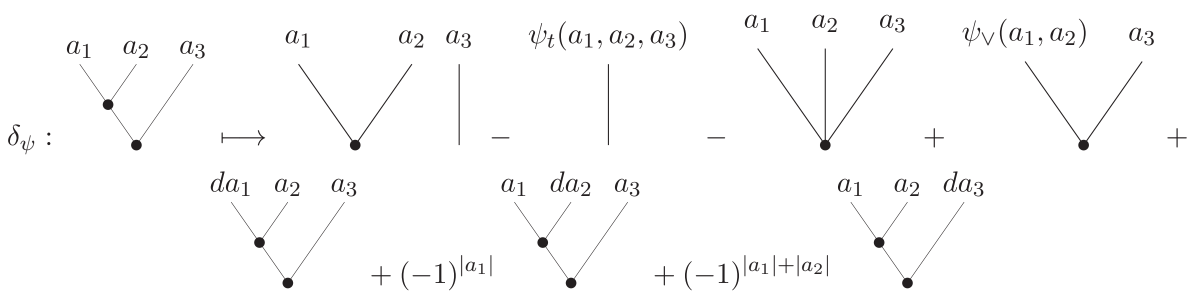
1. For the tree $t = \vee[a_1, a_2]$ with one root and two leaves decorated by elements $a_1, a_2 \in \mathfrak{M}_\bullet$ of degree ≥ 2:



2. For the tree $t = \vee[a_1, a_2]$ with one root and two leaves decorated by elements a_1, a_2 $\mathfrak{M}_\bullet$ of degree 1 and ≥ 2 respectively, we have:



3. For the decorated tree $t[a_1, a_2, a_3]$ with t as below, decorated by the three elements a_1, a_2, a_3 of degree ≥ 2, a direct computation gives:



Compatibility conditions to ensure $\delta_\psi^2 = 0$

Let $(\mathfrak{M}, d)$ be a free $\mathcal{O}$-resolution $(\mathfrak{M}, d)$ of $\mathcal{O}/\mathcal{I}$ as in (14). Let δ_ψ be the derivation of $S(\mathcal{T}ree[\mathfrak{M}_\bullet])$ associated to some arborescent operations $\psi \colon t \mapsto \psi_t$ as in Section 6.2.

Proposition 6.16. *Let $i \geq 1$. Assume that the arborescent operations $t \mapsto \psi_t$ have been chosen such that δ_ψ^2 vanishes on elements of degree less or equal to i. The square of δ_ψ is a derivation of $S(\mathcal{T}ree[\mathfrak{M}_\bullet])$ whose restriction to $\mathcal{T}^{\,\geq 2}[\mathfrak{M}_\bullet]_{i+1}$ takes values in $| \otimes \mathfrak{M}_{i-1}$, and, for any decorated tree $t[a_1, \ldots, a_n]$ of degree i, we have: $\delta_\psi^2(t[a_1, \ldots, a_n]) = | \otimes \spadesuit(t[a_1, \ldots, a_n])$ with*

$$\spadesuit(t[a_1, \ldots, a_n]) = \left(\sum_{A \in InnerVertex(t)} \left(\psi_{t_{\downarrow A}} \circ_A \psi_{t_{\uparrow A}} - \psi_{\partial_A t}\right) - [d, \psi_t] + \pi^1 \circ d \right) (a_1 \otimes \cdots \otimes a_n),$$

where

$$\begin{array}{rcl} \left(\psi_{t_{\downarrow A}} \circ_A \psi_{t_{\uparrow A}}\right) : \otimes^n(\mathfrak{M}) & \to & \mathfrak{M} \\ a_1 \otimes \cdots \otimes a_n & \mapsto & (-1)^{W_A} \psi_{t_{\downarrow A}}(a_1, \ldots, \psi_{t_{\uparrow A}}(a_A), \ldots, a_n), \end{array}$$

$$[d, \psi_t](a_1, \ldots, a_n) \coloneqq d \circ \psi_t(a_1, \ldots, a_n) + \sum_{i=1}^{n} (-1)^{W_i} \psi_t[a_1, \ldots, da_i, \ldots, a_n],$$

$$\psi_{\partial_A t}(a_1, \ldots, a_n) \coloneqq (-1)^{W_A} \psi(\partial_A t[a_1, \ldots, a_n])$$

and where π^1 stands for the projection $\sum_{j=1}^{\infty} \otimes^j \mathfrak{M} \to \mathfrak{M}$, so that the term $\pi^1 \circ d([a_1 \otimes \cdots \otimes a_n])$ is zero unless $n = 2$, and $a_1 \in \mathfrak{M}_1, a_2 \in \mathfrak{M}_{\geq 2}$, or $a_1 \in \mathfrak{M}_{\geq 2}, a_2 \in \mathfrak{M}_1$, or $a_1, a_2 \in \mathfrak{M}_1$, in which cases it is equal to $d(a_1)a_2, (-1)^{|a_1|} d(a_2)a_1$, and $d(a_1)a_2 - a_1 da_2$, respectively.

Proof. Let us use the recursive description of the differential δ_ψ given in Section 6.2, and the decomposition: $\delta_\psi = \tau^{\,\geq 2} + \tau^1_\vee + \tau^1_|$ as in Equation (23). Let us describe their compositions through a direct computation:

(22) $\tau^{\,\geq 2} \circ \tau^{\,\geq 2} = \mathrm{p}^{\,\geq 2} \circ \delta_\psi \circ \rho^{-1}$,

($\vee$2) $\tau^1_\vee \circ \tau^{\,\geq 2} = \mathrm{p}^1_\vee \circ \delta_\psi \circ \rho^{-1}$,

($|$2) $\tau^1_| \circ \tau^{\,\geq 2} = \mathrm{p}^1_| \circ \delta_\psi \circ \rho^{-1}$

and

(2$|$) $\tau^{\,\geq 2} \circ \tau^1_| = 0$,

($\vee|$) $\tau^1_\vee \circ \tau^1_| = 0$,

($||$) $\tau^1_| \circ \tau^1_| = -| \otimes_{\mathbb{K}} (d \circ \psi)$, and

(2∨) $\tau^{\geq 2} \circ \tau^1_\vee = -\mathrm{p}^{\geq 2} \circ \delta_\psi \circ \rho^{-1}$,

(∨∨) $\tau^1_\vee \circ \tau^1_\vee = \rho \circ \mathrm{p}^{\geq 2} \circ \delta^2_\psi \circ \rho^{-1} - \mathrm{p}^1_\vee \circ \delta_\psi \circ \rho^{-1}$,

(|∨) $\tau^1_| \circ \tau^1_\vee = | \otimes_{\mathbb{K}} \left(\psi \circ \rho \circ \mathrm{p}^{\geq 2} \circ \delta_\psi \circ \rho^{-1}\right)$.

The differential δ^2_ψ is the sum of the three maps valued in $S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet])$, $\mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet]$ and $|\otimes \mathfrak{M}$ respectively, obtained by adding up (22) + (2∨) + (2|), (∨2) + (∨∨) + (∨|), and (|2) + (|∨) + (||) respectively. The formula above describes these three sums:

- $\mathrm{p}^{\geq 2} \circ \delta^2_\psi = 0$,
- $\mathrm{p}^1_\vee \circ \delta^2_\psi = \rho \circ \mathrm{p}^{\geq 2} \circ \delta^2_\psi \circ \rho^{-1} = 0$ (Here we use the assumption that δ^2_ψ vanishes on the elements of degree $\leq i$),
- $\mathrm{p}^1_| \circ \delta^2_\psi = \tau^1_| \circ \delta_\psi = \mathrm{p}^1_| \circ \delta_\psi \circ \rho^{-1} + | \otimes_{\mathbb{K}} \left(\psi \circ \rho \circ \mathrm{p}^{\geq 2} \circ \delta_\psi \circ \rho^{-1} - d \circ \psi\right)$.

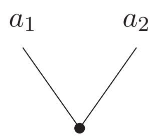
This implies that δ^2_ψ, restricted on $\mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet]_{i+1}$, is valued in $| \otimes \mathfrak{M}_{i-1}$, hence $\delta^2_\psi(t[a_1, \ldots, a_n])$ is of the form $| \otimes \spadesuit(t[a_1, \ldots, a_n])$ where

$$\spadesuit = \mathrm{p}^1_| \circ \delta_\psi \circ \rho^{-1} + | \otimes_{\mathbb{K}} \left(\psi \circ \rho \circ \mathrm{p}^{\geq 2} \circ \delta_\psi \circ \rho^{-1} - d \circ \psi\right).$$

For $n \geq 3$, with the help of the formula giving the differential δ_ψ in Proposition 6.15, we identify the summand $| \otimes_{\mathbb{K}} \left(\psi \circ \rho \circ \mathrm{p}^{\geq 2} \circ \delta_\psi \circ \rho^{-1} - d \circ \psi\right)$ as:

$$| \otimes \left(\sum_{A \in InnerVertex(t)} \left(\psi_{t_{\downarrow A}} \circ_A \psi_{t_{\uparrow A}} - \psi_{\partial_A t}\right) - [d, \psi_t] \right). \tag{26}$$

For $n = 2$, an exceptional term appears, for $\mathrm{p}^1_| \circ \delta_\psi \circ \rho^{-1}$ is non-zero while being evaluated on trees of the form

$\vee[a_1, a_2] =$ 

with $|a_1|$ or $|a_2|$ being equal to 1. This allows us to identify $\mathrm{p}^1_| \circ \delta_\psi \circ \rho^{-1}(t[a_1, \ldots, a_n])$ with $| \otimes \pi^1 \circ d(a_1 \otimes \cdots \otimes a_n)$. This concludes the proof of the proposition.

□

Corollary 6.17. *Assume that the arborescent operations have been chosen such that δ^2_ψ vanishes on elements of degree less or equal to i, then for any element $t[a_1, \ldots, a_n] \in \mathcal{T}ree[\mathfrak{M}_\bullet]_{i+1}$, we have $\delta^2_\psi(t[a_1, \ldots, a_n]) = | \otimes \spadesuit$ with $\spadesuit$ being an element in $\mathfrak{M}_{i-1}$ that satisfies $d(\spadesuit) = 0$.*

Proof. The recursion assumption implies $\delta^2_\psi \circ \delta_\psi(t[a_1, \ldots, a_n]) = 0$, so that $\delta_\psi \circ \delta^2_\psi(t[a_1, \ldots, a_n]) = \delta_\psi(| \otimes \spadesuit) = | \otimes d(\spadesuit) = 0$. This proves the result. □

Corollary 6.18. *There exists a choice of arborescent operations $t \mapsto \psi_t$ such that $\delta^2_\psi = 0$.*

Proof. The proof consists in constructing the recursion operation ψ degree by degree. More precisely, we intend to define $\psi_t[a_1 \ldots, a_n]$ for every decorated tree $t[a_1, \ldots, a_n]$ of degree $i + 1$ provided it is already defined on all decorated trees of degree less or equal to i. If the relation $\delta^2_\psi = 0$ holds on decorated trees of degree $\leq i$, Proposition 6.16 states that the restriction to elements of degree $i+1$ of δ^2_ψ is of the form $| \otimes \spadesuit$ for some $\mathcal{O}$-linear map $\spadesuit$ from $\mathcal{T}ree[\mathfrak{M}]_{i+1} \rightarrow$

$\mathfrak{M}_{i-1}$. Corollary 6.17 states that ♠ is valued into d-cocycles of $\mathfrak{M}_{i-1}$. By exactness of the complex $(\mathfrak{M}_\bullet, d)$, the $\mathcal{O}$-linear map ♠ is therefore valued into d-coboundaries. Since the $\mathcal{O}$-module $(\mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet])_{i+1}$ is free (hence projective) by Proposition 6.3, there exists a $\mathcal{O}$-linear map

$$\psi\colon (\mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet])_{i+1} \to \mathfrak{M}_i$$

which gives back ♠ when composed with $d\colon \mathfrak{M}_i \to \mathfrak{M}_{i-1}$. The corresponding arborescent operations $\psi : t \mapsto \psi_t$ now induce a differential δ_ψ that satisfies $\delta_\psi^2 = 0$ on all elements of degree $i+1$. The construction continues by recursion. □

6.3 Arborescent Koszul-Tate resolutions

We are now ready to reply to Question 5.3. The next Theorem uses free $\mathcal{O}$-resolutions, but extends trivially to the case where $\mathfrak{M}_i$ for all i are the sheaves of sections of a vector bundle, and $\mathcal{O}$ is a sheaf of smooth of holomorphic functions as in Remark 6.10.

Theorem 6.19. *Let $\mathcal{I}$ be an ideal in a commutative algebra $\mathcal{O}$. For any free $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$*

$$\dots \longrightarrow \mathfrak{M}_k \xrightarrow{d} \dots \xrightarrow{d} \mathfrak{M}_1 \xrightarrow{d} \mathcal{O} \longrightarrow \mathcal{O}/\mathcal{I}$$

there exists arborescent operations $\psi\colon t \mapsto \psi_t$ such that symmetric algebra in the free $\mathcal{O}$-module $\mathcal{T}[\mathfrak{M}_\bullet]$ of decorated trees is a Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$ when equipped with the differential δ_ψ described by Proposition 6.15.

This theorem allows to give the following definition.

Definition 6.20. *For $(\mathfrak{M}_\bullet, d)$ a free $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$, we call* arborescent Koszul-Tate $\mathcal{O}$-resolution *of $\mathcal{O}/\mathcal{I}$ attached to the arborescent operations $t \mapsto \psi_t$ any Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$ as in Theorem 6.19.*

Proof of Theorem 6.19. In view of Corollary 6.18, we know that we can choose arborescent operations $\psi\colon t \mapsto \psi_t$ such that $\delta_\psi^2 = 0$. In order to prove Theorem 6.19, it suffices to check that for any such a choice, the homology of δ_ψ is zero in every positive degree and is $\mathcal{O}/\mathcal{I}$ in degree 0.

Since $S(\mathcal{T}ree[\mathfrak{M}_\bullet])_1 = | \otimes \mathfrak{M}_1$, since $\delta_\psi(| \otimes a) = da$ for every $a \in \mathfrak{M}_1$, and since the image of d is $\mathcal{I}$, the degree 0 homology of δ_ψ is $\mathcal{O}/\mathcal{I}$.

We now prove that any closed element of degree $n \geq 1$ is exact. Let $a \in S^\bullet(\mathcal{T}ree[\mathfrak{M}_\bullet])_n$ be a δ_ψ-closed element of degree n. Let us decompose $a \in S^\bullet(\mathcal{T}ree[\mathfrak{M}_\bullet])$ as a sum of two terms:

$$a = a^{\geq 2} + a^1 \tag{27}$$

with $a^{\geq 2} \in S^{\geq 2}(\mathcal{T}ree[\mathfrak{M}_\bullet])$ and $a^1 \in S^1(\mathcal{T}ree[\mathfrak{M}_\bullet]) \simeq \mathcal{T}ree[\mathfrak{M}_\bullet]$. Consider $a - \delta_\psi \circ \rho(a^{\geq 2})$.

Applying δ_ψ to $\rho(a_{\geq 2})$, we get two components:

1. The first component is in $S^{\geq 2}(\mathcal{T}ree[\mathfrak{M}_\bullet])$ and is by definition $\tau^{\geq 2}(\rho(a_{\geq 2})) = a_{\geq 2}$.
2. The remaining components are obtained by applying $\tau^1_\vee(\rho(a_{\geq 2}))$ and $\tau^1_|(\rho(a_{\geq 2}))$. Those are valued in $S^1(\mathcal{T}ree[\mathfrak{M}_\bullet]) \simeq \mathcal{T}ree[\mathfrak{M}_\bullet]$.

As a consequence, by adding the δ_ψ-coboundary $-\delta_\psi \circ \rho(a^{\geq 2})$ to a, we obtain an element $a' \in S^1(\mathcal{T}ree[\mathfrak{M}_\bullet]) \simeq \mathcal{T}ree[\mathfrak{M}_\bullet]$. By construction, a' is δ_ψ-closed. Since a' is δ_ψ-closed, it belongs to the kernel of τ_u, which is $|\otimes \mathfrak{M}_n$. It is therefore of the form $| \otimes b$ for some $b \in \mathfrak{M}_n$. Since $\delta_\psi(| \otimes b) = | \otimes db$, this element b has to be d-closed. By exactness of $\mathfrak{M}_\bullet$, $b = db'$ for some $b' \in \mathfrak{M}_{n+1}$. Since

$$\delta_\psi(| \otimes b') = | \otimes b = a',$$

this proves that

$$a = \delta_\psi(\rho(a_{\geq 2}) + | \otimes b')$$

is exact, so that the homology of δ_ψ is zero. □

Remark 6.21. Again, all results of this section extend to the context of sheaves as in Remark 6.10.

6.4 $\mathfrak{M}$ as a homotopy retract of its arborescent Koszul-Tate resolution

The homotopy retract

Since $(S(\mathcal{T}ree[\mathfrak{M}_\bullet]), \delta)$ is also a free $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ by Proposition 6.3, it has to be homotopy equivalent (over $\mathcal{O}$) to $(\mathfrak{M}, d)$ as an $\mathcal{O}$-module. In this section, we describe a completely explicit expression for this $\mathcal{O}$-linear homotopy equivalence and show that it is even a homotopy retract.

Remark 6.22. Notice that even if the free or projective $\mathcal{O}$-resolution $(\mathfrak{M}_\bullet, d)$ has finite length N, the $\mathcal{O}$-module $\mathcal{T}ree[\mathfrak{M}_\bullet]$ that generate the arborescent Koszul-Tate $\mathcal{O}$-resolution always has infinitely many generators.

Consider an arborescent Koszul-Tate $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ associated to a free or projective $\mathcal{O}$-resolution $(\mathfrak{M}_\bullet, d)$ with arborescent operations ψ and differential δ_ψ. The $\mathcal{O}$-linear morphism:

$$\begin{array}{rcl} \mathfrak{M} & \to & S(\mathcal{T}ree[\mathfrak{M}]) \\ a & \mapsto & \mid[a] \\ F & \mapsto & F, \quad F \in \mathcal{O} \end{array}$$

is a chain map denoted by Incl.

To find its homotopy inverse, we define a $\mathcal{O}$-linear morphism:

$$\mathrm{Proj}_\psi \colon S(\mathcal{T}ree[\mathfrak{M}_\bullet]) \to \mathfrak{M}$$

by mapping

1. F to F, for all $F \in \mathcal{O}$,
2. $\mid[a]$ to a for all $a \in \mathfrak{M}$,
3. $t(a_1, \ldots, a_n)$ to 0 if $n \geq 2$, i.e. if t is not a trivial tree,
4. an element $S \in S^{\geq 2}(\mathcal{T}ree[\mathfrak{M}_\bullet])$ is mapped to $\tau_{\mid}^1 \circ \rho(S)$ (the image of $\tau_{\mid}^1$ being $\mid[\mathfrak{M}] \simeq \mathfrak{M}$).

   Said differently, this last operation is as follows: for all $i \geq 2$, Proj_ψ maps

   $$t_1[a_1^1, \ldots, a_{n_1}^1] \odot \cdots \odot t_i[a_1^i, \ldots, a_{n_i}^i]$$

   to

   $$\psi_{\rho(t_1,\ldots,t_i)}(a_1^1, \ldots, a_{n_1}^1, \ldots, a_1^i, \ldots, a_{n_i}^i)$$

   where $\rho(t_1, \ldots, t_i)$ is obtained by rooting the trees $t_1, \ldots, t_i$ altogether.

Proposition 6.23. *Consider an arborescent Koszul-Tate $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ associated to a free or projective $\mathcal{O}$-resolution $(\mathfrak{M}_\bullet, d)$ with arborescent operations ψ and differential δ_ψ. The $\mathcal{O}$-linear maps*

$$S(\mathcal{T}ree[\mathfrak{M}]) \underset{\mathrm{Incl}}{\overset{\mathrm{Proj}_\psi}{\rightleftarrows}} \mathfrak{M}$$

are homotopy inverse one to the other. More precisely

$$\begin{cases} \mathrm{Proj}_\psi \circ \mathrm{Incl} &= \mathrm{id} \\ \mathrm{Incl} \circ \mathrm{Proj}_\psi &= \mathrm{id} - (h \circ \delta_\psi + \delta_\psi \circ h) \end{cases}$$

where h is given by the sequence of $\mathcal{O}$-linear maps:

$$h := \rho \circ \mathrm{p}^{\geq 2}.$$

It is, moreover, a deformation retract which satisfies the side relations [EM53]:

$$h^2 = 0\,,\ h \circ \mathrm{Incl} = 0\,,\ \mathrm{Proj}_\psi \circ h = 0. \tag{28}$$

Proof. A direct computation gives the three relations listed in Equation (28). The relation $\mathrm{Proj}_\psi \circ \mathrm{Incl} = \mathrm{id}$ is also an obvious consequence of the definition of these maps. We are left with the task of proving:

$$\mathrm{Incl} \circ \mathrm{Proj}_\psi = \mathrm{id} - (h \circ \delta_\psi + \delta_\psi \circ h)$$

Recall that $\mathrm{p}^{\geq 2}$, $\mathrm{p}^1_\vee$, and $\mathrm{p}^1_|$ stand for the projections on $S(\mathcal{T}ree[\mathfrak{M}_\bullet]) = S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet]) \oplus \mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet] \oplus |\,[\mathfrak{M}_\bullet]$ respectively as in Equation (22). An element $a \in S(\mathcal{T}ree[\mathfrak{M}_\bullet])$ now reads to a triple $\left(a^{\geq 2},\, a_\vee,\, a_|\right)$ with $a^{\geq 2}, a_\vee, a_|$ the images of a through these three projections. To prove Proposition 6.23, it suffices to prove the following relation:

$$(\delta_\psi \circ h + h \circ \delta_\psi)\left(a^{\geq 2},\, a_\vee,\, a_|\right) = \left(a^{\geq 2},\, a_\vee,\, \mathrm{Proj}_\psi(a^{\geq 2})\right). \tag{29}$$

Indeed, if we prove Equation (29), we obtain as a consequence that $\mathrm{Incl} \circ \mathrm{Proj}_\psi$ is as in (6.23), and we also obtain that Proj_ψ is a chain map.

Let us prove (29) for each one of the components.

1. For $a^{\geq 2} \in S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet])$, the explicit description of the differential δ_ψ (see Section 6.2) establishes the following equality:

$$\begin{aligned}
&(h \circ \delta_\psi + \delta_\psi \circ h)\,(a^{\geq 2},\, 0,\, 0) && \\
&= (\rho \circ \mathrm{p}^{\geq 2} \circ \delta_\psi + \rho \circ \tau^{\geq 2} + \tau^1_\vee \circ \rho + \tau^1_| \circ \rho)\,(a^{\geq 2},\, 0,\, 0) && \text{by def. of } \delta_\psi \text{ and } h \\
&= (\rho \circ \mathrm{p}^{\geq 2} \circ \delta_\psi + id + \tau^{\geq 2} \circ \rho + \tau^1_| \circ \rho)\,(a^{\geq 2},\, 0,\, 0) && \text{since } \tau^{\geq 2} = \rho^{-1} \\
&= \Big(a^{\geq 2},\, (\rho \circ \mathrm{p}^{\geq 2} \circ \delta_\psi + \tau^1_\vee \circ \rho)(a^{\geq 2}),\, \tau^1_| \circ \rho(a^{\geq 2})\Big) && \text{by def. of } \rho, \tau^1_| \text{ and } \tau^1_\vee \\
&\Big(a^{\geq 2},\, 0,\, \tau^1_| \circ \rho(a^{\geq 2})\Big) && \text{by def. of } \tau^1_\vee \\
&\big(a^{\geq 2},\, 0,\, \mathrm{Proj}_\psi(a^{\geq 2})\big) && \text{since } \mathrm{Proj}_\psi = \tau^1_| \circ \rho \text{ on } S^{\geq 2}(\mathcal{T}[\mathfrak{M}_\bullet])
\end{aligned}$$

2. For any $a_\vee \in \mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet]$ the only non-zero contribution of $(\delta_\psi \circ h + h \circ \delta_\psi)$ comes from $h \circ \delta_\psi$ which consists of un-rooting a_r and then re-rooting it again, so this operation is an identity on $\mathcal{T}^{\geq 2}[\mathfrak{M}_\bullet]$.

3. The $a_|$ component is clearly in the kernel of both $\delta_\psi \circ h$ and $h \circ \delta_\psi$.

This completes the proof. □

Transferred A_∞-algebra structure on $\mathfrak{M}$.

Consider an $\mathcal{O}$-module resolution $(\mathfrak{M}, d)$ of a quotient algebra $\mathcal{O}/\mathcal{I}$ as in (13). Since *(i)* any two resolutions of the same $\mathcal{O}$-module are homotopy equivalent over $\mathcal{O}$, *(ii)* any Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$ is a graded commutative algebra, and therefore a (very particular) A_∞-algebra, and *(iii)* since A_∞-algebra structures can be transferred through homotopy equivalence [LV12], it is no surprise that $(\mathfrak{M}, d)$ comes equipped with an A_∞-structure. We show in this section that such A_∞-structure can be described through the arborescent operations $t \mapsto \psi_t$ that appeared when constructing the differential δ_ψ of the arborescent Koszul-Tate $\mathcal{O}$-resolution. We denote by Y_n the set of planar binary trees.

Proposition 6.24. *Consider an $\mathcal{O}$-module resolution $(\mathfrak{M}_\bullet, d)$ of $\mathcal{O}/\mathcal{I}$ as in (13). Let $t \mapsto \psi_t$ be the arborescent operations of one of its arborescent Koszul-Tate $\mathcal{O}$-resolutions.*

The bilinear map $\mu_2 \colon \otimes_{\mathcal{O}} (\mathcal{O} \oplus \mathfrak{M}) \to \mathcal{O} \oplus \mathfrak{M}$ defined for all $F, G \in \mathcal{O}, a, b \in \mathfrak{M}$ by

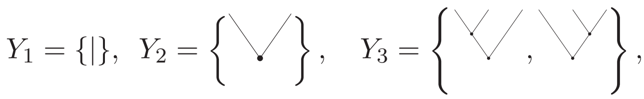
$$\mu_2(F + a, G + b) = FG + Fb + Ga + \psi_\vee(a, b)$$

is the binary product of an A_∞-algebra structure on $(\mathcal{O} \oplus \mathfrak{M}_\bullet, -d)$ whose higher products are given by the sequence of $\mathcal{O}$-linear operations given for all $n \geq 3$ by

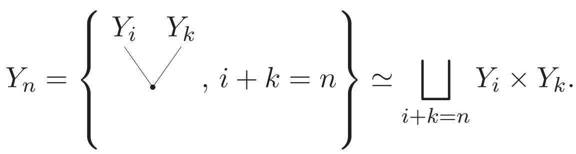
$$\begin{array}{rccc} \mu_n \colon & \otimes_{\mathcal{O}} (\mathcal{O} \oplus \mathfrak{M}_\bullet) & \to & \mathcal{O} \oplus \mathfrak{M}_\bullet \\ & a_1, \ldots, a_n & \mapsto & \sum_{t \in Y_n} (-1)^{P(t[a_1,\ldots,a_n]) + \sum_{i=1}^n (n-i)|a_i|} \psi_t(a_1, \ldots, a_n). \end{array} \tag{30}$$

(The value of $P(t[a_1, \ldots, a_n]) \in \mathbb{N}_0$ will be defined in the proof).

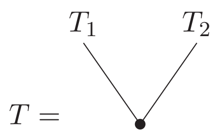
Proposition 6.24 requires some considerations about binary trees (we consider the trivial tree as a binary tree). Let $Y = \bigcup_{k \geq 1} Y_k$ with Y_K the set of planar binary trees with k leaves (see e.g. [LR02]):



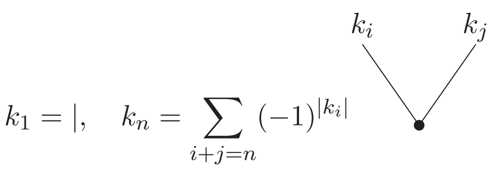
The following recursive relation holds



Any planar binary trees with n leaves can be expressed in an unique way as a rooting of two uniquely defined binary trees:



On the $\mathbb{K}$-vector space freely generated by planar binary trees, we define a family $K := \{k_1, \ldots k_n, \ldots\}$ as follows:



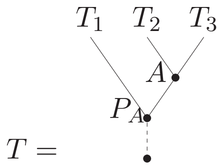
Here $| \ldots |$ stands for the degree of trees defined in Section 6.1. An immediate recursion shows that each k_i is a linear combination of trees having the same degree, so that $|k_i|$ is well-defined. From the definition of K and Y_n it follows that k_n is a sum of all binary trees with n leaves and a prefactor ± 1.

Lemma 6.25. *Each element of K lies in the $Ker(\partial)$, i.e. for all $n \geq 1$:*

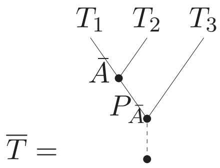
$$\partial k_n = 0.$$

where δ is the differential of $Tree$ defined in Section 6.2.

Proof. To any pair (T, A), with $T \in Y_{\geq 3}$ and A an inner vertex of T corresponds an unique dual pair $(\overline{T}, \bar{A})$ as follows: Without any loss of generality we can assume the location of its parent vertex P_A of A is on the left to A as in the picture:



with T_1, T_2, T_3 being binary trees. Then $(\overline{T}, \bar{A})$ is



with the rest of the trees $\overline{T}$ and T being identical.

Now, recall that for any tree t

$$\partial t = \sum_{A \in InnerVertex(t)} (-1)^{W_A} \partial_A t$$

so that, for all $n \geq 1$, ∂k_n is a sum over pairs (T, A), with $T \in Y_n$ and A an inner vertex of T:

$$\sum_{(T,A)} (-1)^{|W_A| + P(T)} \partial_A(T)$$

where $P(T)$ is a sign of a tree T entering k_n. We claim that the terms corresponding to (T, A) and $(\overline{T}, \bar{A})$ cross out. To start with, it is obvious that they give the same tree, which is a binary except for one inner vertex which has three descendants, and is obtained by erasing the edges from A and $\bar{A}$ to their parents. We therefore just have to check that their signs are opposite. In the expression of k_n, the trees T and $\overline{T}$ appear with the following relative signs:

$$(-1)^{|T_1| + |T_2|} T \text{ and } (-1)^{2|T_1| + |T_2| + 1} \overline{T}.$$

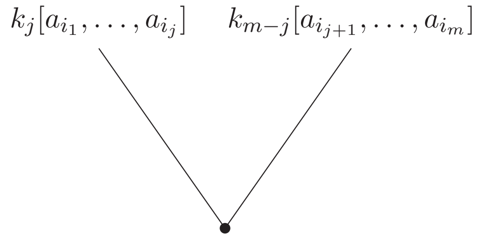
Also, the relative signs of the weights W_A and $W_{\bar{A}}$ are respectively given by $|T_1|$ and 0. As a consequence, these two terms cancel each other when applying ∂_A and $\partial_{\bar{A}}$ to these trees. Summing over all pairs (T, A), we get the statement of the Lemma. □

The structure of the proof of Proposition 6.24 is then as follows:

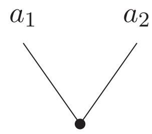
1. We introduce $k_n[a_1, \ldots, a_n]$ - an analog to k_n for decorated planar binary trees.

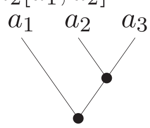
2. We show that the $\mathcal{O}$-linear maps $\mu_n : a_1 \otimes \cdots \otimes a_n \mapsto (-1)^{\sum_{i=1}^{n} |a_i|(n-i)} \psi(k_n[a_1, \ldots, a_n])$ satisfy the relations defining an A_∞-algebra

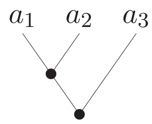
For a given family $\mathcal{D} := (a_1, \dots, a_n)$ of elements in $\mathfrak{M}$, we define a family of *ordered* decorated binary trees by $K_{\mathcal{D}} := \{k_j[a_{i_1}, \dots, a_{i_j}],\ j \in [\![1, n]\!], \ , 1 \le i_1 < \cdots < i_j \le n\}$. The following relation is satisfied for all $m \le n$

$$k_m[a_{i_1}, \dots a_{i_m}] = \sum_{j=1}^{m-1} (-1)^{|k_j[a_{i_1},\dots,a_{i_j}]|}$$ 

with $1 \le i_1 < i_2 < \cdots < i_m \le n$

For instance, $k_1[a_i] = | \otimes a_i$ and $k_2[a_1, a_2] = (-1)^{|a_1|}$ 

$k_3[a_1, a_2, a_3] = (-1)^{|a_1|+|a_2|}$ 

$+ (-1)^{2|a_1|+|a_2|+1}$ 

Lemma 6.26. *$K_{\mathcal{D}} \subset Ker(\partial)$. In particular, $\partial(k_n[a_1, \dots, a_n]) = 0$ for all $n \ge 1$ and all $a_1, \dots, a_n \in \mathfrak{M}$.*

Proof. The proof is analogous to the one for the family $K = \{k_1, \dots, k_n, \dots\}$ in Lemma 6.25 and is left to the reader. □

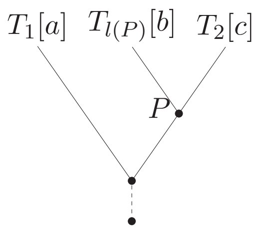
Lemma 6.27. *$k_n[a_1, \dots, a_n]$ is a linear combination of all binary trees decorated by $a_1, \dots, a_n$, with coefficients equal to -1 or $+1$.*

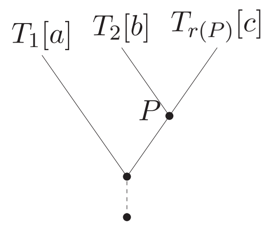
Proof. An obvious recursion on n shows that for any $\mathcal{D} = \{a_1, \dots, a_n\}$ with n elements there is a one-to-one correspondence between monomial summands of $K_{\mathcal{D}}$ with n-leaves and Y_n. □

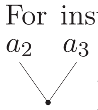
Let us determine the sign prefactor $P(t[a_1, \dots, a_n]) \in \pm 1$ of any monomial summand as in Lemma 6.27, that is:

$$k_n[a] = \sum_{T \in Y_n} (-1)^{P(T[a])} T[a]$$

(with a being from now a shorthand for $a_1, \dots, a_n$). We introduce a set of left and right subtrees, $\mathcal{C}_l(T[a])$ and $\mathcal{C}_r(T[a])$ respectively. A set of all subtrees $\mathcal{C}(T[a])$ of a decorated tree $T[a]$ is defined in a obvious manner. Then the set $\mathcal{C}_{l(r)}(T[a])$ comprises of all subtrees $T_{l(P)}[b]$ ($T_{r(P)}[b]$ respectively) located to the left(right) of the parent vertex P, for all vertices P:

the left subtree of a vertex P: 

, the right subtree of a vertex P: 

For instance, for a first summand in $k_3[a_1, a_2, a_3]$, $\mathcal{C}_l = \{| \otimes a_1, | \otimes a_2\}$, and $\mathcal{C}_r = \{| \otimes a_3,$ 

$\}$.

This notion allows us to establish the following lemma by an immediate recursion.

Lemma 6.28. $P(T[a]) = \sum_{T[b]\in\mathcal{C}_l(T[a])} |T[b]|$.

Proof of Proposition 6.24. We have to check that the collection of $\mathcal{O}$-linear maps $\mu_1 = -d$ and μ_n as in Proposition 6.24 satisfy the axioms of A_∞-algebras, i.e. the sequence of properties:

$$\sum_{i+j+k=n} (-1)^{i+jk} \mu_{n-j+1}(\mathrm{id}^{\otimes i} \otimes \mu_j \otimes \mathrm{id}^{\otimes k}) = 0. \tag{31}$$

For $n = 1$, this simply means that $d^2 = 0$. For $n = 2$, the relation can be checked directly, and means that μ_2 is a chain map.

Let us assume $n \geq 3$ and that the axioms are satisfied up to order $n-1$. Let us check it at order n. First let us verify the case where $a_l = F \in \mathcal{O}$ for some index l. Convention 6.4 implies that $\mu_k(a_1, \dots, a_n) = 0$ for $k \geq 3$ while $\mu_2(F, a_{l+1}) = Fa_{l+1}$ and $\mu_2(a_{l-1}, F) = Fa_{l-1}$ the A_∞-axioms are easily verified in that case. The non-zero summands entering the A_∞ relations have the following form:

$$(-1)^{l-1}\mu_{n-1}(a_1, \dots, a_{l-1}, \mu_2(F, a_{l+1}), \dots, a_n) + (-1)^{l-2}\mu_{n-1}(a_1, \dots, a_{l-2}, \mu_2(a_{l-1}, F), \dots, a_n)$$

whenever $l \neq 1, n$. For $l = 1$ the non-zero summands are:

$$\mu_{n-1}(\mu_2(F, a_2), a_3, \dots, a_n) - \mu_2(F, \mu_{n-1}(a_2, \dots, a_n))$$

and for $l = n$:

$$(-1)^{n-2}\mu_{n-1}(a_1, \dots, \mu_2(a_{n-1}, F)) + (-1)^{n-1}\mu_2(\mu_{n-1}(a_1, \dots, a_{n-1}), F)$$

Since $\mu_2(a, F) = aF$ and $\mu_\bullet$ is $\mathcal{O}$-linear, the pairs listed above cross each other out. The results holds in this case.

Recall from Proposition 6.16 that the relation $\delta_\psi^2 = 0$ implies that for any tree t and any $a = (a_1, \dots, a_n) \in \mathfrak{M}$, $n \geq 3$ we have:

$$\left(\sum_{A \in InnerVertex(T)} \psi_{t_{\downarrow A}} \circ_A \psi_{t_{\uparrow A}} - \psi_{\partial_A t} - [d, \psi_t] \right)(a_1, \dots, a_n) = 0 \tag{32}$$

Adding up all previous expressions over all summands of $k_n[a_1, \dots, a_n]$ yields

$$\sum_{t \in Y_n} (-1^{P(t[a])} \left(\sum_{A \in InnerVertex(T)} \psi_{t_{\downarrow A}} \circ_A \psi_{t_{\uparrow A}} - \psi_{\partial_A t} - [d, \psi_t] \right)(a_1, \dots, a_n) = 0.$$

The terms containing ∂_A cross out due to Lemma 6.26. Recall that by definition (see Section 6.2), $\psi_{t_{\downarrow A}} \circ_A \psi_{t_{\uparrow A}}(a_1, \dots, a_n) = (-1)^{W_A}\psi_{t_{\downarrow A}}(a_1, \dots, \psi_{t_{\uparrow A}}(a_A), \dots, a_n)$.As a consequence:

$$\sum_{T \in Y_n} \sum_{A \in InnerVertex(T)} (-1)^{P(t[a]) + W_A}\psi_{T\downarrow A}(a_1, \dots, \psi_{T\uparrow A}(a_A), \dots, a_n) -$$
$$(-1)^{P(t[a])} d\psi_T(a_1, \dots, a_n) - (-1)^{P(t[a]) + W_A}\psi_T(a_1, \dots, da_A, \dots, a_n) = 0$$

For further convenience we enumerate vertices A by pairs $(i+1, i+j)$ where $i+1$ and $i+j$ are the positions of first and last leaves of $T_{\uparrow A}$. If there is no vertex $(i+1, i+j)$ we simply set $T_{\uparrow A}[a_A]$ to zero. We employ the following identities that follow easily from its definitions ($a_{\uparrow(i+1,i+j)}$ being the set of leaves that descent from the vertex $i+1, i+j$):

$$W_{(i+1,i+j)} = i + |a_1| + \dots + |a_i| + b_r(i+1, i+j)$$
$$P(T[a]) = P(T_{\downarrow(i+1,i+j)}[a_{\downarrow(i+1,i+j)}]) + P(T_{\uparrow(i+1,i+j)}[a_{\uparrow(i+1,i+j)}]) + b_r(i+1, i+j)|T_{\uparrow(i+1,i+j)}[a_{\uparrow(i+1,i+j)}]|$$

Here $b_r(i+1, i+j)$ is the number of subtrees of $T[a]$ attached to the right of the path to $(i+1, i+j)$. Recombining terms in the above expressions, we obtain

$$\Big(\sum_{i+j+k=n}(-1)^{i+|a_1|+\ldots|a_i|}\psi(k_{n+1-j}[a_1,\ldots,a_i,\psi(k_j[a_{i+1},\ldots,a_{i+j}]),a_{i+j+1},\ldots,a_{i+j+k}])\Big)$$
$$-d\psi(k_n[a_1,\ldots,a_n]) - \sum_{i=0}^{n-1}(-1)^{i+|a_1|+\ldots|a_i|}\psi(k_n[a_1,\ldots,da_{i+1},\ldots,a_n]) = 0$$

Above, it must be understood that $\psi(k_n[a_1,\ldots,a_n]) = \sum_{T\in Y_n}(-1)^{P(T[a])}\psi_T[a_1,\ldots,a_n]$. Implementing a change of variables:

$$\psi(k_j[b_1,\ldots,b_j]) \mapsto (-1)^{\sum_{i=1}^{j}|b_i|(j-i)}\mu_j(b_1,\ldots,b_j), \quad -d(b_1)\mapsto \mu_1(b_1)$$

we obtain

$$\sum_{i+j+k=n}(-1)^{i+jk+(j-2)(|a_1|+\cdots+|a_i|)}\mu_{n-j+1}(a_1,\ldots,a_i,\mu_j(a_{i+1},\ldots,a_{i+j}),a_{i+j+1},\ldots,a_{i+j+k}) = 0$$

Or, using Koszul sign convention

$$\sum_{i+j+k=n}(-1)^{i+jk}\mu_{n-j+1}(\mathrm{id}^{\otimes i}\otimes\mu_j\otimes\mathrm{id}^{\otimes k}) = 0.$$

This completes the proof. □

Alternative proof of Proposition 6.24. There exists an alternative proof using explicit Homotopy Transfer Theorem, written first as a particular summation over planar binary trees in [KS00], see also [Mar04] or [LV12]. This proof is shorter, but it uses statements which lack precision about signs.

In general, for

$$(S,d)\underset{\mathrm{i}}{\overset{\mathrm{pr}}{\rightleftarrows}}(\mathfrak{M},d)$$

a homotopy retract, with $(S,\star,d_S)$ a graded associative algebra, a A_∞-algebra structure on $(\mathfrak{M}_\bullet,d)$ is given by the following formulas:

$$\begin{array}{rccc}\mu_n: & \otimes_{\mathcal{O}}\mathfrak{M}_\bullet & \to & \mathfrak{M}_\bullet\\ & a_1,\ldots,a_n & \mapsto & \sum_{t\in Y_n}\pm\mathrm{pr}\circ\chi_t(i(a_1),\ldots,i(a_n))\end{array}$$

where χ_t is described by the following procedure:

1. Any inner edge corresponds to applying h,
2. To any vertex, we apply the product.

For instance, the tree $\vee$ corresponds to the map $(a,b,c)\mapsto \mathrm{pr}(h(i(a)\star i(b))\star i(c))$.

Proposition 6.23 implies that we can apply these methods here, with S the symmetric algebra , $pr = Proj_\psi$, i the natural inclusion, and $h=\rho$. It follows from the explicit descriptions of the homotopy ρ and projection Proj_ψ that

$$\mathrm{Proj}_\psi\circ\chi_t(i(a_1),\ldots,i(a_n)) = \psi_t(a_1,\ldots,a_r).$$

For instance, for the tree $\vee$ above, for all $a,b,c\in\mathfrak{M}$:

$$\chi_{\vee}(i(a),i(b),i(c)) = {}_{\vee}[a,b]\odot |[c]$$

so that applying Proj_ψ we obtain $\psi_{\vee}[a,b,c]$. Up to signs, the products $(\mu_n)_{n\geq 2}$ obtained by this method match the ones of Proposition 6.24. However, signs conventions are unclear in the referenced items. □

Proposition 6.24 makes sense of the following definition.

Definition 6.29. *Let $(\mathfrak{M}_\bullet, d)$ be a projective $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$, and $\psi\colon t \mapsto \psi_t$ a choice of arborescent operations such that $\delta_\psi^2 = 0$. We call* arborescent A_∞-algebra structure *the A_∞-algebra brackets on $\mathcal{O} \oplus \mathfrak{M}$ described in Proposition 6.24.*

Example 6.30. In particular, $a_1 \star a_2 := \mu_2(a_1, a_2)$ satisfies $a_2 \star a_1 = (-1)^{|a_1||a_2|} a_2 \star a_1$ and

$$(a_1 \star a_2) \star a_3 - a_1 \star (a_2 \star a_3) = d\mu_3(a_1, a_2, a_3) + \mu_3(d(a_1), a_2, a_3) + \\ (-1)^{|a_1|}\mu_3(a_1, d(a_2), a_3) + (-1)^{|a_1|+|a_2|}\mu_3(a_1, a_2, d(a_3))$$

where

$$\mu_3 = -\psi_{\vee}(a_1, a_2, a_3) + (-1)^{|a_1|}\psi_{\vee}(a_1, a_2, a_3).$$

Last, we state an immediate consequence of the second proof of Proposition 6.24:

Corollary 6.31. *The arborescent A_∞-algebra structure and the arborescent Koszul-Tate resolution are homotopy equivalent.*

Proof. The arborescent A_∞-algebra structure is obtained by a homotopy transfer procedure. The outcome of such a transfer is homotopy equivalent to the graded commutative differential algebra one started from, see [LV12]. □

7 Examples of arborescent resolutions

The most famous example of a Koszul-Tate resolution is the Koszul resolution of an ideal generated by a regular sequence. It belongs to the class of $\mathcal{O}$-module resolutions that happen to come with a natural graded commutative associative algebra structure, which we study at the end of this section, but it is not an arborescent resolution, as we will see.

We start with some more elementary examples of arborescent resolutions.

7.1 Angular momentum and non-complete intersection examples

Let us first illustrate the construction of the arborescent Koszul-Tate resolution on four seemingly different examples:

(I) "*Angular Momentum*" $\mathcal{O}$ is the polynomial algebra in 6 variables $x_1, x_2, x_3, p^1, p^2, p^3$, and the ideal $\mathcal{I} \subset \mathcal{O}$ is generated by the three "angular momenta"

$$L_1 = x_2p^3 - x_3p^2\,,\; L_2 = x_3p^1 - x_1p^3\,,\; L_3 = x_1p^2 - x_2p^1.$$

The physical meaning of this example is quite clear. The orthogonal group acts on the cotangent space of $\mathbb{R}^3$. This action is Hamiltonian, and the three components of its momentum mapping $\mu : T^*\mathbb{R}^3 \longrightarrow \mathbb{R}^6 \longleftarrow \mathrm{so}_3^* \simeq \mathbb{R}^3$. The Koszul-Tate resolution is therefore associated to the singular subset $\mu^{-1}(0)$.

(II) "*Quadratic Functions*" $\mathcal{O}$ is the polynomial algebra in 2 variables x, y, and $\mathcal{I} \subset \mathcal{O}$ is the ideal generated by quadratic elements, i.e. by x^2, xy, y^2.

(III) $\mathcal{O}$ is the polynomial algebra in 3 variables x, y, z and the ideal $\mathcal{I} \subset \mathcal{O}$ is the ideal generated by x^2, xy, y^2, xz.

(IV) $\mathcal{O}$ is the polynomial algebra in 4 variables x, y, z, w and the ideal $\mathcal{I} \subset \mathcal{O}$ is generated by x^2, xy, y^2z^2, zw, w^2 as in [Kat19].

Let us find the $\mathcal{O}$-module resolution in each one of these four cases, i.e. let us describe all arborescent operations.

(I) For the angular momentum, an $\mathcal{O}$-module resolution has a length two:

$$0 \longrightarrow \underbrace{\mathfrak{M}_2}_{rk=2} \xrightarrow{d} \underbrace{\mathfrak{M}_1}_{rk=3} \xrightarrow{d} \mathcal{O} \longrightarrow\!\!\!\!\rightarrow \mathcal{O}/\mathcal{I}. \tag{33}$$

This resolution is minimal at $0 \in \mathbb{R}^3$, i.e. the differential is 0 at 0. We denote the generators of $\mathfrak{M}_1, \mathfrak{M}_2$ as $\pi_1^a,\ \ a = 1,2,3$ and $\pi_2, \bar{\pi}_2$ respectively. The differential d is defined as follows:

$$d\pi_1^a = L_a,\ \text{for}\ a = 1,2,3$$
$$d\pi_2 = \sum_{a=1}^{3} x_a \pi_1^a, \qquad d\bar{\pi}_2 = \sum_{a=1}^{3} p^a \pi_1^a.$$

Let us construct an associated arborescent Koszul-Tate resolution $S(\mathcal{T}ree[\mathfrak{M}_\bullet], \delta_\psi)$. Recall that the only freedom one has in constructing our Koszul-Tate differential is when choosing the arborescent operations, i.e. the maps $t \mapsto \psi_t$. Since the resolution has length 2, the only on-trivial term is $\psi_\vee$ associated to the tree with one root an two leaves. Due to the injectivity of $d : \mathfrak{M}_2 \to \mathfrak{M}_1$, the map $\psi_\vee$ is even unique. A routine computation shows that it is given by for all $a, b, c \in \{1,2,3\}$ by:

$$\psi_\vee(\pi_1^a, \pi_1^b) = \epsilon_{abc} p^c \pi_2 - \epsilon_{abc} x^c \bar{\pi}_2.$$

The ranks of $\mathcal{T}[\mathfrak{M}]_i$ for $i = 1, \ldots, 5$ can be easily computed:

$\mathcal{T}ree[\mathfrak{M}_\bullet]_1$	$\mathcal{T}ree[\mathfrak{M}_\bullet]_2$	$\mathcal{T}ree[\mathfrak{M}_\bullet]_3$	$\mathcal{T}ree[\mathfrak{M}_\bullet]_4$	$\mathcal{T}ree[\mathfrak{M}_\bullet]_5$
3	2	3	7	18

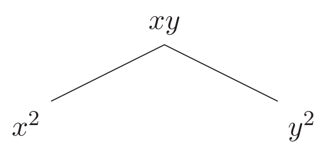
The *Tate algorithm* (see the introduction) gives, however, modules of smaller ranks:

$\mathcal{E}_1^{KT}$	$\mathcal{E}_2^{KT}$	$\mathcal{E}_3^{KT}$	$\mathcal{E}_4^{KT}$	$\mathcal{E}_5^{KT}$
3	2	3	6	11

(II) For quadratic functions, there exists an $\mathcal{O}$-module resolution of length two:

$$0 \longrightarrow \underbrace{\mathfrak{M}_2}_{rk=2} \xrightarrow{d} \underbrace{\mathfrak{M}_1}_{rk=3} \xrightarrow{d} \mathcal{O} \longrightarrow\!\!\!\!\rightarrow \mathcal{O}/\mathcal{I}$$

and coincides with the algebraic Scarf complex of



This $\mathcal{O}$-module resolution and the $\mathcal{O}$-module resolution in (33) look similar: this is not a coincidence, since it is possible to construct an embedding $\iota : \mathbb{R}^2 \hookrightarrow \mathbb{R}^6$ such that the pullback $\iota^*\mathcal{O}_{\mathbb{R}^6}$ of the algebra $\mathcal{O}_{\mathbb{R}^6}$ of polynomial functions on $\mathbb{R}^6$ and the angular momentum ideal $\iota^*\mathcal{I}_{\mathbb{R}^6}$ are identified with $\mathcal{O}$ and $\mathcal{I}$ respectively. This is given explicitly by

$$\begin{aligned} \iota^* x_1 = x, \qquad \iota^* x_2 = -y, \qquad \iota^* x_3 = 0 \\ \iota^* p_1 = 0, \qquad \iota^* p_2 = x, \qquad \iota^* p_3 = -y \end{aligned} \tag{34}$$

The arborescent Koszul-Tate resolution can be obtained by pulling back the arborescent Koszul-Tate resolution constructed for the momentum mapping (I) with the help of the algebra morphism (34). It amounts to taking tensor product of that arborescent Koszul-Tate resolution by $\mathcal{O}\otimes_{\mathcal{O}_{\mathbb{R}^6}}$ (where $\mathcal{O}$ is seen as a $\mathcal{O}_{\mathbb{R}^6}$-module with the help of the algebra morphism (34)) to describe the arborescent Koszul-Tate resolution obtained in (II).

[III] In Example III, the ideal $\mathcal{I}$ admits a free $\mathcal{O}$-resolution of length 3:

$$0 \longrightarrow \underbrace{\mathfrak{M}_3}_{rk=1} \xrightarrow{d} \underbrace{\mathfrak{M}_2}_{rk=4} \xrightarrow{d} \underbrace{\mathfrak{M}_1}_{rk=4} \xrightarrow{d} \mathcal{O} \longrightarrow\!\!\!\!\rightarrow \mathcal{O}/\mathcal{I}$$

We denote generators or $\mathfrak{M}_1$ by $\pi^{xx}, \pi^{xy}, \pi^{yz}, \pi^{yy}$, generators of degree 2 by $\pi^{xyy}, \pi^{yxx}, \pi^{zxx}, \pi^{zxy}$. These choices of indices are natural in view of the differential d that we now introduce:

$$\begin{array}{ll}
\text{Degree 1} & d\pi^{xx} = x^2, \quad d\pi^{xy} = xy, \quad d\pi^{xz} = xz, \quad d\pi^{yy} = y^2, \\
\text{Degree 2} & d\pi^{xyy} = x\pi^{yy} - y\pi^{xy}, \quad d\pi^{yxx} = y\pi^{xx} - x\pi^{xy}, \\
\text{Degree 2} & d\pi^{zxx} = z\pi^{xx} - x\pi^{xz}, \quad d\pi^{zxy} = z\pi^{xy} - y\pi^{xz}, \\
\text{Degree 3} & d\pi = z\pi^{yxx} - y\pi^{zxx} + x\pi^{zxy}.
\end{array}$$

Since $\mathfrak{M}_i \neq 0$ for $i = 1, 2, 3$ only, all maps ψ_t in the arborescent Koszul-Tate resolution vanish except for $t = \vee$ (=the tree with one root and two leaves) and $t = \curlyvee$ (=the tree with one root and three leaves). Those can be computed explicitly:

$$\begin{array}{ll}
\psi_\vee(\pi^{xx}, \pi^{xy}) = -x\,\pi^{yxx}, & \psi_\vee(\pi^{xx}, \pi^{xz}) = -x\,\pi^{zxx} \\
\psi_\vee(\pi^{xx}, \pi^{yy}) = x\,\pi^{xyy} - y\,\pi^{yxx}, & \psi_\vee(\pi^{xy}, \pi^{xz}) = -x\,\pi^{zxy} \\
\psi_\vee(\pi^{xz}, \pi^{yy}) = z\,\pi^{xyy} + y\,\pi^{zxy}, & \psi_\vee(\pi^{xy}, \pi^{yy}) = y\,\pi^{xyy} \\
\\
\psi_\vee(\pi^{xx}, \pi^{zxy}) = x\,\pi, & \psi_\vee(\pi^{xy}, \pi^{zxx}) = -x\,\pi \\
\psi_\vee(\pi^{xz}, \pi^{yxx}) = x\,\pi, & \psi_\vee(\pi^{yy}, \pi^{zxx}) = -y\,\pi \\
\psi_\curlyvee(\pi^{xx}, \pi^{xy}, \pi^{xz}) = -x^2\,\pi, & \psi_\curlyvee(\pi^{xx}, \pi^{xz}, \pi^{yy}) = xy\,\pi.
\end{array}$$

In this example, μ_2 is associative (which was obvious of degree reasons), and $\mu_n = 0$ for all $n \geq 3$.

[IV] A free $\mathcal{O}-$resolution of length 4 is given by:

$$0 \longrightarrow \underbrace{\mathfrak{M}_4}_{rk=1} \xrightarrow{d} \underbrace{\mathfrak{M}_3}_{rk=5} \xrightarrow{d} \underbrace{\mathfrak{M}_2}_{rk=8} \xrightarrow{d} \underbrace{\mathfrak{M}_1}_{rk=5} \xrightarrow{d} \mathcal{O} \longrightarrow\!\!\!\!\rightarrow \mathcal{O}/\mathcal{I}$$

This example was studied in [Kat19], where it was shown that the resolution does not admit a differential graded algebra structure.

The generators and the differential d are given by

$$\begin{array}{ll}
\text{Degree 1} & d\pi^a = x^2, \quad d\pi^b = xy, \quad d\pi^c = y^2z^2, \quad d\pi^d = zw, \quad d\pi^e = w^2, \\
\text{Degree 2} & d\pi^{ab} = x\pi^b - y\pi^a, \quad d\pi^{ad} = x^2\pi^d - zw\pi^a, \quad d\pi^{ae} = x^2\pi^e - w^2\pi^a, \\
\text{Degree 2} & d\pi^{bc} = x\pi^c - yz^2\pi^b, \quad d\pi^{bd} = xy\pi^d - zw\pi^b, \quad d\pi^{be} = xy\pi^e - w^2\pi^b, \\
\text{Degree 2} & d\pi^{cd} = y^2z\pi^d - w\pi^c, \quad d\pi^{de} = z\pi^e - w\pi^d, \\
\text{Degree 3} & d\pi^{abd} = zw\pi^{ab} + x\pi^{bd} - y\pi^{ad}, \quad d\pi^{abe} = w^2\pi^{ab} + x\pi^{be} - y\pi^{ae}, \quad d\pi^{ade} = x^2\pi^{de} + w\pi^{ad} - z\pi^{ae}, \\
\text{Degree 3} & d\pi^{bde} = xy\pi^{de} + w\pi^{bd} - z\pi^{be}, \quad d\pi^{bcd} = w\pi^{bc} + x\pi^{cd} - yz\pi^{bd}, \\
\text{Degree 4} & d\pi^{abde} = w\pi^{abd} - z\pi^{abe} + y\pi^{ade} - x\pi^{bde}.
\end{array}$$

The arborescent operations satisfying (26) correspond to the multiplication on $(\mathfrak{M}, d)$ constructed in [Kat19]. More precisely, $\psi_\vee$ can be chosen to be the 2-ary product on $(\mathfrak{M}, d)$ as in

[Kat19]. This product is not associative, and [Kat19] describes the associator, and shows that for instance:

$$\psi_{\vee}(\pi^a, \psi_{\vee}(\pi^c, \pi^e)) - \psi_{\vee}(\psi_{\vee}(\pi^a, \pi^c), \pi^e) = -yzd\pi^{abde}.$$

In view of (32), we also have

$$\psi_{\vee}(\pi^a, \psi_{\vee}(\pi^c, \pi^e)) - \psi_{\vee}(\psi_{\vee}(\pi^a, \pi^c), \pi^e) = d \circ (\psi_{\vee\!\!/} + \psi_{\backslash\!\!\vee})(\pi^a, \pi^c, \pi^e).$$

A straightforward calculation shows that $\psi_{\vee\!\!/}$ and $\psi_{\backslash\!\!\vee}$ evaluated on (π^a, π^c, π^e) can be chosen as follows:

$$\psi_{\vee\!\!/}(\pi^a, \pi^c, \pi^e) = 0, \quad \psi_{\backslash\!\!\vee}(\pi^a, \pi^c, \pi^e) = -yz\pi^{abcd}.$$

Last, $\psi_{\vee\!\!\!|}$ can be derived from the relation:

$$\psi_{\vee\!\!\!|}(\pi^a, \pi^c, \pi^e) = +\psi_{\vee}(\pi^a, \psi_{\vee}(\pi^c, \pi^e)) + d\psi_{\vee\!\!/}(\pi^a, \pi^c, \pi^e) = yz^2\pi^{abe} + wx\pi^{bcd} + xyz\pi^{bde}.$$

For degree reasons, there are no non-zero arborescent operations corresponding to trees with ≥ 2 vertices, and we obtain a complete description of the arborescent Koszul-Tate $\mathcal{O}$-resolution. The corresponding arborescent A_∞-algebra admits a non-zero term μ_3 while all μ_n are zero for $n \geq 4$.

7.2 Resolutions having a DGCA structure

In the literature, quite a few $\mathcal{O}$-resolutions of $\mathcal{O}/\mathcal{I}$ admit a differential graded commutative algebra (=DGCA) structure. Their arborescent resolutions are quite trivial, in view of the following statement.

Proposition 7.1. *Let $(\mathfrak{M}_\bullet, d)$ be a free $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ equipped with graded commutative associate multiplication $\star$. Then there is an arborescent Koszul-Tate resolution $(S(\mathcal{T}ree[\mathfrak{M}_\bullet]), \delta_\psi)$ where the arborescent operations $\psi\colon t \mapsto \psi_t$ are all equal to zero except when t is a tree with $n \geq 2$ leaves and one root*

$t_n =$ 

in which case for all $a_1, \ldots, a_n \in \mathcal{O} \oplus \mathfrak{M}_\bullet$:

$$\psi_{t_n}(a_1 \otimes \cdots \otimes a_n) = a_1 \star \cdots \star a_n.$$

Let us see several natural instances of such resolutions for which the arborescent operations are described as in Proposition 7.1.

Koszul complex

Let us consider Assume that an ideal $\mathcal{I} \subset \mathcal{O}$ is generated, as an $\mathcal{O}$-module, by r elements $\varphi = \varphi_1, \ldots, \varphi_r$. In that case, the pair $(K(\varphi), \delta)$ given by $K(\varphi) := \mathcal{O} \otimes_{\mathbb{K}} \wedge V^*$, with V a vector space of dimension r with basis $e_1, \ldots, e_r$, and $\delta = \sum_{i=1}^r \varphi_i \otimes \mathrm{i}_{e_i}$ is a differential graded commutative algebra called *Koszul complex.* The following result is well-known:

Proposition 7.2. *The Koszul complex $(K(\varphi), \delta)$ is a Koszul-Tate $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ if the r-tuple $\varphi_1, \ldots, \varphi_r \in \mathcal{O}$ is a regular sequence [Eis95], i.e. if ϕ_1 is not a zero divisor in $\mathcal{O}$, ϕ_i is not a zero divisor in $\mathcal{O}/\langle \phi_1, \ldots \phi_{i-1} \rangle$ for all $i > 1$ and $(\varphi_1, \ldots, \varphi_r)\mathcal{O} \neq \mathcal{O}$.*

Example 7.3. For $\mathcal{I}$ an ideal generated by a non-zero divisor φ, for degree reasons, all foliated trees $t[\phi, \ldots, \phi]$ are equal to zero, except for the trivial tree. As a consequence, both the Koszul complex and the arborescent Koszul-Tate $\mathcal{O}$-resolution are isomorphic to $\mathcal{O} \oplus \mathcal{O}\eta$ with η a degree $+1$ variable and $d = \varphi \frac{\partial}{\partial \eta}$.

Example 7.4. For $\mathcal{O} = \mathbb{C}[x_1, \ldots, x_n]$, the ideal of functions vanishing on an affine variety which is a complete intersection in the sense of [Sha13] is an example of an ideal generated by a regular sequence.

Taylor resolutions of monomial ideals

Let $\mathcal{O}$ be the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ and denote by $\mathcal{J}$ the maximal (irrelevant) ideal generated by $x_1, \ldots, x_n$. Let $\varphi = \{\varphi_1, \ldots, \varphi_k\}$ be a minimal set of generators of a proper ideal $\mathcal{I} \subset \mathcal{O}$. We assume that we can choose them to be monomials, we then say that $\mathcal{I}$ is a *monomial ideal*. The $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ underlines particular simplicial or cellular constructions, see e.g. [Pee10]. We will briefly describe the *Taylor resolution* of $\mathcal{O}/\mathcal{I}$.

Construction 7.5. *Construction of the Taylor complex.*

The Taylor complex is a chain complex

$$\cdots \xrightarrow{d} C_i(\varphi) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{C}_0(\varphi) \xrightarrow{0} 0.$$

of free $\mathcal{O}$-modules $C_i(\varphi)$ equipped with a differential d of degree -1. It is constructed as follows:

- Choose any order on φ and view φ as a set of vertices of $(k-1)$-dimensional simplex Δ_φ. Let $F_i(\Delta_\varphi)$ be a set of i-dimensional faces of Δ_φ and denote by $|F_i(\Delta_\varphi)|$ its cardinality. Then we set $C_i(\varphi) = \mathcal{O}^{|F_{i-1}(\Delta_\varphi)|}$. This means that a basis of $C_i(\varphi)$ is parametrized by $(i-1)$-dimensional faces of σ.
- Let τ be a subset of ϕ and let us denote $m_\tau = \mathrm{lcm}\{\varphi_j \,|\, \varphi_j \in \tau\}$. Let σ be a face of Δ_φ and e_σ - a basis element of $\mathcal{O}^{|F_{i-1}(\Delta_\varphi)|}$ that corresponds to σ. Then d is defined in the following way:
$$d(e_\sigma) = \sum_{\alpha \in \varphi} \mathrm{sign}(\alpha, \sigma) \frac{m_\sigma}{m_{\sigma \backslash \alpha}} e_{\sigma \backslash \alpha},$$
where $\mathrm{sign}(\alpha, \sigma)$ is $(-1)^{p_\alpha + 1}$, with p_α being the position of α in the ordered subset σ.

It can be proven that $C(\varphi)$ is a $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$, see details in [Pee10]. Therefore we may call the complex $C(\varphi)$ the *Taylor resolution*. If φ is a regular sequence, then the Taylor resolution $C(\varphi)$ coincides with the Koszul complex $(K(\varphi), \delta)$ of $\mathcal{O}/\mathcal{I}$.

As explained in [BPS98], the Taylor resolution is equipped with an associative and graded-commutative multiplication $\star$. This product is given by

$$e_\sigma \star e_\tau = \mathrm{sign}(\sigma, \tau) \frac{m_\sigma m_\tau}{m_{\sigma \cup \tau}} e_{\sigma \cup \tau}, \quad \text{if } \tau \cap \sigma = \varnothing \text{ and is 0 otherwise},$$

where $\mathrm{sign}(\sigma, \tau)$ is a permutation sign of a concatenation $\sigma \cdot \tau$ with respect to ordered set $\sigma \cup \tau$. Moreover, the multiplication satisfies the Leibniz rule:

$$d(e_\sigma \star e_\tau) = d(e_\sigma) \star e_\tau + (-1)^{|\sigma|} e_\sigma \star d(e_\tau)$$

which turns the Taylor resolution into a DG-algebra.

Example 7.6. Here is an other example, which is not of one of the previous type. Consider $\mathcal{O} = \mathbb{K}[x, y]/\langle xy \rangle$ and $\mathcal{I}$ is the ideal generated by x. A $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ is given by $\mathfrak{M}_i = \mathcal{O}$ for all $i \geq 1$. Calling by $(e_i)_{i \geq 1}$ the generators of $\mathfrak{M}_i$, we define a differential by $d(e_i) = y e_{i-1}$ for even integer i and $d(e_i) = x e_{i-1}$ for any odd integer i. It is a matter of computation to show that the product defined by $\mu_2(e_i, e_j) = \mu_{i,j} e_{i+j}$ with

$$\mu_{i,j} = \begin{cases} 0 & \text{if } i, j \text{ odd} \\ \begin{pmatrix} i+j \\ j \end{pmatrix} & \text{if } i, j \text{ are even} \\ \begin{pmatrix} i+j \\ j-1 \end{pmatrix} & \text{if } i \text{ is even and } j \text{ is odd} \end{cases}$$

is a DGCA product.

8 Applications

8.1 Complexity

An arborescent Koszul-Tate resolution might be "huge" in the sense that it has a lot of generators of every degree, but it is "small" in the sense that finitely many computations often suffices to construct it. The lines below intend to make this statement precise and rigorous.
We say that an algebra $\mathcal{O}$ is a *Syzygy Noetherian algebra* if any finite rank $\mathcal{O}$-module has a free or projective $\mathcal{O}$-resolution of finite length. For example, by Hilbert Syzygy's theorem and its generalizations [Eis95], polynomial ring in N variables, germs of holomorphic or real analytic functions at $0 \in \mathbb{K}^N$, algebras of formal power series at 0 over $\mathbb{R}$ or $\mathbb{C}$ [Tou76] are Syzygy Noetherian algebras: moreover there exists resolutions of length less than or equal to $N + 1$.
Let us call *linear $\mathcal{O}$-operations* the following two kind of operations:

1. "*First type*" Finding the kernel of a $\mathcal{O}$-linear map $\mathcal{N}_1 \to \mathcal{N}_2$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are finite rank projective $\mathcal{O}$-modules.

2. "*Second type*" Finding a map ψ making the following diagram of projective $\mathcal{O}$-modules commutative:

$$\begin{array}{ccc} \mathcal{N}_1 & \overset{\psi}{\dashrightarrow} & \mathcal{N}_2 \\ & \underset{\chi_1}{\searrow} & \downarrow \chi_2 \\ & & \mathcal{N}_3 \end{array}$$

   for any $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ finite rank projective $\mathcal{O}$-modules, and any $\mathcal{O}$-linear maps χ_1, χ_2 with χ_2 surjective.

Corollary 8.1. *For any ideal $\mathcal{I}$ of a Syzygy Noetherian algebra $\mathcal{O}$, an arborescent Koszul-Tate resolution can be determined by a finite number of linear $\mathcal{O}$-operations.*

Proof. For $\mathcal{O}$ a Noetherian algebras, the n first terms of a free resolution can be computed for all n using finitely many linear $\mathcal{O}$-operations of the first type. If the algebra is Syzygy, then this free $\mathcal{O}$-resolution $(\mathfrak{M}_\bullet, d)$ may be chosen of finite length. Since $(\mathfrak{M}_\bullet, d)$ has finite length, there exists only finitely many trees t such that ψ_t is non-zero. Since each of these maps is computed through linear $\mathcal{O}$-operations of the second type, this proves the assertion. □

Remark 8.2. In contrast, the Tate algorithm also uses $\mathcal{O}$-linear operations of the first and second type, but will use infinitely many of them each time that the algorithm requires the introduction of variables of degree n for all $n \leq 0$, see, e.g. Example of monomial ideals below. Although the resolution obtained out of the Tate algorithm may be defined out of modules of smaller ranks, it requires an infinite number of computations, even for a Syzygy algebra.

For several algebras, like $\mathcal{O} = \mathbb{C}[x_1, \ldots, x_n]$, a given linear $\mathcal{O}$-operation of type I and II can be solved by finitely many algebraic operations. Corollary 8.1 implies therefore a stronger statement:

Corollary 8.3. *For any ideal $\mathcal{I}$ of $\mathcal{O} = \mathbb{C}[x_1, \ldots, x_n]$, an arborescent Koszul-Tate resolution can be determined by finitely many operations.*

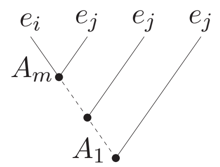
Let us an example for which the Tate algorithm will require infinitely many operations: monomial ideals which are not generated by a regular family. Let $\mathcal{I}$ be a monomial ideal, generated as a family φ, as in Section 7.2. The Taylor resolution $C(\varphi)$ admits a DGCA structure, so that $(S(\mathcal{T}ree[C(\varphi)]), \delta)$ comes equipped with the arborescent operations as in Proposition 7.1. Let us then prove the following statement:

Proposition 8.4. *If the generating family φ of the monomial ideal $\mathcal{I}$ is not a regular sequence, then for any Koszul-Tate resolution $(S(\mathcal{E}), \delta)$ the submodule $\mathcal{E}_k \subset \mathcal{E}$ is different from 0 for every $k \geq 1$.*

Proof. We prove the statement by showing that the reduced complex at $\mathcal{J}$ (see Section 9.3) has homology at arbitrary large degree. For this we make use of the arborescent Koszul-Tate resolution $(S(\mathcal{T}ree[C(\varphi)]), \delta)$ associated to the Taylor resolution $(C(\varphi), d)$.

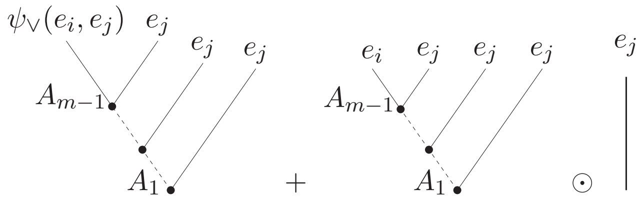
Since φ is not regular, there exists two generators of φ such that $\mathrm{lcm}\{\varphi_i, \varphi_j\} \neq \varphi_i \cdot \varphi_j$. Therefore $\mathrm{lcm}\{\varphi_i, \varphi_j\}$ divides $\varphi_i \cdot \varphi_j$ by some monomial a which is of non-zero polynomial degree. Consider a tree T_m with m inner vertices of the following form:

$T_m =$



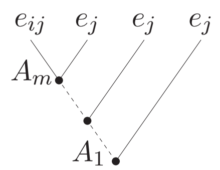
We claim that $\delta T_m \in \mathcal{J}\mathcal{T}ree[C(\varphi)] \oplus S^{\geq 2}(\mathcal{T}ree[C(\varphi)])$. This follows from a direct computation:

$\delta T_m = (-1)^m$



Here we used the arborescent operations chosen as in Proposition 7.1 in order to employ only arborescent operations $\{\psi_t\}$ with t being a tree with 1 vertex. From the DGCA product on the Taylor resolution we deduce that $\psi_v(e_i, e_j)$ is valued in $\mathcal{J}C_2(\varphi)$. Therefore, T_m is a closed element in a reduced complex $(\mathcal{T}ree[C(\varphi)] \otimes \mathcal{O}/\mathcal{I}) \otimes \mathcal{O}/\mathcal{J}, \delta)$. Note that T_m can not be exact, since the image of δ is made of linear combinations of trees which either are non-binary trees, or trees with d-exact decorations exact or decorations in the image of ψ_t (thus in homological degree > 1), see Section 6.2. This is not the case with T_m. Therefore, $H_\bullet(\delta, (\mathcal{T}ree[C(\varphi)] \otimes \mathcal{O}/\mathcal{I}) \otimes \mathcal{O}/\mathcal{J})$ is not zero in degree $2m+1$. Since m is arbitrary, we deduce that submodules $\mathcal{E}_{2m+1} \subset \mathcal{E}$ are not zero. In order to show that $\mathcal{E}_{2m}$ is not zero, we replace e_i by a degree 2 generator e_{ij}. Then

$T'_m =$



Note $\psi_\vee(e_{ij}, e_j) = 0$, thus we deduce in a similar way T'_m is a closed element in the reduced complex. T'_m can not be exact for the same reasons as T_m. Thus we deduce that $\mathcal{E}_{2m} \subset \mathcal{E}$ are not zero for arbitrary $m > 0$. □

Corollary 8.5. *For any ideal as in Proposition 8.4, the Tate algorithm requires infinitely many steps for constructing a Koszul-Tate resolution 5.2 of $\mathcal{O}/\mathcal{I}$.*

A comparison of Corollaries 8.3 and 8.5 shows in which sense the arborescent Koszul-Tate resolution is better than the one obtained from the Tate algorithm.

8.2 Equivariance with respect to a finite group action

Here is an other application of the construction.

Corollary 8.6. *Let K be a finite group acting on an algebra $\mathcal{O}$ over $\mathbb{R}$ or $\mathbb{C}$ and $\mathcal{I}$ a K-invariant ideal. There exists a K-equivariant Koszul-Tate resolution.*

Proof. The proof goes in two steps.
Step 1. Let us show that a K-invariant ideal $\mathcal{I}$ admits a K-equivariant free $\mathcal{O}$-resolution.

We denote by $(h, F) \mapsto h \triangleright F$ the action of $h \in K$ on $F \in \mathcal{O}$. Since K acts on $\mathcal{O}$, it acts on any free $\mathcal{O}$-module. Let $\mathfrak{M}_1$ be a free $\mathcal{O}$-module and $d \colon \mathfrak{M}_1 \to \mathcal{O}$ a $\mathcal{O}$-module morphism whose image is $\mathcal{I}$. In general, d is not K-equivariant. Consider the tensor product

$$\mathfrak{M}'_1 := \mathfrak{M}_1 \otimes_{\mathbb{K}} \mathbb{K}[K]$$

with $\mathbb{K}[K]$ being the free vector space generated by the group G. There is natural K-action on this space by

$$h \triangleright (a \otimes g) := (h \triangleright a) \otimes hg \text{ for all } a \in \mathfrak{M}_1, g, h \in K\ .$$

With respect to that K-action, the $\mathcal{O}$-linear morphism

$$d'(a \otimes g) = g \triangleright d\left(g^{-1} \triangleright a\right) \text{ for all } a \in \mathfrak{M}_1, g \in K \tag{35}$$

is K-equivariant. Its image is still $\mathcal{I}$. Since d is K-equivariant, its kernel is preserved under the K-action, and for all $g \in K$

$$a \mapsto g \triangleright a$$

is an invertible endomorphism of $\mathrm{Ker}(d)$. Let $\mathfrak{M}_2$ be a free $\mathcal{O}$-module equipped with a surjective $\mathcal{O}$-linear morphism $\mathfrak{M}_2 \longrightarrow \mathrm{Ker}(d)$, then consider

$$\mathfrak{M}'_2 := \mathfrak{M}_2 \otimes_{\mathcal{K}} \mathbb{K}[K].$$

The map d' constructed as in (35) is K-equivariant and its image is $\mathrm{Ker}(d)$, so that

$$\mathfrak{M}'_2 \xrightarrow{d'} \mathfrak{M}'_2 \xrightarrow{d'} \mathcal{O} \longrightarrow \mathcal{O}/\mathcal{I}$$

are the first terms of a K-equivariant free $\mathcal{O}$-resolution of $\mathcal{I}$. The procedure continues by recursion.

Step 2. Let $(\mathfrak{M}_\bullet, d')$ be an K-equivariant free $\mathcal{O}$- resolution as obtained out of Step 1. In the construction of its arborescent Koszul-Tate $\mathcal{O}$-resolution, all operations described in Section 6 are K-equivariant except maybe those that involve the arborescent operations $t \mapsto \psi_t$, so that δ_ψ will be K-equivariant if the maps ψ_t can be chosen to be so.

We prove by recursion on the degree k that such choices exist. Out of the proof of Proposition 6.16, we derived a recursion relation for the arborescent operations (see (26) or (32)):

$$d' \circ \psi_t(a_1, \ldots, a_n) = \left(\sum_{A \in InnerVertex(T)} \psi_{t_{\downarrow A}} \circ_A \psi_{t_{\uparrow A}} - \psi_{\partial_A t} - \psi_t \circ d' \right)(a_1, \ldots, a_n). \tag{36}$$

More precisely, while proving Theorem 6.19 (and more precisely Corollary 6.18), we showed that $\delta_\psi^2 = 0$ if and only if the ψ_t are recursilevy constructed so that they satisfy (36). If one assumes that the assignment $(a_1, \ldots, a_n) \mapsto \psi_t[a_1, \dot{;} a_n]$ is K-equivariant for all tree t and all $a_1, \ldots, a_n \mathfrak{M}$ such that the sum $|t| + |a_1| + \cdots + |a_n|$ of all degrees is $\leq k$, then the right hand side of the previous equation is K-equivariant. Since d' is K-equivariant, one may redefine $\psi_t(a_1, \ldots, a_n)$ and all trees t and elements $a_1, \ldots, a_n \in FM$ whose degrees add up to $k+1$ by

$$\tilde{\psi}_t[a_1, \ldots, a_n] \mapsto \frac{1}{|K|} \sum_{g \in K} g^{-1} \triangleright \psi_t(g \triangleright a_1, \ldots, g \triangleright a_n)$$

and Equation (36) still holds with this new definition. The construction of K-equivariant recursion operations $t \mapsto \psi_t$ then continues by recursion. □

Remark 8.7. The Tate algorithm could also be made K-equivariant by using similar constructions. However, the description with the arborescent Koszul-Tate is more natural.

9 Finite and infinite Koszul-Tate resolutions

Throughout this section, $(\mathfrak{M}, d)$ is a $\mathcal{O}$-module resolution of $\mathcal{O}/\mathcal{I}$, and $(S(\mathcal{T}ree[\mathfrak{M}]), \delta_\psi)$ is the arborescent Koszul-Tate resolution obtained through the choice of some arborescent operations $\psi\colon t \mapsto \psi_t$.

9.1 The Koszul resolution as a quotient of its arborescent Koszul-Tate resolution

Although the arborescent Koszul-Tate resolution has the advantages listed in the introduction, it is certainly not minimal. For instance, for $\mathcal{I} \subset \mathcal{O}$ a complete intersection with r generators, the Koszul $\mathcal{O}$-module resolution $((\mathfrak{M}_i = \wedge^i \mathcal{O}^r)_{i\geq 1}, d = \sum_{i=1}^n \phi_i \iota_{e_i})$ is by construction an $\mathcal{O}$-module resolution with finitely many generators. One can therefore consider the associated arborescent Koszul-Tate resolution. This resolution is of course "way too big", since the Koszul resolution itself is already a Koszul-Tate resolution. Let us explain how one can quotient it to get the initial Koszul resolution back. Let us start with a general statement, valid for any arborescent Koszul-Tate resolution:

Proposition 9.1. *Let $\mathfrak{J}$ be a graded differential ideal of an arborescent Koszul-Tate resolution $(S(\mathcal{T}ree[\mathfrak{M}_\bullet]), \delta_\psi)$, which is h-stable, δ_ψ-stable and contained in the kernel of* Proj*, then the quotient $(S(\mathcal{T}ree[\mathfrak{M}_\bullet]), \delta_\psi)/\mathfrak{J}$ is a Koszul-Tate $\mathcal{O}$-resolution of $\mathcal{O}/\mathcal{I}$ provided it is a symmetric algebra.*

Proof. We only need to check that the quotient has no cohomology except in degree 0 where it is $\mathcal{O}/\mathcal{I}$. The short exact sequence

$$0 \longrightarrow (\mathfrak{J}, \delta_\psi) \longrightarrow (S(\mathcal{T}ree[\mathfrak{M}_\bullet]), \delta_\psi) \longrightarrow (S(\mathcal{T}ree[\mathfrak{M}_\bullet])/\mathfrak{J}, \delta_\psi) \longrightarrow 0$$

yields a long exact sequence

$$\cdots H^i(\mathfrak{J}, \delta_\psi) \longrightarrow H^i(S(\mathcal{T}ree[\mathfrak{M}_\bullet]), \delta_\psi) \longrightarrow H^i(S(\mathcal{T}ree[\mathfrak{M}_\bullet])/\mathfrak{J}, \delta_\psi) \longrightarrow H^{i-1}(\mathfrak{J}, \delta_\psi) \longrightarrow \cdots$$

It suffices to show that $H^i(\mathfrak{J}, \delta_\psi) = 0$. Since $\mathfrak{J}$ is in the kernel of Proj and h-stable, it is a homotopy retract on zero, i.e. $(\delta_\psi \circ h + h \circ \delta_\psi - \mathrm{id})\mathfrak{J} = 0$. We conclude that $H^i(\mathfrak{J}, \delta_\psi) = 0$. ☐

Of course, Proposition 9.1 makes many assumptions on the ideal $\mathfrak{J}$. For the Koszul resolution of an ideal generated by a regular sequence as in section 7.2, let $\mathfrak{J}$ be the kernel of Proj_ψ. It is obviously h-stable and δ_ψ-stable. It is easy to checked that $\mathfrak{J}$ coincides with with the ideal generated by:

$$\mathfrak{J} := \langle \mathcal{T}ree^{\geq 2}[\mathfrak{M}_\bullet], \overset{a_1}{\Big|} \odot \cdots \odot \overset{a_k}{\Big|} - \overset{a_1 \wedge \cdots \wedge a_k}{\Big|} \rangle_{S(\mathcal{T}ree[\mathfrak{M}_\bullet])}$$

In particular, $\mathfrak{J}$ is an ideal (it is not the case in general). The quotient $S(\mathcal{T}ree[\mathfrak{M}_\bullet])/\mathfrak{J}$ is by definition isomorphic to the Koszul-resolution. All the requirements of Proposition 9.1 are therefore satisfied in this case. We have therefore described the Koszul resolution of an ideal generated by a regular sequence as a quotient of its arborescent Koszul-Tate resolution. The next section extrapolate on this idea.

9.2 Minimal Koszul-Tate resolutions

Let $\mathcal{J}$ be a maximal ideal containing $\mathcal{I}$. A free or projective resolution $(\mathfrak{M}_\bullet, d)$ of $\mathcal{O}/\mathcal{I}$ is said to be *minimal at $\mathcal{J}$* if $d(\mathfrak{M}_{i+1}) \subset \mathcal{J}\mathfrak{M}_i$ for all $i \geq 1$.

This notion of minimality at $\mathcal{J}$ of free or projective resolution conflicts with an other notion which is naturally addressed in the Tate algorithm of see Construction 5.1. At step $i+1$ we may want to choose a "minimal" set of generators of the kernel of $d\colon \mathfrak{M}_{i+1} \longrightarrow \mathfrak{M}_i$, thus forming a "smallest possible" free $\mathcal{O}$-module $\mathfrak{M}_{i+1}$.

Both notions of minimality in two distinct cases:

Case i. $\mathcal{O}$ is a local Noetherian ring with maximal ideal $\mathcal{J}$ and residue field $\mathbb{K}$.

Case ii. $\mathcal{O}$ is a polynomial ring $\mathbb{K}[x_1, \dots, x_n]$, $\mathcal{J}$ is the irrelevant maximal ideal $\langle x_1, \dots, x_n \rangle$. All $\mathcal{O}$-modules M in question are assumed to be graded with respect to the polynomial degree pol(.). Generators of M are chosen to be homogeneous according to pol(.). The differential $d\colon M \longrightarrow N$ between two graded $\mathcal{O}$-modules is chosen to be of polynomial degree 0.

where the following theorem holds (see e.g. [Eis95, Pee10]):

Theorem 9.2. *Let M be a finitely generated $\mathcal{O}$-module and $\mathcal{J}$ a maximal ideal of $\mathcal{O}$ containing $\mathcal{I}$. Then*

1. *If $\bar{e} = \{\bar{e}_1, \dots \bar{e}_j\}$ is a basis of a vector space $M/\mathcal{J}M$, then its preimage in M is a minimal set of generators of M.*

2. *Every minimal set of generators is obtained through 1.*

3. *In Case i., if $e = \{e_1, \dots e_j\}$ and $f = \{f_1, \dots, f_j\}$ are two minimals sets of generators of M and $e_i = \sum_{k=1}^{j} A_{ij} f_j$ for some matrix (A_{ij}), then (A_{ij}) is invertible. In Case ii. one asks additionally for e and f being homogeneous and (A_{ij}) can be modified to have homogeneous entries, see details in Theorem 2.12 of [Pee10].*

When Theorem 9.2 holds, the resolutions $(\mathfrak{M}_\bullet, d)$ obtained through Construction 5.1 are minimal at $\mathcal{J}$ if and only if at each step of Construction 5.1 we choose a minimal set of generators of $\mathrm{Ker}(d_i)$. Also, in this case, a minimal free resolution is unique up to isomorphisms, and any free resolution can be decomposed as a direct sum of a trivial complex and a minimal free resolution.

We give the following definition of minimality at $\mathcal{J}$ of Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$:

Definition 9.3. *Let $\mathcal{J}$ be a maximal ideal containing $\mathcal{I}$. A Koszul-Tate resolution $(S(\mathcal{E}), \delta)$ is said to be minimal at $\mathcal{J}$ if $\delta(\mathcal{E}_{i+1}) \subset \mathcal{J}\mathcal{E}_i \oplus S_i^{\geq 2}(\mathcal{E})$ for all $i \geq 1$.*

When studying resolution obtained through Tate algorithm (see Construction 5.2), this notion of minimality coincides with the naive one. More precisely,

Proposition 9.4. *A Koszul-Tate resolution $(S(\mathcal{E}), \delta)$ of Construction 5.2 is minimal at $\mathcal{J}$ if and only if at each degree $i \geq 1$ we choose a minimal set of generators of $H_i(S(\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_i), \delta_i)$.*

Proof. Let as assume that $(S(\mathcal{E}), \delta)$ is not minimal. This means that $\exists i \in \mathbb{N}$ such that $\delta(\mathcal{E}_{i+2}) \not\subset \mathcal{J}\mathcal{E}_{i+1} \oplus S_{i+1}^{\geq 2}(\mathcal{E})$. We omit the case $i = -1$, since we always have $\delta(\mathcal{E}_1) \subset \mathcal{I} \subset \mathcal{J}$. Denote by $\chi = \{\chi_1, \dots, \chi_k\}$ the set of generators of $H_i(S(\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_i), \delta_i)$ used in constructing a basis $e_\chi = \{e_{\chi_1}, \dots, e_{\chi_k}\}$ of $\mathcal{E}_{i+1}$. Then there exist $f \in \mathcal{E}_{i+2}$ such that

$$\delta f = \sum_{j \in \chi} a_j e_j \mod S_{i+1}^{\geq 2}(\mathcal{E}),$$

with at least one of a_j being a unit. After multiplying this equality by a_j^{-1} and renumbering the elements of χ we obtain

$$\delta f = e_{\chi_1} + \sum_{j \in \chi/\chi_1} b_j e_j \mod S_{i+1}^{\geq 2}(\mathcal{E}),$$

On the level of homology at degree i, this translates into:

$$\chi_1 + b_2\chi_2 + \cdots + b_k\chi_k = 0$$

Therefore, χ is not a minimal set of generators of $H_i(S(\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_i), \delta_i)$.

Now, if χ is not a minimal set of generators of $H_i(S(\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_i), \delta_i)$, then, after relabeling the generators, we have a relation of the form

$$\chi_1 + b_2\chi_2 + \cdots + b_k\chi_k = 0$$

saying that χ is redundant. On the level of generators of $\mathcal{E}_{i+1}$ this equality reads as

$$\left\{ e_{\chi_1} + \sum_{j \in \chi/\chi_1} b_j e_j + \ldots \right\} \text{ is closed}$$

Here $\ldots$ stands for terms in $S^{\geq 2}_{i+1}(\mathcal{E})$ which are governed by a choice of representatives of $H_i(S(\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_i), \delta_i)$. Since $\delta S^{\geq 2}(\mathcal{E}) \cap \mathcal{E} \subset \mathcal{I}\mathcal{E}$, this cycle is not exact in $(S(\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{i+1}), \delta_{i+1})$. Hence there exists $f \in \mathcal{E}_{i+2}$ such that $\delta f = e_{\chi_1} + \sum_{j \in \chi/\chi_1} b_j e_j + \ldots$. Therefore, the resolution $(S(\mathcal{E}), \delta)$ is not a minimal Koszul-Tate resolution. □

Let us explain how a Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$ minimal at $\mathcal{J}$ can be obtained as a quotient of an arbitrary Koszul-Tate resolution $(S(\mathcal{E}), \delta)$ of $\mathcal{O}/\mathcal{I}$. Let us interpret these conditions in terms of the *reduced complex* (see Section 9.3), i.e. the graded the vector space defined for for all $i \geq 1$ by

$$\mathcal{E}_i \otimes \mathcal{O}/\mathcal{I}$$

equipped with its induced differential $\delta_{\mathcal{I}}$. Let us evaluate at $\mathcal{J}$ the reduced complex , i.e. consider the graded the vector space defined for for all $i \geq 1$ by

$$\mathcal{E}_i \otimes \mathcal{O}/\mathcal{J}$$

with induced differential $\delta_{\mathcal{J}}$.

The next straightforward lemma restates in terms of the reduced complex the condition of minimality.

Lemma 9.5. *A Koszul-Tate resolution is minimal at $\mathcal{J}$ if and only if the reduced complex $(\mathcal{E} \otimes \mathcal{O}/\mathcal{I}, \delta_{\mathcal{I}})$ is minimal at $\mathcal{J}$, i.e. if and only if the differential $\delta_{\mathcal{J}}$ of the evaluation at $\mathcal{J}$ of the reduced complex is zero.*

Remark 9.6. Geometrically, when $\mathcal{J}$ is the ideal of polynomial functions vanishing at 0, the equivalent conditions of the previous lemma mean that the differential of the reduced complex vanishes at 0.

By construction, the evaluation at $\mathcal{J}$ of the reduced complex is a complex of $\mathcal{O}/\mathcal{J} \simeq \mathbb{K}$-module, i.e. a complex of vector spaces. It therefore admits a decomposition, for all $i \geq 1$:

$$E_i = C_i \oplus H_i \oplus C_{i-1}$$

(with the understanding that $C_0 = 0$) such that the differential is given by $d_{\mathcal{J}}(c, h, c') = (0, 0, c)$ for any $i \geq 1$ and any $c \in C_i$, $h \in H_i$, $c' \in C_{i-1}$. In particular, H_i is the i-the homology of the reduced complex evaluated at $\mathcal{J}$. For any $i \geq 1$, there exists degree i elements $(f_i^1, \ldots, f_i^{\dim(C_i)}) \in \mathcal{E}_{i+1}$, respectively $h_i^1, \ldots, h_i^{\dim(H_i)}$, that induce for any $i \geq 1$ a basis of C_i, respectively H_i.

Proposition 9.7. *Assume $\mathcal{O}, \mathcal{I}$ and $\mathcal{J}$ are as in Cases (i) and (ii) in the beginning of this section. The quotient of $S(\mathcal{E})$ by the ideal generated by the families*

$$\left\{ f_i^j \,\middle|\, i \geq 2, j = 1, \ldots, \dim(C_i) \right\} \text{ and } \left\{ \delta(f_{i+1}^j) \,\middle|\, i \geq 1, j = 1, \ldots, \dim(C_{i+1}) \right\}$$

is a minimal Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$.

Proof. Let $\mathcal{R} \subset S(\mathcal{E})$ be the sub-algebra generated by the aforementioned elements. The differential δ restricts to $\mathcal{R}$, and that the quotient $S(\mathcal{E})/\mathcal{R}$ admits an induces differential δ'. Also, let $\delta_{\mathcal{R}}$ be the restriction of the differential. Let us show that the homology of $\delta_{\mathcal{R}}$ is zero in every degree. First, let us notice that $\mathcal{R}$ is a free symmetric graded algebra: since its generators have images in $\mathcal{E} \otimes \mathcal{O}/\mathcal{J}$ which are independent, they are also independent over $\mathcal{O}$ by Theorem 9.2. It makes therefore sense to consider a derivation h of $\mathcal{R}$ defined on generators by:

$$h(f_i^j) = 0 \text{ and } h(\delta(f_i^j)) = f_i^j.$$

We have $\delta \circ h(F) + h \circ \delta(F) = kF$ for every F of arity k in the generators $f_i^j, \delta(f_i^j)$. This implies that the homology of $\mathcal{R}$ is zero on every degree except in degree 0 where it is $\mathcal{O}$.

By item 1 in Theorem 9.2, $\mathcal{E}_i$ is generated by the family

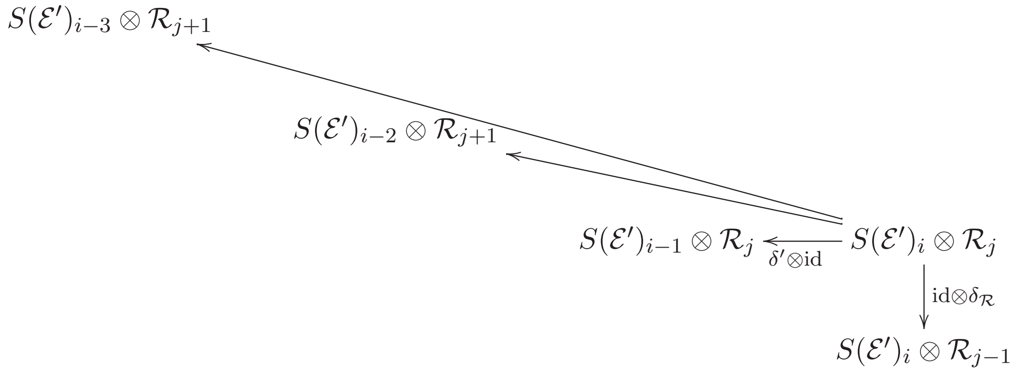
$$\left\{ \delta(f_{i+1}^j), h_i^k, f_i^l \,\middle|\, j = 1, \ldots, \dim(C_{i+1}), k = 1, \ldots, \dim(H_i), l = 1, \ldots, \dim(C_i) \right\}$$

In turn, this implies that there is a graded algebra isomorphism:

$$S(\mathcal{E}) = S(\mathcal{E}') \otimes \mathcal{R}$$

where $\mathcal{E}' \subset \mathcal{E}$ stands for the submodule generated by all the h_i^j's.

Under this isomorphism, the differential δ decomposes as follows, if we consider the bi-grading induced by the previous decomposition:



Since the homology of all vertical lines is zero in every degree except in degree 0 where it is $\mathcal{O}$, the homology of $(S(\mathcal{E}), \delta)$ coincides with the homology $S(\mathcal{E}')$ with respect to the induced degree -1 derivation δ'). Hence $(S(\mathcal{E}'), \delta')$ is again a Koszul-Tate resolution of $\mathcal{O}/\mathcal{I}$. Last, the reduced complex of $(S(\mathcal{E}'), \delta')$ satisfies the required assumption by construction, since its reduced complex at $\mathcal{J}$ is by construction the graded vector space $H_\bullet = \mathbb{K} \oplus \sum_{i \geq 1} H_i$, equipped with trivial differential. □

9.3 The reduced complex of the arborescent Koszul-Tate resolution

We show in this Section that any two Koszul-Tate resolutions are homotopy equivalent, in the sense of Q-manifolds.

Definition 9.8. *Two differential graded commutative algebras* $(\mathcal{A}, \delta)$ *and* $(\mathcal{A}', \delta')$ *are said to be homotopic, if there exist two DGA morphisms* $\Phi : (\mathcal{A}, \delta) \longrightarrow (\mathcal{A}', \delta')$ *and* $\Phi' : (\mathcal{A}', \delta') \longrightarrow (\mathcal{A}, \delta)$ *such that the compositions* $\Phi' \circ \Phi$, $\Phi' \circ \Phi$ *are homotopic to identity.*

For two homotopic commutative DGAs we shall write $(\mathcal{A}, \delta) \sim (\mathcal{A}', \delta')$.

Remark 9.9. Homotopy equivalence is obviously an equivalence relation.

Lemma 9.10. *Let* $(S(\mathcal{E}), \delta)$, $(S(\mathcal{E}'), \delta')$ *be two Koszul-Tate resolutions of* $\mathcal{O}/\mathcal{I}$. *There exists a DGA morphism* $\Phi_\bullet : (S(\mathcal{E}), \delta) \longrightarrow (S(\mathcal{E}'), \delta')$ *covering identity map on* $\mathcal{O}$.

Proof. A standard proposition in a homological algebra course states that between two resolutions of $\mathcal{O}/\mathcal{I}$ there exists an $\mathcal{O}$-linear chain map $f_\bullet$, such that $f_0 = \text{id}$ on $\mathcal{O}$:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathcal{E}_2 \oplus \mathcal{E}_1 \odot \mathcal{E}_1 & \xrightarrow{\delta} & \mathcal{E}_1 & \xrightarrow{\delta} & \mathcal{O} \\
 & \downarrow f_2 & & \downarrow f_1 & & \downarrow \text{id} \\
\cdots \longrightarrow & \mathcal{E}'_2 \oplus \mathcal{E}'_1 \odot \mathcal{E}'_1 & \xrightarrow{\delta'} & \mathcal{E}'_1 & \xrightarrow{\delta'} & \mathcal{O}
\end{array}$$

We define the map $\Phi_\bullet$ recursively as follows:

- $\Phi_0 = \text{id}$, $\Phi_1 = f_1$
- Given a collection of chain maps $\Phi_1, \ldots, \Phi_{k-1}$, $k \geq 1$ we set Φ_k to be an $\mathcal{O}$-linear map satisfying

$$\begin{array}{ccccc}
\cdots \longrightarrow & \mathcal{E}_k & \xrightarrow{\delta} & S_{k-1}(\mathcal{E}) \\
 & \downarrow \Phi_k & & \downarrow \Phi_{k-1} \\
\cdots \longrightarrow & S_k(\mathcal{E}') & \xrightarrow{\delta'} & S_{k-1}(\mathcal{E}')
\end{array}$$

  and on $S^{\geq 2}_k(\mathcal{E})$ Φ_k is a graded algebra morphism of the following form: any monomial a in $S^{\geq 2}_k(\mathcal{E})$ has a decomposition $a = a_1 \odot a_2 \odot \cdots \odot a_j$ for $a_i \in \mathcal{E}$. Then $\Phi_k(a) = \Phi_{|a_1|}(a_1) \odot \cdots \odot \Phi_{|a_j|}(a_j)$. The only thing left to check is that Φ_k is a chain map:

$$\text{For any element } a \in \mathcal{E}_k\text{:} \quad \delta' \Phi_k(a) = \Phi_{k-1}(\delta a)$$

$$\text{For elements } a_1 \odot a_2 \odot \cdots \odot a_j \in S^{\geq 2}_k(\mathcal{E})\text{:} \quad \begin{aligned} \delta' \Phi_k(a_1 \odot a_2 \odot \cdots \odot a_j) &= \\ \delta'(\Phi_{|a_1|}(a_1) \odot \cdots \odot \Phi_{|a_j|}(a_j)) &= \\ \Phi_{k-1} \circ \delta(a_1 \odot a_2 \odot \cdots \odot a_j) \end{aligned}$$

□

We now show that given two Koszul-Tate $(S(\mathcal{E}), \delta)$, $(S(\mathcal{E}'), \delta')$ and two DGA morphisms covering identity $F : (S(\mathcal{E}), \delta) \longrightarrow (S(\mathcal{E}'), \delta')$ and $G : (S(\mathcal{E}'), \delta') \longrightarrow (S(\mathcal{E}), \delta)$. the composition $G \circ F$ is homotopic to identity, i.e.

$$G \circ F - \text{id} = \delta \circ H + H \circ \delta$$

with H being an $\mathcal{O} -$ *linear* map on $S(\mathcal{E})$ with *a non-negative arity*, i.e. it maps $S^{\geq k}(\mathcal{E})$ to $S^{\geq k}(\mathcal{E})$. To do so, we first proof the following statement:

Proposition 9.11. *There exists a smooth family of DGA morphisms $\{\Phi^t, t \in [0,1]\}$ satisfying $\Phi^0 = \mathrm{id}$, $\Phi^1 = G \circ F$ and covering identity on $\mathcal{O}$.*

Proof. At degree 1, we set $\Phi_1^t = \Phi_1^0 + t(\Phi_1^1 - \Phi_1^0)$. The existence of the family $\{\Phi^t\}$ follows from Lemma 9.10. The smoothness of Φ_{k+1}^t in parameter t follows from smoothness of Φ_k^t and linearity of δ in the parameter t. □

Now, given such a family we show that there exists a family of $\mathcal{O}$-linear degree $+1$ maps $\{h_t, t \in [0,1]\}$ such that

- h_t is a Φ_t-derivation, i.e. $h_t(ab) = h_t(a)\Phi_t(b) + (-1)^{|a|}\Phi_t(a)h_t(b)$ for all homogeneous $a, b \in \mathcal{E}$.
- $\frac{\partial}{\partial t}\Phi_t = \delta \circ h_t + h_t \circ \delta$.

At degree 0 and 1:
$\frac{\partial}{\partial t}\Phi_t$ is zero on $\mathcal{O}$, therefore there exists a map $h_t : \mathcal{E}_1 \Longrightarrow \mathcal{E}_2 \oplus \mathcal{E}_1 \odot \mathcal{E}_1$ such that $\delta \circ h_t = \frac{\partial}{\partial t}\Phi_t$.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathcal{E}_2 \oplus \mathcal{E}_1 \odot \mathcal{E}_1 & \xrightarrow{\delta} & \mathcal{E}_1 & \xrightarrow{\delta} & \mathcal{O} \\
 & & \downarrow{\scriptstyle \frac{\partial}{\partial t}\Phi_t} & \swarrow{\scriptstyle h_t} & \downarrow{\scriptstyle \frac{\partial}{\partial t}\Phi_t} & & \downarrow{\scriptstyle 0} \\
\cdots & \longrightarrow & \mathcal{E}_2 \oplus \mathcal{E}_1 \odot \mathcal{E}_1 & \xrightarrow{\delta} & \mathcal{E}_1 & \xrightarrow{\delta} & \mathcal{O}
\end{array}$$

At degree $k > 1$: Assume that h_t satisfies listed above properties on $S_{<k}(\mathcal{E})$. Since $\frac{\partial}{\partial t}\Phi_t$ is a chain map, for all $a \in \mathcal{E}_k$, $\frac{\partial}{\partial t}\Phi_t(a) - h_t \circ \delta(a)$ is $\delta-$closed. Therefore, we extend h_t to $\mathcal{E}_k$ by choosing a preimage of $\Phi_t(\mathcal{E}_k) - h_t \circ \delta(\mathcal{E}_k)$. It remains to check that h_t can be extended as a Φ_t-derivation on $S_k^{\geq 2}(\mathcal{E})$: By evaluation $\frac{\partial}{\partial t}\Phi_t - h_t \circ \delta$ on a monomial of degree k, $a = a_1 \odot a_2 \odot \cdots \odot a_j$ we find

$$(\frac{\partial}{\partial t}\Phi_t - h_t \circ \delta)(a) = \delta(\sum_{m=1}^{j} (-1)^{|a_1|+\cdots+|a_{m-1}|}\Phi_t(a_1) \odot \Phi_t(a_2) \odot \ldots h_t(a_m) \odot \Phi_t(a_j)).$$

Therefore, it is legitimate to extend h_t to $S_k^{\geq 2}(\mathcal{E})$ as a Φ_t-derivation. Now, integrating $\frac{\partial}{\partial t}\Phi_t = \delta \circ h_t + h_t \circ \delta$. over the interval $[0,1]$ we obtain:

$$G \circ F - \mathrm{id} = \delta \circ H + H \circ \delta$$

Remark 9.12. The homotopy H has a non-negative arity by construction.

Having found such homotopy we can make a following quotient construction: Take a quotient of $(S(\mathcal{E}), \delta)$ by $\mathcal{I}\mathcal{E} \oplus S^{\geq 2}(\mathcal{E})$. By construction $\delta(\mathcal{I}\mathcal{E} \oplus S^{\geq 2}(\mathcal{E})) \subset \mathcal{I}\mathcal{E} \oplus S^{\geq 2}(\mathcal{E})$, therefore the quotient is well defined. This complex is isomorphic to $(\mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}/\mathcal{I}, \delta)$, where by abusing the notations we denote the differential of the complex again by δ. Since H is of non-negative arity, it preserves $\mathcal{I}\mathcal{E} \oplus S^{\geq 2}(\mathcal{E})$ as well. Therefore any two such complexes $(\mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}/\mathcal{I}, \delta)$ and $(\mathcal{E}' \otimes_{\mathcal{O}} \mathcal{O}/\mathcal{I}, \delta')$ are homotopy equivalent as chain complexes.

Definition 9.13. *For a Koszul-Tate resolution $(S(\mathcal{E}), \delta)$ of $\mathcal{O}/\mathcal{I}$, a chain complex $(\mathcal{E} \otimes \mathcal{O}/\mathcal{I}, \delta)$ is called a reduced complex of $\mathcal{O}/\mathcal{I}$.*

10 Background on differential graded algebras

We begin with reviewing some material on (co)associative (co)algebras. The tensor products $\otimes, \boxtimes$ are over field $\mathbb{K}$. We reserve $\otimes$ whenever we discuss associative algebras and $\boxtimes$ for coassociative coalgebras. An *associative* algebra is a vector space A over $\mathbb{K}$ equipped with a linear map $\mu : A \otimes A \longrightarrow A$ satisfying $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$. The linear map μ is called multiplication or product. Any $a \in V$ we may view as $1_{\mathbb{K}} \otimes a = a \otimes 1_{\mathbb{K}}$. An associative algebra is *unital*, if there exists a map $u : \mathbb{K} \longrightarrow A$ such that $\mu \circ (u \otimes \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \otimes u)$. This means in particular that $u(1_{\mathbb{K}}) = 1_A$. The associativity can be nicely encoded in the following commutative diagram:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}\otimes\mu} & A \otimes A \\ \downarrow{\scriptstyle \mu\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\quad\mu\quad} & A \end{array}$$

A unital associative algebra A is called *augmented* if it can be decomposed as $A = \mathbb{K}1_A \oplus \bar{A}$. A typical example is the tensor algebra $T(V) = \mathbb{K} \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus \ldots$, where the assotiative product is given by a concatenation: for any $a, b \in \mathbb{N}$

$$\mu : V^{\otimes a} \otimes V^{\otimes b} \longrightarrow V^{\otimes(a+b)}$$
$$v_1 v_2 \ldots v_a \otimes w_1 w_2 \ldots w_b \mapsto v_1 v_2 \ldots v_a w_1 w_2 \ldots w_b$$

Definition 10.1. *A free associative algebra over a vector space V is an associative algebra $F(V)$ together with a linear map $\iota : V \longrightarrow F(V)$, such that for any associative algebra W and a linear map ϕ there exists a unique algebra morphism $\tilde{\phi} : F(V) \longrightarrow W$. This condition can be written as the following commutative diagram:*

$$\begin{array}{ccc} V & \xrightarrow{\iota} & F(V) \\ & {\scriptstyle \phi}\searrow & \downarrow{\scriptstyle \exists!\tilde{\phi}} \\ & & W \end{array}$$

Remark 10.2. *The tensor algebra $T(V)$ is an example of a free associative algebra*

A derivation d is a linear map $d : A \longrightarrow A$ satisfying the Leibniz rule, $d(\mu(a, b)) = \mu(da, b) + \mu(a, db)$. Any linear map $d : V \longrightarrow T(V)$ can be extended to a derivation of $T(V) \longrightarrow T(V)$.

A *coassociative coalgebra* is a $\mathbb{K}$-vector space C together with a linear operation $\Delta : C \longrightarrow C \boxtimes C$ such that $(\Delta \boxtimes \mathrm{id}) \circ \Delta = (\mathrm{id} \boxtimes \Delta) \circ \Delta$. Δ is called a comultiplication of a coproduct. A coassociative coalgebra is *unital* if there exists a map $\epsilon : C \longrightarrow \mathbb{K}$ such that $(\epsilon \boxtimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \boxtimes \epsilon) \circ \Delta$. Like in the case of algebra, a coassociative coalgebra is called *coaugmented* if $C = \mathbb{K}1_C \oplus \bar{C}$.

A typical example of a coalgebra is the tensor coalgebra $T_c(V)$: as a vector space, it is isomorphic to $T(V)$. The coproduct, called deconcatenation, is then given by a map $\Delta \colon V^{\otimes p} \longrightarrow \oplus_{i=0}^{p} V^{\otimes i} \boxtimes V^{\otimes(p-i)}$ for any $p \in \mathbb{N}$, $v_1 \ldots v_p \mapsto 1 \boxtimes v_1 \ldots v_p + v_1 \boxtimes v_2 \ldots v_p + \cdots + v_1 \ldots v_{p-1} \boxtimes v_p + v_1 \ldots v_p \boxtimes 1$. $T_c(V)$ is coaugmented, since it can be written as $T_c(V) = \mathbb{K} \oplus \overline{T}_c(V)$. $\overline{T}_c(V)$ is called the *reduced* tensor coalgebra. The reduced tensor coalgebra coproduct $\bar{\Delta}$, called the *reduced coproduct*, is defined as $\bar{\Delta}(x) = \Delta(x) - 1 \boxtimes x - x \boxtimes 1$. Observe that the reduced coproduct is coassociative, that is $(\bar{\Delta} \boxtimes \mathrm{id}) \circ \bar{\Delta} = (\mathrm{id} \boxtimes \bar{\Delta}) \circ \bar{\Delta}$.

Definition 10.3. *A coderivation $d \colon C \longrightarrow C$ of a coalgebra C is a linear map such that $\Delta \circ d = (d \boxtimes \mathrm{id}) \circ \Delta + (\mathrm{id} \boxtimes d) \circ \Delta$*

Lemma 10.4. *A coderivation $d \colon \overline{T}_c(V) \longrightarrow \overline{T}_c(V)$ is completely determined by its restriction to $\mathrm{p}_1 \circ d \colon \overline{T}_c(V) \longrightarrow V$.*

Proof. Let us write p_n for a projection of $\overline{T}_c(V)$ to its arity n component $V^{\otimes n}$. For the convenience, let us define recursively an n-ary coproduct $\bar{\Delta}^n = \underbrace{\mathrm{id} \boxtimes \mathrm{id} \dots \mathrm{id}}_{n-1 \text{ times}} \boxtimes \bar{\Delta} \circ \bar{\Delta}^{(n-1)}$, with $\bar{\Delta}^1 = \bar{\Delta}$. Observe that due to the coassociativity this definition does not depend on the position of $\bar{\Delta}$ in the tensor product. Furthermore, for any element $v = v_1 \dots v_n$ of $T_c(V)$ of arity n, $\bar{\Delta}^{(n-1)}(v_1 \dots v_n) = v_1 \boxtimes \dots \boxtimes v_n \in \underbrace{T_c(V) \boxtimes \dots \boxtimes T_c(V)}_{n \text{ times}}$. By induction on n one can easily show that

$$\bar{\Delta}^{(n-1)} \circ d(x) = \sum_{i=1}^{n} \underbrace{(\mathrm{id} \boxtimes \dots \boxtimes \mathrm{id} \boxtimes d \otimes \mathrm{id} \boxtimes \dots \boxtimes \mathrm{id})}_{d \text{ is at } i\text{-th position}} \circ \bar{\Delta}^{(n-1)}(x)$$

Then the arity n component $\mathrm{p}_n \circ d(x)$ can be deduced in the following way:

$$\bar{\Delta}^{(n-1)} \circ \mathrm{p}_n \circ d(x) = \sum_{i=1}^{n} \underbrace{(\mathrm{id} \boxtimes \dots \boxtimes \mathrm{id} \boxtimes \mathrm{p}_1 \circ d \boxtimes \mathrm{id} \boxtimes \dots \boxtimes \mathrm{id})}_{d \text{ is at } i\text{-th position}} \circ \bar{\Delta}^{(n-1)}(x)$$

Thus if write x as a deconcatenation into all possible n terms, $x = \sum_{x=\prod_1^n x_{(i)}} x_{(1)} \otimes \dots \otimes x_{(n)}$, $\mathrm{p}_n \circ d(x)$ is given by

$$\mathrm{p}_n \circ d(x) = \sum_{x=\prod_1^n x_{(i)}} \sum_{j=1}^{n} x_{(1)} \otimes \dots \otimes \mathrm{p}_1 \circ d(x_{(j)}) \otimes \dots \otimes x_{(n)} \tag{37}$$

□

10.1 Graded algebras and coalgebras

Definition 10.5. *A graded vector space V is a family of vector spaces $\{V_i, | i \in \mathbb{Z}\}$.*

Any element of $x \in V_n$ is said to have a degree n, denoted by $\deg(x) = n$.

Definition 10.6. *A differential graded vector space V is a chain complex, i.e. it is a collection $\{(V_i, d_i), | i \in \mathbb{Z}\}$ of vector spaces V_i and linear maps $d_i \colon V_i \longrightarrow V_{i-1}$.*

Definition 10.7. *A morphism of degree k between two graded vector spaces V, W is a a family of maps $\{f_n \colon V_n \longrightarrow W_{n+r} | n \in \mathbb{Z}\}$.*

Definition 10.8. *A chain map of degree k of two chain complexes V, W is a linear map $f : V_\bullet \longrightarrow W_{\bullet+k}$ satisfying*

$$d_W \circ f = (-1)^k f \circ d_V.$$

Definition 10.9. *Let f be a morphism of degree k between two differential graded vector spaces V, W. A derivative of f is a morphism of degree $(k-1)$ $\partial(f)$ defined as:*

$$\partial(f) = d_W \circ f - (-1)^k f \circ d_V.$$

In particular, $\partial(f) = 0$ if and only if f is a chain map. Observe that $\partial^2(f) = 0$.

A tensor product $V \otimes W$ of two vector spaces V, W is a graded vector space, whose component of degree n is given by

$$(V \otimes W)_n = \oplus_{i+j=n} V_i \otimes W_j.$$

Whenever we have morphisms of definite degree $f \colon V \longrightarrow V'$ and $g \colon W \longrightarrow W'$, one defines $f \otimes g : V \otimes W \longrightarrow V' \otimes W'$ using the following convention (*the Koszul sign rule*):

$$(f \otimes g)(v \otimes w) = (-1)^{\deg(g)\deg(v)} f(v) \otimes g(w).$$

In order to define a graded symmetric tensor product $\odot$ let us introduce a switching map, which is a linear map $\tau : V \otimes W \longrightarrow W \otimes V$, given for elements of definite degree by $\tau(v \otimes w) = (-1)^{\deg(v)\deg(w)} w \otimes v$ and extended by linearity to the whole $V \otimes W$. The graded vector space $V \odot W$ is then defined as a quotient $V \otimes W/\mathcal{I}$, where $\mathcal{I} = \{v \otimes w - \tau(v \otimes w) \,|v \in V, w \in W\}$.

Let $\mathbb{K}s$, $\mathbb{K}s^{-1}$ be graded vector spaces of dimension 1 spanned by a basis element s of degree 1 and s^{-1} of degree -1 respectively.

Definition 10.10. *A suspension of a graded vector space V is a graded vector space sV defined as $sV := \mathbb{K}s \otimes V$. A desuspension of V, written as $s^{-1}V$, is by definition $\mathbb{K}s^{-1} \otimes V$.*

From this definition it follows that $(sV)_n = V_{n-1}$ and $(s^{-1}V)_n = V_{n+1}$.

Definition 10.11. *A graded associative algebra is a graded vector space V equipped with a linear map $\mu\colon A \otimes A \longrightarrow A$ of degree 0, satisfying $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$.*

Definition 10.12. *A differential graded associative algebra A (or DGA) is graded associative algebra equipped with a differential $d\colon A_\bullet \longrightarrow A_{\bullet-1}$ compatible with the multiplication:*

$$d\mu(a \otimes b) = \mu(d(a) \otimes b) + (-1)^{\deg(a)}\mu(a \otimes db).$$

One has similar definitions for coalgebras in the graded setting:

Definition 10.13. *A graded coassociative coalgebra is a graded vector space C equipped with a degree 0 coassosiative coproduct $\Delta\colon C_p \longrightarrow \oplus_{i+j=p} C_i \boxtimes C_j$ for all $p \in \mathbb{Z}$.*

Definition 10.14. *A differential graded coassociative coalgebra is a a graded coassociative coalgebra equipped with a differential $d\colon C_\bullet \longrightarrow C_{\bullet-1}$ such that d is a coderivation of C, $\Delta \circ d = (\mathrm{id} \boxtimes d \oplus d \boxtimes \mathrm{id}) \circ \Delta$.*

We consider mechanical systems on T^*M with possibly irregular and reducible first class constraints linear in the momenta, which thus correspond to singular foliations on M. According to a recent result, the latter ones have a Lie-infinity algebroid $(\mathcal{M}, Q)$ covering them, where we restrict to the case of Lie-2 algebroids. We propose to consider $T^*\mathcal{M}$ as a potential BFV extended phase space of the constrained system, such that the canonical lift of the nilpotent vector field Q yields automatically a solution to the BFV master equation. We show that in this case, the BFV extension of the Hamiltonian, providing a second corner stone of the BFV formalism, may be obstructed. We identify the corresponding complex governing this second extension problem explicitly (the first extension problem was circumvented by means of the lift of the Lie-2 algebroid structure). We repeatedly come back to the example of angular momenta on $T^*\mathbb{R}^3$: in this procedure, the standard free Hamiltonian does not have a BFV extension—while it does so on $T^*(\mathbb{R}^3\backslash\{0\})$, with a relatively involved ghost contribution singular at the origin.

11 Introduction

One of the main assumptions in textbooks about constrained systems—see, e.g., [HT92]—is that the constraint functions G_a are regular on the unconstrained symplectic manifold. Important existence theorems about BRST-BV [BV84, BV83a] or BFV [BV77, BF83] extensions have then been proven in this setting only—see, however, [FKS14], [Mue17], [BHP] for a notable exception. Physically realistic systems often do not satisfy the regularity conditions. In realistic situations, group actions can have fixed points and there the corresponding constraints are not regular. And in non-abelian Yang-Mills gauge theories on compact spaces, for example, the constraints become non-regular at reducible connections. Other examples from physics where non-regular constraints appear can be found in [HKG10, HT89].

On the other hand, without irreducibility and, in particular, regularity assumptions, often the BV and BFV extensions become hard mathematical problems. Let us illustrate this statement in the finite dimensional setting: Suppose you are given a set v_a of vector fields on a manifold M satisfying

$$[v_a, v_b] = C^c_{ab} v_c \tag{38}$$

for some functions C^c_{ab}. Then the corresponding "BRST differential" Q is readily written down as

$$Q = \xi^a v_a - \tfrac{1}{2} C^a_{bc}\, \xi^b \xi^c \tfrac{\partial}{\partial \xi^a} \tag{39}$$

where ξ^a are the "ghosts", odd coordinates on an extended graded manifold. At first sight, a short calculation, using (38) together with the Jacobi identity for the Lie bracket of vector fields, seems to establish $Q^2 = 0$. However, if the vector fields v_a are not linarly independent at each $x \in M$, there exist functions t^a_I such that

$$t^a_I\, v_a \equiv 0. \tag{40}$$

This happens already for the standard three vector fields L_a generating rotations in $\mathbb{R}^3$ where $x^a L_a \equiv 0$. Then the structure functions C^a_{bc} entering (38) are not unique:

$$C^a_{bc} \sim C^a_{bc} + t^a_I B^I_{bc}. \tag{41}$$

The question if, within this equivalence class, there *exists* a choice of the structure functions C^c_{ab} such that Q as defined in (39) squares to zero is, by the work of Vaintrob [Vai97], equivalent to the question if the singular foliation generated by the vector fields v_a comes from a Lie algebroid covering it.[2] And this question is, as of this day, an unsolved mathematical problem.

In the present paper, we want to consider a Hamiltonian constrained system defined on a cotangent bundle T^*M, with first class [Dir58] constraints G_a linear in the momenta and a Hamiltonian quadratic in them. Linear constraints are of the form

$$G_a(x,p) \equiv v^i_a(x) p_i \,. \tag{42}$$

So we can identify them with the vector fields v_a on M above, since the first class property of the constraints G_a is tantamount to precisely (38). We do not restrict to merely regular constraints, as mostly done in [IS19]—otherwise one deals with regular foliations (on M as well as on T^*M)—but we permit leaves of different dimensions a priori. A quadratic Hamiltonian has the form

$$\mathrm{Ham} = \tfrac{1}{2} g^{ij}(x) p_i p_j. \tag{43}$$

We assume the coefficient matrix g^{ij} to be non-degenerate, thus Ham is the standard Hamiltonian corresponding to a metric g on M. We require the usual compatibility of the constraints with the dynamics,

$$\{G_a, \mathrm{Ham}\} = \omega^b_a \, G_b \tag{44}$$

for some functions ω^b_a on T^*M. This equips the base manifold M with the structure of a singular Riemannian foliation [KS19, IS19, NS22].

We assume here that (40) captures all dependencies of the vector fields and there are no dependencies between the functions t^a_I on M. In more mathematical terms, this means that one has a sequence of vector bundles

$$0 \to F \xrightarrow{t} E \xrightarrow{\rho} TM \,, \tag{45}$$

such that $0 \to \Gamma(F) \to \Gamma(E) \to \mathcal{F} \to 0$ is exact. Here $\mathcal{F}$ denotes the subset of vector fields obtained as the image of ρ, a generating set of which corresponds to the vector fields $v_a = \rho(e_a)$. Likewise, the map t gives rise to the functions t^a_I after the choice of local bases in e_a and b_I in E and F, respectively.

And now the situation is different: Under these assumptions, it has been proven recently [LLS20] that there always exists a homological vector field Q, an extension of (39), which describes a Lie 2-algebroid covering the singular foliation generated by the vector fields v_a.

Note that while (40) yields the reducibility conditions

$$t^a_I G_a \equiv 0, \tag{46}$$

the above does not imply automatically that there are no further reducibilities between the constraints (42). This is because to find all reducibilities, we need to find all independent functions t^a_I satisfying (46) on T^*M—and not just on M.

[2]In the case of the angular momenta L_a, there is a preferred choice of structure functions, namely the constant ones. They can be identified with the structure constants of the Lie algebra $so(3)$. Indeed this example of rotations comes from a simple Lie algebroid, namely the action Lie algebroid $so(3) \times \mathbb{R}^3$.

Consider, e.g., $G_a = \varepsilon_{abc} x_b p_c$, angular momenta.[3] Then besides $x^a G_a \equiv 0$, we have the additional dependency $p_a G_a \equiv 0$. But furthermore, there are dependencies between these dependencies and in the end one finds an infinite set of reducibility conditions leading to an *infinite* tower of ghosts for ghosts in its BFV description. We will illustrate the beginning of this procedure at the end of this article and provide a complete description elsewhere. In principle, having such a complete description of the tower of dependencies between the constraints (42) on T^*M, one can construct a Koszul-Tate resolution, which then guarantees the existence of a BFV extension also in the singular context. However, this description can be very cumbersome for practical purposes apparently and one may be interested in potential shortcuts.

As such the following procedure proposes itself: Let us return to the setting described by (45). As mentioned above, there is a homological vector field Q associated naturally to (45). If we consider its Hamiltonian lift to the cotangent bundle of the graded space underlying Q, we get a function $\mathcal{Q}_{BFV}$ of degree one. Due to $Q^2 \equiv \frac{1}{2}[Q, Q] = 0$, this function satisfies automatically the BFV master equation

$$(\mathcal{Q}_{BFV}, \mathcal{Q}_{BFV})_{BFV} = 0. \tag{47}$$

Also it satisfies some other necessary requirements posed on the BFV charge, like the correct appearance of the constraints (42) and the redundancies (40) at the lowest orders in the ghosts. We may thus consider regarding $\mathcal{Q}_{BFV}$ as the BFV charge of our constrained system.

A price to pay is that now, in general, it is no more guaranteed that the $\mathcal{Q}_{BFV}$-cohomology describes the functions on the (possibly singular) reduced phase space correctly. There is another famous example, however, where such a deficiency is normally disregarded: The AKSZ formalism [ASZK97, CF01, Roy07] yields a BV extension of the Poisson sigma model or the Chern-Simons gauge theory likewise in a much faster way than when following the usual extension procedure. If one compares its cohomology at degree zero with functions on the solutions space to the Euler-Lagrange equations modulo gauge symmetries, however, the cohomology one finds in the AKSZ framework is in general too large.[4]

We describe the construction of the BFV phase space obtained in this way in more detail in the subsequent section. However, there is—in contrast to the Lagrangian BV formalism—one more essential ingredient of the cohomological approach on the Hamiltonian level: One needs an extension $\mathcal{H}_{BFV}$ of the Hamiltonian (43) such that

$$(\mathcal{Q}_{BFV}, \mathcal{H}_{BFV})_{BFV} = 0\,. \tag{48}$$

Equations (47) and (48) are the two indispensable corner stones of the BFV formalism. Under the given assumptions underlying the shortcut for the construction of $\mathcal{Q}_{BFV}$, the existence of the extension of the Hamiltonian is in general not guaranteed. Its potential obstructions is the main subject of the present article.

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[3]On the physical side, implementing the angular momentum as a constraint means the following: If either $\vec{x}$ or $\vec{p}$ are non-zero, one requires them to be colinear. And then different directions are to be identified. Thus, effectively, the system is reduced to a particle on the line. However, at the spatial origin special things happen and the reduced phase space is not regular there.

[4]To see this explicitly, consider a $U(1)$-Chern-Simons theory on a connected 3-manifold Σ where the space of solutions to the Euler-Lagrange equations consists of closed 1-forms and gauge symmetries correspond to adding exact 1-forms to them, thus leading to simply $H^1_{dR}(\Sigma)$. In the AKSZ formalism, on the other hand, one has, in addition to the ghosts c and the classical fields A—0-forms and 1-forms, respectively—their antifields $A^\dagger$ and $c^\dagger$, which are 2-forms and 3-forms, respectively. The BV-BRST differential s acts as follows on them: $sc = 0$, $sA = \mathrm{d}c$, $sA^\dagger = \mathrm{d}A$, and $sc^\dagger = \mathrm{d}A^\dagger$. From this one concludes that $\oint_\gamma A$, the integral of A around a closed loop γ, is s-closed; deforming γ homotopically changes the result by something s-exact on the other hand (using Stokes' theorem). This generates thus precisely $H^1_{dR}(\Sigma)$. But in addition we also have $c(\sigma_0) \oint_S A^\dagger$ which is easily seen to be s-closed. Here σ_0 is a chosen point and S a chosen closed 2-surface in Σ. Homotopic changes of σ_0 and S again do not change the s-cohomology class. Thus, we also find generators of $H^2_{dR}(\Sigma)$ inside $H^0(s)$, thus doubling the classical cohomology incorrectly. In this simple case of an abelian Chern-Simons theory, the discrepancy can be, in the Hamiltonian framework, traced back to a global missing ghosts-for-ghost which would take care of the fact that the integral over the constraint $\mathrm{d}A(\sigma)$ is identically zero.

12 BFV phase space and charge

Much as (39) defines a Lie algebroid on E [Vai97], every homological vector field Q of degree 1 on

$$\mathcal{M} := E[1] \oplus F[2], \tag{49}$$

which is necessarily of the form

$$\begin{aligned} Q &= \xi^a \rho_a^i \frac{\partial}{\partial x^i} - \left(\eta^I t_I^a + \tfrac{1}{2}\xi^b\xi^c C_{bc}^a\right)\frac{\partial}{\partial \xi^a} \\ &\quad + \left(\tfrac{1}{6}\xi^a\xi^b\xi^c h_{abc}^I - \Gamma_{Ja}^I \eta^J \xi^a\right)\frac{\partial}{\partial \eta^I}\,, \end{aligned} \tag{50}$$

defines a Lie 2-algebroid structure on the vector bundle $E \oplus F$. Here E and F are the vector bundles appearing in (45) and the numbers in the brackets denote the degrees carried by the local fiber-linear coordinates ξ^a and η^I on E and F, respectively, in the graded description of the Lie 2-algebroid.

The coefficient functions entering (50) all have some algebraic and/or geometrical meaning: One way of viewing them is that the terms linear in (ξ, η) give rise to a complex (45)—which in our case we require, in addition, to be exact on the level of sections—, those quadratic in them are 2-brackets, which are not tensorial but satisfy a Leibniz rule, and the cubic one, which is tensorial, is a 3-bracket, which, if non-zero, reflects the fact that the 2-bracket between sections of E then does not satisfy the Jacobi identity, giving rise to a Lie 2-algebra. For some introduction to Lie 2-algebras see [BC04], for the more general L_∞-algebras see [LM94, LS93]. Alternatively, we may view, e.g., the coefficients Γ_{Ja}^I as a local expression for an E-covariant derivative on F. The tensor $h \in \Gamma(F \otimes \Lambda^3 E^*)$ then satisfies ${}^E\mathrm{D}h = 0$, where ${}^E\mathrm{D}$ denotes the corresponding exterior covariant E-derivative.

There is a vast literature on the geometry of Lie algebroids, see, e.g., [Mac87, DSW99, MM03]. For further details about the geometry of Lie 2-algebroids see [GS14, JL20]. For a general Lie 2-algebroid, the complex (45) is not exact (also not on the level of sections). Under the condition of exactness of the sequence on the level sections, thus providing a resolution of the $C^\infty(M)$-module of the vector fields generated by v_a, i.e. by the image of $\rho\colon E \to TM$, the existence of a Lie 2-algebroid structure extending a given sequence (45) was proven in [LLS20]. As mentioned, this is tantamount to the existence of a vector field (50) on the corresponding $\mathcal{M}$ of (49) which squares to zero.

Given these data, we now construct the BFV phase space as the cotangent bundle $T^*\mathcal{M}$ of (49). This extends the classical variables $(x^i, p_i) \in T^*M$ by a ghost ξ^a (odd and of degree one) for each of the constraints and a ghost-for-ghosts η^I (even and of degree two) for each of the dependencies (46). Both of the latter two ghost families are accompanied by their momenta of opposite degrees. All this comes together with the BFV symplectic form

$$\omega_{BFV} = \mathrm{d}x^i \wedge \mathrm{d}p_i + \mathrm{d}\xi^a \wedge \mathrm{d}\pi_a + \mathrm{d}\eta^I \wedge \mathrm{d}\mathcal{P}_I\,. \tag{51}$$

It is of degree zero and thus so also the graded Poisson bracket $(\cdot,\cdot)_{BFV}$ it induces, which takes the following form:

$$\{p_i, x^j\} = \delta_i^j, \;\; \{\pi_a, \xi^b\} = \delta_a^b, \;\; \{\mathcal{P}_I, \eta^J\} = \delta_I^J.$$

Regarding the vector field (50) as a function on the BFV phase space $T^*\mathcal{M}$ (by replacing the derivatives simply by the corresponding momenta), one has

$$\begin{aligned} \mathcal{Q}_{BFV} &= \left(\xi^a \rho_a^i\right) p_i - \left(\eta^I t_I^a + \tfrac{1}{2}\xi^b\xi^c C_{bc}^a\right)\pi_a \\ &\quad + \left(\tfrac{1}{6}\xi^a\xi^b\xi^c h_{abc}^I - \Gamma_{Ja}^I \eta^J \xi^a\right)\mathcal{P}_I\,, \end{aligned} \tag{52}$$

where we use the notation $\rho_a^i \equiv v_a^i$. We notice that, as required by the standard BFV procedure, it indeed extends $\xi^a G_a$ by terms such that (47) holds true.

The classical constrained system on T^*M considered here is already uniquely determined by the underlying singular foliation. This is in contrast to the coefficient functions entering (52): they are constrained by $Q^2 = 0$, but, at each step of an extension, they are not unique. This should be also reflected on the BFV level: Indeed, e.g. a change (41) of the almost Lie bracket on E can be obtained by lifting the degree preserving diffeomorphism

$$\eta^I \mapsto \eta^I + \tfrac{1}{2} B^I_{ab} \xi^a \xi^b \tag{53}$$

from $\mathcal{M}$ to $T^*\mathcal{M}$. Such a change of the structure functions C^a_{bc} does not come for free: also other quantities entering $\mathcal{Q}_{BFV}$ are then changed correspondingly. E.g., h receives an additive contribution by the E-covariant exterior derivative of $B \in \Gamma(F \otimes \Lambda^2 E^*)$, $h \mapsto h + {}^E\mathrm{D}B + \ldots$, where the dots denote terms quadratic in B (see [GS14]).

13 Geometrical interpretation of $\mathcal{H}_{BFV}$

We first observe that the BFV bracket decreases the polynomial degree pol of momenta $(p_i, \pi_a, \mathcal{P}_I)$ by one. Since, in addition, both $\mathcal{Q}_{BFV}$ and Ham are homogeneous with respect to pol, we may assume always that the extension $\mathcal{H}_{BFV} = \mathrm{Ham} + \ldots$ satisfies $\mathrm{pol}(\mathcal{H}_{BFV}) = 2$. Identifying momenta with vector fields, we thus may view $\mathcal{H}_{BFV}$ as a graded-symmetric 2-vector field on (49),

$$H \equiv H_{BFV} \in \Gamma(S^2 T\mathcal{M}) \tag{54}$$

and re-interpret the second master equation (48) as

$$\mathcal{L}_Q H = 0 \tag{55}$$

where $Q = Q_{Lie2} \in \Gamma(T\mathcal{M})$. For later use, we remark that a basis of $\Gamma(S^2 T\mathcal{M})$ is spanned by

$$\frac{\partial}{\partial \eta^I}\frac{\partial}{\partial \eta^J},\ \frac{\partial}{\partial \eta^I}\frac{\partial}{\partial \xi^a},\ \frac{\partial}{\partial \eta^I}\frac{\partial}{\partial x^i},\ \frac{\partial}{\partial \xi^a}\frac{\partial}{\partial \xi^b},\ \frac{\partial}{\partial \xi^a}\frac{\partial}{\partial x^i},\ \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} \tag{56}$$

where a graded symmetrization is understood and the degrees are -4, -3, -2, -2, -1, and 0, respectively. H being of degree zero, one needs coefficient functions on $\mathcal{M}$ of the respective opposite degrees when using this basis. A graded manifold equipped with a degree 1 vector field is called a Q-manifold [Sch93]; if there is, in addition, (54) of degree 0 satisfying (55), we call it an HQ-manifold.

Already an H-structure alone, that is a degree zero tensor (54) without any compatibility conditions, contains interesting geometrical data, at least if the coefficients of the very last term in (56) form a non-degenerate matrix—in which case we then call the H-structure non-degenerate. It then defines a metric g on M (subject to no further conditions). But this is not all: Consider the terms linear in $p_i \sim \partial_i$. Due to the non-degeneracy of g^{ij}, we may absorb all such terms by completing the square with a redefinition of the momenta $p_i \to p_i^\nabla$, where

$$p_i^\nabla := p_i - \Gamma^a_{bi} \xi^b \pi_a - (\Gamma^I_{Ji} \eta^J - \tfrac{1}{2} \gamma^I_{abi} \xi^a \xi^b) \mathcal{P}_I \,. \tag{57}$$

Then the 2-tensor (54), rewritten as a function $\mathcal{H}$ in $C^\infty(T^*\mathcal{M})$, takes the form

$$\mathcal{H} = \tfrac{1}{2} g^{ij} p_i^\nabla p_j^\nabla + \tfrac{1}{2} \Sigma^{ab}_{(2)} \pi_a \pi_b + \Sigma^{aI}_{(3)} \pi_a \mathcal{P}_I + \tfrac{1}{2} \Sigma^{IJ}_{(4)} \mathcal{P}_I \mathcal{P}_J \tag{58}$$

where $\Sigma^{\cdot\cdot}_{(k)}$ are functions on $\mathcal{M}$ of degree k. Thus, for example,

$$\Sigma^{ab}_{(2)} = \tfrac{1}{2} \Sigma^{ab}_{cd} \xi^c \xi^d + \Sigma^{ab}_I \eta^I \tag{59}$$

for appropriate tensors Σ^{ab}_{cd} and Σ^{ab}_I. The coefficients in (57) have a geometrical interpretation: Consider a coordinate transformation $\xi^a \mapsto M^a_b(x) \xi^b$ (all other coordinates unchanged). Lifting it to $T^*\mathcal{M}$ so as to leave (51) invariant—or, equivalently, considering its tangent map—yields $\pi_a \mapsto (M^{-1})^b_a \pi_b$ together with $p_i \mapsto p_i + (M^{-1})^a_c M^c_{b,i}\xi^b\pi_a$ (all other momenta unchanged). Requiring (57) to remain form-invariant under such transformations shows that $\Gamma^a_b = \Gamma^a_{bi}\mathrm{d}x^i$ are the local 1-forms representing a connection ∇ on E. Similarly, $\Gamma^I_J = \Gamma^I_{Ji}\mathrm{d}x^i$ corresponds to a covariant derivative ∇ on F. $\gamma \in \Gamma(F \otimes \Lambda^2 E^* \otimes T^*M)$ is needed, finally, to also establish form covariance of (57) w.r.t. (53); it is part of a connection in a graded sense.

An almost Q-structure (a degree one vector field Q which does not necessarily square to zero) equips the graded manifold (49) with the structure of a (not necessarily exact) sequence of vector bundles as in (45) together with 2-brackets and one 3-bracket. A non-degenerate H-structure equips it in the lowest order with a metric g on the base and in the next order—the terms linear in the momenta p_i (see also (57))—with a connection defined on all of (45) (note that we can always equip TM with the canonical Levi-Civita connection of g, moreover).

Given an almost HQ-structure, where we have both, an almost Q-structure Q and an H as above, (55) determines compatibility conditions. To obtain tensorial formulas for these on the nose, it is useful to re-express (52) in terms of the variables (57). Then, e.g., the new coefficient of the term quadratic in ξ is $-C^a_{bc} + \rho^i_b\Gamma^a_{ci} - \rho^i_c\Gamma^a_{bi} =: {}^E T^a_{bc}$ and has the geometric interpretation of what is called an E-torsion ${}^E T \in \Gamma(E \otimes \Lambda^2 E^*)$ [KS19], generalizing the ordinary torsion on a tangent bundle to E. The bracket of two covariant vector fields (57) yields the curvatures of ∇, both on E and F (as well as a contribution proportional to $\mathrm{D}\gamma$). To lowest orders we find:

$$\begin{aligned}\mathcal{L}_Q H &= \tfrac{1}{2}\xi^a \left({}^E\nabla_{e_a} g\right)^{ij} \partial^\nabla_i \partial^\nabla_j \\ &\quad -\tfrac{1}{2}\xi^b\xi^c \left(S^a_{bci} - t^a_I \gamma^I_{bci} - \Sigma^{da}_{bc}\rho^k_d g_{ki}\right) g^{ij}\partial^\nabla_j \tfrac{\partial}{\partial\xi^a} \\ &\quad + b^I \left(t^a_{I;j} g^{ji} + \Sigma^{ba}_I \rho^i_b\right) \partial^\nabla_i \tfrac{\partial}{\partial\xi^a} + \ldots\,. \end{aligned} \tag{60}$$

Here ∂^∇_i is the vector field corresponding to (57) and ${}^E\nabla$ denotes the E-covariant derivative acting on TM according to ${}^E\nabla_s v = [\rho(s), v] + \rho(\nabla_v s)$, where $s \in \Gamma(E)$ and $v \in \Gamma(TM)$. $\Sigma^{ab}_{(2)}$ has been decomposed according to (59), a semicolon denotes a covariant derivative, and we introduced the abbreviation

$$S^a_{bci} = {}^E T^a_{bc;i} + \rho^j_b R^a_{cji} - \rho^j_c R^a_{bji}. \tag{61}$$

The tensor $S \in \Gamma(E \otimes \Lambda^2 E^* \otimes T^*M)$ measures the compatibility of the 2-bracket and the connection ∇ on E, see [KS19]. As for Jacobi identities of the brackets in a higher Lie algebroid, also here we do not expect (55) to ensure S to vanish on the nose, but to instead do so up to some appropriate boundary term only; what this means precisely is subject of the next section.

Let us mention that the current method is efficient in obtaining Bianchi type of identites for quantities such as S. $Q^2 = 0$, which we have when Q comes from a Lie 2-algebroid, implies that the application of another $\mathcal{L}_Q$ to (60) vanishes identically. From this one can read off, for example, that (61) satisfies

$${}^E\mathrm{D}S - \nabla\langle t, h\rangle_F = 0\,. \tag{62}$$

Here ${}^E\mathrm{D}$ denotes the E-exterior covariant derivative associated naturally to ${}^E\nabla$: on TM it acts as specified above, on E according to ${}^E\nabla_s s' = [s, s']_E + \nabla_{\rho(s')} s$ for every $s, s' \in \Gamma(E)$, and on elements in $\Lambda^\bullet E^*$ by a straightforward generalization of the Cartan formula for the de Rahm differential ([GS14, KS19]). The identity (62)—specialized to a Lie algebroid, where $h = 0$—was of essential importance in the construction of the BV-extension [IS21] and the above derivation constitutes a significant simplification of the one provided there.

14 Complex governing the extension

In the construction of (52), the sequence of vector bundles (45) and its exactness on the level of sections plays a crucial role. There is a similar sequence which governs the extension problem

(43),(44),(48), but, despite being constructed out of the previous one, this one is in general no longer exact on the level of sections.

Let us denote the complex (45) by $\mathcal{E}^\bullet$ (so, in particular, $\mathcal{E}^0 = TM$, $\mathcal{E}^{-1} = E$, and $\mathcal{E}^{-2} = F$) and tensor it with itself shifted to the right, $\mathcal{F}^\bullet := \mathcal{E}^\bullet \otimes \mathcal{E}^\bullet[-1]$, where, by definition, $\mathcal{E}^i[-1] = \mathcal{E}^{i-1}$. At degree zero, e.g., one has $\mathcal{F}^0 = TM \otimes E \oplus E \otimes TM$, where in the first term both factors carry degree 0, while in the second one, elements in E enter with degree -1 and those in TM with degree $+1$, again adding up to 0.

By a standard construction (see, e.g., [Eis95]), $\mathcal{F}^\bullet$ is again a complex. It is, however, not yet the sequence $\mathcal{G}^\bullet$ we are interested in:

$$\mathcal{G}^\bullet := 0 \xrightarrow{\delta} S^2F \xrightarrow{\delta} F \otimes E \xrightarrow{\delta} F \otimes TM \oplus \Lambda^2 E \xrightarrow{\delta} E \otimes TM \xrightarrow{\delta} S^2 TM\,, \tag{63}$$

where the degrees are such that $\mathcal{G}^0 = S^2TM$, $\mathcal{G}^{-1} = E \otimes TM$, etc. Now one observes that $\mathcal{G}^\bullet \hookrightarrow \mathcal{F}^\bullet[1]$ (essentially it is embedded as a graded symmetrization). For example, typical elements $\Phi \in \mathcal{F}^0$ and $\varphi \in \mathcal{G}^{-1}$ are of the form $\Phi = \Phi^{ia}\,\partial_i \otimes e_a + \bar{\Phi}^{ai}\, e_a \otimes \partial_i$ and $\varphi = \varphi^{ai}\, e_a \otimes \partial_i$, respectively, and then $\mathcal{G}^{-1}$ is embedded into $\mathcal{F}^0$ by the diagonal map, $\Phi^{ia} := \varphi^{ai}$, $\bar{\Phi}^{ai} := \varphi^{ai}$. The codifferential δ is easily identified with

$$\delta := \rho^i_a p_i \tfrac{\partial}{\partial \pi_a} + t^a_I \pi_a \tfrac{\partial}{\partial \mathcal{P}_I} \tag{64}$$

when replacing the vector fields (56), which provide a basis for $\mathcal{G}^\bullet$, by the corresponding momenta.

In general, for the cohomology of a tensor product of two complexes there is a Künneth formula. It says that even if the cohomology of each of the complexes is trivial, which is the case here, there can still be a non-zero contribution called “torsion”. Below we will provide an example that, in general, $\mathcal{G}^\bullet$ is not exact, even not on the level of sections.

It is remarkable that (64) also generates the coboundary operator of the complex (45). In fact, if all the independent relations between the constraints $G_a = \rho^i_a p_i$ on T^*M are provided by the functions $t^a_I(x)$ (these are all such dependencies that depend on x only, but there could be further ones that also depend on momenta p in principle), then the operator (64) acting on $C^\infty(T^*\mathcal{M})$ would be the Koszul-Tate differential [HT92]. And in that case, it would have no cohomology on $C^\infty(T^*\mathcal{M})$ and neither so when acting on (63), which is the restriction of $(C^\infty(T^*\mathcal{M}), \delta)$ to functions quadratic in the momenta—such as the complex (45) can be identified with its restriction to the subcomplex linear in the momenta. But in general, the true Koszul-Tate complex is much bigger, as we will illustrate by means of an example at the end of this article.

The main point here is, however, that it is precisely the cohomology of (63) that governs the extendability problem $\mathcal{H}_{BFV} = \mathrm{Ham} + \ldots$ satisfying (48) for $\mathcal{Q}_{BFV}$ as in (52).

If $\Gamma(\mathcal{G}^\bullet)$ is exact, the existence of the BFV extension H_{BFV} is guaranteed. This follows from a standard consideration: $(\mathcal{Q}_{BFV}, \cdot) = \delta + X_{rest}$. At each step when adding terms from the right to the left in (63), one finds an expression that is already δ-closed. Now, δ having no cohomology, one can always add a δ-exact contribution from the next level so as to cancel it. In general, the resulting expression for H_{BFV} will contain all possible terms compatible with degrees, as is also the case for (52); together they then define an HQ-structure on (49).

The converse is certainly not true: The complex $\Gamma(\mathcal{G}^\bullet)$ can have non-trivial cohomology, but the extension problem for a particular Hamiltonian (43) may still lead to trivial cohomology classes and thus be unobstructed.

The absence of a cohomology also leads to remarkable, purely geometrical formulas. To illustrate this, assume that

$$H^{-1}(\Gamma(\mathcal{G}^\bullet), \delta) = 0. \tag{65}$$

Then for every $\varphi \in \Gamma(\mathcal{G}^{-1})$ such that $\delta\varphi = 0$, there is some $\psi \equiv \Psi^{Ii} b_I \otimes \partial_i + \frac{1}{2} M^{ab} e_a \wedge e_b \in \Gamma(\mathcal{G}^{-2})$ such that $\varphi = \delta\psi$. Concretely,

$$\rho^i_a\, \varphi^{aj} + \rho^j_a\, \varphi^{ai} = 0 \quad \Rightarrow \quad \varphi^{ai} = t^a_I \Psi^{Ii} + \rho^i_b M^{ab}\,, \tag{66}$$

where $M^{ab}(x) = -M^{ba}(x)$. The exactness at a given degree is preserved if one tensors a complex with a fixed $C^\infty(M)$-module; this permits adding spectator indices to the quantities in (66).

As a first application, let us return to the initial compatibility condition (44). It can be shown [NS22] that (44) holds true in arbitrary local patches iff there exists a connection ∇ on E such that its induced E-connection annihilates the metric, ${}^E\nabla g = 0$. If Γ^b_{ai} denote the connection coefficients of ∇, we can choose $\omega^b_a := \Gamma^b_{ai}(x)g^{ij}p_j$. The difference between two connections is a tensor field and then (44) implies that this tensor is δ-closed. This provides the following equivalence for the choice of the connection coefficients

$$\omega^{ai}_b \sim \omega^{ai}_b + t^a_I \Psi^{Ii}_b + \rho^i_c M^{ac}_b, \tag{67}$$

where $M^{ac}_b = -M^{ca}_b$, corresponding to δ-exact contributions. If (65) holds true, these are *all* the ambiguities in the choice of the connection on E such that ${}^E\nabla g = 0$.

Even more remarkable is the following decomposition that one finds for the tensor S defined in (61). Using a connection ∇ such that ${}^E\nabla g = 0$, the geometrical quantity S is always δ-closed and therefore, assuming (65), it can be decomposed into some $\gamma \in \Gamma(F \otimes \Lambda^2 E^* \otimes T^*M)$ and $\Sigma \in \Gamma(\Lambda^2 E \otimes \Lambda^2 E^*)$ as follows:

$$S^a_{bci} = t^a_I \gamma^I_{bci} + \rho^j_d g_{ij} \Sigma^{da}_{bc}\,. \tag{68}$$

There are, in general, no explicit expressions for these tensors. In fact, they are even not uniquely determined: We can change γ and Σ simultaneously by

$$\begin{aligned} \Sigma^{ab}_{cd} &\mapsto \Sigma^{ab}_{cd} + t^a_I\, V^{Ib}_{cd} - t^b_I\, V^{Ia}_{cd} \\ \gamma^I_{abi} &\mapsto \gamma^I_{abi} + g_{ij}\rho^j_c\, V^{Ic}_{ab} \end{aligned} \tag{69}$$

for any $V \in \Gamma(F \otimes E \otimes \Lambda^2 E^*)$ without changing S—and these are all such ambiguities, if also $H^{-2}_\delta(\Gamma(\mathcal{G})) = 0$.

Finally, the identity $\{t^a_I G_a, H\}$ yields

$$t^a_{I;i} g^{ij} \rho^k_a\, p_j p_k = 0\,. \tag{70}$$

Thus, $t^{a;i}_I \equiv t^a_{I;j} g^{ij}$ satisfies the condition of (66) and, if (65) holds true, there exist tensors $\tilde\Psi$ and $\tilde M$ such that

$$t^{a;i}_I = t^a_J \tilde\Psi^{Ji}_I + \rho^i_b \tilde M^{ab}_I\,. \tag{71}$$

These considerations can be considered as holding true on the purely geometrical level, without any application to physical models.

On the other hand, as explained above, the δ-cohomology governs the physical extension problem here. For example, the second line in (60) says that S needs to be δ-exact (and, by a general feature of the procedure, also shows implicitly that it is already δ-closed), $S = \delta(\gamma + \Sigma)$, where γ and Σ are the quantities entering $\mathcal{H}$, see (57), (58), and (59). In deriving (68) we only demanded S to be δ-exact with respect to some tensors, but in a slight abuse of notation we denoted them already by γ and Σ, the quantities we need to choose for the expansion (58). We have not done so in (71), which can be rewritten as $\nabla t = \delta(\tilde\Psi + \tilde M)$ and one may wonder where the analogue of the first term on its right hand side is in comparison to the last line in (60), while evidently $\tilde M^{ab}_I$ can be identified with Σ^{ab}_I. In fact, this depends on the connection on F: if one uses a different connection than the one appearing in (57), then their difference is a tensor, which can be identified with $\tilde\Psi^{Ji}_I$. So also the last line in (60) says that ∇t should be δ-exact.

15 Examples with $t = 0$

The prototype of a singular constraint $T^*\mathbb{R} \cong \mathbb{R}^2$ is $G = xp$. It is singular at the origin, the constraint surface has the shape of a cross. The corresponding singular foliation on $M = \mathbb{R}$ is generated by $v = x\partial_x$ or, by putting $E = \mathbb{R} \times M$ with an anchor such that $\rho(1) = v$. The bundle F then has rank zero and the exact sequence is of length one only. The corresponding BFV charge is simply $\mathcal{Q}_{BFV} = \xi x p$.

There is no compatible non-degenerate Hamiltonian coming from a metric g in this case, since $\mathbb{R}$ with a non-regular leaf structure does not admit a singular Riemannian foliation. However, if we start with a Hamiltonian $\mathrm{Ham} = \frac{1}{2}h(x)p^2$, dropping the condition that $h = 1/g$, then a BFV extension exists as long as h vanishes at least quadratically near the origin; it then takes the form: $\mathcal{H}_{BFV} = \frac{1}{2}h(x)p^2 + x^2\left(\frac{h}{x^2}\right)'\xi\pi p$.

More generally, if $t = 0$, the anchor map $\rho\colon E \to TM$ needs to be injective on a dense open subset of M: this follows from injectivity of the induced map from $\Gamma(E)$ to $\Gamma(TM)$ when $t = 0$ (see the exactness condition expressed in the sentence containing (45)). And if one insists on a non-degenerate metric g satisfying (44), then $t = 0$ implies injectivity of the anchor map $\rho\colon E \to TM$ everywhere. This in turn implies that the foliation on M and the constraints (42) on T^*M are regular.

Regular foliations can still yield geometrically interesting examples. To illustrate this, consider even $E = TM$, $\rho = \mathrm{id}$. While the BFV charge is very simple in this case, $\mathcal{Q}_{BFV} = \xi^i p_i$, non-trivial geometry remains in the BFV extension of the Hamiltonian: To satisfy the compatibility (44), we need a connection ∇ on TM such that

$$g_{ij;k} + T_{ijk} + T_{jik} = 0, \tag{72}$$

where $T_{ijk} = g_{il}T^l_{jk}$; here T^l_{jk} denote the components of the torsion tensor. The Levi-Civita connection satisfies this condition evidently. The ambiguity (67) translates into the freedom of the choice of the torsion of the connection, but maintaining (72) instead of metricity. One then finds

$$\mathcal{H}_{BFV} = \tfrac{1}{2}g^{ij}p_i^\nabla p_j^\nabla + \tfrac{1}{2}R^k{}_{ij}{}^l\xi^i\xi^j\pi_k\pi_l + \tfrac{1}{4}T^k{}_{ij}{}^{;l}\xi^i\xi^j\pi_k\pi_l \tag{73}$$

where $R^i{}_{jkl}$ denote the components of the Riemann tensor. If we choose the Levi-Civita connection for $\Gamma^i{}_{jk}$ in $p_i^\nabla = p_i + \Gamma^j{}_{ki}\xi^k\pi_j$, then (72) is satisfied identically and the last term in (73) disappears, but one is still left with the curvature term.

Inspection of (61) shows that the last two terms in (73) combine into the tensor S for $E = TM$. In [IS19], such a BFV extension was provided under the assumption that S vanishes. The extension (73) of the Hamiltonian generalizes this result, with the interesting contributions of curvature and torsion which combe into the geometrical quantity S, which, for $E = TM$, in turn can be identified also with the curvature of a dual connection, see [KS19].

16 Angular Momentum I: Example of an obstruction

Let us now consider the phase space $T^*\mathbb{R}^3$ with constrained angular momentum, $G_a = \varepsilon_{abc}x^b p^c$ or $\vec{G} = \vec{x} \times \vec{p}$. Then one has

$$\mathcal{Q}_{BFV} = \vec{\xi}\cdot(\vec{x}\times\vec{p}) + \tfrac{1}{2}(\vec{\xi}\times\vec{\xi})\cdot\vec{\pi} - \eta\,\vec{x}\cdot\vec{\pi}\,. \tag{74}$$

The second term is the (cotangent lift of the) Chevalley-Eilenberg differential of the Lie algebra $so(3)$, the first two terms the Chevalley-Eilenberg differential for $so(3)$ acting on $\mathbb{R}^3$, which then corresponds to the BRST charge (39) of the action Lie algebroid $E = so(3) \times \mathbb{R}^3$. The last term takes care of the dependency $\vec{x}\cdot\vec{G} \equiv 0$ of the constraints. It is easy to verify that the expression (74) satisfies the master equation (47).

In this example, $M = \mathbb{R}^3$, $E = \mathbb{R}^3 \times M$, $F = \mathbb{R} \times M$, and the maps ρ and t in (45) can be identified with the sections $\rho = \varepsilon_{abc} x^b e^a \otimes \frac{\partial}{\partial x^c}$ and

$$t = x^a e_a \otimes b^* \,, \tag{75}$$

respectively; here $(e^a)_{a=1}^3$ denotes a basis in E^* and b^* a basis in F^*. The kernel of ρ is one-dimensional outside the origin, while it is all of $E_{\vec{x}=0} = \mathbb{R}^3$ at the origin $0 \in M$. The map t spans the radial vectors in Kerρ for all $\vec{x} \neq 0$, but, for continuity reasons, vanishes at 0. Thus, the complex (45) has no cohomology outside the origin, but $H^{-1}_{\vec{x}=0}(\mathcal{E}, \delta) = \mathbb{R}^3$. And still—again for continuity reasons (every radial vector field has to vanish at the origin)—it is exact on the level of sections: $H^\bullet(\Gamma(\mathcal{E}^\bullet), \delta) = 0$. Correspondingly, the extension problem for the BFV charge (74) when taking into account (only) the x-dependent dependencies of the constraints, has not been obstructed.

Let us now turn to the extension problem of the Hamiltonian (43), the main subject of this paper, for the standard metric on $M = \mathbb{R}^3$,

$$\mathrm{Ham} = \tfrac{1}{2} \vec{p} \cdot \vec{p} \,. \tag{76}$$

For this purpose we first equip the bundles E and F with their canonical flat connections (for what concerns E, this is also motivated by the fact that (44) is satisfied with $\omega_a^b = 0$). Then one has

$$\nabla t = \mathrm{d}x^a \otimes e_a \otimes b^* \stackrel{\sim}{=} \tfrac{\partial}{\partial x^a} \otimes e_a \otimes b^* \,, \tag{77}$$

where in the second equality we used the standard metric of M for the identification, then yielding an element in $\Gamma(E \otimes TM) \otimes_{C^\infty(M)} \Gamma(F^*)$. It is easy to see that $\delta(\nabla t) = 0$ which is equivalent to (70): applying ρ to e_a and symmetrizing over the two ensuing entries in TM gives zero due to the contraction with the ε-tensor.

Now the main observation of this short section: ∇t cannot be δ exact, i.e. it cannot be of the form (71), since both t and ρ vanish at the origin. This shows, on the one hand, that here

$$H^{-1}(\Gamma(\mathcal{G}^\bullet), \delta) \neq 0, \tag{78}$$

and, on the other hand, that the BFV extension of (76) is indeed obstructed: As we learn from the last line in (60), we *need* ∇t to be exact for the BFV-extension of the Hamiltonian within the present framework.

Note also that any other choice of connections on E and F would not help, since their contribution to ∇t vanish at $x = 0$ as well. In fact, the cohomology class of ∇t does not depend on such choices.

17 Examples with Lie-2 gauge symmetry

As a technically more involved example, we truncate the expansion for the BFV extension of the Hamiltonian (while keeping the general form for $\mathcal{Q}_{BFV}$ as in (52)): Putting Σ_3, Σ_4, and Σ_I^{ab} to zero in (58), one obtains[5]

$$\mathcal{H}_{BFV} = \tfrac{1}{2} g^{ij} p_i^\nabla p_j^\nabla + \tfrac{1}{4} \Sigma_{cd}^{ab} \xi^c \xi^d \pi_a^\nabla \pi_b^\nabla \,, \tag{79}$$

where p_i^∇ is given by (57) and $\pi_a^\nabla = \pi_a - \lambda_{ab}^I \xi^b \mathcal{P}_I$ for some $\lambda \in \Gamma(\Lambda^2 E^* \otimes F)$. This extension of π_a to π_a^∇ has the effect of making (79) covariant with respect to both, changes of frames as

[5]This example is an improvement of a result of [IS19] obtained in the context of Cartan-Lie algebroids, where the Σ-term in (79) is absent. During the preparation of the present work, we were informed by N. Ikeda that he found a similar extension in the context of Courant algebroids, see [Ike21].

well as (53), where then $\lambda \mapsto \lambda + B$. Now we learn from (60) that, under these assumptions, *necessary* conditions for the validity of (48) are

$$ {}^E\nabla g = 0\,, \quad \nabla t = 0\,, \quad S = \delta(\gamma + \Sigma). \tag{80} $$

Note that the second condition implies that t needs to have constant rank. On the other hand, the last condition in (80) is not overly restrictive, since always $\delta S = 0$.

A priori, there are many more conditions to be satisfied for (48) to hold, but with the following trick, one may show that they can be all reduced to one only: For this purpose, we first remark that within (80) we can always change the two quantities γ and Σ—which are then to enter the extension (79)—according to (69). Since t has a constant rank, we can choose some $C \subset E$ such that $E = C \oplus t(F)$. One may now see that the transformations (69) can be used to assure $\Sigma \in \Gamma(\Lambda^2 C \otimes \Lambda^2 E^*)$. Then the only remaining condition is:

$$ {}^E\mathrm{D}_\lambda \Sigma = 0\,. \tag{81} $$

Here ${}^E\mathrm{D}_\lambda$ is defined as ${}^E\mathrm{D}$ but replacing the 2-bracket on E by $[s, s']_\lambda = [s, s'] + t(\lambda(s, s'))$, where λ is the tensor entering π^∇_a.

We finally remark that from (62) one can conclude an equation similar to (81): One first observes that the operators ${}^E\mathrm{D}$ and (64) commute, $[{}^E\mathrm{D}, \delta] = 0$. Using (62), one then finds that $\delta({}^E\mathrm{D}(\gamma + \Sigma)) = \langle t, \nabla h\rangle_F$, where the h-contribution only adds to ${}^E\mathrm{D}\gamma$. If $H^{-2}(\Gamma(\mathcal{G}^\bullet), \delta) = 0$ holds, moreover, one finds that ${}^E\mathrm{D}\Sigma$ is part of a coboundary for some $\tilde{V} \in \Gamma(\Lambda^3 E^* \otimes E \otimes F)$. (81) then translates into the condition that $\tilde{V}$ needs to be a contraction of λ with Σ.

18 Angular Momentum II: Outside of the spatial origin

We want to illustrate the above formulas by means of the example of the angular momentum we discussed already before, but this time excluding the origin $\vec{x} = 0$.

On the parts of $T^*\mathbb{R}^3$ where $\vec{x} \neq 0$, the dependency $\vec{x} \cdot \vec{G} \equiv 0$ is in fact already sufficient to describe all the dependencies of the constraints: this is the case since, assuming $\vec{x} \neq 0$, the constraint surface $\vec{G} = \vec{x} \times \vec{p} \approx 0$ implies $\vec{p} \approx \lambda \vec{x}$ for some $\lambda \in \mathbb{R}$. Thus the dependency $\vec{p} \cdot \vec{G} \equiv 0$ is automatically implemented when $\vec{x} \cdot \vec{G} \equiv 0$ is taken care of.

Therefore now, even if one follows the Koszul-Tate procedure, one does not need to introduce an additional ghost of degree two to take into account all the dependencies of the constraints: the BFV charge (74) is the one that gives the correct cohomology of observables when the $\vec{x}$-origin is excluded. This is, on the other hand, not the case, if the last term in (74) is supressed—as one finds it sometimes in the literature in the treatment of mechanical models with rotational invariance in the BRST-B(F)V formalism.

Excluding the $\vec{x}$-origin, there are now no more obstructions to BFV-extend the classical Hamiltonian (76) to an $\mathcal{H}_{BFV}$ satisfying the second master equation (48). And, as it turns out, it can even be put into the special form (79). As we learn from (80), we need for this that the tensor (75) is covariantly constant. Let us choose for this purpose the connections on E and F by means of

$$ \nabla e_a = -\frac{x_a}{r^2}\, \mathrm{d}x^b \otimes e_b \tag{82} $$

and $\nabla b = 0$, respectively. Note that the first choice implies that the radial section $x^a e_a$ is covariantly constant, $\nabla(x^a e_a) = 0$. Together these choices indeed yield $\nabla t = 0$. It was this condition that we could not satisfy previously; here it is possible also only due to the singularity of the connection ∇ on E when approaching the origin.

One also verifies easily, that this choice of the connection on E guarantees ${}^E\nabla g = 0$, a necessary condition for the BFV extension even without the restricting $\mathcal{H}_{BFV}$ to be of the form

(79), see the first line in (60). The tensor S looks as follows with the above choices for the connection:

$$S = \frac{\epsilon_{bce}x^e}{r^2} e_a \otimes e^b \otimes e^c \otimes \mathrm{d}x^a \,. \tag{83}$$

S now has a contribution coming from the tensor γ in its decomposition (68), which enters the covariantized momentum (57):

$$\vec{p}^\nabla = \vec{p} - \frac{1}{r^2}\vec{\pi}(\vec{\xi}\cdot\vec{x}) + \frac{\vec{x}}{2r^4}\vec{\xi}\cdot(\vec{\xi}\times\vec{x})\mathcal{P}\,. \tag{84}$$

The complete BFV extension of the Hamiltonian then takes the form

$$\mathcal{H}_{BFV} = \tfrac{1}{2}\,\vec{p}^\nabla\!\cdot\vec{p}^\nabla - \frac{1}{4r^4}\big[\vec{\xi}\cdot(\vec{\xi}\times\vec{x})\big]\,\big[\vec{\pi}\cdot(\vec{\pi}\times\vec{x})\big]. \tag{85}$$

This corresponds to the following tensor Σ in (79),

$$\Sigma^{ab}_{cd} = -\frac{\varepsilon_{abe}\varepsilon_{cdg}x^e x^g}{r^4}\,, \tag{86}$$

which satisfies the consistency condition (81) for $\lambda = 0$. Together with the above choice for γ, this completes the decomposition (68) of (83), which then satisfies the last condition in (80). According to the discussion following (79), this then proves the validity of (48), which certainly one can also verify directly to hold true for (74) and (85).

The extension of (76) to $\mathcal{H}_{BFV}$ is not unique, also not for a fixed choice of the BFV charge (74). One can, at each stage, change contributions to $\mathcal{H} = \mathcal{H}_{BFV}$ in (58) by δ-exact terms (once more underlining the importance of the complex $\mathcal{G}^\bullet$ for the given extension problem). For a given Hamiltonian, the first such an ambiguity is provided by the choice of the connection on E, which we found in (67). It is now not difficult to see that with the choice $\Psi^i_b = \frac{1}{r^2}\delta^i_b$, $M^{ac}_b = \frac{1}{r^2}\epsilon_{acb}$, the connection coefficients corresponding to (82) can be made to vanish: $\nabla e_a = 0$ for this modified, equivalent connection. Now the connection on F, if taken as the corresponding coefficient in (57), is fixed by requiring the vanishing of the last line in (60), i.e. by requiring $\nabla t = \rho^i_b \Sigma^{ab} e_a \otimes \mathrm{d}x^i$: One finds

$$\nabla b = \frac{x_a \mathrm{d}x^a}{r^2} \otimes b\,, \tag{87}$$

which, in particular, has the property that the section $\frac{1}{r}b$ is covariantly constant, $\nabla(\frac{1}{r}b) = 0$. After the dust clears, one then finds the following, relatively short BFV-Hamiltonian

$$\mathcal{H}_{BFV} = \tfrac{1}{2}\vec{p}^\nabla\cdot\vec{p}^\nabla + \frac{\vec{\pi}\cdot(\vec{\pi}\times\vec{x})}{2r^2}\eta \tag{88}$$

where

$$\vec{p}^\nabla = \vec{p} - \frac{\vec{x}}{r^2}\eta\mathcal{P}\,. \tag{89}$$

In the extension (85), the last term in the decomposition (59) vanishes. In (88), on the other hand, it is the first term that does so, with the η-coefficient $\Sigma^{ab}_I \equiv \Sigma^{ab}$ taking the form

$$\Sigma_{ab} = \varepsilon_{abc}\frac{x^c}{r^2}\,. \tag{90}$$

19 Angular Momentum III: An infinite Koszul-Tate resolution

Before ending this article, we want to address briefly what happens if one follows the usual BFV procedure in the singular context, not taking the shortcut by the resolution of the singular foliation as advocated here. We want to sketch this at the example of the angular momentum when not excluding the spatial origin $\vec{x} = 0$. As mentioned above, in this case, one needs to take both dependencies $\vec{x} \cdot \vec{G} = 0$ and $\vec{p} \cdot \vec{G} = 0$ into account separately. As a preparation for the construction of the BFV charge and the BFV Hamiltonian, one now determines a Koszul-Tate (KT) resolution of the ideal generated by the constraints G_a. The graded coordinates introduced for this purpose, called the anti-ghosts in [HT92], subsequentely serve as the momenta to the ghosts that one needs for the BFV charge. For the angular momentum, we have three anti-ghosts π_a of degree -1, which provide the generators of the ideal after application of the KT differential: $\delta_{KT}(\pi_a) = G_a$. For the two dependencies between these constraints, we introduce two anti-ghosts of degree -2, $\mathcal{P}$ and $\bar{\mathcal{P}}$, such that:

$$\delta_{KT}\mathcal{P} = x^a \pi^a \,, \quad \delta_{KT}\bar{\mathcal{P}} = p^a \pi^a \,. \tag{91}$$

Note that δ_{KT} coincides with δ on the ghost momenta we introduced before, see (64), but now we have more of them. And it does not stop with the additional $\bar{\mathcal{P}}$. The reducibility functions x^a and p^a are not independent on-shell: $x^a p^b - x^b p^a = \epsilon^{abc} G_c \approx 0$. This implies that one needs additional anti-ghots $\mathcal{P}_3^a$ of degree -3, which then give

$$\delta_{KT}\mathcal{P}_3^a = x^a \bar{\mathcal{P}} - p^a \mathcal{P} + \tfrac{1}{2}\varepsilon_{abc}\pi^b \pi^c \,. \tag{92}$$

Here the last term is needed to ensure $(\delta_{KT})^2 = 0$. However, on the space generated by $(x^a, p_a, \pi_a, \mathcal{P}, \bar{\mathcal{P}}, \mathcal{P}_3^a)$ one now finds non-trivial δ_{KT}-cohomology classes at degree -3. This in turn requires the introduction of six anti-ghosts of degree -4 such that

$$\delta\mathcal{P}_4^a = \varepsilon_{abc} x^b \mathcal{P}_3^c + \pi^a \mathcal{P}, \qquad \delta\bar{\mathcal{P}}_4^a = \varepsilon_{abc} p^b \mathcal{P}_3^c + \pi^a \bar{\mathcal{P}} \,. \tag{93}$$

And this procedure does not stop. [6] By construction, there is no cohomology of δ_{KT} except at degree zero. Therefore, with this starting point, $\mathcal{Q}_{BFV}$ and the extension $\mathcal{H}_{BFV}$ of (43) always exist in principle. But there is a price to be paid: the underlying space of ghosts and anti-ghosts consists of an infinite tower of them.

20 Appendix: Angular Momentum IV

In this appendix we want to return once more to the example of the angular momentum in $\mathbb{R}^3$. While the coordinates and momenta enter the constraint surface $\vec{x} \times \vec{p} \approx 0$ symmetrically, this is not the case for the Hamiltonian, see (76). Now, in a region where $\vec{p} \neq 0$, excluding the origin in momentum space, one may use the BFV charge

$$\bar{\mathcal{Q}}_{BFV} = \vec{\xi} \cdot (\vec{x} \times \vec{p}) + \tfrac{1}{2}(\vec{\xi} \times \vec{\xi}) \cdot \vec{\pi} - \bar{\eta}\, \vec{p} \cdot \vec{\pi} \,, \tag{94}$$

where we introduced a ghost-for-ghost pair $(\bar{\eta}, \bar{\mathcal{P}})$ to implement the dependency $\vec{p} \cdot (\vec{x} \times \vec{p}) \equiv 0$. Like when excluding the spatial origin $\vec{x} \neq 0$, there is no obstruction for the BFV extension of the Hamiltonian, but in this case it even agrees with the classical one, $\bar{\mathcal{H}}_{BFV} = \frac{1}{2}\vec{p} \cdot \vec{p}$.

One may wonder, if and how one might obtain the much more involved BFV extensions (85) and (88) from this simple solution to the extension problem in regions where both $\vec{x}$ and $\vec{p}$ are non-vanishing.

We only have a partial, semi-rigorous answer to this question: Introduce a BFV charge $\mathcal{Q}'$ that incorporates the two dependencies given by $\vec{x}$ and $\vec{p}$ in a symmetrical fashion,

$$\mathcal{Q}' = \vec{\xi} \cdot (\vec{x} \times \vec{p}) + \tfrac{1}{2}(\vec{\xi} \times \vec{\xi}) \cdot \vec{\pi} - \eta\, \vec{x} \cdot \vec{\pi} - \bar{\eta}\, \vec{p} \cdot \vec{\pi}, \tag{95}$$

with now two conjugate ghost-for-ghost pairs $(\eta, \mathcal{P})$ and $(\bar{\eta}, \bar{\mathcal{P}})$. The charge (95) squares to zero in the obvious canonical bracket, $(\mathcal{Q}', \mathcal{Q}')' = 0$. It also agrees with a truncation of the

[6] We will come back to a complete description of this resolution elsewhere.

honestly constructed BFV charge in an infinite tower expansion when following the Koszul-Tate procedure mentioned at the end of the main text above. But being such a truncation, where one drops all higher ghost-for-ghost contributions, it does not have the correct cohomology, however—therefore, the argument is only semi-rigorous at this stage.

There now is a $\mathcal{Q}'$-closed extension of the classic Hamiltonian on this extended phase space and it is even globally defined: The simple

$$\mathcal{H}' = \tfrac{1}{2}\vec{p}\cdot\vec{p} - \eta\bar{\mathcal{P}} \tag{96}$$

is readily seen to satisy $(\mathcal{Q}', \mathcal{H}')' = 0$.

We may obtain the two sought-for BFV formulations from the above one by two different reductions in the extended phase space, where one has the canonical coordinates $(\vec{x}, \vec{p}, \vec{\xi}, \vec{\pi}, \eta, \mathcal{P}, \bar{\eta}, \bar{\mathcal{P}})$. The first one follows from the evident choice

$$\eta := 0\,,\; \mathcal{P} := 0. \tag{97}$$

It reproduces the BFV formulation $(\bar{\mathcal{Q}}_{BFV}, \bar{\mathcal{H}}_{BFV})$ valid on regions with $\vec{p} \neq 0$.

Note that the elimination of a canonical pair, here $(\eta, \mathcal{P})$, does not affect the brackets between the remaining variables. But one still needs to ensure that the new quantities satisfy the two master equations after constraining to (97). One way of doing this is to see if they remain to have vanishing brackets when replacing the original BFV bracket $(\cdot,\cdot)'$ by a Dirac bracket (more precisely, by an adaptation of the Dirac bracket, introduced originally for second class constraints [Dir58], to the extended phase space): Restraining a graded symplectic space by putting two even functions α and β which satisfy $(\alpha, \beta)' = 1$ to zero, one replaces the original bracket $(f, g)'$ between two functions f and g by

$$(f, g)_D := (f, g)' - (f, \alpha)'\,(\beta, g)' + (f, \beta)'\,(\alpha, g)'\,. \tag{98}$$

Restriction to the subspace given by $\alpha = \beta = 0$ and inverting the restricted graded symplectic form, in the end is reproduced by this bracket directly. Since both, (95) and (96) commute with η in the $(\cdot,\cdot)'$ bracket, in view of (98), it is clear that the two master equations still hold for the induced bracket when implementing (97).

The situation changes, however, if one wants to proceed in the same way by replacing (97) with the corresponding barred equations. The reason is that now $(\bar{\eta}, \mathcal{H}')'$ does not vanish, nor does $(\mathcal{Q}', \bar{\mathcal{P}})'$, this then would impede the vanishing of $(\mathcal{Q}', \mathcal{H}')_D$. To take care of this, one may search for deforming the condition $\bar{\mathcal{P}} = 0$ by $\bar{\mathcal{P}} = F$ for some function F of the unbarred variables. This procedure permits to find the function F by an expansion in the ghosts, which can be chosen such that the deformed conditions become

$$\bar{\eta} := 0\,,\; \bar{\mathcal{P}} := \frac{1}{2r^2}\left(2\vec{p}\cdot\vec{x}\,\mathcal{P} - \eta\mathcal{P}^2 - \vec{\pi}\cdot(\vec{\pi}\times\vec{x})\right). \tag{99}$$

With this "gauge" the two master equations remain valid and one obtains precisely the BFV data $\mathcal{Q}_{BFV}$ and $\mathcal{H}_{BFV}$) as given by (74) and (88), respectively. The data corresponding to (74) and (85), on the other hand, result from these by an additional canonical transformation.

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Méthodes homologiques pour les théories de jauge avec singularités

Résumé. Cette thèse est consacrée à l'étude et à l'amélioration des outils de l'algèbre homologique utilisés dans les méthodes de quantification des théories de jauge, en particulier le formalisme Hamiltonien-BRST ou BFV. Dans la première partie, nous passons en revue les résultats classiques de l'algèbre homologique sur les résolutions libres des anneaux commutatifs. Nous approfondissons ces résultats lorsque l'anneau commutatif $\mathcal{O}$ est un anneau de polynômes et $\mathcal{I}$ est un idéal monomial propre. Dans ce cadre, nous décrivons les résolutions simpliciales de $\mathcal{O}/\mathcal{I}$.

Dans la deuxième partie, étant donné une algèbre commutative $\mathcal{O}$ et un idéal propre $\mathcal{I}$, nous introduisons, à partir de toute résolution de $\mathcal{O}/\mathcal{I}$ par des $\mathcal{O}$-modules projectifs, une résolution explicite de Koszul-Tate. Nous l'appelons résolution de Koszul-Tate arborescente car elle est indexée par des arbres décorés. Pour les anneaux à Syzygies, nous montrons qu'un nombre fini d'opérations suffit pour construire la résolution arborescente de Koszul-Tate, tandis que l'algorithme de Tate original en nécessite un nombre infini si $\mathcal{I}$ n'est pas une intersection complète. Nous sommes capables de munir la résolution projective initiale de $\mathcal{O}$ d'une A_∞-algèbre explicite. Pour $\mathcal{O}$ étant un anneau de polynômes et $\mathcal{I}$ un idéal monomial, nous utilisons une partie de la structure arborescente de Koszul-Tate pour donner une démonstration élémentaire d'un théorème d'Avramov : si l'idéal $\mathcal{I}$ n'est pas engendré par une séquence régulière, toute résolution de Koszul-Tate a un nombre infini de générateurs. Pour $\mathcal{O}$ étant un anneau local noethérien ou un anneau de polynômes gradué, nous donnons une définition d'une résolution minimale de Koszul-Tate et étudions ses propriétés.

Dans la troisième partie, nous étudions le formalisme BFV pour les systèmes mécaniques. Nous examinons le lien entre les données universelles de Lie-∞-algébroïde et leur contrepartie BFV sur quelques exemples.

Mots-clés : Algèbre commutative, structures Q, résolutions libres, homotopie.

Homological methods for Gauge Theories with singularities

Abstract. This thesis is devoted to studying and enhancing the tools of homological algebra used in quantization methods of gauge theories, in particular the Hamiltonial-BRST or BFV formalism. In the first part we review the standard results in homological algebra on free resolutions of commutative rings. We go into more details when the commutative ring $\mathcal{O}$ is a polynomial ring and $\mathcal{I}$ is a proper monomial ideal. In this setting we describe the simplicial resolutions of $\mathcal{O}/\mathcal{I}$.

In the second part, given a commutative algebra $\mathcal{O}$ and a proper ideal $\mathcal{I}$ we introduce, out of any resolution of $\mathcal{O}/\mathcal{I}$ by projective $\mathcal{O}$-modules, an explicit Koszul-Tate resolution. We call it the arborescent Koszul-Tate resolution since it is indexed by decorated trees. For Syzygy rings, we show that finitely many operations are needed to construct the arborescent resolution Koszul-Tate resolution, while the original Tate algorithm requires infinitely many if $\mathcal{I}$ is not a complete intersection. We are able to equip the initial projective $\mathcal{O}$-resolution with an explicit A_∞-algebra. For $\mathcal{O}$ being a polynomial ring and $\mathcal{I}$ a monomial ideal, we use part of the arborescent Koszul-Tate structure to give an elementary proof of a theorem by Avramov: for the ideal $\mathcal{I}$ not generated by a regular sequence, any Koszul-Tate resolution has infinitely many generators. For $\mathcal{O}$ being a local Noetherian ring or a graded polynomial ring we give a definition of a minimal Koszul-Tate resolution and we study its properties.

In the third part we study BFV formalism for mechanical systems. We investigate the connection between the universal Lie-∞ algebroid data and the BFV counterpart on some examples.

Keywords: Commutative algebra, Q-structures, free resolutions, homotopy.

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