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R-Symmetries from the Orbifolded Heterotic String

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We examine the geometric origin of discrete R-symmetries in heterotic orbifold compactifications. By analysing the symmetries of the worldsheet instanton solutions and the underlying geometry, we obtain a scheme that allows us to systematically explore the R-symmetries arising in these compactifications. Applying this scheme to a classification of orbifold geometries, we are able to find all R-symmetries of heterotic orbifolds with Abelian point groups. We show that in the vast majority of cases, the R-symmetries found satisfy anomaly universality constraints, as required in heterotic orbifolds. Then we examine the implications of the presence of these R-symmetries on a class of phenomenologically attractive orbifold compactifications known as the heterotic mini-landscape. We use the technique of Hilbert bases in order to analyse the properties of a vacuum configuration. We find that phenomenologically viable models remain and the main attractive features of the mini-landscape are unaltered.

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Contents

CHAPTER 1

Introduction

Presently we know of four kinds of fundamental interactions of elementary particles. Three of these are described within the context of quantum field theory by the *Standard Model of particle physics* (SM) [1, 2]. It is a gauge field theory with gauge group

$$G_{\mathrm{SM}} = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_Y \,,$$

where the Lie group $\mathrm{SU}(3)$ corresponds to the strong and $\mathrm{SU}(2) \times \mathrm{U}(1)_Y$ describes electroweak interactions. The fundamental matter is represented by chiral spin-$1/2$ fermions that transform as (bi-) fundamental representations of the gauge group. It comes in three copies, so-called *families*, of the same set of particles which is given by[1]

$$\underset{\substack{\uparrow \\ Q}}{(\mathbf{3},\,\mathbf{2})_{1/6}} + \underset{\substack{\uparrow \\ \bar{u}}}{(\bar{\mathbf{3}},\,\mathbf{1})_{-2/3}} + \underset{\substack{\uparrow \\ \bar{d}}}{(\bar{\mathbf{3}},\,\mathbf{1})_{1/3}} + \underset{\substack{\uparrow \\ L}}{(\mathbf{1},\,\mathbf{2})_{-1/2}} + \underset{\substack{\uparrow \\ \bar{e}}}{(\mathbf{1},\,\mathbf{1})_{1}} \,,$$

where we have given the $U(1)_Y$ charge as a subscript and $Q = (u\ d)$ and $L = (e\ \nu)$ denote the quark and the lepton doublet respectively. The couplings, on the other hand, are mediated by spin-1 gauge bosons which transform as adjoint representations of G_{SM}. By construction all fundamental particles of the Standard Model are necessarily massless, which is in conflict with experimental observations. Their masses can however be generated dynamically by the *Higgs mechanism* [3, 4]. By giving a non-trivial vacuum expectation value (vev) to an additional scalar $\mathrm{SU}(2)$ doublet $(\mathbf{1}, \mathbf{2})_{1/2}$ the Yukawa couplings of that field with the Standard Model matter give rise to effective mass terms. At the same time the gauge symmetry of the Standard Model is spontaneously broken according to $G_{\mathrm{SM}} \to \mathrm{SU}(3) \times U(1)_{\mathrm{em}}$. The experimental evidence for the new massive spin-0 particle that is introduced in the Standard Model, was lacking for a long time. Very recently, however, two experiments at the Large Hadron Collider, ATLAS and CMS, finally announced the observation of a "Higgs-like resonance" corresponding to a mass of around $125\,\mathrm{GeV}$ [5, 6]. The experimental confirmation of the existence of the Higgs particle, about 50 years after it was first predicted, displays the latest of a long list of successes of the Standard Model.

[1] The right-chiral fields are represented by the corresponding left-chiral CPT conjugate fields which transform as the conjugate representation.

The nature of the fourth fundamental interaction, gravity, is completely different. While the interactions described by the Standard Model arise from the invariance of the theory under local gauge transformations, the origin of gravity is the nature of spacetime itself. It is described by Einstein's theory of *general relativity*, which in contrast to the Standard Model is a classical theory. The unification of general relativity with quantum mechanics is one of the fundamental problems of contemporary theoretical physics.

Physics Beyond the Standard Model

Despite its many successes the Standard Model also has drawbacks. There are some experimental hints which clearly require an extension of the Standard Model. The first one is the observation of neutrino oscillations [7] which requires small but non-vanishing masses for the neutrinos. However, not only has the existence of a mass to be explained but also its value, which needs to be extremely small compared to the masses of the SM matter. A possible way out could be the existence of one or more right-handed neutrinos together with the *see-saw mechanism*, which explains the small value of the mass of the SM neutrinos by the huge mass of the right-handed neutrinos. The second experimental hint is the evidence for *dark matter*, which is needed to explain several astronomical and cosmological observations, such as the rotational curves of spiral galaxies or the formation of structure in the universe. In order to explain the origin of dark matter, a new kind of fundamental particle is required, which only interacts very weakly with the SM particles. A third experimental hint on physics beyond the Standard Model comes from the energy dependence of the strengths of the gauge couplings. More specifically, the coupling constants of strong, weak and electromagnetic interactions come very close to each other at an energy scale of about $10^{15}\,\text{GeV}$ which suggests that there could be a simpler theory, the gauge group of which contains G_{SM} and is spontaneously broken at that energy scale. Such theories are called *Grand Unified Theories* (GUTs) [8].

Furthermore, the Standard Model also faces some theoretical challenges. Within the SM, the one-loop contribution to the Higgs mass turns out to be quadratically divergent. As a consequence quantum effects would drive the Higgs mass up to the natural cut-off scale, i.e. the Planck mass which is of order $10^{18}\,\text{GeV}$. In order to cancel the contributions to the Higgs mass an enormous amount of fine-tuning of parameters would be needed. This problem can be rephrased as the *hierarchy problem* which describes the lack of an explanation for the fact that the electroweak scale is about 16 orders of magnitude smaller than the Planck scale or, equivalently, why the weak force is about 10^{32} times stronger than gravity.

The most obvious drawback of the Standard Model is the fact that it does not include gravity, i.e. that it does not contain a quantum theory of gravity which unifies the description of all known forces. A quantum theory of gravity is important to describe physical processes at energies at which the strength of gravity becomes comparable to those of the other forces. Such processes need to be understood in order to describe the dynamics of black holes or the big bang. Furthermore there is experimental evidence for a small, positive cosmological constant which drives the accelerated expansion of our universe and which contributes about 70% to the total energy density of the universe [9]. The value of this constant vacuum energy cannot be explained within general relativity and the attempt to explain its presence by the ground state energy of the SM quantum vacuum leads to a value which differs from the measured value by a factor of 10^{120}.

Many of the questions about physics beyond the Standard Model can be explained within grand unified theories, supersymmetry (SUSY) and extra dimensions, which we briefly comment on in the following.

Grand Unified Theories

The most prominent GUT groups are $\mathrm{SU}(5)$ and $\mathrm{SO}(10)$, since each of them serves to unify the gauge interactions of the Standard Model within one simple group. Furthermore the Standard Model families fit nicely into the lowest dimensional representations of these groups. Namely the anti-fundamental representation $\bar{\mathbf{5}}$ together with the anti-symmetric tensor representation $\mathbf{10}$ of $\mathrm{SU}(5)$ contain exactly one family of quarks and leptons, as can be seen from the decomposition of these representations under the breaking $\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$,

$$\begin{aligned} \bar{\mathbf{5}} &\longrightarrow (\bar{\mathbf{3}},\, \mathbf{1})_{1/3} + (\mathbf{1},\, \mathbf{2})_{-1/2}\,, \\ \mathbf{10} &\longrightarrow (\mathbf{3},\, \mathbf{2})_{1/6} + (\bar{\mathbf{3}},\, \mathbf{1})_{-2/3} + (\mathbf{1},\, \mathbf{1})_{1}\,. \end{aligned}$$

Intriguingly, the embedding of $\mathrm{U}(1)_Y$ into the non-Abelian GUT group therefore yields an explanation for the quantization of the electric charge. In this picture, the Standard Model Higgs boson arises as a further $\mathbf{5}$ of $\mathrm{SU}(5)$, which however leads to the presence of an unwanted triplet $(\mathbf{3},\, \mathbf{1})_{1/3}$ in the spectrum of the GUT. The difficulty to decouple this state by making it very heavy and at the same time keeping the doublet light is known as the *doublet triplet splitting problem*. Finally, the gauge bosons of the Standard Model can be embedded into the adjoint $\mathbf{24}$ of $\mathrm{SU}(5)$, which however contains two additional bosons which are called *leptoquarks*. One can go further and embed the $\mathrm{SU}(5)$ into $\mathrm{SO}(10)$. In that case a complete Standard Model family can be allocated in the spinor representation $\mathbf{16}$ of $\mathrm{SO}(10)$, as can be seen from the decomposition of this representation under the breaking $\mathrm{SO}(10) \to \mathrm{SU}(5)$,

$$\mathbf{16} \longrightarrow \mathbf{10} + \bar{\mathbf{5}} + \mathbf{1}\,.$$

It is interesting that the additional singlet that is contained in the $\mathbf{16}$ carries exactly the quantum numbers of a right-handed neutrino. Similar to the case of $\mathrm{SU}(5)$ the Higgs can be obtained from the fundamental $\mathbf{10}$ of $\mathrm{SO}(10)$, which decomposes according to $\mathbf{10} \to \mathbf{5} + \bar{\mathbf{5}}$ and therefore gives rise to several exotic fields that need to be decoupled from the spectrum in order to obtain the Standard Model.

While the picture of GUTs is very appealing, a generic feature of such theories is rapid proton decay, which is mediated by the exotic particles. As the lifetime of the proton is experimentally bound to be bigger than $5.9 \cdot 10^{33}$ years [10], the corresponding couplings need to be heavily suppressed. However, generically GUTs lack a mechanism to explain this.

Supersymmetry

Due to a no-go theorem by Coleman and Mandula [11] supersymmetry is the only possible extension of spacetime Poincaré symmetry within the Standard Model [12]. It is fundamentally different from all other symmetries, as it transforms bosons and fermions to each other. In principle it is possible to write down theories with $\mathcal{N} = 1, 2, 4$ and 8 supersymmetries. However only the $\mathcal{N} = 1$ theory gives rise to chiral fermions. The simplest and most well studied supersymmetric extension of the Standard Model is the *Minimal Supersymmetric Standard Model* (MSSM) in which every Standard Model particle gets exactly one superpartner of the correspondingly other statistics but with all other quantum numbers equal. However, in addition this model needs a second Higgs doublet in order for it to be possible to give masses to all

fermions [13].
When supersymmetry is exact, the masses of superpartners are necessarily equal. Therefore it must be broken at some higher scale. Only then can the fact be explained, that the massive superpartners of the Standard Model particles have not been observed yet. If SUSY is softly broken at around the TeV scale, it offers a natural solution to the hierarchy problem. The superpartners contribute to the Higgs one-loop amplitude in such a way, that they exactly cancel the contributions of the Standard Model fermions and in that way stabilize the electroweak scale. Furthermore the additional particles vary the running of the coupling constants such that they actually nearly meet in one point at a scale of about 10^{16}GeV. This makes supersymmetric GUTs very appealing.

It is intriguing that by making supersymmetry local one automatically obtains a theory which involves gravity, called *supergravity* (SUGRA). Although this theory turns out to be non-renormalizable, it plays an important role in the context of string theory.

Just like GUTs, the MSSM suffers from the problem of rapid proton decay caused by dimension four and five operators. Many of these operators can be forbidden by imposing *matter parity*, that is a $\mathbb{Z}_2$ discrete symmetry which implies that Standard Model particles and their superpartners can only be created in pairs [14]. The presence of this symmetry automatically renders the *lightest supersymmetric particle* (LSP) stable, making it a good candidate for dark matter [15].

Further problems of the MSSM are concerned with the mass term of the SUSY partners of the Higgs bosons (μ *problem*), the appearance of flavour changing neutral currents and CP violation and are mostly problems of understanding the parameters of the theory.

Extra Dimensions

Already a few years after the foundations of general relativity were laid by Einstein, Kaluza and Klein realised the possibility to unify four dimensional gravity and electromagnetism by a five dimensional theory of gravity. It was the first example of a mechanism tracing back the features of a lower dimensional theory to the properties of the compactification of a simpler, higher dimensional theory. Assuming large extra dimensions it is for instance possible to explain the weakness of gravity [16–18]. While the SM interactions and fields are confined to a four dimensional *brane* embedded in a higher dimensional space (*bulk*), gravity spreads over the whole space. Therefore its coupling strength is diluted over the bulk and looks much smaller from a four dimensional perspective.

More complicated set-ups allow extra dimensional models explaining supersymmetry breaking, the breaking of GUT groups or the origin of discrete symmetries.

String Theory

String theory is a theory combining all of the previously mentioned ideas. It is based on the assumption that the fundamental degrees of freedom describing our world are not point-like particles, but extended, one dimensional objects called *strings*. At a length scale much bigger than the string length, the excitations of strings effectively look like particles. As the spectrum of string theory necessarily contains an excitation of spin 2, it automatically includes a theory of quantum gravity. The string length then introduces a natural cut-off scale and serves as a regulator of the divergences that would otherwise render the theory unphysical.

It is intriguing that consistency requirements make it mandatory to combine string theory with supersymmetry and further fix the number of spacetime dimensions to be ten. There are five different ten dimensional *superstring theories*, which are connected by a web of dualities. This has lead to the interpretation that they are all limiting descriptions of a more fundamental theory called *M-theory*, for which a description in terms of fundamental degrees of freedom, however, is lacking. In the low energy regime all five string theories are approximated by 10D supergravity theories, while the effective theory of M-theory is the unique eleven dimensional SUGRA.

In order to make contact with the four dimensional world that we observe, string theory needs to be compactified on a six-dimensional space. Requiring the resulting four dimensional theory to preserve exactly $\mathcal{N} = 1$ supersymmetry requires any smooth compactification space to be of a special kind, named *Calabi-Yau* [19]. While generic Calabi-Yau spaces are extremely complicated, some have singular limits which are called *orbifolds*, in which the curvature of the space gets concentrated to a finite number of points. In this work we will deal with orbifold compactifications [20, 21] of the heterotic string theory [22–24] with gauge group $E_8 \times E_8$. These theories have the advantage that an exact description in terms of a superconformal field theory is known [25], such that quantities such as the couplings of string states are, in principle, exactly computable. Consequently orbifolds have turned out to serve as a very successful patch for building (semi-) realistic models within the landscape of string compactifications. The most prominent set of such models is the heterotic *mini-landscape* [26–28].

As described above, within the context of model building, global discrete symmetries can solve numerous problems by constraining possible couplings. In this way they can lead to natural explanations of why certain effects are sub dominant or why parameters are small. As an example we already mentioned matter parity in the context of the MSSM. A special class of discrete symmetries are those under which fermions and bosons do not transform in the same way. These discrete symmetries, which do not commute with supersymmetry, are called discrete R-symmetries. They have been proven to be particularly useful as symmetries forbidding dimension four and five proton decay operators [29–31], but their presence is also linked to the breaking of supersymmetry itself [32, 33]. In compactifications of string theory the R-symmetries of the resulting low energy effective field theory arise as remnants of the Lorentz symmetry of the compactified directions. In this work we discuss the origin of discrete R-symmetries within the orbifolded heterotic string. We give a complete exploration of such symmetries arising from orbifolds for which the point group is Abelian. Then we discuss the phenomenological consequences of these R-symmetries and further discrete symmetries in the context of the $\mathbb{Z}_6$-II orbifold. Using the concept of Hilbert bases we are able to deduce the superpotential to all orders in the fields.

Outline

This thesis is organized as follows.

In chapter 2 we review the bosonic construction of heterotic string theory, emphasizing especially the superconformal field theory describing the dynamics on the worldsheet. Then we introduce toroidal orbifolds and review a classification scheme for space groups of such orbifolds. Using these two building blocks we discuss orbifold compactifications of the heterotic string, where we highlight the consequences of orbifolding for the conformal field theory. We conclude this section by discussing the $\mathbb{Z}_3$ orbifold as the simplest possible example.

Chapter 3 deals with the calculation of string correlation functions and the coupling selection rules that can be deduced from them. We consider general correlation functions that give rise to terms in the superpotential of the corresponding low energy effective field theory. From the structure of these correlation functions, it is possible to extract several different selection rules. We start by reviewing how gauge invariance, H-momentum conservation and the space group selection rule arise. Then, specializing to a subclass of orbifolds, we discuss the world-sheet instanton contributions to the couplings and explain how further selection rules can arise from symmetries of these instantons. Remarkably we find the instantons to be the origin of the discrete nature of R-symmetries. Motivated by this finding we start an exploration of all possible R-symmetries arising in orbifolds with Abelian point groups. We present a scheme that allows us to infer the R-symmetries from the isometry groups of the orbifold. Then we apply this scheme to the classification of all space groups leading to six dimensional orbifolds with $\mathcal{N} = 1$ SUSY, which was performed in [34]. In this way we are able to present a complete list of R-symmetries arising in these theories. As a consistency check we calculate the anomalies of each of the R-symmetries for a vast set of models. We find that for 101 out of the 107 classes of orbifolds that are non-trivial, the anomalies fulfil the required universality conditions and therefore pass the consistency check.

In chapter 4 we make use of the previously identified R-symmetries for the $\mathbb{Z}_6$-II orbifold and discuss their impact on a model of the mini-landscape. We review the technique of Hilbert bases which allows to find a complete basis of solutions of a given set of diophantine equations. Then we apply this technique to our model, in order to deduce the superpotential to all orders and discuss its phenomenology.

In the last chapter we conclude and discuss possible extensions of this work.

Note, that throughout this work we use units in which $\alpha' = 2$, $l = \pi$, $\hbar = 1$.

CHAPTER 2

Heterotic Orbifolds

2.1 Bosonic Construction of the Heterotic String

In this section we briefly review the bosonic construction of heterotic string theory [22–24]. It is a theory of closed strings which consists of two parts, namely

- a *right-moving* 10 dimensional *superstring* with bosonic and fermionic degrees of freedom described by $X^\mu_{\mathrm{R}}(\tau-\sigma)$ and the Majorana-Weyl fermions $\psi^\mu_{\mathrm{R}}(\tau-\sigma)$ respectively, and
- a *left-moving* 26 dimensional *bosonic string* with bosonic degrees of freedom denoted by $X^\mu_{\mathrm{L}}(\tau+\sigma)$ and $X^I_{\mathrm{L}}(\tau+\sigma)$,

where $\mu = 0,\ldots,9$ and $I = 1,\ldots,16$.
The right-moving degrees of freedom are related by $\mathcal{N}=1$ worldsheet supersymmetry and the left-moving theory is compactified on a 16 dimensional torus. Modular invariance of the partition function requires the torus lattice to be even, self-dual and Euclidean. The only lattices satisfying these requirements in 16 dimensions are the root lattice of $\mathrm{E}_8\times\mathrm{E}_8$ and the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$[1], which contains the root lattice of $\mathrm{SO}(32)$ [35]. For the remainder of this work we restrict ourselves to the case of $\mathrm{E}_8\times\mathrm{E}_8$. As a consequence, the internal momenta of the left-moving strings are elements of the root lattice of $\mathrm{E}_8\times\mathrm{E}_8$, which we denote by $\Gamma_{\mathrm{E}_8\times\mathrm{E}_8}$. We will see later, how the 16 left-moving bosons X^I_{L} give rise to a gauge theory.

2.1.1 Equations of Motion and Mode Expansions

In superconformal gauge, the classical action of the heterotic string theory takes the form[2] [35, 36]

$$S = -\frac{1}{2\pi}\int \mathrm{d}^2z\,\left(\partial X\cdot\bar\partial X + \psi_{\mathrm{R}}\cdot\partial\psi_{\mathrm{R}} + \delta_{IJ}\partial X^I_{\mathrm{L}}\bar\partial X^J_{\mathrm{L}}\right)\,, \tag{2.1}$$

[1] $\mathrm{Spin}(32)$ has centre $\mathbb{Z}_2\times\mathbb{Z}_2$. Here we divide by one of these $\mathbb{Z}_2$ such that the resulting lattice contains the root lattice of $\mathrm{Spin}(32)$ as well as the weights of the spinor representation. Dividing by the other $\mathbb{Z}_2$ would result in a lattice containing the roots as well as the weights of the cospinor representation, while dividing by a diagonal $\mathbb{Z}_2$ would result in a lattice containing the roots as well as the weights of the vector representation.

[2] Note that in writing the action we have assumed the anti-symmetric tensor background field $B_{\mu\nu}$ as well as the background gauge field A^I_μ to vanish.

with the additional constraint

$$\bar{\partial} X_{\mathrm{L}}^{I} = 0 \tag{2.2}$$

ensuring that the 16 internal bosons are indeed purely left-moving. Here we have made use of the Wick rotated, complex worldsheet coordinates

$$z = \mathrm{e}^{2(\tau + i\sigma)}\,, \qquad \bar{z} = \mathrm{e}^{2(\tau - i\sigma)}\,, \tag{2.3}$$

written shorthand $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$ and have combined the ten dimensional left- and right moving bosonic fields according to $X(z,\bar{z}) = X_{\mathrm{L}}(z) + X_{\mathrm{R}}(\bar{z})$.

Using the action we can deduce the classical Euler-Lagrange equations of motion for the fields. They are given by

$$\begin{aligned} \partial\bar{\partial} X^{\mu} &= 0\,, \\ \partial \psi_{\mathrm{R}} &= 0\,, \\ \partial\bar{\partial} X_{\mathrm{L}}^{I} &= 0\,. \end{aligned} \tag{2.4}$$

The first equation tells us that X^{μ} indeed splits into a left- and a right-moving part and the second one dictates the fermionic field ψ_{R} to be anti-chiral as expected. The equation of motion for X^{I} is actually trivial once the additional constraint (2.2) is imposed.

Now one can write down the mode expansions of the fields that solve the classical equations of motion, as well as the closed string boundary conditions

$$\begin{aligned} X^{\mu}(\mathrm{e}^{2\pi\mathrm{i}} z, \mathrm{e}^{-2\pi\mathrm{i}}\bar{z}) &= X^{\mu}(z,\bar{z})\,, \\ \psi_{\mathrm{R}}^{\mu}(\mathrm{e}^{-2\pi\mathrm{i}}\bar{z}) &= \begin{cases} -\psi_{\mathrm{R}}^{\mu}(\bar{z}) & (\mathcal{R}) \\ \psi_{\mathrm{R}}^{\mu}(\bar{z}) & (\mathcal{NS}) \end{cases}\,, \\ X_{\mathrm{L}}^{I}(\mathrm{e}^{2\pi\mathrm{i}} z) &= X_{\mathrm{L}}^{I}(z) + 2\pi\lambda^{I}\,, \quad \text{with} \quad \lambda \in \Gamma_{\mathrm{E}_8\times\mathrm{E}_8}\,, \end{aligned} \tag{2.5}$$

where we abbreviate the Ramond (Neveu-Schwarz) sector by $\mathcal{R}$ ($\mathcal{NS}$). They are given by[3]

$$X_{\mathrm{L}}^{\mu}(z) = \frac{1}{2}x^{\mu} + p^{\mu}\ln z + \mathrm{i}\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}_n^{\mu}\, z^{-n}\,, \tag{2.6}$$

$$X_{\mathrm{R}}^{\mu}(\bar{z}) = \frac{1}{2}x^{\mu} + p^{\mu}\ln \bar{z} + \mathrm{i}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^{\mu}\, \bar{z}^{-n}\,, \tag{2.7}$$

$$\psi_{\mathrm{R}}^{\mu}(\bar{z}) = \begin{cases} \displaystyle\sum_{n\in\mathbb{Z}} b_n^{\mu}\, \bar{z}^{-n-\frac{1}{2}} & (\mathcal{R}) \\ \displaystyle\sum_{r\in\mathbb{Z}+1/2} b_r^{\mu}\, \bar{z}^{-r-\frac{1}{2}} & (\mathcal{NS}) \end{cases}\,, \tag{2.8}$$

$$X_{\mathrm{L}}^{I}(z) = x^{I} + 2P^{I}\ln z + \mathrm{i}\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}_n^{I}\, z^{-n}\,. \tag{2.9}$$

[3] Note that for writing the mode expansion of the left-moving internal bosons we have implicitly used the fact that the compactification lattice is self-dual [23].

The (on-shell) $\mathcal{N} = 1$ worldsheet supersymmetry acts on the right-moving fields according to

$$\delta_\epsilon X^\mu_\mathrm{R} = -\mathrm{i}\epsilon\psi^\mu_\mathrm{R}\,, \qquad \delta_\epsilon \psi^\mu_\mathrm{R} = \mathrm{i}\epsilon X^\mu_\mathrm{R}\,, \tag{2.10}$$

where ϵ is an infinitesimal Grassmann parameter.

2.1.2 Light-Cone Gauge Quantization and Spectrum

Let us now quantize the theory in light-cone gauge by introducing canonical (anti-)commutation relations. Using the light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^1\right)\,, \tag{2.11}$$

and the mode expansions (2.6)-(2.9) we impose for the transverse coordinates

$$\begin{aligned} [x^i, p^j] = \mathrm{i}\,\delta^{ij}\,, \qquad [\alpha^i_n, \alpha^j_m] &= [\tilde\alpha^i_n, \tilde\alpha^j_m] = n\,\delta_{n+m}\delta^{ij}\,, \\ [\alpha^i_n, \tilde\alpha^j_m] &= 0\,, \\ \{b^i_r, b^j_s\} &= \delta^{ij}\delta_{r+s}\,, \end{aligned} \tag{2.12a}$$

where $i = 2, \ldots, 9$. In the case of the internal left-moving bosons we have to take the secondary class constraint $\bar\partial X^I_\mathrm{L} = 0$ into account. Hence the quantization proceeds via Dirac brackets, which results in the commutation relations

$$\begin{aligned} [\tilde\alpha^I_n, \tilde\alpha^J_m] &= n\,\delta_{n+m}\delta^{IJ}\,, \\ [x^I, P^J] &= \frac{1}{2}\mathrm{i}\,\delta^{IJ}\,. \end{aligned} \tag{2.12b}$$

The $\mathcal{R}$ and $\mathcal{NS}$ ground states of the theory are defined as the states that are annihilated by all positive modes, that is

$$\begin{aligned} \alpha^\mu_m\,|0\rangle_{\mathcal{R}/\mathcal{NS}} = \tilde\alpha^\mu_m\,|0\rangle_{\mathcal{R}/\mathcal{NS}} &= 0\,, & \forall m > 0\,, \\ \tilde\alpha^I_m\,|0\rangle_{\mathcal{R}/\mathcal{NS}} &= 0\,, & \forall m > 0\,, \\ b^u_r\,|0\rangle_{\mathcal{R}/\mathcal{NS}} &= 0\,, & \forall r > 0\,. \end{aligned}$$

Note that while the $\mathcal{NS}$ ground state is unique, the states of the form $b^\mu_0\,|0\rangle_\mathcal{R}$ all have the same mass eigenvalue and hence the $\mathcal{R}$ ground state is degenerate. Since the fermionic zero modes fulfil the Clifford algebra, $\{b^\mu_0, b^\nu_0\} = \eta^{\mu\nu}$, we can represent the states as

$$b^\mu_0\,|\alpha\rangle_\mathcal{R} = \frac{1}{\sqrt{2}}(\gamma^\mu)^\alpha{}_\beta\,|\beta\rangle_\mathcal{R}\,,$$

where α, β are $\mathrm{SO}(9,1)$ spinor indices and γ^μ denote the Dirac matrices in 10 dimensions and satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

In order to get the spectrum of the theory we have to gauge fix the remaining symmetries of the worldsheet theory. This is done by going to the light-cone gauge choice, which is given by

$$\begin{aligned} X^+ &= x^+ + p^+ \ln z\bar{z}\,, \\ \psi^+ &= 0\,, \end{aligned} \tag{2.13}$$

and completely fixes the reparametrisation as well as local supersymmetry invariance. This breaks Lorentz symmetry to its little group $\mathrm{SO}(8)$. The equations of motion for the metric then result in the constraints

$$\begin{aligned} \alpha_n^- &= \frac{1}{2p^+}\sum_i \left(\sum_m :\alpha_m^i \alpha_{n-m}^i: + \sum_{r+\phi} \left(\frac{n}{2} - r\right) :b_r^i b_{n-r}^i: + a_\mathrm{R}\,\delta_m \right)\,, \\ \tilde{\alpha}_n^- &= \frac{1}{2p^+}\sum_m \left(\sum_i :\tilde{\alpha}_m^i \tilde{\alpha}_{n-m}^i: + \sum_I :\tilde{\alpha}_m^I \tilde{\alpha}_{n-m}^I: + a_\mathrm{L}\,\delta_m \right)\,, \\ b_r^- &= \frac{1}{p^+}\sum_q \sum_i \alpha_{r-q}^i b_q^i\,, \end{aligned} \tag{2.14}$$

where we have used the notation $\alpha_0^i = \tilde{\alpha}_0^i = \frac{1}{2}p^i$, $\alpha_0^I = P^I$ and for the $\mathcal{R}$ ($\mathcal{NS}$) sector ϕ takes the value 0 ($\frac{1}{2}$). The normal ordering constants are given by [36]

$$a_\mathrm{L} = -1\,, \qquad a_\mathrm{R} = \begin{cases} -\frac{1}{2} & (\mathcal{NS}) \\ 0 & (\mathcal{R}) \end{cases}\,. \tag{2.15}$$

Similarly p^+, the generator of τ-translations conjugate to X^+ is determined. Using this we can write down the mass operator $M^2 = M_\mathrm{L}^2 + M_\mathrm{R}^2 = 2p^+p^- - (p_\mathrm{trans})^2$,

$$2M_\mathrm{R}^2 = \mathcal{N}_\mathrm{R} + a_\mathrm{R} \quad \text{and} \quad 2M_\mathrm{L}^2 = \frac{1}{2}P^2 + \mathcal{N}_\mathrm{L} - 1\,, \tag{2.16}$$

where $P^2 = \sum_I P^I P^I$ and the number-operators are given by

$$\begin{aligned} \mathcal{N}_\mathrm{R} &= \sum_{i=2}^{9} \left(\sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=\phi}^{\infty} r\, b_{-r}^i b_r^i \right)\,, \\ \mathcal{N}_\mathrm{L} &= \sum_{n=1}^{\infty} \left(\sum_{i=2}^{9} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \sum_{I=1}^{16} \tilde{\alpha}_{-n}^I \tilde{\alpha}_n^I \right)\,. \end{aligned} \tag{2.17}$$

The physical states of the theory are tensor products of states of the Fock states of the left- and right-moving strings fulfilling the level-matching condition

$$M_\mathrm{L}^2 = M_\mathrm{R}^2\,, \tag{2.18}$$

which follows from the requirement of invariance of the states under translations in the σ direction. Due to this condition the lowest mass state of the theory has mass eigenvalue zero, such that there is no tachyon in the spectrum.

Now we can write down the massless spectrum of the heterotic string on a 10D Minkowski spacetime. It consists of the following states:

- $\tilde{\alpha}^i_{-1}\left|0\right\rangle \otimes b^j_{-1/2}\left|0\right\rangle_{NS}$ graviton, antisymmetric tensor and dilaton,
- $\tilde{\alpha}^I_{-1}\left|0\right\rangle \otimes b^j_{-1/2}\left|0\right\rangle_{NS}$ 16 Cartan generators of $\mathrm{E}_8 \times \mathrm{E}_8$,
- $\left|P^2 = 2\right\rangle \otimes b^j_{-1/2}\left|0\right\rangle_{NS}$ 480 generators of $\mathrm{E}_8 \times \mathrm{E}_8$ corresponding to the root vectors,

as well as their supersymmetry partners which are obtained by replacing $b^j_{-1/2}\left|0\right\rangle_{NS}$ by the Ramond vacuum which, as we have seen above, transforms as a Majorana-Weyl spinor of the transverse SO(8). The quantum numbers of the bosonic massless states are summarized in table 2.1.

State	$\mathrm{E}_8 \times \mathrm{E}_8$	SO(8)
$\tilde{\alpha}^I_{-1}\left|0\right\rangle \otimes b^j_{-1/2}\left|0\right\rangle_{NS}$	$(\mathbf{248}, \mathbf{248})$	$\mathbf{8}_v$
$\left|P^2 = 2\right\rangle \otimes b^j_{-1/2}\left|0\right\rangle_{NS}$		$\mathbf{8}_v$
$\tilde{\alpha}^i_{-1}\left|0\right\rangle \otimes b^j_{-1/2}\left|0\right\rangle_{NS}$	singlet	$\mathbf{8}_v \times \mathbf{8}_v = \mathbf{1} + \mathbf{28} + \mathbf{35}_v$

Table 2.1: Quantum numbers of the bosonic massless states of heterotic string theory on 10D Minkowski spacetime.

2.1.3 Worldsheet Conformal Field Theory

In the superconformal gauge, superstring theory becomes a superconformal field theory (SCFT) on the worldsheet. Here we want to briefly describe the basic features of the heterotic string worldsheet superconformal field theory. We will not give a general introduction to conformal field theory here but instead refer to [36, 37]. However, in order to settle our notation let us start by writing down some basic objects.
Here and in the remainder of this work we use Wick rotated spacetime coordinates, such that the ten dimensional metric is given by $\delta^{\mu\nu}$ and the Lorentz group is SO(10).

Superconformal Algebra

The heterotic SCFT splits into a bosonic left-moving and a supersymmetric right-moving conformal field theory. The generators of (super)conformal transformations are the conserved energy-momentum tensor $T(z)$, $\bar{T}(\bar{z})$ and the conserved fermionic supercurrent $\bar{T}_{\mathrm{F}}(\bar{z})$. They are expanded as

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} z^{-n-2} L_n\,, \\ \bar{T}(\bar{z}) &= \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n\,, \\ \bar{T}_{\mathrm{F}}(\bar{z}) &= \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi} \bar{z}^{-\frac{3}{2}-r} \bar{G}_r\,. \end{aligned} \tag{2.19}$$

Here $\phi = 0$ corresponds to the $\mathcal{R}$ sector and $\phi = \frac{1}{2}$ corresponds to the $\mathcal{NS}$ sector. The modes satisfy

$$(L_n)^\dagger = L_{-n}\,, \quad (\bar{L}_n)^\dagger = \bar{L}_{-n}\,, \quad (\bar{G}_r)^\dagger = \bar{G}_{-r}\,, \tag{2.20}$$

as well as the algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{8}(m^3 - m)\delta_{m+n}\,, \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{8}(m^3 - m)\delta_{m+n}\,, \\ [\bar{L}_m, \bar{G}_r] &= \left(\frac{1}{2}m - r\right)\bar{G}_{m+r}\,, \\ \{\bar{G}_r, \bar{G}_s\} &= 2\bar{L}_{r+s} + \frac{\bar{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s}\,. \end{aligned} \tag{2.21}$$

We call c, $\bar{c}$ the *central charges* of the theory. For $\phi = \frac{1}{2}$ the right-moving algebra has a finite dimensional subalgebra, generated by $\bar{L}_0$, $\bar{L}_{\pm 1}$, and $\bar{G}_{\pm\frac{1}{2}}$. This is the super-algebra $\mathfrak{osp}(1|2)$. Meanwhile, the left-moving algebra has the finite-dimensional subalgebra $\mathfrak{sl}(2,\mathbb{R})$ generated by L_0 and $L_{\pm 1}$.

Vacuum Structure

The vacuum of the theory is described, just as any other state in the theory, as a product of the left- and right-moving vacuum,

$$|0\rangle = |0\rangle_{\mathrm{L}} \otimes |0\rangle_{\mathrm{R}}\ .$$

Note that a unitary CFT always has a unique vacuum[4], while we have seen that the right-moving vacuum falls into two distinct sectors. This issue is solved by introducing the so called *spin fields* $S^{\pm}_{\alpha}(\bar{z})$ which are related by [38, 39]

$$\bar{T}_{\mathrm{F}}(\bar{w})S^{+}_{\alpha}(\bar{z}) = \frac{\frac{1}{2}}{(\bar{w}-\bar{z})^{\frac{3}{2}}}S^{-}_{\alpha}(\bar{z}) + \dots\,.$$

Then the $\mathcal{R}$ vacuum is given in terms of the $\mathcal{NS}$ vacuum by[5]

$$|\alpha\rangle_{\mathcal{R}} = S^{+}_{\alpha}(0)\,|0\rangle_{\mathcal{NS}} \tag{2.22}$$

and has conformal dimension $\bar{h} = \frac{1}{16}\bar{c}$. The presence of the spin fields introduces *branch cuts* into the operator algebra. This means that when an $\mathcal{NS}$ fermion is transported around the spin field, it feels the branch cut and changes sign, i.e.

$$\begin{aligned} \psi^{\mu}(\mathrm{e}^{-2\pi\mathrm{i}}\bar{z})S^{\alpha}(0) &= -\psi^{\mu}(\bar{z})S^{\alpha}(0)\,, \\ \psi^{\mu}(\bar{z})S^{\alpha}(\bar{w}) &= \frac{1}{(\bar{z}-\bar{w})^{\frac{1}{2}}}(\gamma^{\mu})^{\alpha}{}_{\beta}S^{\beta}(\bar{w}) + \dots\,. \end{aligned} \tag{2.23}$$

[4] This follows from the fact that the unit operator is the only field with $h = \bar{h} = 0$. For details see for instance [36].

[5] The state $S^{-}_{\alpha}\,|0\rangle_{\mathcal{NS}}$ is a null-state in the case of unbroken worldsheet SUSY. Hence we will ignore it and suppress the label $+$ from now on.

Note that in principle this renders the theory non-local. Locality is restored by certain requirements on the spectrum, which in the case of the heterotic string arise from the level-matching condition (and the restrictions on the lattice). The unique vacuum of the heterotic worldsheet SCFT is now the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{OSP}(1|2)$ invariant state

$$|0\rangle = |0\rangle_{\mathrm{L}} \otimes |0\rangle_{\mathrm{R}\,\mathcal{NS}} \;, \tag{2.24}$$

that is annihilated by all positive frequency operators,

$$\begin{aligned} L_0\,|0\rangle &= L_{\pm1}\,|0\rangle = 0\,,\\ \bar{L}_0\,|0\rangle &= \bar{L}_{\pm1}\,|0\rangle = \bar{G}_{\pm\frac{1}{2}}\,|0\rangle = 0\,,\\ L_m\,|0\rangle &= 0\,, \qquad \forall m>0\,,\\ \bar{L}_m\,|0\rangle &= 0\,, \qquad \forall m>0\,,\\ \bar{G}_r\,|0\rangle &= 0\,, \qquad \forall\, r\,>0\,. \end{aligned} \tag{2.25}$$

Field Content

We have seen above that the fields involved in the heterotic SCFT are X^μ, ψ^μ and X^I_{L}. We can deduce their two-point functions using the commutation relations (2.12) and find[6]

$$\begin{aligned} \langle X^\mu(z,\bar z)X^\nu(w,\bar w)\rangle &= -\delta^{\mu\nu}\log(z-w)(\bar z-\bar w)\,,\\ \left\langle X^I_{\mathrm{L}}(z)X^J_{\mathrm{L}}(w)\right\rangle &= -\delta^{IJ}\log(z-w)\,,\\ \left\langle\psi^\mu_{\mathrm{R}}(\bar z)\psi^\nu_{\mathrm{R}}(\bar w)\right\rangle_{\mathcal{NS}} &= \frac{\delta^{\mu\nu}}{\bar z-\bar w}\,,\\ \left\langle\psi^\mu_{\mathrm{R}}(\bar z)\psi^\nu_{\mathrm{R}}(\bar w)\right\rangle_{\mathcal{R}} &= \frac{1}{2}\delta^{\mu\nu}\frac{1}{\bar z-\bar w}\left(\sqrt{\frac{\bar z}{\bar w}}+\sqrt{\frac{\bar w}{\bar z}}\right)\,, \end{aligned} \tag{2.26}$$

where (here and always) we assume radial ordering, that is $|z|>|w|$, $|\bar z|>|\bar w|$. The energy-momentum tensor and fermionic supercurrent following from the action (2.1) are given by

$$\begin{aligned} T(z) &= -\frac{1}{2}:\partial X\cdot\partial X(z): -\frac{1}{2}\,\delta_{IJ}\;:\partial X^I_{\mathrm{L}}\partial X^J_{\mathrm{L}}(z):\,,\\ \bar T(\bar z) &= -\frac{1}{2}:\bar\partial X\cdot\bar\partial X(\bar z): -\frac{1}{2}:\psi_{\mathrm{R}}\cdot\bar\partial\psi_{\mathrm{R}}(\bar z):\,,\\ \bar T_{\mathrm{F}}(\bar z) &= \;\;\mathrm{i}:\psi_{\mathrm{R}}\cdot\bar\partial X(\bar z):\;. \end{aligned} \tag{2.27}$$

Note that since the right-moving sector of the theory is supersymmetric, the corresponding fields combine into anti-chiral superfields according to

$$\Psi^\mu_{\mathrm{R}}(\bar z) = X^\mu_{\mathrm{R}}(\bar z) + \bar\theta\psi^\mu_{\mathrm{R}}(\bar z)\,, \tag{2.28}$$

[6] Here we suppress some unphysical regulators in the two-point functions, which are not important for our purposes [36].

where $\bar{\theta}$ is the fermionic superspace coordinate. The components of the superfields are related by

$$\begin{aligned} \bar{T}_{\mathrm{F}}(\bar{z}) X_{\mathrm{R}}^{\mu}(\bar{w}) &= \frac{\frac{1}{2}\psi_{\mathrm{R}}^{\mu}(\bar{w})}{\bar{z}-\bar{w}} + \ldots\,, \\ \bar{T}_{\mathrm{F}}(\bar{z}) \psi_{\mathrm{R}}^{\mu}(\bar{w}) &= \frac{h X_{\mathrm{R}}^{\mu}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\frac{1}{2}\partial X_{\mathrm{R}}^{\mu}(\bar{w})}{\bar{z}-\bar{w}} + \ldots\,, \end{aligned} \tag{2.29}$$

where the dots denote terms that are finite as $z \to w$, $\bar{z} \to \bar{w}$.

Using equations (2.26) and (2.27) we can calculate the OPEs of the fields with the energy-momentum tensor to determine the conformal primaries and their conformal weights h and $\bar{h}$. This can be done employing the identities

$$\begin{aligned} T(z)\Phi(w) &= \frac{h}{(z-w)^2}\Phi(w) + \frac{\partial\Phi(w)}{z-w} + \ldots\,, \\ \bar{T}(\bar{z})\phi_0(\bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi_0(\bar{w}) + \frac{\bar{\partial}\phi_0(\bar{w})}{\bar{z}-\bar{w}} + \ldots\,, \\ \bar{T}(\bar{z})\phi_1(\bar{w}) &= \frac{\left(\bar{h}+\frac{1}{2}\right)}{(\bar{z}-\bar{w})^2}\phi_1(\bar{w}) + \frac{\bar{\partial}\phi_1(\bar{w})}{\bar{z}-\bar{w}} + \ldots\,, \end{aligned} \tag{2.30}$$

which hold for any conformal field $\Phi(z)$ or superconformal field $\phi(\bar{z}) = \phi_0(\bar{z}) + \bar{\theta}\phi_1(\bar{z})$.

When looking at the field content of the heterotic SCFT, the first observation is that the OPEs of the fields X^μ with themselves involve logarithms which means that, although they fulfil (2.30) with $h = 0$, they cannot be highest weight states of a representation of the conformal algebra [39]. Their derivatives ∂X^μ and $\bar{\partial} X^\mu$, however are conformal primaries with conformal dimensions $(h, \bar{h}) = (1, 0)$ and $(h, \bar{h}) = (0, 1)$. Meanwhile, the ψ_{R} are themselves conformal primary fields of dimension $(h, \bar{h}) = (0, \frac{1}{2})$. Further primaries can be constructed from normal ordered exponentials[7] of X^μ. They fulfil

$$\begin{aligned} :\mathrm{e}^{\mathrm{i}p\cdot X(w,\bar{w})}::\mathrm{e}^{\mathrm{i}q\cdot X(z,\bar{z})}: = |z-w|^{2p\cdot q} \Big\{ &: \mathrm{e}^{\mathrm{i}(p+q)\cdot X(w,\bar{w})} : \\ &+ \mathrm{i}\,p\,(z-w) : \partial X(w) \mathrm{e}^{\mathrm{i}(p+q)\cdot X(w,\bar{w})} : \\ &+ \mathrm{i}\,p\,(\bar{z}-\bar{w}) : \bar{\partial} X(\bar{w}) \mathrm{e}^{\mathrm{i}(p+q)\cdot X(w,\bar{w})} : \Big\} \\ + \mathcal{O}\left((z-w)^{p\cdot q+2}\right)\,, & \\ \partial X^\mu(z) : \mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar{w})} : = -\frac{\mathrm{i}\,p^\mu}{z-w} : \mathrm{e}^{\mathrm{i}\,p\cdot X(w,\bar{w})} + \ldots & \end{aligned} \tag{2.31}$$

from which one can deduce that their conformal dimensions are given by $(h, \bar{h}) = (\frac{1}{2}k^2, \frac{1}{2}k^2)$.

Physical string states are created from the vacuum by means of Vertex operators, the construction of which we will discuss below. We have already seen that in order to create string states in the $\mathscr{R}$ sector, we need to make use of the spin fields (2.22). These have a complicated expression in terms of ψ^μ [40]. A much simpler representation can be constructed using bosonization on the worldsheet.

[7] In the supersymmetric language, one can further construct normal ordered exponentials of $\Psi^\mu(z,\bar{z},\bar{\theta}) = X^\mu(z,\bar{z}) + \bar{\theta}\psi^\mu(\bar{z})$. However, due to $\bar{\theta}$ being Grassmann, i.e. $\bar{\theta}^2 = 0$, there are no terms beyond linear order in ψ and $\mathrm{e}^{\mathrm{i}k\cdot\Psi} = (1 + \mathrm{i}\bar{\theta}k\cdot\psi)\mathrm{e}^{\mathrm{i}k\cdot X}$.

Bosonization

The main idea of bosonization is that two conformal field theories are indistinguishable if all their correlation functions are identical. However, the correlation functions are completely determined by the OPEs of the fields. Hence, different representations of a physical field are interchangable provided they have the same OPEs among themselves and with all other fields of the theory.

Consider the right-moving fermion sector of the heterotic worldsheet SCFT. It is given by the fields $\psi^{\mu}_{\mathrm{R}}(\bar{z})$ which we represent by the complexified fermions

$$\begin{aligned} \psi^{m}_{\mathrm{R}} &= \frac{1}{\sqrt{2}}\left(\psi^{2m}_{\mathrm{R}} + \mathrm{i}\,\psi^{2m+1}_{\mathrm{R}}\right) , \\ \bar{\psi}^{m}_{\mathrm{R}} &= \frac{1}{\sqrt{2}}\left(\psi^{2m}_{\mathrm{R}} - \mathrm{i}\,\psi^{2m+1}_{\mathrm{R}}\right) , \end{aligned} \tag{2.32}$$

where $m = 0, \ldots, 4$. Then their non-vanishing mutual OPEs are given by

$$\psi^{m}_{\mathrm{R}}(\bar{z})\bar{\psi}^{n}_{\mathrm{R}}(\bar{w}) = \frac{\delta^{mn}}{\bar{z} - \bar{w}} + \ldots .$$

Let us introduce five right-moving bosonic coordinates H^m, which fulfill

$$H^m(\bar{z})H^n(\bar{w}) = -\delta^{mn}\log\left(\bar{z} - \bar{w}\right) + \ldots .$$

Now we can represent the fermions ψ^{m}_{R} by

$$\begin{aligned} \psi^{m}_{\mathrm{R}}(\bar{z}) &=: \mathrm{e}^{\mathrm{i}H^m(\bar{z})} : c_m , \\ \bar{\psi}^{\bar{m}}_{\mathrm{R}}(\bar{z}) &=: \mathrm{e}^{-\mathrm{i}H^m(\bar{z})} : c_m , \end{aligned} \tag{2.33}$$

where the c_m are so called *cocycle factors*, which ensure that the fermion representations with $m \neq n$ actually anti-commute. They are given by [40]

$$c_m = (-1)^{N^1 + \cdots + N^{m-1}} , \tag{2.34}$$

where N^m is the fermion number operator for the m^{th} fermion. In terms of the bosonized coordinates the N^m can be expressed as

$$N^m = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}\bar{z}\,\bar{\partial}H^m(\bar{z}) = \bar{\partial}H^m_0 ,$$

i.e. by the zero modes of $\bar{\partial}H^m$. It is easy to verify that indeed

$$[\bar{\partial}H^m_0, : \mathrm{e}^{\mathrm{i}kH^n(\bar{z})} :] = k\delta^{mn} : \mathrm{e}^{\mathrm{i}kH^n(\bar{z})} : .$$

In the light of (2.31), it is clearly possible to write down more general operators of the form

$$O_\lambda =: \mathrm{e}^{\mathrm{i}\lambda\cdot H} : c_\lambda , \tag{2.35}$$

where λ is a five dimensional vector with integer valued entries and c_λ is the corresponding cocycle operator. These cocycle operators consists of products of the primitive operators (2.34) and can be written as [40]

$$c_\lambda = \exp\left(\mathrm{i}\pi\lambda \cdot M\partial H_0\right) \,, \tag{2.36}$$

where M is a constant, integer-valued, lower-triangular 5×5 matrix. The fusion rule (2.31) then becomes

$$\begin{aligned} O_\lambda(\bar z) O_{\lambda'}(\bar w) &= (\bar z - \bar w)^{\lambda\cdot\lambda'} \mathrm{e}^{\mathrm{i}\pi\lambda\cdot M\lambda'} \mathrm{e}^{\mathrm{i}(\lambda\cdot H(\bar z) + \lambda'\cdot H(\bar w))} c_{\lambda+\lambda'} &(2.37)\\ &= (\bar z - \bar w)^{\lambda\cdot\lambda'} O_{\lambda+\lambda'}(\bar w)\left(1 + \mathcal{O}(\bar z - \bar w)\right) \,. &(2.38)\end{aligned}$$

Closure of the operator product algebra requires that the vertex operator $O_{\lambda+\lambda'}$ exists and generates a state $|\lambda+\lambda'\rangle$ of the theory. In this way one can associate the set of operators $\{O_\lambda\}$ to a lattice. It is not a coincidence that the lattice generated by the fermions (2.33) contains the vector weight lattice and the root lattice of $\mathrm{SO}(10)$. Indeed we can construct the currents

$$\begin{aligned} &J^{\overleftrightarrow{m}\overleftrightarrow{n}}(\bar z) =: \mathrm{e}^{\pm\mathrm{i}H^m(\bar z)\pm\mathrm{i}H^n(\bar z)} : c_m c_n \qquad m \neq n \,,\\ &J^{m\bar m}(\bar z) = \mathrm{i}\partial H^m(\bar z) \,, \end{aligned} \tag{2.39}$$

which fulfil the $\mathfrak{so}(10)$ current algebra and where $J^{\overleftrightarrow{m}\overleftrightarrow{n}}$ with $m \neq n$ correspond to the root and $J^{m\bar m}$ to the Cartan generators. Their operator product expansions with the fermions (2.33) show that the Ψ^m transform in the vector representation of this algebra.

This is an example of the Frenkel-Kač-Segal construction of a Kač-Moody algebra, in which a set of free bosons compactified on a torus represent the generators of the algebra, which is determined by the torus lattice. The idea is then, that the fact that the generators of the algebra commute with the mass operator means, that the string states form a representation of the algebra at each mass level, which proves the existence of the corresponding symmetry [41]. In precisely this way one can describe the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge symmetry arising from the toroidal compactification of the 16 internal left-moving bosons.

In this framework it is now much easier to describe the spin fields (2.22). We have seen in (2.23) that their OPEs with the fermions Ψ^m have square-root branch-cuts. Using the fusion rule (2.37), it is easy to guess the form of the spin fields in the bosonic language. They are associated with lattice vectors with half-integer entries,

$$S^\alpha(\bar z) =: \mathrm{e}^{\mathrm{i}\lambda\cdot H(\bar z)} : \,, \qquad \text{with} \qquad \lambda = (\pm1/2\,, \pm1/2\,, \ldots\,, \pm1/2) \,, \tag{2.40}$$

where we have suppressed the cocycle operators for convenience[8]. In fact the set splits into two subsets containing the vectors with an even or odd number of minus signs, corresponding to the spinor or cospinor weight lattice of $\mathrm{SO}(10)$ respectively.

Superconformal Ghost System

In order to write down the states and vertex operators of the quantized theory, one more ingredient is missing. Namely we need to gauge-fix the worldsheet symmetries. This is done in a manifestly Lorentz invariant way by introducing the Faddeev-Popov superdeterminant compensating the Jacobian from reparametrisation and local supersymmetry transformations. This determinant may be represented by the path integral over

[8] The cocycle operators of the spin fields can be found in [40].

- a *left-moving* conjugate pair of dimension $(-1,0)$, $(2,0)$ ghost fields

$$c^z\,, \qquad\qquad b_{zz}\,,$$

- a *right-moving* conjugate pair of dimension $(0,-1)$, $(0,\frac{3}{2})$ ghost superfields

$$C^{\bar z} = c^{\bar z} + \bar\theta\gamma^{\bar z}\,, \qquad B_{\bar z\bar\theta} = \beta_{\bar z\bar\theta} + \bar\theta b_{\bar z\bar z}\,,$$

with action

$$S_{\rm gh} = \frac{1}{2\pi}\int \mathrm{d}^2 z\,\left(b_{zz}\bar\partial c^z + b_{\bar z\bar z}\partial c^{\bar z} + \beta_{\bar z\bar\theta}\partial\gamma^{\bar z}\right)\,. \tag{2.41}$$

Note that β,γ (b,c) (anti-)commute. They have OPEs[9]

$$\begin{aligned} c^z(z)b_{zz}(w) &= \frac{1}{z-w} + \ldots\,, \\ c^{\bar z}(\bar z)b_{\bar z\bar z}(\bar w) = \gamma^{\bar z}(\bar z)\beta_{\bar z\bar\theta}(\bar w) &= \frac{1}{\bar z-\bar w} + \ldots \end{aligned} \tag{2.42}$$

and the energy-momentum tensor is given by

$$\begin{aligned} T_{\rm gh}(z) &=: -2b_{zz}\partial c^z - (\partial b_{zz})c^z :\,, \\ \bar T_{\rm gh}(\bar z) &=: -2b_{\bar z\bar z}\bar\partial c^{\bar z} - (\bar\partial b_{\bar z\bar z})c^{\bar z} - \frac{3}{2}\beta_{\bar z\bar\theta}\bar\partial\gamma^{\bar z} - \frac{1}{2}(\bar\partial\beta_{\bar z\bar\theta})\gamma^{\bar z} :\,, \\ \bar T_{\rm F\,gh}(\bar z) &=: \frac{1}{2}b_{\bar z\bar z}\gamma^{\bar z} - (\bar\partial\beta_{\bar z\bar\theta})c^{\bar z} - \frac{3}{2}\beta_{\bar z\bar\theta}\bar\partial c^{\bar z} :\,. \end{aligned} \tag{2.43}$$

It is straightforward to check that the ghost system fulfils the left- (right-) moving (super)conformal algebra (2.21) with central charges $c_{\rm gh} = -26$ and $\bar c_{\rm gh} = -10$. These charges cancel precisely the central charges arising from the matter fields so that the superconformal symmetry of heterotic string theory is anomaly free.

The ghost fields can be expanded in modes according to

$$\begin{aligned} c^z(z) &= \sum_{n\in\mathbb{Z}} z^{-n+1}\tilde c_n\,, & b_{zz}(z) &= \sum_{n\in\mathbb{Z}} z^{-n-2}\tilde b_n\,, \\ c^{\bar z}(\bar z) &= \sum_{n\in\mathbb{Z}} \bar z^{-n+1}c_n\,, & b_{\bar z\bar z}(\bar z) &= \sum_{n\in\mathbb{Z}} \bar z^{-n-2}b_n\,, \\ \beta_{\bar z\bar\theta}(\bar z) &= \sum_{r\in\mathbb{Z}+\phi} z^{-r-\frac{3}{2}}\beta_r\,, & \gamma^{\bar z}(\bar z) &= \sum_{r\in\mathbb{Z}+\phi} z^{-r+\frac{1}{2}}\gamma_r\,, \end{aligned} \tag{2.44}$$

where, as above, $\phi = 0$ ($\frac{1}{2}$) in the $\mathcal{R}$ ($\mathcal{NS}$) sector. Then the OPEs (2.42) are equivalent to

$$\{\tilde c_m, \tilde b_n\} = \{c_m, b_n\} = \delta_{m+n}\,, \qquad [\beta_r, \gamma_s] = \delta_{r+s}\,.$$

and the modes of $\beta_{\bar z\bar\theta}$ are anti-Hermitean while all other ghost modes are Hermitean.

[9] Note that here '=' only means equal up to non-singular terms.

The action (2.41) is invariant under two global $\mathrm{U}(1)$s consisting of

$$\begin{pmatrix} c^z \\ b_{zz} \end{pmatrix} \to \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha} & c^z \\ \mathrm{e}^{-\mathrm{i}\alpha} & b_{zz} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C^{\bar{z}} \\ B_{\bar{z}\bar{z}} \end{pmatrix} \to \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha} & C^{\bar{z}} \\ \mathrm{e}^{-\mathrm{i}\alpha} & B_{\bar{z}\bar{z}} \end{pmatrix} \tag{2.45}$$

respectively. The chiral Noether currents corresponding to these transformations are given by

$$\begin{aligned} j(z) &= -: b_{zz}(z) c^z(z) := \sum_n z^{-n-1} \tilde{j}_n \,, \qquad \tilde{j}_n = \sum_m : \tilde{c}_{n-m} \tilde{b}_m : \,, \\ \bar{j}(\bar{z}) &= \bar{j}_{bc}(\bar{z}) + \bar{j}_{\beta\gamma}(\bar{z}) \,, \end{aligned} \tag{2.46a}$$

where

$$\begin{aligned} \bar{j}_{bc}(\bar{z}) &= -: b_{\bar{z}\bar{z}}(\bar{z}) c^{\bar{z}}(\bar{z}) := \sum_n \bar{z}^{-n-1} j_n^{(bc)} \,, \qquad j_n^{(bc)} = \sum_m : c_{n-m} b_m : \,, \\ \bar{j}_{\beta\gamma}(\bar{z}) &= -: \beta_{\bar{z}\bar{\theta}}(\bar{z}) \gamma^{\bar{z}}(\bar{z}) := \sum_n \bar{z}^{-n-1} j_n^{(\beta\gamma)} \,, \qquad j_n^{(\beta\gamma)} = \sum_r : \gamma_{n-r} \beta_r : \,. \end{aligned} \tag{2.46b}$$

Note that due to the form of the action and the energy-momentum tensor we can define the two auxiliary currents $\bar{j}_{bc}$, $\bar{j}_{\beta\gamma}$. However only their sum $\bar{j}$ is a Noether current. The OPEs of the ghost fields b and c (β and γ) with the j reflect the fact that they have charges -1 and 1 respectively. The OPEs of the currents with the energy-momentum tensor are given by

$$\begin{aligned} T(z) j(w) &= \frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial j(w)}{z-w} + \dots \qquad \text{with} \qquad Q = -3 \,, \\ \bar{T}(\bar{z}) \bar{j}_{bc}(\bar{w}) &= \frac{\bar{Q}_{bc}}{(\bar{z}-\bar{w})^3} + \frac{\bar{j}_{bc}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial \bar{j}_{bc}(\bar{w})}{\bar{z}-\bar{w}} + \dots \qquad \text{with} \qquad \bar{Q}_{bc} = -3 \,, \\ \bar{T}(\bar{z}) \bar{j}_{\beta\gamma}(\bar{w}) &= \frac{\bar{Q}_{\beta\gamma}}{(\bar{z}-\bar{w})^3} + \frac{\bar{j}_{\beta\gamma}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial \bar{j}_{\beta\gamma}(\bar{w})}{\bar{z}-\bar{w}} + \dots \qquad \text{with} \qquad \bar{Q}_{\beta\gamma} = 2 \,. \end{aligned} \tag{2.47}$$

As Q and $\bar{Q} = \bar{Q}_{bc} + \bar{Q}_{\beta\gamma}$ are non-vanishing, the $\mathrm{U}(1)$s are anomalous, which implies that the ghost number currents j and $\bar{j}$ are not conserved. The anomalies are related to the existence of ghost zero modes, the number of which may be calculated using the Riemann-Roch theorem. The result reads [39]

$$\begin{aligned} N_c - N_b = N_{\bar{c}} - N_{\bar{b}} &= -3(g-1) \,, \\ N_\gamma - N_\beta &= -2(g-1) \,, \end{aligned} \tag{2.48}$$

where g is the genus of the worldsheet[10]. As a consequence of the anomalies, one finds from the OPEs (2.47) that while $j_n^\dagger = -j_n$ for all three currents, the zero modes are not (anti-)Hermitean. Instead they fulfil

$$\left(j_0^{(a)} \right)^\dagger = -j_0^{(a)} - Q_a \,,$$

[10] As we will only be dealing with tree-level amplitudes, we restrict to $g = 0$ for the remainder of this work.

where a labels the three currents. Let $\left|q_{(a)}\right\rangle$ be a state of charge $q_{(a)}$, i.e. $j_0^{(a)}\left|q_{(a)}\right\rangle = q_{(a)}\left|q_{(a)}\right\rangle$ and let O be an operator of charge $q_{(a)}^O$, i.e. $[j_0^{(a)}, O] = q_{(a)}^O O$. We find

$$q_{(a)}^O \left\langle q'_{(a)}\right| O \left|q_{(a)}\right\rangle = (-q'_{(a)} + q_{(a)} + Q_a) \left\langle q'_{(a)}\right| O \left|q_{(a)}\right\rangle . \tag{2.49}$$

Only for $q_{(a)}^O = (-q'_{(a)} + q_{(a)} + Q_a)$ the result is non-vanishing, i.e. the operator insertions have to cancel the *background charges* Q_a.

Just as the right-moving fermions, the superconformal ghost system may be bosonized according to[11]

$$\begin{aligned} b_{zz}(z) &=: \mathrm{e}^{-\phi_{bc}(z)} :, & c^z(z) &=: \mathrm{e}^{\phi_{bc}(z)} :, \\ b_{\bar z\bar z}(\bar z) &=: \mathrm{e}^{-\bar\phi_{bc}(\bar z)} :, & c^{\bar z}(\bar z) &=: \mathrm{e}^{\bar\phi_{bc}(\bar z)} :, \\ \beta_{\bar z\bar\theta}(\bar z) &=: \mathrm{e}^{-\bar\phi_{\beta\gamma}(\bar z)}\mathrm{e}^{\bar\chi(\bar z)}\bar\partial\bar\chi(\bar z) :, & \gamma^{\bar z}(\bar z) &=: \mathrm{e}^{-\bar\chi(\bar z)}\mathrm{e}^{\bar\phi_{\beta\gamma}(\bar z)} :, \end{aligned}$$

where

$$\begin{aligned} \phi_{bc}(z)\phi_{bc}(w) &= \log(z-w)\,, & \bar\phi_{bc}(\bar z)\bar\phi_{bc}(\bar w) &= \log(\bar z - \bar w)\,, \\ \bar\phi_{\beta\gamma}(\bar z)\bar\phi_{\beta\gamma}(\bar w) &= -\log(\bar z - \bar w)\,, & \bar\chi(\bar z)\bar\chi(\bar w) &= \log(\bar z - \bar w) \end{aligned}$$

and we have as well "bosonized" the Bose fields β,γ. Note that the bosonized representations of β,γ are in the $\mathcal{NS}$ sector. To create fields in the $\mathcal{R}$ sector ghost spin fields $\mathrm{e}^{\pm\frac{1}{2}\bar\phi_{\beta\gamma}}$ have to be inserted.
In terms of the bosonized fields the currents are given by

$$j(z) = \partial\phi_{bc}(z)\,, \quad \bar j_{bc}(\bar z) = \bar\partial\bar\phi_{bc}(\bar z)\,, \quad \bar j_{\beta\gamma}(\bar z) = -\bar\partial\bar\phi_{\beta\gamma}(\bar z)\,.$$

Further conformal primaries may be constructed as exponentials of $\phi_a \in \{\phi_{bc}, \bar\phi_{bc}, \bar\phi_{\beta\gamma}\}$. They fulfil

$$\begin{aligned} j(z) : \mathrm{e}^{q\phi_{bc}(w)} : &= \frac{q}{z-w} : \mathrm{e}^{q\phi_{bc}(w)} : + \ldots\,, \\ \bar j_a(\bar z) : \mathrm{e}^{q\bar\phi_a(\bar w)} : &= \frac{q}{\bar z - \bar w} : \mathrm{e}^{q\bar\phi_a(\bar w)} : + \ldots\,, \\ T(z) : \mathrm{e}^{q\phi_{bc}(w)} : &= \frac{\frac{1}{2}q(q+Q)}{(z-w)^2} : \mathrm{e}^{q\phi_{bc}(w)} : + \frac{1}{z-w} : \partial_w \mathrm{e}^{q\phi_{bc}(w)} : + \ldots\,, \\ \bar T(\bar z) : \mathrm{e}^{q\bar\phi_a(\bar w)} : &= \frac{\frac{1}{2}\epsilon q(q+\bar Q_a)}{(\bar z - \bar w)^2} : \mathrm{e}^{q\bar\phi_a(\bar w)} : + \frac{1}{\bar z - \bar w} : \bar\partial_{\bar w} \mathrm{e}^{q\bar\phi_a(\bar w)} : + \ldots\,, \end{aligned} \tag{2.50}$$

where $\epsilon = +1$ (-1) for the b,c (β,γ) system and $a \in \{bc, \beta\gamma\}$. These operators hence shift the ghost charges of states by q units. They are vertex operator for states[12]

$$|q_a = q\rangle =: \mathrm{e}^{q\phi_a(0)} : |0\rangle \,, \tag{2.51}$$

[11] Analogously to the discussion of bosonization above, cocycle factors are necessary to achieve the correct (anti-)commutation relations of the fields. For simplicity we suppress them in the following expressions.

[12] Note that the states $|q_a\rangle$ can have a lower energy than the vacuum state $|0\rangle$ and for the Bose fields β,γ the energy is not even bounded from below. For a discussion of why this does not lead to an instability of the vacuum we refer to [39]. Note further that L_{-1} annihilates only the state $|0\rangle$, so the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{OSP}(1|2)$ invariant vacuum remains unique.

where $q \in \mathbb{Z}$ for the b,c systems and the $\mathcal{NS}$ sector of the β,γ system and $q \in \mathbb{Z} + \frac{1}{2}$ for the $\mathcal{R}$ sector of the β,γ system. We find

$$\langle 0| : \mathrm{e}^{3\phi_{bc}(0)+3\bar{\phi}_{bc}(0)-2\bar{\phi}_{\beta\gamma}(0)} : |0\rangle = 1 \, .$$

As the fields of the b,c ghost systems follow Fermi statistics there is only a finite number of states (2.51) of different ghost charge. For the β,γ system however, the fact that the fields follow Bose statistics leads to the existence of infinitely many such states.
Note that regularity of the energy-momentum tensor implies that the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{OSP}(1|2)$ invariant vacuum $|0\rangle$ fulfils

$$\begin{aligned}
\tilde{b}_n\,|0\rangle &= b_n\,|0\rangle = 0\,, && n \geq -1\,,\\
\tilde{c}_n\,|0\rangle &= c_n\,|0\rangle = 0\,, && n \geq 2\,,\\
\beta_r\,|0\rangle &= 0\,, && r \geq -\frac{1}{2}\,,\\
\gamma_r\,|0\rangle &= 0\,, && r \geq \frac{3}{2}\,.
\end{aligned}$$

BRST and Covariant Vertex Operators

In the framework of BRST quantization the physical asymptotic states of the theory are obtained as cohomology classes of the BRST operator, i.e. the vertex operators must be BRST closed but not exact. They have to fulfil

$$\begin{aligned}
Q_{\mathrm{BRST}}\,|\mathrm{phys}\rangle = \bar{Q}_{\mathrm{BRST}}\,|\mathrm{phys}\rangle &= 0\,,\\
|\mathrm{phys}\rangle &\neq Q_{\mathrm{BRST}}\,|\mathrm{phys}'\rangle\\
&\neq \bar{Q}_{\mathrm{BRST}}\,|\mathrm{phys}'\rangle\,.
\end{aligned} \tag{2.52}$$

The BRST operator of heterotic string theory is given by [39]

$$\begin{aligned}
Q_{\mathrm{BRST}} &= \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \left\{ c^z(z) \left(T(z) + \frac{1}{2} T_{\mathrm{gh}}(z) \right) \right\}\,,\\
\bar{Q}_{\mathrm{BRST}} &= \oint \frac{\mathrm{d}\bar{z}}{2\pi\mathrm{i}} \left\{ c^{\bar{z}}(\bar{z}) \left(\bar{T}(\bar{z}) + \frac{1}{2} \bar{T}_{\mathrm{gh}}(\bar{z}) \right) - \gamma^{\bar{z}}(\bar{z}) \left(\bar{T}_{\mathrm{F}}(\bar{z}) + \frac{1}{2} \bar{T}_{\mathrm{F\,gh}}(\bar{z}) \right) \right\}\,,
\end{aligned} \tag{2.53}$$

where the energy-momentum tensors of the matter and ghost systems are given in (2.27) and (2.43). The BRST transformations of the fields can be straightforwardly calculated. The results read

$$\begin{aligned}
\left[Q_{\mathrm{BRST}},\, X^\mu_{\mathrm{L}}(z)\right] &= c^z \partial X^\mu_{\mathrm{L}}(z)\,, & \left[Q_{\mathrm{BRST}},\, X^I_{\mathrm{L}}(z)\right] &= c^z \partial X^I_{\mathrm{L}}(z)\,,\\
\left\{Q_{\mathrm{BRST}},\, c^z(z)\right\} &= c^z \partial c^z(z)\,, & \left\{Q_{\mathrm{BRST}},\, b_{zz}(z)\right\} &= T(z) + T_{\mathrm{gh}}(z)
\end{aligned}$$

and

$$\begin{aligned}
\left[\bar{Q}_{\mathrm{BRST}},\, X^\mu_{\mathrm{R}}(\bar{z})\right] &= c^{\bar{z}} \bar{\partial} X^\mu_{\mathrm{R}}(\bar{z}) + \frac{\mathrm{i}}{2} \gamma^{\bar{z}} \psi^\mu_{\mathrm{R}}(\bar{z})\,,\\
\left\{\bar{Q}_{\mathrm{BRST}},\, \psi^\mu_{\mathrm{R}}(\bar{z})\right\} &= \frac{1}{2} (\bar{\partial} c^{\bar{z}}) \psi^\mu_{\mathrm{R}}(\bar{z}) + \frac{1}{2} c^{\bar{z}} \bar{\partial} \psi^\mu_{\mathrm{R}}(\bar{z}) - \frac{\mathrm{i}}{2} \gamma^{\bar{z}} \bar{\partial} X^\mu_{\mathrm{R}}(\bar{z})\,,
\end{aligned}$$

as well as

$$\{\bar{Q}_{\rm BRST},\, c^{\bar z}(\bar z)\} = c^{\bar z}\bar\partial c^{\bar z}(\bar z) - \frac{1}{4}(\gamma^{\bar z})^2(\bar z)\,, \qquad \{\bar{Q}_{\rm BRST},\, b_{\bar z\bar z}(\bar z)\} = \bar T(\bar z) + \bar T_{\rm gh}(\bar z)\,,$$
$$[\bar{Q}_{\rm BRST},\, \gamma^{\bar z}(\bar z)] = -\frac{1}{2}(\bar\partial c^{\bar z})\gamma^{\bar z}(\bar z) + c^{\bar z}\bar\partial\gamma^{\bar z}(\bar z)\,, \qquad [\bar{Q}_{\rm BRST},\, \beta_{\bar z\bar\theta}(\bar z)] = -\bar T_{\rm F}(\bar z) - \bar T_{\rm F\,gh}(\bar z)\,.$$

Further $Q^2_{\rm BRST} = \bar{Q}^2_{\rm BRST} = 0$ and the BRST operator (anti-)commutes with the total energy-momentum tensor (fermionic supercurrent) so that the algebra closes.

Note that BRST invariant operators come in different *ghost pictures*. This is a consequence of the existence of ghost zero modes, just analogous to the existence of the states (2.51). For the b,c systems the situation is rather simple. Conformal invariance requires the vertex operators to be conformal dimension $h = \bar h = 1$ operators that are either integrated over, or multiplied by c and $\bar c$, i.e.

$$\int \mathrm{d}^2 z\, V_{h=\bar h=1}(z,\bar z) \qquad \text{or} \qquad c(z)\bar c(z) V_{h=\bar h=1}(z,\bar z)\,. \tag{2.55}$$

Note that in a tree-level L-point function, exactly three of the operators have to be of the latter form in order to cancel the background ghost charges. However, the situation in the β,γ system is more complicated. There each vertex operator has infinitely many representations of different ghost picture, which we label by a subscript on the operator. We will see below that states in the $\mathcal{NS}$ ($\mathcal{R}$) sector have (half-)integral ghost picture charge. The β,γ ghost picture of a vertex operator can be changed by means of a *picture changing* operation. The only such operation we will need in the following is the one changing the picture from -1 to 0. This can be done using an insertion of the fermionic supercurrent,

$$(V_{-1}\mathrm{e}^{\bar\phi_{\beta\gamma}}\bar T_{\rm F})(z,\bar z) = V_0(z,\bar z)\,. \tag{2.56}$$

Note that, as is required, picture changing does not change the conformal dimension h, $\bar h$ of the vertex operator. Correlation functions are independent of the ghost pictures of the vertex operators they contain, as long as the background ghost charge is cancelled [39].

Let us start our discussion of vertex operators corresponding to massless physical states with the $\mathcal{NS}$ sector. Note that since the superconformal ghost fields are associated to $\bar T_{\rm F}$ they need to have the same periodicity as that operator. Hence they need to have the same periodicity as the $\psi_{\rm R}$.
The general BRST invariant, massless $\mathcal{NS}$ vertex operator in *canonical ghost picture*[13] has the form $V_{-1} = \int \mathrm{d}^2 z V_{-1}(z,\bar z)$, where[14] [36, 39]

$$V_{-1}(z,\bar z) = \mathrm{e}^{-\bar\phi_{\beta\gamma}(\bar z)}\, \zeta^{(q)}_{\mu/I}\partial X^{\mu/I}_{\rm L}(z)\, \mathrm{e}^{\mathrm{i}q\cdot H(\bar z)}\, \mathrm{e}^{\mathrm{i}P_I X^I_{\rm L}(z)}\, \mathrm{e}^{\mathrm{i}p\cdot X(z,\bar z)}\,. \tag{2.57}$$

Here q is an $\mathrm{SO}(10)$ vector weight lattice vector, P is an $\mathrm{E}_8\times\mathrm{E}_8$ vector weight lattice vector and p denotes the momentum of the state. The q-dependent polarization of the state is described by $\zeta^{(q)}_{\mu/I}$. We have refrained from writing possible right-moving oscillators be-

[13] Canonical ghost picture just means that it is the ghost picture in which the vertex operator takes its simplest form. Of course it can be brought to any other ghost picture by means of picture changing.
[14] Note that for the remainder of this work we suppress cocycle factors arising from the bosonization of various fields wherever they occur, as they will not be important for our purposes.

cause vertex operators containing such oscillators can not have conformal dimension $(1,1)$ due to the contribution of the ghosts to $\bar{h}$. There are two kinds solutions to the constraint $h = \bar{h} = 1$. Namely $p^2 = 0$, $q^2 = 1$ and $(P^2, N) = (0,1)$ or $(P^2, N) = (2,0)$, where N is zero or one, depending on whether the oscillator ∂X^μ is present or not. The corresponding vertex operators are[15]

$$\begin{aligned} V_{-1}(z,\bar{z}) &= \mathrm{e}^{-\bar{\phi}_{\beta\gamma}(\bar{z})}\, \zeta_\mu^{(q)} \partial X_\mathrm{L}^\mu(z)\, \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})}\,, \\ V_{-1}(z,\bar{z}) &= \mathrm{e}^{-\bar{\phi}_{\beta\gamma}(\bar{z})}\, \zeta_I^{(q)} \partial X_\mathrm{L}^I(z)\, \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})}\,, \\ V_{-1}(z,\bar{z}) &= \mathrm{e}^{-\bar{\phi}_{\beta\gamma}(\bar{z})}\, \mathrm{e}^{\mathrm{i}P_I X_\mathrm{L}^I(z)}\, \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})}\,. \end{aligned}$$

These vertex operators are in one-to-one correspondence with the $\mathcal{NS}$ states we identified in section 2.1.2. The first class corresponds to graviton, anti-symmetric tensor and dilaton, the second to the 16 Cartan generators of $\mathrm{E}_8 \times \mathrm{E}_8$ and the third one to the 480 root generators of $\mathrm{E}_8 \times \mathrm{E}_8$.

Let us now turn to the $\mathcal{R}$ sector. We have seen above that the $\mathcal{R}$ vacuum is created out of the $\mathcal{NS}$ vacuum by the spin fields S_α of conformal dimension $\bar{h} = \frac{5}{8}$. As the ghost fields have to have $\mathcal{R}$ boundary conditions as well, we need an additional twist field that creates the ghost $\mathcal{R}$ vacuum. We have seen above that such a ghost spin field is given by[16] $\mathrm{e}^{-\frac{1}{2}\bar{\phi}_{\beta\gamma}}$ of conformal dimension $\bar{h} = \frac{3}{8}$. With this information at hand, we can write down the general BRST invariant, massless $\mathcal{R}$ vertex operator in its canonical ghost picture. It is given by $V_{-\frac{1}{2}} = \int \mathrm{d}^2 z V_{-\frac{1}{2}}(z,\bar{z})$, where

$$V_{-\frac{1}{2}}(z,\bar{z}) = \mathrm{e}^{-\frac{1}{2}\bar{\phi}_{\beta\gamma}(\bar{z})}\, \zeta_{\mu/I}^{(q)} \partial X_\mathrm{L}^{\mu/I}(z)\, \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})}\, \mathrm{e}^{\mathrm{i}P_I X_\mathrm{L}^I(z)}\, \mathrm{e}^{\mathrm{i}p\cdot X(z,\bar{z})}\,,$$

where now q is a vector in the $\mathrm{SO}(10)$ spinor or cospinor weight lattice. Solving the constraints $h = \bar{h} = 1$, we arrive at the spacetime $\mathcal{N} = 1$ superpartners of the massless states from the $\mathcal{NS}$ sector,

$$\begin{aligned} V_{-\frac{1}{2}}(z,\bar{z}) &= \mathrm{e}^{-\frac{1}{2}\bar{\phi}_{\beta\gamma}(\bar{z})}\, \zeta_\mu^{(q)} \partial X_\mathrm{L}^\mu(z)\, \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})}\,, \\ V_{-\frac{1}{2}}(z,\bar{z}) &= \mathrm{e}^{-\frac{1}{2}\bar{\phi}_{\beta\gamma}(\bar{z})}\, \zeta_I^{(q)} \partial X_\mathrm{L}^I(z)\, \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})}\,, \\ V_{-\frac{1}{2}}(z,\bar{z}) &= \mathrm{e}^{-\frac{1}{2}\bar{\phi}_{\beta\gamma}(\bar{z})}\, \mathrm{e}^{\mathrm{i}P_I X_\mathrm{L}^I(z)}\, \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})}\,. \end{aligned}$$

2.2 Toroidal Orbifolds

Toroidal orbifolds form the class of spaces one obtains by dividing the action of a discrete symmetry group G out of a d dimensional flat torus,

$$O = \frac{T^d}{G}\,.$$

[15] We suppress normal ordering symbols in the vertex operators.

[16] Another candidate would be $\mathrm{e}^{\frac{1}{2}\bar{\phi}_{\beta\gamma}}$, which however has conformal dimension $\bar{h} = -\frac{3}{8}$. Hence the combination $S_\alpha \mathrm{e}^{\frac{1}{2}\bar{\phi}_{\beta\gamma}}$ of conformal dimension $\bar{h} = \frac{1}{4}$ cannot be combined with other right-moving fields to yield a vertex operator of $\bar{h} = 1$.

Here G is a finite discrete subgroup of the isomorphism group of the torus and is called *orbifolding group*. We will only consider the case $d = 6$, in which case the six dimensional torus can be described by the six dimensional flat space modulo a lattice Λ, $T^6 = \mathbb{R}^6/2\pi\Lambda$. This leads to the definition of the *space group* S,

$$O = \frac{T^d}{G} = \frac{\mathbb{R}^6}{S}\,. \tag{2.58}$$

Consider as an example the two dimensional orbifold $T^2_{\mathrm{SU}(3)}/\mathbb{Z}_2$, where $T^2_{\mathrm{SU}(3)}$ is the torus obtained by taking the quotient of $\mathbb{R}^2$ by the root lattice of $\mathrm{SU}(3)$. As depicted in figure 2.1, the fundamental region of this orbifold forms a tetrahedron [25].

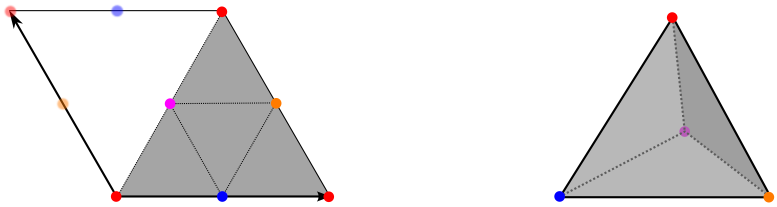

Figure 2.1: The shaded region is the fundamental region of the two dimensional orbifold $T^2_{\mathrm{SU}(3)}/\mathbb{Z}_2$, which forms a tetrahedron. The arrows display the basis vectors of the torus lattice and the coloured dots mark the loci of fixed points. Those with the same colour are equivalent on the orbifold.

Any space group element may be written as a pair (ϑ, λ), where $\vartheta \in \mathrm{O}(6)$ and λ is a translation. The action on a point X is

$$S \ni (\vartheta, \lambda): \quad \begin{array}{l} \mathbb{R}^6 \longrightarrow \mathbb{R}^6\,, \\ X \longmapsto \vartheta X + 2\pi\lambda \end{array}, \tag{2.59}$$

from which the group product can be determined to be $(\vartheta, \lambda)\cdot(\vartheta', \lambda') = (\vartheta\vartheta', \vartheta\lambda' + \lambda)$. As a consequence the orbifold may be described by points X in $\mathbb{R}^6$ with the identification

$$X \cong \vartheta X + 2\pi\lambda\,, \qquad \forall\, (\vartheta, \lambda) \in S\,. \tag{2.60}$$

The subset of space group elements of the form $(\vartheta, 0)$ forms a finite discrete group, called the *point group* P. If the point group is a subgroup of the space group then S is a semi-direct product of P and Λ, $S = P \ltimes \Lambda$ and $G = P$. This is the case if the space group does not contain *roto-translations*, i.e. elements of the form (ϑ, λ) such that $(\vartheta, 0) \notin S$.
Note that if $S = P \ltimes \Lambda$ the generators of the space group are given by the generators of P, denoted ϑ_k, and the basis vectors of the lattice, denoted by e_i,

$$S = \langle (\vartheta_1,\, 0),\, \ldots,\, (\vartheta_N,\, 0),\, (\mathbb{1},\, e_1),\, \ldots,\, (\mathbb{1},\, e_6) \rangle\,,$$

where $N = \mathrm{rank}(P)$. In the presence of roto-translations however, the space group can contain elements with fractional lattice vectors, $(\vartheta, n^i e_i)$ where $n^i \notin \mathbb{Z}$.

The action of the space group on $\mathbb{R}^6$ is in general not free, that means its elements can have non-trivial fixed hypersurfaces $\mathcal{F}$,

$$\vartheta X + 2\pi\lambda = X \qquad \forall X \in \mathcal{F}\,. \tag{2.61}$$

The space group element $g = (\vartheta, \lambda)$ is called the *constructing element* of $\mathcal{F}$. In the following we will only be dealing with fixed hypersurfaces of even dimension, that is with *fixed points* X_f,

$$g\,X_\mathrm{f} = \vartheta X_\mathrm{f} + 2\pi\lambda = X_\mathrm{f}\,, \tag{2.62}$$

and *fixed tori*.
Let g be the constructing element of the fixed hypersurface $\mathcal{F}$ and let h be another space group element, such that $hg \neq gh$. Then

$$(hgh^{-1})\,hX = h\,gX = hX\,, \qquad \forall\, hX \in h\mathcal{F}\,,$$

i.e. $h\mathcal{F}$ is a fixed hypersurface of hgh^{-1}. Hence fixed hypersurfaces are in one-to-one correspondence to *conjugacy classes*[17] $[g]$ of S, given by

$$[(\vartheta, \lambda)] = \left\{ \left(\vartheta, (1-\vartheta)\tilde{\lambda} + \tilde{\vartheta}\lambda\right) \,\middle|\, (\tilde{\vartheta}, \tilde{\lambda}) \in S \right\}. \tag{2.63}$$

Unlike manifolds, orbifolds can have discrete holonomy groups. Indeed the holonomy group of a toroidal orbifold is given by its point group, as can be seen as follows. Take two points x and $y = \vartheta x + 2\pi\lambda$ on the orbifold and a vector v in the tangent space of x as displayed in figure 2.2. By the action of (ϑ, λ) the vector v is mapped to ϑv in the tangent space of y. Now parallel transport the vector along a path γ from y to x. As the points are identified, this is a closed loop on the orbifold and as the torus is flat, the vector ϑv does not change along the path.

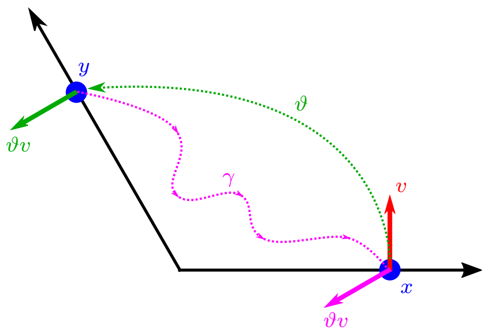

Figure 2.2: Orbifolds have discrete holonomy groups.

As a result, on the orbifold space, the vector v has been mapped to ϑv by means of parallel transport along a closed loop.

2.2.1 Classification of Space Groups

There are different classification schemes for space groups [42]. We choose the classification into $\mathbb{Q}$, $\mathbb{Z}$ and *affine classes* as depicted in figure 2.3. As we will see later, this classification allows for a physical interpretation in the context of heterotic orbifolds.

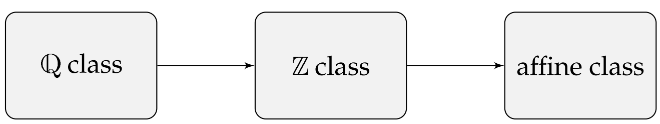

Figure 2.3: Classification of space groups into $\mathbb{Q}$, $\mathbb{Z}$ and affine classes.

Let S_1 and S_2 be two space groups with associated point groups P_1 and P_2 respectively.

[17] Two space group elements g and g' are conjugate, $g \sim g'$, iff there is an element $h \in S$ such that $hgh^{-1} = g'$.

Then S_1 and S_2 belong to the same $\mathbb{Q}$ class iff

$$\exists V \in \mathrm{GL}(6,\mathbb{Q}) \quad \text{s.t.} \quad VP_1V^{-1} = P_2\,.$$

As a consequence space groups within the same $\mathbb{Q}$ class have identical holonomy groups.
Further, two space groups belong to the same $\mathbb{Z}$ class, iff

$$\exists V \in \mathrm{GL}(6,\mathbb{Z}) \quad \text{s.t.} \quad VP_1V^{-1} = P_2\,.$$

As the elements of P_i can be represented by $\mathrm{GL}(6,\mathbb{Z})$ matrices themselves, this means that the point groups are related by a change of basis. Equivalently the $\mathbb{Z}$ class specifies the lattice Λ of the space group. If two space groups belong to the same $\mathbb{Z}$ class, they clearly belong to the same $\mathbb{Q}$ class as well.
If not only the point groups are equal up to a change of basis but the space groups are related by an affine transformation $A : \mathbb{R}^6 \to \mathbb{R}^6$,

$$AS_1A^{-1} = S_2\,,$$

the space groups belong to the same affine class. The action of any affine transformations can be written as $A:\ x \mapsto Mx + b$, where M is a linear transformation and b is a translation vector. Hence the action on the space group elements is given by

$$\forall g = (\vartheta, \lambda) \in S \qquad AgA^{-1} = (M\vartheta M^{-1},\, M\lambda + (\mathbb{1} - M\vartheta M^{-1})b)\,.$$

The classification of the space groups relevant for the compactification of the heterotic string to four dimensions was performed in [34]. We will review the classification of space groups with abelian point groups, which is the case we will restrict ourselves to for the remainder of this work, in section 3.2. We will see that the point groups that are of interest to us, are $\mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_M$ subgroups of $\mathrm{SU}(3)$. Let us denote the point group generators by θ (θ and ω for the case of $\mathbb{Z}_N \times \mathbb{Z}_M$). As they can be embedded in the Cartan subgroup of $\mathrm{SO}(6)$, we may write them as

$$\begin{aligned} \theta &= \mathrm{e}^{2\pi \mathrm{i}(v_\theta^1 J_{12} + v_\theta^2 J_{34} + v_\theta^3 J_{56})}\,, \\ \omega &= \mathrm{e}^{2\pi \mathrm{i}(v_\omega^1 J_{12} + v_\omega^2 J_{34} + v_\omega^3 J_{56})}\,, \end{aligned} \tag{2.64}$$

where J_{12}, J_{34} and J_{56} are the Cartan generators of $\mathrm{SO}(6)$. We call $(v_1,\, v_2,\, v_3)$ and $(w_1,\, w_2,\, w_3)$ the *twist vectors* of the point group. Consequently every point group element ϑ may be represented by a twist vector v_ϑ. If there is a basis of the lattice, such that the matrix representation of θ and ω takes the form

$$M = \begin{pmatrix} M_{12} & 0 & 0 \\ 0 & M_{34} & 0 \\ 0 & 0 & M_{56} \end{pmatrix}$$

in that basis, the orbifold is called *factorizable*.

2.3 Compactification of the Heterotic String on Toroidal Orbifolds

We will consider heterotic string theory compactified on six dimensional orbifolds, such that the spacetime background is

$$\mathcal{M}^{3,1} \times O = \mathcal{M}^{3,1} \times \frac{\mathbb{R}^6}{S}\,.$$

As the orbifold geometry is flat everywhere but at the loci of fixed hypersurfaces, the heterotic SCFT remains free and hence exactly solvable. Therefore many of the features of the heterotic string on a ten dimensional Minkowski spacetime, which we discussed in section 2.1, remain valid.

There are restrictions on the space groups that give rise to well-defined string theories. For the representation of the point group generators (2.64) on the $\mathscr{R}$ spinors to be well-defined it is required that

$$N_{(a)} \sum_{i=1}^{3} v^i_{(a)} = 0 \mod 2\,, \tag{2.65}$$

where $a \in \{\theta, \omega\}$. Further, in order to preserve (at least) $\mathcal{N} = 1$ supersymmetry, the holonomy group, that is the point group, should be a discrete subgroup of $\mathrm{SU}(3)$ [19]. For the point group generators this means [21],

$$\sum_{i=1}^{3} |v^i_{(a)}| = 0\,,$$

where it is always possible to choose the signs of the $v^i_{(a)}$ such that

$$\sum_{i=1}^{3} v^i_{(a)} = 0\,. \tag{2.66}$$

Hence the condition (2.65) is trivial once (2.66) is fulfilled.

2.3.1 Gauge Embedding

It is possible, and we will see that it is required in most of the cases, to extend the action of the space group to the 16 internal gauge degrees of freedom. If the point group is abelian, the action can always be realized by a shift in the gauge coordinates [21], such that the group homomorphism reads[18]

$$\varphi:\ g = (\vartheta, n^\alpha e_\alpha) \longmapsto V_g = V_\vartheta + n^i W_i\,, \tag{2.67}$$

where V_ϑ is called the *shift embedding* and W_α are six discrete *Wilson lines*, which correspond to a constant gauge background. This means that a transformation of the compactified coordinates by a space group element g, $X \mapsto gX$, is always accompanied by a shift in the gauge coordinates, $X^I_\mathrm{L} \mapsto X^I_\mathrm{L} + \pi V^I_g$. Note that the embedding breaks the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group down

[18] More complicated actions of the space group on the gauge coordinates are possible and lead to continuous Wilson lines [43]. We will, however, not consider such embeddings.

to a subgroup of the same rank. The map φ is a homomorphism, provided

$$\begin{aligned} n_g V_g &\in \Gamma_{\mathrm{E}_8\times\mathrm{E}_8}\,, & \forall g &\in S\,,\\ (\mathbb{1}-\vartheta)_\alpha{}^\beta W_\beta &\in \Gamma_{\mathrm{E}_8\times\mathrm{E}_8}\,, & \forall \vartheta &\in P\,, \end{aligned} \tag{2.68}$$

where n_g is the order of the element g, $g^n = \mathbb{1}$, and $\vartheta_\alpha{}^\beta e_\beta = (\vartheta e)_\alpha$. The shift embedding vectors are subject to further consistency conditions arising from modular invariance. They read [21]

$$N_{(a)}\left(V_{(a)}^2 - v_{(a)}^2\right) = 0 \mod 2\,. \tag{2.69}$$

As for many point groups $v_{(a)}^2 \neq 0 \mod 2$, it is in general necessary to extend the point group action non-trivially into the gauge coordinates.

2.3.2 Boundary Conditions, Mode Expansions and Hilbert Space

In the following discussion we will use complexified coordinates for the bosons,

$$\begin{aligned} Z^m &= \frac{1}{\sqrt{2}}\left(X^{2m} + \mathrm{i}\,X^{2m+1}\right)\,,\\ \bar{Z}^m &= \frac{1}{\sqrt{2}}\left(X^{2m} - \mathrm{i}\,X^{2m+1}\right)\,, \end{aligned} \tag{2.70}$$

where $m = 0,\dots,4$, as well as the complexified fermions (2.32). We will often use the index $i = 1,2,3$ to denote the internal directions, i.e. $i = 1,\dots$ corresponds to $m = 2,\dots$ in (2.32). In this basis, the action of the point group elements on the internal string coordinates is given by

$$Z^i \overset{\vartheta}{\longmapsto} \mathrm{e}^{2\pi \mathrm{i} v_\vartheta^i} Z^i\,, \quad \bar{Z}^i \overset{\vartheta}{\longmapsto} \mathrm{e}^{-2\pi \mathrm{i} v_\vartheta^i} \bar{Z}^i\,, \quad \psi_\mathrm{R}^i \overset{\vartheta}{\longmapsto} \mathrm{e}^{2\pi \mathrm{i} v_\vartheta^i} \psi_\mathrm{R}^i\,, \quad \bar{\psi}_\mathrm{R}^i \overset{\vartheta}{\longmapsto} \mathrm{e}^{-2\pi \mathrm{i} v_\vartheta^i} \bar{\psi}_\mathrm{R}^i\,.$$

The most immediate consequence of the identification (2.60) is that the closed string boundary conditions (2.5) of the internal degrees of freedom get generalized to

$$\begin{aligned} Z^i(\mathrm{e}^{2\pi\mathrm{i}}z, \mathrm{e}^{-2\pi\mathrm{i}}\bar{z}) &= \mathrm{e}^{2\pi\mathrm{i}v_\vartheta^i} Z^i(z,\bar{z}) + 2\pi n^\alpha e_\alpha^i\,,\\ \psi_\mathrm{R}^i(\mathrm{e}^{-2\pi\mathrm{i}}\bar{z}) &= \begin{cases} -\mathrm{e}^{2\pi\mathrm{i}v_\vartheta^i}\psi_\mathrm{R}^i(\bar{z}) & (\mathcal{R})\\ \mathrm{e}^{2\pi\mathrm{i}v_\vartheta^i}\psi_\mathrm{R}^i(\bar{z}) & (\mathcal{NS}) \end{cases}\,,\\ X_\mathrm{L}^I(\mathrm{e}^{2\pi\mathrm{i}}z) &= X_\mathrm{L}^I(z) + 2\pi(V_\vartheta^I + n^\alpha W_\alpha^I) + 2\pi\gamma^I\,, \end{aligned} \tag{2.71}$$

where $\{e_\alpha\,, \alpha = 1,2,3\}$ denotes a complexified basis of the torus lattice and $\gamma \in \Gamma_{\mathrm{E}_8\times\mathrm{E}_8}$. Similar boundary conditions hold for $\bar{Z}^i, \bar{\psi}^i$. If a string state fulfils such a boundary condition it is called a *twisted string* and $(\vartheta, n^\alpha e_\alpha)$ is called the *constructing element* of the string. Strings, the constructing elements of which are given by the identity element, are called *untwisted*. Twisted and untwisted strings on an orbifold are visualized in figure 2.4.

The mode expansion of Z^i fulfilling these boundary conditions is given by

$$\begin{aligned} Z^i(z,\bar z) &= z^i + p^i \ln z\bar z + \mathrm{i}\sum_{n\in\mathbb{Z}}\left(\frac{1}{n-w^i_\vartheta}\,\tilde\alpha^i_{n-w^i_\vartheta}\, z^{-n+w^i_\vartheta} + \frac{1}{n+w^i_\vartheta}\,\alpha^i_{n+w^i_\vartheta}\,\bar z^{-n-w^i_\vartheta}\right),\\ \bar Z^i(z,\bar z) &= \bar z^i + \bar p^i \ln z\bar z + \mathrm{i}\sum_{n\in\mathbb{Z}}\left(\frac{1}{n+w^i_\vartheta}\,\bar{\tilde\alpha}^i_{n+w^i_\vartheta}\, z^{-n-w^i_\vartheta} + \frac{1}{n-w^i_\vartheta}\,\bar\alpha^i_{n-w^i_\vartheta}\,\bar z^{-n+w^i_\vartheta}\right), \end{aligned} \tag{2.72}$$

where $w^i_\vartheta = v^i_\vartheta - \lfloor v^i_\vartheta \rfloor$. If $g = \mathbb{1}$, the mode expansion reduces to (2.6), (2.7).

If the string is twisted, we find that the boundary conditions require the momentum to vanish, $p = \bar p = 0$, and the center of mass coordinate of the string to be a fixed point[19], $z^i = z^i_f = \mathrm{e}^{2\pi \mathrm{i} v^i_\vartheta} z^i_f + 2\pi n^\alpha e^i_\alpha$. That means, twisted strings are localized at the fixed points and cannot propagate away. Furthermore twisted strings involve fractional oscillators and hence the eigenvalues of the number operator $\mathcal{N}$ are no longer necessarily integers. We have seen above that inequivalent fixed points are in one-to-one correspondence with the conjugacy classes of the space group.

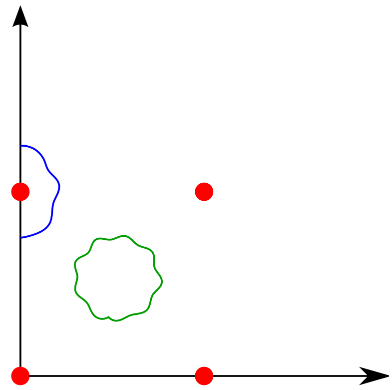

Figure 2.4: An untwisted and a twisted string on a 2D $\mathbb{Z}_2$ orbifold.

Similarly let g be the constructing element of a string state and h another space group element such that $gh \neq hg$. Then under the action of h the initial point of the string is mapped to $hZ(z)$ while the final point is mapped to $hgZ(z) = hgh^{-1}\,hZ(z)$, i.e. the image of the string has constructing element hgh^{-1}. Of course the string Hilbert space $\mathcal{H}$ must be space group invariant. Therefore it is constructed as [20, 21]

$$\mathcal{H} = \oplus_{[g]}\ \mathcal{H}_{[g]}\,, \tag{2.73}$$

where $\mathcal{H}_{[g]}$ are the Hilbert spaces of states, the constructing elements g' of which are conjugate to g. They are space group invariant themselves and are constructed as sums

$$\mathcal{H}_{[g]} = \oplus_{g'\in g}\ P_{h\in C_S(g')}(\mathcal{H}_{g'})\,,$$

where $\mathcal{H}_{g'}$ denotes the Hilbert space of states with constructing element g', $C_S(g')$ is the centralizer of g' in S and $P_{h\in C_S(g')}$ projects onto the subspace invariant under space group elements h that commute with g'.

Instead of first performing the light-cone gauge quantization as for the uncompactified heterotic string, we will directly discuss the modifications of the superconformal field theory that are required for the description of the orbifolded heterotic string. For this discussion we will restrict ourselves to orbifolds with $\mathbb{Z}_N$ point group and assume that there are no rototransla-

[19] As ϑ might not have full rank, the more general statement is that the center of mass of twisted string is fixed to the locus of the fixed hyperplane of g. We will, for simplicity, often make statements about fixed points which straightforwardly generalize to fixed hypersurfaces.

tions, i.e. that the space group is a semi-direct product of the point group with the lattice.

2.3.3 Orbifold Superconformal Field Theory

We have seen above that the main new feature arising in orbifold compactifications of the heterotic string is the presence of twisted sectors. In the worldsheet SCFT these sectors are, analogously to the $\mathcal{R}$ sector, created by *twist fields* [25].

Twist Fields

While the $\mathrm{SL}(2,\mathbb{R})$ invariant vacuum is in the untwisted sector, twisted string states are created by acting with the various untwisted fields on the twisted vacuum states

$$|g\rangle = \Sigma_g(0,0)\,|0\rangle \ .$$

Unlike the spin fields that create the $\mathcal{R}$ ground state out of the vacuum, the twist fields twist all physical degrees of freedom of the string, i.e. the worldsheet bosons Z^i and X^I as well as the fermions ψ^μ. The ghost fields are not twisted because that would correspond to a change of the worldsheet topology [25]. The total twist field Σ may be split into components which only act on one kind of worldsheet fields each. We will denote the field twisting the bosonic, fermionic and gauge degrees of freedom by σ_g, s_g and $\tilde{s}_g$ respectively.

The fields most complicated to deal with are the bosonic twist fields σ_g. In the presence of a twist field $\sigma_g(0,0)$ at the origin, $Z(z,\bar{z})$ is subject to the *global monodromy* condition

$$Z^i(\mathrm{e}^{2\pi\mathrm{i}}z, \mathrm{e}^{-2\pi\mathrm{i}}\bar{z}) = \mathrm{e}^{2\pi\mathrm{i}kv^i_\theta} Z^i(z,\bar{z}) + 2\pi\lambda^i \,, \tag{2.74}$$

when transported along a closed loop around the origin. Here we have written $g = (\theta^k, \lambda)$. If the field $Z(z,\bar{z})$ is split into a classical and a quantum part $Z = Z_{\mathrm{cl}} + Z_{\mathrm{qu}}$, then only the classical part is subject to the full global monodromy, while the quantum part ignores the shift

$$Z^i_{\mathrm{qu}}(\mathrm{e}^{2\pi\mathrm{i}}z, \mathrm{e}^{-2\pi\mathrm{i}}\bar{z}) = \mathrm{e}^{2\pi\mathrm{i}kv^i_\theta} Z^i_{\mathrm{qu}}(z,\bar{z}) \,. \tag{2.75}$$

This is intuitively clear, as only the classical part of the field knows about the center of mass of the string. The same is true for the derivatives ∂Z and we can write down the OPEs with the twist fields that reflect these *local monodromies* [25]

$$\begin{aligned}
\partial Z^i(z)\sigma^i_g(w,\bar{w}) &\sim (z-w)^{-(1-w^i_g)}\,\tau^i_g(w,\bar{w}) + \dots \,,\\
\partial \bar{Z}^i(z)\sigma^i_g(w,\bar{w}) &\sim (z-w)^{-w^i_g}\qquad \tau'^i_g(w,\bar{w}) + \dots \,,\\
\bar{\partial} Z^i(z)\sigma^i_g(w,\bar{w}) &\sim (\bar{z}-\bar{w})^{-w^i_g}\qquad \tilde{\tau}'^i_g(w,\bar{w}) + \dots \,,\\
\bar{\partial} \bar{Z}^i(z)\sigma^i_g(w,\bar{w}) &\sim (\bar{z}-\bar{w})^{-(1-w^i_g)}\,\tilde{\tau}^i_g(w,\bar{w}) + \dots \,,
\end{aligned} \tag{2.76}$$

where $w^i_g = v^i_g - \lfloor v^i_g \rfloor$ and we have factorized the twist fields σ_g into three two-dimensional twist fields twisting only one component of the Z^i each. While this in general fails for non-factorizable orbifolds because of the lattice part in the global monodromy condition, the local monodromy conditions, that the ∂Z^i fulfil, do not involve the lattice. Hence in writing the OPEs it makes sense to use the factorization. The OPEs define four different *excited twist fields* τ. While the primes distinguish fields of different conformal weights, the tildes denote fields that are related by complex conjugation. The conformal dimensions of these fields can be calculated rather indirectly, by calculating the expectation value of the energy-momentum tensor in the presence of the twist fields. The results are summarized in table 2.2.

As we have seen in section 2.1, the discussion of the spin fields S is considerably simplified after going to a bosonized description. The same holds true for the twist fields s_g. They can be represented as[20]

$$s_g(\bar z) = \mathrm{e}^{\mathrm{i}\sum_i v_g^i H^i(\bar z)}\,, \tag{2.77}$$

so that their action on the bosonized worldsheet fermions $\mathrm{e}^{\mathrm{i}q\cdot H}$ can be described by a shift

$$q \mapsto q_{\mathrm{sh}} = q + \left(0, 0, v_g^1, v_g^2, v_g^3\right)\,. \tag{2.78}$$

Just as in the bosonic case, the OPEs of the fermion twist fields with the ψ^i

$$\begin{aligned} \psi^i(\bar z) s_g^i(\bar w) &= (\bar z - \bar w)^{v_g^i} \;\; t'^{\,i}_g(\bar w)\,,\\ \bar\psi^i(\bar z) s_g^i(\bar w) &= (\bar z - \bar w)^{-v_g^i}\, t_g^i(\bar w) + \dots\,, \end{aligned} \tag{2.79}$$

define excited twist fields which can be represented as $t'^{\,i}_g = \mathrm{e}^{\mathrm{i}(v_g^i+1)H^i}$, $t_g^i = \mathrm{e}^{\mathrm{i}(v_g^i-1)H^i}$. Note that there is no singularity in the OPE of ψ^i and s_g^i, so ψ annihilates the twisted vacuum state created by s_g.

Finally, since the twisted sector boundary conditions can be realized as a shift in the gauge coordinates, the fields $\tilde s_g$, take the form

$$\tilde s_g(z) = \mathrm{e}^{\mathrm{i}\sum_I V_g^I X_{\mathrm{L}}^I(z)}\,, \tag{2.80}$$

where V_g is the embedding of g defined in (2.67). Hence, the twist in the gauge coordinates can be described by a shift in their gauge momentum P,

$$P \mapsto P_{\mathrm{sh}} = P + V_g\,. \tag{2.81}$$

The conformal dimensions of the twist fields are summarized in table 2.2.

	σ_g^i	τ_g^i	$\tau'^{\,i}_g$	s_g^i	t_g^i	$t'^{\,i}_g$	$\tilde s_g^I$
h	$\frac12 w_g^i(1-w_g^i)$	$\frac12 w_g^i(3-w_g^i)$	$\frac12 w_g^i(-1-w_g^i)+1$	0	0	0	$\frac12\left(V_g^I\right)^2$
$\bar h$		$\frac12 w_g^i(1-w_g^i)$	$\frac12 w_g^i(1-w_g^i)$	$\frac12\left(v_g^i\right)^2$	$\frac12\left(v_g^i-1\right)^2$	$\frac12\left(v_g^i+1\right)^2$	0

Table 2.2: Conformal dimensions of the twist fields σ_g, s_g, $\tilde s_g$ acting on X^i, ψ^i, X_{L}^I and their excitations.

The OPE with the fermionic supercurrent, $\bar T_{\mathrm{F}}(\bar z) = \mathrm{i} : (\bar\partial Z \cdot \bar\psi + \bar\partial \bar Z \cdot \psi)$:, shows that the susy partner of Σ_g is given by $\tilde\Sigma_g = -\frac12 \tau_g t_g \tilde s_g$.

We have seen above that the twisted Hilbert space splits into a sum of sectors

$$\mathcal{H}_{[g]} = \oplus_{g'\in[g]}\, P_{h\in C_S(g')}(\mathcal{H}_{g'})\,.$$

That means, if we want to describe a twisted string we have to take a linear combination of

[20] The proper bosonized description of the twist fields s_g involves cocycle factors, which we suppress.

twist fields within the same conjugacy class,

$$\Sigma_{[g]} = \sum_{g' \in [g]} \mathrm{e}^{2\pi \mathrm{i}\tilde{\gamma}(g')} \Sigma_{g'} \,, \tag{2.82}$$

where $\tilde{\gamma}$ are phases to be determined. Then under the space group automorphism $g \to hgh^{-1}$ the sum changes according to

$$\Sigma_{[g]} \to \sum_{g' \in S} \mathrm{e}^{2\pi \mathrm{i}\tilde{\gamma}(g')} \Sigma_{hg'h^{-1}} = \sum_{g' \in S} \mathrm{e}^{2\pi \mathrm{i}(\tilde{\gamma}(g') - \tilde{\gamma}(hg'h^{-1}))} \mathrm{e}^{2\pi \mathrm{i}\tilde{\gamma}(hg'h^{-1})} \Sigma_{hg'h^{-1}} \,.$$

That means, the phases have to fulfil

$$\tilde{\gamma}(g) - \tilde{\gamma}(hgh^{-1}) = \tilde{\gamma}(g') - \tilde{\gamma}(hg'h^{-1}) \mod 1 \qquad \forall h,\, g \in S,\; g' \in [g]$$

for the $\Sigma_{[g]}$ to transform with a phase under $g \to hgh^{-1}$. Thus we are lead to the definition

$$\gamma(g,h) = \tilde{\gamma}(g) - \tilde{\gamma}(hgh^{-1}) \mod 1 \tag{2.83}$$

and we find the transformation behaviour of $\Sigma_{[g]}$ under the automorphism $g \to hgh^{-1}$ to be

$$\Sigma_{[g]} \to \mathrm{e}^{2\pi \mathrm{i}\gamma(g,h)} \Sigma_{[g]} \,. \tag{2.84}$$

Note, that we were able to express the twist fields s_g and $\tilde{s}_g$ in terms of the free bosonic fields H^m and X^I_{L}. As $s_g = \mathrm{e}^{\mathrm{i}\sum_i v^i_g H^i}$ and (for abelian point groups) the twist vectors of g and hgh^{-1} are always equal, $v_g = v_{hgh^{-1}}$, we can move s_g out of the sum and write

$$\Sigma_{[g]} = s_g \sum_{g' \in [g]} \mathrm{e}^{2\pi \mathrm{i}\tilde{\gamma}(g')} \tilde{s}_{g'} \sigma_{g'} \,.$$

Further, as we have seen in (2.81) the space group acts on the gauge coordinates by a shift, i.e. the action is abelian. Hence $V_{hgh^{-1}} \cong V_h + V_g + V_{h^{-1}} \cong V_h + V_{h^{-1}} + V_g \cong V_g$, where $\cong$ means identification up to $\mathrm{E}_8 \times \mathrm{E}_8$ lattice vectors. Vertex operators with non-vanishing gauge momentum always have a contribution $\mathrm{e}^{2\pi \mathrm{i} P^I_{\mathrm{sh}} X^I_{\mathrm{L}}}$ with a fixed P^2_{sh} (because $h = \frac{1}{2}P^2_{\mathrm{sh}}$). If we take two elements of the sum in $\Sigma_{[g]}$ corresponding to g and hgh^{-1} respectively, then their P_{sh} are

$$\begin{aligned} P_{\mathrm{sh}} &= P + V_g \,, \\ P'_{\mathrm{sh}} &= P + V_{hgh^{-1}} = P + V_g + \gamma_{\mathrm{E}_8 \times \mathrm{E}_8} = P' + V_g \,, \end{aligned}$$

that is the difference in the gauge embedding vectors can be reabsorbed in the untwisted momentum P, such that P_{sh} remains the same for both contributions to the state. As a consequence we can as well move $\tilde{s}_g$ out of the sum in Σ_g,

$$\begin{aligned} \Sigma_{[g]} &= s_g \tilde{s}_g \sum_{g' \in [g]} \mathrm{e}^{2\pi \mathrm{i}\tilde{\gamma}(g')} \sigma_{g'} \\ &= s_g \tilde{s}_g \sigma_{[g]} \,. \end{aligned} \tag{2.85}$$

Note, however, that this identity only holds within a physical vertex operator.

Vertex Operators

Now we can write down the vertex operators corresponding to physical massless states of the orbifolded heterotic string.

Let us start with the untwisted sector. It corresponds to compactifying the string theory on a six-dimensional torus. As all winding modes are massive, the vertex operators take the same form as in the uncompactified theory,

$$\begin{aligned} V_{-1}(z,\bar{z}) &= \mathrm{e}^{-\bar{\phi}_{\beta\gamma}(\bar{z})}\ \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})}\ \mathrm{e}^{\mathrm{i}P_I X_\mathrm{L}^I(z)} \times \mathrm{osc.}\,, \\ V_{-\frac{1}{2}}(z,\bar{z}) &= \mathrm{e}^{-\frac{1}{2}\bar{\phi}_{\beta\gamma}(\bar{z})}\ \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})}\ \mathrm{e}^{\mathrm{i}P_I X_\mathrm{L}^I(z)} \times \mathrm{osc.}\,, \end{aligned} \tag{2.86}$$

where we have used the abbreviation osc. for possible contributions of $\partial X_\mathrm{L}^{\mu/I}(z)$ and $\partial \overset{(-)}{Z}{}^i(z)$. We have seen above that the requirement $h = \bar{h} = 1$ restricts the quantum numbers of the bosonic states to fulfil $q^2 = 1$ and either $(P^2, N) = (0,1)$ or $(P^2, N) = (2,0)$. Provided the breaking of the 10D Lorentz group, the fields however split into various 4D components which can be distinguished by the first two components of the SO(10) weight vector q. Namely[21] $(q^0, q^1) = (0,0)$, $(q^0, q^1) = (\underline{\pm 1, 0})$ and $(q^0, q^1) = (\pm\frac{1}{2}, \pm\frac{1}{2})$ correspond to spacetime scalars, vectors and fermions respectively. From the left-moving part of the string there can be oscillator contributions in addition, which modify the 4D Lorentz group representation of the states depending on whether they are excited along the internal or external directions. We will use the index $\mu = 0,1,2,3$ to label 4D spacetime directions and $i = 1,2,3$ for the complexified compact dimensions.

As we have seen in the discussion of the Hilbert space in section 2.3.2, we further have to require invariance of the states under general space group elements $h \in S$. From the considerations above it follows that the untwisted fields transform under $h = (\vartheta, \lambda)$ according to

$$\begin{aligned} Z^i(z,\bar{z}) &\overset{h}{\mapsto} \mathrm{e}^{2\pi\mathrm{i}v_\vartheta^i} Z^i(z,\bar{z}) + 2\pi\lambda\,, & \bar{Z}^i(z,\bar{z}) &\overset{h}{\mapsto} \mathrm{e}^{-2\pi\mathrm{i}v_\vartheta^i} \bar{Z}^i(z,\bar{z}) + 2\pi\lambda\,, \\ \partial Z^i(z) &\overset{h}{\mapsto} \mathrm{e}^{2\pi\mathrm{i}v_\vartheta^i} \partial Z^i(z)\,, & \partial \bar{Z}^i(z) &\overset{h}{\mapsto} \mathrm{e}^{-2\pi\mathrm{i}v_\vartheta^i} \partial \bar{Z}^i(z)\,, \\ \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})} &\overset{h}{\mapsto} \mathrm{e}^{2\pi\mathrm{i}q_i v_\vartheta^i} \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})}\,, & \mathrm{e}^{\mathrm{i}P_I X_\mathrm{L}^I(z)} &\overset{h}{\mapsto} \mathrm{e}^{\mathrm{i}P_I V_h^I} \mathrm{e}^{\mathrm{i}P_I X_\mathrm{L}^I(z)}\,. \end{aligned}$$

Hence the space group invariant, conformal dimension $(1,1)$ vertex operators in the **untwisted sector** are the following:

- $\mathrm{e}^{-\bar{\phi}_{\beta\gamma}(\bar{z})}\, \zeta_m^{(q)} \partial X^m(z)\, \mathrm{e}^{\mathrm{i}q\cdot H(\bar{z})}$ $\qquad q = (\underline{\pm 1,\ 0},\ 0,\ 0,\ 0)\,,\ m = 0,1\,,$

  4D SUGRA multiplet

- $\mathrm{e}^{-\bar{\phi}_{\beta\gamma}(\bar{z})}\, \zeta_i^{(q)} \partial \overset{(-)}{Z}{}^i(z)\, \mathrm{e}^{\mathrm{i}q_{(\pm i)}\cdot H(\bar{z})}$ $\qquad q^j_{(\pm i)} = \pm\delta^j_i\,,$

  complex structure and Kähler moduli

[21] In our notation $(\underline{a_1, \ldots, a_n}, a_{n+1})$ abbreviates all vectors obtained by permuting the components a_1 to a_n of the vector $(a_1, \ldots, a_n, a_{n+1})$.

- $e^{-\bar{\phi}_{\beta\gamma}(\bar{z})}\,\zeta_I^{(q)}\partial X_{\rm L}^I(z)\,e^{iq\cdot H(\bar{z})}$ $\qquad q = (\underline{\pm1,\,0},\,0,\,0,\,0)\,,$

  16 Cartan generators

- $e^{-\bar{\phi}_{\beta\gamma}(\bar{z})}\,e^{iq\cdot H(\bar{z})}\,e^{iP_I X_{\rm L}^I(z)}$ $\qquad q = (\underline{\pm1,\,0},\,0,\,0,\,0)$, $\quad P\cdot V_h = 0 \mod 1 \quad \forall h\in S\,,$

  root generators

- $e^{-\bar{\phi}_{\beta\gamma}(\bar{z})}\,e^{iq\cdot H(\bar{z})}\,e^{iP_I X_{\rm L}^I(z)}$ $\qquad q = (0,\,0,\,\underline{\pm1,\,0,\,0})$, $\quad P\cdot V_h + q\cdot v_h = 0 \mod 1 \quad \forall h\in S\,,$

  chiral multiplets

Here we list the vertex operators in the $\mathcal{NS}$ sector. The $\mathcal{R}$ operators can be obtained from them by substituting

$$e^{-\bar{\phi}_{\beta\gamma}(\bar{z})} \to e^{-\frac{1}{2}\bar{\phi}_{\beta\gamma}(\bar{z})} \qquad \text{and} \qquad q \to q \pm (\pm\tfrac{1}{2},\,\pm\tfrac{1}{2},\,\tfrac{1}{2},\,\tfrac{1}{2},\,\tfrac{1}{2})\,, \tag{2.87}$$

where the sign in the second substitution is to be taken such that $q^2 = \frac{5}{4}$.

The twisted sector vertex operators are obtained by additional insertions of the twist fields $\Sigma_{[g]}$, i.e. we can write them as

$$\begin{aligned} V_{-1}(z,\bar{z}) &= e^{-\bar{\phi}_{\beta\gamma}(\bar{z})}\;e^{iq_{\rm sh}\cdot H(\bar{z})}\,e^{iP_{\rm sh}\cdot X_{\rm L}(z)}\,\sigma_{[g]}\times\text{osc.}\,,\\ V_{-\frac{1}{2}}(z,\bar{z}) &= e^{-\frac{1}{2}\bar{\phi}_{\beta\gamma}(\bar{z})}\,e^{iq_{\rm sh}\cdot H(\bar{z})}\,e^{iP_{\rm sh}\cdot X_{\rm L}(z)}\,\sigma_{[g]}\times\text{osc.}\,. \end{aligned} \tag{2.88}$$

Note that due to the presence of bosonic twist fields, the oscillator contributions are modified. As we have seen in (2.72), the oscillator expansions of Z involve twisted oscillators. Hence the left-moving oscillator terms of Z and $\bar{Z}$ in the sectors twisted by $e^{2\pi i v_g}$, are given by $\frac{\partial}{\partial z^{w_g^i+n}}Z^i(z,\bar{z})$ and $\frac{\partial}{\partial z^{n-w_g^i}}\bar{Z}^i(z,\bar{z})$ such that their conformal dimension are $(w_g^i+n,0)$ and $(n-w_g^i,0)$ respectively [44].
The conditions for V_{-1} to be a conformal dimension $(1,1)$ field are

$$\begin{aligned} 1 &= \frac{1}{2}P_{\rm sh}^2 + h(\text{osc.}) + \frac{1}{2}\sum_i w_g^i(1-w_g^i)\,,\\ 1 &= \frac{1}{2} + \frac{1}{2}q_{\rm sh}^2 + \frac{1}{2}\sum_i w_g^i(1-w_g^i)\,. \end{aligned} \tag{2.89}$$

In the $\mathcal{R}$ sector, the contribution of $\frac{1}{2}$ to $\bar{h}$ from the ghost sector is replaced by $\frac{3}{8}$. As in the untwisted case, conformal dimension $(1,1)$ vertex operators cannot have right-moving oscillators. Also the possible contributions from left-moving oscillators are rather constrained, the only ones allowed being $\partial Z/\partial z^{w_g^i}$ and $\partial\bar{Z}/\partial z^{1-w_g^i}$. Thus let us define the oscillator number operators

$$N_{\rm L} = \sum_i \left(w_g^i\,\mathcal{N}_{\rm L}^i + (1-w_g^i)\,\bar{\mathcal{N}}_{\rm L}^i\right)\,, \tag{2.90}$$

such that $h(\text{osc.}) = N_{\rm L}$ and $\mathcal{N}_{\rm L}^i$ ($\bar{\mathcal{N}}_{\rm L}^i$) count the number of oscillators ∂Z ($\partial\bar{Z}$) present in the vertex operator.
In the light of the fact that the conditions (2.89) are far more model dependent than the ones

in the untwisted sector, we will discuss their solutions for the example of a $\mathbb{Z}_3$ orbifold in the next section.

Provided the condition (2.66) the spectrum of heterotic orbifolds is $\mathcal{N}=1$ spacetime supersymmetric[22]. However there may be subsectors of the Hilbert space which enjoy enhanced $\mathcal{N}=2$ spacetime supersymmetry[23]. In addition the spectrum enjoys CPT invariance: Given a massless state twisted by g with quantum numbers $(q_{\text{sh}}, P_{\text{sh}}, \mathcal{N}_{\text{L}}, \bar{\mathcal{N}}_{\text{L}})$, in the sector twisted by g^{-1} there is a state with $(-q_{\text{sh}}, -P_{\text{sh}}, \bar{\mathcal{N}}_{\text{L}}, \mathcal{N}_{\text{L}})$ that solves the condition (2.89).

Orbifold GSO Projectors

The set of solutions to (2.89) has to be projected onto space group invariant subspaces in order to obtain physical string states. The constituents of the vertex operators transform according to

$$
\begin{aligned}
Z^i(z,\bar z) &\overset{h}{\mapsto} \mathrm{e}^{2\pi \mathrm{i} v^i_\vartheta} Z^i(z,\bar z) + 2\pi\lambda\,, &\qquad \bar Z^i(z,\bar z) &\overset{h}{\mapsto} \mathrm{e}^{-2\pi \mathrm{i} v^i_\vartheta} \bar Z^i(z,\bar z) + 2\pi\lambda\,,\\
\partial_{z^a} Z^i(z) &\overset{h}{\mapsto} \mathrm{e}^{2\pi \mathrm{i} v^i_\vartheta} \partial_{z^a} Z^i(z)\,, & \partial_{z^a} \bar Z^i(z) &\overset{h}{\mapsto} \mathrm{e}^{-2\pi \mathrm{i} v^i_\vartheta} \partial_{z^a} \bar Z^i(z)\,,\\
\mathrm{e}^{\mathrm{i} q_{\text{sh}}\cdot H(\bar z)} &\overset{h}{\mapsto} \mathrm{e}^{2\pi \mathrm{i} q_{\text{sh}}\cdot v_\vartheta} \mathrm{e}^{\mathrm{i} q_{\text{sh}}\cdot H(\bar z)}\,, & \mathrm{e}^{\mathrm{i} P_{\text{sh}}\cdot X_{\text{L}}(z)} &\overset{h}{\mapsto} \mathrm{e}^{\mathrm{i} P_{\text{sh}}\cdot V_h} \mathrm{e}^{\mathrm{i} P_{\text{sh}}\cdot X_{\text{L}}(z)}\,.
\end{aligned}
\tag{2.91}
$$

under the action a space group element h. We still need to determine the transformation behaviour of the twist fields $\sigma_{[g]}$. For the case of non-commuting elements $hg \neq gh$ we have already discussed it above. Namely the global transformation of spacetime by the space group element h induces the space group automorphism $g \to hgh^{-1}$, such that $gX \mapsto (hgh^{-1})hX = hgX$. The twist fields transform according to (2.84). The requirement that a given vertex operator be invariant under the action of h then fixes the phase $\gamma(g,h)$,

$$
V_a \overset{h}{\mapsto} \mathrm{e}^{2\pi \mathrm{i}\left((q_{\text{sh}} + \mathcal{N}_{\text{L}} - \bar{\mathcal{N}}_{\text{L}})\cdot v_h + P_{\text{sh}}\cdot V_h + \gamma(g,h)\right)} V_a \overset{!}{=} V_a\,,
\tag{2.92}
$$

where $a \in \{-1, -\frac{1}{2}\}$ denotes the ghost picture. Note that the gamma phases are in general different for different states. As a consequence there is not only one twist field $\sigma_{[g]}$ for every space group conjugacy class, but one such field for every possible set of values of $\gamma(g,h)$.

For elements $h \in S$ commuting with the constructing element g of a state, the gamma phase is zero. The twist fields might however still transform with a phase, called *vacuum phase*, under such elements,

$$
\sigma_{[g]} \overset{h}{\mapsto} \mathrm{e}^{2\pi \mathrm{i}\Phi_{\text{vac}}(g,h)} \sigma_{[g]}\,, \qquad \forall h \in S \quad \text{s.t.} \quad hg = gh\,,
\tag{2.93}
$$

which arises as a collective phase of each of the *auxiliary twist fields* $\sigma_g \overset{h}{\mapsto} \mathrm{e}^{2\pi \mathrm{i}\Phi_{\text{vac}}(g,h)} \sigma_g$. For non-commuting elements we absorbed a similar phase into the gamma phases. Modular invariance of the one-loop partition function requires the vacuum phase to be [46]

$$
\Phi_{\text{vac}}(g,h) = -\frac{1}{2}\left(V_g \cdot V_h - v_g \cdot v_h\right) \quad \mod 1\,.
\tag{2.94}
$$

Provided this information we can write down the transformation behaviour of the vertex op-

[22] The existence of $\mathcal{N}=1$ spacetime supersymmetry can be traced back to the worldsheet theory being globally $N=2$ supersymmetric. We will not discuss this issue further but refer to [45].

[23] If a twisted sector contains a fixed torus, then in that sector the compactification looks like $\mathcal{O}_4 \times T^2$, where $\mathcal{O}_4$ is a four dimensional orbifold. As a consequence the theory is $\mathcal{N}=1$ supersymmetric in six dimensions, that is $\mathcal{N}=2$ supersymmetric in 4D.

erators under commuting elements $h \in S$ and the corresponding projection condition for twisted string states,

$$V_a \xrightarrow{h} \mathrm{e}^{2\pi\mathrm{i}\left((q_{\mathrm{sh}}+\mathcal{N}_{\mathrm{L}}-\bar{\mathcal{N}}_{\mathrm{L}})\cdot v_h + P_{\mathrm{sh}}\cdot V_h + \Phi_{\mathrm{vac}}(g,h)\right)} V_a \stackrel{!}{=} V_a \,. \tag{2.95}$$

These conditions are called *orbifold GSO projection* conditions.

2.3.4 Example: $\mathbb{Z}_3$ Orbifold

To illustrate the general results obtained above, let us discuss the $\mathbb{Z}_3$ orbifold and determine its spectrum for two different gauge embeddings of the space group.

The unique (up to equivalence) $\mathbb{Z}_3$ space group giving rise to a $\mathcal{N}=1$ supersymmetric spectrum is generated by the twist $v_\theta = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ which acts on the torus obtained by quotiening $\mathbb{R}^6$ by three copies of the $\mathrm{SU}(3)$ root lattice. The lattice basis can be written as

$$e_{2i-1} = \hat{e}_{2i-1} - \hat{e}_{2i}\,, \qquad e_{2i} = \frac{-1+\sqrt{3}}{2}\hat{e}_{2i-1} + \frac{1+\sqrt{3}}{2}\hat{e}_{2i}\,,$$

where $\hat{e}_i{}^j = \delta_i^j$ denotes the standard orthonormal basis of $\mathbb{R}^6$. The action of the point group generator θ on this basis is given by

$$\theta e_{2i-1} = e_{2i}\,, \qquad \theta e_{2i} = -e_{2i-1} - e_{2i}\,,$$

so the orbifold is obviously factorizable. The action of θ does not possess fixed tori but only fixed points, which are given by the $3^3 = 27$ combinations

$$\left(X_{\mathrm{f}}^{2i-1}, X_{\mathrm{f}}^{2i}\right) \in \{(0,\,0),\,(\tfrac{1}{3},\,\tfrac{2}{3}),\,(\tfrac{2}{3},\,\tfrac{1}{3})\}\,, \qquad i = 1,2,3$$

and are depicted in figure 2.5. As $\theta^2 = \theta^{-1}$ the fixed point content of the sectors twisted by θ and θ^2 is equal.

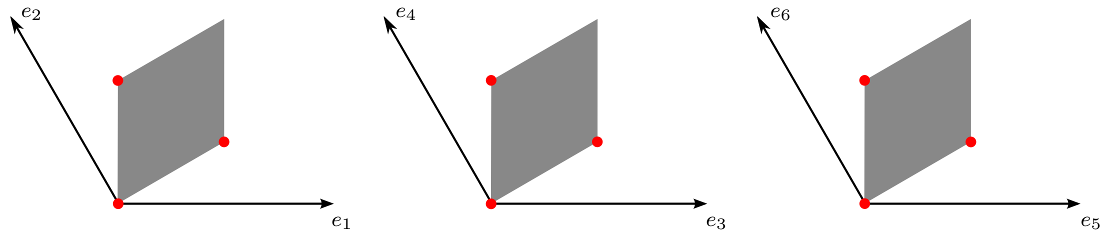

Figure 2.5: The $\mathbb{Z}_3$ orbifold and its fixed points. The fundamental regions of the orbifolded tori are highlighted in gray.

In order to have a well-defined orbifold theory, we still have to specify the embedding of the space group into the gauge group. Let us first consider the *standard embedding*. It is defined as

$$\begin{aligned} V_\theta &= (v_\theta^1, v_\theta^2, v_\theta^3, 0,0,0,0,0)(0,0,0,0,0,0,0,0)\,,\\ W_\alpha &= (0,0,0,0,0,0,0,0)(0,0,0,0,0,0,0,0)\,, \qquad \alpha = 1,\ldots,6\,. \end{aligned} \tag{2.96}$$

The untwisted root generators surviving the projection $P \cdot V_\theta = 0 \mod 1$ split into two sets:

- The states fulfilling $\pm(P^1, P^2, P^3) \in \{(1,-1,0), (0,1,-1), (1,0,-1)\}$ give rise to an $\mathrm{SU}(3)$ gauge group.

- The states fulfilling $(P^1, P^2, P^3) = (0, 0, 0)$ or $(P^1, P^2, P^3) = \pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ give rise to an $E_6 \times E_8$ gauge group

In addition, the untwisted sector contains three chiral multiplets fulfilling the condition $P \cdot V + q \cdot v = 0 \mod 1$, that transform as a $(\overline{27}, 3, 1)$ under the gauge group.
There are two kinds of solutions to the condition $h = \bar{h} = 1$ in the sector twisted by θ:

- The states with $q_{\text{sh}}^2 = \frac{1}{3}$, $P_{\text{sh}}^2 = \frac{4}{3}$, $N_{\text{L}} = 0$, fulfilling the projection condition $q_{\text{sh}} \cdot v_\theta + P_{\text{sh}} \cdot V_\theta = 0 \mod 1$, give rise to a chiral $(\overline{27}, 1, 1)$ multiplet.
- The states with $q_{\text{sh}}^2 = \frac{1}{3}$, $P_{\text{sh}}^2 = \frac{2}{3}$, $N_{\text{L}} = \frac{2}{3}$, surviving the projector give rise to three chiral $(1, \overline{3}, 1)$ multiplets.

As no Wilson lines are switched on, the location of the states enters neither the conditions (2.89) nor the orbifold GSO projection conditions (2.95). Hence each fixed points allocates the same matter content and the $\mathbb{Z}_3$ orbifold in standard embedding contains 3 $(\overline{27}, 3, 1)$, 27 $(\overline{27}, 1, 1)$ and 81 $(1, \overline{3}, 1)$ chiral matter multiplets.

Let us break the fixed point degeneracy by switching on a Wilson line in the first plane[24],

$$W_1 = W_2 = \frac{1}{3}(0, 0, 0, 0, 0, 0, 0, 0)(0, 0, 1, 1, 1, 1, 1, 1)\,.$$

This breaks the gauge group further down to $E_6 \times SU(3) \times E_6 \times SU(3)$. It also breaks the degeneracy in the spectra of different fixed points. Namely, as visualized in figure 2.6, the fixed points split into three groups depending on their location in the first plane:

- Those with $(X_{\text{f}}^1, X_{\text{f}}^2) = (0, 0)$ support one $(\overline{27}, 1, 1, 1)$ and three $(1, \overline{3}, 1, 1)$ each.
- Those with $(X_{\text{f}}^1, X_{\text{f}}^2) = (\frac{2}{3}, \frac{1}{3})$ support one $(1, \overline{3}, 1, 3)$ each.
- Those with $(X_{\text{f}}^1, X_{\text{f}}^2) = (\frac{1}{3}, \frac{2}{3})$ support one $(1, \overline{3}, 1, \overline{3})$ each.

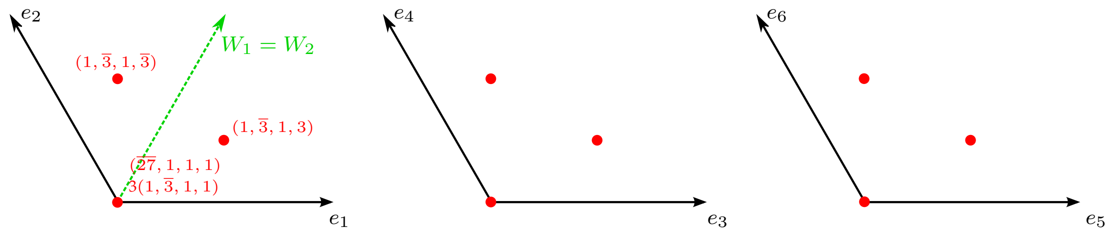

Figure 2.6: The Wilson line configuration $W_1 = W_2 \neq 0$ breaks the degeneracy between the fixed points. They split into three groups depending on their location in the first plane.

While this orbifold model does not at all look phenomenologically appealing it serves as an easy example. We will present more realistic orbifold models based on a $\mathbb{Z}_6$ point group in chapter 4.

[24] Note that since $\theta e_1 = e_2$, the Wilson lines W_1 and W_2 are identified up to lattice vectors as can be seen from (2.68).

CHAPTER 3

Coupling Selection Rules

In order to determine the phenomenology of a given string compactification it is important to study its dynamics, i.e. the couplings of physical states. As we have seen the conformal field theory describing the world sheet dynamics of orbifold compactifications of the heterotic string is free and hence it is in principle possible to calculate any correlation function explicitly. However, many phenomenological questions can be addressed without knowing the explicit coupling strengths, only using the information about whether a specific coupling is possible or not. This information is encoded in so called *selection rules*. In this chapter we want to discuss the stringy origin of such selection rules in heterotic orbifold compactifications. We will restrict ourselves to couplings corresponding to terms in the superpotential of the four dimensional low energy effective field theory (EFT). As the superpotential of an $\mathcal{N} = 1$ supersymmetric field theory is a $\mathrm{d}^2\theta$ integral of a holomorphic polynomial $\mathcal{P}$ in the superfields Ψ_α,

$$W = \int \mathrm{d}^2\theta \, \mathcal{P}(\{\Psi_\alpha\}) \, ,$$

the relevant couplings in the string theory are L-point correlators of two chiral fermions and $L-2$ scalar bosons,

$$\langle \psi_1 \psi_2 \phi_1 \ldots \phi_{L-2} \rangle \ .$$

Holomorphicity of the superpotential has a far reaching consequence. Namely it does not get any corrections from string loop amplitudes [47], so that we only need to consider tree-level correlators.

We will pay special attention to the origin of R-symmetries of the low energy theory within the orbifolded string theory. As bosonic and fermionic components of the same superfield have different charges under R-symmetries, it is natural to expect them to arise from remnants of the Lorentz symmetry of the compactified directions, i.e. from isometries of the compactification space. We will identify the relevant isometries of the orbifold geometry and deduce the corresponding R-charge conservation rules. As we will see, a special role is played by the worldsheet instanton solutions which are the reason that the resulting R-symmetries are not continuous but discrete.

3.1 Correlation Functions

We will write the vertex operators (2.86), (2.88) as

$$V_a = \mathrm{e}^{a\bar{\phi}_{\beta\gamma}}\,(\partial Z^i)^{\mathcal{N}_\mathrm{L}^i}\,(\partial\bar{Z}^i)^{\bar{\mathcal{N}}_\mathrm{L}^i}\,\mathrm{e}^{\mathrm{i}q_\mathrm{sh}\cdot H}\,\mathrm{e}^{\mathrm{i}P_\mathrm{sh}\cdot X_\mathrm{L}}\,\sigma_{[g]}\,, \tag{3.1}$$

where $a \in \{-1, -\frac{1}{2}\}$ denotes the ghost picture, ∂Z^i is short for $\frac{\partial Z^i}{\partial z^{w_g^i}}$ and analogous for $\partial\bar{Z}$. Untwisted string states are obtained by replacing shifted momenta by unshifted ones and $\sigma_{[g]}$ by $\mathbb{1}$. Note that according to (2.56) the bosonic vertex operator in the 0-picture is obtained as[1]

$$V_0 = \mathrm{i}\sum_m \left(\bar{\partial}Z^m\,\mathrm{e}^{\mathrm{i}\,q_{(-m)}\cdot H} + \bar{\partial}\bar{Z}^m\,\mathrm{e}^{\mathrm{i}\,q_{(m)}\cdot H}\right) V_{-1}\,, \tag{3.2}$$

where $q^n_{(\pm m)} = \pm\delta^n_m$. As we have seen in section 2.1.3, the operators in a string tree-level correlation function have to cancel the background ghost charge $\bar{Q}_{\beta\gamma} = 2$ on the sphere. Hence the correlators we need to discuss are of the form

$$\mathcal{F} = \left\langle V_{-\frac{1}{2}}V_{-\frac{1}{2}}V_{-1}V_0\ldots V_0\right\rangle\,. \tag{3.3}$$

As the presence of picture changed vertex operators introduces right-moving oscillators into the correlation function, let us introduce the number operators

$$N_\mathrm{R} = w_g^i\,\mathcal{N}_\mathrm{R}^i + (1-w_g^i)\,\bar{\mathcal{N}}_\mathrm{R}^i\,, \tag{3.4}$$

such that, just as in the left-moving case $\bar{h}(\mathrm{osc.}) = N_\mathrm{R}$ and $\mathcal{N}_\mathrm{R}^i$ ($\bar{\mathcal{N}}_\mathrm{R}^i$) count the number of oscillators $\bar{\partial}Z$ ($\bar{\partial}\bar{Z}$) present in the vertex operator. We will sometimes use $\tilde{q}_\mathrm{sh}$ to denote the total H-momentum of a vertex operator, that is q_sh together with the possible contribution from picture changing, such that any vertex operator just includes one factor $\mathrm{e}^{i\tilde{q}_\mathrm{sh}\cdot H}$.

The crucial feature of correlation functions (3.3) is that they factorize, such that the OPE of any pair of fields involved in different pieces of the correlator vanish. This means we can write

$$\mathcal{F} = \mathcal{F}_\mathrm{gh}(\phi_{bc},\bar{\phi}_{bc},\bar{\phi}_{\beta\gamma}) \times \mathcal{F}_H(H^m) \times \mathcal{F}_\mathrm{gauge}(X_\mathrm{L}^I) \times \mathcal{F}_\mathrm{bos}(\partial Z,\sigma) \tag{3.5}$$

Note that, as explained in (2.55), three of the vertex operators have their worldsheet position fixed (we choose 0, 1 and ∞) and contribute the fields ϕ_{bc} and $\bar{\phi}_{bc}$ to the correlator. All other vertex operators are integrated over their worldsheet insertions. The ghost part of the correlator $\mathcal{F}_\mathrm{gh}$ then integrates to factors related to the infinite volume of the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{OSP}(1|2)$ invariance group of the worldsheet [25]. These factors get cancelled by similar ones arising from other parts of the correlator such that the correlation function is finite[2].

[1] Note that when an insertion of $\bar{T}_\mathrm{F}$ is used to picture change a twisted state vertex operator, the internal right-moving oscillators contained in $\bar{T}_\mathrm{F}$ are twisted as well.

[2] We will ignore such factors in the following. For a discussion see e.g. [36, 41].

3.1.1 Gauge Invariance and H-momentum conservation

The contributions of $\mathcal{F}_H(H^m)$ and $\mathcal{F}_{\text{gauge}}(X^I_{\text{L}})$ are easily dealt with since they only contain exponentials of free fields. Hence they contribute factors [41]

$$\begin{aligned}\mathcal{F}_H(H^m) &\sim \left\langle : \mathrm{e}^{2\pi\mathrm{i}\tilde{q}_{sh\,1}\cdot H(\bar{z}_1)} : \cdots : \mathrm{e}^{2\pi\mathrm{i}\tilde{q}_{sh\,L}\cdot H(\bar{z}_L)} : \right\rangle \\ &\sim \delta^{(5)}\left(\sum_\alpha \tilde{q}_{\text{sh}\,\alpha}\right) \prod_{\alpha<\beta} (\bar{z}_\alpha - \bar{z}_\beta)^{\tilde{q}_{\text{sh}\,\alpha}\cdot\tilde{q}_{\text{sh}\,\beta}}\end{aligned} \tag{3.6}$$

and

$$\mathcal{F}_{\text{gauge}}(X^I) \sim \delta^{(16)}\left(\sum_\alpha P_{\text{sh}\,\alpha}\right) \prod_{\alpha<\beta} (z_\alpha - z_\beta)^{P_{\text{sh}\,\alpha}\cdot P_{\text{sh}\,\beta}} . \tag{3.7}$$

From these expressions follow the first two selection rules. Namely any coupling of the form (3.3) vanishes unless

$$\sum_{\alpha=1}^{L} \tilde{q}^m_{\text{sh}\,\alpha} = 0 \qquad\qquad (H\textit{-momentum conservation})\,, \tag{3.8}$$

$$\sum_{\alpha=1}^{L} P^I_{\text{sh}\,\alpha} = 0 \qquad\qquad (\textit{gauge invariance})\,. \tag{3.9}$$

We will rewrite the H-momentum conservation rule for the compact directions in a more useful way. First we split the sum into the part coming from the quantum numbers of the states and the one that is specific to the correlator,

$$\sum_{\alpha=1}^{L} \tilde{q}^i_{\text{sh}\,\alpha} = \sum_{\alpha=1}^{L} q^i_{\text{sh}\,\alpha} + \sum_{\alpha=4}^{L} \left(\bar{\mathcal{N}}^i_{\text{R}\,\alpha} - \mathcal{N}^i_{\text{R}\,\alpha}\right)$$

While gauge invariance is really a selection rule, the situation with H momentum conservation is more complicated, as it depends on the unphysical right-moving oscillators in the correlator. Now observe that for chiral superfields of positive chirality, which is the case of our interest, the sum of q_{sh} over the compact directions is always -1 for the bosons and $\frac{1}{2}$ for the fermions. That is

$$\sum_{i=1}^{3}\sum_{\alpha=1}^{L} \tilde{q}^i_{\text{sh}\,\alpha} = -(L-2) + 1 + \sum_{i=1}^{3}\sum_{\alpha=4}^{L} \left(\bar{\mathcal{N}}^i_{\text{R}\,\alpha} - \mathcal{N}^i_{\text{R}\,\alpha}\right) .$$

Using the fact that $\sum_{i=1}^{3}\sum_{\alpha=4}^{L}\left(\bar{\mathcal{N}}^i_{\text{R}\,\alpha} + \mathcal{N}^i_{\text{R}\,\alpha}\right) = L-3$ we find

$$\mathcal{N}^i_{\text{R}} = 0\,, \qquad \sum_{i=1}^{3} \bar{\mathcal{N}}^i_{\text{R}} = L-3\,, \tag{3.10}$$

i.e. each of the picture changed vertex operator contains a right-moving oscillator $\bar{\partial}\bar{Z}^i$ in one of the internal directions. Finally, one might rewrite the shifted right-moving momenta q_{sh} of the fermions contributing to the correlator in terms of those of the bosons from the same left-chiral 4D multiplet. Using the relation (2.87), we find the internal H-momentum conservation rule

$$\sum_{\alpha=1}^{L} q_{\text{sh}\,\alpha}^{(boson)\,i} = 1 - \sum_{\alpha=4}^{L} \bar{\mathcal{N}}_{\text{R}\,\alpha}^{i} \,. \tag{3.11}$$

So the question of whether a coupling is allowed by H-momentum conservation is the question of whether it is possible to find a set of $\bar{\mathcal{N}}_{\text{R}\,\alpha}^{i}$ such that (3.11) as well as the constraints $\bar{\mathcal{N}}_{\text{R}\,\alpha}^{i} \in \{0,1\}$ and $\sum_i \bar{\mathcal{N}}_{\text{R}\,\alpha}^{i} = 1$ are fulfilled.

3.1.2 Space Group Selection Rule

The bosonic piece of the correlation function (3.5) takes the form

$$\mathcal{F}_{\text{bos}} \sim \prod_{\alpha}\prod_{i} \left(\partial Z^i(z_\alpha)\right)^{\mathcal{N}_{\text{L}\,\alpha}^{i}} \left(\partial \bar{Z}^i(z_\alpha)\right)^{\bar{\mathcal{N}}_{\text{L}\,\alpha}^{i}} \left(\bar{\partial} \bar{Z}^i(z_\alpha)\right)^{\bar{\mathcal{N}}_{\text{R}\,\alpha}^{i}} \sigma_{[g_\alpha]}(z_\alpha, \bar{z}_\alpha) \,. \tag{3.12}$$

Since a general operator expansion of the twist fields is not known, the calculation of this correlator is usually performed in the path integral formalism. Furthermore, as the action of the theory is quadratic in the fields ∂Z, we can split them into a classical and a quantum part $\partial Z = \partial Z_{\text{cl}} + \partial Z_{\text{qu}}$, where ∂Z_{cl} is a solution to the classical equation of motion with the correct monodromy (2.74) around the locations of the twist fields [25]. Then the correlator can be expanded as[3]

$$\begin{aligned}\mathcal{F}_{\text{bos}} \sim \sum_{Z_{\text{cl}}} \mathrm{e}^{-S_{\text{cl}}} \prod_{i=1}^{3} \sum_{r=0}^{\mathcal{N}_{\text{L}}^{i}} \sum_{s=0}^{\bar{\mathcal{N}}_{\text{L}}^{i}} \sum_{t=0}^{\bar{\mathcal{N}}_{\text{R}}^{i}} & \binom{\mathcal{N}_{\text{L}}^{i}}{r} \binom{\bar{\mathcal{N}}_{\text{L}}^{i}}{s} \binom{\bar{\mathcal{N}}_{\text{R}}^{i}}{t} \\ & \times \left(\partial Z_{\text{cl}}^{i}\right)^{\mathcal{N}_{\text{L}}^{i}-r} \left(\partial \bar{Z}_{\text{cl}}^{i}\right)^{\bar{\mathcal{N}}_{\text{L}}^{i}-s} \left(\bar{\partial} \bar{Z}_{\text{cl}}^{i}\right)^{\bar{\mathcal{N}}_{\text{R}}^{i}-t} \\ & \times \int \mathcal{D}X_{\text{qu}}\, \mathrm{e}^{-S_{\text{qu}}^{i}} \left(\partial Z_{\text{qu}}^{i}\right)^{r} \left(\partial \bar{Z}_{\text{qu}}^{i}\right)^{s} \left(\bar{\partial} \bar{Z}_{\text{qu}}^{i}\right)^{t} \sigma_{[g_1]}^{i} \ldots \sigma_{[g_L]}^{i} \,,\end{aligned} \tag{3.13}$$

The quantum part is independent of the classical solutions and the fields involved are subject only to the local monodromy conditions (2.75). Note that the non-trivial monodromy conditions mean that the fields ∂Z are multi-valued fields on the worldsheet. Therefore one has to go to a surface that covers the worldsheet multiple times, such that the fields become single-valued [44]. We will not go into the details of the calculation of these amplitudes that have been carried out in [25, 48, 49] and more recently in [50], but analyse the structure of the correlator (3.13) to find further selection rules.

Of particular importance are the classical solutions ∂Z_{cl}, which correspond to worldsheet instantons. As the complete correlator $\mathcal{F}_{\text{bos}}$ is proportional to a sum over these solutions, we find that it can only be non-zero if such solutions exist. The worldsheet coordinate dependence

[3] Here $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ denote the binomial coefficients.

of the classical solutions is determined by the local monodromy conditions to be [25, 48, 51]

$$\begin{aligned} \partial Z_{\text{cl}}^i(z) &= \left(\bar\partial \bar Z_{\text{cl}}^i\right)^* = \sum_{l=1}^{L-M^i-1} {a_l}^i\, {h_l}^i(z)\,, \\ \bar\partial Z_{\text{cl}}^i(z) &= \left(\partial \bar Z_{\text{cl}}^i\right)^* = \sum_{l=1}^{M^i-1} {b_l}^i\, {\bar h_l}^i(\bar z)\,, \end{aligned} \tag{3.14}$$

where $M^i = \sum_\alpha w^i_{g_\alpha}$. Here, the basis functions are

$$\begin{aligned} {h_l}^i(z) &= z^{l-1} \prod_\alpha (z - z_\alpha)^{w^i_{g_\alpha}-1}\,, \\ {\bar h_l}^i(\bar z) &= \bar z^{l-1} \prod_\alpha (\bar z - \bar z_\alpha)^{-w^i_{g_\alpha}}\,. \end{aligned} \tag{3.15}$$

The $L-2$ coefficients a_l, b_l are determined by the global monodromy conditions

$$\begin{aligned} \int_{\gamma_p} \mathrm{d}z\, \partial Z^i_{\text{cl}} + \int_{\gamma_p} \mathrm{d}\bar z\, \bar\partial Z^i_{\text{cl}} &= \nu^i_p\,, \\ \int_{\gamma_p} \mathrm{d}z\, \partial \bar Z^i_{\text{cl}} + \int_{\gamma_p} \mathrm{d}\bar z\, \bar\partial \bar Z^i_{\text{cl}} &= \bar\nu^i_p\,, \end{aligned} \tag{3.16}$$

where γ_p are zero-twist loops, i.e. they close on the N-fold covering of the worldsheet. This must be the case, since a loop that encircles all twist insertions on the sphere can be pulled off to infinity, i.e. it must have trivial homology. On the other hand, however, consider a loop starting at a base point p. As the loop encircles the insertion points of all twist fields and comes back to p, it has been multiplied by $g_1 g_2 \dots g_L$. As a consequence such loops can only exist, provided

$$\prod_\alpha [g_\alpha] = [\mathbb{1}]\,. \tag{3.17}$$

This is the *space group selection rule*. It can also be understood from a more intuitive point of view [44]. Suppose $|g\rangle$ is a string in the Hilbert space $\mathcal{H}_{[g]}$ and $|h\rangle$ in the Hilbert space $\mathcal{H}_{[h]}$. If the two strings join, as displayed in figure 3.1, they form a string in the Hilbert space $\mathcal{H}_{[hg]}$. More generally, a process

$$|g_1 g_2 \dots g_m\rangle \to |h_1 h_2 \dots h_n\rangle$$

is possible only if $[g_1][g_2]\dots[g_m] = [h_1][h_2]\dots[h_n]$, i.e. $[g_1][g_2]\dots[g_m][g_{m+1}][g_{m+2}]\dots[g_{m+n}] = [\mathbb{1}]$, where we have substituted the outgoing string by the CPT conjugate ingoing state using the replacement $g_{m+j} = h^{-1}_{n-j+1}$.

As we may write the space group elements g_α as $(\vartheta_\alpha, \lambda_\alpha)$ the space group selection rule may be split into a rule for the twist part,

$$\vartheta_1\, \vartheta_2 \dots \vartheta_L = \mathbb{1}\,, \tag{3.18}$$

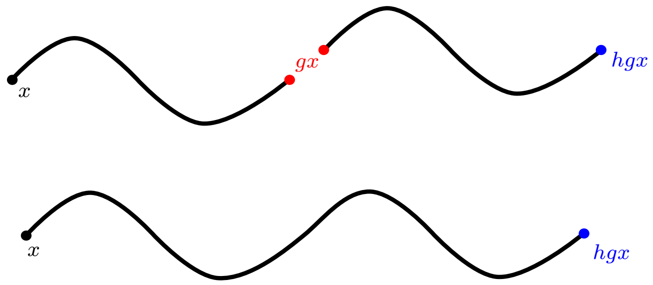

Figure 3.1: A twisted string with constructing element g and one with constructing element h join to form a string with constructing element hg.

called the *point group selection rule* and a lattice part which can be written as

$$\sum_{\alpha=1}^{L} \left(\prod_{\beta=\alpha+1}^{L} \vartheta_\beta \right) \left(\tilde{\vartheta}_\alpha \lambda_\alpha + (1 - \vartheta_\alpha) \tilde{\lambda}_\alpha \right) = 0\,, \tag{3.19}$$

where the selection rule rule is fulfilled iff L space group elements $h_\alpha = (\tilde{\vartheta}_\alpha, \tilde{\lambda}_\alpha)$ can be found such that (3.19) is satisfied. Often the space group selection rule can be rewritten as a set of simpler discrete symmetries. For the case of the $\mathbb{Z}_3$ orbifold discussed above one finds the four discrete symmetries [52, 53]

$$\begin{aligned} \sum_\alpha k_\alpha &= 0 \quad \mod 3\,, \\ \sum_\alpha k_\alpha n_i &= 0 \quad \mod 3\,, \qquad i = 1,2,3\,, \end{aligned} \tag{3.20}$$

where we have written $\vartheta_\alpha = \theta^{k_\alpha}$ and, in each plane, have labelled the three fixed points $(0,0)$, $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$ by $n = 0$, 1 and 2.

3.1.3 Classical Solutions and their Symmetries

Let us investigate the classical solutions (3.14) a bit further. The coefficients a_l and b_l can be obtained from (3.16) as

$$\begin{aligned} a_l{}^i &= \nu_p{}^i (W^{-1})^i{}_l{}^p\,, \\ l_l{}^i &= \nu_p{}^i (W^{-1})^i{}_{L-M^i-1+l}{}^p\,, \end{aligned} \tag{3.21}$$

where we have used the *period matrices* [54]

$$\begin{aligned} W^i{}_p{}^l &= \int_{\gamma_p} \mathrm{d}z\, h_l{}^i(z)\,, \qquad l = 1, \ldots, L - M^i - 1\,, \\ W^i{}_p{}^{L-M^i-1+l} &= \int_{\gamma_p} \mathrm{d}\bar{z}\, \bar{h}_l{}^i(\bar{z})\,, \qquad l = 1, \ldots, M^i - 1\,. \end{aligned}$$

The sum over classical solutions in (3.13) therefore becomes a sum over the *coset vectors* ν_p. The calculation of the coefficients a_i and b_i proceeds as follows[4]. One chooses a basis of all possible

[4] Here we restrict ourselves to space groups with $\mathbb{Z}_N$ point group.

zero twist loops enclosing the twist fields, which has $(L-2)$ [54] elements. One such basis [48, 49, 54, 55] is given by the loops that encircle the fixed point f_p clockwise r_p times followed by the fixed point f_{p+1} counter-clockwise s_p times, where r_p and s_p are the smallest integers such that $r_p k_p = s_p k_{p+1} \mod N$. The corresponding coset vectors can be written as

$$\nu_p = (1-\theta^{r_p k_p})(f_{p+1} - f_p + \lambda)\,, \tag{3.22}$$

where λ is some lattice vector of the internal torus. As it turns out, the homologically not linear independent $(L-1)^{\text{th}}$ loop leads to a consistency condition and thereby to a further restriction on the coset vectors [49]. For instance, for the three-point function the ν_p are restricted as

$$\nu = \left(1-\theta^{r_1 k_1}\right)\left(f_2 - f_1 - \tilde{\lambda}_2 + \tilde{\lambda}_1 + \frac{1-\theta^{k_1+k_2}}{1-\theta^{\gcd(k_1,k_2)}}\lambda\right)\,, \qquad \lambda \in \Lambda\,, \tag{3.23}$$

where $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ are the lattice vectors that appear in the space group selection rule (3.19).

However, before we proceed to discuss the properties of the coset vectors, there is one more selection rule to be discussed. Namely suppose we are considering a coupling of states that are all located at the same fixed point. Then for every coset vector ν, its $\mathbb{Z}_2$ image $-\nu$ is a coset vector as well. If the order of the $\mathbb{Z}_N$ point group generator θ is odd, this symmetry is not contained in the point group and the classical solutions of such couplings enjoy an enhanced $\mathbb{Z}_N \times \mathbb{Z}_2$ symmetry that can forbid couplings. This selection rule, called *Rule 4* [56], however possesses a rather strange feature. It was shown [52] that the way in which this rule forbids couplings cannot be explained by effective symmetries from the point of view of a low energy theory, i.e. it is not possible to define effective quantum numbers, the conservation of which explains the vanishing of the couplings. As a consequence one might expect that couplings forbidden by Rule 4 may be induced by perturbative and non-perturbative effects when going from the string to the electroweak scale.

Let us go back to the general case. Having obtained the explicit form of the coset vectors we can investigate their symmetries. Recall that the bosonic part of the correlator is proportional to powers of the classical solutions and hence to powers (of sums) of the coset vectors. Any isometry of the internal space that maps the set of allowed coset vectors to itself should manifest as a symmetry of the correlation function and hence lead to a selection rule[5]. Inspection of (3.22) and (3.23) reveals that the relevant symmetries are automorphisms ϱ of the torus lattice Λ that commute with all point group elements: If f_i and f_j are fixed points and $\tilde{\lambda}_i$ and $\tilde{\lambda}_j$ are the corresponding lattice vectors from the space group selection rule, then there is a loop γ with corresponding coset vector $\nu(\lambda)$. At the same time ϱf_i and ϱf_j are fixed points and $\varrho\tilde{\lambda}_i$ and $\varrho\tilde{\lambda}_j$ are the corresponding lattice vectors from the space group selection rule. So there exists a loop γ' with corresponding coset vector $\nu'(\lambda')$. However, as the automorphism ϱ commutes with all point group elements, we find $\nu'(\lambda') = \varrho\nu(\lambda)$.

Note that this result is in particular intriguing because it shows that the continuous R-symmetries that one would expect in smooth compactifications of heterotic string theory get broken to discrete ones by worldsheet instanton effects on the orbifold. The quantum part of the correlator (3.13) does not know about the relative positions of the fixed points at which the states live. Hence, if there were no instantons one could indeed find three independent U(1) R-symmetries, just in the way we will deduce the discrete R-symmetries in section 3.3.2. The

[5] There are further restrictions, as we will see. The isometries also have to respect the fact that different fixed points in general support different states and hence cannot be exchanged.

information about the presence of the lattice and its fixed points is encoded in the classical solutions, that is the worldsheet instantons that mediate the couplings between states located at different (copies of) fixed points. Although we have shown it explicitly only for the case of three-point couplings in $\mathbb{Z}_N$ orbifolds, we will use this result in the following sections as a motivation to identify the R-symmetries for all heterotic orbifolds with Abelian space groups.

Before we can discuss the systematics of our exploration of R-symmetries, we need to examine the classification of all heterotic orbifolds with Abelian point groups leading to $\mathcal{N} = 1$ supersymmetry in four dimensions.

3.2 Abelian Orbifolds with $\mathcal{N} = 1$ Supersymmetry

This classification has been performed in [34] according to the scheme introduced in section 2.2.1. There we saw that the possible space groups split into $\mathbb{Q}$, $\mathbb{Z}$ and affine classes, which we can now interpret more physically. As we have seen, the $\mathbb{Q}$ class of the space group specifies the point group, i.e. the holonomy group of the orbifold and thereby the amount of supersymmetry of the resulting 4D theory. Furthermore the number of geometrical moduli is fixed by the $\mathbb{Q}$ class. Since the $\mathbb{Z}$ class specifies the lattice of the space group, it fixes the nature of the moduli and by determining the automorphism group of the lattice, restricts the isometries of the internal space. However, two representatives of the same $\mathbb{Z}$ class can be inequivalent because of different actions of the orbifolding group on the lattice, i.e. depending on the presence of roto-translations. This is determined by specifying the affine class of the space group which fixes many phenomenological features of the model, such as the nature of gauge symmetry breaking (c.f. section 4.1). Affine transformations correspond to moving in the moduli space of the orbifold theory, so that only one representative of each affine class is needed in the classification.

The program of the classification is now to find all $\mathbb{Q}$ classes of space groups such that the point group is a subgroup of $\mathrm{SU}(3)$ and then to construct all $\mathbb{Z}$ and affine classes contained in these $\mathbb{Q}$ classes. A catalogue of all possible affine classes of space groups in up to six dimensions was created in [57] and the results can be accessed using the computer program `Carat` [58]. A nice way to determine the $\mathbb{Q}$ classes leading to $\mathcal{N} = 1$ supersymmetry proceeds via representation theory [34]. Clearly all point groups are discrete subgroups of $\mathrm{O}(6)$. Having selected those ones, the generators of which have determinant 1, one can make use of the breaking of the vector representation **6** of $\mathrm{SO}(6)$ into the representations of $\mathrm{SU}(3)$, which takes the form

$$\mathbf{6}_{\mathrm{SO}(6)} \to \mathbf{3}_{\mathrm{SU}(3)} \oplus \overline{\mathbf{3}}_{\mathrm{SU}(3)} \,.$$

The six-dimensional representation of the point group P is reducible in general and hence splits into irreducible components according to $\mathbf{6}_P = \oplus_k n^{(k)} \mathbf{r}_P^{(k)}$. Using the character table of the point group P and the relation

$$n^{(k)} = \frac{1}{|P|} \sum_{g \in P} \chi_{\mathbf{r}_P^{(k)}}(g) \, \left(\chi_{\mathbf{6}_P}(g)\right)^*$$

one can determine the multiplicities $n^{(k)}$ and check whether the sum splits according to $\mathbf{6}_P = \mathbf{a}_P \oplus \bar{\mathbf{a}}_P$, where $\mathbf{a}_P$ denotes the representation originating from $\mathbf{3}_{\mathrm{SU}(3)}$ and $\bar{\mathbf{a}}_P$ is its conjugate. If this is the case and the determinant of the matrix representation of $\mathbf{a}_P$ has determinant 1, the point group is indeed a subgroup of $\mathrm{SU}(3)$. Then one can check whether or not the breaking of $\mathbf{3}_{\mathrm{SU}(3)}$ into irreducible representations of P contains the trivial representation and in this way whether the orbifolded theory preserves more than one spacetime supersymmetry or not.

As a result, out of the 7103 $\mathbb{Q}$ classes of six dimensional space groups, 52 lead to heterotic orbifolds preserving exactly $\mathcal{N} = 1$ supersymmetry. Out of these 17 contain Abelian point groups and split into 60 $\mathbb{Z}$ classes and 138 affine classes [34].

This set of 138 affine classes provides us with the starting point for the determination of R-symmetries for all Abelian orbifolds with $\mathcal{N} = 1$ supersymmetry that we will perform in the next section.

3.3 R-Symmetries

As discussed above, R-symmetries of the effective theory of massless states can originate from symmetries of the internal space, i.e. from its isometries. In this section we want to describe the isometries of orbifold compactifications and discuss the coupling selection rules that we can deduce from them.

3.3.1 Orbifold Isometries

The isometries of six dimensional flat tori split into two classes. The continuous isometries form the first class and are given by translations along the cycles of the torus. They form the group $[\mathrm{U}(1)]^6$. The second class contains discrete isometries that are given by the automorphisms of the torus lattice. As the continuous isometries are clearly broken by orbifolding, we have to determine the lattice automorphisms which are compatible with taking the quotient of the torus and the orbifolding group. Let us therefore consider various subgroups [59] of the lattice automorphism group, Aut, as depicted in figure 3.2.

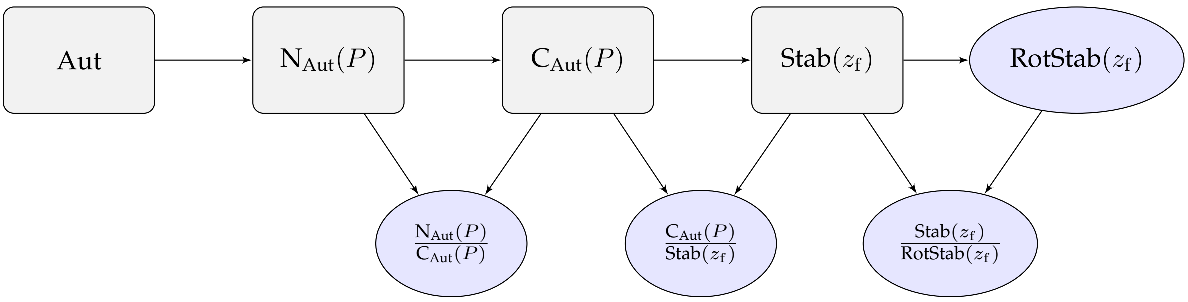

Figure 3.2: Decomposition of the automorphism group Aut of the torus lattice Λ into the subgroups described in the text.

- $\mathrm{N}_{\mathrm{Aut}}(P) \subset \mathrm{Aut}$ denotes the *normalizer* of the point group in the automorphism group, i.e. the subgroup

$$\mathrm{N}_{\mathrm{Aut}}(P) = \left\{ \varrho \in \mathrm{Aut} \;\middle|\; \forall \vartheta \in P \;\; \varrho\vartheta\varrho^{-1} \in P \right\} . \tag{3.24}$$

- $\mathrm{C}_{\mathrm{Aut}}(P) \subset \mathrm{N}_{\mathrm{Aut}}(P)$ denotes the *centralizer* of the point group in the automorphism group, i.e. the subgroup

$$\mathrm{C}_{\mathrm{Aut}}(P) = \left\{ \varrho \in \mathrm{Aut} \;\middle|\; \forall \vartheta \in P \;\; \varrho\vartheta\varrho^{-1} = \vartheta \right\} . \tag{3.25}$$

- $\mathrm{Stab}(z_{\mathrm{f}}) \subset \mathrm{C}_{\mathrm{Aut}}(P)$ denotes the "stabilizer" of the fixed point conjugacy classes, i.e. the subgroup

$$\mathrm{Stab}(z_{\mathrm{f}}) = \{ \varrho \in \mathrm{Aut} \mid \forall z_{\mathrm{f}} \text{ fixed point of } S,\; \exists h \in S \text{ s.t. } \varrho z_{\mathrm{f}} = h z_{\mathrm{f}} \} . \tag{3.26}$$

- $\mathrm{RotStab}(z_{\mathrm{f}}) \subset \mathrm{Stab}(z_{\mathrm{f}})$ denotes the subset of the stabilizer, the elements of which have determinant 1, i.e. the subgroup

$$\mathrm{RotStab}(z_{\mathrm{f}}) = \{ \varrho \in \mathrm{Stab}(z_{\mathrm{f}}) \mid \det \varrho = 1 \} . \tag{3.27}$$

As can be straightforwardly shown the subgroups defined above fulfil the normalcy relation

$$\mathrm{N}_{\mathrm{Aut}}(P) \;\triangleright\; \mathrm{C}_{\mathrm{Aut}}(P) \;\triangleright\; \mathrm{Stab}(z_{\mathrm{f}}) \;\triangleright\; \mathrm{RotStab}(z_{\mathrm{f}}) , \tag{3.28}$$

such that we can define the quotient groups $\mathrm{N}_{\mathrm{Aut}}(P)/\mathrm{C}_{\mathrm{Aut}}(P)$, $\mathrm{C}_{\mathrm{Aut}}(P)/\mathrm{Stab}(z_{\mathrm{f}})$ and $\mathrm{Stab}(z_{\mathrm{f}})/\mathrm{RotStab}(z_{\mathrm{f}})$.

Clearly only those automorphisms of the lattice, that are compatible with the point group, i.e. which are elements of the normalizer of the point group in the automorphism group survive as isometries. Let us discuss the action of the different elements in the light of heterotic orbifold compactifications. In order to distinguish the different types of elements it is more useful to discuss the quotient groups instead of the subgroups themselves.

- The elements of $\mathrm{N}_{\mathrm{Aut}}(P)/\mathrm{C}_{\mathrm{Aut}}(P)$ map different point group elements to each other.
  As we have seen for the $\mathbb{Z}_3$ orbifold, such automorphisms may indeed be a symmetry of the spectrum of the orbifolded string theory. For instance a symmetry mapping the $\mathbb{Z}_3$ generator to its inverse would actually interchange twisted states with their CPT conjugates in the inverse twisted sector.

- Elements of $\mathrm{C}_{\mathrm{Aut}}(P)/\mathrm{Stab}(z_{\mathrm{f}})$ map different fixed point conjugacy classes to each other that are twisted by the same point group element, i.e. $\varrho : (\vartheta, \lambda) \mapsto (\vartheta, \lambda')$ such that $(\vartheta, \lambda) \not\sim (\vartheta, \lambda')$.
  Such isometries can in general be symmetries of the spectrum as well, as we have seen in section 2.3.4 for the $\mathbb{Z}_3$ example. There, if no Wilson lines are switched on, the spectrum of each inequivalent fixed point is the same, so that linear combinations of states with equal quantum numbers which are located at different fixed points can be eigenstates of such automorphisms. We will discuss these isometries further in section 3.3.3.

- The elements of $\mathrm{Stab}(z_{\mathrm{f}})/\mathrm{RotStab}(z_{\mathrm{f}})$ are isometries that leave the fixed point structure invariant but are not rotations. Hence, although they might manifest as symmetries of the theory, those symmetries will never be R.

Finally, the elements of $\mathrm{RotStab}(z_{\mathrm{f}})$ are symmetries of the orbifold geometry in all twisted sectors and of the spectrum. Hence they are the kind of isometries which give rise to R-symmetries in the low energy effective theory. We will discuss these lattice automorphisms in detail in the next section.

Before we proceed a comment is in order. Many lattices give rise to unfixed moduli, such as the lengths of basis vectors and angles between them. One example are the relative sizes of the tori in the $\mathbb{Z}_3$ orbifold. If such moduli are fixed to special values the symmetry of the geometry is enhanced, which manifests itself in the appearance of outer automorphisms of the lattice. We do not consider such enhanced symmetries in our exploration, because the moduli would be charged under the respective R-symmetries and therefore moduli stabilization would generically break these symmetries.

3.3.2 R-Symmetries from Orbifold Isometries

Now let us investigate the selection rules that arise from orbifold isometries in the group $\mathrm{RotStab}(z_\mathrm{f})$. As we have seen in section 3.1.3 such isometries are symmetries of the classical solutions and hence we expect the correlation functions to be invariant under the corresponding transformations. Therefore we will investigate the transformation behaviour of each of the vertex operators involved in the correlators and then deduce the selection rule by asking the complete correlator to be invariant.

Recall that $\varrho \in \mathrm{RotStab}(z_\mathrm{f})$ maps any given fixed point z_f with constructing element g to a conjugate one, i.e. there is a space group element h_g such that $\varrho z_\mathrm{f} = h_g z_\mathrm{f}$. On the corresponding constructing element $g = (\vartheta, \lambda)$, ζ acts according to

$$\varrho : \,(\vartheta, \lambda) \mapsto (\vartheta, \varrho\lambda)\,. \tag{3.29}$$

The space group element h_g fulfils $\rho(g) = h_g g h_g^{-1}$. If we write $h_g = (\vartheta_{h_g}, \lambda_{h_g})$, it is given by a solution to the equation

$$(1-\vartheta)\lambda_{h_g} = (\rho - \vartheta_{h_g})\lambda\,. \tag{3.30}$$

As we will see later, it turns out that for all cases considered, every element of $\mathrm{RotStab}(z_\mathrm{f})$ can be embedded into the Cartan subgroup of $\mathrm{SO}(6)$. Hence it can be written as

$$\varrho = \mathrm{e}^{2\pi\mathrm{i}(\xi^1 J_{12} + \xi^2 J_{34} + \xi^3 J_{56})}\,. \tag{3.31}$$

Using this information it is easy to write down the transformation behaviour of vertex operators of the form (3.1) once we know how the bosonic twist fields transform. Consider the linear combination (2.85). We can make use of the "local" identity $\rho(g) = h_g g h_g^{-1}$, to deduce that under $g \mapsto \varrho(g)$ the twist fields transform with the gamma phases,

$$\sigma_{[g]} \mapsto \mathrm{e}^{2\pi\mathrm{i}\gamma(g,\,h_g)}\sigma_{[g]}\,. \tag{3.32}$$

Therefore the transformation behaviour of the vertex operators (3.1) is determined to be

$$V_a \overset{\varrho}{\mapsto} \mathrm{e}^{2\pi\mathrm{i}\left[\,\xi_i\left(\mathcal{N}_\mathrm{L}^i - \bar{\mathcal{N}}_\mathrm{L}^i + q_\mathrm{sh}^i\right) + \gamma(g,\,h_g)\,\right]}\, V_a\,. \tag{3.33}$$

Now, correlation functions involve not only vertex operators of the form (3.1) but also picture changed bosonic operators (3.2). Observe however, that these are constructed using insertions of $\mathrm{e}^{\phi_{\beta\gamma}} T_\mathrm{F}$ which are manifestly $\mathrm{SO}(6)$ invariant. Therefore vertex operators in the 0-picture transform just according to (3.33).

Asking the correlation function to be invariant under ϱ transformations we arrive at the selection rule

$$\sum_{\alpha}\left(\sum_{i=1}^{3}\xi^{i}\left[\mathcal{N}^{i}_{\mathrm{L}\,\alpha}-\bar{\mathcal{N}}^{i}_{\mathrm{L}\,\alpha}+q^{i}_{\mathrm{sh}\,\alpha}\right]+\gamma(g_{\alpha},\,h_{g_{\alpha}})\right)=0\mod 1\,.$$

Note however, just as we saw when we discussed the H-momentum rule, that two of the q_{sh} are those of the fermionic part of a supermultiplet while the rest are those of the bosons. Let us therefore rewrite the selection rule in terms of purely bosonic q_{sh},

$$\sum_{\alpha}\left(\sum_{i=1}^{3}\xi^{i}\left[\mathcal{N}^{i}_{\mathrm{L}\,\alpha}-\bar{\mathcal{N}}^{i}_{\mathrm{L}\,\alpha}+q^{(\text{boson})\,i}_{\mathrm{sh}\,\alpha}\right]+\gamma(g_{\alpha},\,h_{g_{\alpha}})\right)=\sum_{i=1}^{3}\xi^{i}\mod 1\,. \tag{3.34}$$

Let M be the smallest integer such that $M\xi^i$ is an integer for all i and define

$$R=M\sum_{i=1}^{3}\xi^{i} \tag{3.35}$$

Then we can rewrite (3.34) according to

$$\sum_{\alpha=1}^{L}r_{\alpha}=R\mod M\,,\qquad\text{with}$$
$$r_{\alpha}=\sum_{i=1}^{3}M\xi^{i}\left[\mathcal{N}^{i}_{\mathrm{L}\,\alpha}-\bar{\mathcal{N}}^{i}_{\mathrm{L}\,\alpha}+q^{(\text{boson})\,i}_{\mathrm{sh}\,\alpha}\right]+M\gamma(g_{\alpha},\,h_{g_{\alpha}})\,. \tag{3.36}$$

Here r_α is the charge of the bosonic component of the α^{th} superfield.

If $\sum_{i=1}^{3}\xi^i\notin\mathbb{Z}$ this corresponds to an R-charge selection rule. Thus, by asking the correlation function (3.3) to be invariant under transformations $\varrho\in\mathrm{RotStab}(z_{\mathrm{f}})$, we have derived a selection rule that can be interpreted as a $\mathbb{Z}_M^R$ discrete symmetry from the point of view of the low energy effective theory.

Observe that we could have derived exactly the same selection rule by determining the transformation behaviour of the correlation function (3.12) under ϱ and asking the correlator to be invariant. In that way it is obvious that we have to require that ϱ indeed be a symmetry of the classical solutions and thereby of the coset vectors. Otherwise the classical solutions would not transform with a phase and it would be impossible to make the correlation function invariant.

Examples

Let us discuss the symmetries $\varrho\in\mathrm{RotStab}(z_{\mathrm{f}})$ that lead to R-charge selection rules for two examples. The first one is the $\mathbb{Z}_3$ orbifold which we already discussed in section 2.3.4. There one finds the following symmetry generators[6]:

$$\varrho_1=\left(\tfrac{1}{3},\,0,\,0\right)\,,\qquad\varrho_2=\left(0,\,\tfrac{1}{3},\,0\right)\,,\qquad\varrho_3=\left(0,\,0,\,\tfrac{1}{3}\right)\,.$$

[6] Clearly not all three R-symmetries are independent, as their combination leads to the orbifold generator θ which, by construction, acts trivially on all states.

We find that the action of the orbifold generator θ on each of the planes separately gives rise to an R-symmetry. Consequently one finds the selection rules

$$\sum_{\alpha=1}^{L} r_\alpha^i = 1 \mod 3\,, \qquad r_\alpha^i = \left[\mathcal{N}_{\mathrm{L}\,\alpha}^i - \bar{\mathcal{N}}_{\mathrm{L}\,\alpha}^i + q_{\mathrm{sh}\,\alpha}^{(\mathrm{boson})\,i}\right] + 3\gamma(g_\alpha,\, h_{g_\alpha}) \tag{3.37}$$

for all three internal directions $i = 1, 2, 3$. Note that if there are no Wilson lines switched on, the γ phases are all vanishing and the R-charge conservation rules reduce to those already discussed in [52].
The second example is the $\mathbb{Z}_4$ orbifold with twist vector $v_\theta = (\frac{1}{4}, \frac{1}{4}, -\frac{1}{2})$ based on the $\mathrm{SO}(4)^2 \times \mathrm{SU}(2)^2$ root lattice. There one finds the $\mathrm{RotStab}(z_\mathrm{f})$ generators [60]

$$\varrho_1 = \left(\tfrac{1}{4},\, \tfrac{1}{4},\, 0\right)\,, \qquad \varrho_2 = \left(\tfrac{1}{2},\, 0,\, 0\right)\,, \qquad \varrho_3 = \left(0,\, 0,\, \tfrac{1}{2}\right)\,.$$

Therefore from the exploration of the lattice automorphisms we find that the generalization of our result in the $\mathbb{Z}_3$ orbifold is not straightforward: Not always does the action of the orbifold generator θ on each of the planes separately gives rise to an R-symmetry. We can deduce the following selection rules,

$$\begin{aligned} &\sum_{\alpha=1}^{L} r_\alpha = 2 \mod 4\,, \qquad r_\alpha = \sum_{i=1}^{2}\left[\mathcal{N}_{\mathrm{L}\,\alpha}^i - \bar{\mathcal{N}}_{\mathrm{L}\,\alpha}^i + q_{\mathrm{sh}\,\alpha}^{(\mathrm{boson})\,i}\right] + 4\gamma(g_\alpha,\, h_{g_\alpha})\,,\\ &\sum_{\alpha=1}^{L} r_\alpha^i = 1 \mod 2\,, \qquad r_\alpha^i = \left[\mathcal{N}_{\mathrm{L}\,\alpha}^i - \bar{\mathcal{N}}_{\mathrm{L}\,\alpha}^i + q_{\mathrm{sh}\,\alpha}^{(\mathrm{boson})\,i}\right] + 2\gamma(g_\alpha,\, h_{g_\alpha}) \qquad i = 1, 3\,. \end{aligned} \tag{3.38}$$

The complete exploration of R-symmetries of Abelian orbifolds will be discussed in section 3.3.6. Before we attempt this let us discuss further R-symmetries that can arise from isometries of the orbifold.

3.3.3 Further R-Symmetries

In the previous section we have considered only symmetries $\varrho \in \mathrm{RotStab}(z_\mathrm{f})$, that is symmetries acting trivially on the fixed point conjugacy classes. However we have seen in section 2.3.4 that, depending on the Wilson line configuration of the model, it can happen that distinct fixed points allocate precisely the same twisted matter. If that is the case, isometries $\zeta \in {}^{C_{\mathrm{Aut}}(P)}/_{\mathrm{Stab}(z_\mathrm{f})}$ can actually be symmetries of the theory if one considers states that are superpositions of the copies of matter located at the fixed points that get mapped to each other. In this way those ζ that are rotations can lead to R-symmetries of the theory. Here we will make use of the fact that, as it turns out, it is always possible to write any of the relevant ζ in terms of the Cartan generators of SU(3) such that we can represent it by a twist vector $\eta_\zeta = (\eta_\zeta^1, \eta_\zeta^2, \eta_\zeta^3)$.
Assume $\zeta \in {}^{C_{\mathrm{Aut}}(P)}/_{\mathrm{Stab}(z_\mathrm{f})}$ interchanges the two conjugacy classes $[g]$ and $[g']$, i.e.

$$\zeta(g) = h_{g'} g' h_{g'}^{-1}\,, \qquad \zeta(g') = h_g g h_g^{-1}\,, \tag{3.39}$$

where such h_g and $h_{g'}$ exist for any pair of representatives g and g' of the conjugacy classes. Let V and V' be the vertex operators of two states with identical quantum numbers, except that V has constructing element g and V' has constructing element g'. Let $\sigma_{[g]}$ and $\sigma_{[g']}$ the corresponding twist fields. Then we find that under ζ these fields transform according to

$$\begin{aligned}\sigma_{[g]} &\overset{\zeta}{\mapsto} \mathrm{e}^{2\pi\mathrm{i}[\tilde{\gamma}(g)-\tilde{\gamma}'(\zeta(g))]}\,\sigma_{[g']}\,,\\ \sigma_{[g']} &\overset{\zeta}{\mapsto} \mathrm{e}^{2\pi\mathrm{i}[\tilde{\gamma}'(g')-\tilde{\gamma}(\zeta(g'))]}\,\sigma_{[g]}\,.\end{aligned} \tag{3.40}$$

Now let us consider the linear combinations $\tilde{V}^{(s)}$ of V and V',

$$\tilde{V}^{(s)} = V + \mathrm{e}^{2\pi\mathrm{i}(\delta+s)}V'\,, \qquad s = 0, \tfrac{1}{2}\,, \tag{3.41}$$

where δ is to be determined such that $\tilde{V}^{(s)}$ are eigenstates of ζ. We can make use of the explicit form of ζ and (3.40) to write down the transformation behaviour of V and V',

$$\begin{aligned}V &\overset{\zeta}{\mapsto} \mathrm{e}^{2\pi\mathrm{i}\left[\eta_i\left(\mathcal{N}^i_{\mathrm{L}}-\bar{\mathcal{N}}^i_{\mathrm{L}}+q^i_{\mathrm{sh}}\right)+\tilde{\gamma}(g)-\tilde{\gamma}'(\zeta(g))\right]}\,V'\,,\\ V' &\overset{\zeta}{\mapsto} \mathrm{e}^{2\pi\mathrm{i}\left[\eta_i\left(\mathcal{N}^i_{\mathrm{L}}-\bar{\mathcal{N}}^i_{\mathrm{L}}+q^i_{\mathrm{sh}}\right)+\tilde{\gamma}'(g')-\tilde{\gamma}(\zeta(g'))\right]}\,V\end{aligned} \tag{3.42}$$

and use this information to determine δ to be

$$\delta = \frac{1}{2}\left(-\tilde{\gamma}'(g') + \tilde{\gamma}(h_g g h_g^{-1}) + \tilde{\gamma}(g) - \tilde{\gamma}'(h_{g'} g' h_{g'}^{-1})\right)\,. \tag{3.43}$$

Then the transformation behaviour of the ζ eigenstates is given by

$$\tilde{V}^{(s)} \overset{\zeta}{\mapsto} \mathrm{e}^{2\pi\mathrm{i}\left[\eta_i\left(\mathcal{N}^i_{\mathrm{L}}-\bar{\mathcal{N}}^i_{\mathrm{L}}+q^i_{\mathrm{sh}}\right)+\frac{1}{2}\left(\gamma(g,h_g)+\gamma(g',h_{g'})\right)+s\right]}\,\tilde{V}^{(s)}\,, \tag{3.44}$$

where we have used

$$\begin{aligned}\gamma(g,h_g) &= \tilde{\gamma}(g) - \tilde{\gamma}(h_g g h_g^{-1})\,,\\ \gamma'(g',h_{g'}) &= \tilde{\gamma}'(g') - \tilde{\gamma}'(h_{g'} g' h_{g'}^{-1})\end{aligned}$$

and the fact that, as the physical gamma phases $\gamma'(g', h_{g'})$ are determined by the other quantum numbers of the state, which are equal for V and V', we can suppress the prime on $\gamma'(g', h_{g'})$.

Now consider correlators of A states, the constructing elements of which are mapped to their conjugates by ζ, and B linear combinations of states of the form (3.41),

$$\left\langle V_{(1)} \dots V_{(A)} V^{(s_1)}_{(1)} \dots V^{(s_B)}_{(B)} \right\rangle\,. \tag{3.45}$$

Such correlators[7] are supposed to be invariant under the action of ζ and we can deduce an R-charge selection rule. Observe that the states $V_{(\alpha)}$ transform under these isometries just as in (3.33) so their charges are as in (3.36). Then it follows from (3.44) that the states $\tilde{V}^{(s)}_{(\alpha)}$ have R-charges

$$r_{\alpha\,s} = \sum_{i=1}^{3} M\eta^i\left[\mathcal{N}^i_{\mathrm{L}\,\alpha} - \bar{\mathcal{N}}^i_{\mathrm{L}\,\alpha} + q^{(\mathrm{boson})\,i}_{\mathrm{sh}\,\alpha}\right] + \frac{1}{2}M\left(\gamma(g_\alpha, h_{g_\alpha}) + \gamma(g'_\alpha, h_{g'_\alpha})\right) + Ms\,. \tag{3.46}$$

It is remarkable that we were able to deduce further R-charge selection rules from isomet-

[7] Note that correlation functions of this form make sense, because if a coupling $\langle V_{(1)} \dots V_{(L-1)} V\rangle$ is allowed by all selection rules, so is its partner $\langle V_{(1)} \dots V_{(L-1)} V'\rangle$.

ries that do not leave all fixed point conjugacy classes invariant. Note that these symmetries may in general be broken by a Wilson line configuration which spoils the degeneracy in the spectrum of fixed points that are mapped to each other by ζ. It can however happen that this degeneracy is protected due to the restrictions (2.68) on the Wilson lines, which do not allow them to be switched on in such a way. In our exploration of possible R-symmetries we will also consider those isometries from $^{\mathrm{C}_{\mathrm{Aut}}(P)}/_{\mathrm{Stab}(z_{\mathrm{f}})}$ for which exactly this happens[8]. Therefore the R-symmetries we give in section 3.3.6 are valid for generic Wilson line configurations.
Before we go on, let us discuss an example of an R-symmetry arising from an isometry $\zeta \in {}^{\mathrm{C}_{\mathrm{Aut}}(P)}/_{\mathrm{Stab}(z_{\mathrm{f}})}$.

Example

Let us consider the $\mathbb{Z}_4$ orbifold of the previous section. The fixed point/torus structure is depicted in figure 3.3. As we will see, exploration of the orbifold isometries yields the generators

$$\zeta_1 = \left(\tfrac{1}{4},\, 0,\, 0\right) \,, \qquad \zeta_2 = \left(0,\, \tfrac{1}{4},\, 0\right) \,,$$

of $^{\mathrm{C}_{\mathrm{Aut}}(P)}/_{\mathrm{Stab}(z_{\mathrm{f}})}$. As they are not independent we will consider only $\zeta = \zeta_1$, which interchanges the two fixed tori

$$z_{\mathrm{f}} = \frac{e_2 + e_3}{2} \overset{\zeta}{\longleftrightarrow} z_{\mathrm{f}}' = \frac{e_2 + e_4}{2}$$

of the sector twisted by θ^2, which are generated by g and g' respectively. All other conjugacy classes of fixed points/tori are left invariant by ζ.

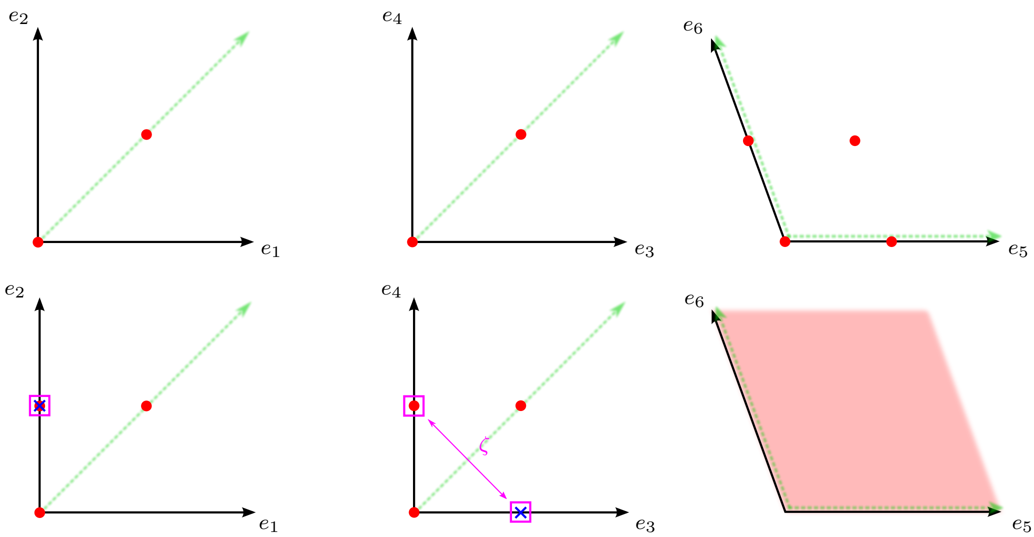

Figure 3.3: Fixed point/torus structure of the θ and θ^2 twisted sector of the $\mathrm{SO}(4)^2 \times \mathrm{SU}(2)^2$ $\mathbb{Z}_4$ orbifold. The fixed points and tori are given by the combinations of red dots. In the θ^2 sector there is an additional fixed torus located at the blue crosses. The fixed tori that get interchanged by ζ are marked by a purple box. Possible Wilson lines are indicated by blurred green arrows.

[8] Note that the conditions on the Wilson lines are identifications up to lattice vectors. The degeneracy is protected in general only in the cases in which the identified Wilson lines are taken to be exactly equal. Shifts of only one out of two (or more) identified Wilson lines, by a lattice vector, can therefore in general break these R-symmetries.

Note that because of the Wilson line identifications, $W_1 = W_2$ and $W_3 = W_4$, these two fixed tori support exactly the same states. Indeed if we consider the embedding

$$V_\theta = \frac{1}{4}(1,1,-2,0,0,0,0,0)(0,0,0,0,0,0,0,0)\,,$$
$$W_3 = W_4 = \frac{1}{4}(0,0,0,0,0,0,0,0)(0,0,0,0,1,1,1,1)\,,$$

the gauge group breaks down to $[\mathrm{E}_6 \times \mathrm{SU}(3)] \times [\mathrm{SO}(16) \times \mathrm{U}(1)]$ and both fixed tori support one $(\mathbf{1},\mathbf{2},\mathbf{16})_0$. The space group elements h_g and $h_{g'}$ that fulfil (3.39) are given by

$$h_g = (\theta,\, e_3) \quad \text{and} \quad h_{g'} = (\theta,\, 0)\,.$$

In the last section we found that in this $\mathbb{Z}_4$ orbifold, not all restrictions of the point group generator to individual planes give rise to an R-symmetry, because those in the first two planes exchange fixed point conjugacy classes. Now we see that, due to the identifications of Wilson lines, those symmetries are actually there and give rise to R-symmetries. The reason is that the spectra of states located at the fixed points being interchanged are equal. Therefore the states coming from the two fixed points are indistinguishable from the low energy point of view. We can however form linear combinations $\tilde{V}$ of those states, such that they become eigenstates of ζ and have different quantum numbers under the corresponding R-symmetry.

3.3.4 Universal R-Symmetry Anomalies

We have discussed the identification of R-symmetries in the last sections quite generally. However we have explicitly discussed three-point functions of $\mathbb{Z}_N$ orbifolds only. Therefore a consistency check of our findings which works more generally is needed. Such a check is provided by the cancellation of discrete anomalies.

The R-symmetries we have deduced in the previous sections are clearly classical symmetries of the geometry and can in principle be broken by quantum effects, i.e. be anomalous. Just as for continuous symmetries this can be most easily analysed by determining the transformation behaviour of the path integral measure [61, 62],

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \to \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{A}}\mathcal{D}\psi\mathcal{D}\bar{\psi}\,, \tag{3.47}$$

where the anomaly function $\mathcal{A}$ is a sum of gauge and gravity contributions. Namely it is given by

$$\mathcal{A} = -\frac{2\pi}{M}\left(\sum_a A_{G_a^2-\mathbb{Z}_M^R}\cdot\frac{1}{16\pi^2}\int \mathrm{tr}\left[\mathcal{F}_a\wedge\mathcal{F}_a\right] + A_{\mathrm{grav}^2-\mathbb{Z}_M^R}\cdot\frac{1}{384\pi^2}\int \mathrm{tr}\left[\mathcal{R}\wedge\mathcal{R}\right]\right)\,, \tag{3.48}$$

for the case of discrete $\mathbb{Z}_M$ R-symmetries [63–66]. Here $\mathcal{F}_a$ is the field strength corresponding to the gauge group G_a and $\mathcal{R}$ denotes the Riemann tensor. The anomaly coefficients are given by [67]

$$\begin{aligned} A_{G_a^2-\mathbb{Z}_M^R} &= C_2(G_a)\frac{R}{2} + \sum_\alpha \left(r_\alpha - \frac{R}{2}\right) T(\boldsymbol{R}_a^\alpha)\,, \\ A_{\text{grav}^2-\mathbb{Z}_M^R} &= \left(-21 - 1 - N_T - N_U + \sum_a \dim[\text{adj}(G_a)]\right)\frac{R}{2} \\ &\qquad + \sum_\alpha \left(r_\alpha - \frac{R}{2}\right)\dim(\boldsymbol{R}^\alpha)\,. \end{aligned} \tag{3.49}$$

Here α labels the left-chiral states in the spectrum, $C_2(G_a)$ denotes the quadratic Casimir of G_a and $T(\boldsymbol{R}_a^\alpha)$ is the Dynkin index of the representation $\boldsymbol{R}_a^\alpha$ of the state α under G_a. N_T and N_U denote the number of twisted and untwisted modulini of the theory and the contributions of -21 and -1 are due to the gravitino and dilatino respectively. Recall that the Pontryagin indices

$$\frac{T(\boldsymbol{F}_a)}{16\pi^2}\int \text{tr}\left[\mathcal{F}_a \wedge \mathcal{F}_a\right] \quad \text{and} \quad \frac{1}{2}\frac{1}{384\pi^2}\int \text{tr}\left[\mathcal{R}\wedge\mathcal{R}\right] \tag{3.50}$$

are integer valued, where $\boldsymbol{F}_a$ denotes the fundamental representation of the gauge group G_a.

It is well-known that, if present, such anomalies can be cancelled via the *Green-Schwarz mechanism* [68]. The idea is to introduce a term into the Lagrangian that is explicitly not invariant under the anomalous symmetry and the transformation of which cancels the transformation of the path integral measure. This proceeds via fields that transform with a shift under the respective symmetry. However in orbifold compactifications of the heterotic string the only field able to cancel anomalies via this mechanism is the four-dimensional two-form b_2 arising from the reduction of the Kalb-Ramond field B_2, that is dual to the imaginary part of the axio-dilaton. As a consequence, the anomaly coefficients of any $\mathbb{Z}_N^R$ symmetry have to fulfil the *universality conditions*,

$$\begin{aligned} A_{G_a^2-\mathbb{Z}_M^R} \mod MT(\boldsymbol{F}_a) &= A_{G_b^2-\mathbb{Z}_M^R} \mod MT(\boldsymbol{F}_b)\,, \\ A_{G_a^2-\mathbb{Z}_M^R} \mod MT(\boldsymbol{F}_a) &= \frac{1}{24}\left(A_{\text{grav}^2-\mathbb{Z}_M^R} \mod \frac{M}{2}\right) \end{aligned} \tag{3.51}$$

if the anomalies are to be cancelled by the Green-Schwarz mechanism.

A non-trivial check of whether the R-symmetries we identify do actually make sense at a quantum level is therefore given by anomaly universality which can be checked using the conditions (3.51). Note however that since the anomalies discussed above are chiral, this consistency check is only available when considering models with a chiral spectrum.

3.3.5 Family Symmetries

Arguments similar to the one we have used in section 3.3.3 have been applied to identify non-Abelian family symmetries of the low energy theory [69]. One can consider permutation symmetries of the fixed points with equal spectra and form linear combinations of the states, just as in (3.41), which transform in a given representation of the permutation group. Furthermore, as we have seen in (3.20), one can often rewrite the space group selection rule in terms of a set of $\mathbb{Z}_N$ symmetries. These discrete symmetries can then combined with the permutation symmetries to form non-Abelian flavour groups of the effective theory.

One would expect that those permutation symmetries arise from symmetries in $\text{C}_{\text{Aut}}(P)/\text{Stab}(z_f)$

that are not rotations and therefore not R. However, comparison with the results in [69], reveals that, geometrically, many of the permutation symmetries are realised by translations and therefore not covered by our scheme. Note that in the sector twisted by a given orbifolding group element $o \in G$, translations by any fixed point of o are a symmetry of the geometry. However this does not necessarily hold for sectors twisted by a different orbifolding group element $o' \in G$.

Let us illustrate this by an example. Consider again the $\mathbb{Z}_3$ orbifold. As we have seen in section 2.3.4, in the absence of Wilson lines, all fixed points in a given sector allocate exactly the same states. Therefore the low energy theory can be argued to be invariant under the relabelling of fixed points, i.e. an S_3 permutation symmetry arises from each of the three planes. Further we have seen that the space group selection rule can be written according to (3.20) as a discrete $(Z_3)^4$ symmetry. As a consequence the flavour symmetry group is generically

$$(S_3 \times S_3 \times S_3) \cup (Z_3)^4\,,$$

but may be enhanced further if the angles between and lengths of the basis vectors take special values. Let us try to identify the geometric symmetries that lead to the S_3 permutation symmetry. For simplicity we restrict ourselves to one plane. If we label the fixed points $(0,0)$, $(\frac{2}{3},\frac{1}{3})$ and $(\frac{1}{3},\frac{2}{3})$ by 1, 2 and 3 respectively, the permutation group S_3 can be generated by the two elements

$$s = (123) \qquad \text{and} \qquad m = (132)\,. \tag{3.52}$$

They can be realized geometrically by a shift by $s = \frac{2}{3}e_1 + \frac{1}{3}e_2$ and by a reflection about the axis $e_1 + e_2$ respectively. This is depicted in figure 3.4.

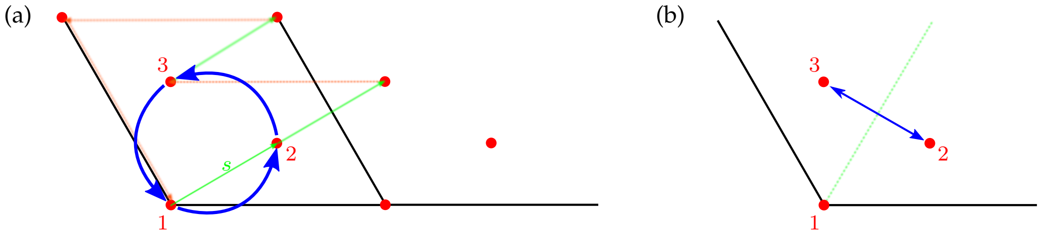

Figure 3.4: Permutation symmetry generators of one $\mathbb{Z}_3$ plane: (a) shows a shift by $s = \frac{2}{3}e_1 + \frac{1}{3}e_2$. Because of lattice identifications, the fixed points get interchanged periodically, i.e. this corresponds to $(123) \in S_3$. (b) shows the reflection about the axis $e_1 + e_2$. The fixed points 2 and 3 are interchanged, i.e. this corresponds to $(23) \in S_3$.

As discussed above, we do not cover shifts in our exploration of isometries, so let us consider the reflection m. Indeed m is an automorphism of the lattice. However its action on the restriction of the point group generator θ to the first plane, $\theta_1 = \mathrm{e}^{2\pi\mathrm{i}\frac{1}{3}}$ is non-trivial. Namely,

$$m\,\theta_1\,m^{-1} = (\theta_1)^{-1}\,, \tag{3.53}$$

that means, the action of m is not only to interchange the fixed points 2 and 3, but the images of the fixed points are in the inverse twisted sector. Therefore states located at the fixed point 2 get mapped to CPT conjugates of their partners at the fixed point 3. This is clearly not what we expect from a family symmetry. Moreover, the six-dimensional point group generator θ does not even get mapped to another point group element by m. Therefore it seems like the flavour symmetries, at least in this example, cannot be understood by isometries of the compactification space in this way. However inspiration of (3.23) shows that these lattice automorphisms are still symmetries of the instanton solutions. Note, that the flavour symmetries we just discussed, unlike the extra R-symmetries of section 3.3.3, can be broken by generic Wilson line configurations.

3.3.6 Catalogue of R-Symmetries for Abelian Orbifolds

Employing the scheme we developed in sections 3.3.1, 3.3.2 and 3.3.3 we can find all R-symmetries of six-dimensional toroidal orbifold compactifications of the heterotic string, the space group of which contains an Abelian point group. We use the results of the classification of such orbifolds performed in [34] and discussed in section 3.2.

Provided these results, for every affine class we follow the steps:

1. Find the generators of the automorphism group of the lattice.
2. Generate the complete automorphism group of the lattice.
3. Calculate the subgroups $\mathrm{N}_{\mathrm{Aut}}(P)$, $\mathrm{C}_{\mathrm{Aut}}(P)$, $\mathrm{Stab}(z_{\mathrm{f}})$, $\mathrm{RotStab}(z_{\mathrm{f}})$ as well as the quotient groups ${}^{\mathrm{N}_{\mathrm{Aut}}(P)}/{}_{\mathrm{C}_{\mathrm{Aut}}(P)}$, ${}^{\mathrm{C}_{\mathrm{Aut}}(P)}/{}_{\mathrm{Stab}(z_{\mathrm{f}})}$ and ${}^{\mathrm{Stab}(z_{\mathrm{f}})}/{}_{\mathrm{RotStab}(z_{\mathrm{f}})}$.
4. Find a set of generators of $\mathrm{RotStab}(z_{\mathrm{f}}) \cup ({}^{\mathrm{C}_{\mathrm{Aut}}(P)}/{}_{\mathrm{Stab}(z_{\mathrm{f}})} \cap \mathrm{SU}(3))$.
5. For each of the generators calculate the space group elements h_g defined in (3.30) and (3.39).

The first step is done using the software `Carat` [58] which uses an algorithm introduced in [70]. Steps 2-5 are performed using code we developed within `Mathematica` [71].

The results can be found in appendix A. In order to perform the consistency check of anomaly universality for the R-symmetries identified in this way, we use the `Orbifolder` [72] which we modified such that it can calculate the charges of the fields under the R-symmetries using the space group elements h_g. For each affine class we perform a scan over 10000 randomly generated gauge embeddings and check the universality conditions for each of the R-symmetries. The results can be found in appendix A. Note that out of the 138 Abelian orbifolds we consider, there are 23 which do not have a chiral spectrum. Hence there are no chiral R-symmetry anomalies and consequently we cannot perform the consistency check.

Recall that every single massless string state of the orbifold compactification which has a non-vanishing R-charge contributes to the chiral anomalies of the R-symmetries, which makes this consistency check far from trivial. It is an intriguing result, that although we have motivated our search strategy by the explicit form of the correlation functions of three-point couplings in $\mathbb{Z}_N$ orbifolds only, nearly all of the R-symmetries identified in this way pass the check of anomaly universality. Namely there are only four $\mathbb{Z}_2 \times \mathbb{Z}_2$ and two $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifolds in which it fails. These are listed in table 3.1. It is interesting that those $\mathbb{Z}_2 \times \mathbb{Z}_2$, for which the R-symmetries do not have universal anomalies are exactly those which can be described by modding a freely acting involution out of a different geometry. For instance the $\mathbb{Z}_2 \times \mathbb{Z}_2$-5-1 space group, which was discussed in [73], is obtained by modding the involution generated by the shift $\tau = \frac{1}{2}(e_2 + e_4 + e_6)$ out of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-1-1 geometry [34]. One might speculate that the reason the R-symmetries do not seem to be good symmetries of the quantum theory in these cases is due to the fact, that two states located at two fixed points of $\mathbb{Z}_2 \times \mathbb{Z}_2$-1-1 which get identified under the shift τ transform with different phases under the R-symmetries. This, as well as whether and how the R-symmetries might be repaired and finally whether this way of arguing extends to the problematic $\mathbb{Z}_2 \times \mathbb{Z}_4$ cases remains to be understood and is under current investigation.

$\mathbb{Q}$ class	$\mathbb{Z}$ class	affine class	R-symmetries		
			ρ	M	$-R$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	5	1	$\left(0,\ 0,\ \frac{1}{2}\right)$	2	-1
			$\left(\frac{1}{2},\ 0,\ 0\right)$	2	-1
			$\left(0,\ \frac{1}{2},\ 0\right)$	2	-1
	9	1	$\left(0,\ 0,\ \frac{1}{2}\right)$	2	-1
			$\left(\frac{1}{2},\ 0,\ 0\right)$	2	-1
			$\left(0,\ \frac{1}{2},\ 0\right)$	2	-1
	10	1	$\left(0,\ \frac{1}{2},\ 0\right)$	2	-1
			$\left(0,\ 0,\ \frac{1}{2}\right)$	2	-1
			$\left(\frac{1}{2},\ 0,\ 0\right)$	2	-1
	12	1	$\left(0,\ \frac{1}{2},\ 0\right)$	2	-1
			$\left(0,\ 0,\ \frac{1}{2}\right)$	2	-1
			$\left(\frac{1}{2},\ 0,\ 0\right)$	2	-1
$\mathbb{Z}_2 \times \mathbb{Z}_4$	1	6	$\left(0,\ 0,\ \frac{1}{4}\right)$	4	-1
			$\left(\frac{1}{4},\ 0,\ 0\right)$	4	-1
	2	4	$\left(0,\ \frac{1}{2},\ 0\right)$	2	-1

Table 3.1: R-symmetries, the chiral anomalies of which are not universal.

One more comment on the R-symmetries from a point of view of the low energy effective theory is in order. Note that the property of being R is conserved if an R-symmetry mixes with a non-R-symmetry. Sources of such non-R-symmetries may be family symmetries or discrete remnants of broken $\mathrm{U}(1)$ gauge symmetries (c.f. e.g. [74, 75]). Furthermore, multiple R-symmetries may be combined to form different R and non-R-symmetries. Therefore using the results we obtained here one can engineer many different R-symmetries in a phenomenological model. Recall also that in $\mathcal{N} = 1$ supersymmetric field theories there can only be one linear independent R-symmetry.

CHAPTER 4

R-Symmetries and a Heterotic MSSM

In this chapter we will make use of the R-symmetries we identified to evaluate a phenomenologically attractive heterotic orbifold model based on the $\mathbb{Z}_6$-II orbifold[1]. We start by reviewing the heterotic mini-landscape that was developed in [26–28] and more recently discussed and extended in [76, 77]. Then we discuss the impact of the R-symmetries identified in section 3.3.6 on these models[2]. We choose a configuration of vacuum expectation values (VEVs) of Standard Model singlets in the spectrum and analyse it using the technique of Hilbert bases [79].

4.1 The Heterotic Mini-Landscape

We consider the $\mathbb{Z}_6$-II orbifold based on the root lattice of $G_2 \times \mathrm{SU}(3) \times \mathrm{SU}(2)^2$, which in terms of the classification we discussed in section 3.2, corresponds to $\mathbb{Z}_6$-II_1_1. The point group generator θ with shift vector $v_\theta = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$ acts on the basis vectors according to

$$\theta e_1 = 2e_1 + 3e_2\,, \qquad \theta e_2 = -e_1 - e_2\,,$$
$$\theta e_3 = e_4\,, \qquad \theta e_4 = -e_3 - e_4\,,$$
$$\theta e_5 = -e_5\,, \qquad \theta e_6 = -e_6\,.$$

The fixed point/torus structure is depicted in figure 4.1.

Space Group and Family Symmetries

As the space group is factorizable we might split the lattice part of the space group selection rule (3.19) into three rules, one for each plane[3]. We have already seen that the selection rule of

[1] Note that, in order for our results to be comparable to the standard literature we choose a different convention for the right-moving momenta in this chapter. The q_{sh} that occur in the formulas of this chapter are obtained from the q_{sh} in the previous chapters by multiplication with -1.

[2] For the case of the $\mathbb{Z}_6$-II orbifold that we consider here, the R-symmetries were identified independently also in [78].

[3] Note that while the lattice factorizes according to $\Lambda = \Lambda_{\mathrm{G}_2} \times \Lambda_{\mathrm{SU}(3)} \times \Lambda_{\mathrm{SU}(2)^2}$, the orbifold does not factorize according to $O \neq (T_{\mathrm{G}_2}/\mathbb{Z}_6) \times (T_{\mathrm{SU}(3)}/\mathbb{Z}_3) \times (T_{\mathrm{SU}(2)^2}/\mathbb{Z}_2)$ because the point group generator θ acts on all three planes simultaneously.

the $T_{\mathrm{SU}(3)}/\mathbb{Z}_3$ plane can be written in the simple form (3.20). The same holds true for the $T_{\mathrm{SU}(2)^2}/\mathbb{Z}_2$ plane. For the $T_{\mathrm{G}_2}/\mathbb{Z}_6$ plane however it does not seem to be possible to identify a simple discrete symmetry resembling the space group selection rule in that plane [53]. Therefore we find the following discrete symmetries arising from the space group selection rule

$$\begin{aligned} \sum_\alpha k_\alpha &= 0 \mod 6\,, \\ \sum_\alpha k_\alpha m_\alpha &= 0 \mod 3\,, \\ \sum_\alpha n^5_\alpha &= 0 \mod 2\,, \\ \sum_\alpha n^6_\alpha &= 0 \mod 2\,, \end{aligned} \tag{4.1}$$

where the constructing elements of the states involved in the coupling are written as $g_\alpha = (\theta^{k_\alpha}, \lambda_\alpha)$ and the relation between their locations and the quantum numbers (m, n^5, n^6) are displayed in figure 4.1. Note that states in the θ^2 (θ^3) sector have charge $n^5 = n^6 = 0$ ($m = 0$).

The space group allows for an order three Wilson line, namely $W_3 = W_4$ and two order two Wilson lines, namely W_5 and W_6. Note that, as we have seen before, in the absence of Wilson lines all fixed points allocate the same massless matter. This however does not hold for the fixed tori in the sectors twisted by θ^2, θ^3 and θ^4. Some of them, e.g. the θ^2 fixed torus located at $2e_1/3$, are not fixed under θ, i.e. the space group element $(\theta, 0)$ does not commute with their constructing elements. Therefore for states supported at these "special" fixed tori the number of space group elements giving rise to projection conditions (2.95) is reduced as compared to those of the states that are located at "ordinary" fixed tori.

We have discussed the non-Abelian flavour structure of $T_{\mathrm{SU}(3)}/\mathbb{Z}_3$ already in section 3.3.5. While the $T_{G_2}/\mathbb{Z}_6$ plane does not enjoy any family symmetry, the $T_{\mathrm{SU}(2)^2}/\mathbb{Z}_2$ plane possesses a further $(D_4 \times D_4)/\mathbb{Z}_2$ symmetry [69]. As the orders of the restrictions of θ to these two planes are coprime the complete family symmetry of the $\mathbb{Z}_6$-II orbifold we consider, in the absence of Wilson lines, is given by

$$\Delta(54) \times (D_4 \times D_4)/\mathbb{Z}_2\,. \tag{4.2}$$

However realistic models need non-trivial Wilson line configurations. We have seen in section 2.3.4 how the presence of a non-trivial Wilson line breaks the degeneracy of the fixed point spectra and therefore the permutation symmetry $S_3 \subset \Delta(54)$ in the second plane. Therefore, if the order three Wilson line is switched on, $\Delta(54)$ breaks according to

$$\Delta(54) \xrightarrow{W_3 = W_4 \neq 0} \mathbb{Z}_3 \times \mathbb{Z}_3\,. \tag{4.3}$$

Furthermore the presence of the order two Wilson line W_5 and/or W_6 breaks $(D_4 \times D_4)/\mathbb{Z}_2$ according to

$$(D_4 \times D_4)/\mathbb{Z}_2 \xrightarrow{W_5 \neq 0} D_4 \times \mathbb{Z}_2 \xrightarrow{W_6 \neq 0} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\,. \tag{4.4}$$

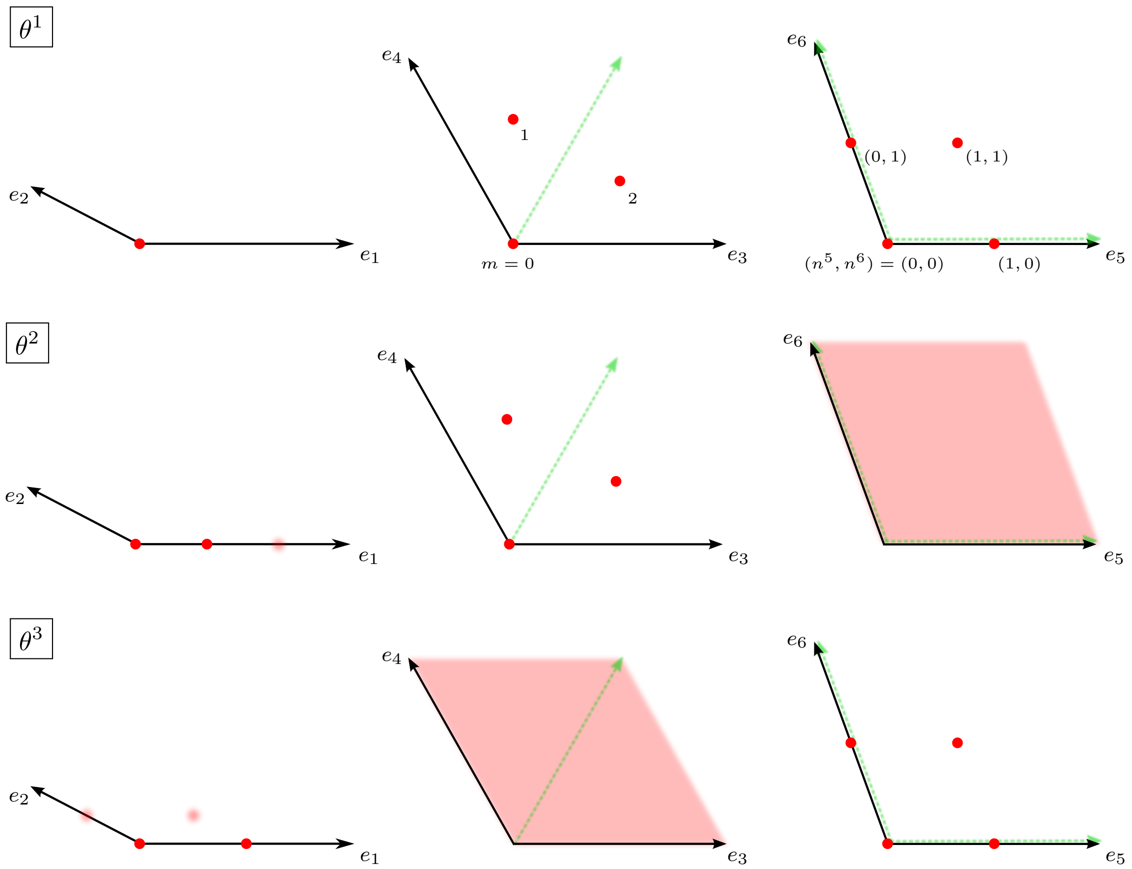

Figure 4.1: The sectors twisted by θ, θ^2 and θ^3 of the $\mathbb{Z}_6$-II orbifold we are considering. The possible Wilson lines are depicted by dotted green arrows, copies of fixed tori in the fundamental domain of the torus are marked by blurred red dots. In the θ sector the charges (m, n^5, n^6) of states, that are located at the respective fixed points, under the symmetries (4.1) are displayed.

Local GUTs and Gauge Embedding

The search strategy of the Mini-Landscape is based on the concept of *local GUTs* [26]: Recall that the conditions for a state to be massless, (2.89), and the projection conditions (2.95) in general depend on the fixed loci of twisted fields. That is, locally, the gauge group can look different at different fixed points and states located at these fixed points come in representations of the respective local gauge group. The untwisted sector states on the other hand are subject to projection conditions arising from all space group elements and are hence charged only under the *bulk gauge group* which is the intersection of the local gauge groups of all fixed points.

A local GUT is obtained by selecting embeddings of the space group into the gauge coordinates (2.67), such that they admit a local SO(10) or E_6 GUT structures. This means that there are fixed points in the sector twisted by θ^1 or θ^5 (T^1 or T^5), the massless spectra of which contain **16**- or **27**-plets respectively. The Wilson lines are chosen such that the untwisted sector gauge group breaks down to the Standard Model gauge group, in such a way that

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_Y \subset \mathrm{SU}(5) \subset \mathrm{SO}(10) \text{ or } \mathrm{E}_6\,. \tag{4.5}$$

There are two shift embeddings leading to a local $\mathrm{SO}(10)$ GUT, namely

$$V_\theta^{\mathrm{SO}(10),\,1} = (\tfrac{1}{3},\tfrac{1}{2},\tfrac{1}{2},0,0,0,0,0)(\tfrac{1}{3},0,0,0,0,0,0,0)\,,$$
$$V_\theta^{\mathrm{SO}(10),\,2} = (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0,0,0,0,0)(\tfrac{1}{6},\tfrac{1}{6},0,0,0,0,0,0)$$

and two shift embeddings leading to a local E_6 GUT, namely [26]

$$V_\theta^{\mathrm{E}_6,\,1} = (\tfrac{1}{2},\tfrac{1}{3},\tfrac{1}{6},0,0,0,0,0)(0,0,0,0,0,0,0,0)\,,$$
$$V_\theta^{\mathrm{E}_6,\,2} = (\tfrac{2}{3},\tfrac{1}{3},\tfrac{1}{3},0,0,0,0,0)(\tfrac{1}{6},\tfrac{1}{6},0,0,0,0,0,0)\,.$$

In principle there are two straightforward ways of obtaining three-family models from such orbifolds. The first one is to switch on the Wilson lines W_5 and W_6 such that T^1 (T^5) contains at least three $\mathbf{16}$- or $\mathbf{27}$-plets. However such models always contain chiral exotics [26]. The second strategy is to switch on one order two Wilson line, i.e. W_5 or W_6, as well as the order three Wilson line $W_3 = W_4$ such that T^1 (T^5) contains two $\mathbf{16}$- or $\mathbf{27}$-plets. The reason for exactly two families to survive the projection conditions arising from the Wilson lines is the D_4 family symmetry (4.4). The third family then arises from states of the untwisted or other twisted sectors. Note that while the two Standard Model families arising as $\mathbf{16}$- or $\mathbf{27}$-plets are true GUT multiplets, the third family only has the SM quantum numbers of an additional $\mathbf{16}$- or $\mathbf{27}$-plet. Therefore the third family is often called *patchwork* [76].

Exploring the Mini-Landscape

The program of the mini-landscape exploration was composed of the following steps:

1. Generate inequivalent[4] Wilson line configurations fulfilling (4.5)
2. Select models with three $(\mathbf{3},\,\mathbf{2})$
3. Select models with non-anomalous $\mathrm{U}(1)_Y \subset \mathrm{SU}(5)$
4. Select models with 3 SM families + Higgses + vector-like exotics

As a result 223 models, 218 of which are based on the $\mathrm{SO}(10)$ shift embeddings, were obtained. Provided this set of models one can proceed by specifying a supersymmetric VEV configuration and discussing the phenomenological properties that depend on Yukawa couplings, such as

- quark and lepton masses,
- the Higgs sector,
- proton decay,
- decoupling of exotics,
- the possibility for hidden sector gaugino condensation and susy breaking.

We will address these model dependent questions within a specific VEV configuration in section 4.3. Before we proceed in that direction, let us review the technique of Hilbert bases, which we will use as a very efficient tool for analysing the VEV configuration.

[4] Models are considered inequivalent if they have identical spectra with respect to the non-Abelian gauge groups and the same number of non-Abelian singlets.

4.2 Hilbert Bases

We have seen that selection rules are an efficient tool to determine whether or not a given term occurs in the superpotential of the low energy effective theory. While the relative coefficients of the terms cannot be determined in this way, still a lot can be learned about the structure of the superpotential.

Assume we are given a set of fields, which we denote by $\Phi_1, \dots \Phi_N$ and a set of selection rules. Then we expect the superpotential to contain every monomial $\Phi_{i_1} \dots \Phi_{i_m}$ that satisfies all of these selection rules. However, if such terms are allowed by the selection rules, so will be all their powers and products with each other[5]. Then it is a natural question to ask, whether there is a set of monomials, such that any allowed combination of fields can be written as a product of these "basis monomials". For those selection rules, which can be expressed by conditions that are linear in the charges of the fields the answer is positive and the basis monomials are given by the *Hilbert basis* [79–82].

4.2.1 Hilbert Basis for U(1)s

Consider, for simplicity, a set of M $\mathrm{U}(1)$ symmetries, under which the fields $\Phi_1, \dots \Phi_N$ carry charges $Q_1^{(a)}, \dots Q_N^{(a)}$, $a = 1, \dots, M$. Then a monomial $\Phi_1^{n_1} \dots \Phi_N^{n_N}$ is allowed by the selection rules if and only if

$$Qn = \begin{pmatrix} Q_1^{(1)} & \dots & Q_N^{(1)} \\ \vdots & \ddots & \vdots \\ Q_1^{(M)} & \dots & Q_N^{(M)} \end{pmatrix} \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix} = 0\,. \tag{4.6}$$

This displays a set of linear diophantine equations, and as the superpotential is holomorphic in the fields, we are interested in solutions for which $n_\alpha \in \mathbb{N}_0$, $\alpha = 1, \dots N$.

Therefore, Mathematically speaking, the problem of finding the basis monomials is the problem of finding the set of minimal solutions of the monoid[6] of non-negative, integral solutions of a system of linear diophantine equations. It is a non-trivial statement that such a minimal set exists and that it has only finitely many elements. While, in general, the Hilbert basis cannot be determined analytically, efficient algorithms have been developed, that make use of the geometrical interpretation of the set of solutions [83–87].
We will not go into details about how these algorithms work, but use the publicly available computer program `normaliz` [88, 89] to compute the Hilbert bases $\mathcal{H}$ of systems of diophantine equations.

4.2.2 Extension to Other Symmetries

The method described above can be extended to other kinds of symmetries for which the selection rules can be written as conditions that are linear in the charges of the fields.

[5] Properly speaking this is true only for the non-R-symmetries. We will discuss this issue in detail later.
[6] A monoid is a set together with a binary operations which is associative and for which a neutral element exists. If for every element of the set there exists an inverse with respect to the binary operation, the monoid forms a group. As the natural numbers form a monoid under addition, the set of allowed monomials forms a monoid under multiplication.

Non-Abelian Gauge Symmetries

Let us start with the non-Abelian gauge symmetries of the theory. A monomial of fields is invariant under the gauge group G, if and only if it transforms trivially under all $\mathrm{U}(1)$s generated by the Cartan subalgebra of the group [79]. Therefore, we can write the invariance condition in the way (4.6) with $\mathrm{rank}(G)$ diophantine equations. However, there is a subtlety. There can be monomials which are zero by construction, because of the way symmetric and anti-symmetric indices are contracted. An example is the monomial Φ^2, where Φ transforms in the fundamental representation of $\mathrm{SU}(2)$. Therefore elements of the Hilbert basis which have this property must be eliminated from the basis a posteriori.

Discrete Non-R Symmetries

Let us turn to discrete symmetries. For non-R-symmetries we can straightforwardly write a condition similar to (4.6) by exchanging equations with congruences. Consider, for simplicity, a single discrete symmetry, under which the fields carry charges $Q_1, \dots Q_N$. Without loss of generality we can assume these charges to be positive. Then a monomial $\Phi_1^{n_1} \dots \Phi_N^{n_N}$ is allowed by the corresponding selection rule if and only if

$$\begin{pmatrix} Q_1 & \dots & Q_N \end{pmatrix} \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix} = 0 \mod m\,. \tag{4.7}$$

This congruence relation can be rephrased as a diophantine equation by introducing an auxiliary variable a and asking the monomial to fulfil

$$\begin{pmatrix} -m & Q_1 & \dots & Q_N \end{pmatrix} \begin{pmatrix} a \\ n_1 \\ \vdots \\ n_N \end{pmatrix} = 0\,. \tag{4.8}$$

Again allowed monomials are characterized by non-negative, integral solutions to this diophantine equation, the first entry of the solution vectors being unphysical. Accordingly, for the case of M discrete symmetries, the system of linear congruences is rewritten in the form

$$\begin{pmatrix} -m_1 & \dots & 0 & Q_1^{(1)} & \dots & Q_N^{(1)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -m_M & Q_1^{(M)} & \dots & Q_N^{(M)} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_M \\ n_1 \\ \vdots \\ n_N \end{pmatrix} = 0\,. \tag{4.9}$$

Discrete R-Symmetries

For the case of R-symmetries the situation is more involved, as we are dealing with an inhomogeneous system of linear congruences. It is possible to rewrite any such system as a system of homogeneous and inhomogeneous diophantine equations by the method described above. Let us therefore consider an inhomogeneous system of linear diophantine equations instead. Observe that we can rewrite the inhomogeneous system

$$\begin{pmatrix} Q_1^{(1)} & \dots & Q_N^{(1)} \\ \vdots & \ddots & \vdots \\ Q_1^{(M)} & \dots & Q_N^{(M)} \end{pmatrix} \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix}, \tag{4.10}$$

as the homogeneous system

$$\begin{pmatrix} -x_1 & Q_1^{(1)} & \dots & Q_N^{(1)} \\ \vdots & & \ddots & \vdots \\ -x_M & Q_1^{(M)} & \dots & Q_N^{(M)} \end{pmatrix} \begin{pmatrix} n_0 \\ n_1 \\ \vdots \\ n_N \end{pmatrix} = 0\,. \tag{4.11}$$

We are interested in solutions with $n_0 = 1$ or $n_0 = 0$, because adding any solution of the homogeneous analogue of (4.10) to a solution of the inhomogeneous system again yields a solution of the inhomogeneous system. Assume $\mathcal{H} = \{h_1, \dots, h_D\}$ is the Hilbert basis of the system (4.11), i.e. any solution of this system can be written as a linear combination $n = \sum_{i=1}^{D} \lambda_i h_i$ with positive integral coefficients, $\lambda_i \in \mathbb{N}_0$. Then it is clear that any solution of (4.11), for which $n_0 = 1$, can be written as a linear combination of those Hilbert basis elements, whose first component is 0 or 1. Therefore

$$\begin{aligned} \mathcal{H}_{\text{inhom}} &= \mathcal{H}_{\text{inhom}}^{(0)} \cup \mathcal{H}_{\text{inhom}}^{(1)}\,, \qquad && \text{where} \\ \mathcal{H}_{\text{inhom}}^{(a)} &= \left\{h \in \mathcal{H} \,\middle|\, h^0 = a\right\} && a \in \{0,1\}\,, \end{aligned} \tag{4.12}$$

serves as a basis of solutions of the system (4.10). Each solution to this system may be written as a linear combination

$$n = h^{(1)} + \sum_i \lambda_i h_i^{(0)}\,, \tag{4.13}$$

where $h^{(1)} \in \mathcal{H}_{\text{inhom}}^{(1)}$, $h_1^{(0)}, \dots, h_K^{(0)} \in \mathcal{H}_{\text{inhom}}^{(0)}$ and $\lambda_1, \dots, \lambda_K \in \mathbb{N}_0$ are positive, integral coefficients.

Combinations of Different Symmetries

We have motivated the concept of Hilbert bases, as a method to find the set of basis monomials of fields that are allowed by the selection rules of a given theory. These monomials of course have to fulfil all selection rules at the same time. Therefore one has to combine the systems of equations of the different kinds of symmetries we have discussed above into a single set of diophantine equations and compute the Hilbert basis of this set. We will discuss the resulting system for the $\mathbb{Z}_6$-II model under consideration in the following section.

4.3 A Semi-Realistic String Model

The model we will consider is leaned against model I of [27], i.e. we choose the gauge embedding

$$\begin{aligned} V &= (\ \tfrac{1}{3}, -\tfrac{1}{2}, -\tfrac{1}{2},\ 0,\ 0,\ 0,\ 0,\ 0)(\ \tfrac{1}{2}, -\tfrac{1}{6}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2},\ \tfrac{1}{2})\,, \\ W_3 = W_4 &= (-\tfrac{1}{2}, -\tfrac{1}{2},\ \tfrac{1}{6},\ \tfrac{1}{6},\ \tfrac{1}{6},\ \tfrac{1}{6},\ \tfrac{1}{6},\ \tfrac{1}{6})(\ \tfrac{1}{3},\ 0,\ 0,\ \tfrac{2}{3},\ 0,\ \tfrac{5}{3}, -2,\ 0)\,, \\ W_5 &= (\ 0, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ 0,\ 0)(\ 4, -3, -\tfrac{7}{2}, -4, -3, -\tfrac{7}{2}, -\tfrac{9}{2},\ \tfrac{7}{2})\,. \end{aligned} \tag{4.14}$$

This embedding breaks the gauge group to

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{U}(1)_{\text{anom}} \times \mathrm{U}(1)_Y \times \mathrm{U}(1)^7\,, \tag{4.15}$$

where we have chosen the generator of one of the nine $\mathrm{U}(1)$s such that it gives the desired hypercharge of the Standard Model. Furthermore, we have rotated the Abelian anomaly into a single $\mathrm{U}(1)$. Note that this anomaly is cancelled, just as the discrete R-symmetry anomaly, via the Green-Schwarz mechanism by a shift in the axion field.

4.3.1 Spectrum

The complete spectrum of the model is presented in appendix C. We use the same labels for the fields as in [27], in order to make comparisons simpler. There are three families of quarks and leptons, two of which arise as local $\mathrm{SO}(10)$ multiplets in the θ^5 sector at the fixed points 0 and $\frac{1}{2}e_6$. In agreement with the general statement in section 4.1, the third family constitutes of states in the bulk (q_3, $\bar{u}_3$, $\bar{e}_3$) as well as from fixed tori in the θ^2 and θ^4 sectors. Further, there is a single pair of massless Higgs fields, ϕ_1 and $\bar{\phi}_1$ in the bulk.

In addition to these Standard Model states there are several vector-like exotics in the spectrum which need to be decoupled.

4.3.2 Couplings and Hilbert Basis

In order to discuss further phenomenological features which arise from the allowed couplings, let us discuss the selection rules in the model. According to our exploration of section 3.3.6 the geometry gives rise to a $\mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ R-symmetry, which corresponds to the charge conservation laws [60, 78]

$$\begin{aligned} \sum_{\alpha=1}^{L} r_\alpha^1 &= \sum_{\alpha=1}^{L} \left[q_{\text{sh}\,\alpha}^{(\text{boson})\,1} - \mathcal{N}_{\text{L}\,\alpha}^1 + \bar{\mathcal{N}}_{\text{L}\,\alpha}^1 - 6\gamma(g_\alpha, h_{g_\alpha}^{\theta_1}) \right] = -1 \mod 6\,, \\ \sum_{\alpha=1}^{L} r_\alpha^2 &= \sum_{\alpha=1}^{L} \left[q_{\text{sh}\,\alpha}^{(\text{boson})\,2} - \mathcal{N}_{\text{L}\,\alpha}^2 + \bar{\mathcal{N}}_{\text{L}\,\alpha}^2 - 3\gamma(g_\alpha, h_{g_\alpha}^{\theta_2}) \right] = -1 \mod 3\,, \\ \sum_{\alpha=1}^{L} r_\alpha^3 &= \sum_{\alpha=1}^{L} \left[q_{\text{sh}\,\alpha}^{(\text{boson})\,3} - \mathcal{N}_{\text{L}\,\alpha}^3 - \bar{\mathcal{N}}_{\text{L}\,\alpha}^3 - 2\gamma(g_\alpha, h_{g_\alpha}^{\theta_3}) \right] = -1 \mod 2\,, \end{aligned} \tag{4.16}$$

where the space group elements $h_{g_\alpha}^{\theta_i}$ needed to compute the charges are listed in appendix B. These conditions differ from the ones used in [27], which we will refer to as the "old rules", by the contributions of the γ phases. We will devote the remainder of this section to studying the phenomenological consequences of these modified selection rules. We will not aim at finding a vacuum configuration that fulfils all phenomenological constraints. Instead, we choose a benchmark VEV configuration which is semi-realistic and allows us to check, whether candidates for realistic models within the mini-landscape remain at all.

Differences due to the new R-symmetries arise already at order 3 in the superpotential. Several new terms, which were forbidden by the old rules, appear. In the following we show a simple example. The R-charges of the considered fields are displayed in table 4.1. We immediately see that the coupling $s^0_{13}s^0_{14}s^0_{30}$ was forbidden by the old rules but is allowed by the new R-symmetries.

	r^1_{old}	r^2_{old}	r^3_{old}	r^1_{new}	r^2_{new}	r^3_{new}
s^0_{13}	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$
s^0_{14}	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$
s^0_{30}	$-\frac{5}{3}$	$-\frac{1}{3}$	0	$\frac{4}{3}$	$-\frac{1}{3}$	0

Table 4.1: Differences between our R-charges of the fields s^0_{13}, s^0_{14} and s^0_{30} and the ones in [27].

This example shows that the modified R-symmetries can have important phenomenological implications. If all fields s^0_{13}, s^0_{14} and s^0_{30} get a VEV in a given configuration, the superpotential W gets a VEV as well. Since the superpotential is linked to the μ-term in such models, the μ-term will be generically too large. We will discuss this issue for our VEV configuration later in more detail. There are also couplings forbidden by the new R-symmetries which were allowed by the old ones. The first example occurs at order 4 in the superpotential.

We have discussed the space group selection rule of the model in (4.1). The gauge symmetry is given in (4.15). As the fields obtaining non-trivial VEVs are singlets under the non-Abelian gauge group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(4)$ and either singlets or doublets under the remaining $\mathrm{SU}(2)$, we will only consider the non-anomalous $\mathrm{U}(1)$ symmetries to calculate the Hilbert basis. The basis elements vanishing due to the way the $\mathrm{SU}(2)$ indices are contracted, will be removed a posteriori. The same holds for the monomials forbidden by the space group selection rule in the $T_{G_2}/\mathbb{Z}_6$ plane, as we were not able to rewrite it as a diophantine equation. The last missing ingredient are the R-symmetries discussed in (4.16).

Therefore the system of diophantine equations, that is fulfilled by any monomial $\Phi_1^{n_1}\ldots\Phi_L^{n_L}$ allowed by the selection rules, is given by

$$\begin{pmatrix} r_1^1 & \ldots & r_L^1 \\ r_1^2 & \ldots & r_L^2 \\ r_1^3 & \ldots & r_L^3 \\ q_1^{\mathbb{Z}_6} & \ldots & q_L^{\mathbb{Z}_6} \\ q_1^{\mathbb{Z}_3} & \ldots & q_L^{\mathbb{Z}_3} \\ q_1^{\mathbb{Z}_2} & \ldots & q_L^{\mathbb{Z}_2} \\ q_1^{\mathbb{Z}_2'} & \ldots & q_L^{\mathbb{Z}_2'} \\ q_1^{\mathrm{U}(1)^8} & \ldots & q_L^{\mathrm{U}(1)^8} \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ \vdots \\ n_L \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{mod} \quad \begin{pmatrix} 6 \\ 3 \\ 2 \\ 6 \\ 3 \\ 2 \\ 2 \\ 0 \end{pmatrix} . \tag{4.17}$$

4.3.3 VEV Configuration and Phenomenology

We proceed by choosing a VEV configuration, which, as we will see, gives rise to a semi-realistic model. Namely, we give the 14 Standard Model singlets

$$\tilde{s} = \{h_1, h_2, h_3, h_4, s_3^0, s_4^0, s_9^0, s_{10}^0, s_{12}^0, s_{21}^0, s_{24}^0, s_{28}^0, s_{29}^0, s_{30}^0\} \tag{4.18}$$

a non-trivial VEV. As the h_i are charged under the hidden $\mathrm{SU}(2)$, this VEV configuration breaks the gauge group of the model to $G_{\mathrm{SM}} \times \mathrm{SU}(4)$.

In order to determine the phenomenological properties of this configuration, we consider those coupling of the $\tilde{s}$ with the other fields which give rise to the quark and lepton Yukawa couplings, give masses to the exotics and neutrinos or mediate proton decay. In contrast to [27], we do not calculate the couplings order-by-order in the singlet fields, but instead construct the corresponding Hilbert bases. In this way we obtain the complete information about the phenomenologically relevant effects for all orders in the singlet fields.

The Superpotential and the μ-Term

Let us exemplify how we determine the Hilbert basis of couplings corresponding to a specific phenomenological property by looking at the μ-Term. It gets induced by couplings $\phi_1\bar{\phi}_1 f(\tilde{s})$, so we need a basis for allowed monomials of the form $\phi_1^{n_1}\bar{\phi}_1^{n_2}\tilde{s}_{i_1}^{n_3}\ldots\tilde{s}_{i_M}^{n_{M+2}}$, where only solutions with $n_1 = n_2 = 1$ are physical. We can make use of this requirement to absorb the inhomogeneity arising from the R-symmetries in a redefinition of the R-charges of these fields. If we write $\tilde{r}^i = r_1^i + r_2^i + 1$, the system (4.17) takes the form

$$\begin{pmatrix} \tilde{r}^1 & r_3^1 & \cdots & r_{M+2}^1 \\ \tilde{r}^2 & r_3^2 & \cdots & r_{M+2}^2 \\ \tilde{r}^3 & r_3^3 & \cdots & r_{M+2}^3 \\ q_1^{\mathbb{Z}_6} + q_2^{\mathbb{Z}_6} & q_3^{\mathbb{Z}_6} & \cdots & q_{M+2}^{\mathbb{Z}_6} \\ q_1^{\mathbb{Z}_3} + q_2^{\mathbb{Z}_3} & q_3^{\mathbb{Z}_3} & \cdots & q_{M+2}^{\mathbb{Z}_3} \\ q_1^{\mathbb{Z}_2} + q_2^{\mathbb{Z}_2} & q_3^{\mathbb{Z}_2} & \cdots & q_{M+2}^{\mathbb{Z}_2} \\ q_1^{\mathbb{Z}_2'} + q_2^{\mathbb{Z}_2'} & q_3^{\mathbb{Z}_2'} & \cdots & q_{M+2}^{\mathbb{Z}_2'} \\ q_1^{\mathrm{U}(1)^8} + q_2^{\mathrm{U}(1)^8} & q_3^{\mathrm{U}(1)^8} & \cdots & q_{M+2}^{\mathrm{U}(1)^8} \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_3 \\ \vdots \\ n_{M+2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mod \begin{pmatrix} 6 \\ 3 \\ 2 \\ 6 \\ 3 \\ 2 \\ 2 \\ 0 \end{pmatrix} . \tag{4.19}$$

After determining the Hilbert basis $\mathcal{H}$, in order to obtain physical viable solutions, we split it into a homogeneous and an inhomogeneous part, similar as in (4.12). Elements with $n_1 = 1$ are assigned to $\mathcal{H}^{(1)}_{\text{inhom}}$ and basis elements with $n_1 = 0$ are assigned to $\mathcal{H}^{(0)}_{\text{inhom}}$. Then, the physical solutions are given by

$$\left\{ n^{(1)}_{\text{inhom}} + m\, n^{(0)}_{\text{inhom}} \,\middle|\, n^{(1)}_{\text{inhom}} \in \mathcal{H}^{(1)}_{\text{inhom}},\ n^{(0)}_{\text{inhom}} \in \mathcal{H}^{(0)}_{\text{inhom}},\ m \in \mathbb{N}_0 \right\} . \tag{4.20}$$

Note that, as $\phi_1\bar{\phi}_1$ is a complete singlet under all symmetries of the theory, the terms inducing the μ-term also give a VEV to the superpotential itself. This is known as *gauge-Higgs unification* and was identified as one of the features rendering the models of the mini-landscape so phenomenologically attractive [76]. It was argued that this feature is a consequence of the presence of R-symmetries in the model [27, 31, 90]. This argument is not altered by the modification of the definition of the R-charges, as the Higgs fields are bulk fields and therefore the contribution of the γ phases to their R-charges vanishes.

We find that up to order 10 in $\tilde{s}$, the superpotential is given by[7]

$$W_{\tilde{s}} = (M_{1,\text{inhom}} + M_{2,\text{inhom}} + M_{3,\text{inhom}})(1 + M_{1,\text{hom}} + M_{2,\text{hom}}) \tag{4.21}$$

with

$$\begin{aligned} M_{1,\text{inhom}} &= s_3^0 s_4^0 s_{29}^0, & M_{1,\text{hom}} &= (s_{21}^0 s_{31}^0)^3 \\ M_{2,\text{inhom}} &= s_9^0 s_{10}^0 s_{29}^0, & M_{2,\text{hom}} &= h_1 h_2 s_{21}^0 s_{24}^0 s_{28}^0 s_{31}^0 \\ M_{3,\text{inhom}} &= h_3 h_4 s_{21}^0 s_{31}^0. \end{aligned} \tag{4.22}$$

In principle we know all Hilbert basis elements and therefore all monomials which means we know the exact form of W to all orders.

Quark and Lepton Yukawa Couplings

Employing a similar strategy to the one outlined above, we are able to identify the Yukawa interactions

$$W_{\text{Yuk}} = Y_u(\tilde{s}) q \bar{u} \bar{\phi}_1 + Y_d(\tilde{s}) q \bar{d} \phi_1 + Y_e(\tilde{s}) l \bar{e} \phi_1 \tag{4.23}$$

where the Yukawa matrices $Y_i(\tilde{s})$ depend on the fields $\tilde{s}$. At lowest order in the singlets, the results read

$$Y_u = \begin{pmatrix} M_1 & M_2 & M_3 + M_4 \\ M_2 & M_1 & M_5 + M_6 \\ M_7 + M_8 + M_9 + M_{10} & M_{11} + M_{12} + M_{13} + M_{14} & 1 \end{pmatrix} \tag{4.24}$$

with

$$\begin{aligned} M_1 &= h_1 h_3 s_4^0 s_{21}^0 s_{29}^0 s_{31}^0, & M_2 &= h_1 h_3 s_{10}^0 s_{21}^0 s_{29}^0 s_{31}^0 \\ M_3 &= h_1 h_3 s_3^0 s_{10}^0 s_{21}^0 s_{29}^0 s_{31}^0 = s_3^0 M_2, & M_4 &= h_1 h_3 s_4^0 s_9^0 s_{21}^0 s_{29}^0 s_{31}^0 = s_9^0 M_1 \\ M_5 &= h_1 h_3 s_3^0 s_4^0 s_{21}^0 s_{29}^0 s_{31}^0 = s_3^0 M_1, & M_6 &= h_1 h_3 s_9^0 s_{10}^0 s_{21}^0 s_{29}^0 s_{31}^0 = s_9^0 M_2 \\ M_7 &= h_1 h_3 s_4^0 s_{12}^0 s_{21}^0 s_{29}^0 s_{31}^0 = s_{12}^0 M_1, & M_8 &= s_3^0 s_4^0 s_{10}^0 s_{21}^0 (s_{29}^0)^2 s_{31}^0 \\ M_9 &= s_9^0 (s_{10}^0)^2 s_{21}^0 (s_{29}^0)^2 s_{31}^0, & M_{10} &= (s_4^0)^2 s_9^0 s_{21}^0 (s_{29}^0)^2 s_{31}^0 \\ M_{11} &= h_1 h_3 s_{10}^0 s_{12}^0 s_{21}^0 s_{29}^0 s_{31}^0 = s_{12}^0 M_2, & M_{12} &= s_4^0 s_9^0 s_{10}^0 s_{21}^0 (s_{29}^0)^2 s_{31}^0 \\ M_{13} &= s_3^0 (s_{10}^0)^2 s_{21}^0 (s_{29}^0)^2 s_{31}^0, & M_{14} &= s_3^0 (s_4^0)^2 s_{21}^0 (s_{29}^0)^2 s_{31}^0. \end{aligned} \tag{4.25}$$

As well as for the down quarks and leptons,

$$Y_d = \begin{pmatrix} M_1 & M_2 & M_3 + M_4 \\ M_2 & M_1 & M_5 \\ M_6 & M_7 & M_8 + M_9 \end{pmatrix}, \qquad Y_e = \begin{pmatrix} M_1 & M_2 & M_{10} + M_{11} \\ M_2 & M_1 & M_{12} \\ M_{13} & M_{14} & M_{15} + M_{16} \end{pmatrix} \tag{4.26}$$

[7] For practical purposes it seems reasonable to truncate the solution at some finite order in the fields $\tilde{s}$.

with

$$\begin{aligned}
M_1 &= h_1 h_2 s_9^0 s_{12}^0 s_{29}^0, & M_2 &= h_1 h_2 s_3^0 s_{12}^0 s_{29}^0 \\
M_3 &= h_1 h_2 (s_9^0)^2 s_{12}^0 s_{29}^0 = s_9^0 M_1, & M_4 &= h_1 h_2 (s_3^0)^2 s_{12}^0 s_{29}^0 = s_3^0 M_2 \\
M_5 &= h_1 h_2 s_3^0 s_9^0 s_{12}^0 s_{29}^0 = s_3^0 M_1, & M_6 &= h_1 h_2 s_9^0 (s_{12}^0)^2 s_{21}^0 s_{29}^0 = s_{12}^0 s_{21}^0 M_1 \\
M_7 &= h_1 h_2 s_3^0 (s_{12}^0)^2 s_{21}^0 s_{29}^0 = s_{12}^0 s_{21}^0 M_2, & M_8 &= h_1 h_2 (s_9^0)^2 (s_{12}^0)^2 s_{21}^0 s_{29}^0 = s_{12}^0 s_{21}^0 M_3 \\
M_9 &= h_1 h_2 (s_3^0)^2 (s_{12}^0)^2 s_{21}^0 s_{29}^0 = s_{12}^0 s_{21}^0 M_2, & M_{10} &= (s_9^0)^2 s_{12}^0 s_{21}^0 s_{29}^0 \\
M_{11} &= (s_3^0)^2 s_{12}^0 s_{21}^0 s_{29}^0, & M_{12} &= s_3^0 s_4^0 s_{12}^0 s_{21}^0 s_{29}^0 \\
M_{13} &= h_1 h_2 s_9^0 (s_{12}^0)^2 s_{29}^0, & M_{14} &= h_1 h_2 s_3^0 (s_{12}^0)^2 s_{29}^0 \\
M_{15} &= (s_9^0)^2 (s_{12}^0)^2 s_{21}^0 s_{29}^0 = s_{12}^0 M_{10}, & M_{16} &= (s_3^0)^2 (s_{12}^0)^2 s_{21}^0 s_{29}^0 = s_{12}^0 M_{10}.
\end{aligned} \tag{4.27}$$

Note, that as expected, two further properties of the mini-landscape that were identified as crucial in [76] become apparent here. First, the top quark coupling is at order one. The reason for this behaviour is the connection of the coupling to the higher dimensional gauge coupling [91], guaranteeing a realistic top quark mass. Second, there is a D_4 symmetry for the first two generations which is a consequence of the localization of these fields in the extra dimensional space [69, 92]. Note, that this symmetry needs to be broken at a lower scale to explain the difference between the first and second generation [93].

While the impact of the D_4 symmetry and gauge-top unification remain unaltered by the modification of the R-symmetries, the detailed form of the Yukawa matrices is varied noticeably. We have switched on VEVs for 14 Standard Model singlets, as compared to 32 in the VEV configuration IA of [27]. At the same time, the mass matrices we obtain are more densely populated with allowed monomials in the singlets.

Decoupling of exotics

The decoupling of exotics by giving them high masses remains possible with the new R-symmetries. We will discuss one example in detail and skip the details for the other exotics. As can be seen from table 4.2, the charges of y_1 and y_2 under the R-symmetry in the third plane have changed due to the modification of the R-symmetries. However, as the two fields always appear in pairs and the R-charge selection rule in the third plane reads

	r^1_{old}	r^2_{old}	r^3_{old}	r^1_{new}	r^2_{new}	r^3_{new}
y_1	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$
y_2	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$

Table 4.2: Differences between the R-charges for y_1 and y_2 from model 1 of [27].

$$\sum_\alpha r_\alpha^3 = -1 \mod 2\,, \tag{4.28}$$

the mass matrix remains unchanged. Similarly, we find that all exotics get a high mass and therefore decouple from the low energy particle spectrum. Hence, the massless spectrum of our model is precisely that of the MSSM.

Proton decay

The situation concerning proton decay does not differ substantially from the previous results described in [27]. We find that $qqql$ operators as well as couplings with massive exotic triplets like $q_1 l_1 \bar{\delta}_4$ and $q_1 q_1 \delta_4$ are allowed. Furthermore, after integrating out the exotic triplets, the corresponding trilinear operators induce rapid proton decay, which displays a problem of these models.

Supersymmetry

As the VEVs of the fields $\tilde{s}$ take values at the string scale, these configurations should not break supersymmetry. Instead supersymmetry breaking proceeds via gaugino condensation in the hidden $\mathrm{SU}(4)$ gauge group. However, we have seen in (4.21) that the superpotential develops a VEV. Therefore we need to check for F-flatness, i.e. we need to search for solutions to $F = 0$. Once such solutions are found, $F = 0$ and $D = 0$ can be simultaneously fulfilled by making use of complexified gauge transformations [53, 94].

We use the technique of Gröbner bases, which is known in computational algebraic geometry and was discussed in the context of high energy physics and string theory for example in [95–97]. We are looking for solutions to the system of equations

$$F_i = \frac{\partial W}{\partial \tilde{s}_i} = 0, \qquad \forall i. \tag{4.29}$$

To find a solution we have to truncate the superpotential W at a given order. We start with W up to order 10 in Standard Model singlets which was given in (4.21). Note that, for simplicity, we set all coupling coefficients to unity. We use `Singular` [98] to compute the Gröbner basis of the ideal generated by the F-term equations. Then we compute its primary decomposition and search for F-flat solutions $F_i = 0$.

We find only one branch of solutions, which is in agreement with our assumption that all fields $\tilde{s}$ develop non-trivial VEVs. This branch can be solved for example by

$$\left\langle s^0_{28}\right\rangle = -\frac{1+\left\langle s^0_{21}\right\rangle^3\left\langle s^0_{31}\right\rangle^3}{\langle h_1\rangle\langle h_2\rangle\left\langle s^0_{21}\right\rangle\left\langle s^0_{24}\right\rangle\left\langle s^0_{31}\right\rangle}, \qquad \left\langle s^0_{29}\right\rangle = -\frac{\langle h_3\rangle\langle h_4\rangle\left\langle s^0_{21}\right\rangle\left\langle s^0_{31}\right\rangle}{\left\langle s^0_{3}\right\rangle\left\langle s^0_{4}\right\rangle+\left\langle s^0_{9}\right\rangle\left\langle s^0_{10}\right\rangle}, \tag{4.30}$$

which results in $F_i = 0$. That means we can solve all 14 F-term equations simultaneously by fixing only two VEVs. This is nearly the opposite behaviour to the one discussed in [82], where a remnant $\mathbb{Z}_4^R$ symmetry has been used to restrict the superpotential. There it seems to be more fertile to look for minima in which all singlets get fixed by the F-term equations. It remains as an interesting open question how the symmetries of the superpotential determine the solution structure of the F-term equations. At this stage we are satisfied by finding a consistent, non-trivial solution. Note however, that the systematics of generating a potential that dynamically fixes the VEVs of the fields remains to be understood.

Remarkably, the requirement of F-flatness implies that the VEV of the superpotential, $\langle W\rangle$, vanishes. As the μ-term is directly linked to the VEV of the superpotential, this implies a vanishing μ-term for the supersymmetric minima. Note, that since we know the superpotential not only to order 10 in singlets, but to all orders, it seems to be possible to address the question of F-flatness to all orders. Due to computational restrictions, we have not been able to find such a solution. Already at order 11 in the singlets, the Gröbner basis consists of extremely many polynomials and we have not been able to find a primary decomposition.

Our exploration of the impact of the modified definition of the R-symmetries on the models of the $\mathbb{Z}_6$-II mini-landscape shows, that their main appealing features remain unaltered. The "three lessons for successful model building" that were identified[8] in [76, 77] remain unaltered because the modification of the R-symmetries does not affect the R-charges of the Higgs doublet. In general, more couplings enter the mass matrices and Yukawa couplings. This makes it easier to give masses to the Standard Model particles and the exotics. On the other hand a μ term gets induced already at order 3 in the singlets. At the same time, however, F-flatness implies a vanishing of the superpotential VEV. As these models face gauge-Higgs unification this sets the μ term to zero up to order 10 in the singlets. Using the technique of Hilbert bases, we were able to compute the VEV of the superpotential to all orders in the singlets attaining a VEV. Furthermore it enabled us to determine the mass matrices for the exotics as well as the Yukawa couplings of Standard Model fields with much less computational effort as compared to calculating the allowed couplings order-by-order in the singlets. Therefore we find that Hilbert bases serve as a very powerful tool to analyse the phenomenology of such models.

[8] The fourth lesson of the mini-landscape deals with supersymmetry breaking, which we do not discuss here.

CHAPTER 5

Conclusions

In this thesis we discussed the geometric origin of discrete R-symmetries within orbifold compactifications of the heterotic string. We examined the results of exact CFT calculations of the string correlation functions corresponding to superpotential terms of the low energy effective theory for a subclass of heterotic orbifolds. In this way we were able to determine the symmetries of the worldsheet instanton solutions. We found that the origin for the discrete nature of the R-symmetries lies in the form of these solutions. Based on this result we presented a scheme allowing us to identify the symmetries of the orbifold space groups that lead to R-charge selection rules in the low energy theory. Then, by determining the transformation behaviour of the string states under such symmetries we were able to calculate the R-charges of the corresponding fields.

We found two kinds of automorphisms of the space group lattice that give rise to R-symmetries. Those that leave the fixed point structure of the theory invariant are immediate symmetries of the theory, under which the string states are eigenstates. By asking the correlators to transform trivially under these symmetries, the precise form of the R-charge selection rule can be deduced. Additionally, the compactification geometry can in some cases lead to a degeneracy in the spectrum. Namely two or more fixed points might allocate exactly the same string states. As these states are indistinguishable from the point of view of the low energy effective theory, isometries that exchange such fixed points lead to further R-symmetries. Linear combinations of the states from different fixed points then transform with a definite R-charge under these symmetries.

We applied our scheme to a recent classification of all space groups leading to theories with $\mathcal{N} = 1$ supersymmetry in 4D. In this way we were able to give a complete exploration of all R-symmetries arising in orbifolds that are based on space groups, for which the point group is Abelian.

As a consistency check of the results we obtained, we calculated the anomalies of the R-symmetries. As heterotic orbifold compactifications contain only a single axion, these anomalies must fulfil certain universality requirements if they are to be cancelled by a discrete Green-Schwarz mechanism. We modified the C++ orbifolder such that it is able to calculate the correct R-charges of the string states and the corresponding anomaly coefficients. Then we checked anomaly universality for all R-symmetries that we obtained in random sets of 10000 orbifold models each. We found that out of the 107 geometries for which the anomaly check is non-trivial, only in 6 cases the R-symmetry anomalies were non-universal. A first analysis showed that these orbifolds might be related to geometries in which a freely acting involution is divided out. If that is the case, our algorithms may need refinement in order to take the effects of a non-trivial fundamental group into account. Most notably, among these orbifolds is the $\mathbb{Z}_2 \times \mathbb{Z}_2\_5\_1$ orbifold which has been shown to be a fertile patch for model building. Furthermore, the R-symmetries that were assumed to arise in these models have been used to provide a string theoretical realisation of the phenomenologically attractive $\mathbb{Z}_4^R$ [82]. Therefore it is an important open question whether these R-symmetries can be made anomaly universal. It would further be interesting to try to extend the results to the non-Abelian case. However much less is known about the details of the CFT that describe these string theories.

As the R-symmetries can have important consequences on the phenomenological properties of string models we reconsidered a successful class of string models. Choosing a vacuum configuration, we analysed the impact of the new R-symmetries on a representative of this $\mathbb{Z}_6$-II mini-landscape using the technique of Hilbert bases. This method allowed us to examine phenomenological features such as Yukawa matrices for the standard model particles or mass matrices for the exotics to all orders in the standard model singlets that develop a vacuum expectation value. We were able to obtain the precise MSSM spectrum and to find F-flat directions.

Our analysis showed that the main attractive features of the models from the mini-landscape remain unaltered by the new R-symmetries. The lessons for successful model building that were identified in the context of these models are not affected by the R-symmetries. Using the technique of Hilbert bases enabled us to determine the mass matrices for the exotics as well as the Yukawa couplings of standard model fields with much less computational effort as compared to calculating the allowed couplings order-by-order in the singlets.

It would be interesting to study how, in such models, discrete remnants of the broken $\mathrm{U}(1)$ symmetries and non-R-symmetries mix with the R-symmetries we identified, to give a phenomenologically attractive R-symmetry for the low energy theory.

APPENDIX A

Catalogue of R-Symmetries for Abelian Orbifolds

The results of our exploration of R-symmetries in abelian orbifold compactifications of the heterotic string are presented in the following table. For each affine class we list all R-symmetry generators together with the quanta M and R that enter the R-charge conservation rule (3.36). Those generated marked with an asterisk originate from the group $C_{\mathrm{Aut}}(P)/\mathrm{Stab}(z_{\mathrm{f}})$ and therefore interchange fixed points allocating the same twisted matter, as explained in section 3.3.3. For each R-symmetry we further display the result of the check of universal chiral anomalies which has been performed by a scan over 10000 randomly generated gauge embeddings. Those affine classes which do not have an entry in this column do not possess a chiral spectrum, so that the check cannot be performed.

$\mathbb{Q}$ class (twist)	$\mathbb{Z}$ class	affine class	R symmetries ρ	M	$-R$	Anomaly Universality
$\mathbb{Z}_3$	1	1	$(0,\ \frac{1}{3},\ 0)$	3	-1	✓
$(\frac{1}{3},\ \frac{1}{3},\ -\frac{2}{3})$			$(0,\ 0,\ \frac{1}{3})$	3	-1	✓
			$(\frac{1}{3},\ 0,\ 0)$	3	-1	✓
$\mathbb{Z}_4$	1	1	$(\frac{1}{4},\ 0,\ 0)^*$	4	-1	✓
$(\frac{1}{4},\ \frac{1}{4},\ -\frac{1}{2})$			$(0,\ \frac{1}{4},\ 0)^*$	4	-1	✓
			$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
	2	1	$(0,\ \frac{1}{4},\ \frac{1}{2})$	4	-3	✓
			$(\frac{1}{4},\ 0,\ 0)$	4	-1	✓
	3	1	$(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{2})$	4	-4	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
$\mathbb{Z}_6$-I	1	1	$(\frac{1}{6},\ \frac{1}{6},\ 0)$	6	-2	✓
$(\frac{1}{6},\ \frac{1}{6},\ -\frac{1}{3})$			$(0,\ 0,\ \frac{1}{3})$	3	-1	✓
	2	1	$(\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3})$	3	-6	✓

$\mathbb{Q}$ class	$\mathbb{Z}$ class	affine class	R symmetries			Anomaly
(twist)			ρ	M	$-R$	Universality
			$(0,\ \frac{1}{2},\ 0)^*$	2	-1	✓
			$(\frac{1}{6},\ 0,\ 0)^*$	6	-1	✓
$\mathbb{Z}_6$-II	1	1	$(0,\ \frac{1}{3},\ 0)$	3	-1	✓
$(\frac{1}{6},\ \frac{1}{3},\ -\frac{1}{2})$			$(\frac{1}{6},\ 0,\ 0)$	6	-1	✓
			$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
	2	1	$(\frac{1}{6},\ \frac{1}{3},\ 0)$	6	-3	✓
			$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
	3	1	$(\frac{1}{6},\ 0,\ \frac{1}{2})$	6	-4	✓
			$(\frac{1}{3},\ 0,\ 0)$	3	-1	✓
			$(0,\ \frac{1}{3},\ 0)$	3	-1	✓
	4	1	$(\frac{1}{6},\ \frac{1}{3},\ \frac{1}{2})$	6	-6	✓
$\mathbb{Z}_7$ $(\frac{1}{7},\ \frac{2}{7},\ -\frac{3}{7})$	1	1				
$\mathbb{Z}_8$-I $(\frac{1}{8},\ \frac{1}{4},\ -\frac{3}{8})$	1	1	$(0,\ \frac{1}{4},\ 0)^*$	4	-1	✓
	2	1				
	3	1				
$\mathbb{Z}_8$-II	1	1	$(\frac{1}{8},\ \frac{3}{8},\ 0)$	8	-4	✓
$(\frac{1}{8},\ \frac{3}{8},\ -\frac{1}{2})$			$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
	2	1	$(\frac{1}{8},\ \frac{3}{8},\ \frac{1}{2})$	8	-8	✓
$\mathbb{Z}_{12}$-I	1	1	$(\frac{5}{12},\ 0,\ \frac{11}{12})$	12	-16	✓
$(\frac{1}{12},\ \frac{1}{3},\ -\frac{5}{12})$			$(0,\ \frac{1}{3},\ 0)$	3	-1	✓
	2	1				
$\mathbb{Z}_{12}$-II	1	1	$(\frac{1}{12},\ \frac{5}{12},\ 0)$	12	-6	✓
$(\frac{1}{12},\ \frac{5}{12},\ -\frac{1}{2})$			$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
$\mathbb{Z}_2 \times \mathbb{Z}_2$	1	1	$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
$(0,\ \frac{1}{2},\ -\frac{1}{2})$			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
$(\frac{1}{2},\ 0,\ -\frac{1}{2})$			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		2	$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
		3	$(0,\ \frac{1}{2},\ 0)$	2	-1	
			$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	

$\mathbb{Q}$ class	$\mathbb{Z}$ class	affine class	R symmetries			Anomaly
(twist)			ρ	M	$-R$	Universality
		4	$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
	2	1	$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
			$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		2	$(\frac{1}{2},\ 0,\ \frac{1}{2})$	2	-2	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
		3	$(\frac{1}{2},\ 0,\ \frac{1}{2})$	2	-2	
			$(0,\ \frac{1}{4},\ 0)$	4	-1	
		4	$(\frac{1}{2},\ 0,\ 0)$	2	-1	
			$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
		5	$(\frac{1}{2},\ \frac{1}{2},\ 0)$	2	-2	
			$(0,\ 0,\ \frac{1}{2})$	2	-1	
		6	$(\frac{1}{2},\ 0,\ 0)$	2	-1	
			$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
	3	1	$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
			$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
		2	$(0,\ \frac{1}{2},\ 0)$	2	-1	
			$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
		3	$(0,\ \frac{1}{2},\ 0)$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
		4	$(0,\ \frac{1}{2},\ 0)$	2	-1	
			$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
	4	1	$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
			$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		2	$(\frac{1}{2},\ 0,\ 0)$	2	-1	
			$(0,\ 0,\ \frac{1}{2})$	2	-1	

$\mathbb{Q}$ class (twist)	$\mathbb{Z}$ class	affine class	R symmetries ρ	M	$-R$	Anomaly Universality
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
	5	1	$(0,\ 0,\ \frac{1}{2})$	2	-1	✗
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✗
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✗
		2	$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
		3	$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
		4	$(\frac{1}{2},\ \frac{1}{2},\ 0)$	2	-2	
			$(0,\ 0,\ \frac{1}{2})$	2	-1	
		5	$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
	6	1	$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
			$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
		2	$(0,\ \frac{1}{2},\ 0)$	2	-1	
			$(0,\ 0,\ \frac{1}{2})$	2	-1	
		3	$(\frac{1}{2},\ 0,\ 0)$	2	-1	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
	7	1	$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		2	$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
	8	1	$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
			$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
	9	1	$(0,\ 0,\ \frac{1}{2})$	2	-1	✗
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✗
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✗

$\mathbb{Q}$ class (twist)	$\mathbb{Z}$ class	affine class	R symmetries			Anomaly Universality
			ρ	M	$-R$	
		2	$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
		3	$(0,\ 0,\ \frac{1}{2})^*$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)^*$	2	-1	
			$(0,\ \frac{1}{2},\ 0)$	2	-1	
	10	1	$(0,\ \frac{1}{2},\ 0)$	2	-1	✗
			$(0,\ 0,\ \frac{1}{2})$	2	-1	✗
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✗
		2	$(0,\ \frac{1}{2},\ 0)$	2	-1	
			$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
	11	1	$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
	12	1	$(0,\ \frac{1}{2},\ 0)$	2	-1	✗
			$(0,\ 0,\ \frac{1}{2})$	2	-1	✗
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✗
		2	$(0,\ \frac{1}{2},\ 0)$	2	-1	
			$(0,\ 0,\ \frac{1}{2})$	2	-1	
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	
$\mathbb{Z}_2 \times \mathbb{Z}_4$	1	1	$(0,\ 0,\ \frac{1}{4})^*$	4	-1	✓
$(0,\ \frac{1}{2},\ -\frac{1}{2})$			$(\frac{1}{4},\ 0,\ 0)^*$	4	-1	✓
$(\frac{1}{4},\ 0,\ -\frac{1}{4})$			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		2	$(0,\ 0,\ \frac{1}{4})$	4	-1	✓
			$(\frac{1}{4},\ 0,\ 0)$	4	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		3	$(\frac{1}{4},\ 0,\ \frac{1}{4})$	4	-2	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		4	$(\frac{1}{4},\ \frac{1}{2},\ 0)^*$	4	-3	✓
			$(0,\ 0,\ \frac{1}{4})^*$	4	-1	✓
		5	$(\frac{1}{4},\ \frac{1}{2},\ 0)$	4	-3	✓
			$(0,\ 0,\ \frac{1}{4})$	4	-1	✓

$\mathbb{Q}$ class (twist)	$\mathbb{Z}$ class	affine class	R symmetries ρ	M	$-R$	Anomaly Universality
		6	$(0,\ 0,\ \frac{1}{4})^*$	4	-1	✗
			$(\frac{1}{4},\ 0,\ 0)^*$	4	-1	✗
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
	2	1	$(\frac{1}{4},\ 0,\ \frac{1}{4})$	4	-2	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		2	$(\frac{1}{4},\ 0,\ \frac{1}{4})$	4	-2	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		3	$(\frac{1}{4},\ 0,\ \frac{1}{4})$	4	-2	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		4	$(0,\ \frac{1}{2},\ 0)$	2	-1	✗
		5	$(\frac{1}{4},\ 0,\ \frac{1}{4})$	4	-2	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		6	$(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4})$	4	-4	✓
	3	1	$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
			$(\frac{1}{4},\ 0,\ 0)$	4	-1	✓
			$(0,\ 0,\ \frac{1}{4})$	4	-1	✓
		2	$(\frac{1}{4},\ \frac{1}{2},\ 0)$	4	-3	✓
			$(0,\ 0,\ \frac{1}{4})$	4	-1	✓
		3	$(\frac{1}{4},\ \frac{1}{2},\ 0)^*$	4	-3	✓
			$(0,\ 0,\ \frac{1}{4})^*$	4	-1	✓
		4	$(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4})$	4	-4	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
		5	$(0,\ 0,\ \frac{1}{4})$	4	-1	✓
		6	$(0,\ 0,\ \frac{1}{4})$	4	-1	✓
	4	1	$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
			$(\frac{1}{4},\ 0,\ 0)^*$	4	-1	✓
			$(0,\ 0,\ \frac{1}{4})^*$	4	-1	✓
		2	$(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4})$	4	-4	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
		3	$(\frac{1}{4},\ 0,\ \frac{1}{4})^*$	4	-2	✓
			$(0,\ \frac{1}{2},\ 0)^*$	2	-1	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓

$\mathbb{Q}$ class (twist)	$\mathbb{Z}$ class	affine class	R symmetries ρ	M	$-R$	Anomaly Universality
		4	$(\frac{1}{4},\ 0,\ \frac{1}{4})^*$	4	-2	✓
			$(0,\ \frac{1}{2},\ 0)^*$	2	-1	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
		5	$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
	5	1	$(\frac{1}{4},\ 0,\ \frac{1}{4})$	4	-2	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		2	$(\frac{1}{4},\ 0,\ \frac{1}{4})$	4	-2	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
	6	1	$(\frac{1}{4},\ 0,\ \frac{1}{4})$	4	-2	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
		2	$(0,\ \frac{1}{2},\ \frac{1}{4})$	4	-3	✓
			$(\frac{1}{4},\ 0,\ 0)$	4	-1	✓
		3	$(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4})$	4	-4	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
		4	$(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4})$	4	-4	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
		5				✓
	7	1	$(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4})$	4	-4	✓
			$(\frac{1}{2},\ \frac{1}{2},\ 0)$	2	-2	✓
		2	$(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4})$	4	-4	✓
		3				✓
	8	1	$(\frac{1}{4},\ 0,\ \frac{1}{4})$	4	-2	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
		2	$(\frac{1}{2},\ \frac{1}{2},\ 0)$	2	-2	✓
			$(0,\ 0,\ \frac{1}{2})$	2	-1	✓
		3	$(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2})$	2	-3	✓
	9	1	$(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4})$	4	-4	✓
			$(\frac{1}{2},\ \frac{1}{2},\ 0)$	2	-2	✓
		2	$(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4})$	4	-4	✓
		3	$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
	10	1	$(\frac{1}{2},\ \frac{1}{2},\ 0)$	2	-2	✓

$\mathbb{Q}$ class (twist)	$\mathbb{Z}$ class	affine class	R symmetries ρ	M	$-R$	Anomaly Universality
			$(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$	4	-4	✓
		2	$(0, \frac{1}{2}, \frac{1}{2})$	2	-2	✓
$\mathbb{Z}_2 \times \mathbb{Z}_6$-I $(0, \frac{1}{2}, -\frac{1}{2})$ $(\frac{1}{6}, 0, -\frac{1}{6})$	1	1	$(\frac{2}{3}, 0, \frac{1}{3})$	3	-3	✓
			$(0, \frac{1}{2}, 0)$	2	-1	✓
			$(0, 0, \frac{1}{2})$	2	-1	✓
			$(\frac{1}{2}, 0, 0)$	2	-1	✓
		2	$(\frac{1}{6}, \frac{1}{2}, 0)$	6	-4	✓
			$(0, 0, \frac{1}{3})$	3	-1	✓
	2	1	$(\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$	6	-6	✓
			$(0, 0, \frac{1}{2})$	2	-1	✓
			$(\frac{1}{2}, 0, 0)$	2	-1	✓
		2	$(\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$	6	-6	✓
$\mathbb{Z}_2 \times \mathbb{Z}_6$-II $(0, \frac{1}{2}, -\frac{1}{2})$ $(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})$	1	1	$(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$	6	-6	✓
	2	1	$(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$	6	-6	✓
			$(\frac{1}{6}, 0, \frac{1}{6})$	6	-2	✓
	3	1	$(\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$	6	-6	✓
			$(\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$	6	-6	✓
	4	1	$(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$	6	-6	✓
			$(\frac{1}{2}, 0, \frac{1}{2})$	2	-2	✓
$\mathbb{Z}_3 \times \mathbb{Z}_3$ $(0, \frac{1}{3}, -\frac{1}{3})$ $(\frac{1}{3}, 0, -\frac{1}{3})$	1	1	$(0, \frac{1}{3}, 0)$	3	-1	✓
			$(0, 0, \frac{1}{3})$	3	-1	✓
			$(\frac{1}{3}, 0, 0)$	3	-1	✓
		2	$(0, \frac{1}{3}, 0)$	3	-1	✓
			$(0, 0, \frac{1}{3})$	3	-1	✓
			$(\frac{1}{3}, 0, 0)$	3	-1	✓
		3	$(\frac{1}{3}, 0, \frac{1}{3})$	3	-2	✓
			$(0, \frac{1}{3}, 0)$	3	-1	✓
		4	$(0, \frac{1}{3}, 0)$	3	-1	✓
			$(0, 0, \frac{1}{3})$	3	-1	✓
			$(\frac{1}{3}, 0, 0)$	3	-1	✓
	2	1	$(\frac{1}{3}, 0, 0)$	3	-1	✓
			$(0, 0, \frac{1}{3})$	3	-1	✓
			$(0, \frac{1}{3}, 0)$	3	-1	✓
		2	$(\frac{1}{3}, 0, \frac{1}{3})$	3	-2	✓

$\mathbb{Q}$ class (twist)	$\mathbb{Z}$ class	affine class	R symmetries			Anomaly Universality
			ρ	M	$-R$	
			$(0,\ \frac{1}{3},\ 0)$	3	-1	✓
		3	$(\frac{1}{3},\ 0,\ \frac{1}{3})$	3	-2	✓
			$(0,\ \frac{1}{3},\ 0)$	3	-1	✓
		4	$(\frac{1}{3},\ 0,\ \frac{1}{3})$	3	-2	✓
			$(0,\ \frac{1}{3},\ 0)$	3	-1	✓
	3	1	$(\frac{2}{3},\ 0,\ \frac{1}{3})$	3	-3	✓
			$(0,\ \frac{2}{3},\ \frac{1}{3})$	3	-3	✓
		2	$(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3})$	3	-3	✓
		3	$(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3})$	3	-3	✓
	4	1	$(0,\ \frac{1}{3},\ \frac{2}{3})$	3	-3	✓
			$(\frac{2}{3},\ \frac{1}{3},\ 0)$	3	-3	✓
		2	$(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3})$	3	-3	✓
		3	$(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3})$	3	-3	✓
	5	1	$(0,\ \frac{2}{3},\ \frac{1}{3})$	3	-3	✓
			$(\frac{2}{3},\ \frac{1}{3},\ 0)$	3	-3	✓
$\mathbb{Z}_3 \times \mathbb{Z}_6$ $(0,\ \frac{1}{3},\ -\frac{1}{3})$ $(\frac{1}{6},\ 0,\ -\frac{1}{6})$	1	1	$(\frac{1}{6},\ 0,\ \frac{1}{2})$	6	-4	✓
			$(0,\ \frac{1}{3},\ 0)$	3	-1	✓
			$(0,\ 0,\ \frac{1}{3})$	3	-1	✓
		2	$(\frac{1}{6},\ \frac{1}{3},\ \frac{1}{2})$	6	-6	✓
	2	1	$(\frac{1}{6},\ 0,\ \frac{1}{2})$	6	-4	✓
			$(0,\ 0,\ \frac{1}{3})$	3	-1	✓
			$(0,\ \frac{1}{3},\ 0)$	3	-1	✓
		2	$(\frac{1}{6},\ \frac{1}{3},\ \frac{1}{2})$	6	-6	✓
$\mathbb{Z}_4 \times \mathbb{Z}_4$ $(0,\ \frac{1}{4},\ -\frac{1}{4})$ $(\frac{1}{4},\ 0,\ -\frac{1}{4})$	1	1	$(0,\ 0,\ \frac{1}{4})$	4	-1	✓
			$(\frac{1}{4},\ 0,\ 0)$	4	-1	✓
			$(0,\ \frac{1}{4},\ 0)$	4	-1	✓
		2	$(\frac{1}{4},\ 0,\ \frac{1}{4})$	4	-2	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		3	$(\frac{1}{4},\ 0,\ \frac{1}{4})$	4	-2	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓
			$(0,\ \frac{1}{2},\ 0)$	2	-1	✓
		4	$(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4})$	4	-4	✓
			$(\frac{1}{2},\ 0,\ 0)$	2	-1	✓

$\mathbb{Q}$ class (twist)	$\mathbb{Z}$ class	affine class	R symmetries			Anomaly Universality
			ρ	M	$-R$	
	2	1	$\left(\frac{1}{4},\ 0,\ 0\right)^*$	4	-1	✓
			$\left(0,\ 0,\ \frac{1}{4}\right)^*$	4	-1	✓
			$\left(0,\ \frac{1}{4},\ 0\right)^*$	4	-1	✓
		2	$\left(\frac{1}{4},\ 0,\ \frac{1}{4}\right)$	4	-2	✓
			$\left(0,\ \frac{1}{4},\ 0\right)^*$	4	-1	✓
		3	$\left(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4}\right)$	4	-4	✓
			$\left(\frac{1}{2},\ 0,\ 0\right)$	2	-1	✓
		4	$\left(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4}\right)$	4	-4	✓
	3	1	$\left(0,\ \frac{1}{4},\ 0\right)^*$	4	-1	✓
			$\left(\frac{1}{4},\ 0,\ 0\right)^*$	4	-1	✓
			$\left(0,\ 0,\ \frac{1}{4}\right)^*$	4	-1	✓
		2				✓
	4	1	$\left(\frac{1}{4},\ \frac{1}{4},\ 0\right)$	4	-2	✓
			$\left(\frac{1}{4},\ 0,\ \frac{1}{4}\right)$	4	-2	✓
			$\left(\frac{1}{2},\ 0,\ 0\right)$	2	-1	✓
		2	$\left(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4}\right)$	4	-4	✓
			$\left(\frac{1}{2},\ 0,\ 0\right)$	2	-1	✓
		3	$\left(\frac{1}{2},\ 0,\ \frac{1}{2}\right)$	2	-2	✓
	5	1	$\left(\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4}\right)$	4	-4	✓
			$\left(\frac{1}{2},\ \frac{1}{4},\ \frac{1}{4}\right)$	4	-4	✓
		2				✓
$\mathbb{Z}_6 \times \mathbb{Z}_6$	1	1	$\left(\frac{1}{6},\ 0,\ \frac{1}{6}\right)$	6	-2	✓
$\left(0,\ \frac{1}{6},\ -\frac{1}{6}\right)$			$\left(\frac{1}{6},\ \frac{1}{6},\ 0\right)$	6	-2	✓
$\left(\frac{1}{6},\ 0,\ -\frac{1}{6}\right)$			$\left(\frac{1}{3},\ 0,\ 0\right)$	3	-1	✓

APPENDIX B

Space Group Embedding of R-Symmetry Generators in $\mathbb{Z}_6$-II

Here we list the space group elements h_g fulfilling $\varrho(g) = h_g g h_g^{-1}$ for all constructing elements g of the fixed points/tori and all R-symmetries ϱ of the $\mathbb{Z}_6$-II orbifold discussed in chapter 4.

g	$h_g^{\theta_1}$	$h_g^{\theta_2}$	$h_g^{\theta_3}$
$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(0,0,1,1,1,1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,-1,-1,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$
$\theta,(0,0,1,1,1,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,-1,-1,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$
$\theta,(0,0,1,1,0,1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,-1,-1,0,0)$	$\mathbb{1},(0,0,0,0,0,-1)$
$\theta,(0,0,1,1,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,-1,-1,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(0,0,0,0,1,1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$
$\theta,(0,0,0,0,1,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$
$\theta,(0,0,0,0,0,1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,-1)$
$\theta,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta,(0,0,1,0,1,1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$
$\theta,(0,0,1,0,1,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$
$\theta,(0,0,1,0,0,1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,0,-1)$
$\theta,(0,0,1,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,-1,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^2,(-1,1,0,2,0,0)$	$\theta,(0,0,2,2,0,0)$	$\mathbb{1},(0,0,-2,-2,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^2,(-1,1,0,0,0,0)$	$\theta,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^2,(-1,1,0,1,0,0)$	$\theta,(0,0,1,1,0,0)$	$\mathbb{1},(0,0,-1,-1,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^2,(0,0,0,2,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,-2,-2,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^2,(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^2,(0,0,0,1,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,-1,-1,0,0)$	$\mathbb{1},(0,0,0,0,0,0)$
$\theta^3,(1,0,0,0,1,1)$	$\theta,(0,0,0,0,1,1)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,-1)$
$\theta^3,(1,0,0,0,1,0)$	$\theta,(0,0,0,0,1,0)$	$\mathbb{1},(0,0,0,0,0,0)$	$\mathbb{1},(0,0,0,0,-1,0)$

g	$h_g^{\theta_1}$	$h_g^{\theta_2}$	$h_g^{\theta_3}$
$\theta^3, (1,0,0,0,0,1)$	$\theta, (0,0,0,0,0,1)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,-1)$
$\theta^3, (1,0,0,0,0,0)$	$\theta, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$
$\theta^3, (0,0,0,0,1,1)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,-1,-1)$
$\theta^3, (0,0,0,0,1,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,-1,0)$
$\theta^3, (0,0,0,0,0,1)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,-1)$
$\theta^3, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$
$\theta^4, (-1,1,1,1,0,0)$	$\theta, (0,0,1,1,0,0)$	$\mathbb{1}, (0,0,-1,-1,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$
$\theta^4, (-1,1,0,0,0,0)$	$\theta, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$
$\theta^4, (-1,1,1,0,0,0)$	$\theta, (0,0,1,0,0,0)$	$\mathbb{1}, (0,0,-1,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$
$\theta^4, (0,0,1,1,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,-1,-1,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$
$\theta^4, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$
$\theta^4, (0,0,1,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,-1,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$
$\theta^5, (0,0,0,2,1,1)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,-2,-2,0,0)$	$\mathbb{1}, (0,0,0,0,-1,-1)$
$\theta^5, (0,0,0,2,1,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,-2,-2,0,0)$	$\mathbb{1}, (0,0,0,0,-1,0)$
$\theta^5, (0,0,0,2,0,1)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,-2,-2,0,0)$	$\mathbb{1}, (0,0,0,0,0,-1)$
$\theta^5, (0,0,0,2,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,-2,-2,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$
$\theta^5, (0,0,0,0,1,1)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,-1,-1)$
$\theta^5, (0,0,0,0,1,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,-1,0)$
$\theta^5, (0,0,0,0,0,1)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,-1)$
$\theta^5, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$
$\theta^5, (0,0,0,1,1,1)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,-1,-1,0,0)$	$\mathbb{1}, (0,0,0,0,-1,-1)$
$\theta^5, (0,0,0,1,1,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,-1,-1,0,0)$	$\mathbb{1}, (0,0,0,0,-1,0)$
$\theta^5, (0,0,0,1,0,1)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,-1,-1,0,0)$	$\mathbb{1}, (0,0,0,0,0,-1)$
$\theta^5, (0,0,0,1,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$	$\mathbb{1}, (0,0,-1,-1,0,0)$	$\mathbb{1}, (0,0,0,0,0,0)$

APPENDIX C

Spectrum of the $\mathbb{Z}_6$-II Model

Here we present the spectrum of the orbifold model considered in 4.3. We use the notation of [27]. Our results differ from the ones obtained in [27] by the modified definition of the R-charges of the fields. Therefore we highlight those R-charges that differ from the old ones by red colour.

label	(k,λ)	R_1	R_2	R_3	representation	q_Y	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{anom}	q_{B-L}
$\bar{n}_3$	$0,\ (0,0,0,0,0,0)$	-1	0	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{2}$	0	0	0	0	$-\frac{1}{3}$	-1
$\bar{e}_3$	$0,\ (0,0,0,0,0,0)$	-1	0	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	$\frac{2}{3}$	-1
$\bar{u}_3$	$0,\ (0,0,0,0,0,0)$	-1	0	0	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{2}{3}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	$\frac{2}{3}$	$\frac{1}{3}$
$\bar{f}_1$	$0,\ (0,0,0,0,0,0)$	-1	0	0	$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$	0	0	0	0	0	$-\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{3}$	1
f_1	$0,\ (0,0,0,0,0,0)$	-1	0	0	$(\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1})$	0	0	0	0	0	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{2}{3}$	-1
ϕ_1	$0,\ (0,0,0,0,0,0)$	0	0	-1	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	0	0	-1	1	0	0	0	0	2	0
$\bar{\phi}_1$	$0,\ (0,0,0,0,0,0)$	0	0	-1	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	0	0	1	-1	0	0	0	0	-2	0
s^0_2	$0,\ (0,0,0,0,0,0)$	0	-1	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	0	0	0	-1	0	-1	0	$-\frac{5}{3}$	0
s^0_1	$0,\ (0,0,0,0,0,0)$	0	-1	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	0	0	0	-1	0	1	0	$\frac{7}{3}$	0
q_3	$0,\ (0,0,0,0,0,0)$	0	-1	0	$(\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1})$	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	$\frac{4}{3}$	$-\frac{1}{3}$
n_{12}	$2,\ (-1,1,0,2,0,0)$	$-\frac{2}{3}$	$\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{5}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$-\frac{1}{9}$	1
$\bar{f}_4$	$2,\ (-1,1,0,2,0,0)$	$-\frac{2}{3}$	$\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{8}{9}$	0
δ_6	$2,\ (-1,1,0,2,0,0)$	$\frac{7}{3}$	$-\frac{1}{3}$	0	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$-\frac{1}{9}$	$\frac{2}{3}$
$\bar{n}_9$	$2,\ (-1,1,0,2,0,0)$	$\frac{7}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{2}{3}$	$\frac{2}{3}$	0	$-\frac{2}{3}$	$-\frac{1}{9}$	-1
$\bar{\eta}_3$	$2,\ (-1,1,0,2,0,0)$	$\frac{7}{3}$	$-\frac{4}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$-\frac{2}{3}$	$-\frac{1}{9}$	-1
$\bar{d}_3$	$2,\ (-1,1,0,2,0,0)$	$-\frac{2}{3}$	$-\frac{4}{3}$	0	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{8}{9}$	$\frac{1}{3}$
s^0_{31}	$2,\ (-1,1,0,2,0,0)$	$\frac{7}{3}$	$\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{8}{9}$	0
δ_4	$2,\ (-1,1,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	-1	$\frac{2}{3}$	0	0	0	$-\frac{7}{9}$	$\frac{2}{3}$
h_8	$2,\ (-1,1,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$\frac{2}{3}$	0	0	0	$-\frac{1}{3}$	-1	0	0	$\frac{2}{9}$	0
$\bar{\delta}_4$	$2,\ (-1,1,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0	1	$\frac{2}{3}$	0	0	0	$-\frac{1}{9}$	$-\frac{2}{3}$
h_7	$2,\ (-1,1,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$\frac{2}{3}$	0	0	0	$-\frac{1}{3}$	1	0	0	$\frac{8}{9}$	0
s^0_{25}	$2,\ (-1,1,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{2}{3}$	0	0	0	$-\frac{1}{3}$	0	-1	0	$-\frac{13}{9}$	0
s^0_{24}	$2,\ (-1,1,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{2}{3}$	0	0	0	$-\frac{1}{3}$	0	1	0	$\frac{23}{9}$	0
s^0_{30}	$2,\ (-1,1,0,0,0,0)$	$\frac{4}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{2}{3}$	0	0	0	$\frac{2}{3}$	0	0	0	$\frac{2}{9}$	0
s^0_{26}	$2,\ (-1,1,0,0,0,0)$	$-\frac{2}{3}$	$\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{2}{3}$	0	0	0	$\frac{2}{3}$	0	0	0	$\frac{2}{9}$	0
$\bar{\delta}_6$	$2,\ (-1,1,0,1,0,0)$	$\frac{7}{3}$	$\frac{2}{3}$	0	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{9}$	$-\frac{2}{3}$

label	(k, λ)	R_1	R_2	R_3	representation	q_Y	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{anom}	q_{B-L}
$\bar{n}_{11}$	$2,\ (-1,1,0,1,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$-\frac{1}{3}$	$\frac{4}{3}$	0	$-\frac{1}{3}$	$\frac{5}{9}$	-1
$\bar{n}_{10}$	$2,\ (-1,1,0,1,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{5}{6}$	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{14}{9}$	-1
$\bar{\eta}_4$	$2,\ (-1,1,0,1,0,0)$	$-\frac{2}{3}$	$\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{9}$	-1
$\bar{f}_6$	$2,\ (-1,1,0,1,0,0)$	$\frac{7}{3}$	$-\frac{4}{3}$	0	$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{7}{9}$	0
s^0_{32}	$2,\ (-1,1,0,1,0,0)$	$\frac{7}{3}$	$-\frac{4}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{2}{9}$	0
$\bar{l}_1$	$2,\ (-1,1,0,1,0,0)$	$\frac{7}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{4}{9}$	-1
$\bar{n}_{16}$	$2,\ (-1,1,0,1,0,0)$	$\frac{4}{3}$	$-\frac{4}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{9}$	-1
$\bar{n}_{12}$	$2,\ (-1,1,0,1,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{9}$	-1
n_{13}	$2,\ (0,0,0,2,0,0)$	$-\frac{2}{3}$	$\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{5}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$-\frac{1}{9}$	1
$\bar{f}_5$	$2,\ (0,0,0,2,0,0)$	$-\frac{2}{3}$	$\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{8}{9}$	0
$\bar{d}_4$	$2,\ (0,0,0,2,0,0)$	$-\frac{2}{3}$	$-\frac{4}{3}$	0	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{8}{9}$	$\frac{1}{3}$
δ_5	$2,\ (0,0,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	-1	$\frac{2}{3}$	0	0	0	$-\frac{7}{9}$	$\frac{2}{3}$
h_{10}	$2,\ (0,0,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$\frac{2}{3}$	0	0	0	$-\frac{1}{3}$	-1	0	0	$\frac{2}{9}$	0
$\bar{\delta}_5$	$2,\ (0,0,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0	1	$\frac{2}{3}$	0	0	0	$-\frac{1}{9}$	$-\frac{2}{3}$
h_9	$2,\ (0,0,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$\frac{2}{3}$	0	0	0	$-\frac{1}{3}$	1	0	0	$\frac{8}{9}$	0
s^0_{28}	$2,\ (0,0,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{2}{3}$	0	0	0	$-\frac{1}{3}$	0	-1	0	$-\frac{13}{9}$	0
s^0_{27}	$2,\ (0,0,0,0,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{2}{3}$	0	0	0	$-\frac{1}{3}$	0	1	0	$\frac{23}{9}$	0
s^0_{29}	$2,\ (0,0,0,0,0,0)$	$-\frac{2}{3}$	$\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{2}{3}$	0	0	0	$\frac{2}{3}$	0	0	0	$\frac{2}{9}$	0
$\bar{n}_{14}$	$2,\ (0,0,0,1,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$-\frac{1}{3}$	$\frac{4}{3}$	0	$-\frac{1}{3}$	$\frac{5}{9}$	-1
$\bar{n}_{13}$	$2,\ (0,0,0,1,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{5}{6}$	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{14}{9}$	-1
$\bar{\eta}_5$	$2,\ (0,0,0,1,0,0)$	$-\frac{2}{3}$	$\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{9}$	-1
$\bar{n}_{15}$	$2,\ (0,0,0,1,0,0)$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{9}$	-1
s^+_{14}	$3,\ (1,0,0,0,1,1)$	$-\frac{5}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0	$-\frac{5}{6}$	0
s^-_{14}	$3,\ (1,0,0,0,1,1)$	$-\frac{5}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	0	0	0	1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{5}{6}$	0
s^+_{13}	$3,\ (1,0,0,0,1,1)$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	0	0	0	-1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{6}$	0
s^-_{13}	$3,\ (1,0,0,0,1,1)$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	0	0	0	1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{5}{6}$	0
$\bar{f}^+_2$	$3,\ (1,0,0,0,1,1)$	$\frac{3}{2}$	0	$\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$	$-\frac{1}{2}$	0	0	0	-1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
$\bar{f}^-_2$	$3,\ (1,0,0,0,1,1)$	$\frac{3}{2}$	0	$\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1})$	$\frac{1}{2}$	0	0	0	1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
s^+_{11}	$3,\ (1,0,0,0,1,0)$	$-\frac{5}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0	$-\frac{5}{6}$	0
s^-_{11}	$3,\ (1,0,0,0,1,0)$	$-\frac{5}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	0	0	0	1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{5}{6}$	0
s^+_9	$3,\ (1,0,0,0,1,0)$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	0	0	0	-1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{6}$	0
s^-_9	$3,\ (1,0,0,0,1,0)$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	0	0	0	1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{5}{6}$	0
$\bar{f}^+_1$	$3,\ (1,0,0,0,1,0)$	$\frac{3}{2}$	0	$\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$	$-\frac{1}{2}$	0	0	0	-1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
$\bar{f}^-_1$	$3,\ (1,0,0,0,1,0)$	$\frac{3}{2}$	0	$\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1})$	$\frac{1}{2}$	0	0	0	1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
h_6	$3,\ (1,0,0,0,0,1)$	$\frac{3}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	1	0	0	$-\frac{1}{2}$	0
h_5	$3,\ (1,0,0,0,0,1)$	$\frac{3}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	-1	0	0	$\frac{1}{2}$	0
χ_2	$3,\ (1,0,0,0,0,1)$	$\frac{3}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	1	$-\frac{1}{2}$	2
χ_1	$3,\ (1,0,0,0,0,1)$	$\frac{3}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	-1	$\frac{1}{2}$	-2
h_4	$3,\ (1,0,0,0,0,0)$	$\frac{3}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	1	0	0	$-\frac{1}{2}$	0
h_3	$3,\ (1,0,0,0,0,0)$	$\frac{3}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	-1	0	0	$\frac{1}{2}$	0
χ_4	$3,\ (1,0,0,0,0,0)$	$\frac{3}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	1	$-\frac{1}{2}$	2
χ_3	$3,\ (1,0,0,0,0,0)$	$\frac{3}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	-1	$\frac{1}{2}$	-2
s^+_{12}	$3,\ (0,0,0,0,1,1)$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	0	0	0	-1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{6}$	0
s^-_{12}	$3,\ (0,0,0,0,1,1)$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	0	0	0	1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{5}{6}$	0
s^+_{10}	$3,\ (0,0,0,0,1,0)$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	0	0	0	-1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{6}$	0
s^-_{10}	$3,\ (0,0,0,0,1,0)$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	0	0	0	1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{5}{6}$	0

label	(k,λ)	R_1	R_2	R_3	representation	q_Y	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{anom}	$q_{\mathrm{B-L}}$
s^0_{22}	4, $(-1,1,1,1,0,0)$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$-\frac{2}{9}$	0
f_5	4, $(-1,1,1,1,0,0)$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{7}{9}$	0
l_3	4, $(-1,1,1,1,0,0)$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{4}{9}$	1
n_{10}	4, $(-1,1,1,1,0,0)$	$\frac{8}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{1}{6}$	$-\frac{1}{2}$	$\frac{5}{6}$	$-\frac{5}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$-\frac{14}{9}$	1
n_9	4, $(-1,1,1,1,0,0)$	$\frac{8}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{1}{3}$	$-\frac{4}{3}$	0	$\frac{1}{3}$	$-\frac{5}{9}$	1
η_5	4, $(-1,1,1,1,0,0)$	$\frac{8}{3}$	$\frac{4}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{9}$	1
δ_2	4, $(-1,1,1,1,0,0)$	$-\frac{1}{3}$	$\frac{4}{3}$	0	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{2}{3}$
n_7	4, $(-1,1,1,1,0,0)$	$\frac{2}{3}$	$\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{1}{9}$	1
n_{11}	4, $(-1,1,1,1,0,0)$	$\frac{8}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{1}{9}$	1
h_1	4, $(-1,1,0,0,0,0)$	$\frac{8}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$-\frac{2}{3}$	0	0	0	$\frac{1}{3}$	1	0	0	$-\frac{2}{9}$	0
$\bar{\delta}_1$	4, $(-1,1,0,0,0,0)$	$\frac{8}{3}$	$-\frac{2}{3}$	0	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	1	$-\frac{2}{3}$	0	0	0	$\frac{7}{9}$	$-\frac{2}{3}$
h_2	4, $(-1,1,0,0,0,0)$	$\frac{8}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$-\frac{2}{3}$	0	0	0	$\frac{1}{3}$	-1	0	0	$-\frac{8}{9}$	0
δ_1	4, $(-1,1,0,0,0,0)$	$\frac{8}{3}$	$-\frac{2}{3}$	0	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	-1	$-\frac{2}{3}$	0	0	0	$\frac{1}{9}$	$\frac{2}{3}$
s^0_{18}	4, $(-1,1,0,0,0,0)$	$\frac{8}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{2}{3}$	0	0	0	$\frac{1}{3}$	0	-1	0	$-\frac{23}{9}$	0
s^0_{17}	4, $(-1,1,0,0,0,0)$	$\frac{8}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{2}{3}$	0	0	0	$\frac{1}{3}$	0	1	0	$\frac{13}{9}$	0
s^0_{15}	4, $(-1,1,0,0,0,0)$	$\frac{2}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{2}{3}$	0	0	0	$-\frac{2}{3}$	0	0	0	$-\frac{2}{9}$	0
s^0_{19}	4, $(-1,1,0,0,0,0)$	$\frac{8}{3}$	$\frac{4}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{2}{3}$	0	0	0	$-\frac{2}{3}$	0	0	0	$-\frac{2}{9}$	0
s^0_{20}	4, $(-1,1,1,0,0,0)$	$-\frac{1}{3}$	$\frac{4}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{5}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{8}{9}$	0
d_1	4, $(-1,1,1,0,0,0)$	$\frac{8}{3}$	$\frac{1}{3}$	0	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{8}{9}$	$-\frac{1}{3}$
η_3	4, $(-1,1,1,0,0,0)$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{9}$	1
f_4	4, $(-1,1,1,0,0,0)$	$\frac{8}{3}$	$\frac{4}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{8}{9}$	0
$\bar{n}_8$	4, $(-1,1,1,0,0,0)$	$\frac{8}{3}$	$\frac{4}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{5}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{1}{9}$	-1
n_5	4, $(-1,1,1,0,0,0)$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{1}{9}$	1
$\bar{\delta}_2$	4, $(-1,1,1,0,0,0)$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{1}{9}$	$-\frac{2}{3}$
s^0_{23}	4, $(0,0,1,1,0,0)$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$-\frac{2}{9}$	0
f_6	4, $(0,0,1,1,0,0)$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{7}{9}$	0
l_4	4, $(0,0,1,1,0,0)$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{4}{9}$	1
δ_3	4, $(0,0,1,1,0,0)$	$-\frac{1}{3}$	$\frac{4}{3}$	0	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{2}{3}$
n_8	4, $(0,0,1,1,0,0)$	$\frac{2}{3}$	$\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{1}{9}$	1
s^0_{16}	4, $(0,0,0,0,0,0)$	$\frac{2}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{2}{3}$	0	0	0	$-\frac{2}{3}$	0	0	0	$-\frac{2}{9}$	0
s^0_{21}	4, $(0,0,1,0,0,0)$	$-\frac{1}{3}$	$\frac{4}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{5}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{8}{9}$	0
η_4	4, $(0,0,1,0,0,0)$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{9}$	1
n_6	4, $(0,0,1,0,0,0)$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{1}{9}$	1
$\bar{\delta}_3$	4, $(0,0,1,0,0,0)$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{1}{9}$	$-\frac{2}{3}$
s^+_4	5, $(0,0,0,2,1,1)$	$-\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{37}{18}$	0
ν_2	5, $(0,0,0,2,1,1)$	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{6}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{18}$	$\frac{2}{3}$
s^-_3	5, $(0,0,0,2,1,1)$	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{17}{18}$	-1
m_6	5, $(0,0,0,2,1,1)$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{19}{18}$	1
s^+_3	5, $(0,0,0,2,1,1)$	$-\frac{1}{6}$	$-\frac{4}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{19}{18}$	0
s^-_4	5, $(0,0,0,2,1,1)$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{25}{18}$	1
s^+_2	5, $(0,0,0,2,1,0)$	$-\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{37}{18}$	0
ν_1	5, $(0,0,0,2,1,0)$	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{6}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{18}$	$\frac{2}{3}$
s^-_1	5, $(0,0,0,2,1,0)$	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{17}{18}$	-1
m_5	5, $(0,0,0,2,1,0)$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{5}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{19}{18}$	1
s^+_1	5, $(0,0,0,2,1,0)$	$-\frac{1}{6}$	$-\frac{4}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{19}{18}$	0
s^-_2	5, $(0,0,0,2,1,0)$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{25}{18}$	1

label	(k, λ)	R_1	R_2	R_3	representation	q_Y	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	$q_{\rm anom}$	$q_{\rm B-L}$
η_2	5, (0, 0, 0, 2, 0, 1)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$	0	$\frac{1}{6}$	0	$-\frac{2}{3}$	$-\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{19}{18}$	1
$\bar{n}_5$	5, (0, 0, 0, 2, 0, 1)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{2}{3}$	$-\frac{5}{6}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{19}{18}$	-1
n_2	5, (0, 0, 0, 2, 0, 1)	$-\frac{1}{6}$	$-\frac{4}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{2}{3}$	$-\frac{5}{6}$	$\frac{2}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{19}{18}$	1
η_1	5, (0, 0, 0, 2, 0, 0)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$	0	$\frac{1}{6}$	0	$-\frac{2}{3}$	$-\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{19}{18}$	1
$\bar{n}_4$	5, (0, 0, 0, 2, 0, 0)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{2}{3}$	$-\frac{5}{6}$	$-\frac{1}{3}$	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{19}{18}$	-1
n_1	5, (0, 0, 0, 2, 0, 0)	$-\frac{1}{6}$	$-\frac{4}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{2}{3}$	$-\frac{5}{6}$	$\frac{2}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{19}{18}$	1
y_2	5, (0, 0, 0, 0, 1, 1)	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})$	0	$-\frac{1}{3}$	0	0	0	$\frac{1}{6}$	0	$\frac{1}{2}$	0	$\frac{13}{18}$	0
m_3	5, (0, 0, 0, 0, 1, 1)	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	0	0	0	$\frac{1}{6}$	1	$-\frac{1}{2}$	0	$-\frac{17}{18}$	0
m_4	5, (0, 0, 0, 0, 1, 1)	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	0	0	0	$\frac{1}{6}$	-1	$-\frac{1}{2}$	0	$-\frac{29}{18}$	0
y_1	5, (0, 0, 0, 0, 1, 0)	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})$	0	$-\frac{1}{3}$	0	0	0	$\frac{1}{6}$	0	$\frac{1}{2}$	0	$\frac{13}{18}$	0
m_1	5, (0, 0, 0, 0, 1, 0)	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	0	0	0	$\frac{1}{6}$	1	$-\frac{1}{2}$	0	$-\frac{17}{18}$	0
m_2	5, (0, 0, 0, 0, 1, 0)	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{2}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	0	0	0	$\frac{1}{6}$	-1	$-\frac{1}{2}$	0	$-\frac{29}{18}$	0
$\bar{d}_1$	5, (0, 0, 0, 0, 0, 1)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$-\frac{1}{3}$	$\frac{1}{6}$	0	0	$-\frac{3}{2}$	$-\frac{1}{3}$	0	0	0	$-\frac{5}{18}$	$\frac{1}{3}$
$\bar{e}_1$	5, (0, 0, 0, 0, 0, 1)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	-1	$\frac{1}{6}$	0	0	$\frac{1}{2}$	$-\frac{1}{3}$	0	0	0	$\frac{7}{18}$	-1
$\bar{u}_1$	5, (0, 0, 0, 0, 0, 1)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$\frac{2}{3}$	$\frac{1}{6}$	0	0	$\frac{1}{2}$	$-\frac{1}{3}$	0	0	0	$\frac{7}{18}$	$\frac{1}{3}$
l_1	5, (0, 0, 0, 0, 0, 1)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$\frac{1}{2}$	$\frac{1}{6}$	0	0	$-\frac{3}{2}$	$-\frac{1}{3}$	0	0	0	$-\frac{5}{18}$	1
q_1	5, (0, 0, 0, 0, 0, 1)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$-\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{2}$	$-\frac{1}{3}$	0	0	0	$\frac{7}{18}$	$-\frac{1}{3}$
$\bar{n}_1$	5, (0, 0, 0, 0, 0, 1)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$\frac{1}{6}$	0	0	$\frac{5}{2}$	$-\frac{1}{3}$	0	0	0	$\frac{19}{18}$	-1
s^0_{12}	5, (0, 0, 0, 0, 0, 1)	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$\frac{2}{3}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$-\frac{5}{18}$	0
s^0_{11}	5, (0, 0, 0, 0, 0, 1)	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$\frac{2}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$\frac{25}{18}$	0
s^0_{14}	5, (0, 0, 0, 0, 0, 1)	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$\frac{13}{18}$	0
s^0_{13}	5, (0, 0, 0, 0, 0, 1)	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$-\frac{17}{18}$	0
s^0_{10}	5, (0, 0, 0, 0, 0, 1)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$\frac{13}{18}$	0
s^0_9	5, (0, 0, 0, 0, 0, 1)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$-\frac{17}{18}$	0
$\bar{d}_2$	5, (0, 0, 0, 0, 0, 0)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$-\frac{1}{3}$	$\frac{1}{6}$	0	0	$-\frac{3}{2}$	$-\frac{1}{3}$	0	0	0	$-\frac{5}{18}$	$\frac{1}{3}$
$\bar{e}_2$	5, (0, 0, 0, 0, 0, 0)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	-1	$\frac{1}{6}$	0	0	$\frac{1}{2}$	$-\frac{1}{3}$	0	0	0	$\frac{7}{18}$	-1
$\bar{u}_2$	5, (0, 0, 0, 0, 0, 0)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$\frac{2}{3}$	$\frac{1}{6}$	0	0	$\frac{1}{2}$	$-\frac{1}{3}$	0	0	0	$\frac{7}{18}$	$\frac{1}{3}$
l_2	5, (0, 0, 0, 0, 0, 0)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$\frac{1}{2}$	$\frac{1}{6}$	0	0	$-\frac{3}{2}$	$-\frac{1}{3}$	0	0	0	$-\frac{5}{18}$	1
q_2	5, (0, 0, 0, 0, 0, 0)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$-\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{2}$	$-\frac{1}{3}$	0	0	0	$\frac{7}{18}$	$-\frac{1}{3}$
$\bar{n}_2$	5, (0, 0, 0, 0, 0, 0)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$\frac{1}{6}$	0	0	$\frac{5}{2}$	$-\frac{1}{3}$	0	0	0	$\frac{19}{18}$	-1
s^0_6	5, (0, 0, 0, 0, 0, 0)	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$\frac{2}{3}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$-\frac{5}{18}$	0
s^0_5	5, (0, 0, 0, 0, 0, 0)	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$\frac{2}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$\frac{25}{18}$	0
s^0_8	5, (0, 0, 0, 0, 0, 0)	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$\frac{13}{18}$	0
s^0_7	5, (0, 0, 0, 0, 0, 0)	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$-\frac{17}{18}$	0
s^0_4	5, (0, 0, 0, 0, 0, 0)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$\frac{13}{18}$	0
s^0_3	5, (0, 0, 0, 0, 0, 0)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	0	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{3}$	0	0	0	$-\frac{17}{18}$	0
$\bar{\nu}_2$	5, (0, 0, 0, 1, 1, 1)	$-\frac{1}{6}$	$-\frac{4}{3}$	$-\frac{1}{2}$	$(\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$\frac{1}{6}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{25}{18}$	$-\frac{2}{3}$
s^-_8	5, (0, 0, 0, 1, 1, 1)	$-\frac{1}{6}$	$-\frac{4}{3}$	$\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{11}{18}$	0
s^+_7	5, (0, 0, 0, 1, 1, 1)	$-\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{2}{3}$	$-\frac{17}{18}$	1
m_8	5, (0, 0, 0, 1, 1, 1)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{19}{18}$	-1
s^-_7	5, (0, 0, 0, 1, 1, 1)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$\frac{1}{2}$	$\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{31}{18}$	0
s^+_8	5, (0, 0, 0, 1, 1, 1)	$\frac{5}{6}$	$-\frac{4}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{13}{18}$	-1
$\bar{\nu}_1$	5, (0, 0, 0, 1, 1, 0)	$-\frac{1}{6}$	$-\frac{4}{3}$	$-\frac{1}{2}$	$(\bar{\mathbf{3}}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$\frac{1}{6}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{25}{18}$	$-\frac{2}{3}$
s^-_6	5, (0, 0, 0, 1, 1, 0)	$-\frac{1}{6}$	$-\frac{4}{3}$	$\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{11}{18}$	0
s^+_5	5, (0, 0, 0, 1, 1, 0)	$-\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{2}{3}$	$-\frac{17}{18}$	1
m_7	5, (0, 0, 0, 1, 1, 0)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{19}{18}$	-1

label	(k,λ)	R_1	R_2	R_3	representation	q_Y	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_anom	$q_\mathrm{B-L}$
s_5^-	5, (0, 0, 0, 1, 1, 0)	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$\frac{1}{2}$	$\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{31}{18}$	0
s_6^+	5, (0, 0, 0, 1, 1, 0)	$\frac{5}{6}$	$-\frac{4}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{13}{18}$	-1
f_3	5, (0, 0, 0, 1, 0, 1)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{1}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{18}$	0
$\bar{f}_3$	5, (0, 0, 0, 1, 0, 1)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{1}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{37}{18}$	0
$\bar{\eta}_2$	5, (0, 0, 0, 1, 0, 1)	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$\frac{1}{6}$	0	$-\frac{1}{3}$	$\frac{5}{6}$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{19}{18}$	-1
n_4	5, (0, 0, 0, 1, 0, 1)	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{1}{3}$	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{19}{18}$	1
$\bar{n}_7$	5, (0, 0, 0, 1, 0, 1)	$\frac{5}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{1}{3}$	$\frac{5}{6}$	$\frac{2}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{7}{18}$	-1
f_2	5, (0, 0, 0, 1, 0, 0)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{1}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{18}$	0
$\bar{f}_2$	5, (0, 0, 0, 1, 0, 0)	$-\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\bar{\mathbf{4}},\mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{1}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{37}{18}$	0
$\bar{\eta}_1$	5, (0, 0, 0, 1, 0, 0)	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})$	0	$\frac{1}{6}$	0	$-\frac{1}{3}$	$\frac{5}{6}$	$-\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{19}{18}$	-1
n_3	5, (0, 0, 0, 1, 0, 0)	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{1}{3}$	$\frac{5}{6}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{19}{18}$	1
$\bar{n}_6$	5, (0, 0, 0, 1, 0, 0)	$\frac{5}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0	$\frac{1}{6}$	0	$-\frac{1}{3}$	$\frac{5}{6}$	$\frac{2}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{7}{18}$	-1

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