Emergent Geometry in a Gauge-Invariant Matrix Quantum Mechanics

by

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Abstract

Matrix quantum mechanics provides an example of spacetime emergence where the underlying system is non-geometric. Understanding the mechanism for this emergence, and the way in which entanglement between the quantum mechanical degrees of freedom relates to the emergent field theory, remains a key challenge in quantum gravity. Previous work has examined this relation in the context of a single matrix theory, and its emergent theory upon the noncommutative sphere. By using direct calculations of entanglement entropy of the matrix degrees of freedom, construction of the emergent fuzzy geometry is examined. This thesis attempts to extend this work to the case of a three matrix theory. The theory contains gauge-invariance, and a possible prescription is explored for restricting the full theory to the gauge-invariant configuration space. The implications for locality in the emergent theory, and thus potential for entanglement entropy definition, are evaluated and discussed.

Lay Summary

This thesis investigates physical questions related to the geometry of fuzzy spaces. In a fuzzy space, it is not possible to define specific points in the normal sense. Instead, the notion of a point is replaced with a small region of fuzziness, where we cannot differentiate locations specifically at scales smaller than that of the fuzzy regions. Fuzzy spaces are of particular interest to the study of quantum gravity, a theory which seeks to unify quantum and gravitational physics. Certain theories of quantum gravity, describing physics in space-time, appear to be somehow encoded within base mathematical descriptions which themselves do not describe space, nor time. The physics of these theories is said to be 'emergent' from these alternate mathematical descriptions. Here we have explored how specific aspects of these base descriptions correspond to the structure of the space that emerges from them. In general, the emergence itself is a poorly understood subject in quantum gravity research, and this work aims to address this gap in our knowledge.

Preface

This dissertation is original, unpublished, independent work by the author, J. Berean-Dutcher, with supervision by J. Karczmarek.

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Acknowledgments

I wish to ackowledge and pay respect to the indigenous people, past and present, as the traditional owners of the land on which I conducted this research. Namely, this is the unceded territory of the Coast Salish Peoples, including the territories of the xwməθkwəýəm (Musqueam), Skwxwú7mesh (Squamish), Stó:lō and Səĺílətaʔ/Selilwitulh (Tsleil-Waututh) Nations.

I would like to thank Joanna Karczmarek for providing me with this project and for her supervision throughout a difficult time in all our lives.

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Chapter 1

Introduction

The effort to discover a quantum mechanical description of gravity has led us to the intriguing notion that spacetime itself may be an emergent phenomenon. Indeed, among the most important developments in our present understanding of quantum gravity was the discovery of how theories of gravitating spacetime can emerge from the entanglement of underlying quantum mechanical degrees of freedom [3, 4]. In fact, there are examples of this emergence wherein the underlying quantum theory describes no space at all, and is nothing more than a quantum mechanics arranged within large super-symmetric matrices. If emergence is to be foundational to an eventually consistent quantum gravity theory, then a quantitative understanding of its operation is a theoretical necessity. Attempts at formulating emergent gravity have called for the study of noncommutative geometry, that is, geometry wherein coordinates are nonabelian and described, for example, by noncommuting matrices. By studying nonabelian objects, we hope to further understand the potential utility of noncommutative geometry to viable theories of emergent quantum gravity.

Recently, mathematical tools in the field of quantum information theory have been of great utility in probing the structure of gravitational theories. Perhaps the most well-known example of this is the gauge/gravity duality, where the gravitational theory of a spacetime is somehow encoded in a different quantum mechanical theory that lives on the spacetime's boundary. Calculations of the entropy of quantum entanglement, and other quantities that are commonplace in quantum information theory, have played a key role in uncovering how exactly this encoding works [5]. However, in much of the existing study, the physical theory on the boundary is a local one, where physics at any given point is directly influenced only by its immediate surroundings. In contrast, it is expected that the theory connected to our own spacetime is highly non-local [6]. Furthermore, noncommutative, non-local theories have been argued to be related to black hole horizons and information scrambling [7]. Studying the entanglement entropy in nonlocal theories can allow us to understand which properties of the entropy are dependent on locality and how.

This thesis is concerned with an emergent quantum field theory on the noncommutative sphere. The noncommutative, or 'fuzzy' sphere, is the space of a two-dimensional spherical surface but with coordinates that are noncommutative in the sense described earlier [8]. This particular field theory is emergent from an underlying matrix quantum mechanics theory [9]. Matrix quantum mechanics theories originally arose from considerations in string theory, purporting to present a potential definition of M-theory, the latter being a possible framework for a unified theory of all fundamental physical forces [10].

Recently, investigations into the nature of entanglement entropy in noncommutative theories have produced qualitative disagreements. In studies of the matrix quantum mechanics of a single matrix, it has been demonstrated that for small sub-regions of the emergent geometric space, entanglement entropy obeys an extensive growth law, proportional to a region's volume [11, 12]. This differs significantly from the situation in ordinary, commutative theories, where entanglement entropy grows with the size of a region's boundary. The difference can be understood qualitatively as arising from the blurring of coordinates by noncommutativity, such that more distant points, more distant than those directly across the boundary from each other, may become entangled. These 'volume-law' results are in agreement with observations obtained from parallel studies of these same theories using holographic techniques [13–16]. Conversely, an alternative computational approach involving variational wavefunctions has found that the non-local character of noncommutative field theory does not preclude partitions into sub-regions possessing boundary-law growth of entanglement entropy [17]. This most recent work involved a theory of three bosonic matrices, possessing invariance under $U(N)$ transformations of the matrices, which in turn corresponds to invariance under gauge, or spatially dependent, transformations in the emergent field theory. In addition, this recent work utilized a partitioning of the emergent spacetime that differed in construction to the methods of the single matrix work referenced above. It is not known what particular aspects of these differences, of model and methodology, is resulting in the differing entanglement entropy growth law results. Teasing out the cause of these observations is a primary motivation for the present work.

The aforementioned work on the noncommutative sphere relied on methods for discriminating between degrees of freedom inside and outside of spherical cap sub-regions [11, 12]. Studying entanglement entropy in a model with gauge invariance will require comparable methods, but in a context where the correct geometric assignation of degrees of freedom will be less obvious. In this thesis, an approach is made to the identification of a projection onto the space of gauge-invariant states, where a geometric interpretation of different collections of degrees of freedom can be retained. Analogous procedures have been described in gauge field theory more generally, allowing computation of various classes of entanglement entropies [18–21].

Chapter 2

Strings, branes, and matrices

In this chapter the study of quantum mechanical theories of $N \times N$ matrices will be motivated from a starting point in string theory. Beginning with a single string propagating in flat spacetime, we will make a series of generalizations so as to eventually consider theories that govern the dynamical hypersurfaces to which the string's endpoints may be adhered. Matrix theories will arise when we aim to write down physical descriptions of these hypersurfaces, with particular dimensionality.

2.1 Open strings and D-branes

String theory is the study of the physics of 1d extended objects called 'strings'. Strings can either be 'closed' or 'open'. For closed strings, the end-points of the string are connected such that the object we're dealing with is a 1d loop, propagating in spacetime. For open strings the end-points are not connected up, and this has specific consequences for their propagation in spacetime. The physics of a single open string propagating in D dimensional flat spacetime is captured by the Polyakov action,

$$S = -\frac{T}{2} \int d^2\sigma \ \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{2.1}$$

Here T is the string tension, σ^a for $a = 0, 1$ are the coordinates of the string worldsheet, and X^μ for $\mu = 0, ..., D$ are the coordinates for the flat spacetime in which the string is propagating. The worldsheet of the string is the 2d surface swept out by the string as it propagates, pictured in Figure 2.1. In (2.1) the background spacetime is flat, as specified by the metric $\eta_{\mu\nu}$. The string worldsheet is a submanifold of the background spacetime, and we have used the gauge symmetry of the Polyakov action (2.1) to set the metric upon this submanifold to likewise be flat, as specified by η^{ab}. From the perspective of the worldsheet, the $X^\mu(\sigma)$ are a set of D scalar fields living on the worldsheet's surface. It's worth emphasizing that the Polyakov action describes both: a physical theory of fields on the worldsheet surface, and simultaneously, the physics of a 1d object - the string itself - propagating through the D dimensional spacetime. Consider the variation of (2.1),

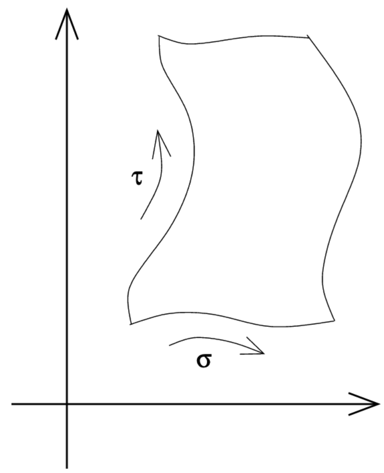


Figure 2.1: As an open string propagates through spacetime, it sweeps out a worldsheet parameterized by a timelike coordinate, $\sigma^0 \equiv \tau$, and a spacelike coordinate, $\sigma^1 \equiv \sigma$. Figure taken from [1].

$$\delta S = -\frac{T}{2}\int d^2\sigma \ \left(\partial_a\left(\delta X^\mu\right)\partial^a X_\mu + \partial_a X^\mu \partial^a\left(\delta X_\mu\right)\right) \tag{2.2}$$

$$= T\int d^2\sigma \ \left(\left(\partial_a\partial^a X_\mu\right)\delta X^\mu - \partial_a\left(\partial^a X_\mu \delta X^\mu\right)\right). \tag{2.3}$$

The variational principle applied to the first term yields the equations of motion,

$$\partial_a\partial^a X_\mu = 0, \tag{2.4}$$

i.e. that the string coordinates satisfy $D-1$ dimensional wave equations. In typical variational problems the second term vanishes by the argument that the variation falls off sufficiently fast at infinity in the coordinates being integrated over. However, considering the integration of this term explicitly, the requirement that the variation of the action vanishes yields,

$$0 = \int_{\tau=\tau_i}^{\tau=\tau_f} d\tau \, \Big[\partial_\sigma X_\mu \delta X^\mu\Big]_{\sigma=0}^{\sigma=\pi} - \int_{\sigma=0}^{\sigma=\pi} d\sigma \, \Big[\partial_\tau X_\mu \delta X^\mu\Big]_{\tau=\tau_i}^{\tau=\tau_f}. \tag{2.5}$$

We have chosen the conventional $\sigma \in [0, \pi]$ parameterization of the worldsheet's spacelike coordinate. The second term in Equation 2.5 vanishes by the argument given above as we consider bounds of infinite τ. Had this been a closed string, the first term would vanish by virtue of periodic identification of the string endpoints at 0 and π. For an open string, this is not the case, and thus we must impose additional constraints in order to have vanishing variation. We are presented with two distinct choices of boundary condition at $\sigma = 0, \pi$,

$$\partial_\sigma X_\mu = 0 \quad \text{(Neumann)}, \tag{2.6}$$

$$\delta X^\mu = 0 \quad \text{(Dirichlet)}. \tag{2.7}$$

Notice that we may utilize a combination of the Neumann and Dirichlet conditions in satisfying (2.5), electing a different choice for each of the background spacetime dimensions μ. This restricts the string to have completely fixed endpoints (Dirichlet) in some dimensions, while remaining free in others (Neumann). Suppose we choose we such an admixture of boundary conditions in the following way,

$$\partial_\sigma X^a = 0, \quad a = 0, 1, ..., p \tag{2.8}$$

$$\delta X^I = 0, \quad I = p+1, ..., D-1. \tag{2.9}$$

This restricts the strings motion so as to break the pre-existing Lorentz invariance of its description within the background spacetime. Specifically we have,

$$SO(1, D-1) \to SO(1, p) \times SO(D-p-1). \tag{2.10}$$

The open string's endpoints are restricted to a Lorentz invariant $p + 1$ dimensional hypersurface. This hypersurface is called a 'D-brane', where the D stands for Dirichlet. So as to specify it's spatial dimension, it may be referred to as a 'Dp-brane'. The X^a directions coordinatize the D-brane, and the remaining X^I directions are transverse to it. To this point, the D-brane hypersurface is merely a fixed submanifold of the spacetime, existing physically only in so far as it is the anchoring space for the open string endpoints. Now though we will consider an alternative theory in which we consider the D-brane to be a dynamical object in its own right. It is at this juncture that string theory becomes the study of a much wider collection of objects, with myriad dimensionality as defined through conditions (2.9).

2.2 D-brane actions

Our starting point for a D-brane action is the Dirac action for a generic membrane,

$$S = -T_p \int d^{p+1}\xi \ \left[-\det\left(\gamma_{ab}\right)\right]^{1/2}, \tag{2.11}$$

where ξ^a are the coordinates of the D-brane hypersurface or worldvolume, and T_p is the D-brane tension. Here the term 'membrane' is used to refer to any objected extended in spacetime. In this way, a string is a 1-brane, and a point particle is a 0-brane. Note that generic membranes are an independent object from D-branes, as the latter specifically refers to those objects upon which open strings end. The metric γ_{ab} is the pull-back of the background spacetime metric - again taken to be flat, $\eta_{\mu\nu}$ - to the D-brane worldvolume,

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu}. \tag{2.12}$$

Consider a choice of parameterization of the background spacetime coordinates, X^μ, so as to coincide with the D-brane coordinates, ξ^a, upon the worldvolume,

$$\begin{aligned} X^a &= \xi^a, \quad a = 0, 1, ..., p \\ X^I &= 2\pi\alpha'\phi^I, \quad I = p+1, ..., D-1. \end{aligned} \tag{2.13}$$

Here α' is a constant related to the string tension, and appears in this rescaling on dimensional grounds. In these coordinates the action (2.11) becomes,

$$S = -T_p \int d^{p+1}\xi \ \left[-\det\left(\eta_{ab} + (2\pi\alpha')^2\partial_a\phi^I\partial_b\phi^J\delta_{IJ}\right)\right]^{1/2}. \tag{2.14}$$

In the quantization of the open string, the details of which are beyond the scope of this discussion, two types of quantum field are identified whose quanta are the states of the string. The physics of these fields are contingent upon the boundary conditions (2.9) that define the D-brane, such that they can be viewed as field theories on the brane,

$$A_a(\xi), \quad a = 0, 1, ..., p \tag{2.15}$$

$$\phi^I(\xi), \quad I = p+1, ..., D-1. \tag{2.16}$$

The first of these fields, A_a carries the a index of the D-brane hypersurface coordinates, and corresponds to a set of massless string states that transform as a vector under the $SO(1,p)$ Lorentz symmetry of the D-brane. Hence, A_a is a photon, $U(1)$ gauge field living on the D-brane. The second, ϕ^I, carries the index I of the coordinates in the space transverse to the D-brane. These are states that transform as a scalar under the $SO(1,p)$, but as a vector under the rotational group $SO(D-p-1)$ of the transverse space. Notice that these fields are, for the D-brane, analogous to the X^μ scalars of the string worldsheet. For the string, the Polyakov action (2.1) described scalar fields $X^\mu(\sigma)$ living on the string worldsheet, which simultaneously described coordinates of the string's motion in the background spacetime. Here, the ϕ^I fields are scalars on the D-brane worldvolume and we simultaneously understand them to parameterize the motion of the D-brane hypersurface in the $D-p-1$ dimensional space transverse to it. It is in this sense that the D-brane is formulated as a dynamical object, with its motion in the transverse space described by fields whose quanta are excited open string states. In introducing the coordinates (2.13) we deliberately make contact with these scalar fields, emphasizing that this yields a dynamical description of the membrane in the action (2.14).

So far we have described aspects of the physics of a single open string and the D-brane upon which its endpoints are confined to move. Now we consider cases of multiple D-branes in spacetime, beginning with two. The situation is depicted in Figure 2.2. For a background spacetime dimension of $D = 4$, the possible open strings ending on a pair of parallel D2-branes are pictured. The coordinates of the D-branes are x^a with $a = 0, 1, 2$, and x^I with $I = 3$ is the sole transverse coordinate. Distinct cases, which are referred to as sectors of the given theory, are shown. Strings could have both their endpoints lying on a single of the branes, or they could have one endpoint on the first brane and the second endpoint on the opposing brane. Notice that this latter case, of strings stretched between the branes, is further distinguished by arrows indicating the two possible orientations of the string. As we summarized earlier, quantization of the string yields the gauge field A_a and the scalar fields ϕ^I. In the parallel brane context we'll have distinct fields labelled by the sector of the string whose excited states are identified with the field quanta. We'll introduce a labelling $[i, j]$, where $i, j = 1, 2$ indexing the two branes upon which string endpoints could lie.

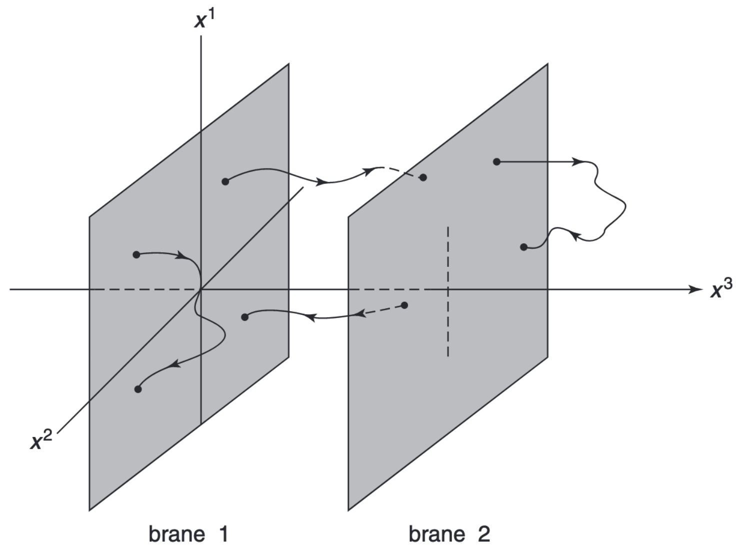


Figure 2.2: Open strings ending on distinct, parallel D-branes. Figure taken from [2].

These indices are known as Chan-Paton factors, and here they indicate the four open string sectors in the theory with two D-branes. Recall that for the gauge field A_a, its carrying of the brane index a implied that it was a field on the D-brane. Let us denote $(A_a)^i{}_j$ as the gauge field of the $[i, j]$ sector. For the $[1, 1]$ and $[2, 2]$ sectors the interpretation is as before, with $(A_a)^1{}_1$ and $(A_a)^2{}_2$ living on their respective branes. We are confounded by the fields $(A_a)^1{}_2$ and $(A_a)^2{}_1$ though, and perhaps are led to an interpretation where the fields each live simultaneously on both of the branes. This weirdness is our first exposure to variety of indications that D-branes experiences spacetime in an unfamiliar way.

Returning now to D-brane actions, there are further steps we can take beyond the Dirac actions of (2.11) and (2.14). In a chronicle that is beyond the scope of this chapter, renormalization of the field theory on the open string worldsheet leads to the derivation of an effective action for D-branes. The equations of motion for the effective action coincide with the vanishing of the beta function for the open string endpoint coupling to the A_a on the D-brane. This coupling in particular will be revisited in the next section. This effective action is known as the Born-Infeld action,

$$S = -T_p \int d^{p+1}\xi \; \left[-\det\left(\eta_{ab} + 2\pi\alpha' F_{ab}\right)\right]^{1/2}, \tag{2.17}$$

where α' is a constant related to the string tension, and F_{ab} is the field strength associated to the

1-form gauge field A_a. In comparing this to (2.14) we see that the metric on the D-brane is flat, the scalar fields ϕ^I have been set to 0, and the dynamics of the brane are now coupled to the gauge field. We may consider the case of low field strengths, $F_{ab} \ll \frac{1}{\alpha'}$, and expand (2.17),

$$S = -T_p \int d^{p+1}\xi \ \left[1 + \frac{(2\pi\alpha')^2}{4} F_{ab}F^{ab} + \cdots\right]. \tag{2.18}$$

We see that in this limit there is a $U(1)$ gauge field theory on the brane, i.e. the physics will be governed by Maxwell's equations. This makes contact with our earlier discussion of string excitations producing an A_a photon field that is interpreted as living upon the brane worldvolume. Terms of higher order in (2.18) are suppressed in α', and for higher field strengths these non-linear contributions would be governed by the full Born-Infeld action of (2.17). An analogous renormalization calculation can be performed for the scalar fields ϕ^I and the implied effective action is one which unifies the Dirac action (2.11) with the Born-Infeld,

$$S = -T_p \int d^{p+1}\xi \ \left[-\det\left(\gamma_{ab} + 2\pi\alpha' F_{ab}\right)\right]^{1/2}. \tag{2.19}$$

Sensibly, this is referred to as the Dirac-Born-Infeld (DBI) action. The only difference from the Born-Infeld action is that the pull-back metric (2.12) has reappeared. Once again switching to the coordinates (2.13), and expanding in the limit of low field strengths F_{ab} as well as small derivatives $\partial_a\phi^I$, yields,

$$S = -(2\pi\alpha')^2 T_p \int d^{p+1}\xi \ \left[\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial_a\phi^I\partial_b\phi^J\delta_{IJ} + \cdots\right]. \tag{2.20}$$

The effective theory under this expansion is a $U(1)$ gauge theory coupled to the scalar fields ϕ^I.

The DBI action describes the low-energy dynamics of the D-brane in a flat background space-time. This description is arrived at by considering both the physics of a general propagating (mem)brane, and the dynamical fields arising from the quantization of the open string. The effective theory is that of a $U(1)$ gauge theory coupled to scalars, and it is perhaps natural to consider a generalization of this action to the case of a non-Abelian gauge theory upon its worldvolume. In the next section, we will motivate a physical situation in which such a generalization might be desirable.

2.3 N coincident D-branes

Let's generalize our earlier discussion of parallel D-branes to the case where there are N-many of them. The Chan-Paton factors are $i, j = 1, ..., N$, and so we have N^2 sectors. A particularly interesting case is that in which all the N D-branes are coincident in spacetime. Then, although the Chan-Paton factors meaningfully distinguish which fields end on each brane, the fields themselves must all live upon a single, common hypersurface. It is natural that we package the N^2 gauge fields and the N^2 scalar fields into $N \times N$ Hermitian matrices,

$$(A_a)^i{}_j, \quad (\phi^I)^i{}_j. \tag{2.21}$$

Here, diagonal elements are those fields arising from strings that begin and end on the same brane. Conversely, strings beginning and ending on differently labelled branes correspond to fields that are off-diagonal elements. The matrix $(A_a)^i{}_j$ has the structure of a $U(N)$ gauge connection, and this is indeed how it should be interpreted. The A_a field of any single open string is a $U(1)$ gauge field, and we have understood this as a field that lives on the D-brane. Suppose that we switch perspectives and consider the action for the string's endpoint, a 0-dimensional brane in its own right, coupled to the $U(1)$ gauge field on the D-brane it is confined to move upon,

$$S = \int d\tau A_a \frac{\partial X^a}{\partial \tau}. \tag{2.22}$$

This is the pull-back of A_a to the worldline - parameterized by the proper time τ - of the 0-brane. We also recognize this to be the action for a charged particle coupled to the $U(1)$ gauge field of electromagnetism. In the context of our N^2 coincident D-branes, this means that the endpoints of each string are charged under the various $U(1)$ gauge fields as indicated by their Chan-Paton indices. But, we will see that it does not mean that there is simply a $U(1)^{N^2}$ symmetry in the theory. For the diagonal elements of (2.21) there is nothing unfamiliar afoot, we have a $U(1)^N$ gauge symmetry in considering these fields all together. However, consider, explicitly for $N = 2$ and $(A_a)^i{}_j$, a $U(1)$ adjoint action upon the first D-brane,

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (A_a)^1{}_1 & (A_a)^1{}_2 \\ (A_a)^2{}_1 & (A_a)^2{}_2 \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (A_a)^1{}_1 & e^{i\theta}(A_a)^1{}_2 \\ e^{-i\theta}(A_a)^2{}_1 & (A_a)^2{}_2 \end{pmatrix}. \tag{2.23}$$

Under the action on the second D-brane we'd of course end up with swapped signs. It's clear that under the two original $U(1)$ symmetries, the $(A_a)^1{}_2$ and $(A_a)^2{}_1$ fields carry $(+1, -1)$ and $(-1, +1)$

charge, respectively. This type of structure can't be made sense of by $U(1)^4$, but is precisely the structure of non-Abelian $U(2)$ symmetry. We ought not be surprised by this, we have N identical D-brane hypersurfaces all coincident in spacetime, and N^2 field labels organized into Hermitian matrices. The Lie group $U(N)$ is precisely an N^2-dimensional manifold and its adjoint action implements a relabelling of all these matrix-embedded fields. The upshot here is that the field theory living on the N coincident D-branes is a $U(N)$ non-Abelian, or Yang-Mills, theory coupled to the scalars $(\phi^I)^i{}_j$, the latter transforming in the $U(N)$ adjoint representation.

This leads us back to considering a possible non-Abelian generalization of the DBI action (2.19). In order to correctly - i.e. $U(N)$ gauge covariantly - describe the dynamics of N coincident D-branes, such a generalization will be required. Unfortunately there are several reasons why such a generalization is a subtle and perhaps illogical proposition. For the present work, it will be sufficient to once again consider the low-energy effective theory, which we postulate as having the form,

$$S = -(2\pi\alpha')^2 T_p \int d^{p+1} \ \mathrm{tr} \left[\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} \mathcal{D}_a \phi^I \mathcal{D}_b \phi^J \delta_{IJ} - \frac{1}{4} \sum_{I \neq J} [\phi^I, \phi^J]^2 \cdots \right]. \tag{2.24}$$

Here the field strength is that of a non-Abelian gauge theory,

$$F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b], \tag{2.25}$$

with the gauge group representation indices suppressed. The kinetic term for the scalars has been generalized to utilize the gauge covariant derivative,

$$\mathcal{D}_a \phi^I = \partial_a \phi^I + i[A_a, \phi^I]. \tag{2.26}$$

Lastly, we have introduced a potential term for the ϕ^I. We will not pursue a derivation of this term, but will consider its implications for the dynamics of the N D-branes. First, note that it is clearly positive semi-definite,

$$V(\phi) \equiv -\frac{1}{4}\sum_{I\neq J} \mathrm{tr}\left([\phi^I,\phi^J]^2\right) \tag{2.27}$$

$$= -\frac{1}{4}\sum_{I\neq J} \mathrm{tr}\left([\phi^I,\phi^J][\phi^I,\phi^J]\right) \tag{2.28}$$

$$= +\frac{1}{4}\sum_{I\neq J} \mathrm{tr}\left([\phi^I,\phi^J]^\dagger[\phi^I,\phi^J]\right), \tag{2.29}$$

a fact made manifest in the final line using anti-Hermiticity of the commutator of two Hermitian matrices. This potential is a function of the $D-p-1$ Hermitian matrix fields ϕ^I. The set of configurations for which $V(\phi)$ is vanishing corresponds to those in which all the matrices commute with one another, causing all the commutators in (2.27) to be 0. A set of mutually commuting matrices are all simultaneously diagonalizable by single similarity transformation, $\phi^I \to U\phi^I U^{-1}$. Notice that in this context, such a transformation is equivalently a $U(N)$ gauge transformation in the adjoint representation. So, there is a choice of gauge that would bring these matrices into a basis where they are all diagonal,

$$\phi^I = \begin{pmatrix} \phi^I_1 & & & \\ & \phi^I_2 & & \\ & & \ddots & \\ & & & \phi^I_N \end{pmatrix}. \tag{2.30}$$

Recall that these fields physically parameterize the fluctuations of the D-branes in the transverse $D-p-1$-dimensional space. Then for these special, commuting, $V(\phi)=0$ field configurations, the various elements $\phi^I_i(\xi)$ have a clear interpretation of the I^{th} transverse coordinate of the position of the i^th D-brane. As long as these ϕ^I matrices remain mutually commuting, we can freely translate the D-branes in the transverse space at no potential energy cost. In terms of the full, potentially non-commuting, space of possible ϕ^I configurations these represent a subspace of transformations under which $V(\phi)$ is invariant. But what are we to make of all the field configurations in which the ϕ^I are non-commuting? The evident conclusion is that the spacetime locations of the D-branes are described by non-commuting matrices, rather than commuting coordinate functions. As before when we tried to reckon with the domain of the $(A)^1{}_2$ and $(A)^2{}_1$ fields in case of two parallel D-branes, we again see hints of a potentially novel physical description of spacetime locality. In the next section we will make a series of simplifications to the DBI action, and approach a model in which we can study this non-locality more acutely.

2.4 A D0-brane toy model

Suppose that we consider the case of $p = 0$, that is, a set of N coincident $D0$-branes. Geometrically, these are 0-dimensional objects, i.e. particles. The action of (2.24) is reduced to an integral over the coincident worldline. The field strength $F_{ab}F^{ab}$ vanishes because there is is no such object in a theory defined on a 1d space. In other words, there is no two-form F that is exact for the 1-form gauge field A living on the 1d manifold of the world-line. Schematically, this gives us an action of the form,

$$S \sim \int dt \ \mathrm{tr} \left(\frac{1}{2} \left(\mathcal{D}_t \phi^I \right) + \frac{1}{4} \sum_{I \neq J} [\phi^I, \phi^J]^2 \right), \tag{2.31}$$

with $I = 1, ..., D$. In super-symmetric string theory, where $D = 10$, the theory with an action of the form (2.31) forms the bosonic part of what is called the BFSS model [10]. It has been conjectured that the BFSS model is mathematically dual to a theory of quantum gravity. We may further simplify our theory by supposing that the $D0$-branes are restricted to move within a lower-dimensional sub-manifold of the transverse, target space. In the models discussed in this work, we will consider this submanifold to be three-dimensional. Thus, our theories will be of the form (2.31), but with $I = 1, 2, 3$. These simplified theories will serve as toy models that will allow us to make practical investigations of the noncommutative spacetime structure discussed in the preceding sections.

To be consistent with notation in the recent literature, we will write down a modified toy model action in terms of the three matrices X^1, X^2, X^3,

$$S = \int dt \ \mathrm{tr} \left(\left(\dot{X}^I \right)^2 + \frac{1}{4} \sum_{I \neq J} [X^I, X^J]^2 + \mu^2 \left(X^I \right)^2 \right), \tag{2.32}$$

with $\dot{X}^I \equiv \mathcal{D}_t X^I$. Note that the action continues to possess invariance under the adjoint $U(N)$ action on the three matrices, now constituting a relabeling of the D0-branes. There is also an $SO(3)$ symmetry of the X^i as a vector, which is a remnant of the $SO(D-p-1)$ symmetry of the ϕ^I in the transverse space for Dp-branes. In this model, the noncommutative structure interacts with the dynamics in a specific way. With the presence of the quadratic mass term, we have the following potential,

$$V(X^I) = \mathrm{tr}\left(-\frac{1}{4}\sum_{I\neq J}[X^I, X^J][X^I, X^J] - \mu^2\sum_{I=1}^{3}(X^I)^2\right). \tag{2.33}$$

It can be shown - see Appendix A - that a minimum for this potential, and hence a solution for the equations of motion, is provided by any three matrices satisfying the μ-deformed $\mathfrak{su}(2)$ algebra,

$$[X^i, X^j] = i\mu\epsilon^{ijk}X^k. \tag{2.34}$$

These set of matrices defines a noncommutative, of 'fuzzy' sphere, an example of noncommutative geometry. If we perform a background field expansion about this classical solution of 0-dimensional fields, the resulting effective theory may be viewed as being defined upon this noncommutative geometric space [22]. This brings to the forefront questions about the structure of the fuzzy sphere, and the behaviour of objects defined upon it. We are naturally confused about how N coincident D0 branes could experience spacetime as a noncommutative manifold. By studying the types of field theories that are emergent - via this background field expansion - upon the fuzzy sphere, we may gain insight into how this structure plays out.

It's worth pausing to reflect on how we've arrived at this point. From considering the boundary conditions required to solve the open string's variational problem, we reasoned through to a very strange type of physical behaviour for sub-manifolds known as D-branes. To study that behaviour, we've simplified and modified the description greatly to that of a toy model (2.32) upon which we'll make an expansion about the noncommutative solution to the theory.

In the next chapter we will discuss noncommutative geometry in more formal terms, and consider how a map from our matrices X^I to the emergent theory on such a manifold is to be achieved mathematically.

Chapter 3

Fuzzy geometry and noncommutative field theory

In this chapter noncommutative geometry will be discussed more generally. We'll introduce the fuzzy sphere and describe a mapping from matrix theory to noncommutative field theory on the sphere. We'll see that latter, image field theory, is in fact a $U(N)$ gauge theory.

3.1 Noncommutative geometry

In noncommutative geometry, the ordinarily commuting coordinates of a space are replaced with operators that are generally noncommuting. This ruptures our ability to specify position exactly within the space. The coordinates themselves follow the logic of the Heisenberg uncertainty principle, and only along one coordinate axis at a time can location be exactly specified. The notion of a point in space is replaced with a minimal area that can be resolved, in direct analogy with the cell of size $\hbar$ in the phase space of ordinary single particle quantum mechanics. As an illustrative example, we consider the noncommutative plane. First define coordinate operators,

$$X = \frac{\theta}{2}(a^\dagger + a), Y = i\frac{\theta}{2}(a^\dagger - a).$$

Here $a, a^\dagger$ are the usual harmonic oscillator raising and lowering operators with $[a, a^\dagger] = 1$. The coordinate operators then have the commutation relation,

$$[X, Y] = i\frac{\theta^2}{2}$$

and for any state, the generalized uncertainty principle gives us

$$\langle \Delta X \Delta Y \rangle \geq \frac{\theta^2}{4},$$

i.e. the minimal area cell as discussed above. In this setup, we wish to define states which we can associate to points, or rather fuzzy areas about them, on the two-dimensional plane. A sensible way of doing this is to define coherent states $|\alpha\rangle$ that provide a symmetric saturation of the uncertainty bound,

$$\langle \alpha | \Delta X | \alpha \rangle = \langle \alpha | \Delta Y | \alpha \rangle = \frac{\theta}{2}.$$

These states are exactly those coherent states of the harmonic oscillator,

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle.$$

They are eigenstates of the lowering operator,

$$a|\alpha\rangle = \alpha|\alpha\rangle,$$

and so in these states,

$$\langle X \rangle = \theta[\mathrm{Re}(\alpha)], \quad \langle Y \rangle = \theta[\mathrm{Im}(\alpha)].$$

As $\alpha = x + iy$ is an arbitrary complex number, the coherent states cover the entirety of the complex plane. This map is not injective since the states are overcomplete:

$$|\langle \beta | \alpha \rangle|^2 = e^{-|\alpha - \beta|^2}$$

This construction allows for the definition of functions living on the noncommutative plane, and can be generalized so as to define quantum field theory on various noncommutative manifolds. For a pedagogical review see [23].

3.2 The fuzzy sphere

In the previous chapter we discussed a toy model (2.32) with the classical solution of a fuzzy sphere (2.34). If we perform a background field expansion about this solution, then we can understand the effective theory to be defined upon the fuzzy sphere as a noncommutative base manifold. Before addressing those details, let's first consider how it is that the $\mathfrak{su}(2)$ algebra can be said to define the fuzzy sphere. Functions on an ordinary sphere form a commutative algebra. The noncommutative, or fuzzy, sphere is what we get when we take this algebra of functions to be noncommutative. In the commutative case, we can write any function in terms of the spherical harmonics, Y_{jm}, i.e.

$$f(\theta, \phi) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} a_{jm} Y_{jm}. \tag{3.1}$$

Here a_{jm} are generally complex coefficients. In the noncommutative case, functions are generated by the same spherical harmonics, but truncated at some $j = j_{\text{max}}$. When taking the product of two functions, f and g, we truncate the resulting sum by omitting any terms with total j exceeding j_{max}. By choosing to enact this truncation in a way that is dependent upon the ordering of f and g in the product, we replace an infinite dimensional algebra of ordinary functions on the sphere with a j^2 dimensional algebra of noncommutative functions on the fuzzy sphere. Such an algebra is explicitly realized by setting the Cartesian coordinates of 3d space to be in correspondence with the three matrices forming the basis for a representation of $\mathfrak{su}(2)$. What we are describing is a map, wherein any $N \times N$ matrix will correspond to a function on the noncommutative sphere, and these three specific $\mathfrak{su}(2)$ basis matrices are special in that they map to the coordinate functions x^i of points (θ, ϕ) on the sphere. In the next section we'll detail how this map, and thus realization of the noncommutative sphere, is brought about.

3.3 A correspondence between matrices, and fields on an emergent sphere

We can define a linear map from the space of $N \times N$ matrices to the space of scalar fields on an emergent sphere. Here we will closely follow the exposition of Appendix C of [17]. In defining this map, we will require that it establishes an equality between inner products defined on the two spaces. In other words, the usual inner product between matrices will be preserved as an inner product between functions on the sphere,

$$\frac{1}{N}\mathrm{tr}\left(A^{\dagger}A'\right) = \frac{1}{4\pi}\int d\Omega\; f_A^*(\theta,\phi) f_{A'}(\theta,\phi). \tag{3.2}$$

We will further require this map to maintain the $\mathfrak{su}(2)$ adjoint action,

$$f_{[L^i,A]}(\theta,\phi) = J^i f(\theta,\phi), \tag{3.3}$$

where L^i are the generators of $SU(2)$ in the N-dimensional irreducible representation, and J^i are the differential operators generating rotations on the sphere,

$$J^i = -i\epsilon^{ijk} x_j \frac{\partial}{\partial x_k}. \tag{3.4}$$

In light of this last requirement, we will use a basis of functions on the sphere provided by the spherical harmonics, $Y_{jm}(\theta,\phi)$. Under the $\mathfrak{su}(2)$ adjoint action, these satisfy,

$$\sum_{i=1}^{3} J^i J^i Y_{jm} = j(j+1)Y_{jm}, \quad J^3 Y_{jm} = mY_{jm}, \tag{3.5}$$

and are orthonormal with respect to the inner product defined in (3.2),

$$\frac{1}{4\pi}\int d\Omega\; Y_{jm}^*(\theta,\phi) Y_{j'm'}(\theta,\phi) = \delta_{j,j'}\delta_{m,m'}. \tag{3.6}$$

On the matrix side, we will define a basis of 'matrix spherical harmonics', denoted as $\hat{Y}_{jm}$, which we will find to be orthonormal as required. We begin by recalling the following property of the spherical harmonics,

$$Y_{j,m\pm 1} = \frac{J^{\pm} Y_{jm}}{\sqrt{(j-1)(j+m+1)}}, \tag{3.7}$$

where $J^{\pm} = L^1 \pm iL^2$. To maintain the $\mathfrak{su}(2)$ adjoint action we will require of our basis matrices that they satisfy the corresponding relation,

$$\hat{Y}_{j,m\pm 1} = \frac{[L^{\pm}, \hat{Y}_{jm}]}{\sqrt{(j-1)(j+m+1)}}, \tag{3.8}$$

with $L^{\pm} = L^1 \pm iL^2$. Provided that we have $\hat{Y}_{j,-j}$, these raising and lowering relations can take us to all the remaining basis matrices. The spherical harmonics also satisfy $J^+ Y_{jj} = 0$ and $J^- Y_{j,-j} = 0$. Likewise, for the matrices we'll have $[L^+, \hat{Y}_{jj}] = 0$ and $[L^-, \hat{Y}_{j,-j}] = 0$. From this we may conclude that,

$$\hat{Y}_{j,-j} = C \left(L^-\right)^j , \tag{3.9}$$

where C is a numerical constant. Note that for $j \geq N$, the right hand side vanishes, $(L^-)^N = 0$. This is the first sign of the fact that expansions of matrices in terms of matrix spherical harmonics will, unlike the expansion on the function side of the map, truncate at $j = N-1$. We'll choose the constant C such that,

$$\frac{1}{N} \mathrm{tr} \left(\hat{Y}^{\dagger}_{j,-j} \hat{Y}_{j,-j} \right) = 1. \tag{3.10}$$

This and the above conditions turns out to not fix the sign of C, which we'll take to be positive so as to be consistent with the function spherical harmonics for which $Y_{j,-j} \propto (x^1 - ix^2)^j$. It can be shown that the matrix spherical harmonics satisfy,

$$\sum_{i=1}^{3} [L^i, [L^i, \hat{Y}_{jm}]] = j(j+1)\hat{Y}_{jm}, \quad [L^3, \hat{Y}_{jm}] = m\hat{Y}_{jm}. \tag{3.11}$$

We observe here that the $\hat{Y}_{jm}$ form an eigenbasis for the adjoint actions of both L^3 and the Casimir $(L^i)^2$. It follows that this basis is orthogonal for the indices j and m, and it is normalized due to (3.10). We conclude that this is an orthonormal basis with respect to the inner product from the left hand side of (3.2). This completes the construction of a linear map $A \mapsto f_A$, and the $j \leq N-1$ requirement on our matrix basis translates into an angular momentum cutoff on the fields,

$$f(\theta, \phi) = \sum_{j=0}^{N-1} \sum_{m=-j}^{j} a_{jm} Y_{jm}. \tag{3.12}$$

The relation (3.2) connects matrix observables to averages of functions over the sphere. This is how the identification of the 3d Cartesian coordinates with $\mathfrak{su}(2)$ basis representation matrices is achieved. We interpret $X^i = \mu L^i$ to be those coordinate matrices, in that they map to coordinate functions x^i taking values given by the averages of the functions $f_{X^i}(\theta, \phi)$ via (3.2). Such an identification allows us to compute the radius of the fuzzy sphere,

$$r^2 = \frac{1}{4\pi} \sum_{i=1}^{3} \int d\Omega \ \left(f_{X^i}(\theta, \phi)\right)^2 \tag{3.13}$$

$$= \frac{1}{N} \sum_{i=1}^{3} \mathrm{tr} \left(\left[X^i\right]^\dagger X^i \right) \tag{3.14}$$

$$= \frac{\mu^2}{N} \mathrm{tr} \left(\sum_{i=1}^{3} \left(L^i\right)^2 \right) \tag{3.15}$$

$$= \frac{\mu^2}{N} \frac{N^2 - 1}{4} \mathrm{tr}(\mathbb{1}) \tag{3.16}$$

$$= \frac{\mu^2 (N^2 - 1)}{4}. \tag{3.17}$$

Note that in the computations of Chapter 5 there emergent sphere's radius will be set to $r = 1$, by normalizing the coordinate matrices using this $\mathfrak{su}(2)$-Casimir-derived value. Due to the inherent non-commutativity of matrix multiplication, our map implies that the corresponding fields on the sphere must obey a noncommutative product. For two functions $f(\theta, \phi)$ and $g(\theta, \phi)$ this is given by,

$$(f \star g)(\theta, \phi) = \frac{1}{N} \sum_{jm} \mathrm{tr} \left(\hat{Y}_{jm} \hat{f} \hat{g} \right) Y_{jm}(\theta, \phi). \tag{3.18}$$

This definition of the product is referred to as the 'star product'. Presuming that the matrix product $(\hat{f}\hat{g})$ possesses an expansion in the spherical harmonic matrices $\hat{Y}_{jm}$, we see that the trace pulls out the requisite expansion coefficient for the spherical harmonic function Y_{jm}. The sum over j is upper bounded by $j = N - 1$, as defined above, and this implies a truncation of the expansion of $(\hat{f}\hat{g})$. These are matrices, hence the non-commutativity of this expression.

3.4 Noncommutative gauge theory on the fuzzy sphere

With our map from the space of $N \times N$ Hermitian matrices in hand, we are ready to consider the image of our matrix model (2.32) under it. We will not be interested in any computations within the image field theory, but will state the form of the expansion and identify the emergent gauge symmetry. In Chapter 4 we will revisit the consequences of this gauge symmetry.

We perform a background field expansion, about the matrix configurations that define the fuzzy sphere,

$$X^I = L^I + \alpha x^I, \tag{3.19}$$

where $\alpha \ll 1$, and x^I are matrices parameterizing the fluctuations about the classical solution given by L^I. We will work in the basis where L^3 is diagonal. It will be sufficient that we consider solely the expansion of the potential $V(X^I)$,

$$V(L^I + \alpha x^I) = \mathrm{tr}\left(\sum_{ij} \frac{1}{4}[L^I + \alpha x^I, L^J + \alpha x^J][L^I + \alpha x^I, L^J + \alpha x^J] + \sum_{I=1}^{3}\left(L^I + \alpha x^I\right)^2\right). \tag{3.20}$$

Using the $\hat{Y}_{jm} \mapsto Y_{jm}$ map, this matrix potential becomes a potential for fields wherein all the products are noncommutative and defined by (3.18).

We recall that the $U(N)$ gauge invariance of the DBI action (2.19), which relabelled our stack of N coincident D-branes, continues to exist in our dimensionally restricted, mass-deformed, D0-brane model of (2.32). We will see that upon mapping to fields on the sphere, that this $U(N)$ invariance is manifest as a gauge symmetry in the noncommutative field theory.

Let $U = \exp(iM) \approx 1 + iM$ where $U \in U(N)$, $M \in \mathfrak{su}(N)$, and therefore M is a Hermitian matrix. The adjoint $U(N)$ action on the background expansion, to linear order in α, is given by,

$$(1 + iM)(L^I + \alpha x^I)(1 - iM) = L^I + \alpha x^I + i[M, L^I] + i\alpha[M, x_i].$$

From this we recognize a transformation law for the dynamical fluctuation matrices x^I,

$$x^I \to x^I - \frac{i}{\alpha}[L^I, M] + i[M, x^I].$$

Which, upon mapping to fields becomes,

$$\chi_i \to \chi_i - \frac{i}{\alpha} J^I \mathcal{M} + i[\mathcal{M}, \chi^I]_\star,$$

where the fields associated to the matrices M and x^I are denoted by $\mathcal{M}$ and χ^I, respectively. We have mapped the adjoint action on M to the action of the generator of rotation on the sphere, following the map property (3.3). Here $[\cdot, \cdot]_\star$ denotes a commutator where multiplication is carried out with the star product of (3.18). We see that the transformation law for the fluctuation field x^I

possesses the form of a gauge symmetry. The emergent quantum field theory on the sphere will be a $U(N)$ gauge theory, albeit one defined upon a noncommutative manifold.

Chapter 4

Entanglement entropy in matrix quantum mechanics

In this chapter we will review review the use of entanglement entropy as a tool for interrogating the geometry emergent from matrix theories. This involves a recasting of matrix theory with a quadratic potential as a system of coupled harmonic oscillators. Using this method, previous works have computed matrix entanglement and analyzed its dependance upon the emergent geometry [11, 12]. We will review the complications arising upon considering the analogous calculation in the three matrix model of (2.32).

4.1 Entanglement entropy

Given two quantum systems, A and B, with two respective Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, we can construct a tensor product space that describes the bipartite system, i.e. $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We refer to a state in $\mathcal{H}_{AB}$ as a product state if it can be written as a tensor product of states from the separate Hilbert spaces of the two systems, i.e.

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle, \quad |\psi_i\rangle \in \mathcal{H}_i. \tag{4.1}$$

States that cannot be factorized in this way are called entangled. Given an operator $\mathcal{O}_A$ on $\mathcal{H}_A$, it's expectation value in a product state will depend only upon the factor from $\mathcal{H}_A$, i.e.

$$\langle\psi|\mathcal{O}_A|\psi\rangle = \langle\psi|\left(\mathcal{O}_A \otimes \mathbb{1}_B\right)|\psi\rangle = \langle\psi_A|\mathcal{O}_A|\psi_A\rangle. \tag{4.2}$$

In the more general case of entangled states, this will not be the case. However, we can always find an ensemble of states for the A subsystem such that the expectation values of $\mathcal{O}_A$ may be reproduced. We consider an ensemble defined by the collection $\{(|\psi_i\rangle, p_i)\}$ of orthogonal states and associated classical probabilities. The expectation value for an operator in the ensemble is defined as a weighted average over the expectation values for the individual states,

$$\langle\mathcal{O}\rangle_{\text{ensemble}} \equiv \sum_i p_i \langle\psi_i|\mathcal{O}|\psi_i\rangle. \tag{4.3}$$

And so, in a general state of the bipartite system, we can recover expectation values for an operator $\mathcal{O}_A$ by finding the right ensemble, i.e.

$$\langle\psi|\mathcal{O} \otimes \mathbb{1}|\psi\rangle = \sum_i p_i \langle\psi_i^A|\mathcal{O}_A|\psi_i^A\rangle. \tag{4.4}$$

With an ensemble of states, $\{(|\psi_i\rangle, p_i)\}$, we can define a new operator,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{4.5}$$

called the density matrix. To compute the expectation value of an operator with the density matrix we take a trace,

$$\langle\mathcal{O}\rangle = \mathrm{tr}(\mathcal{O}\rho) = \sum_i p_i \langle\psi_i|\mathcal{O}|\psi_i\rangle \tag{4.6}$$

In terms of density matrices, we can reproduce the expectation values of operators $\mathcal{O}_A$ on $\mathcal{H}_A$ via

$$\mathrm{tr}(\rho\mathcal{O}_A) = \mathrm{tr}_A(\rho_A\mathcal{O}_A) \tag{4.7}$$

where $\rho_A = \mathrm{tr}_B\rho$ is the reduced density matrix, and tr_A is the partial trace over the A subsystem. In this way the expectation value of operators in a subspace can be reproduced at the cost of introducing a classical uncertainty if the original state was entangled. This uncertainty may be quantified by computing the von Neumann entropy of the reduced density matrix, and this provides a natural measure of the degree to which a state is entangled. We refer to this as the entanglement entropy:

$$S(A) = -\mathrm{tr}_A(\rho_A \mathrm{log} \rho_A). \tag{4.8}$$

If ρ were a pure state in $\mathcal{H}_{AB}$ then the entanglement entropy computed using either subspace must be equivalent, i.e. $S(A) = S(B)$. Any classical uncertainty in the reduced density matrices upon the respective subspaces arises from the entanglement alone. In the case of an ensemble of states, this is no longer necessarily an equality, as the classical uncertainty may be distributed in arbitrary ways between the two subsystems.

Now if our system is a field theory, then the partitioning into subsystems may naturally take the form of a partitioning of the theory base space. We decompose $\mathcal{H}$ into those degrees of freedom associated to a region in space, and those in its complement. Then the entanglement entropy we compute is associated to a spatial subregion. When performing this decomposition in a gauge theory, there are known ambiguities related to the non-locality of the physical gauge-invariant Hilbert space. Prescriptions aimed at the resolution of these ambiguities are abundant in the literature [19–21]. Note that the entropic ambiguities alluded to here are present in all gauge theories. This is separate from the non-local ambiguity inherent in noncommutative theories. In the ground state of local relativistic field theories, the entanglement entropy typically has an area law divergence [24]. In d dimensions,

$$S(A) = C\frac{|\partial A|}{\epsilon^{d-1}} + \cdots , \quad (d > 2), \tag{4.9}$$

$$S(A) = C\mathrm{log}\epsilon + \cdots , \quad (d = 2) \tag{4.10}$$

where ∂A is the boundary of the subregion, $|\partial A|$ is the boundary's ares, C is a regulator dependent constant, and ϵ is a UV regulator such as a lattice spacing. Area law divergence is a manifestation of the locality of the theory, as it reflects the short-range entanglement of degrees of freedom directly adjacent to one another across the partitioning boundary.

The matrix theory that is the object of this work (2.32) can be considered to be a 0+1 dimensional field theory. Therefore, there is no base space to be partitioned. Instead, we will define partitions of the vector space of matrix configurations, grouping collections of matrix degrees of freedom onto either side of the defined partition. We will see that this can be be utilized to perform an entanglement entropy computation in the matrix theory following a reformulation of the theory as a system of coupled oscillators. It is in this way that entanglement entropy will provide a useful calculational tool for understanding how the physics upon noncommutative manifolds is structured. If we can define a sensible entanglement entropy quantity in the matrix configuration space, then it's growth behaviour will yield insights into the geometric structure of the emergent field theory.

4.2 Entanglement entropy of N coupled harmonic oscillators

Here we review the calculation of entropy from [25], which we will reconsider later in the context of matrix quantum mechanics. Suppose we have a system of N-coupled harmonic oscillators described by the Hamiltonian,

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i,j=1}^{N} x_i K_{ij} x_j, \tag{4.11}$$

where K is a symmetric $N \times N$ matrix providing the couplings between the various oscillators. The quantum mechanical ground state of this theory is,

$$\psi_0(x) = \pi^{-N/4} \left(\det \Omega\right)^{1/4} \exp\left(-x^T \cdot \Omega \cdot x\right), \tag{4.12}$$

where x here is a vector of the oscillators, and Ω is the square root of K in the sense: for $K = UK_DU^T$ with U orthogonal and K_D diagonal, then $\Omega = UK_D^{1/2}U^T$. To compute the entropy associated with a partitioning into two groups of oscillators, labelled as 'inside' and 'outside', we integrate out the degrees of freedom corresponding to the 'inside' oscillators,

$$\begin{aligned}\rho_{\text{out}}(x_{n+1}, ..., x_N; x'_{n+1}, ..., x'_N) = \int \prod_{i=1}^{n} dx_i \ \psi_0(x_1, ..., x_n, x_{n+1}, ..., x_N) \\ \times \psi_0^*(x_1, ..., x_n, x'_{n+1}, ..., x'_N),\end{aligned} \tag{4.13}$$

to construct a reduced density matrix, ρ_{out}, for the 'outside' subsystem. We can introduce the sub-matrices,

$$\Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \tag{4.14}$$

such that (4.13) is clearly organized into Gaussian integrals. We arrive at,

$$\rho_{\rm out}(x;x') \sim \exp\left(-\left(c^T\cdot\gamma\cdot x + x'^T\cdot\gamma\cdot x'\right)/2 + x^T\cdot\beta\cdot x'\right) \tag{4.15}$$

where $\beta = \frac{1}{2}B^T A^{-1} B$, and $\gamma = C - \beta$. Here x and x' are $(N-n)$-dimensional vectors. We need not keep track of the overall normalization, since we may later invoke the fact that the eigenvalues of $\rho_{\rm out}$ must sum to 1, given that it is a density matrix. Next consider the eigenvalue equation,

$$\int dx' \rho_{\rm out}(x;x') f_n(x') = p_n f_n(x), \tag{4.16}$$

where we wish to identify the p_n since in terms of them the entropy is, $S = -\sum_n p_n \log p_n$. Consider the transformation of (4.16) under a change of variables, $x \to Gx$ for some non-singular matrix G:

$$\int dx' (\det G) \rho_{\rm out}(Gx;Gx') f_n(Gx') = p_n f_n(Gx). \tag{4.17}$$

From this we conclude that $(\det G)\rho_{\rm out}(x;x')$ and $\rho_{\rm out}(x;x')$ possess the same eigenvalues. First, we change variables with $\gamma = V^T \gamma_D V$, where V is an orthogonal matrix and γ_D is diagonal, and $x = V^T \gamma_D^{-1/2} y$, yielding,

$$\rho_{\rm out}(x;x') \sim \exp\left(-\left(y^T\cdot y + y'^T\cdot y'\right)/2 + y^T\cdot\beta'\cdot y'\right), \tag{4.18}$$

with $\beta' = \gamma_D^{-1/2} V \beta V^T \gamma^{-1/2}$. Then we perform a second change of variables, $z = Wy$ with W an orthogonal matrix such that $W^T \beta' W$ is diagonal, yielding,

$$\rho_{\rm out}(x;x') \sim \prod_{i=1}^{N-n} \exp\left(-\left(z_i^2 + z_i'^2\right)/2 + \beta_i' z_i z_i'\right), \tag{4.19}$$

where β_i' are the eigenvalues of β'. The reduced density matrix of (4.19) takes the form of a product of reduced density matrices for independent two oscillator systems, wherein one of the oscillators has been traced out. For this simplest, $N = 2, n = 1$, of cases, the entropy can be shown to be,

$$S(\xi) = -\log(1-\xi) - \frac{\xi}{1-\xi}\log\xi, \tag{4.20}$$

where ξ is a function of the couplings between the two oscillators. In the general N oscillator case, the entropy associated to the tracing out of the inside n oscillators is given by the sum,

$S = \sum_i S(\xi_i)$, with $\xi_i = \beta_i'/(1 + (1 - \beta_i')^{1/2})$.

An upshot from this calculation, that will be central to our later discussion, is that if the coupling matrix K were to have a null eigenvalue then it may imply a divergent entropy. Specifically, if K has a null eigenvalue then the matrix A may not be invertible. As any eigenvalue of A approaches zero, the corresponding eigenvalue of A^{-1} diverges, which carried through the transformations just outlined, would ultimately lead to divergent ξ_i, and therefore divergent entropy. We will now outline how this divergence appears in our attempts to compute entropies in matrix quantum mechanics, and what we may do in response.

4.3 Entanglement entropy in a scalar matrix-field theory on the noncommutative sphere

Understanding the individual matrix elements to define linearly independent dynamical variables, we can view a quadratic matrix theory as defining a system of N coupled harmonic oscillators. The following Hamiltonian defines a matrix theory with a single matrix, and a noncommutative scalar field theory on an emergent fuzzy sphere,

$$H = \mathrm{tr}\left(\dot{\phi}^2 + \sum_{I=1}^{3}[L^I, \phi]^2 + \mu^2\phi^2\right) \tag{4.21}$$

The Hamiltonian (4.21) possesses the following potential,

$$V(\phi;\mu) = \mathrm{tr}\left(\sum_{I=1}^{3}[L^I, \phi]^2 + \mu^2\phi^2\right), \tag{4.22}$$

where ϕ is an $N \times N$ complex, Hermitian matrix, the L^I are again the generators of $SU(2)$ in the N-dimensional irreducible representation, and μ is again a mass parameter. The argument to the trace appearing in the potential is quadratic in the matrix ϕ, and so the potential itself consists of a series of summands that are all quadratic in the matrix entries. The space of complex Hermitian matrices forms an N^2 dimensional vector space and thus the potential (4.22) may be recast in the form of the N-coupled harmonic oscillator potential appearing in (4.11),

$$V(\phi;\mu) = \frac{1}{2}\sum_{ab}\phi_a K^{(\mu)}\phi_b, . \tag{4.23}$$

In writing ϕ_a with a single index we assume a prescription for labelling the N^2 real degrees of freedom of the matrix ϕ with $a = 1, ..., N^2$. Here the coupling matrix $K^{(\mu)}$ is explicitly given by,

$$K^{(\mu)} = \frac{1}{2} \frac{\partial^2 V(\phi; \mu)}{\partial \phi_a \partial \phi_b} \tag{4.24}$$

In light of (4.23), we see that entropies associated with sub-sets of the matrix configuration space can be computed using the technique described in Section 4.2.

Separately consider the potential in the case that $\mu = 0$,

$$V(\phi; 0) = \mathrm{tr}\left(\sum_{I=1}^{3} [L^I, \phi]^2 \right). \tag{4.25}$$

The classical minimum configuration of ϕ is obtained when all these commutators all vanish, and may in fact be identified as any matrix where $\phi \propto \mathbb{1}$. This 1D non-trivial space of minimum potential energy configurations implies that the matrix $K^{(0)}$ will have a corresponding null eigenvalue. A proof of this implication is provided in Appendix C.

As discussed at the end of Section 4.2, this leads to divergent entropy. We see clearly the utility of turning on the mass term, in that the $V(\phi; \mu)$ with μ non-zero lifts the degeneracy of this classical minimum. The minimal potential energy configuration for $\mu \neq 0$ is given uniquely by the zero matrix, upon which the adjoint $U(N)$ acts trivially. In this sense, the mass term regulates the entropy computation for this potential.

In [11], entanglement entropy for the scalar field theory model was computed. The coupling matrix for a system of N coupled harmonic oscillators was constructed in the manner of (4.24).

For this theory it was noticed that under a different formulation - from the spherical harmonic based approach detailed in Section 3.3 - of the map from matrices to functions, there is an explicit association of matrix degrees of freedom with certain geometric regions on the fuzzy sphere. Specifically, elements of the matrix ϕ in the set, $\{\langle m_1|\phi|m_2\rangle \mid m_1 + m_2 > 2j(\cos\theta)\}$, are associated to spherical cap region centred upon the north pole and delimited by the polar angle θ. Here $j = (N-1)/2$ is the spin of the $SU(2)$ representation, and the m_i are our basis elements, or equivalently, $SU(2)$ weights. The association is presented pictorially in Figure 4.1. With this geometric interpretation in hand, computation of a geometric entanglement entropy was carried out by selecting the oscillator degrees of freedom in the shaded triangle of Figure 4.1 as the 'inside', and tracing those degrees of freedom out to produce a reduced density matrix for the 'outside'

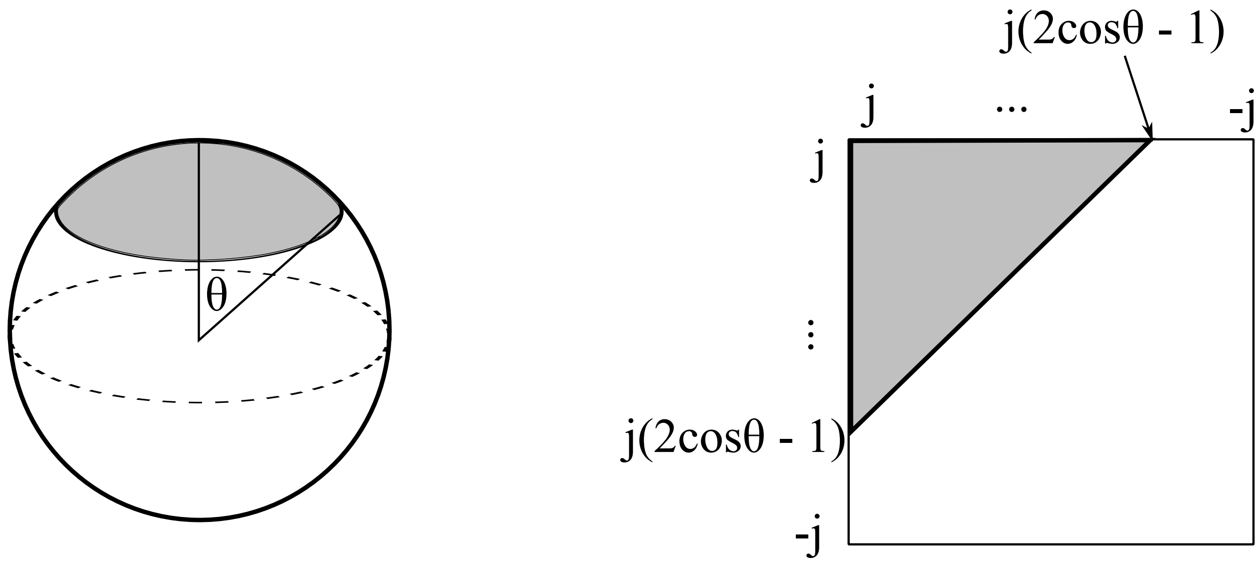


Figure 4.1: Schematic illustration of the association between degrees of freedom in the matrix ϕ and the scalar field theory on the fuzzy sphere.

oscillators. For caps representing a small fractional area, a, of the sphere, entanglement entropy was found to have an extensive character,

$$S \sim \frac{\text{area}(C)}{4\pi R^2} = \sin^2(\theta/2). \tag{4.26}$$

where R is the radius of the sphere, and C denotes the cap region delimited by the angle θ.

As the fractional area became larger, reaching the order of the size of the sphere itself, a cross-over was observed to boundary-law behaviour, consistent with a commutative, local field theory. In this theory, the fundamental length scale of the non-commutativity is denoted by $\sqrt{\theta}$, with $\sqrt{\theta} \sim R/\sqrt{N}$. Comparatively, the UV cutoff of the theory, ϵ, is a separate length scale determined by the number of degrees of freedom and the full spherical area. There should be one degree of freedom in an area of ϵ^2 size, and so we determine that $\epsilon \sim R/N$. We observe that ϵ is parametrically smaller than $\sqrt{\theta}$. Naively, we might expect that for regions on the sphere of size larger than the scale $\sqrt{\theta}$ that non-commutativity would not be relevant to calculations of entanglement entropy, and that the results would coincide with expectations for scalar field theory on a commutative sphere.

However, there is a property of noncommutative field theories known as UV/IR mixing, whereby physics at vastly different length scales become interdependent, that leads to the deviation from commutative expectations, as was observed in this work. Specifically, UV/IR mixing is expected to be relevant in the range of region sizes from $\sqrt{\theta}$ to $\theta/\epsilon \sim R$ (i.e. the size of the sphere, or equivalently the IR cutoff) [26]. This coincides with the results observed, extensive behaviour for small fractional areas, up until a cross-over to boundary law behaviour once the considered areas approach the order of the sphere itself. In [12], the effective scale of this nonlocality is adjusted by introducing a UV cutoff to this calculation with the same model of (4.21). The same crossover to boundary law entropy was observed, with the UV cutoff magnitude controlling the fractional area at which the crossover is observed.

One peculiarity of the association presented in Figure 4.1 is that the number of degrees of freedom above the anti-diagonal matrix partition does not grow proportionally with area. This is a fundamentally non-local characteristic of the theory. Specifically, the number of matrix degrees of freedom is proportional to $2j^2(1-\cos\theta)^2$, i.e. proportional to the square of the fractional area. A possible interpretation of this is that these degrees of freedom are not uniformly distributed within the spherical cap, and are subject to some form of organization that leads to the boundary law behaviour for the entropy once the critical cap size is considered.

4.4 Entanglement entropy in a three matrix quantum mechanics

A natural extension of the single matrix work described in the previous section is to consider models of multiple matrices. This will necessarily introduce a few complications. Once more we consider the potential in the three matrix model of (2.32),

$$V(X^I) = \mathrm{tr}\left(-\frac{1}{4}\sum_{I\neq J}[X^I, X^J][X^I, X^J] - \mu^2 \sum_{I=1}^{3}(X^I)^2\right). \tag{4.27}$$

Note that $V(X^I)$ is quartic in the matrix degrees of freedom due to the $[X^I, X^J]^2$ term. The technique of computing entanglement entropy from Section 4.2 relied on the potential being quadratic, and therefore will not be viable here. Thus, we will choose to truncate our background expansion at quadratic order α, so as to retain an approximation to the theory that is still tractable for our entanglement entropy technique. Explicitly, the quadratic expansion to the expanded potential is found to be

$$V_2(L^I, x^I) = \mathrm{tr}\left(-\sum_{i\neq j}[L^I + \alpha x^I, L^I + \alpha x^I]^2 - \mu^2 \sum_{I=1}^{3}(L^I + \alpha x^I)^2\right) \tag{4.28}$$

$$\begin{aligned} &\equiv 2\alpha^2 \ \mathrm{tr}\Big(iL^3x^1x^2 - iL^3x^2x^1 + (x^1)^2 - L^1x^2x^2L^1 - 2L^2x^1x^1L^2 + 2L^1x^2L^1x^2 \\ &+ L^2x^1L^2x^1 + x^1L^2L^1x^2 + L^2x^1x^2L^1 - L^2x^1L^1x^2 - x^1L^2x^2L^1 \\ &+ \text{cyclic permutations}\Big). \end{aligned} \tag{4.29}$$

In Appendix B this expansion is carried out more explicitly. Here in the second line the use of '$\equiv$' reflects the fact that we are defining $V_2(L^I, x^I)$ to be the expression with all terms $\mathcal{O}(\alpha^3)$ and greater dropped. We have also dropped constant terms, and set $\mu = 1$ as we will not be concerned with it as a parameter. Recall that for the single scalar field theory model we observed that the vanishing of the mass term introduced a divergence in the entanglement entropy computation. Here a different divergence will appear, descending from the fact that the potential of (4.27) was invariant under the $U(N)$ adjoint action. Let us parameterize a general $U(N)$ transformation as,

$$U = e^{i\alpha M} \tag{4.30}$$

for M an $N \times N$ Hermitian matrix, i.e. $M \in \mathfrak{u}(N)$, and again $\alpha \ll 1$. We have inserted α into the exponential map defining U so as to consider transformations parameterized by $U(N)$ elements that are of the same infinitesimal order as our fluctuations αx^I. Then we consider an adjoint action with this $U(N)$ matrix, upon the expanded matrices $X^I = L^I + \alpha x^I$,

$$L^I + \alpha x^I \to UL^IU^{-1} + \alpha U x^I U^{-1} \approx L^I - i\alpha[L^I, M] + \alpha x^I + i\alpha^2[M, x^I] + \mathcal{O}(\alpha^3), \tag{4.31}$$

where this approximation is consistent with the truncation at quadratic order that we have made in defining $V_2(L^I, x^I)$. Viewed as a transformation in the configuration space of the x^I we have,

$$x^I \to x^I - i[L^I, M] + i\alpha[M, x^I]. \tag{4.32}$$

Since the $U(N)$ action was an invariance of the un-expanded potential, it follows that at this order we have (4.31) as an invariance of $V_2(L^I, x^I)$. The N^2 degrees of freedom parameterizing the Hermitian M give rise to N^2 flat directions to $V_2(L^I, x^I)$. These flat directions are the chief computational roadblock to computing entanglement entropy in this theory. In Chapter 5 the primary object of our computations will be to find a prescription for handling these flat directions, and thus regulating the divergent entanglement entropy.

As discussed in Section 3.4, the matrix transformation (4.32) is a gauge symmetry of the emergent quantum field theory. Divergent entanglement entropy of the sort seen here is characteristic of gauge theories more generally, and implies the need of alternative approaches for defining partitions of the base space of the theory.

It's not at all clear how potential prescriptions for addressing these flat directions will affect the entanglement entropy growth behaviour. In recent works, three matrix models have been investigated with prescriptions offered for computing entanglement entropy. In [17], a restriction is enforced to the space of gauge-invariant physical states, and the computation is made possible by using a variational approach to finding wavefunctions in a quartic matrix potential. The authors find boundary law entanglement growth throughout the entire range of fractional areas on the sphere. There is no cross-over from area law behaviour observed. While this clearly conflicts with [12], as described in Section 4.3, it is not clear what aspect of their considered model is causing the conflict. In going from a single matrix to three matrices, the authors of [17] considered an alternatively quartic potential, and used a different prescription for defining partitions of matrix configuration space. This underscores the motivation for considering the simplified three matrix potential of (4.27) and expanding as described above. In defining this model and considering it at quadratic order, we may gain insight into specifically the role played by the descendant $U(N)$ gauge invariance of (4.32), in so far as it may be modifying the entropic growth behaviour in the theory.

Chapter 5

Matrix configurations and geometry in a gauge-invariant model

At the end of the last section we saw how the invariance of the expanded three matrix quantum mechanics Hamiltonian under the adjoint $U(N)$ action led to a divergence in the computation of entanglement entropy. In this section we will discuss possible prescriptions for ameliorating this divergence, and interrogating the nature of geometric emergence in this model.

5.1 Considering matrix configuration space as a vector space

In this chapter we will be considering various subspaces of the full configuration space of our three-matrix model. To begin, recall that the set of all complex Hermitian matrices form a vector space over $\mathbb{R}$. Our model possesses three such matrices, and so we have a real, $3N^2$ dimensional vector space associated to the configuration space of these $N \times N$ matrices. This vector space will be denoted as V_{3N^2}. To make manifest the fact that the field over which V_{3N^2} is defined is $\mathbb{R}$, we will map the complex Hermitian x^I matrices to a set of matrices with real-valued entries, $\tilde{x}^I$. For each such matrix x we carry out the mapping,

$$x \to \tilde{x} = \frac{1}{2}\left(x + x^T\right) + \frac{1}{2i}\left(x - x^T\right). \tag{5.1}$$

Now we will introduce a specific basis for V_{3N^2} which identifies individual matrix elements of the three real-valued $\tilde{x}^I$ with standard basis vectors for V_{3N^2}. Explicitly, we are re-organizing the $3N^2$-

many matrix elements into a $3N^2$ dimensional column vector under the following prescription. Labelling the main diagonals of each of the $\tilde{x}^I$ as '0^I', we proceed to label the successive upper diagonals by '1^I', '2^I', ..., '$(N-1)^I$'. Similarly, the successive lower diagonals are labelled by '-1^I', '-2^I',..., $-(N-1)^I$. We assemble the diagonals, retaining the order of the matrix elements in each diagonal, into a vector in V_{3N^2} as

$$(0^1, 0^2, 0^3, 1^1, 1^2, 1^3, -1^1, -1^2, -1^3, 2^1, ..., -(N-1)^1, -(N-1)^2, -(N-1)^3)^T. \tag{5.2}$$

For $N = 3$ this looks like,

$$\begin{pmatrix} [\tilde{x}^1]_{11} & [\tilde{x}^1]_{12} & [\tilde{x}^1]_{13} \\ [\tilde{x}^1]_{21} & [\tilde{x}^1]_{22} & [\tilde{x}^1]_{23} \\ [\tilde{x}^1]_{31} & [\tilde{x}^1]_{32} & [\tilde{x}^1]_{33} \end{pmatrix}, \begin{pmatrix} [\tilde{x}^2]_{11} & [\tilde{x}^2]_{12} & [\tilde{x}^2]_{13} \\ [\tilde{x}^2]_{21} & [\tilde{x}^2]_{22} & [\tilde{x}^2]_{23} \\ [\tilde{x}^2]_{31} & [\tilde{x}^2]_{32} & [\tilde{x}^2]_{33} \end{pmatrix}, \begin{pmatrix} [\tilde{x}^3]_{11} & [\tilde{x}^3]_{12} & [\tilde{x}^3]_{13} \\ [\tilde{x}^3]_{21} & [\tilde{x}^3]_{22} & [\tilde{x}^3]_{23} \\ [\tilde{x}^3]_{31} & [\tilde{x}^3]_{32} & [\tilde{x}^3]_{33} \end{pmatrix} \rightarrow \begin{pmatrix} [\tilde{x}^1]_{11} \\ [\tilde{x}^1]_{22} \\ [\tilde{x}^1]_{33} \\ [\tilde{x}^2]_{11} \\ \vdots \\ [\tilde{x}^3]_{13} \\ [\tilde{x}^1]_{31} \\ [\tilde{x}^2]_{31} \\ [\tilde{x}^3]_{31} \end{pmatrix}. \tag{5.3}$$

Henceforth we will denote this specific basis - in which real matrix elements $\tilde{x}^I$ are in 1-1 correspondence with column vector entries - for V_{3N^2} as $\{e_i\}$.

Note that when formulated as the vector space of three complex Hermitian matrices, we had the natural inner product given by,

$$\langle v, v \rangle = \sum_{I=1}^{3} \mathrm{tr}\left([x^I]^\dagger [x^I]\right), \tag{5.4}$$

where $v \in V_{3N^2}$ denotes a generic vector; a generic set of the x^I. It is straightforward to convince oneself that following the mapping (5.1), and the subsequent reformulation described above (5.2, 5.3), that the inner product on matrices (5.4) becomes the standard Euclidean inner product on $\mathbb{R}^{3N^2}$.

5.2 The operator $\mathcal{K}$ and its null space

At quadratic order in our expansion parameter, α, the $U(N)$ adjoint action upon the X^I can be expressed instead as a transformation (4.32) of the fluctuation matrices, x^I. Note that this is defined upon the complex Hermitian x^I, not the real-valued $\tilde{x}^I$. The expanded potential, $V_2(x^I)$, is invariant under this transformation. The transformation (4.32) is parameterized by an $N \times N$ Hermitian matrix M, in turn possessing N^2 independent degrees of freedom. These 'directions' in matrix configuration space will be of interest to our study in this chapter. In terms of the vector space V_{3N^2}, these directions collectively span an associated subspace, $V_G \subset V_{3N^2}$. In other words, the potential function $V_2(x^I)$ is flat along any direction lying within the V_G subspace.

To the potential $V_2(x^I)$, we may consider an associated operator, $\mathcal{K} \in \mathcal{L}(V_{3N^2})$, such that,

$$\langle v, \mathcal{K}v \rangle = V_2(x^1, x^2, x^3). \tag{5.5}$$

In the $\{e_i\}$ basis for V_{3N^2} we will denote the matrix representing $\mathcal{K}$ as $\mathbf{K}$. The matrix $\mathbf{K}$ is the three-matrix-model analogue of the $K^{(\mu)}$ matrix of (4.23). Here the (μ) superscript is omitted as we'll only consider the $\mu = 1$ case. In general, we will distinguish operators in $\mathcal{L}(V_{3N^2})$ from their associated $\{e_i\}$ basis matrix representations by use of calligraphic font for the former ($\mathcal{K}$), and bold-face font font for the latter ($\mathbf{K}$). For $3N^2 \times 3N^2$ matrices such as $\mathbf{K}$, the bold-face font is used to distinguish them, in their dimensionality, from the various $N \times N$ matrix objects discussed in the thesis to this point.

Consistent with V_G being the space spanned by the flat directions of $V_2(x^I)$, we anticipate that,

$$\text{null}(\mathcal{K}) = V_G. \tag{5.6}$$

As discussed in Section 4.4, the presence of this null space will lead to divergent entanglement entropy by the computational method outlined in Section 4.2.

If we wish to use this method to study the nature of geometric emergence in this model then we will have to provide a prescription for removing this divergence. In the ensuing sections of this chapter, we will pursue a line of development in which we perform mappings onto the orthogonal complement to V_G, attempting to glean a geometric understanding from the remaining configuration space. Recall that the invariance under (4.32) corresponds to a gauge symmetry in the emergent noncommutative field theory. Thus, via the dual field theory description, the removal of V_G - mapping onto its orthogonal complement - can be equated with the implementation of a gauge-fixing constraint that would remove the symmetry.

5.3 Linear maps for gauge-fixing and height restriction

As in the scalar field model, height on the emergent sphere from this model is associated with degrees of freedom from the three matrices. We may define a 'height operator', $\mathcal{Z} \in \mathcal{L}(V_{3N^2})$, whose action on the x^I is as follows,

$$\mathcal{Z} : (x^1, x^2, x^3) \to \left(\frac{1}{2}\left(L^3 x^1 + x^1 L^3\right), \frac{1}{2}\left(L^3 x^2 + x^2 L^3\right), \frac{1}{2}\left(L^3 x^3 + x^3 L^3\right) \right). \tag{5.7}$$

Notice that as we choose to work in a basis where the $N \times N$ matrix L^3 is diagonal, that it follows that in the $\{e_i\}$ basis for V_{3N^2} the matrix $\mathbf{Z}$ associated to $\mathcal{Z}$ will be likewise diagonal. Eigenvectors of $\mathcal{Z}$ are those $v \in V_{3N^2}$ which are linear combinations of x^I matrix elements all lying within a common anti-diagonal; across the three matrices. Equivalently, through their eigenvalues under $\mathcal{Z}$, such v are all associated to a common height on the emergent sphere, following the illustration of Figure 4.1. The only difference in this model is that there are three matrices each constituting the same anti-diagonal-to-height pattern as was the case for the single matrix ϕ in Chapter 4. The eigenspaces of $\mathcal{Z}$ are degenerate with multiplicity given by the length of the corresponding anti-diagonal in the x^I matrices, multiplied by three for the number of matrices in our model. This degeneracy makes crude the geometric characterization of general matrix configurations, and subspaces of V_{3N^2}. Nonetheless, the eigenbasis of $\mathcal{Z}$ forms an orthonormal basis, trivially taken to be $\{e_i\}$ given that $\mathbf{Z}$ is diagonal therein, with which we may construct orthogonal projections onto certain subspaces interpreted as height-restricted regions on the emergent sphere. Specifically, we will consider orthogonal projection operators $\mathcal{P}_{h_1,h_2} \in \mathcal{L}(V_{3N^2})$ defined as,

$$\mathcal{P}_{h_1,h_2} \equiv \mathcal{P}(h_1 < \mathcal{Z} < h_2), \tag{5.8}$$

where by the right-hand side we mean orthogonal projection onto the subspace formed by the union of those eigenspaces of $\mathcal{Z}$ possessing eigenvalues falling within the range (h_1, h_2). We will refer to this subspace as $V_{h_1,h_2} \subset V_{3N^2}$, and understand it geometrically as consisting of those configurations of x^I which are associated to field degrees of freedom falling within the spherical band (h_1, h_2). For $h_2 = 1$, which we will frequently consider, the projection is onto the region $(h_1, 1)$ of a spherical cap centred upon the north pole. This was the type of geometric region discussed in Chapter 4, and in [11, 12]. A schematic illustration of the interpretation of $\mathcal{P}_{h_1,h_2}$ is provided in Figure 5.1

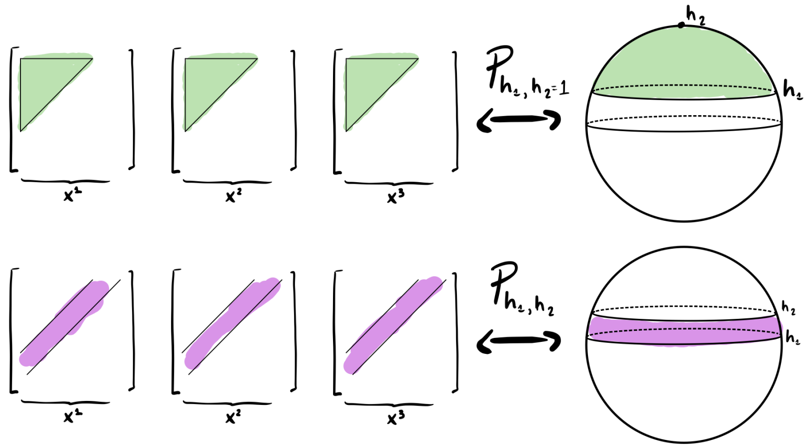


Figure 5.1: The association, realized by the projection operator $\mathcal{P}_{h_1,h_2}$, between matrix degrees of freedom and geometric regions on the emergent sphere.

Next let us consider linear maps involving the space null$(\mathcal{K}) = V_G$ discussed above. We are interested in restricting to the space of gauge-invariant configurations, a restriction we will implement upon V_{3N^2} by projecting onto the orthogonal complement to V_G. Let us denote this orthogonal complement as V_R, i.e.

$$V_R \equiv V_G^{\perp}. \tag{5.9}$$

In similar fashion to the definition (5.8), we define a projection operator $\mathcal{P}_R \in \mathcal{L}(V_{3N^2})$,

$$\mathcal{P}_R \equiv \mathcal{P}(\mathcal{K} \neq 0). \tag{5.10}$$

Consistent with our earlier discussion, this is a projection onto the space spanned by the non-null eigenvectors of $\mathcal{K}$. In practice, such an operator is constructed via the numerical diagonalization of the matrix $\mathbf{K}$.

To compute entanglement entropy by the method of Section 4.2 we will need to construct a basis for V_R. Furthermore this basis must admit a sufficient geometric interpretation. In the quantized model of $3N^2$ harmonic oscillators, we will trace out certain of these basis vectors - certain of these oscillators - and understand this as a tracing out of degrees of freedom associated to geometric regions on the emergent sphere. Gauge-fixing will necessarily disrupt this interpretation, and we can begin to investigate this disruption by considering the composite operator,

$$\mathcal{O}_{h_1,h_2;R} \equiv \mathcal{P}_R \mathcal{P}_{h_1,h_2} \mathcal{P}_R. \tag{5.11}$$

The operator $\mathcal{O}_{h_1,h_2;R}$ is not a projection operator itself. In [12], a similar operator was constructed for the single scalar field matrix theory (4.21) which involved the combination of: (a) geometric restriction to a spherical cap region, and (b) imposition of a UV-cutoff. We can view (5.11) as the analogue of that, with projection onto energy eigenstates below some UV cutoff replaced by projection onto all non-null eigenstates of $\mathcal{K}$. Furthermore, if gauge-fixing is to only minimally disrupt the geometric interpretation of matrix configuration space, then we anticipate that $\mathcal{O}_{h_1,h_2;R}$ will be approximately a projection operator, in that it's spectrum should consist of mostly 0's and 1's.

In Figure 5.2 the eigenvalues of $\mathcal{O}_{h_1,h_2;R}$ are plotted in a histogram. The matrix dimension of

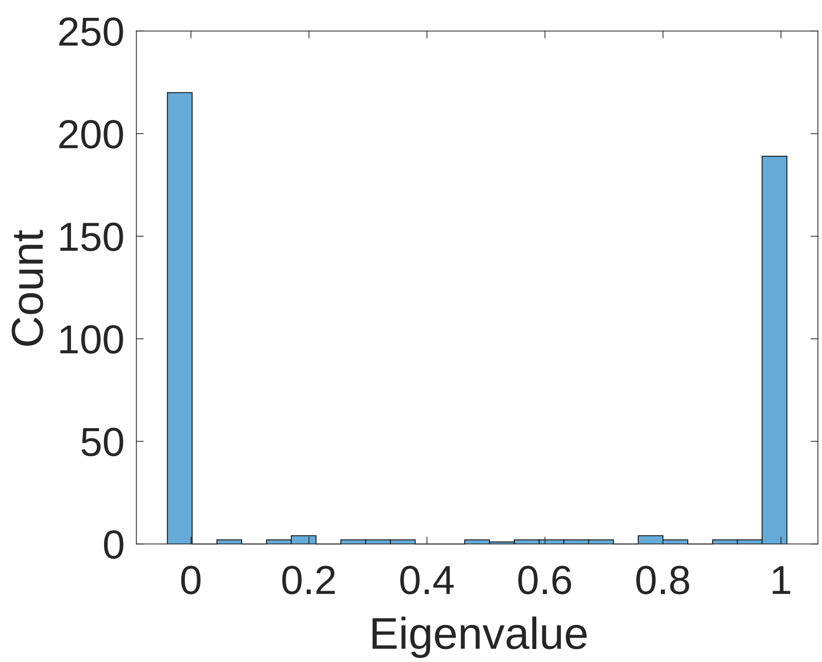


Figure 5.2: Histogram of the eigenvalues of $\mathcal{O}_{h_1,h_2,R} \in \mathcal{L}(V_{3N_2})$ for $h_1 = 0$, $h_2 = 1$ and $N = 15$.

$N = 15$ has been used, and the heights delimiting the spherical band are $h_1 = 0$ and $h_2 = 1$, i.e. it is the upper hemisphere. This preliminary test passes; we see an operator that has spectrum where the vast majority of values are equal to 0 or 1, or very nearly so, with few values filling out the intermediate region.

The extent to which $\mathcal{O}_{h_1,h_2;R}$ fails to be a projector is dependent on the interplay between the non-commutativity of the emergent geometry and the effect of gauge-fixing. We would expect that as we take the x^I matrix size, N, to be large, toward the commutative limit of $N = \infty$, that the fraction of eigenvalues falling in the intermediate region will decrease. In Figure 5.3 the magnitude of this fraction of the spectrum has been plotted as a function of N for various values of h_1, i.e. various spherical caps where we keep $h_2 = 1$. The result broadly confirms our

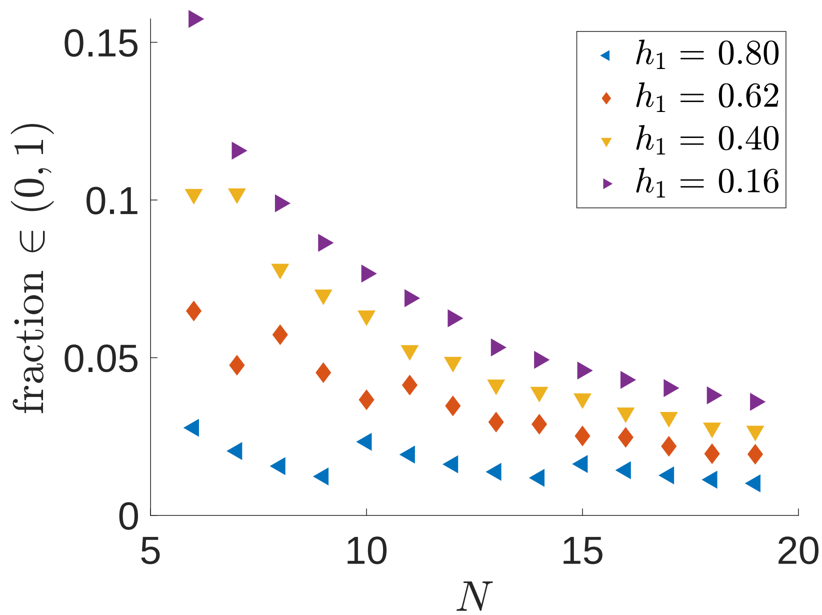


Figure 5.3: The magnitude of the fraction of eigenvalues of $\mathcal{O}_{h_1,h_2,R}$ falling in the range $(0, 1)$ is plotted against N for various spherical cap regions, where h_1 is as indicated and $h_2 = 1$.

expectation, demonstrating that indeed this fraction is decreasing with N as we reduce the scale of the non-commutativity.

In the gauge-fixed theory, we may choose to interpret the eigenspaces of $\mathcal{O}_{h_1,h_2;R}$ with eigenvalue $> \frac{1}{2}$ as associated to degrees of freedom within the (h_1, h_2) band, with all others interpreted as being outside. This is sensible as we have seen that $\mathcal{O}_{h_1,h_2;R}$ nearly has the spectrum of projector, and thus we are not making an unreasonable approximation. With this notion of a rounding procedure in mind, it is worth checking how the number of degrees of freedom within (h_1, h_2) region scales with area on the emergent sphere. We define a new projection operator,

$$\mathcal{P}_{h_1,h_2;R} \equiv \mathcal{P}\left(\mathcal{O}_{h_1,h_2;R} > \frac{1}{2}\right), \tag{5.12}$$

which implements precisely this rounding. Then the quantity we are interested in is rank$(\mathcal{P}_{h_1,h_2;R})$. If gauge-fixing introduces minimal disruption to geometry then this quantity will grow with similar behaviour to that of rank$(\mathcal{P}_{h_1,h_2})$, i.e. similar to what was observed in the single scalar field model as discussed in Section 4.3. Recall that the number of matrix degrees of freedom within an upper triangular partition, as pictured in Figure 4.1, grows proportional to $2j^2(1-\cos\theta)^2$, where j is the spin of the $SU(2)$ representation. This is quadratic in the fractional area of the spherical cap, C, when $\theta \in [0, 2\pi]$. Thus we expect the relationship,

$$\text{rank}\left(\mathcal{P}_{\cos\theta,1;R}\right) \sim \left[\frac{\text{area}(C)}{4\pi}\right]^2 = \sin^4\left(\frac{\theta}{2}\right), \tag{5.13}$$

where we have chosen a specific projection associated to a spherical cap delimited by the angle θ. In Figure 5.4 this relationship is plotted and evidently matches the expectation of (5.13). This provides further evidence for a retention of the geometric interpretation following the gauge-fixing.

At this point, we have obtained preliminary verification that the composite operator $\mathcal{O}_{h_1,h_2;R}$, and its rounded-to-projection version $\mathcal{P}_{h_1,h_2;R}$, possess the characteristics required for use in a tracing out procedure that would produce a geometrically meaningful entanglement entropy. In the next section we will attempt to achieve a more granular view of the situation, and consider how one might identify the most optimally-geometric basis vectors that could span gauge-fixed subspaces such as range$(\mathcal{P}_{h_1,h_2;R})$.

5.4 Searching for a basis for V_R

Our aim in this section is to identify matrix configurations that are associated to well-localized excitations in the theory on the emergent sphere. We will compute the degree of localization associated to given configurations, and use this as a guide for assembling basis vectors that might span the subspace associated to geometric regions.

Here we will again utilize the height operator (5.7). For a given configuration $v \in V_{3N^2}$ we will

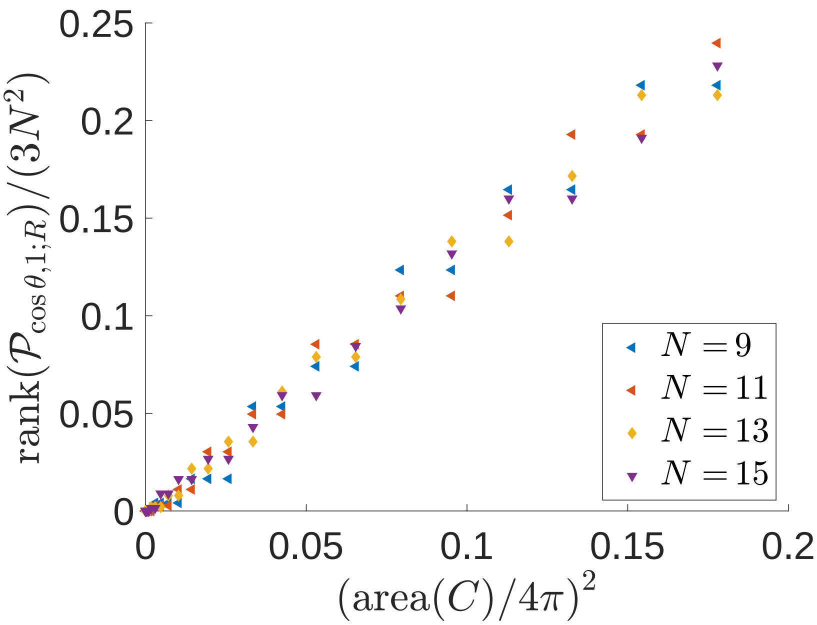


Figure 5.4: The quantity $\mathrm{rank}(\mathcal{P}_{\cos\theta,1;R})/(3N^2)$ is presented as a function of the squared fractional area of the corresponding spherical cap on the noncommutative sphere. The rank has been normalized by $\dim(V_{3N^2})$. The relationship is linear, as anticipated from the arrangement of matrix degrees of freedom and their connection to geometric degrees of freedom on the emergent sphere.

compute the variance,

$$\langle(\Delta\mathcal{Z})^2\rangle = \langle v, \mathcal{Z}^2 v\rangle - (\langle v, \mathcal{Z} v\rangle)^2. \tag{5.14}$$

It is worth stressing here that $\mathcal{Z}$ is not a quantum mechanical operator. The interpretation of $\mathcal{Z}$ as a random variable arises from the identification between the vector space V_{3N^2} and the space of fields on the sphere. In this sense we are viewing the field associated to each v, living on the noncommutative sphere, as a probability density function over the spherical surface. By 'localization' of the matrix configuration, we refer to the extent of the spatial spread of this associated probability density function.

With the ability to compute $\langle(\Delta\mathcal{Z})^2\rangle$, we may iteratively search for well-localized configurations throughout the height of the sphere. We implement this by searching for $v \in V_{3N^2}$ for which $\langle(\Delta\mathcal{Z})^2\rangle$

is minimized, while subject to the constraints,

$$\begin{aligned}\langle v, v\rangle &= 1,\\ \langle v, \mathcal{Z}v\rangle &= \bar{z}.\end{aligned} \tag{5.15}$$

These constraints limit our search to unit normalized configurations, or simply directions in V_{3N^2}, and to configurations that are centred at a height of value $\bar{z} \in \mathbb{R}$ on the emergent sphere. This problem is now phrased as a constrained minimization, for which we may make straightforward application of the `fmincon` built-in routine in MATLAB.

In Figure 5.5, this constrained minimization is carried out for various values of N. The results

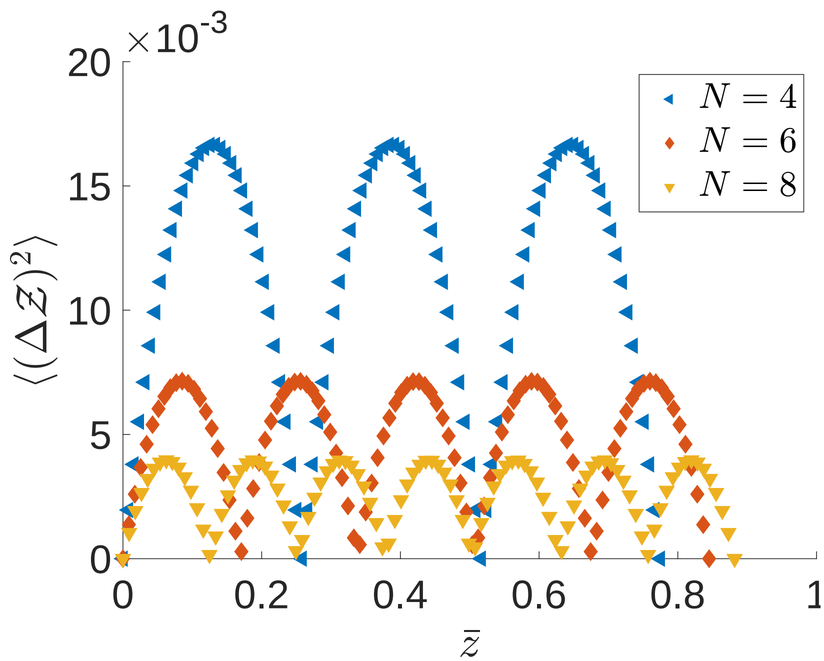


Figure 5.5: Constrained minimization applied to the height operator, $\mathcal{Z}$, in the vector space V_{3N^2}. For a range of centering height constraints, $\bar{z}$, the configuration v having minimal variance, $\langle(\Delta\mathcal{Z})^2\rangle$ is found through MATLAB's `fmincon` routine. The variance for this optimal v is plotted on the vertical axis.

may be understood as follows. For any value of the constraint $\bar{z}$, the optimal v configuration is simply the linear combination of the two eigenvectors of $\mathcal{Z}$ that are nearest to $\bar{z}$ from above and below. This follows from the fact that $\mathcal{Z}$ and $\mathcal{Z}^2$ are commuting operators. In the basis $\{e_i\}$ in which we are performing these numerical computations, both matrices $\mathbf{Z}$ and $\mathbf{Z}^2$ are diagonal. The linear combination is the one which satisfies,

$$\bar{z} = \langle (\alpha_i v_i + \alpha_{i+1} v_{i+1}), \mathcal{Z} (\alpha_i v_i + \alpha_{i+1} v_{i+1}) \rangle, \tag{5.16}$$

$$= (\alpha_i)^2 z_i + (\alpha_{i+1})^2 z_{i+1}, \tag{5.17}$$

and,

$$1 = \langle (\alpha_i v_i + \alpha_{i+1} v_{i+1}), (\alpha_i v_i + \alpha_{i+1} v_{i+1}) \rangle, \tag{5.18}$$

$$= (\alpha_i)^2 + (\alpha_{i+1})^2, \tag{5.19}$$

where α_i and α_{i+1} are the coefficients of the linear combination for the two eigenvectors v_i and v_{i+1}. The corresponding eigenvalues are z_i and z_{i+1}. It is clear that the repeated sinusoidal behaviour seen in Figure 5.5 arises from (5.19).

In fact, the triviality of this procedure foreshadows a potential pitfall of this approach; a pitfall we will encounter when attempting to repeat this analysis in the gauge-fixed model. Specifically, $\mathcal{Z}$ has a highly degenerate spectrum by virtue of the fact that each anti-diagonal, across the three x^I, lies in a shared eigenspace. This follows simply from the definition of $\mathcal{Z}$ in (5.7). When we combine the left and right actions of a diagonal L^3 upon a matrix x^I, each anti-diagonal will have a unique eigenvalue given by $2j - (a + b)$ for a matrix element $[x^I]_{ab}$. Consider the complete collection of vectors appearing for a given N generated by constrained minimization. We now see that every one of these vectors is a linear combination of just two vectors, themselves being eigenvectors from these degenerate subspaces. Optimization in this manner produces a small set of vectors, not nearly sufficient for spanning the full vector space. Although every height on the sphere $\bar{z} \in [0, 1]$ is achievable via linear combinations of these minimal $\langle (\Delta \mathcal{Z})^2 \rangle$ vectors, there are only $2N - 1$ of them, i.e. the number of unique eigenspaces of $\mathcal{Z}$. When we gauge-fix, we will be hoping that these degeneracies will be somehow broken so that the procedure may identify far more potential basis vectors.

As we have seen, this constrained minimization in V_{3N^2} is trivial, and is not the object of our search. Let us now apply this routine in the subspace of interest, V_R. To similarly analyze the emergent height of configurations in V_R, we will again utilize the projection operator $\mathcal{P}_R$ as defined in (5.10). We compute the compressions of the operators $\mathcal{Z}$, and $\mathcal{Z}^2$, to the subspace V_R,

$$\mathcal{Z}_R \equiv \mathcal{P}_R \mathcal{Z} \mathcal{P}_R, \tag{5.20}$$

$$[\mathcal{Z}^2]_R \equiv \mathcal{P}_R \mathcal{Z}^2 \mathcal{P}_R. \tag{5.21}$$

With these we will consider a height variance quantity that is approximate for the gauge-fixed theory,

$$\langle (\Delta \mathcal{Z})^2 \rangle_R \equiv \langle v, [\mathcal{Z}^2]_R v \rangle - \left(\langle v, \mathcal{Z}_R v \rangle\right)^2, \tag{5.22}$$

where now we will be considering only configurations $v \in V_R$. The minimization procedure will be again subject to the constraints (5.15), but with $\mathcal{Z}$ substituted for $\mathcal{Z}_R$.

In Figure 5.6 the results of this constrained minimization are presented for various N. There is

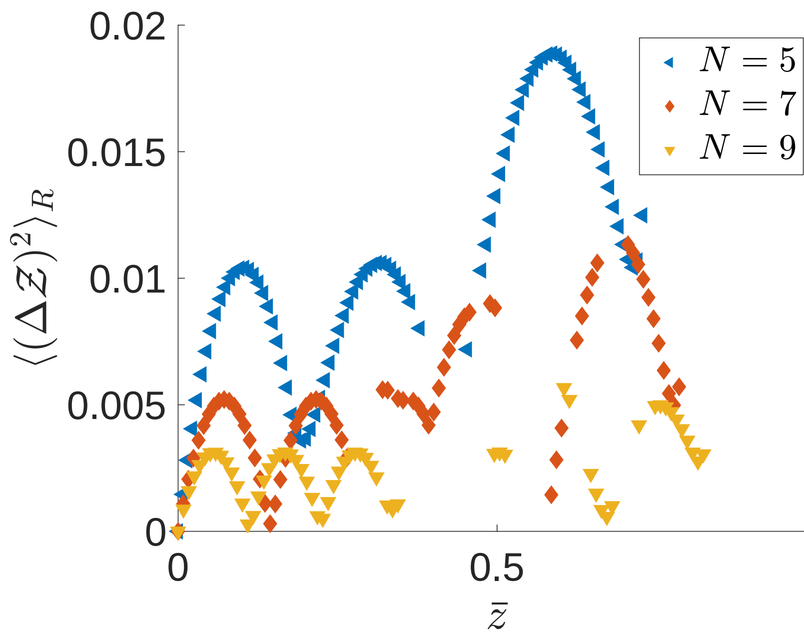


Figure 5.6: Constrained minimization applied to the projected height operators, $\mathcal{Z}$ and $[\mathcal{Z}^2]_R$, in the vector space V_R. For a range of centering height constraints, $\bar{z}$, the configuration v having minimal variance, $\langle (\Delta \mathcal{Z})^2 \rangle_R$ is found through MATLAB's `fmincon` routine. The variance for this optimal v is plotted on the vertical axis.

again some semblance of the repeated sinusoidal behaviour observed in Figure 5.5. It is difficult to ascertain whether or not this procedure is generating sufficiently many well-localized basis vectors to span V_R. Furthermore, this procedure is too computationally intensive to be reasonably extended beyond $N \approx 20$. For these reasons we will take an alternative approach to our search, employing the method of Lagrange multipliers, in the next section.

5.5 The method of Lagrange multipliers

Performing a brute-force constrained minimization to find the matrix configuration of lowest $\langle(\Delta\mathcal{Z})^2\rangle$, or of lowest $\langle(\Delta\mathcal{Z})^2\rangle_R$, is a computationally intensive procedure. Instead, we will set up the optimization problem such that we can search over a larger space of its potential solutions. We define the auxiliary function; the Lagrangian,

$$L = \langle v, [\mathcal{Z}^2]_R v\rangle - (\langle v, \mathcal{Z}_R v\rangle)^2 - \lambda(\langle v, v\rangle - 1) - \mu(\langle v, \mathcal{Z}_R v\rangle - \bar{z}). \tag{5.23}$$

The equations specifying the configuration v which achieves the minimum, while satisfying the constraints, are

$$\frac{\partial L}{\partial v_i} = 2([\mathcal{Z}^2]_R)_{ij} v_j - 4(v_k(\mathcal{Z}_R)_{kl} v_l)(\mathcal{Z}_R)_{ij} v_j - 2\lambda v_i - 2\mu\lambda[\mathcal{Z}_R]_{ij} v_j = 0, \tag{5.24}$$

$$\frac{\partial L}{\partial \lambda} = (v_i v_i - 1) = 0 \implies v_i v_i = 0, \tag{5.25}$$

$$\frac{\partial L}{\partial \mu} = (v_k(\mathcal{Z}_R)_{kl} v_l - \bar{z}) = 0 \implies v_k(\mathcal{Z}_R)_{kl} v_l = \bar{z}. \tag{5.26}$$

Substituting (5.26) into (5.24) yields,

$$2([\mathcal{Z}^2]_R)_{ij} v_j - 4(\bar{z})(\mathcal{Z}_R)_{ij} v_j - 2\lambda v_i - 2\mu(\mathcal{Z}_R)_{ij} v_j = 0 \tag{5.27}$$

$$\implies ([\mathcal{Z}^2]_R - (2\bar{z} + \mu)\mathcal{Z}_R)_{ij}\, v_j = \lambda v_i. \tag{5.28}$$

Note that in the previous section we were iteratively minimizing Δ_R for various fixed $\bar{z}$ values. In fact, we were actually minimizing only the quantity $\langle v, [\mathcal{Z}^2]_R v\rangle$ since the additional summand to Δ_R,

$$(\langle v, \mathcal{Z}_R v\rangle)^2 = \bar{z}^2, \tag{5.29}$$

was fixed as a constraint. The appearance of this term in (5.23) is redundant to the problem, and with it dropped we arrive at the following three revised equations for the minimum Δ_R configuration,

$$([\mathcal{Z}^2]_R - \mu \mathcal{Z}_R)v = \lambda v, \tag{5.30}$$

$$\langle v, \mathcal{Z}_R v \rangle = \bar{z}, \tag{5.31}$$

$$\langle v, v \rangle = 1. \tag{5.32}$$

Here (5.30) takes the form of an eigenvalue problem. This allows us to take a step back from the brute-force approach, and solve this eigenvalue equation iteratively for various values of μ, while relaxing the height constraint (5.31). For every μ, this will produce $\dim(V_R)$ many eigenvectors, v, and we may compute $\bar{z}$ for each of these. What we are doing is searching over the possible solution space - the possible values of the Lagrange multiplier μ - and generating a much broader set of configurations, $\{v\}$. Our hope is that we can implement some binning procedure upon $\{v\}$ according to the computed $\bar{z}$, and perhaps see a structure emerge that provides insight into the as-of-yet ambiguous results of Figure 5.6.

It turns out that no there is no particularly clever binning procedure that can yield results for Δ_R as a function of $\bar{z}$ reproducing those of Figure 5.6. The reason for this is that throughout the range of $\bar{z}$ on the sphere, there exists a single minimizing μ solution for large subsets of that range. We can gain some understanding of why this is the case by considering once more the trivial case of minimization in V_{3N^2} with $\mathcal{Z}$ and $\mathcal{Z}^2$. Suppose we apply the Lagrange multiplier approach in this context, reproducing (5.30-5.32), i.e.

$$(\mathcal{Z}^2 - \mu \mathcal{Z})v = \lambda v, \tag{5.33}$$

$$\langle v, \mathcal{Z} v \rangle = \bar{z}, \tag{5.34}$$

$$\langle v, v \rangle = 1. \tag{5.35}$$

Recall that the solution for v is given by precisely the linear combination $v = \alpha_i v_i + \alpha_{i+1} v_{i+1}$, as characterized in (5.16-5.19). The eigenvalues of $\mathcal{Z}$ are fully determined by our choice of $SU(2)$ representation,

$$\mathrm{spec}(\mathcal{Z}) = \{-(N-1), -(N-1)+1, ..., (N-1)-1, (N-1)\}\,. \tag{5.36}$$

Given the linear independence of the eigenvectors of $\mathcal{Z}$, v_i and v_{i+1}, the optimization equation

(5.33) consists of the two equations,

$$z_i^2 v_i - \mu z_i v_i = \lambda v_i \tag{5.37}$$

$$z_{i+1}^2 v_{i+1} - \mu z_{i+1} = \lambda v_{i+1}, \tag{5.38}$$

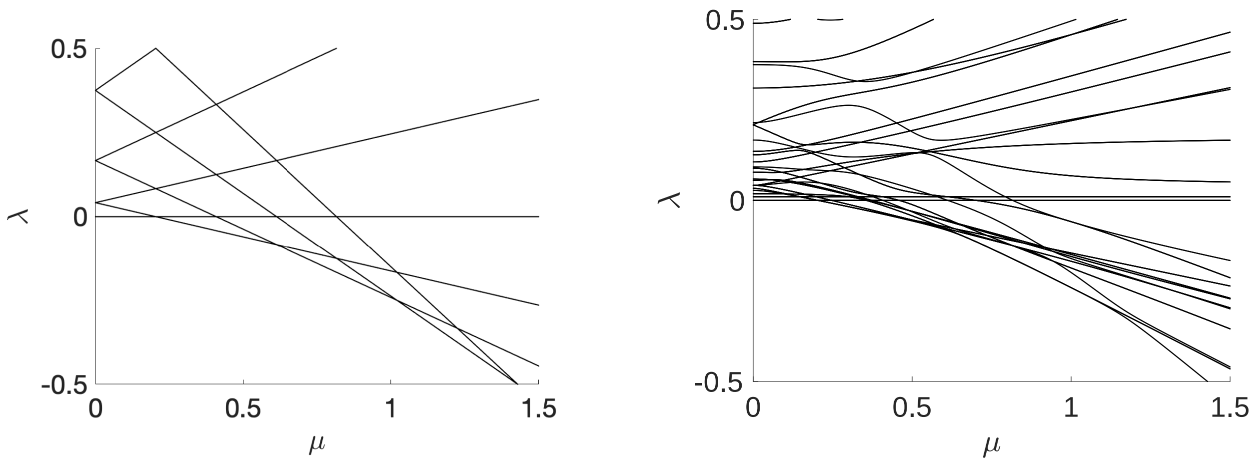
where z_i, $z_{i+1} \in \text{spec}(Z)$, and are the respective eigenvalues of $\mathcal{Z}$ for v_i and v_{i+1}. Let $z_{i+1} = m$, $z_i = m-1$, and suppose we are solving the optimization problem for the constraint of $z_i \leq \bar{z} \leq z_{i+1}$. The explicit solution for Lagrange multipliers λ and μ follows as,

$$\lambda = m(1-m), \tag{5.39}$$

$$\mu = 2m - 1. \tag{5.40}$$

For heights $\bar{z}$ that lie between each adjacent pair of eigenvalues, $(z_i, z_{i+1}) = (m-1, m) \in \text{spec}(\mathcal{Z})$, the optimization problem is solved by these Lagrange multipliers (λ, μ). This provides a different perspective on the degeneracy problem discussed above. There are only $2N-2$ pairs of (λ, μ) from which optimal v can be found for the entire range of $\bar{z}$. This cannot yield a sufficient quantity of basis vectors.

Now we turn to the problem in terms of $\mathcal{Z}_R$ and $[\mathcal{Z}^2]_R$ (5.30-5.32). In Figure 5.7 the Lagrange multiplier search approach is presented by plotting each of $\text{spec}(\mathcal{Z}^2 - \mu\mathcal{Z})$ and $\text{spec}([\mathcal{Z}^2]_R - \mu\mathcal{Z}_R)$ as functions of μ. In each case, the matrix size of $N = 5$ has been used. In Figure 5.7a the evolution of $\text{spec}(\mathcal{Z}^2 - \mu\mathcal{Z})$ with μ is seen to just be a series of intersecting lines. This follows simply from the fact that $\mathcal{Z}^2$ and $\mathcal{Z}$ are commuting operators, and so they share an eigenbasis. At $\mu = 0$, the values of λ are the eigenvalues of $\mathcal{Z}^2$, and as μ increases these values split into lines of positive slope and negative slope, associated with the positive and negative pairs of $\mathcal{Z}$ eigenvalues respectively. When two lines of negative slope intersect, it corresponds to a value of μ - and subsequently λ - where (5.37) and (5.38) are both satisfied. Equivalently, it means that the two lowest eigenvalues of $(\mathcal{Z}^2 - \mu\mathcal{Z})$ have become degenerate, on top of the already discussed degeneracy of this spectrum originating in the anti-diagonal structure of the action on matrices. Since $\mathcal{Z}^2$ and $\mathcal{Z}$ commute, the eigenbasis of $(Z^2 - \mu Z)$ is independent of μ, and so we see that the optimal linear combinations, $v = \alpha_i v_i + \alpha_{i+1} v_{i+1}$, are each eigenvectors of $(\mathcal{Z}^2 - \mu\mathcal{Z})$ lying in these series of degenerate eigenspaces, appearing at the special values of μ given by (5.40). From this perspective, the fact that the optimal configurations for any height $z_i \leq \bar{z} \leq z_{i+1}$ are supported entirely on just the two vectors v_i and v_{i+1} is re-framed as a statement about the range of heights covered by vectors lying in a particular degenerate subspace of $(\mathcal{Z}^2 - \mu\mathcal{Z})$. Each intersection between a



(a) spec($\mathcal{Z}^2 - \mu\mathcal{Z}$).

(b) spec($[\mathcal{Z}^2]_R - \mu\mathcal{Z}_R$).

Figure 5.7: The spectra of the operators spec($\mathcal{Z}^2 - \mu\mathcal{Z}$), and spec($[\mathcal{Z}^2]_R - \mu\mathcal{Z}_R$), are each plotted as functions of μ.

pair of negative slope lines in Figure 5.7a corresponds to a region in $\bar{z}$ where one of the curves of Figure 5.5 increases from 0 in its value of $\langle(\Delta\mathcal{Z})^2\rangle$, reaches a local maximum, and then falls back down to 0.

Conversely, Figure 5.7b does not admit such a determined understanding. Here spec($[\mathcal{Z}^2]_R - \mu\mathcal{Z}_R$) is far less degenerate. The operators $[\mathcal{Z}^2]_R$ and $\mathcal{Z}_R$ are not commuting, and so the eigenbasis for ($[\mathcal{Z}^2]_R - \mu\mathcal{Z}_R$) is dependent on μ. However, we do observe that the solutions to the constrained minimization problem are again obtained from vectors v which lie in a sequence of degenerate eigenspaces of ($[\mathcal{Z}^2]_R - \mu\mathcal{Z}_R$). Like for ($\mathcal{Z}^2 - \mu\mathcal{Z}$), these degenerate eigenspace appear in Figure 5.7b as points where two monotonically decreasing lines intersect, and their corresponding shared eigenvalue is the lowest in spec($[\mathcal{Z}^2]_R - \mu\mathcal{Z}_R$). The range of $\bar{z}$ associated to each of these points of special degeneracy is given by,

$$\langle v_i^{(\mu)}, \mathcal{Z}_R v_i^{(\mu)}\rangle \leq \bar{z} \leq \langle v_{i+1}^{(\mu)}, \mathcal{Z}_R v_{i+1}^{(\mu)}\rangle \tag{5.41}$$

where $v_i^{(\mu)}$ and $v_{i+1}^{(\mu)}$ are eigenvectors of ($[\mathcal{Z}^2]_R - \mu\mathcal{Z}_R$); the superscript denoting that they belong to an eigenbasis that is changing under the parameter μ. There continue to be exactly $2(N-1)$ of these intersection points from which the degenerate eigenspaces collectively possess all of the optimal configurations across the full range in height of the sphere.

We can conclude that this approach will definitely not provide a sufficient basis for V_R. The space of V_G is at least of $\mathcal{O}(N^2)$, coinciding with the space parameterized by the Hermitian matrix M of (4.32). Our investigation here has led to the conclusion that this constrained minimization will consistently identify a set of vectors that is at most $\mathcal{O}(N)$. If we wish to identify an explicit basis for V_R to use in an entanglement entropy calculation, we will need to find a different approach.

Chapter 6

Final remarks

The exact nature of spacetime emergence, and its origin in quantum mechanical entanglement, is fundamentally obscured to us. Matrix models provide a pure example of emergence, where we begin with a fully non-geometrical system, and discover a theory on a continuous spacetime. In this thesis, we have explored the viability of a calculation - of geometrical entanglement entropy - that would provide insight into the nature of of this emergence. In a matrix model that possesses gauge symmetry, the implied null space associated to the pure gauge transformations presents an obstacle for computing entanglement entropy. We have provided a prescription for approaching this calculation while restricted to the gauge-fixed theory, thus side-stepping this obstacle. The operator $\mathcal{P}_{h_1,h_2;R}$ defines a projection onto a subspace of the matrix model's configuration space that can be associated with a geometrical region, the spherical band of height (h_1, h_2), in the space on which the emergent field theory is defined. Importantly the particular subspace, range$(\mathcal{P}_{h_1,h_2;R})$, lies in the space of physically distinct classical solutions, i.e. with the directions of gauge symmetric transformations projected-out. However, there is necessarily some loss in this process. In the definition (5.12), of $\mathcal{P}_{h_1,h_2;R}$, we are implementing a rounding procedure so as to define an orthogonal projector and hence clearly decompose the space of matrix configurations into 'inside' and 'outside' of the geometrical region (h_1, h_2). In our inspections of the before-rounding operator $\mathcal{O}_{h_1,h_2;R}$, as well as of the dimensionality of range$(\mathcal{P}_{h_1,h_2;R})$, we have observed behaviour that is consistent with these linear maps admitting geometrical interpretations to their spectra and ranges. A clear next step for this work is to use $\mathcal{P}_{h_1,h_2;R}$ as the arbiter of a tracing-out procedure for computing entanglement entropy in the quadratic approximation to this matrix model.

Within V_R there are certain vectors that correspond more precisely than others with specific locations on the emergent sphere. This fact is relevant and worth exploring quantitatively because we know that by construction the operator $\mathcal{P}_{h_1,h_2;R}$ introduces some blurring - by rounding - of an already fuzzy geometry. In Section 5.4 and Section 5.5 we sought a method for identifying these well-localized vectors, and to acquire a more granular view of how gauge-fixing reshapes the matrix configuration space.

At the end of Section 4.3 we discussed how regions in the emergent non-commutative geometry have a non-standard relationship to the quantity of matrix model degrees of freedom underlying them. It's possible that degrees of freedom are not distributed uniformly - as is the typical case in local field theory - and are organized in some way that leads to boundary law entropy growth for sufficiently large regions. Matrix configurations are associated to averages of fields on the sphere, and so we might reasonably aim to identify a basis of configurations where each element is associated to a well-localized emergent field. Of course this picture is further complicated when gauge-fixing enters, presumably distorting any geometric identifications. However, in Section 5.4 we saw that an approach to identifying these basis vectors using constrained optimization of the variance in their height, $\langle(\mathcal{Z})^2\rangle$ is in fact a hopeless endeavour before gauge-fixing. The degeneracy of the $\mathcal{Z}$ operator and the fact that it trivially commutes with $\mathcal{Z}^2$ renders us unable to identify sufficiently many well-localized basis elements. When we gauge-fix though, and consider optimization in $\langle(\mathcal{Z})^2\rangle_R$, there is a possibility of breaking that degeneracy and collecting many more elements. Unfortunately, the perspective offered by the Lagrange multiplier approach of Section 5.5 shows that similar limitations continue to arise when optimizing within V_R.

A possible next step for furthering the search for a well-localized basis could be to combine constrained optimization with a height-restriction before gauge-fixing, much as we did in Section 5.3 when constructing $\mathcal{O}_{h_1,h_2;R}$ in (5.11). Suppose a height restriction to the height operator itself, taking the form,

$$\mathcal{Z}_{h_1,h_2} \equiv (\mathcal{P}_{h_1,h_2})\,\mathcal{Z}\,(\mathcal{P}_{h_1,h_2})\,. \tag{6.1}$$

Then, as in (5.20) we may consider the subsequent projection to V_R,

$$[\mathcal{Z}_{h_1,h_2}]_R \equiv (\mathcal{P}_R)\,\mathcal{Z}_{h_1,h_2}\,(\mathcal{P}_R)\,. \tag{6.2}$$

Constrained optimization can be carried out for a pre-gauge-fixing height-restriction with the variance,

$$\langle (\mathcal{Z}_{h_1,h_2})^2 \rangle_R \equiv \langle v, [\mathcal{Z}^2_{h_1,h_2}]_R v \rangle - \left(\langle v, [\mathcal{Z}_{h_1,h_2}]_R v \rangle \right)^2 , \tag{6.3}$$

as its object. These two projections, $\mathcal{P}_{h_1,h_2}$ and $\mathcal{P}_R$ are non-commutative, and thus we are right to anticipate that the results from constrained optimization will be fundamentally different from what we've seen thus far. In some sense, the reverse ordering, of gauge-fixing to V_R followed by restricting height is what is explored in Figure 5.6, if we imagine looking at a limited range of the horizontal, $\bar{z} = \langle v, \mathcal{Z}_R v \rangle$, axis.

This ordering of projections opens up the possibility of generating far more optimized vectors, as we'll be able to run constrained optimization for subsets of (h_1, h_2) height restriction. Suppose we segment the emergent sphere into a sequence of bands, and then run constrained optimization for $\langle (\mathcal{Z}_{h_1,h_2})^2 \rangle_R$ in each band. The projection to a single band involves discarding a large fraction of configuration space, and this could certainly be a limitation upon finding a sufficiently basis of well-localized vectors. We also expect that for regions of smaller area that those optimally localized vectors identified will be possess greater variance in height. Still, a less-localized basis is better than no basis, and certainly better than the arbitrariness of whatever basis is returned numerically in the construction of $\mathcal{P}_{h_1,h_2;R}$. It is an avenue worth exploring.

Computation of entanglement entropies in gauge theories more generally is a possible extension to this work. There are various proposals for this general problem [19–21]. However, these investigations inevitably encounter a separate problem that is characteristic to quantum field theories with gauge symmetry, namely that it is non-trivial to define entanglement entropy in the first place. The core issue is that in a gauge theory, there is no unique set of quantum mechanical operators that can be associated with a given region of the space. Thus, there are several natural entropic measures that could be associated to the region. A lack of clarity exists as to which entanglement entropy is the appropriate one to be calculating in different contexts. In [27] an attempt to clarify the entropic quantity computed in each proposal is made, given the sub-algebra choice. In the case of a noncommutative gauge theory emerging from a matrix model, we may be able to realize these choices concretely as subspaces of the matrix model configuration space, and thus provide further clarity. Entanglement entropy in continuum quantum field theory more generally is often obscured by the failure of the Hilbert space to factorize into a direct product of degrees of freedom within a region, and those outside that region [28]. It is possible that the duality with matrix quantum mechanics can provide a way of performing this factorization based on the structure within the matrix theory, and shed light on this important problem in quantum field theory.

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Appendix A

Matrix model equation of motion

In this appendix we will derive the equations of motion for the theory of the three matrix quantum mechanics defined by the Lagrangian,

$$L = \mathrm{tr}\left(\sum_{I=1}^{3}(\dot{X}^I)^2\right) - V(X^I), \tag{A.1}$$

with,

$$V(X^I) = \mathrm{tr}\left(-\frac{1}{4}\sum_{I,J}[X^I, X^J][X^I, X^J] - \mu^2\sum_{I=1}^{3}(X^I)^2\right). \tag{A.2}$$

The equations of motion are given by the Euler-Lagrange equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{X}^I} = \frac{\partial V}{\partial X^I} \tag{A.3}$$

Approaching this piece by piece, for the kinetic term we have,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{X}^I_{\alpha\beta}} = \frac{d}{dt}\frac{\partial}{\partial \dot{X}^I_{\alpha\beta}}\left(\operatorname{tr}\left(\sum_{I=J}^{3}(\dot{X}^J)^2\right)\right) \tag{A.4}$$

$$= \frac{d}{dt}\frac{\partial}{\partial \dot{X}^I_{\alpha\beta}}\left(\operatorname{tr}\left((\dot{X}^J)^\dagger \dot{X}^J\right)\right) \tag{A.5}$$

$$= \frac{d}{dt}\frac{\partial}{\partial \dot{X}^I_{\alpha\beta}}\left(\dot{X}^J_{ab}\dot{X}^J_{ba}\right) \tag{A.6}$$

$$= \frac{d}{dt}\left(\dot{X}^I_{\beta\alpha} + \dot{X}^I_{\beta\alpha}\right) \tag{A.7}$$

$$= 2\ddot{X}^I_{\beta\alpha}. \tag{A.8}$$

The derivative of the mass term in the potential of (A.2) is of the same form,

$$\frac{\partial}{\partial X^I_{\alpha\beta}}\left(-\operatorname{tr}\left(\mu^2\sum_{I=1}^{3}(X^I)^2\right)\right) = -2\mu^2 X^I_{\beta\alpha}. \tag{A.9}$$

This leaves the squared commutator term,

$$\frac{\partial}{\partial X^I_{\alpha\beta}}\left(\operatorname{tr}\left(-\frac{1}{4}\sum_{JK}[X^J,X^k][X^J,X^k]\right)\right) = \frac{\partial}{\partial X^I_{\alpha\beta}}\left(\operatorname{tr}\left(-\frac{1}{2}\sum_{J\neq I}[X^I,X^J][X^I,X^J]\right)\right) \tag{A.10}$$

$$= -\sum_{J\neq I}\operatorname{tr}\left(\frac{\partial}{\partial X^I_{\alpha\beta}}\left([X^I,X^J]\right)[X^I,X^J]\right) \tag{A.11}$$

$$= -\sum_{J\neq I}\left[X^J,\left[X^I,X^J\right]\right]_{\beta\alpha} \tag{A.12}$$

In the second line we have used the product rule and cyclicity of the trace. Putting everything together we have the equation of motion for $X^I_{\alpha\beta}$,

$$2\ddot{X}^I_{\beta\alpha} + \sum_{J\neq I}\left[X^J,\left[X^I,X^J\right]\right]_{\beta\alpha} + 2\mu^2 X^I_{\beta\alpha} = 0. \tag{A.13}$$

We observe that the classical solution is provided by any X^I satisfying,

$$[X^I,X^J] = i\mu\epsilon^{IJK}X^K, \tag{A.14}$$

with the mass parameter deforming the $\mathfrak{su}(2)$ algebra. Explicitly, we can see that this is an extremum of the potential,

$$\frac{\partial V}{\partial X^I} = -\sum_{J\neq I}\left[X^J, \left[X^I, X^J\right]\right] - 2\mu^2 X^I \tag{A.15}$$

$$= +2\mu^2 X^I - 2\mu^2 X^I \tag{A.16}$$

$$= 0 \tag{A.17}$$

Appendix B

Background expansion of the matrix model

In this appendix we will explicitly perform the expansion of the matrix model about the classical fuzzy sphere solution of

$$X^I = \mu L^I \tag{B.1}$$

where L^I are the generators of the N-dimensional irreducible representation of $SU(2)$, and μ is the mass parameter appearing in the Lagrangian for this model. We will set $\mu = 1$. We will parameterize the fluctuations above this solution with Hermitian matrices αx^I where $\alpha \ll 1$.

$$V(L^I + \alpha x^I) = \mathrm{tr}\left(\sum_{IJ} \frac{1}{4}[L^I + \alpha x^I, L^J + \alpha x^J][L^I + \alpha x^I, L^J + \alpha x^J] + \sum_{I=1}^{3} \left(L^I + \alpha x^I\right)^2\right). \tag{B.2}$$

We begin by considering the expanded mass term,

$$\mathrm{tr}\left(\sum_{I=1}^{3} \left(L^I + \alpha x^I\right)^2\right) = \mathrm{tr}\left(\sum_{I=1}^{3} \left((L^I)^2 + 2\alpha L^I x^I + (\alpha x^I)^2\right)\right). \tag{B.3}$$

Consider one of the commutator factors appearing in (B.2),

$$
\begin{aligned}
[L^I + \alpha x^I, L^J + \alpha x^J] &= [L^I, L^J] + \alpha\left([x^I, L^J] - [x^J, L^I]\right) + \alpha^2 [x^I, x^J] && \text{(B.4)}\\
&= i\epsilon^{IJK} L^K + \alpha\left([x^I, L^J] - [x^J, L^I]\right) + \alpha^2 [x^I, x^J] && \text{(B.5)}
\end{aligned}
$$

A single term in the sum over squared commutators becomes,

$$
\begin{aligned}
&\frac{1}{4}\mathrm{tr}\Bigg(\left(i\epsilon^{IJK} L^K + \alpha\left([x^I, L^J] - [x^J, L^I]\right) + \alpha^2 [x^I, x^J]\right) && \text{(B.6)}\\
&\qquad \times \left(i\epsilon^{IJL} L^L + \alpha\left([x^I, L^J] - [x^J, L^I]\right) + \alpha^2 [x^I, x^J]\right)\Bigg)
\end{aligned}
$$

which we truncate at quadratic order in α,

$$
\begin{aligned}
\approx \frac{1}{4}\mathrm{tr}\Bigg(& i\epsilon^{IJK} L^K \left(i\epsilon^{IJL} L^L + \alpha\left([x^I, L^J] - [x^J, L^I]\right) + \alpha^2 [x^I, x^J]\right)\\
&+ \alpha\left([x^I, L^J] - [x^J, L^I]\right)\left(i\epsilon^{IJL} L^L + \alpha\left([x^I, L^J] - [x^J, L^I]\right)\right) + \alpha^2 [x^I, x^J]\left(i\epsilon^{IJL} L^L\right)\Bigg).
\end{aligned}
\tag{B.7}
$$

From this vantage point, we can see that the entire expression will have the following terms that are constant,

$$
\begin{aligned}
\mathcal{O}(\alpha^0) &= \mathrm{tr}\left(-\sum_{IJ} \frac{1}{4}\epsilon^{IJK} L^K \epsilon^{IJL} L^L + \sum_{I=1}^{3} (L^I)^2\right) && \text{(B.8)}\\
&= \mathrm{tr}\left(\sum_{I=1}^{3}\left(-\frac{1}{2}(L^I)^2 + (L^I)^2\right)\right) && \text{(B.9)}\\
&= \frac{1}{2}\mathrm{tr}\left(\sum_{I=1}^{3} (L^I)^2\right) && \text{(B.10)}\\
&= \frac{1}{4}(j^2 - 1)N, && \text{(B.11)}
\end{aligned}
$$

which we will drop from the result. Next we consider the terms at linear order in α,

$$\mathcal{O}(\alpha) = \mathrm{tr}\Bigg(\frac{1}{4}\sum_{IJ}\left(i\epsilon^{IJK}L^K\alpha\left([x^I,L^J]-[x^J,L^I]\right)+\alpha\left([x^I,L^J]-[x^J,L^I]\right)\left(i\epsilon^{IJL}L^L\right)\right) \tag{B.12}$$

$$+\sum_{I=1}^{3}\left(2\alpha L^I x^I\right)\Bigg) \tag{B.13}$$

$$= \alpha\ \mathrm{tr}\left(\frac{i}{2}\sum_{IJ}\left(\epsilon^{IJK}L^K\left([x^I,L^J]-[x^J,L^I]\right)\right)+2\sum_{I=1}^{3}\left(L^I x^I\right)\right) \tag{B.14}$$

$$= \alpha\ \mathrm{tr}\left(\frac{i}{2}\sum_{IJ}\left(\epsilon^{IJK}\left([L^J,L^K]x^I-[L^I,L^K]x^J\right)\right)+2\sum_{I=1}^{3}\left(L^I x^I\right)\right) \tag{B.15}$$

$$= \alpha\ \mathrm{tr}\left(-\frac{1}{2}\sum_{IJ}\left(L^I x^I+L^J x^J\right)+2\sum_{I=1}^{3}\left(L^I x^I\right)\right) \tag{B.16}$$

$$= 0. \tag{B.17}$$

We are left with the terms quadratic in α defining the expanded potential,

$$V_2(x^I) \equiv \mathrm{tr}\Bigg(\frac{1}{4}\sum_{IJ}\Big(i\epsilon^{IJK}L^K\alpha^2[x^I,x^J]+\alpha^2\left([x^I,L^J]-[x^J,L^I]\right)\left([x^I,L^J]-[x^J,L^I]\right) \tag{B.18}$$

$$+\alpha^2[x^I,x^J]\left(i\epsilon^{IJL}L^L\right)\Big)+\sum_{I=1}^{3}(\alpha x^I)^2\Bigg) \tag{B.19}$$

$$= \alpha^2\ \mathrm{tr}\Bigg(\frac{1}{4}\sum_{IJ}\left(2i\epsilon^{IJK}L^K[x^I,x^J]+\left([x^I,L^J]-[x^J,L^I]\right)\left([x^I,L^J]-[x^J,L^I]\right)\right)+\sum_{I=1}^{3}(x^I)^2\Bigg). \tag{B.20}$$

Fully expanded we have,

$$V_2(x^I) = 2\alpha^2\ \mathrm{tr}\Big(iL^3x^1x^2-iL^3x^2x^1+(x^1)^2-L^1x^2x^2L^1-2L^2x^1x^1L^2+2L^1x^2L^1x^2$$
$$+L^2x^1L^2x^1+x^1L^2L^1x^2+L^2x^1x^2L^1-L^2x^1L^1x^2-x^1L^2x^2L^1$$
$$+\text{cyclic permutations}\Big). \tag{B.21}$$

Appendix C

Proof regarding the null space of K

We have shown that for the theory of a scalar field upon the noncommutative sphere, represented by the matrix model described in Section 4.3, there is in the massless case, a 1D subspace of configurations that constitute degenerate classical minima of the theory. These are precisely those matrices proportional to the identity,

$$\phi \propto \mathbb{1}. \tag{C.1}$$

This is readily apparent from the expression for the potential, with $m = 0$,

$$V(\phi) = \mathrm{tr}\left(\sum_{I=1}^{3}[L^I, \phi]^2\right), \tag{C.2}$$

which clearly vanishes for any such matrix ϕ. In this appendix, we will demonstrate that the coupling matrix, $K^{(\mu)}$, will have a null eigenvalue associated to this subspace of configurations with vanishing potential energy. First, let us consider the associated bilinear form to the potential of (C.2), $g : V \times V \to \mathbb{C}$, where V is the real N^2-dimensional configuration space of ϕ. Recalling that the trace defines an inner product on the space of hermitian matrices, we may express the bilinear form g as,

$$g(\phi,\phi') \longleftrightarrow \mathrm{tr}\left(\sum_{I=1}^{3}[L^I,\phi]^\dagger[L^I,\phi']\right) \tag{C.3}$$

$$\longleftrightarrow \mathrm{tr}\left(\phi\left(\sum_{I=1}^{3}\left(-L^I\phi' L^I + \phi' L^I L^I + L^I L^I \phi' - L^I \phi' L^I\right)\right)\right) \tag{C.4}$$

$$\longleftrightarrow \langle \phi, T\phi'\rangle \tag{C.5}$$

where $T \in \mathcal{L}(V)$. The linear operator T is uniquely defined because the matrices L^I are themselves belonging to the unique N-dimensional irreducible representation of $\mathfrak{su}(2)$. It is apparent that T is self-adjoint, or rather symmetric, given that we're working with a real vector space. This follows from the symmetry of g in conjunction with the symmetry of the inner product,

$$g(\phi,\phi') \longleftrightarrow \langle \phi, T\phi'\rangle \tag{C.6}$$

$$\longleftrightarrow \langle T^*\phi, \phi'\rangle \quad \text{(def. of adjoint)} \tag{C.7}$$

$$\longleftrightarrow \langle \phi', T^*\phi\rangle \quad \text{(sym. of inner prod.)} \tag{C.8}$$

$$\longleftrightarrow \langle \phi, T^*\phi'\rangle \quad \text{(sym. of } g\text{)}, \tag{C.9}$$

where T^* is the adjoint of T, and equivalence of (C.6) and (C.9) demonstrate that $T = T^*$. We note that T is positive semi-definite, following from the observation that the argument to the trace appearing in (C.2) is of the form, $A = B^\dagger B$, for $B \in \mathcal{L}(V)$. Furthermore, following from T being self-adjoint, its associated matrix is orthogonally diagonalizable. Thus we can write,

$$T\phi = \sum_j \lambda_j \phi_j \tag{C.10}$$

for any $\phi \in V$. As previously stated, the set of ϕ where $\phi \propto \mathbb{1}$ constitutes a 1D subspace of V, which we'll now denote as W, upon which $V(\phi) = 0$. Correspondingly, for the quadratic form associated to the bilinear g, and for $\phi_w \in W$, we have,

$$g(\phi_w,\phi_w) \longleftrightarrow \langle \phi_w, T\phi_w\rangle = 0 \tag{C.11}$$

$$\longleftrightarrow \sum_j \lambda_j ||\phi_j||^2 = 0. \tag{C.12}$$

In the final line, positive semi-definiteness of T implies that the only consistent possibility is $\lambda_j = 0$ for all j, i.e. that ϕ_w lies in a null eigenspace of T. We conclude that the matrix $K^{(\mu)}$ of this linear map T indeed has a null eigenvalue associated to the 1D subspace of matrix configurations that are $\propto \mathbb{1}$.