

Article

General Relativistic Aberration Equation and Measurable Angle of Light Ray in Kerr–de Sitter Spacetime

Hideyoshi Arakida 

College of Engineering, Nihon University, Koriyama, Fukushima 963-8642, Japan;
arakida.hideyoshi@nihon-u.ac.jp

Abstract: As an extension of our previous paper, instead of the total deflection angle α , we will mainly focus on the discussion of measurable angle of the light ray ψ_P at the position of observer P in Kerr–de Sitter spacetime, which includes the cosmological constant Λ . We will investigate the contribution of the radial and transverse motion of the observer which are connected with radial velocity v^r and transverse velocity bv^ϕ (b is the impact parameter) as well as the spin parameter a of the central object which induces the gravito-magnetic field or frame dragging and the cosmological constant Λ . The general relativistic aberration equation is employed to take into account the influence of motion of the observer on the measurable angle ψ_P . The measurable angle ψ_P derived in this paper can be applicable to the observer placed within the curved and finite-distance region in the spacetime. The equation of light trajectory will be obtained in such a sense that the background is de Sitter spacetime instead of Minkowski one. As an example, supposing the cosmological gravitational lensing effect, we assume that the lens object is the typical galaxy and the observer is in motion with respect to the lensing object at a recession velocity $v^r = bv^\phi = v_H = H_0 D$ (where H_0 is a Hubble constant and D means the distance between the observer and the lens object). The static terms $\mathcal{O}(\Lambda b m, \Lambda b a)$ are basically comparable with the second order deflection term $\mathcal{O}(m^2)$, and they are almost one order smaller than the Kerr deflection $-4ma/b^2$. The velocity-dependent terms $\mathcal{O}(\Lambda b m v^r, \Lambda b a v^r)$ for radial motion and $\mathcal{O}(\Lambda b^2 m v^\phi, \Lambda b^2 a v^\phi)$ for transverse motion are at most two orders of magnitude smaller than the second order deflection $\mathcal{O}(m^2)$. We also find that even when the radial and transverse velocity have the same sign, asymptotic behavior as ϕ approaches 0 is different from each other, and each diverges to opposite infinity.

Keywords: bendin of light ray; cosmological constant; aberration equation; effect of motion of observer

PACS: 95.30.Sf; 98.62.Sb; 04.20.-q; 04.20.Cv



Citation: Arakida, H. General Relativistic Aberration Equation and Measurable Angle of Light Ray in Kerr–de Sitter Spacetime. *Universe* **2021**, *7*, 173. <https://doi.org/10.3390/universe7060173>

Academic Editor: Lorenzo Iorio

Received: 4 May 2021

Accepted: 26 May 2021

Published: 1 June 2021

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1. Introduction

The cosmological constant problem is an old but unsolved issue in astrophysics and cosmology that is closely related to the general theory of relativity; see reviews by, e.g., [1,2]. After the establishment of the general theory of relativity in 1915–1916, Einstein incorporated the cosmological term $\Lambda g_{\mu\nu}$ into the field equation to represent the static Universe. Although the discovery of cosmic expansion by Hubble caused Einstein to withdraw the cosmological term from the field equation, it is currently widely considered that the cosmological constant Λ , or the dark energy in a more general sense, is the most promising candidate for explaining the observed accelerating expansion of the Universe [3–5] despite the fact that its details are not at all clear.

One straightforward way to tackle the cosmological constant problem from another viewpoint is to investigate the effect of the cosmological constant Λ on the light deflection. The deflection of a light ray is the basis of gravitational lensing which is a powerful tool in astrophysics and cosmology; see, e.g., [6,7] and the references therein. Particularly, in

cosmological gravitational lensing, it is important to take into account the cosmological constant to the formula of the total deflection angle α ; then it is expected to provide a new verification of the cosmological constant.

The influence of the cosmological constant Λ on the total deflection angle α had been the subject of a long debate and was investigated mainly under the static and spherically symmetric vacuum solution, namely the Schwarzschild–de Sitter/Kottler solution. See [8–24] and the references therein. However, since celestial objects such as stars and galaxies are generally rotating, then it is necessary to formulate the effect of rotation as well as the effect of the cosmological constant Λ to consider light deflection. To date, several authors have discussed the total light deflection α in Kerr–de Sitter spacetime, which is the stationary and axially symmetric vacuum solution and includes the spin parameter a of the central object and the cosmological constant Λ ; see, e.g., [25–28] and the references therein. The same consideration is further extended to the more general Kerr-type solutions, see, e.g., [29–36].

Moreover, it is also significant to incorporate the effect of the relative motion between the lens object and the observer. In many previous papers, the light deflection is mainly discussed under the assumption that the observer is stationary in space. However, in general, the observer is in motion with respect to the lens object, thus it is important to take velocity effect into account for more accurate formulation. In addition, not only to solve the problem of the cosmological constant but also to verify the theory of gravity using gravitational lensing, it is crucial to clarify how the rotation of celestial body, the cosmological constant, and the velocity of observer as well as the mass of celestial body are connected with each other and contribute to the bending of light ray.

In [37], accounting for the effect of the observer’s velocity, we examined the measurable angle of the light ray ψ_P at the position of the observer P in the Kerr spacetime instead of the discussing on the total deflection angle α . In order to deal with velocity effect, we adopted the general relativistic aberration equation,

$$\cos \psi_P = \frac{g_{\mu\nu} k^\mu w^\nu}{(g_{\mu\nu} u^\mu k^\nu)(g_{\mu\nu} u^\mu w^\nu)} + 1. \quad (1)$$

Equation (1) gives the measurable angle ψ_P at the position P between the tangent vector k^μ of the light ray Γ_k that we investigate and the tangent vector w^μ of the radial null geodesic Γ_w connecting the center O and the position of observer P , and u^μ is the 4-velocity of the observer P . See Figure 1 and for the derivation of Equation (1), refer to [20] and see also [37–39]. Equation (1) enables us to compute the effect of the motion of the observer easily and straightforwardly because the equations of the null geodesic of Γ_k and Γ_w do not depend on the motion of the observer; the velocity effect is incorporated in the formula as the form of the 4-velocity of the observer u^μ .

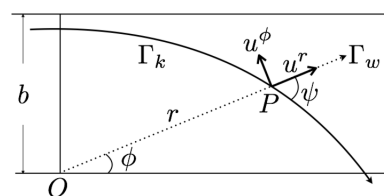


Figure 1. Schematic diagram of light trajectory. Γ_k (bold line) is the trajectory of the light ray which we now investigate, Γ_w (dotted line) is the radial null geodesic connecting the center O and the position of observer P and the measurable angle ψ is the angle of intersection between Γ_k and Γ_w at P . Two bold vectors u^r and u^ϕ at P indicate the directions of the r (radial) and ϕ (transverse) components of the 4-velocity u^μ . The direction of the time component u^t is perpendicular to this schematic plane.

In this paper, we will extend our paper [37] to Kerr–de Sitter spacetime containing the cosmological constant Λ as well as the spin parameter of the central object a . Our purpose is to examine not only the contribution of the cosmological constant Λ and the

spin parameter a of the central object but also the effect of the motion of the observer on the measurable angle ψ_P . The 4-velocity of the observer u^μ appearing in Equation (1) is converted to the coordinate radial velocity $v^r = dr/dt$ and coordinate transverse velocity $bv^\phi = bd\phi/dt$ (b is the impact parameter and $v^\phi = d\phi/dt$ denotes the coordinate angular velocity), respectively. Previously, the radial motion of the observer in Kerr–Newman and Kerr–de Sitter spacetimes have been discussed by [40,41], respectively.

Before closing this section, it is noteworthy to explain why we deal with measurable angle ψ_P instead of the total deflection angle α of light ray in this paper. This is because the concept and definition of the total deflection angle of the light ray α is a counterintuitive and difficult problem. One of the reasons for this is that because the Schwarzschild–de Sitter and Kerr–de Sitter solutions are not asymptotically flat unlike the Schwarzschild and Kerr solutions, it is ambiguous and unclear how the total deflection angle α should be defined in curved spacetime. Although to overcome this difficulty, a method for calculating the total deflection angle is independently investigated and proposed on the basis of the Gauss–Bonnet theorem by e.g., [23,24,42,43], it seems that further consideration is needed to settle the argument. However it is always possible to determine the measurable (local) angle ψ_P at the position of the observer P .

This paper is organized as follows: in Section 2, the trajectory of a light ray in Kerr–de Sitter spacetime is derived from the first-order differential equation of the null geodesic. In Section 3, the measurable angle ψ_P in Kerr–de Sitter spacetime is calculated for the cases of the static observer, the observer in radial motion and the observer in transverse motion. Finally, Section 4 is devoted to presenting the conclusions.

2. Light Trajectory in Kerr–de Sitter Spacetime

The Kerr–de Sitter solution—see Equations of (5.65) and (5.66) in [44], and also, e.g., [25–28]—in Boyer–Lindquist type coordinates (t, r, θ, ϕ) can be rearranged as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -\frac{\Delta_r - \Delta_\theta a^2 \sin^2 \theta}{\rho^2 \Xi^2} dt^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{2a \sin^2 \theta}{\rho^2 \Xi^2} [\Delta_\theta (r^2 + a^2) - \Delta_r] dt d\phi \\ &\quad + \frac{\sin^2 \theta}{\rho^2 \Xi^2} [\Delta_\theta (r^2 + a^2)^2 - \Delta_r a^2 \sin^2 \theta] d\phi^2, \\ \Delta_r &= r^2 + a^2 - 2mr - \frac{\Lambda}{3} r^2 (r^2 + a^2), \\ \Delta_\theta &= 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Xi = 1 + \frac{\Lambda}{3} a^2. \end{aligned} \quad (2)$$

$g_{\mu\nu}$ is the metric tensor; Greek indices, e.g., μ, ν , run from 0 to 3; Λ is the cosmological constant; m is the mass of the central object; $a \equiv J/m$ is a spin parameter of the central object (J is the angular momentum of the central object) and we use the geometrical unit $c = G = 1$ throughout this paper. For the sake of brevity, we restrict the trajectory of the light ray to the equatorial plane $\theta = \pi/2, d\theta = 0$, and rewrite Equation (2) in symbolic form as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + 2C(r)dt d\phi + D(r)d\phi^2, \quad (3)$$

$$A(r) = \left(1 + \frac{\Lambda}{3} a^2\right)^{-2} \left[1 - \frac{2m}{r} - \frac{\Lambda}{3} (r^2 + a^2)\right], \quad (4)$$

$$B(r) = \left[\left(1 - \frac{\Lambda}{3} r^2\right) \left(1 + \frac{a^2}{r^2}\right) - \frac{2m}{r}\right]^{-1}, \quad (5)$$

$$C(r) = -\left(1 + \frac{\Lambda}{3}a^2\right)^{-2} a \left[\frac{2m}{r} + \frac{\Lambda}{3}(r^2 + a^2) \right], \quad (6)$$

$$D(r) = \left(1 + \frac{\Lambda}{3}a^2\right)^{-2} \left[(r^2 + a^2) \left(1 + \frac{\Lambda}{3}a^2\right) + \frac{2ma^2}{r} \right]. \quad (7)$$

Two constants of motion of the light ray, the energy E and the angular momentum L , are given by

$$E = A(r) \frac{dt}{d\lambda} - C(r) \frac{d\phi}{d\lambda}, \quad L = C(r) \frac{dt}{d\lambda} + D(r) \frac{d\phi}{d\lambda}, \quad (8)$$

where λ is the affine parameter. Solving for $dt/d\lambda$ and $d\phi/d\lambda$, we obtain two relations:

$$\frac{dt}{d\lambda} = \frac{ED(r) + LC(r)}{A(r)D(r) + C^2(r)}, \quad \frac{d\phi}{d\lambda} = \frac{LA(r) - EC(r)}{A(r)D(r) + C^2(r)}. \quad (9)$$

From the null condition $ds^2 = 0$, the geodesic equation of the light ray is expressed as

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{A(r)D(r) + C^2(r)}{B(r)[bA(r) - C(r)]^2} [-b^2A(r) + 2bC(r) + D(r)], \quad (10)$$

where b is the impact parameter defined as

$$b \equiv \frac{L}{E}. \quad (11)$$

Replacing r by $u = 1/r$ and using Equations (4)–(7), and (10), the first-order differential equation of the light ray becomes¹

$$\begin{aligned} \left(\frac{du}{d\phi}\right)^2 &= \frac{\left[\left(1 - \frac{\Lambda}{3u^2}\right)(1 + a^2u^2) - 2mu\right]^2}{\left(1 + \frac{\Lambda}{3}a^2\right)^2 \left[b - 2mu(b - a) - \frac{\Lambda}{3u^2}(b - a)(1 + a^2u^2)\right]^2} \\ &\times \left[1 + (a^2 - b^2)u^2 + 2mu^3(b - a)^2 + \frac{\Lambda}{3}(b - a)^2(1 + a^2u^2)\right]. \end{aligned} \quad (12)$$

Expanding Equation (12) up to the second order with respect to m/b , a/b , and Λb^2 , we have

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{b^2} + \frac{\Lambda}{3} - u^2 + 2mu^3 - 2a^2u^4 + \frac{3a^2u^2}{b^2} - \frac{4amu}{b^3} - \frac{2\Lambda a}{3b^3u^2} + \mathcal{O}(\epsilon^3). \quad (13)$$

Note that for the sake of simplicity, we introduced the notation for the small expansion parameters m/b , a/b and Λb^2 as

$$\epsilon \sim m/b \sim a/b \sim \Lambda b^2, \quad \epsilon \ll 1, \quad (14)$$

then $\mathcal{O}(\epsilon^3)$ in Equation (13) denotes combinations of these three parameters, m/b , a/b , and Λb^2 . Henceforth, we use the same notation to represent the order of the approximation and residual terms.

It is instructive to discuss how to choose a zeroth-order solution u_0 of the light trajectory u . If $m = 0$ and $a = 0$, then Equation (13) reduces to the null geodesic equation in de Sitter spacetime

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{b^2} + \frac{\Lambda}{3} - u^2, \quad (15)$$

which can be also derived immediately from Equation (12). Because we assume a nonzero cosmological constant Λ a priori, we cannot take Λ to be zero, and Equation (15) cannot be reduced to the null geodesic equation in Minkowski spacetime; in fact the action

$$S = \int \left[\frac{c^4}{16\pi G} (R - 2\Lambda) + \mathcal{L}_M \right] \sqrt{-g} d^4x, \quad (16)$$

and the field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (17)$$

include the cosmological constant Λ explicitly; here $g = \det(g_{\mu\nu})$, \mathcal{L}_M denotes the Lagrangian for the matter field; $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar, respectively; $T_{\mu\nu}$ is the energy-momentum tensor. Because m and a are the integration or arbitrary constants in the Kerr–de Sitter solution, it is possible to put $m = 0$ and $a = 0$. Therefore, zeroth-order solution of the u of the light ray should be taken as the form

$$u_0 = \frac{\sin \phi}{B}, \quad \frac{1}{B^2} \equiv \frac{1}{b^2} + \frac{\Lambda}{3}, \quad (18)$$

in which we introduced another constant B . Note that Equation (18) should be evaluated in de Sitter spacetime.

Let us obtain the equation of the light trajectory in accordance with the standard perturbation scheme. We take the solution $u = u(\phi)$ of the light trajectory as

$$u = \frac{\sin \phi}{B} + \varepsilon \delta u_1 + \varepsilon^2 \delta u_2, \quad (19)$$

where $\varepsilon \delta u_1$ and $\varepsilon^2 \delta u_2$ are the first order $\mathcal{O}(\varepsilon)$ and second order $\mathcal{O}(\varepsilon^2)$ corrections to the zeroth-order solution $u_0 = \sin \phi / B$, respectively.

According to the standard perturbation method, we insert Equation (19) into Equation (13), and obtain the differential equation for $d\delta u_1/d\phi$ and $d\delta u_2/d\phi$ by expanding with respect to ε and combining the same order terms of ε . Then, integrating $d\delta u_1/d\phi$ and $d\delta u_2/d\phi$ iteratively, δu_1 and δu_2 are obtained, and inserting the δu_1 and δu_2 into Equation (19), the equation of the light trajectory in Kerr–de Sitter spacetime is given up to second-order $\mathcal{O}(\varepsilon^2)$ by

$$\begin{aligned} \frac{1}{r} = & \frac{\sin \phi}{B} + \frac{m}{2B^2} (3 + \cos 2\phi) \\ & + \frac{1}{16B^3b^3} \left\{ b^3 (3m^2 + 2a^2) (3 \sin \phi - \sin 3\phi) + 4b [7b^2m^2 + 6a^2(B^2 - b^2)] \sin \phi \right. \\ & + (30b^3m^2 - 12a^2b^3 + 12B^2a^2b) (\pi - 2\phi) \cos \phi - 32B^3am \left. \right\} \\ & - \frac{\Lambda B^3a(2 \sin^2 \phi - 1)}{3b^3 \sin \phi} + \mathcal{O}(\varepsilon^3), \end{aligned} \quad (20)$$

where the integration constants of δu_1 and δu_2 are chosen so as to maximize u (or to minimize r) at $\phi = \pi/2$.

We note that because the differential equations, $d\delta u_1/d\phi$ and $d\delta u_2/d\phi$, are contain two constants, B and b (see Equations (13) and (19)), and the expansion parameter ε is also related to the constant b instead of B (see Equation (14)). As a consequence, the expression of $u = 1/r$, Equation (20), includes two constants B and b , and leads to a complicated expression for the measurable angle ψ_P . To avoid the complex expression for ψ_P , we expand B in Λ and express B by b and Λ , obtaining

$$\begin{aligned} \frac{1}{r} = & \left(\frac{1}{b} + \frac{b\Lambda}{6} - \frac{b^3\Lambda^2}{72} \right) \sin \phi + \frac{m}{2} (3 + \cos 2\phi) \left(\frac{1}{b^2} + \frac{\Lambda}{3} \right) \\ & + \frac{1}{16b^3} \left\{ m^2 [37 \sin \phi + 30(\pi - 2\phi) \cos \phi - 3 \sin 3\phi] + 8a^2 \sin^3 \phi - 32am \right\} \\ & - \frac{\Lambda a (2 \sin^2 \phi - 1)}{3 \sin \phi} + \mathcal{O}(\epsilon^3). \end{aligned} \quad (21)$$

Note that if we use the approximate solution of Equation (13) used by, e.g., [27], as,

$$u = \frac{\sin \phi}{b} + \delta u_1 + \delta u_2, \quad (22)$$

the following terms in Equation (21) disappear:

$$\left(\frac{b\Lambda}{6} - \frac{b^3\Lambda^2}{72} \right) \sin \phi, \quad \frac{m\Lambda}{6} (3 + \cos 2\phi). \quad (23)$$

The existence of the above terms in Equation (21) reflects the fact that the background spacetime is de Sitter spacetime instead of Minkowski spacetime.

Before concluding this section, it is noteworthy that, unlike Schwarzschild–de Sitter spacetime, the trajectory equation of the light ray in Kerr–de Sitter spacetime depends on the cosmological constant Λ ; from the condition

$$\left. \frac{du}{d\phi} \right|_{u=u_0} = 0, \quad u_0 = \frac{1}{r_0}, \quad (24)$$

and Equation (13), we have following relation:

$$\frac{1}{B^2} = \frac{1}{b^2} + \frac{\Lambda}{3} = \frac{1}{r_0^2} - \frac{2m}{r_0^3} + \frac{2a^2}{r_0^2} - \frac{3a^2}{b^2 r_0^2} + \frac{4am}{b^3 r_0} - \frac{2\Lambda a r_0^2}{3b^3} + \mathcal{O}(\epsilon^3), \quad (25)$$

in which r_0 is the radial coordinate value of the light ray at the point of closest approach (in our case $\phi = \pi/2$), and r_0 can be obtained by the observation in principle as the circumference radius $\ell_0 = 2\pi r_0$. It is also possible to obtain a similar relation from Equations (12) and (13); but the expression becomes more complicated. Equation (25) means that unlike the Schwarzschild–de Sitter case, B cannot be expressed only by r_0 , m , and a ; Λ and b too are required. As a result, the trajectory equation of the light ray in Kerr–de Sitter spacetime depends on the cosmological constant Λ and b ; whereas the equation of the light trajectory in Schwarzschild–de Sitter spacetime is independent of Λ and b ; in fact setting $a = 0$ in Equation (25) yields

$$\frac{1}{B_{\text{Sds}}^2} = \frac{1}{b^2} + \frac{\Lambda}{3} = \frac{1}{r_0^2} - \frac{2m}{r_0^3}. \quad (26)$$

Equation (26) shows that the constant B_{Sds} in Schwarzschild–de Sitter spacetime can be determined without knowing Λ and b .

3. Measurable Angle in Kerr–de Sitter Spacetime

Henceforth, in accordance with the procedure and formulas described in [37], we calculate the measurable angle ψ_P at the position of observer P and in this paper, we only gives the final results of the measurable angle ψ_P . See [37] for details of the calculation.

3.1. Measurable Angle by Static Observer

In the case of the static observer, the component of the 4-velocity of the observer u^μ becomes

$$u^\mu = (u^t, 0, 0, 0), \quad (27)$$

and the condition for the time-like observer $g_{\mu\nu}u^\mu u^\nu = -1$ gives u^t as

$$u^t = \frac{1}{\sqrt{A(r)}}, \quad (28)$$

where we take u^t to be positive.

Expanding up to the order $\mathcal{O}(\varepsilon^2)$, the measurable angle ψ_{stat} is obtained as, for the range $0 \leq \psi \leq \pi/2$,

$$\begin{aligned} \psi_{\text{stat}} = & \phi + \frac{2m}{b} \cos \phi + \frac{1}{8b^2} \left\{ m^2 [15(\pi - 2\phi) - \sin 2\phi] - 16ma \cos \phi \right\} \\ & - \frac{\Lambda b^2}{6} \cot \phi + \frac{\Lambda b}{3} \cos \phi \left[m(1 + \csc^2 \phi) + 2a \csc \phi \right] - \frac{\Lambda^2 b^4}{288} \csc^4 \phi \sin 4\phi \\ & + \mathcal{O}(\varepsilon^3). \end{aligned} \quad (29)$$

and for the range $\pi/2 \leq \phi \leq \pi$

$$\begin{aligned} \psi_{\text{stat}} = & \pi - \phi - \frac{2m}{b} \cos \phi - \frac{1}{8b^2} \left\{ m^2 [15(\pi - 2\phi) - \sin 2\phi] - 16ma \cos \phi \right\} \\ & + \frac{\Lambda b^2}{6} \cot \phi - \frac{\Lambda b}{3} \cos \phi \left[m(1 + \csc^2 \phi) + 2a \csc \phi \right] + \frac{\Lambda^2 b^4}{288} \csc^4 \phi \sin 4\phi \\ & + \mathcal{O}(\varepsilon^3). \end{aligned} \quad (30)$$

As in [37], we divided the expression for ψ_{stat} into two cases, Equations (29) and (30). The purpose of this was to utilize trigonometric identities such as $\sqrt{1 - \sin^2 \phi} = \cos \phi$ for $0 \leq \phi \leq \pi/2$ and $\sqrt{1 - \sin^2 \phi} = -\cos \phi$ for $\pi/2 \leq \phi \leq \pi$. Henceforth we adopt a similar procedure when calculating angle measured by the observer in radial motion, ψ_{rad} and in transverse motion ψ_{trans} below. Although this procedure may not be necessary for computing ψ_{stat} and ψ_{rad} , it is required when computing ψ_{trans} ; see Equation (38) and observe the case for $\phi \rightarrow \pi$.

The first line in Equations (29) and (30) are in agreement with the measurable angle of the static observer in Kerr spacetime derived in [37], and the second line in Equations (29) and (30) are due to the influence of the cosmological constant Λ .

Before concluding this section, we note that if Equation (22) (see also Equation (23)) is adopted, the measurable angle $\tilde{\psi}_{\text{stat}}$ for the range $0 \leq \phi \leq \pi/2$ becomes

$$\begin{aligned} \tilde{\psi}_{\text{stat}} = & \phi + \frac{2m}{b} \cos \phi + \frac{1}{8b^2} \left\{ m^2 [15(\pi - 2\phi) - \sin 2\phi] - 16ma \cos \phi \right\} \\ & - \frac{\Lambda b^2}{6} \csc \phi \sec \phi + \frac{\Lambda b}{3} [m(\cot \phi \csc \phi - \sec \phi) + 2a \cot \phi] \\ & - \frac{\Lambda^2 b^4}{9} \cot 2\phi \csc^2 2\phi + \mathcal{O}(\varepsilon^3), \end{aligned} \quad (31)$$

where the first line is also in agreement with the measurable angle of the static observer in the Kerr spacetime as derived in [37].

Comparing Equations (29) and (31), we find the following: first, in spite of the different correction terms due to the cosmological constant Λ , the measurable angles ψ_{stat} and $\tilde{\psi}_{\text{stat}}$ take a large value rapidly and diverge to negative infinity when ϕ approaches 0. This property is related to the existence of the de Sitter horizon. Second, when $\phi \rightarrow \pi/2$, Equation (29) leads to the result $\psi_{\text{stat}} \rightarrow \pi/2$, which is consistent with the initial condition, $du/d\phi|_{\phi=\pi/2} = 0$. However, Equation (31) diverges, and $\tilde{\psi}_{\text{stat}} \rightarrow \infty$, which contradicts the initial condition, $du/d\phi|_{\phi=\pi/2} = 0$. Therefore, Equation (18) should be employed as the zeroth-order solution of u ; as a consequence, Equation (20) or at least Equation (21) should be used as the trajectory equation of the light ray when investigating light bending in Kerr–de Sitter spacetime. The same holds for Schwarzschild–de Sitter spacetime.

3.2. Measurable Angle by Observer in Radial Motion

The component of the 4-velocity u^μ of the radially moving observer is

$$u^\mu = (u^t, u^r, 0, 0), \quad (32)$$

and from the condition $g_{\mu\nu}u^\mu u^\nu = -1$, u^t can be expressed in terms of u^r as

$$u^t = \sqrt{\frac{B(r)(u^r)^2 + 1}{A(r)}}, \quad (33)$$

where u^t is taken to be positive.

Here, we use the radial velocity v^r converted from the component of the 4-velocity u^r , and impose the slow motion approximation for the radial velocity of the observer, $v^r \ll 1$. Then, the measurable angle ψ_{rad} for the range $0 \leq \phi \leq \pi/2$ is given by up to the order $\mathcal{O}(\varepsilon^2, \varepsilon^2 v^r)$ as,

$$\begin{aligned} \psi_{\text{rad}} = & \phi + v^r \sin \phi + \frac{2m}{b}(\cos \phi + v^r) + \frac{1}{8b^2} \left(m^2[15(\pi - 2\phi) - \sin 2\phi] - 16am \cos \phi \right) \\ & + \frac{v^r}{16b^2} \left\{ m^2[30(\pi - 2\phi) \cos \phi + 95 \sin \phi - \sin 3\phi] \right. \\ & \left. - 16am(1 + \cos 2\phi) - 2a^2(3 \sin \phi - \sin 3\phi) \right\} \\ & - \frac{\Lambda b^2}{6} \cot \phi - \frac{\Lambda b^2 v^r}{12} (\cos 2\phi - 3) + \frac{\Lambda b}{3} \cos \phi \left[m(1 + \csc^2 \phi) + 2a \csc \phi \right] \\ & + \frac{\Lambda b v^r}{3} \csc \phi [a + m \csc \phi + (a - 2m \csc \phi) \cos 2\phi] - \frac{\Lambda^2 b^4}{288} \csc^4 \phi \sin 4\phi \\ & - \frac{\Lambda^2 b^4 v^r}{144} (\cos 2\phi - 7) \cot^2 \phi \csc \phi + \mathcal{O}(\varepsilon^3, (v^r)^2), \end{aligned} \quad (34)$$

and for the range $\pi/2 \leq \phi \leq \pi$

$$\begin{aligned} \psi_{\text{rad}} = & \pi - \phi + v^r \sin \phi + \frac{2m}{b}(-\cos \phi + v^r) - \frac{1}{8b^2} \left(m^2[15(\pi - 2\phi) - \sin 2\phi] - 16am \cos \phi \right) \\ & + \frac{v^r}{16b^2} \left\{ m^2[30(\pi - 2\phi) \cos \phi + 95 \sin \phi - \sin 3\phi] \right. \\ & \left. - 16am(1 + \cos 2\phi) - 2a^2(3 \sin \phi - \sin 3\phi) \right\} \\ & + \frac{\Lambda b^2}{6} \cot \phi - \frac{\Lambda b^2 v^r}{12} (\cos 2\phi - 3) - \frac{\Lambda b}{3} \cos \phi \left[m(1 + \csc^2 \phi) + 2a \csc \phi \right] \\ & + \frac{\Lambda b v^r}{3} \csc \phi [a + m \csc \phi + (a - 2m \csc \phi) \cos 2\phi] + \frac{\Lambda^2 b^4}{288} \csc^4 \phi \sin 4\phi \\ & - \frac{\Lambda^2 b^4 v^r}{144} (\cos 2\phi - 7) \cot^2 \phi \csc \phi + \mathcal{O}(\varepsilon^3, (v^r)^2). \end{aligned} \quad (35)$$

The first three lines in Equations (34) and (35) coincide with the measurable angle in Kerr spacetime which has already been investigated in [37], and the remaining terms, lines 4 to 6, are the correction due to the cosmological constant Λ .

3.3. Measurable Angle by Observer in Transverse Motion

As the third case, let us investigate the observer in transverse motion which is the motion in a direction perpendicular to the radial direction in the orbital plane. The component of the 4-velocity of the observer u^μ is

$$u^\mu = (u^t, 0, 0, u^\phi), \quad (36)$$

and the condition $g_{\mu\nu}u^\mu u^\nu = -1$ gives

$$u^t = \frac{C(r)u^\phi + \sqrt{[C(r)u^\phi]^2 + A(r)[D(r)(u^\phi)^2 + 1]}}{A(r)}, \quad (37)$$

in which we chose the sign of u^t to be positive.

As in the radial case, we convert from the component of 4-velocity u^ϕ to the angular velocity v^ϕ and impose a slow motion approximation $v^\phi \ll 1$. Further, because $v^\phi = d\phi/dt$ is the coordinate angular velocity, we regard bv^ϕ as the coordinate transverse velocity. $bv^\phi \ll 1$. Expanding up to the order $\mathcal{O}(\varepsilon^2, \varepsilon^2 bv^\phi)$, the measurable angle ψ_{trans} becomes for $0 \leq \phi \leq \pi/2$,

$$\begin{aligned} \psi_{\text{trans}} = & \phi + bv^\phi \tan \frac{\phi}{2} + \frac{2m}{b} \cos \phi \left(1 + \frac{bv^\phi}{1 + \cos \phi} \right) \\ & + \frac{1}{8b^2} \left\{ m^2 [15(\pi - 2\phi) - \sin 2\phi] - 16am \cos \phi \right\} \\ & + \frac{bv^\phi}{8b^2(1 + \cos \phi)} \left\{ m^2 \left[15(\pi - 2\phi) - 16 \sin \phi + 7 \sin 2\phi + 16 \tan \frac{\phi}{2} \right] \right. \\ & \left. + 8ma(1 - 2 \cos \phi - \cos 2\phi) \right\} \\ & - \frac{\Lambda b^2}{6} \cot \phi - \frac{\Lambda b^3 v^\phi \cot \phi}{6(1 + \cos \phi)} + \frac{\Lambda b}{3} \cos \phi \left[m(1 + \csc^2 \phi) + 2a \csc \phi \right] \\ & + \frac{\Lambda b^3 v^\phi}{6b(\cos \phi - 1)(1 + \cos \phi)^2} [m(1 - 4 \cos \phi + \cos 2\phi) - 2a(\sin \phi + \sin 2\phi)] \\ & - \frac{\Lambda^2 b^4}{288} \csc^4 \phi \sin 4\phi - \frac{\Lambda^2 b^5 v^\phi}{2304} \cos \phi (1 - 2 \cos \phi + 3 \cos 2\phi) \csc^3 \frac{\phi}{2} \sec^5 \frac{\phi}{2} \\ & + \mathcal{O}(\varepsilon^3, (bv^\phi)^2), \end{aligned} \quad (38)$$

and for $\pi/2 \leq \phi \leq \pi$,

$$\begin{aligned} \psi_{\text{trans}} = & \pi - \phi + bv^\phi \cot \frac{\phi}{2} - \frac{2m}{b} \cos \phi \left(1 + \frac{bv^\phi}{1 - \cos \phi} \right) \\ & - \frac{1}{8b^2} \left\{ m^2 [15(\pi - 2\phi) - \sin 2\phi] - 16am \cos \phi \right\} \\ & + \frac{bv^\phi}{8b^2(1 - \cos \phi)} \left\{ m^2 \left[-15(\pi - 2\phi) - 16 \sin \phi - 7 \sin 2\phi + 16 \cot \frac{\phi}{2} \right] \right. \\ & \left. + 8am(1 + 2 \cos \phi - \cos 2\phi) \right\} \\ & + \frac{\Lambda b^2}{6} \cot \phi + \frac{\Lambda b^3 v^\phi \cot \phi}{6(1 - \cos \phi)} - \frac{\Lambda b}{3} \cos \phi \left[m(1 + \csc^2 \phi) + 2a \csc \phi \right] \\ & - \frac{\Lambda b^3 v^\phi}{6b(\cos \phi - 1)^2(1 + \cos \phi)} [m(1 + 4 \cos \phi + \cos 2\phi) - 2a(\sin \phi - \sin 2\phi)] \\ & + \frac{\Lambda^2 b^4}{288} \csc^4 \phi \sin 4\phi + \frac{\Lambda^2 b^5 v^\phi}{2304} \cos \phi (1 + 2 \cos \phi + 3 \cos 2\phi) \csc^5 \frac{\phi}{2} \sec^3 \frac{\phi}{2} \\ & + \mathcal{O}(\varepsilon^3, (bv^\phi)^2). \end{aligned} \quad (39)$$

As is the case of Equations (34) and (35), the first four lines in Equations (38) and (39) are equivalent to the measurable angle in Kerr spacetime obtained in [37], and the remaining terms, lines 5 to 7, are the additional terms due to the cosmological constant Λ .

3.4. Comparison of Static, Radial and Transverse Cases and Their Properties

Here, let us examine how the cosmological constant Λ , the spin parameter a as well as the radial and transverse velocities, v^r and bv^ϕ , contribute to the measurable angle ψ_P of the light ray. We assume that the observer is located within the range $0 \leq \phi \leq \pi/2$, and we adopt typical galaxy with mass M_{gal} , radius R_{gal} , and angular momentum J_{gal} as the lens object; the impact parameter b is comparable with R_{gal} . See Table 1.

Table 1. Numerical values. We use the following numerical values in this paper. As the value of the total angular momentum J_{gal} , we adopt that of our Galaxy $J_{\text{gal}} \simeq 1.0 \times 10^{67} \text{ kg m}^2/\text{s}$ from [45].

Name	Symbol	Value
Mass of the Galaxy	$M_{\text{gal}} = 10^{12} M_\odot$	$2.0 \times 10^{42} \text{ kg}$
	$m = GM_{\text{gal}}/c^2$	$1.5 \times 10^{15} \text{ m}$
Impact Parameter	$b = R_{\text{gal}} = 26 \text{ kly}$	$2.5 \times 10^{20} \text{ m}$
Angular Momentum of the Galaxy [45]	$J = J_{\text{gal}}$	$1.0 \times 10^{67} \text{ kg m}^2/\text{s}$
Spin Parameter	$a = J/(M_{\text{gal}}c)$	$1.7 \times 10^{16} \text{ m}$
Cosmological Constant	Λ	10^{-52} m^{-2}
Hubble Constant	$H_0 = c\sqrt{\Lambda/3}$	$1.73 \times 10^{-18} \text{ s}^{-1}$
Distance from Lens Object	$D = 1.0 \text{ Gly}$	$9.4 \times 10^{24} \text{ m}$
Recession Velocity	$v_H = H_0 D$	$1.6 \times 10^7 \text{ m/s}$
Radial Velocity	$v^r = v_H/c$	0.05
Transverse Velocity	$bv^\phi = v_H/c$	0.05

In order to clarify the contribution of the cosmological constant Λ to the total deflection angle of light ray in Kerr spacetime, we compute the following terms using the values summarized in Table 1:

$$\frac{4m}{b} \approx 2.4 \times 10^{-5} \text{ rad}, \quad (40)$$

$$\frac{15\pi m^2}{4b^2} \approx 4.2 \times 10^{-10} \text{ rad}, \quad (41)$$

$$-\frac{4ma}{b^2} \approx \mp 1.6 \times 10^{-9} \text{ rad} \quad \text{for} \quad \pm a. \quad (42)$$

First, we deal with the case of the static observer. Because the Kerr contributions appearing in Equations (29) and (30) are examined in [37], we extract the terms concerning the cosmological constant Λ from Equation (29) and put,

$$\psi_{\text{stat}}^1 = \psi_{\text{stat}}(\phi; \Lambda, b) = -\frac{\Lambda b^2}{6} \cot \phi - \frac{\Lambda^2 b^4}{288} \csc^4 \phi \sin 4\phi, \quad (43)$$

$$\psi_{\text{stat}}^2 = \psi_{\text{stat}}(\phi; m, a, \Lambda, b) = \frac{\Lambda b}{3} \cos \phi \left[m(1 + \csc^2 \phi) + 2a \csc \phi \right], \quad (44)$$

Figures 2 and 3 show the ϕ dependences of Equations (43) and (44), respectively. From Figure 2, Equation (43) is a monotonic function of ϕ and increases rapidly, diverging to negative infinity as ϕ approaches 0. This property is due to the existence of the de Sitter horizon at $r_{\text{dS}} \approx \sqrt{3/\Lambda}$. The order $\mathcal{O}(\Lambda b, \Lambda^2 b^2)$ terms in Equation (43) take a negative value which diminishes the measurable angle ψ . Equation (44) contains the order $\mathcal{O}(\Lambda b m, \Lambda b a)$ terms and its magnitude is $\mathcal{O}(10^{-10})$ which is almost comparable to the second order deflection angle, Equation (41). In accordance with the sign of the spin parameter a , Equation (44) takes a different sign; for $a > 0$, Equation (44) is positive and vice versa. However, regardless of the sign of the spin parameter a , Equation (44) diverges to positive infinity as ϕ approaches 0.

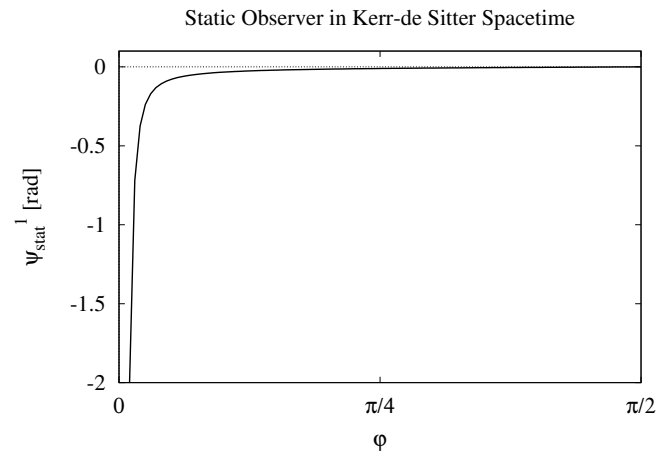


Figure 2. ϕ dependence of Equation (43).

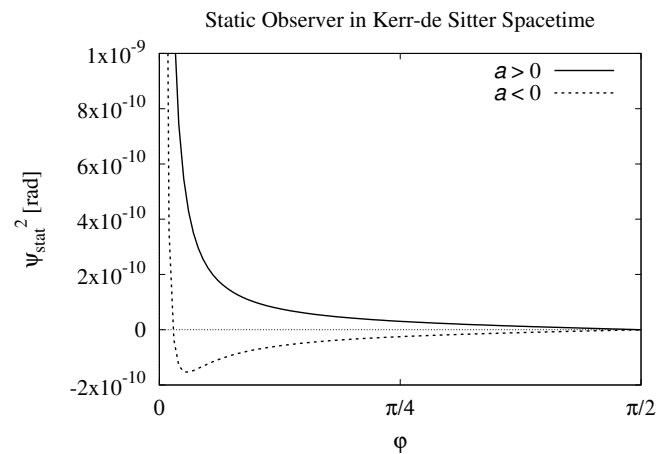


Figure 3. ϕ dependence of Equation (44).

Next, to observe the case of radially moving observer, we extract the coupling terms for cosmological constant Λ and radial velocity v^r from Equation (34) except the part of Equations (43) and (44):

$$\psi_{\text{rad}}^1 = \psi_{\text{rad}}(\phi; \Lambda, b, v^r) = -\frac{\Lambda b^2 v^r}{12} (\cos 2\phi - 3) - \frac{\Lambda^2 b^4 v^r}{144} (\cos 2\phi - 7) \cot^2 \phi \csc \phi, \quad (45)$$

$$\psi_{\text{rad}}^2 = \psi_{\text{rad}}(\phi; m, a, \Lambda, b, v^r) = \frac{\Lambda b v^r}{3} \csc \phi [a + m \csc \phi + (a - 2m \csc \phi) \cos 2\phi]. \quad (46)$$

Because the background spacetime of Kerr–de Sitter is de Sitter spacetime, we assume that radial velocity obeys Hubble's law:

$$v^r \approx v_H = H_0 D, \quad H_0 = c \sqrt{\frac{\Lambda}{3}}, \quad (47)$$

where D is the distance between the lens (central) object O and observer P , and we take $D \approx 1 \text{ Gly} \approx 9.4 \times 10^{24} \text{ m}$ which is the typical distance scale of the galaxy lensing. We note that because we are now concerned with cosmological gravitational lensing effect, it seems appropriate to assume that the expansion of the universe is dominant than the effect of local gravity due to the central (lens) object and the observer is receding from the lens object (and the source) with velocity v_H .²

We mention that we are not only dealing with a receding observer with velocity $v_H > 0$, but also, as an example we also discuss the case where the observer approaches

the lens object with $v_H < 0$ though the observer does not actually approach the lens object at the recession velocity v_H .

Figures 4 and 5 illustrate the ϕ dependence of Equations (45) and (46), respectively. Equation (45) diverges to positive or negative infinity when ϕ approaches 0 depending on the sign of the velocity v^r . On the other hand, when $\phi \rightarrow \pi/2$, Equation (45) approaches $\Lambda b^2 v^r / 3$. For the positive velocity $v^r > 0$, Equation (46) becomes negative infinity when ϕ approaches 0, while for the negative velocity $v^r < 0$, Equation (46) diverges to positive infinity. These properties are independent of the sign of the spin parameter a . Next, when $\phi \rightarrow \pi/2$, Equation (46) converges to $\Lambda b m v^r$ which depends on the radial velocity v^r but is independent of the spin parameter a . The magnitude of the velocity-dependent part, Equation (46), is at most $\mathcal{O}(10^{-12})$ which is two orders of magnitude smaller than the second order contribution $\mathcal{O}(m^2)$ in Equation (41).

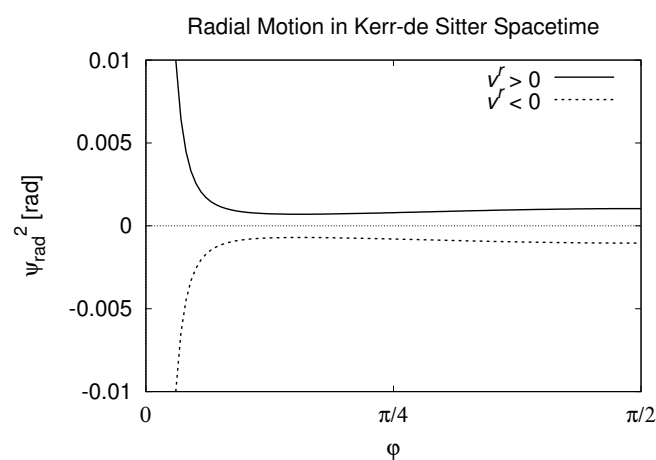


Figure 4. ϕ dependence of Equation (45).

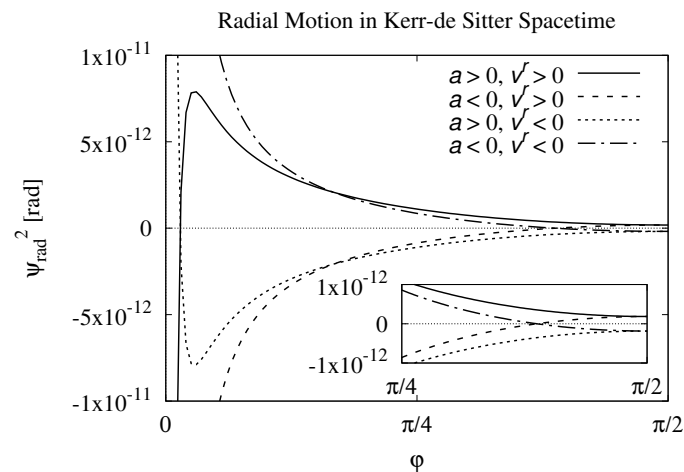


Figure 5. ϕ dependence of Equation (46).

At last, in the case of an observer in transverse motion, The parts depending on the cosmological constant Λ and the transverse velocity bv^ϕ can be extracted from Equations (38) and (39) as,

$$\begin{aligned}\psi_{\text{trans}}^1 &= \psi_{\text{trans}}(\phi; \Lambda, b, bv^\phi) \\ &= -\frac{\Lambda b^3 v^\phi \cot \phi}{6(1 + \cos \phi)} - \frac{\Lambda^2 b^5 v^\phi}{2304} \cos \phi (1 - 2 \cos \phi + 3 \cos 2\phi) \csc^3 \frac{\phi}{2} \sec^5 \frac{\phi}{2},\end{aligned}\quad (48)$$

$$\begin{aligned}\psi_{\text{trans}}^2 &= \psi_{\text{trans}}(\phi; m, a, \Lambda, b, bv^\phi) \\ &= \frac{\Lambda b^3 v^\phi}{6b(\cos \phi - 1)(1 + \cos \phi)^2} [m(1 - 4 \cos \phi + \cos 2\phi) - 2a(\sin \phi + \sin 2\phi)],\end{aligned}\quad (49)$$

and we also assume here that the transverse velocity is comparable with the recessional velocity v_H

$$bv^\phi \approx v_H, \quad (50)$$

see also Table 1.

Figures 6 and 7 show the ϕ dependences of Equations (48) and (49), respectively. Equation (48) consists of $\mathcal{O}(\Lambda b^3 v^\phi, \Lambda^2 b^5 v^\phi)$ terms; however, unlike Equation (45), for the positive transverse velocity $bv^\phi > 0$, Equation (48) diverges to negative infinity when ϕ approaches 0 and vice versa. When $\phi \rightarrow \pi/2$, Equation (48) converges to 0 regardless of the sign of the transverse velocity bv^ϕ . From Figure 7, we find that regardless of the sign of the spin parameter a , Equation (49) becomes positive infinity for the positive transverse velocity $bv^\phi > 0$ and negative infinity for the negative transverse velocity $bv^\phi < 0$ when ϕ approaches 0. When $\phi \rightarrow \pi/2$, Equation (49) converges to $\Lambda b^2 a v^\phi / 3$ which depends on both the spin parameter a and the transverse velocity bv^ϕ unlike Equation (46). As in Equation (46), the magnitude of Equation (49) is at most $\mathcal{O}(10^{-12})$ and it is two orders of magnitude of smaller than the second order contribution $\mathcal{O}(m^2)$ in Equation (41).

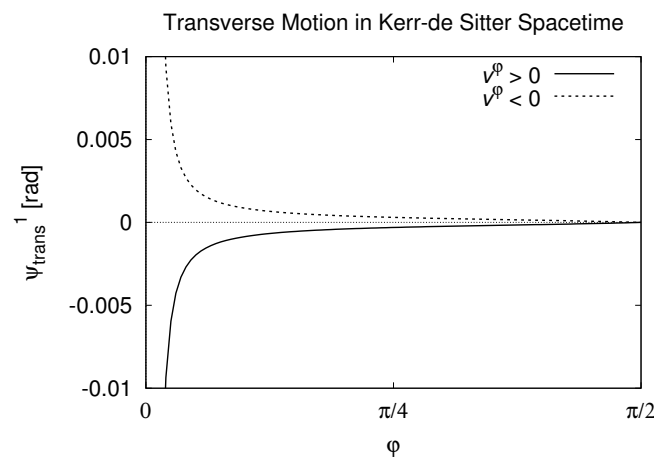


Figure 6. ϕ dependence of Equation (48).

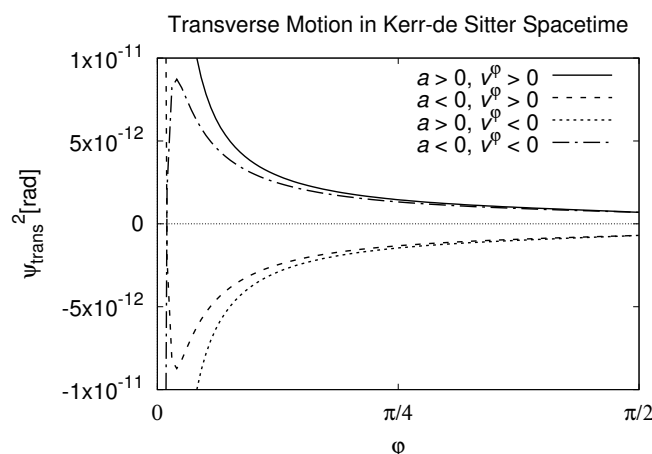


Figure 7. ϕ dependence of Equation (49).

Before closing this section, we summarize the behavior of Equations (43)–(46), (48) and (49) within the range $0 < \phi < \pi/2$ in Table 2.

Table 2. Behavior of Equations (43)–(46), (48) and (49).

Motion of Observer	Equation Number	$\phi \rightarrow 0$	$0 < \phi < \pi/2$	$\phi \rightarrow \pi/2$
Static	Equation (43)	$-\infty$	Negative	0
	Equation (44)	∞	Positive for $a > 0$ Mostly Negative for $a < 0$	0
Radial	Equation (45)	∞ for $v^r > 0$ $-\infty$ for $v^r < 0$	Positive for $v^r > 0$ Negative for $v^r < 0$	$\Lambda b^2 v^r / 3$
	Equation (46)	$-\infty$ for $v^r > 0$	Mostly Positive for $a > 0$ Mostly Negative for $a < 0$	$\Lambda b m v^r$
		∞ for $v^r < 0$	Mostly Negative for $a > 0$ Mostly Positive for $a < 0$	
	Equation (48)	$-\infty$ for $v^\phi > 0$ ∞ for $v^\phi < 0$	Negative for $v^\phi < 0$ Positive for $v^\phi < 0$	0
Transverse	Equation (49)	∞ for $v^\phi > 0$	Positive for $a > 0$ Mostly Negative for $a < 0$	$\Lambda b^2 a v^\phi / 3$
		$-\infty$ for $v^\phi < 0$	Negative for $a > 0$ Mostly Positive for $a < 0$	

4. Conclusions

In this paper, instead of the total deflection angle α , we mainly focused on discussing the measurable angle of the light ray ψ at the position of the observer in Kerr–de Sitter spacetime which includes the cosmological constant Λ . We investigated the contributions of the radial and transverse motions of the observer which are related to the radial velocity v^r and the transverse velocity bv^ϕ as well as the influence of the gravitomagnetic field or frame dragging described by the spin parameter a of the central object and the cosmological constant Λ .

The general relativistic aberration equation was employed to incorporate the effect of the motion of the observer on the measurable angle ψ . To obtain the measurable angle ψ , the equation of the light trajectory was obtained in such a way that the background is de Sitter spacetime instead of Minkowski spacetime. At the end of Section 3.1, we mentioned that the choice of the zeroth-order solution u_0 is important; a zeroth-order solution u_0 in Kerr–de Sitter and Schwarzschild–de Sitter spacetimes should be chosen in such a way that the background is de Sitter spacetime, Equation (18). Further, Equation (20) or at least Equation (21) should be used as the trajectory equation of the light ray.

If we assume that the lens object is the typical galaxy, the static terms $\mathcal{O}(\Lambda bm, \Lambda ba)$ in Equation (44) are basically comparable with the second order deflection term $\mathcal{O}(m^2)$ but almost one order smaller than the Kerr deflection $-4ma/b^2$. The velocity-dependent terms $\mathcal{O}(\Lambda bmv^r, \Lambda bav^r)$ in Equation (46) for radial motion and $\mathcal{O}(\Lambda b^2mv^\phi, \Lambda b^2av^\phi)$ in Equation (49) for transverse motion are at most two orders of magnitude smaller than the second order deflection $\mathcal{O}(m^2)$. Therefore, if the second order deflection term $\mathcal{O}(m^2)$ becomes detectable by gravitational lensing, it may be possible to detect the cosmological constant Λ from the static terms in Equation (44). We also find that even when the radial and transverse velocities have the same sign, their asymptotic behavior when ϕ approaches 0 is differs, and each diverges to the opposite infinity.

Our future work is to formulate the total deflection angle of light ray in Kerr–de Sitter spacetime, taking into account the velocity effect of the observer, and apply it to the cosmological lens equation to discuss the observability of the contribution to the cosmological constant by gravitational lensing. The formulation of the total deflection angle of the light ray in Kerr–de Sitter spacetime may be realized by using the Gauss–Bonnet theorem, and we expect that the results of this study will be useful in the future work.

Funding: This research received no external funding.

Acknowledgments: We would like to acknowledge anonymous referees for reading our manuscript carefully and for giving fruitful comments and suggestions, which significantly improved the quality of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Notes

- ¹ We mention that the term $\left(1 + \frac{\Lambda}{3}a^2\right)^2$ in the denominator is missing in Equation (4) of [27] however, their result is not affected by this missing term because they expanded Equation (4) up to the second order in m , a , and Λ , whereas Λa^2 corresponds to the third order.
- ² The motion of an observer in Kerr–de Sitter spacetime, especially in the radial direction, can be characterized by the static radius r_s [46–48] at which the gravitational interaction due to the central object and the expansion (repulsion) of the universe are balanced. At the position of the static radius r_s , it is possible for the observer to remain stationary. However the observer at the static radius is in unstable equilibrium (the maximum of the effective potential), then observer begins to accelerate up to the velocity v_H which is related to the boundary of the vacuola of Kerr–de Sitter spacetime (In fact, without rotation $a = 0$, Kerr–de Sitter spacetime can be regarded as the Einstein–Strauss–de Sitter vacuola model of the Universe, see, e.g., [49]). Therefore, there is a region where the observer’s velocity v is $0 < v_s < v_H$; however, because we are now considering cosmological gravitational lensing in a region far enough from the lens object, we assume the observer’s velocity to be v_H in this paper.

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