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Article

Quantum-Spacetime Symmetries: A Principle of Minimum Group Representation

Diego J. Cirilo-Lombardo [1,2] and Norma G. Sanchez [3,*]

1 M. V. Keldysh Institute of the Russian Academy of Sciences, Federal Research Center, Institute of Applied Mathematics, Miusskaya sq. 4, 125047 Moscow, Russia; diego777jcl@gmail.com
2 Departamento de Fisica, Instituto de Fisica Interdisciplinaria y Aplicada (INFINA), CONICET-Universidad de Buenos Aires, Buenos Aires C1428EGA, Argentina
3 International School of Astrophysics Daniel Chalonge, Hector de Vega, CNRS, INSU—Institut National des Sciences de l'Univers, Sorbonne University, 75014 Paris, France
* Correspondence: norma.sanchez@orange.fr

Abstract: We show that, as in the case of the principle of minimum action in classical and quantum mechanics, there exists an even more general principle in the very fundamental structure of *quantum spacetime*: this is the principle of *minimal group representation*, which allows us to consistently and simultaneously obtain a natural description of spacetime's dynamics and the physical states admissible in it. The theoretical construction is based on the physical states that are average values of the generators of the metaplectic group $Mp(n)$, the double covering of $SL(2C)$ in a vector representation, with respect to the *coherent states* carrying the spin weight. Our main results here are: (i) There exists a connection between the dynamics given by the metaplectic-group symmetry generators and the physical states (the mappings of the generators through bilinear combinations of the basic states). (ii) The ground states are coherent states of the Perelomov–Klauder type defined by the action of the metaplectic group that divides the Hilbert space into *even* and *odd* states that are mutually orthogonal. They carry spin weight of 1/4 and 3/4, respectively, from which two other basic states can be formed. (iii) The physical states, mapped bilinearly with the basic 1/4- and 3/4-spin-weight states, plus their symmetric and antisymmetric combinations, have spin contents $s = 0,\ 1/2,\ 1,\ 3/2$ and 2. (iv) The generators realized with the bosonic variables of the harmonic oscillator introduce a natural supersymmetry and a superspace whose line element is the geometrical Lagrangian of our model. (v) From the line element as operator level, a coherent physical state of spin 2 can be obtained and naturally related to the metric tensor. (vi) The metric tensor is *naturally discretized* by taking the discrete series given by the basic states (coherent states) in the n number representation, reaching the classical (continuous) spacetime for $n \to \infty$. (vii) There emerges a relation between the eigenvalue α of our coherent-state metric solution and the black-hole area (entropy) as $A_{bh}/4l_p^2 = |\alpha|$, relating the phase space of the metric found, g_{ab}, and the black hole area, A_{bh}, through the Planck length l_p^2 and the eigenvalue $|\alpha|$ of the coherent states. As a consequence of the lowest level of the quantum-discrete-spacetime spectrum—e.g., the ground state associated to $n = 0$ and its characteristic length—there exists a minimum entropy related to the black-hole history.

Keywords: quantum spacetime; fundamental principle; minimum group representation; symmetry; metaplectic group; phase space; quantum coherent states

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1. Introduction

A key concept for a full quantum theory of gravity, as well as for quantum theory on its own, is quantum spacetime. The basic motivation of this paper is to demonstrate that, as in the case of classical and quantum mechanics, in which the minimum action is the regulating and determining principle, there is an even more general principle that intervenes in the fundamental structure of quantum spacetime: this is the interplay between dynamics and symmetry or, alternatively, matter/energy and spacetime. The maximum simplicity by which to achieve this goal is based on the metaplectic group $Mp(n)$, which is the double covering of the $S_p(2C)$ group, and which for the illustrative case that we intend to establish here we fix to $Mp(2)$.

We characterize quantum spacetime as originating from a mapping $P(G, M)$ between the real-spacetime manifold M and the quantum-phase-space manifold of a group G. Once one component of the momentum P operator is identified with the time T, the spacetime metric of M is found, using the metric g_{ab} on the phase-space group manifold.

The group's compactness or noncompactness determines the metric's signature; nonetheless, noncompact groups are necessary for the majority of physical situations of interest because of the real spacetime signature and its hyperbolic structure.

The quantum spacetime established from the harmonic oscillator's phase space represents the more obviously fundamental examples of this development. In the case of the normal (real-frequency) oscillator, refer to Refs. [1–3] and the mapping $(X, P) \to (X, iT)$ or, alternatively, $(X, P) \to (X, T)$ in the situation of the imaginary frequency (the inverted oscillator), which appears in many physical examples—in particular, in cosmology (e.g., in the propagation eqs of classical and quantum perturbations). The quantum-spacetime algebra of non-commutative operator coordinates is the quantum-oscillator algebra. The line element arises from the Hamiltonian (Casimir operator), and its discretization yields the quantum-space levels. The zero-point energy yields the new quantum region, splitting the light-cone origin because the classical generating lines $X = \pm T$ are replaced by the curves $X^2 - T^2 = [X, T]$, which are the quantum hyperbolae due to the non-zero space and time commutators, and generate, in particular, the quantum light cone [4,5].

The inverted oscillator is associated to the hyperbolic spacetime structure, while the normal oscillator yields a Euclidean (imaginary-time) signature (quantum gravitational instantons).

In various forms, the inverted oscillator can be found in a multitude of fascinating physical scenarios, including black holes, particle physics, and contemporary cosmology (which includes inflation and dark energy), Refs. [4–9].

An important point in this principle of minimum group representation is the description of quantum-spacetime symmetries: "algebro-pseudo-differential correspondence" plays a key role. This correspondence establishes that a radical operator (e.g., a Hamiltonian) is equivalent, in the context of the metaplectic description, to a Majorana–Dirac-type operator with internal variables in the oscillator representation. This correspondence is exemplified in the expression in Equation (13), Section 4 in this paper. This algebraic interpretation is significant because it allows for a connection with pseudodifferential operators and semigroup (Fourier-integral) representations, as shown below [10]:

algebraic interpretation

$\nearrow$ $\nwarrow$

pseudo differential operators $\longleftrightarrow$ *semigroup (Fourier − integral representations)*

In this theoretical and physical context, the resulting solution consists of two types: the basic state and the observable physical state, which is bilinear with respect to the basic state (e.g., the mean value). The basic state is a coherent state corresponding to the metaplectic group, which is the double covering of the SL(2C) group, Refs. [10–15].

We use as our example Ref. [10], an N = 1 superspace with an invertible and non-degenerate supermetric, where the unconstrained quantization is precisely carried out using novel techniques based on coherent states and keeping the Hamiltonian form. Thus, from the discrete spectrum of the states themselves, a discrete structure of spacetime automatically arises without any prescription of discretization (Figure 1).

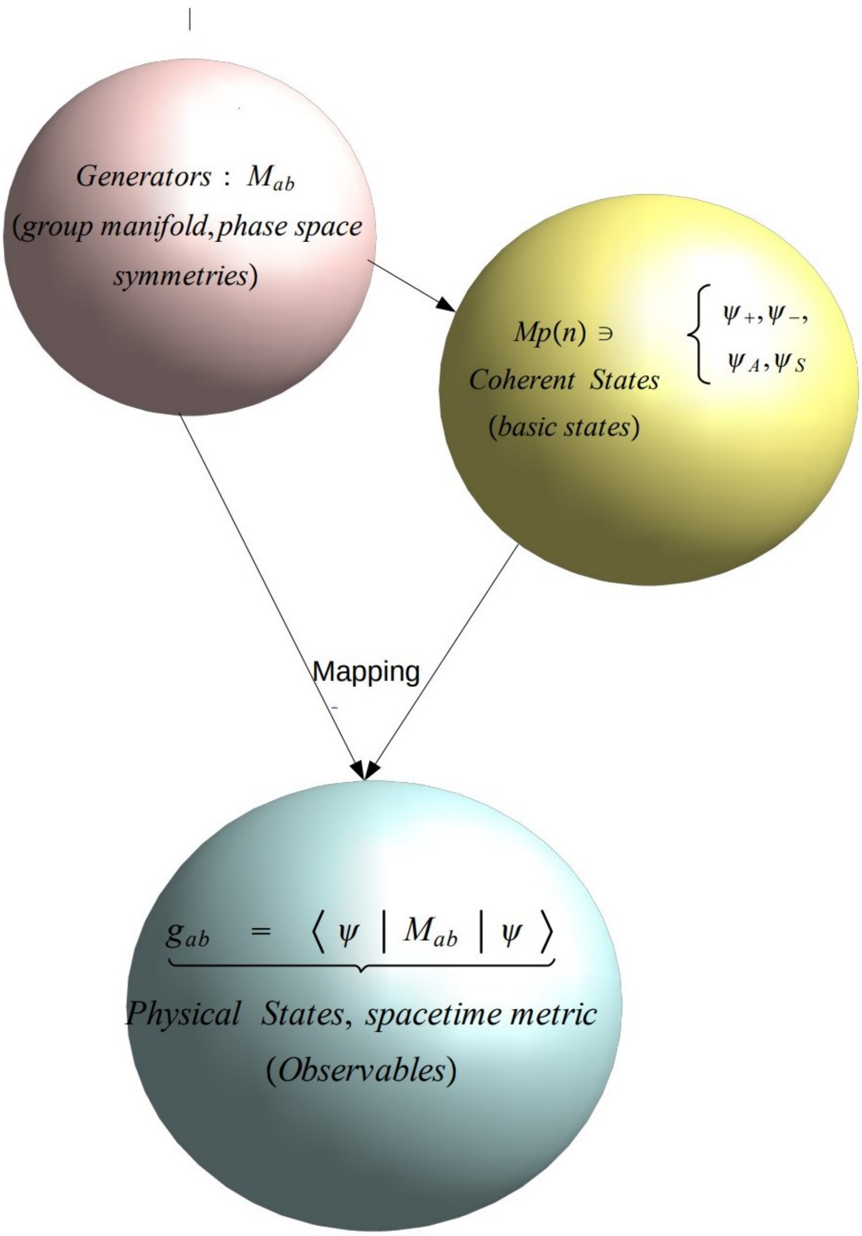

Figure 1. Quantum gravity regime and that of a dynamical quantum microscopic picture (the complete process of black hole emission in all its stages being a clear example).

Due to the metaplectic representation (the double covering of the $SL(2C)$) of the coherent-state solution representing the emergent spacetime, the crossover from the quantum microscopic regime to the macroscopical regime (classical or not) is natural and consistent. This important fact allows us to conciliate such apparently different pictures as that of a macroscopical-quantum-gravity regime and that of a dynamical-quantum-microscopic picture (the complete process of black-hole emission in all its stages being a clear example).

Despite its simplicity, the framework introduced here has provided physically and geometrically significant responses concerning an accurate description of quantum gravity.

It is convenient to think of this kind of coherent state as arising from a Lie group G operating on a Hilbert space $\mathcal{H}$ through a unitary, irreducible representation T. The set of vectors $\psi \in \mathcal{H}$, such that $\psi = T(g)\,\psi_0$ for some $g \in G$, is what we describe as the coherent state system$\{T, \psi_0\}$ for a fixed vector ψ_0. We define the states $|\psi\rangle$ corresponding to these vectors in $\mathcal{H}$ as generalized coherent states.

2. The Metaplectic Group and the Principle of Minimal Representation

Mp(2), SU(1,1) and Sp(2)

We briefly describe now the relevant symmetry group to achieve the realization of the Hamiltonian operator of the problem. Specifically, this group is the metaplectic $Mp(2)$ group; the groups $SU(1,1)$ and $Sp(2)$ are topologically covered by it. In the function of the $(q,\ p)$ operators, or equivalently the operators $(a,\ a^{+})$ of the standard harmonic oscillator, the generators of $Mp(2)$ are

$$\begin{aligned} T_1 &= \frac{1}{4}(q\,p + p\,q) = \frac{i}{4}\left(a^{+2} - a^2\right), \\ T_2 &= \frac{1}{4}\left(p^2 - q^2\right) = -\frac{1}{4}\left(a^{+2} + a^2\right), \\ T_3 &= -\frac{1}{4}\left(p^2 + q^2\right) = -\frac{1}{4}\left(a^{+}a + a\,a^{+}\right), \end{aligned} \tag{1}$$

with the commutation relations,

$$[\,T_3\,,\,T_1\,] = i\,T_2\,; \qquad [\,T_3\,,\,T_2\,] = -\,i\,T_1\,; \qquad [\,T_1, T_2\,] = -\,i\,T_3.$$

The commutation relations can be written as $[T_3, T_1 \pm iT_2] = \pm(T_1 \pm iT_2)$; $[\,T_1 + i\,T_2, T_1 - iT_2] = -2\,T_3$. It is then easy to see that $T_1 + i\,T_2 = -\frac{i}{2}a^2$ and $T_1 - i\,T_2 = \frac{i}{2}a^{+2}$. Therefore, the oscillator states $|\,n\,\rangle$ of the number operator are eigenstates of the T_3 generator:

$$T_3\,|\,n\,\rangle = -\frac{1}{2}\left(n + \frac{1}{2}\right)|\,n\,\rangle.$$

3. The Mp(2) Vector Representation and Its Coverings

The commutation relation that specifies the generators L_i is the main feature of the specific representation that was introduced in [2] :

$$[\,L_i, a^{\alpha}\,] = \frac{1}{2}\,a^{\beta}(\sigma_i)_{\beta}^{\ \alpha}. \tag{2}$$

The representation above is non-compact Lie algebra with the matrix form

$$\sigma_i = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

obeying, in a geometrical way,

$$\sigma_i \wedge \sigma_j = -i\sigma_k, \qquad \sigma_k \wedge \sigma_i = i\sigma_j, \qquad \sigma_j \wedge \sigma_k = i\sigma_i. \tag{4}$$

We want to remark on the following equivalence:

The generators in the representation of Equation (2) fulfill the relation

$$L_i = \frac{1}{2}\,a^{\beta}\,(\sigma_i)_{\beta}^{\ \alpha}a_{\alpha} = T_i, \tag{5}$$

where T_i are the metaplectic generators, namely [10,11]:

$$T_1 = \frac{i}{4}\left(a^{+2} - a^2\right), \tag{6}$$

$$T_2 = -\frac{1}{4}\left(a^{+2} + a^2\right), \tag{7}$$

$$T_3 = -\frac{1}{4}\left(aa^{+} + a^{+}a\right). \tag{8}$$

Proof. We can write the generators L_i in matrix form as

$$L_i = \overline{u}\, \mathbb{M}_i\, v \tag{9}$$

$$\overline{u} \equiv \begin{pmatrix} a^+ & a \end{pmatrix}, \qquad v \equiv \begin{pmatrix} a \\ a^+ \end{pmatrix}.$$

The representation of Equation (2) is faithful. We take into account that σ_k is entered as a "metric" in the sense given in Ref. [16]—that is, it introduces the signature in the quadratic terms in a and a^+ in Equation (9), explicitly giving rise to the expression in Equation (5). Therefore, we have

$$M_1 = \frac{i}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{4}\,\sigma_k\,\sigma_i\,, \tag{10}$$

$$M_2 = -\frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\frac{1}{4}\,\sigma_k\,\sigma_j\,, \tag{11}$$

$$M_3 = -\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{4}\,\sigma_k^2. \tag{12}$$

Consequently, and by inspection, Equation (2) coincides with Equation (9). Thus, the equivalence in Equation (5) is proved. □

4. Symmetry and Dynamics Principle: Steps to Follow

A fundamental component of the dynamic description is the square-root-type Hamiltonian or Lagrangian, which is, in theory, a non-local and non-linear operator. This is because the right physical spectrum is generated by the invariance under reparametrizations, both as a Lagrangian and as a corresponding Hamiltonian. The fundamental principles of our strategy here are based on certain elements that are explicitly mentioned in the following.

4.1. The Invariant Action

(i) Considering the spacetime–matter structure, the geometric Lagrangian (functional action) of the theory is the elementary distance function, which is defined as the positive square root of the line element.

At the least, the line element's symmetry matches that provided by the super-Poincaré or Cartan-Killing form of Osp(1,2), enabling a bosonic realization based on the a and a^+ operators of the conventional harmonic oscillator. This leads to the metric being non-degenerate and having extra odd (Fermionic) coordinates.

4.2. Extended Hamiltonian of the System

The geometric Hamiltonian, which is the fundamental classical–quantum operator, is obtained from (i) in the conventional manner.

From the perspective of the physical states, this universal Hamiltonian (square-root Hamiltonian) has an enlarged phase space because it includes a zero moment P_0 characteristic of the entire phase space at its highest level.

Time "disappears" from the dynamic equations in a proper time system when the evolution coincides with the time coordinate. This is prevented by including a zero-momentum P_0, which would otherwise lead to the arbitrary nullification of the Hamiltonian.

4.3. Relativistic Wave Equation and the Algebraic Interpretation

(iv) The Hamiltonian $\mathcal{H}_s$, rewritten in differential form, defines a new relativistic wave equation of second order and degree $1/2$ (square-root form). This fact can be reinterpreted as a Dirac–Sudarshan-type equation of positive energies and internal variables (e.g., oscillator-type variables) contained as components of the auxiliary or internal vector L_α,

$$\mathcal{H}_s \Psi \equiv \sqrt{\mathcal{F}} \mid \Psi \rangle \leftrightarrow \left\{ [\mathcal{F}]^\alpha_\beta L_\alpha \right\} \Psi^\beta, \tag{13}$$

having the basic solution states of the system, a para-Bose or para-Fermi interpretation of $\mid \Psi \rangle$. This gives rise to the main justification for an algebraic interpretation of the radical operator: we have a clean action at operator level and a consistent number of states of the system (the Lagrange-multiplier method eliminates the square root in a non-physical way, doubling the spectrum of physical states).

4.4. Basic States of Representation and the Spectrum of Physical States

The basic states $\mid \Psi_s \rangle$ belong to the group Mp(n) and have a spin weight s = 1/4, 3/4 in the simplest case, Mp(2): They contain *even* and *odd* sectors (s = 1/4, 3/4) in the number of levels of the Hilbert space, respectively, and, therefore, they span non-dense irreducible spaces.

In this way, states that are bilinear in fundamental functions (corresponding to $\left| \Psi_{s=1/4,\,3/4} \right\rangle$ form the full physical spectrum. In the case of the metaplectic group $Mp(2)$, these fundamental functions are $f_{1/4}$ and $f_{3/4}$, having a spin weight $s = 1/4$ and $3/4$, respectively. A physical-state characteristic of $Mp(2)$ is given by $\Phi_\mu = \langle s \mid L_\mu \mid s' \rangle$, with $(s, s' = 1/4, 3/4)$ and L_μ being the vector representation of one of the generators of $Mp(2)$.

With the Mp(2) interpretation, we can also describe a *complete* multiplet spanning *spins* from $(0, 1/2, 1, 3/2, 2)$. This is a consequence of the fact that with the fundamental states and the allowed vectorial generators, the tower of states is finite and *all* the states involved are *physical*, as it must be in the physical context.

5. Statement of the Problem

Geometrically, we take as the starting point the functional action that will describe the worldline (measure on a superspace) of the superparticle, as follows:

$$S = \int_{\tau_1}^{\tau_2} d\tau\, L(x, \theta, \bar{\theta}) = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{\overset{\circ}{\omega}_\mu \overset{\circ}{\omega}{}^\mu + \mathbf{a}\, \dot{\theta}^\alpha \dot{\theta}_\alpha - \mathbf{a}^* \dot{\bar{\theta}}{}^{\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}}}, \tag{14}$$

where $\overset{\circ}{\omega}_\mu = \dot{x}_\mu - i(\dot{\theta}\, \sigma_\mu \bar{\theta} - \theta\, \sigma_\mu \dot{\bar{\theta}})$ and the dot indicates the derivative, with respect to the parameter τ, as usual. The above Lagrangian was constructed considering the line element (e.g., the measure, the positive square root of the interval) of the non-degenerated supermetric,

$$ds^2 = \omega^\mu \omega_\mu + \mathbf{a}\, \omega^\alpha \omega_\alpha - \mathbf{a}^* \omega^{\dot{\alpha}} \omega_{\dot{\alpha}},$$

where a superspace $(1, 3 \,|1)$ is composed of the bosonic term and the Majorana bispinor with coordinates $(t, x^i, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, being the Maurer–Cartan forms of the supersymmetry group—$\omega_\mu = dx_\mu - i(d\theta\sigma_\mu\bar{\theta} - \theta\sigma_\mu d\bar{\theta})$, $\omega^\alpha = d\theta^\alpha$, $\omega^{\dot{\alpha}} = d\theta^{\dot{\alpha}}$—with evident supertranslational invariance.

As our manifold has been extended to include Fermionic coordinates, it is natural to extend also the concept of the trajectory for a point particle to the superspace. Consequently, we take the coordinates $x(\tau)$, $\theta^\alpha(\tau)$ and $\bar{\theta}^{\dot{\alpha}}(\tau)$, depending on the evolution parameter τ.

The Hamiltonian in square-root form, namely,

$$\sqrt{m^2 - \mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right)} |\Psi\rangle = 0,$$

is constructed defining the supermomenta, as usual, and the Lanczos method for constrained Hamiltonian systems is used, due the nullification of this Hamiltonian.

Therefore, an algebraic realization of the pseudo-differential operator (square root) does exist in the case of an underlying Mp(n) group structure:

$$\sqrt{\mathcal{H}}\,|\Psi\rangle \;\equiv\; \sqrt{m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^{\alpha}\Pi_{\alpha} - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)}\,|\Psi\rangle \;=\; 0 \tag{15}$$

$$\left\{[\,\mathcal{H}\,]^{\alpha}_{\beta}\,(\Psi L_{\alpha}\,)\right\}\Psi^{\beta} \;\equiv\; \left\{\left[m^2 - \mathcal{P}_0\mathcal{P}^0 - \left(\mathcal{P}_i\mathcal{P}^i + \frac{1}{a}\Pi^{\alpha}\Pi_{\alpha} - \frac{1}{a^*}\Pi^{\dot{\alpha}}\Pi_{\dot{\alpha}}\right)\right]^{\alpha}_{\beta}(\Psi L_{\alpha})\right\}\Psi^{\beta} \;=\; 0 \tag{16}$$

Therefore, both structures can be identified: e.g., $\sqrt{\mathcal{H}} \longleftrightarrow [\,\mathcal{H}\,]^{\alpha}_{\beta}\,(\Psi L_{\alpha}\,)$, being the state Ψ, the square root of a spinor Φ (on which the "square root" Hamiltonian operates), in such a manner that it can have the bilinear expression $\Phi = \Psi L_{\alpha}\Psi$.

Equation (15) in the context of our work has its equivalent second-order Dirac-like operator in the expression given by Equation and Equation (16). This type of operator has been developed by Majorana, Dirac (e.g., [17,18]) and others [19], containing internal variables of the harmonic-oscillator type, and in our original and particular case it gives an algebraic interpretation to the radical operator, with two fundamental objectives fulfilled: interpreting the action of the square-root operator and describing the relationship between the physical (bilinear) states and the fundamental (basic) states, as described in detail in Section 6 here.

Equation (16) is nothing more and nothing less than the algebraic interpretation of the radical operator: a Majorana–Dirac-type operator—that is to say, an equation with internal variables in the sense of Dirac, Majorana and others (refs, e.g., [17–19]), with a different spinorial decomposition structure. The curly brackets in Equation (16) define the limit of the equivalence to the radical operator expression given by Equation (15).

The key observation here is that the operability of the pseudo-differential "square root" Hamiltonian can be clearly interpreted if it acts on the square root of the physical states. The square root of a spinor certainly exists in the case of the metaplectic group [16,20], Refs. [17,18] making our interpretation, Equations (15) and (16), fully consistent from both the relativistic and group theoretical viewpoints.

Regarding Equation (16), we want to emphasize that our paper refers to the role of the generator of the metaplectic group both in the dynamics and in the physically admissible states of the model. We stress that the variables of the harmonic oscillator are internal from the point of view of the equations, and the origin of these variables is the faithful and fundamental representation of the symmetry of the generators of the dynamics of the spacetime through the physical states, such as the mappings of the generators in that particular representation, as explained in the paper.

The concept and underlying logic of Equations (15) and (16) are clear: quantum symmetries contain—give rise to—the classical structure. The physical states, as well as the metric (spin 2) are emergent under the action of the symmetry operator via Equation (15). (Moreover, concrete examples of this concept can be found in Refs. [21–24] by these authors). Spin and supersymmetry do not need the Minkowskian structure. A clear example can be seen for the case of spin 2, in Sections 8–10 of this paper.

Our Equation (16) here is fully relativistic and capable of including a complete (super) multiplet spanning spins from $0,\ 1/2,\ 1,\ 3/2,\ 2$ of physical states.

In the next section, we will describe the states (truly spinorial and relativistic ones) coming from the algebraic correspondence.

6. Physical States from Symmetries

Generators (dynamical symmetries) being in an oscillator-like vector representation (spinorial) are mapped through their mean values, with respect to the basic states (the Mp(n) coherent states) giving rise to the observable physical states. That is to say, there is an interrelation between symmetries and physical states. This gives rise to the first important consequence that, taking into account the unobservable basic states, the bilinear states that are observable can only contain spins $(0,\ 1/2,\ 1,\ 3/2,\ 2)$.

Next, we will provide a brief theoretical justification of the above construction, and then, in the following section, we describe the emergent spacetime-discretization mechanism.

It should be noted that the family of representations can be increased—e.g., like those of the Hilbert-space operators in the Weyl representation—for a great variety of groups and asymmetric representations of various forms. In our case, the large group involved is the *metaplectic group* $Mp(2)$ (the covering group of $SL(2C)$). This important group $Mp(2)$ is also closely related to the para-Bose coherent states and squeezed states (CS and SS).

Let us consider the concept of generalized coherent states (CS) based in a Lie group G acting on a Hilbert space H through a unitary, irreducible representation T in the following. The coherent-state system $\{T, \psi_0\}$ is defined as the set of vectors $\psi \in \mathcal{H}$, such that $\psi = T(g)\,\psi_0$ for some $g \in G$, given a fixed vector ψ_0. These vectors' equivalent states in H are known as generalized coherent states (states $|\psi\rangle$).

The following coherent-state-reproducing kernel for any operator A (not necessarily bounded) serves as the foundation for our analysis:

$$K_{\widehat{A}}(\alpha, \alpha'; g) \;=\; e^{\left[\,|\alpha|^2 - |\alpha'|^2\,\right]} \left\langle\, \alpha \mid A \mid \alpha' \,\right\rangle, \tag{17}$$

where α and α' are complex variables that characterize a respective coherent state, and g is an element of $Mp(\,2\,)$. The possible *basic CS states* are classified as

$$\begin{aligned} |\,\Psi_{1/4}\,(t, \xi, q)\,\rangle \;&=\; f(\xi)\,|\,\alpha_+(t)\,\rangle \\ |\,\Psi_{3/4}\,(t, \xi, q)\,\rangle \;&=\; f(\xi)\,|\,\alpha_-(t)\,\rangle, \end{aligned} \tag{18}$$

with the following independent, non-equivalent *symmetric* and *anti-symmetric* combinations:

$$\begin{aligned} \left|\,\Psi^S\,\right\rangle \;&=\; \frac{f(\xi)}{\sqrt{2}}\,(\,|\,\alpha_+\,\rangle \,+\, |\,\alpha_-\rangle\,) \;=\; f(\xi)\left|\,\alpha^S\,(t)\,\right\rangle, \\ \left|\,\Psi^A\,\right\rangle \;&=\; \frac{f(\xi)}{\sqrt{2}}\,(\,|\,\alpha_+\,\rangle \,-\, |\,\alpha_-\rangle\,) \;=\; f(\xi)\left|\,\alpha^A\,(t)\,\right\rangle. \end{aligned} \tag{19}$$

The important fact, in order to evaluate the kernels of Equation (17), is the action of a and a^2 over the states previously defined:

$$a|\Psi_{1/4}\rangle = \alpha|\Psi_{3/4}\rangle;\;\; a|\Psi_{3/4}\rangle = \alpha|\Psi_{1/4}\rangle;\;\; a\left|\Psi^S\right\rangle = \alpha\left|\Psi^S\right\rangle;\;\; a\left|\Psi^A\right\rangle = -\alpha\left|\Psi^A\right\rangle,$$

$$a^2|\Psi_{1/4}\rangle = \alpha^2|\Psi_{1/4}\rangle;\;\; a^2|\Psi_{3/4}\rangle = \alpha^2|\Psi_{3/4}\rangle;\;\; a^2\left|\Psi^S\right\rangle = \alpha^2\left|\Psi^S\right\rangle;\;\; a^2\left|\Psi^A\right\rangle = \alpha^2\left|\Psi^A\right\rangle,$$

and similarly for the states $\overline{\Psi}$.

We have established that the *physical states* are particular representations of the operators L_{ab} and $\mathbb{L}_{ab} \in Mp(2)$ in spinorial form, in the sense of quasi-probabilities (tomograms in the Ψ_s plane) or as mean values, with respect to the basic coherent states, Equations (18) and (19): $|\Psi_\lambda\rangle,\ \lambda \;=\; (1/4,\ 1/2,\ 3/4,\ 1\,)$. There are six possible generalized kernels from Equation (17): two $g\,(\,t,\ s, \pm\,\alpha\,),\ s \,=\, 1, 2$ in the Heinsenberg-Weil-(HW)-oscillator representation corresponding to the symmetric and anti-symmetric states, respectively:

$$g_{ab}\,(\,t,\,2,\,\alpha\,)\,|_{HW} \;=\; \left\langle\, \Psi^S(t) \,\middle|\, L_{ab} \,\middle|\, \Psi^S\,(\,t\,) \,\right\rangle \;=\; \mathcal{F}\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_{(2)\,ab} \tag{20}$$

$$g_{ab}\,(\,t,\,1,\,-\,\alpha\,)\,|_{HW} \;=\; \left\langle\, \Psi^A(t) \,\middle|\, L_{ab} \,\middle|\, \Psi^A(t) \,\right\rangle \;=\; \mathcal{F}\begin{pmatrix} -\alpha \\ -\alpha^* \end{pmatrix}_{(1)\,ab} \tag{21}$$

where

$$\mathcal{F} = e^{\left[-\left(\frac{m}{\sqrt{2}\,|\mathbf{a}|}\right)^2 [\,(\alpha+\alpha^*)-B\,]^2+D\,\right]} e^{[\,\xi\,\varrho\,(\alpha+\alpha^*)\,]}\,|\,f(\,\xi\,)\,|^2,$$

and four $g_{ab}\,(\,t,\,s,\,\alpha^2\,)$, $s = (\,1,\ 2,\ 1/2,\ 3/2\,)$ for $SU(1,1)$, with the symmetric Ψ^S, antisymmetric Ψ^A, and $\Psi_{1/4}$, $\Psi_{3/4}$ states:

$$g_{ab}\left(\,t,\,2,\,\alpha^2\,\right)_{SU(1,1)} = \left\langle \Psi^S(t)\right|\mathbb{L}_{ab}\left|\Psi^S(t)\right\rangle = \mathcal{F}\left(\begin{array}{c}\alpha^2\\ \alpha^{*2}\end{array}\right)_{(2)\,ab} \tag{22}$$

$$g_{ab}\left(\,t,\,1,\,\alpha^2\,\right)_{SU(1,1)} = \left\langle \Psi^A(t)\right|\mathbb{L}_{ab}\left|\Psi^A(t)\right\rangle = \mathcal{F}\left(\begin{array}{c}\alpha^2\\ \alpha^{*2}\end{array}\right)_{(1)\,ab} \tag{23}$$

$$g_{ab}\left(\,t,\,3/2,\,\alpha^2\,\right)_{SU(1,1)} = \langle \Psi_{3/4}(t)|\mathbb{L}_{ab}|\Psi_{3/4}(t)\rangle = \mathcal{F}\left(\begin{array}{c}\alpha^2\\ \alpha^{*2}\end{array}\right)_{(3/2)ab} \tag{24}$$

$$g_{ab}\left(\,t,\,1/2,\,\alpha^2\,\right)_{SU(1,1)} = \langle \Psi_{1/4}(t)|\mathbb{L}_{ab}|\Psi_{1/4}(t)\rangle = \mathcal{F}\left(\begin{array}{c}\alpha^2\\ \alpha^{*2}\end{array}\right)_{(1/2)\,ab} \tag{25}$$

where B and D are given by

$$B = \left(\frac{|\,\mathbf{a}|}{m}\right)^2 c_1, \quad , \quad D = \left(\frac{|\,\mathbf{a}\,|\,c_1}{\sqrt{2}\,m}\right)^2 + c_2, \tag{26}$$

c_1 and c_2 being constants characterizing the solution or its initial conditions.

The dynamical structure of (quantum) spacetime clearly encodes the metric through the coherent basic states, solutions of Equations (15) and (16). Therefore, the spacetime structure defined in this paper through the metrics in Equations (20)–(25) fully and rigorously respect all the properties required in the fundamental quantum regime, as well as in the classical domain.

Equation (25) is expressed in the so-called Sudarshan's-diagonal representation that leads, as an important consequence, to the *physical states* with spin content $\lambda = (\,1/2,\ 1,\ 3/2,\ 2\,)$. Precisely, the generalized coherent states here generate a map that relates the metric solution of the wave equation g_{ab} to the specific subspace of the full Hilbert space where these coherent states live. Moreover, there exists for operators $\in Mp(2)$ an asymmetric—kernel leading, in our case, to the following $\lambda = 1$ state:

$$g_{ab}(\,t,\,1,\,\alpha)\,|_{HW} = \langle \Psi_{3/4}(t)|L_{ab}|\Psi_{1/4}(t)\rangle = \langle \Psi_{1/4}(t)|L_{ab}|\Psi_{3/4}(t)\rangle = \mathcal{F}\left(\begin{array}{c}\alpha\\ \alpha^{*}\end{array}\right)_{(1)\,ab}.$$

This is so because the non-diagonal projector involved in the reconstruction formula of L_{ab} is formed with the $\Psi_{1/4}$ and $\Psi_{3/4}$ states, which span completely the *full* Hilbert space.

Observation 1. *Due to the non-observability of isolated basic states, the spin-zero physical states appear as bounded states* $(\,g\,\overline{g}\,)$*, where* $g_{ab}\,(\,t, s, w\,)$ *and* $\overline{g}_{ab}\,(\,t, s, w\,)$ *are given by the bilinear expressions in Equation (25).*

Observation 2. *Each kernel represents a global physical state composed of fundamental states that separately are basic and unobservable.*

Note that the spectrum of the physical states is labeled not only by their spin content λ, but also by the "eigenspinors" $\left(\begin{array}{c}\alpha\\ \alpha^{*}\end{array}\right)_{(\lambda)\,ab}$ and $\left(\begin{array}{c}\alpha^2\\ \alpha^{*2}\end{array}\right)_{(\lambda)\,ab}$ corresponding to the vector representations of L_{ab} and $\mathbb{L}_{ab}$, respectively (maps over a region of $\mathcal{H}$).

7. Supermetric and Emergent Spacetime

The Lagrangian density from the action of Equation (14) represents a free particle in a superspace with coordinates $z_A \equiv \left(x_\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}\right)$. In these coordinates, the line element of the superspace reads

$$ds^2 \longrightarrow \dot{z}^A \dot{z}_A = \dot{x}^\mu \dot{x}_\mu - 2\,i\,\dot{x}^\mu\,(\,\dot{\theta}\,\sigma_\mu\,\bar{\theta} - \theta\,\sigma_\mu \dot{\bar{\theta}}\,) + (\mathbf{a} - \bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}})\dot{\theta}^\alpha\dot{\theta}_\alpha - (\mathbf{a}^* + \theta^\alpha\theta_\alpha\,)\dot{\bar{\theta}}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}}.$$

It is important to note that, following the steps detailed in Section 4, the quantization is exactly performed, providing the correct physical and mathematical interpretation to the square-root Hamiltonian and the correct spectrum of physical states.

Without loss of generality, and for simplicity, we take the solution of Equation (20) to represent the metric, and with three compactified dimensions ($s = 2$ spin fixed) we have

$$g_{AB}\,(t) = e^{\,A(t)\,+\,\xi\,\varrho\,(t)}\,g_{AB}\,(0), \tag{27}$$

where the initial values of the metric components are given by

$$g_{ab}\,(0) = \langle\psi(0)|\begin{pmatrix} a \\ a^\dagger \end{pmatrix}_{ab}|\psi(0)\rangle, \tag{28}$$

or, explicitly,

$$g_{\mu\nu}(0) = \eta_{\mu\nu}\,, \qquad g_{\mu\alpha}(0) = -\,i\,\sigma_{\mu\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\,, \qquad g_{\mu\dot{\alpha}}(0) = -\,i\,\theta^\alpha\sigma_{\mu\alpha\dot{\alpha}}\,, \tag{29}$$

$$g_{\alpha\beta}(0) = (a - \bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}})\,\epsilon_{\alpha\beta}\,, \qquad g_{\dot{\alpha}\dot{\beta}}(0) = -\,(a^* + \theta^\alpha\theta_\alpha)\,\epsilon_{\dot{\alpha}\dot{\beta}}\,. \tag{30}$$

The bosonic and spinorial parts of the exponent in the superfield solution of Equation (27) are, respectively,

$$\begin{aligned} A(t) &= -\left(\frac{m}{|\mathbf{a}|}\right)^2 t^2 + c_1\,t + c_2, \\ \xi\,\varrho(t) &= \xi(\,\phi_\alpha\,(t) + \bar{\chi}_{\dot{\alpha}}\,(t)\,) \\ &= \theta^\alpha\left(\overset{\circ}{\phi}_\alpha \cos\,(\,\omega\,t/\,2\,) + \tfrac{2}{\omega}\,Z_\alpha\right) - \bar{\theta}^{\dot{\alpha}}\left(-\,\overset{\circ}{\bar{\phi}}_{\dot{\alpha}}\sin\,(\,\omega t/2\,) - \tfrac{2}{\omega}\overline{Z}_{\dot{\alpha}}\right) \\ &= \theta^\alpha\overset{\circ}{\phi}_\alpha\cos(\,\omega t/2\,) + \bar{\theta}^{\dot{\alpha}}\,\overset{\circ}{\bar{\phi}}_{\dot{\alpha}}\,\sin(\,\omega t/\,2\,) + 4\,|\,\mathbf{a}\,|\,Re(\,\theta Z\,), \end{aligned} \tag{31}$$

where $\overset{\circ}{\phi}_\alpha$, Z_α, $\overline{Z}_{\dot{\beta}}$ are constant spinors, $\omega = 1/|\,\mathbf{a}\,|$ and the constant $c_1 \in \mathbb{C}$, due to the obvious physical reasons and the chiral restoration limit of the superfield solution. We see in the next section the associated emerging discrete spacetime structure.

8. Superspace and Discrete Spacetime Structure

Let us see in this section how the discrete spacetime structure emerges naturally from the model under consideration. Expanding on a basis of eigenstates of the number operator,

$$\sum_m |m\rangle\,\langle m| = 1, \tag{32}$$

we have

$$g_{ab}\,(0) = \sum_{n,m}\langle\,\psi(0)\,|m\rangle\,\langle\,m\,|\,L_{ab}\,|\,n\,\rangle\,\langle\,n\,|\,\psi(0)\,\rangle. \tag{33}$$

Then,

$$g_{ab}(t) = \underbrace{e^{A(t)\,+\,\xi\,\rho(t)}}_{f(t)}\sum_{n,m}\langle\psi(0)\,|\,m\rangle\,\langle n\,|\,\psi(0)\,\rangle\,\langle m|\,L_{ab}\,|\,n\rangle$$

$$\langle m\,|\,L_{ab}\,|\,n\rangle = \langle m|\begin{pmatrix} a \\ a^\dagger \end{pmatrix}_{ab}|n\rangle = \begin{pmatrix} \langle m|n-1\rangle\sqrt{n} \\ \langle m|n+1\rangle\sqrt{n+1} \end{pmatrix}_{ab} = \begin{pmatrix} \delta_{m,n-1}\sqrt{m} \\ \delta_{m,n+1}\sqrt{m+1} \end{pmatrix}_{ab} \tag{34}$$

It follows,

$$g_{ab}(0) = \sum_{n,m} \langle\, \psi(0)\,|m\rangle \begin{pmatrix} \delta_{m,n-1}\sqrt{m} \\ \delta_{m,n+1}\sqrt{m+1} \end{pmatrix}_{ab} \langle\, n\,|\,\psi(0)\,\rangle$$

$$g_{ab}(0) = \sum_{n} \sqrt{n}\ \langle\, \psi(0)\,|\,n-1\,\rangle\ \langle n|\,\psi(0)\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{ab} + \sum_{m} \sqrt{n+1}\ \langle\, \psi(0)\,|n+1\rangle\ \langle n|\,\psi(0)\,\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{ab}$$

According to the equation above, the splitting of ψ into the fundamental states of the metapletic representation is the only explanation that makes sense:

$$|\,\psi(0)\,\rangle \;=\; A\,|\alpha_+\rangle + B\,|\alpha_-\rangle. \tag{35}$$

Consequently, at the macroscopic level, the arbitrary constants A and B govern the spectrum's classical behavior. Without losing generality, we assume for the purposes of this discussion that $A = B$, such that $|\psi(0)\rangle = |\alpha_+\rangle + |\alpha_-\rangle$. However, we will come back to this crucial point later.

This is the outcome of the $SO(2,1)$ group's breakdown into two irreducible representations of the metaplectic group $Mp(2)$, spanning even and odd n, respectively.

Let us highlight the important property of the state $|\,\psi(0)\,\rangle = |\,\alpha_+\,\rangle\ +\ |\alpha_-\rangle$, which (if $A = B$) is invariant to the action of the operators a and $a^\dagger$. This is a consequence of the fact that in the metaplectic representation the general behaviors of these states are $a\,|\alpha_+\rangle = a^\dagger\,|\alpha_+\rangle = |\alpha_-\rangle$ and $a\,|\alpha_-\rangle = a^\dagger|\alpha_-\rangle = |\alpha_+\rangle$.

Statistical Distributions and Classical Limit

From the Poissonian distribution for the coherent states, we can see

$$P_\alpha(n) = |\,\langle\, n\,|\,\alpha\,\rangle\,|^2 = \frac{\alpha^n e^{-\alpha}}{n!}$$

fulfilling

$$\sum_{n=0}^{\infty} P_\alpha(n) = 1, \quad \sum_{n=0}^{\infty} n\, P_\alpha(n) \;=\; \alpha.$$

This is different from the individual distributions defined from each one of the two irreducible representations of the metaplectic group Mp(2) (which span even and odd n, respectively):

$$\left.\begin{array}{l} \sum_{n=0}^{\infty} P_{\alpha_+}(2n) \;=\; e^{-\alpha}\cosh(\alpha) \\ \sum_{n=0}^{\infty} P_{\alpha_-}(2n+1) \;=\; e^{-\alpha}\sinh(\alpha) \end{array}\right\} \;\rightarrow\; \sum_{n=0}^{\infty} (P_{\alpha_+}(n) \;+\; P_{\alpha_-}(n)) \;=\; 1. \tag{36}$$

Note that in spite of the different form between the above equations the limit $n \to \infty$ is the same for both—the sum of the two distributions arising from the Mp(2) irreducible representations (IR) and for the $SO(2,1)$ representation, as it must be.

Taking this into account, the explicit form of $|\,\alpha_+\,\rangle$, $|\alpha_-\rangle$ is given by

$$|\,\alpha_+\,\rangle \;\equiv\; |\,\Psi_{1/4}(0,\xi,q)\,\rangle \;=\; \sum_{k=0}^{+\infty} f_{2k}(0,\xi)|2k\rangle \;=\; \sum_{k=0}^{+\infty} f_{2k}(0,\xi\,)\frac{\left(a^\dagger\right)^{2k}}{\sqrt{(2k)!}}\;|0\rangle \tag{37}$$

$$|\alpha_-\rangle \equiv |\,\Psi_{3/4}\,(\,0,\xi,q\,)\,\rangle \;=\; \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)\;|2k+1\rangle \;=\; \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)\frac{\left(a^\dagger\right)^{2k+1}}{\sqrt{(2k+1)!}}|0\rangle,$$

where all the possible odd n dependence is stored in the parameter ξ.

Consequently, $|\alpha_+\rangle$ connects only with the *even* vectors of the basis number and $|\alpha_-\rangle$ with the *odd* vectors of the basis number. Therefore, using the decomposition of

Equation (35), and decomposing the base number $|n\rangle$ into *even* and *odd*, we obtain the following explicit result for the spacetime metric:

$$g_{ab}(t) = \frac{f(t)}{2}\sum_m \Big\{ [\, P_{\alpha_+}(2m)\cdot 2m + P_{\alpha-}(2m+1)\cdot(2m+1)\,] \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{ab} + \\ + \Big[P_{\alpha_+^*}(2m)\cdot 2m + P_{\alpha_-^*}(2m+1)\cdot(2m+1) \Big] \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{ab} \Big\} \quad (38)$$

The expression above is an important pillar of our findings here: in this equation, *the discrete structure of spacetime* is shown explicitly as the fundamental basic feature of a consistent quantum-field theory of gravity.

On the other hand, in the limiting case $n \to \infty$ our solution for metric is the continuum, as it must be:

$$\sum_{n=0}^{\infty} [P_{\alpha_+}(2m)\cdot 2m + P_{\alpha-}(2m+1)\cdot(2m+1)] = \alpha\, e^{-|\alpha|}(\cosh(\alpha)+\sinh(\alpha)) = \alpha$$

Similarly, for the lower part (spinor down) of the above equation, we obtain

$$\sum_{n=0}^{\infty} [P_{\alpha_+}(2m)\cdot 2m + P_{\alpha_-}(2m+1)\cdot(2m+1)] = \alpha^*.$$

Therefore, when the number of discrete levels increases, our metric solution goes to the general-relativistic-continuum-"manifold" behavior:

$$g_{ab}(t)_{n\to\infty} \;\to\; \frac{f(t)}{2}\left\{\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{ab} + \alpha^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{ab}\right\} = f(t)\langle\, \psi(0) \begin{pmatrix} a \\ a^\dagger \end{pmatrix}_{ab} |\, \psi(0)\,\rangle, \quad (39)$$

as expected.

9. The Lowest n = 0 Level and Its Length

It is not difficult to see that for the number $n = 0$ the metric solution takes the value

$$\begin{aligned} g_{ab}(t) &= \frac{f(t)}{2}\left[P_\alpha(1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{ab} + P_{\alpha_-^*}(1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{ab} \right] \\ &= \frac{f(t)}{2}\, e^{-|\alpha|}\left[\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{ab} + \alpha^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{ab} \right]. \end{aligned} \quad (40)$$

This evidently defines an associated characteristic length for the eigenvalues α, α^* because of the metric axioms in a Riemannian manifold. In principle, fundamental symmetries such as the Lorentz symmetry can be preserved at this level of discretization, due to the existence of discrete Poincare subgroups of this supermetric.

10. Implications for Black-Hole Entropy: A Superspace Solution

Black-hole entropy, $S = k_B\, A_{bh}\,/4\, l_P^2$—where A is the horizon area and $l_P \equiv \sqrt{\hbar G/c^3}$ is the Planck length—as is well known, was first found by Bekenstein and Hawking [25,26], using the thermodynamic arguments of the preservation of the first and second laws of thermodynamics.

Also found by Bekenstein was an information-theory proof, in which black-hole entropy is treated as the measure of the "inaccessible" information for an external observer in an actual internal configuration of the black hole in a given state. Such a state is described by the values of mass, charge and angular momentum.

From the statistical-mechanics viewpoint, the entropy is the mean logarithm of the density matrix. About this issue, Bekenstein proposed a model of quantization of the horizon area with the title "Demystifying black hole's entropy proportionality to area", Refs. [27,28].

Following the same reasoning, the horizon is formed by patches or cells of equal area δl_P^2. Consequently, the horizon can be considered as endowed with many fundamental degrees of freedom: one degree per each patch. Therefore, the horizon does appear to be composed of fundamental patches, all having an equal number χ of quantum states.

Then, the horizon has a total number of quantum states given by $\Omega_H = \chi^{A_{bh} / \delta l_P^2}$, and the Boltzmann statistical entropy due to the horizon is $S = k_B \ln \Omega_H = k_B \left(A_{bh} / \delta l_P^2 \right) \ln \chi$.

The choice $\delta = 4 \ln \chi$ yields the expected thermodynamical-black-hole Bekenstein's formula.

Introducing δ into the original black-hole entropy formula, one obtains the Poisson expression for the total number of states:

$$\Omega_H = e^{A_{bh} / 4 l_P^2}. \tag{41}$$

This expression is explicitly *the bridge* with the structure of the emergent coherent-state metric of our approach here. We consider a similar Poissonian expression for the number of states from g_{ab}, namely $e^{|\alpha|}$; therefore, the relation between the coherent-state eigenvalue α corresponding to our coherent-state metric solution and the above equation does appear:

$$A_{bh} / 4 l_P^2 = |\alpha|. \tag{42}$$

This expression links the black-hole area A_{bh} and the phase space of the coherent-state solution metric g_{ab} through the Planck length l_P^2 and the eigenvalue $|\alpha|$ characterizing the coherent states.

11. Implications for Hawking Radiation

- As is known, the area of a black hole is related to its mass. Consequently, the black-hole mass in our approach here is quantized as well. The emitted radiation from the black hole does appear because of the quantum jump from one quantized value of the mass (energy) to a lower quantized value. The decreasing of the black-hole mass occurs because of this process.
- Therefore, (and because radiation is emitted at quantized frequencies corresponding to the differences between energy levels), quantum gravity implies a discretized emission spectrum for black-hole radiation.
- The spectral lines can be very dense in macroscopic regimes, leading physically to no contradiction with Hawking's prediction of a continuous thermal spectrum in the semi-classical regime.
- From the point of view of our approach here:
- If we now suppose simply that the constants A, B in the state solution, e.g., Equations (27) and (35), are different, $A \neq B$, we have

  $$|\psi(0)\rangle = A\,|\,\alpha_+\,\rangle + B\,|\,\alpha_-\,\rangle.$$

- Therefore, the thermal (Hawking) spectrum at the macroscopic or semi-classical level does not appear.
- This fact is clearly explained because an exact balance between the superposition of the two irreducible representations of the metaplectic group is needed.
  This will lead, as a result, to non-classical states of radiation in the sense of [29], as can be easily seen by making, for example, one of the constants, B (or A), equal to zero:

$$g_{ab}(t) = A\ \frac{f(t)}{2} \sum_m [P_{\alpha+}(2m)\cdot(2m) + P_{\alpha-}(2m+1)\cdot(2m+1)] \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{ab} \tag{43}$$

- Note that only the up spinor part survives in this case, and the classical thermal limit is not attained. This is so, even in the continuous limit for this case, in which the number of levels increases accordingly to

$$g_{ab}(t)\ _{n\to\infty}\ \to\ \frac{f(t)}{2}\ A\ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{ab} = A\ f(t)\langle\ \psi(0)\ |\begin{pmatrix} a \\ 0 \end{pmatrix}_{ab} |\ \psi(0)\ \rangle. \tag{44}$$

- Consequently, in such a case, where $A = 0$ (or $B = 0$), the spectrum will take only *even (or odd)* levels, becoming evidently non-thermal.

  Therefore,

- In cases where $A = B$, the thermal Hawking spectrum is attained at the continuum classical gravity level, e.g., the Poissonian behavior of the distribution is complete.
- In cases where $A \neq B$, the spectrum is non-classical, and the quantum properties of gravity are manifest at the macroscopic level.

12. Concluding Remarks

Here, we have demonstrated that there is a principle of minimal group representation that allows us to consistently and simultaneously obtain a natural description of the dynamics of spacetime and the physical states admissible in it.

The theoretical construction is based on the fact that the physical states are, roughly speaking, average values of the generators of the metaplectic group Mp(n) in a vector representation, with respect to the coherent states that are not observable (carrying the weight of spin). Schematically, we have the following picture, where $M_{ab} = \mathbb{L}_{ab}(L_{ab})$:

$$\underbrace{g_{ab}\ =\ \langle\ \psi\ |\ M_{ab}\ |\ \psi\ \rangle}_{\textit{Physical States, spacetime metric (Observables)}}$$

$$\nearrow \qquad\qquad\qquad \nwarrow$$

$$\begin{matrix} Mp(n) \ni \\ \textit{Coherent States} \\ (\textit{basic states}) \end{matrix} \left\{ \begin{matrix} \psi_+, \psi_-, \\ \psi_A, \psi_S \end{matrix} \right. \qquad \longleftrightarrow \qquad \begin{matrix} \textit{Generators}: \ M_{ab} \\ (\textit{group manifold,} \\ \textit{phase space} \\ \textit{symmetries}) \end{matrix}$$

In summary:

(1) We have demonstrated that there is a connection between the dynamics given by the generators of the symmetries and the physically admissible states.

(2) The physically admissible states are mappings of the generators of the relevant symmetry groups covered by the metaplectic group, in the simplest case, according to the chain $Mp(2) \supset SL(2R) \supset SO(1,2)$, through a bilinear combination of basic states.

(3) The ground states are coherent states defined by the action of the metaplectic group (the Perelomov–Klauder type); these states divide the Hilbert space into *even* and *odd* states, and are mutually orthogonal. They carry a weight of spin of $1/4$ and $3/4$, respectively.

(4) From the basic states combined symmetrically and antisymmetrically, two other basic states can be formed. These new states manifest a change of sign under the action of the creation operator a^+.

(5) The physically admissible states, mapped bilinearly with the basic states with spin weight $1/4$ and $3/4$, plus their symmetric and antisymmetric combinations, have spin contents $\mathrm{s}\ =\ 0,\ 1/2,\ 1, 3/2\ \text{and}\ 2$.

(6) A symmetry of the superspace is formed by a realization of the generators with bosonic variables of the harmonic oscillator as Lagrangian. Taking a line element corresponding to such a superspace, a physical state of spin 2 can be obtained and related to the metric tensor.

(7) The metric tensor is discretized simply by taking the discrete series given by the basic states (coherent states) in the number n representation; consequently, the metric tends to the classical (continuous) value when $n \to \infty$.

(8) The results of this paper have implications for the lowest level of the discrete spectrum of spacetime, the ground state associated to $n = 0$ and its characteristic length, in the black-hole history of black-hole evaporation.

(9) Moreover, recently we have successfully applied this general approach in physical scenarios of current interest, obtaining coherent states of quantum spacetime for black holes and de Sitter spacetime, in our Ref. [21].

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