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PROPAGATOR ZEROS, VERTEX POLES AND THE Z → 0 LIMIT*

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ABSTRACT

We prove the cancellation mechanism between the poles of the proper vertex function and the zeros of the propagator even when the propagator satisfies a subtracted Källen-Lehmann representation, so that the vertex pole does not lead to a pole in the scattering amplitude in general. The dynamical origin of the vertex poles is discussed. The renormalization function is constructed in two ways to determine the residues of the pole terms of the inverse propagator in terms of the driving force. Then we show how the compositeness conditions used by other authors follow from our construction of the various field-theoretic and S-matrix quantities. Discussions on the high energy behavior of those functions are also briefly given.

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1. INTRODUCTION

It is generally the case in quantum field theory that the coupling constants and the masses of the interacting particles can assume irrespective values from each other. Within the framework of the S-matrix theory, however, the coupling constant can be expressed by certain equations evaluated at the particle mass. Some time ago, it was argued that[1] the restrictions on the magnitudes of the coupling constants at any given mass follow from the general principles of quantum field theory with no additional assumption. The assumptions made in this discussion are that the propagator satisfies an unsubtracted Lehmann representation[2] and the proper vertex function has no pole in the complex s-plane. It has been pointed out since then by Goebel and Sakita[3] that the assumption of no poles in the proper vertex function has no direct physical significance and that the upper bound on the coupling constants so obtained has likewise no direct physical significance. In particular, they have shown that it is possible for the propagator to have a zero coupled to a pole of the vertex function as well as the dynamical origin of the vertex pole in non-relativistic elastic models. By extending this argument, it has also been shown that[4] a vertex pole does not lead to a pole in the scattering amplitude. A physical interpretation of this cancellation mechanism has been given in the work of Jin and MacDowell[4] by considering the propagator as a continuous function of the coupling constant. However, in the proof, they have again assumed that the propagator obeys an unsubtracted spectral representation. Thus only those zeros existing between the particle pole and the threshold have been considered in this work.

If, however, the propagator needs a subtraction[5], then it necessarily has a zero to the left of the particle pole, and it is not immediately clear if the cancellation theorem still holds for this inevitable zero. This problem is attacked in this note. We prove that the cancellation theorem holds in general. However, the zero of the propagator below the particle pole cannot be explained as emerging from the second Riemann sheet through the elastic branch cut by increasing the coupling strength. Such an interpretation can be used easily to the zero between the particle pole and the threshold. But to apply the same mechanism to the zero below the particle pole, some explanation should be given for the zero to cross over the particle pole. Thus we have simply attributed the zeros in the propagator to the dynamical origin, i.e., the characteristics of the driving force. We note however that one could interpret the zero to the left of the particle pole as coming from $-\infty$ when the coupling strength was increased[6].

Indeed, we shall show if the driving force is such that it generates a pole at some point in the irreducible part of the scattering amplitude, then the proper vertex function and the re-normalization function will have a pole while the propagator will have a zero at the same point. If there are no poles generated in the irreducible part, then the propagator will have no zeros. If a pole is generated between the particle pole and the threshold, then the propagator will have a zero there but if a pole is generated below the particle pole then the propagator function should obey a one-subtracted dispersion relation in such a way to have a zero there. Therefore we will be able to relate the asymptotic behaviors of the various field-theoretic and S-matrix functions to each other. In general, the case with a pole in the proper vertex function below the particle position yields additional sum rule. If the zeros in the propagator are due to the characteristics of the driving force, one naturally expects that the position of the zero and the slope at the zero are not arbitrary parameters. This is also shown by introducing an alternative representation[7] for the inverse propagator function. It is through this representation that we prove the cancellation theorem in general.

Then we show how the limits $Z \to 0$ and $Z_1 \to 0$[8] reproduces the compositeness conditions as defined in the S-matrix theory. We shall see that the existence of a propagator zero is important and moreover such existence is implied by these limits. Also it will be seen how $Z \to 0$ and $Z\delta m^2 \to 0$[9] follow from them. We will briefly sketch the asymptotic behaviors of the various functions for various values of Z and Z_1. This consideration has led some authors to notice the four different situations for the amplitude. In particular, we shall obtain Table 1 which tells for instance that in the limits $Z \to 0$ and $Z_1 \to 0$, not only the irreducible part but also the propagator function does need one subtraction, while the whole amplitude does not.

In Section 2, we obtain the inverse propagators from the Herglotz[10] property. From this, the renormalization function and the mass renormalization are derived. Section 3 deals with the scattering amplitude. By introducing the well-known decomposition of the amplitude[4,8], we relate the field-theoretic quantities to the S-matrix quantities. The proof of the cancellation theorem is given in Section 4. Also, $Z_1 \to 0$, and $Z \to 0$ are investigated in connection with the compositeness conditions.

2. THE RENORMALIZATION FUNCTION Z(s)

Let us consider the Lehmann representation[2] for the propagator with no subtractions

$$\Delta'(s) = \frac{1}{m^2 - s} + \frac{1}{\pi}\int_{s_t}^{\infty} ds' \, \frac{\rho(s')|\Lambda(s')|^2}{s' - s} \tag{1}$$

and with one subtraction

$$\Delta'(s) = \frac{1}{m^2 - s} + \alpha + \frac{s - m^2}{\pi}\int_{s_t}^{\infty} ds' \, \frac{\rho(s')|\Lambda(s')|^2}{(s' - s)(s' - m^2)} \tag{2}$$

The spectral function in Eq. (1) and (2) is positive definite and thus the propagator given by Eq. (1) or (2) is a Herglotz function[10]. It has no complex zeros but it may have a real zero between m^2 and s_t. But the propagator given by Eq. (2) must have a real zero below the particle pole. Notice that the latter kind of zeros is originated from the high energy information of the propagator, while the former is from the low energy behavior. The quantity $\Lambda(s)$ is related to the form factor $F(s)$ by

$$F(s) = (m^2 - s)\ \Lambda(s) \tag{3}$$

and the proper vertex function $\Gamma(s)$ and the propagator by

$$\Lambda(s) = \Gamma(s)\ \Delta'(s) \tag{4}$$

from which it follows that

$$\mathrm{Im}\ \Delta'^{-1}(s) = \rho(s)|\Gamma(s)|^2 \tag{5}$$

Here $\rho(s)$ is the phase-space factor which approaches a constant at infinity.

At this point, we recall some properties of the Herglotz function. If $H(z)$ is a Herglotz function then $-H^{-1}(z)$ is also a Herglotz function and it has a representation

$$H(z) = A + Bz + \frac{z}{\pi}\int \frac{\operatorname{Im} H(x)}{x(x - z)}\, dx + \sum_i \frac{R_i}{z_i - z} \tag{6}$$

with $B \geq 0$, $\operatorname{Im} H(z) \geq 0$, $R_i \geq 0$ and z_i are the CDD poles of $H(z)$. From Eq. (6) it follows that for $\epsilon < \arg z < \pi - \epsilon$,

$$B = \lim_{z \to \infty} H(z)/z \tag{7}$$

and

$$C|z|^{-1} \leq |H(z)| \leq C'|z| \tag{8}$$

Furthermore, Eq. (6) implies that

$$\int \frac{\operatorname{Im} H(x)}{x^2}\, dx < \infty \ , \ \int \frac{\operatorname{Im} H(x)}{x^2\, H(x)^2}\, dx < \infty \tag{9}$$

If $H(z)$ has only one cut on the real axis, then one can further show that[11] the two integrals

$$\int \frac{\operatorname{Im} H(x)}{x}\, dx \qquad \text{and} \qquad \int \frac{\operatorname{Im} H(x)}{x\, H(x)^2}\, dx$$

cannot both diverge at the same time.

It is easy to show for both Eq. (1) and (2) that the inverse propagator has a solution

$$\Delta'^{-1}(s) = (m^2 - s)\left[1 + \frac{s - m^2}{\pi}\int_{s_t}^{\infty} ds'\, \frac{\rho(s')\,|\Gamma(s')|^2}{(s' - m^2)^2(s' - s)} + \sum_i \frac{R_i(s - m^2)}{(s_i - s)(s_i - m^2)^2}\right] \tag{10}$$

The pole terms in Eq. (10) represent the zeros of the propagator. Thus for

Eq. (1), there are two situations, i.e. the inverse propagator has no or one pole between the particle pole and the threshold. For Eq. (2), there are again two situations, either one or two poles because of the inevitable pole below m^2. For both Eq. (1) and (2), we have

$$\int_{s_t}^{\infty} \frac{\rho(s)|\Gamma(s)|^2}{s^2}\, ds < \infty \tag{11}$$

but for Eq. (2) we have further

$$\int_{s_t}^{\infty} \frac{\rho(s)|\Gamma(s)|^2}{s}\, ds < \infty \tag{12}$$

Let us define the renormalization function $Z(s)$ by

$$Z^{-1}(s) = (m^2 - s)\, \Delta'(s) \tag{13}$$

By taking the limit $s \to \infty$, we get the wave function renormalization constant

$$Z = 1 - \frac{1}{\pi} \int_{s_t}^{\infty} ds\, \frac{\rho(s)|\Gamma(s)|^2}{(s - m^2)^2} - \sum_i \frac{R_i}{(s_i - m^2)^2} \tag{14}$$

Because of Eq. (11), the quantity

$$Z(s) - Z = \frac{1}{\pi} \int_{s_t}^{\infty} ds'\, \frac{\rho(s')|\Gamma(s')|^2}{(s' - s)(s' - m^2)} + \sum_i \frac{R_i}{(s_i - m^2)(s_i - s)} \tag{15}$$

does exist both for Eq. (1) and (2). In particular, for Eq. (2)

$$\lim_{s \to \infty} s\,[Z(s)-Z] = -\frac{1}{\pi}\int_{s_t}^{\infty} \frac{\rho(s)\left|\Gamma(s)\right|^2}{s-m^2}\,ds - \sum_i \frac{R_i}{s_i - m^2} \tag{16}$$

exists further because of Eq. (12). Although the integral on the right hand side is positive definite, Eq. (16) can be made zero in this case as there must be a pole below m^2.

If Z vanishes, we obtain from Eq. (14) the sum rule

$$1 = \frac{1}{\pi}\int_{s_t}^{\infty} ds\,\frac{\rho(s)\left|\Gamma(s)\right|^2}{(s-m^2)^2} + \sum_i \frac{R_i}{(s_i - m^2)^2} \tag{17}$$

and from Eq. (15) that

$$Z(s) = \frac{1}{\pi}\int_{s_t}^{\infty} ds'\,\frac{\rho(s')\left|\Gamma(s')\right|^2}{(s'-s)(s'-m^2)} + \sum_i \frac{R_i}{(s_i - m^2)(s_i - s)} \tag{18}$$

which satisfies $Z(m^2) = 1$ due to Eq. (13). We further have from Eq. (16) that

$$-\lim_{s \to \infty} sZ(s) = \frac{1}{\pi}\int_{s_t}^{\infty} ds\,\frac{\rho(s)\left|\Gamma(s)\right|^2}{s-m^2} + \sum_i \frac{R_i}{s_i - m^2} \tag{19}$$

which would result another sum rule if $sZ(s)$ vanishes in the limit $s \to \infty$. Since, however, Eq. (1) has no zeros below m^2 we can never have this second sum rule for the unsubtracted Lehmann representation. It is also obvious that Eq. (16) can never be made zero for Eq. (1) even when a relation like Eq. (12) holds.

We note that the mass renormalization $\delta m^2 \equiv m^2 - m_0^2$ is given by

$$\delta m^2 = \frac{1}{Z} \lim_{s \to \infty} (s - m^2)[Z(s)-Z] \tag{20}$$

thus

$$-Z\delta m^2 = \frac{1}{\pi}\int_{s_t}^{\infty} \frac{\rho(s)|\Gamma(s)|^2}{s - m^2} ds + \sum_i \frac{R_i}{s_i - m^2} \tag{21}$$

from Eq. (16). It should be mentioned that $Z\delta m^2$ always exists for Eq. (2) while it may not for Eq. (1). In particular, $Z\delta m^2$ can never be made zero for Eq. (1). This follows from the fact that $Z\delta m^2$ has the same representation as that of $\lim_{s \to \infty} s[Z(s)-Z]$.

To sum us, if the propagator does not need any subtraction, $Z\delta m^2$ may not be finite in general and can never become zero regardless $Z = 0$ or not, while if the propagator is given by Eq. (2), $Z\delta m^2$ is always finite and can be made zero for both $Z = 0$ and $Z \neq 0$.

Finally, from the Herglotz property of the propagator we notice $-\lim_{s \to \infty} [s\Delta'(s)]^{-1}$ is positive definite. This implies $Z \geq 0$ from Eq. (13) both for Eq. (1) and (2). On the other hand Z is bounded by 1 from Eq. (14), so that we get the familiar result $0 \leq Z \leq 1$ both for Eq. (1) and (2). Furthermore because of Eq. (8), one gets $C'/|s|^2 \leq |Z(s)| \leq C$.

3. THE SCATTERING AMPLITUDE

To see the origin of zeros of the propagator functions, we consider the s-wave amplitude $T(s)$ for the two spinless equal mass particle scattering in the presence of a particle with mass m and coupling constant g^2 having the quantum numbers of this channel. The scattering amplitude can be expressed by a dispersion relation

$$T(s) = \lambda + \frac{s - s_0}{\pi} \int_{s_t}^{\infty} ds' \frac{\mathrm{Im}\, T(s')}{(s' - s_0)(s' - s)} + \frac{s - s_0}{\pi} \int_L \frac{f(s')}{(s' - s_0)(s' - s)} + \frac{g^2 (s - s_0)}{(m^2 - s)(m^2 - s_0)} \tag{22}$$

where $f(s)$ is the given discontinuity across the left hand cut L.

Following the usual N/D method[12], we write the amplitude

$$T(s) = N(s)/D(s) \tag{23}$$

where

$$D(s) = 1 - \frac{s - s_0}{\pi} \int_{s_t}^{\infty} ds' \frac{\rho(s') N(s')}{(s' - s_0)(s' - s)}$$

$$N(s) = \lambda + \frac{s - s_0}{\pi} \int_L ds' \frac{f(s') D(s')}{(s' - s_0)(s' - s)} + \frac{g^2 D(m^2)(s - s_0)}{(m^2 - s)(m^2 - s_0)} \tag{24}$$

Here $\rho(s)$ is the same phase-space factor introduced before and normalizes the amplitude by

$$T(s) = \rho^{-1}(s)\, e^{i\delta(s)} \sin \delta(s) \tag{25}$$

In general, the denominator function may have pole terms corresponding to zeros of $T(s)$. If we, however, assume that the scattering length is negative and the value of $T(s)$ at the left hand branch point is positive then no such pole terms may occur in $D(s)$[13]. In the absence of the particle, the above procedure would have resulted in a unitary amplitude

$$t(s) = \rho^{-1}(s)\, e^{i\delta_0(s)} \sin \delta_0(s) \tag{26}$$

whose N/D solution has the numerator and denominator functions of the form

$$\begin{aligned} d(s) &= 1 - \frac{s - s_0}{\pi} \int_{s_t}^{\infty} ds' \, \frac{\rho(s')n(s')}{(s'- s)(s'- s_0)} \\ n(s) &= \lambda + \frac{s - s_0}{\pi} \int_{L} ds' \, \frac{f(s')d(s')}{(s'- s)(s'- s_0)} \end{aligned} \tag{27}$$

Although T(s), N(s) and n(s) are subtracted, this is done so formally[14] and we shall see that in certain cases they do not require any subtractions. Then one can easily find that[15]

$$T(s) = t(s) + \frac{g^2}{m^2 - s} \frac{D(m^2)d(m^2)}{D(s)d(s)} \tag{28}$$

If we denote the second term in Eq. (28) as

$$a(s) = \frac{g^2}{m^2 - s} \frac{D(m^2)d(m^2)}{D(s)d(s)} \tag{29}$$

its analytic property admits a representation

$$a(s) = \frac{g^2}{m^2 - s} + \frac{1}{\pi} \int_{s_t}^{\infty} ds' \, \frac{\mathrm{Im}\ a(s')}{s' - s} \tag{30}$$

where the absorptive part can be obtained from the unitarity conditions of T(s) and t(s)

$$\mathrm{Im}\ a(s + i\epsilon) = \rho(s)\left|a(s + i\epsilon)\right|^2 + 2\,\rho(s)\,\mathrm{Re}[a^*(s + i\epsilon)t(s)] \tag{31}$$

Although the imaginary part given by Eq. (31) is not positive definite, we can deduce a function, after some usual manipulation[16], whose imaginary part is positive definite. Namely, the quantity defined by

$$A(s) = a(s)d^2(s)/g^2d^2(m^2) = \frac{1}{m^2 - s}\,\frac{D(m^2)}{D(s)}\,\frac{d(s)}{d(m^2)} \tag{32}$$

can be shown to satisfy

$$A(s) = \frac{1}{m^2 - s} + \frac{1}{\pi}\int_{s_t}^{\infty} ds'\,\frac{g^2\rho(s')d^2(m^2)}{s' - s}\left|\frac{A(s')}{d(s')}\right|^2 \tag{33a}$$

or in the one subtracted form

$$A(s) = \beta + \frac{1}{m^2 - s} + \frac{s - m^2}{\pi}\int_{s_t}^{\infty} ds'\,\frac{g^2\rho(s')d^2(m^2)}{(s' - s)(s' - m^2)}\left|\frac{A(s')}{d(s')}\right|^2 \tag{33b}$$

We note that $A(s)$ is a Herglotz function and has the same analytic property as that of the propagator function Eq. (1) or (2). It should also be noticed from Eq. (32) that a zero of $d(s)$ coincides with a zero of $A(s)$.

Now let us compare the scattering amplitude

$$T(s) = t(s) + \frac{g^2d^2(m^2)}{d^2(s)}\,A(s) \tag{34}$$

with the well-known decomposition[4,8]

$$T(s) = t(s) + \Gamma(s)\Delta'(s)\Gamma(s) \tag{35}$$

From Eqs. (3), (4) and (35), we have

$$T(s) = t(s) + \frac{F(s)\Gamma(s)}{m^2 - s} \tag{36}$$

By making use of the unitarity conditions, one can easily verify that[15] $F(s)$ has the same phase as $T(s)$ while $\Gamma(s)$ has the phase of $t(s)$. Thus from Eq. (28) and (36), we can put

$$F(s) = \frac{gD(m^2)}{D(s)} \tag{37}$$

and

$$\Gamma(s) = \frac{gd(m^2)}{d(s)} \tag{38}$$

Upton inserting Eq. (38) into (35) and comparing with Eq. (34), we realize that $A(s)$ so constructed is nothing but the propagation function $\Delta'(s)$. We observe that a zero of $d(s)$ results in a pole of the proper vertex function $\Gamma(s)$ and a zero of the propagator $\Delta'(s)$.
Also it follows that

$$Z(s) = \frac{d(m^2)}{d(s)} \frac{D(s)}{D(m^2)} \tag{39}$$

which ensures $Z(m^2) = 1$. A zero of $d(s)$ thus a zero of $\Delta'(s)$ corresponds to a pole in $Z(s)$. From the Herglotz property of $A(s)$ we find

$$- \lim_{s \to \infty} [sA(s)]^{-1} = \frac{D(\infty)}{d(\infty)} \frac{d(m^2)}{D(m^2)} \geq 0 \tag{40}$$

Notice that a zero of $d(s)$ below m^2 does necessarily imply $Z = 0$. From Eq. (39), if $d(s)$ has a zero at a point to the left of m^2, then the only way to be consistent with Eq. (40) is such that $d(s)$ grows faster than $D(s)$ as $s \to \infty$. Otherwise the limit would be nagative. However, a zero of $d(s)$ between m^2 and the threshold is consistent with $0 \leq Z \leq 1$. Remember that a zero in $d(s)$ of Eq. (27) arises from the dynamics of the driving force. Thus the zeros of the propagator and the poles of the proper vertex function are determined by the given left hand discontinuity.

Hence we consider in what follows the driving force which makes Eq. (27) meaningful. Namely, the left hand cut discontinuity is bounded by

$$|f(s)| < \frac{C}{(\ln s)^2} \tag{41}$$

Then by examining the integral equation

$$d(s) = 1 - \lambda R(s,s_0) - \frac{s - s_0}{\pi}\int_L dx \frac{R(s,x)f(x)d(x)}{(x-s)(x-s_0)} \tag{42}$$

where

$$R(a,b) = \frac{a-b}{\pi}\int_{s_t}^{\infty} dx \frac{\rho(x)}{(x-a)(x-b)}$$

one finds that $d(s)$ grows asymptotically slower than or equal to $\ln s$. Thus we shall consider two types of asymptotic behavior for $d(s)$ in our general discussions, either constant or $\ln s$.

To this end, we mention that if the particle in $T(s)$ is composite then the elementary pole term in Eq. (22) would not appear so that $T(s) \rightarrow [t(s)]_{\text{elementary}} = [T(s)]_{\text{composite}}$ and the particle would be generated by the driving force through

$$D(m^2) = 0, \quad g^2 = -\frac{N(m^2)}{D'(m^2)} \tag{43}$$

When one wants to determine the parameters of the particle by the S-matrix theoretic conditions Eq. (43), it is usually desired to have no other arbitrary parameters in the amplitude such as the subtraction constant λ. Also one should rule out the possibility of having coinciding zeros of $N(s)$ and $D(s)$.

4. THE CANCELLATION THEOREM AND $Z \to 0$ LIMIT

Now, we show in general that the poles of the proper vertex function do not lead to poles in the scattering amplitude $T(s)$. For this purpose, we introduce a function defined by[6,17]

$$G(s) = \frac{g^2 d^2(m^2)}{(s - m^2)} [F(s) - F(m^2) - (s - m^2)F'(m^2)] \tag{44}$$

where

$$F(s) = [n(s)d(s)]^{-1} \tag{45}$$

The function $G(s)$ has two cuts, physical as well as unphysical. The point $s = m^2$ is regular and $G(m^2) = 0$ by the definition Eq. (44). In addition to the cuts, $G(s)$ has poles at the zeros of $d(s)$ and $n(s)$. Remembering that $Z(s)$ has also poles at the zeros of $d(s)$, we can immediately get

$$\begin{aligned} G(s) - \frac{s - m^2}{\pi} \int_L ds' \frac{g^2 d^2(m^2)}{(s' - m^2)^2 (s' - s) d(s')} \operatorname{Im}\left(\frac{1}{n(s')}\right) \\ - \sum_{\ell} \frac{g^2 d^2(m^2)}{(s - s_\ell)(s_\ell - m^2) n'(s_\ell) d(s_\ell)} = Z(s) - 1 \end{aligned} \tag{46}$$

where s_ℓ denotes the possible zeros of $n(s)$. We mention again that $d(s)$ has no more than one zero to the right of m^2 as long as $Z \neq 0$. For $Z = 0$, there can be two zeros, one above and the other below m^2. From Eq. (46), one can determine the residues R_i of $Z(s)$ at a zero s_i of $d(s)$,

$$R_i = - \frac{g^2 d^2(m^2)}{n(s_i) d'(s_i)} \tag{47}$$

which implies

$$n(s_i)d'(s_i) < 0 \tag{48}$$

Notice also from Eq. (46) that $Z(s)$ is independent of s_i. Thus s_i in Eqs. (13)-(21) is in a way a hidden variable in the theory. Moreover if we insert Eq. (47) into (39), we find $D(s)$ is regular at s_i. Also from Eq. (28), the numerator function $N(s)$ of $T(s)$ defined by

$$N(s) = n(s)Z(s) + \frac{g^2 d^2(m^2)}{(m^2 - s)d(s)} \tag{49}$$

is regular at s_i. This completes the proof of our premise.

We also mention that some authors[8] have argued the equivalence between the composite particle in the S-matrix theory, i.e. as defined by $T(s) \to t(s)$ with Eq. (43) and the elementary particle in the field theory with $Z = 0$ and $Z_1 = 0$. Here a definition of Z_1 is made by[18]

$$Z_1 = \lim_{s \to \infty} \frac{1}{g} \Gamma(s) \tag{50}$$

To see this, we consider $Z_1 = 0$ from Eq. (50). This implies

$$d(m^2)/\lim_{s \to \infty} d(s) = 0 \tag{51}$$

Suppose the driving force is such that there is a zero, say, at $s = s_0 > m^2$. We have seen above that if $s_0 < m^2$ this necessarily gives $Z = 0$. In any case, $Z = 0$ results

$$1 = \frac{g^2}{\pi} \int_{s_t}^{\infty} ds \, \frac{\rho(s)}{(s - m^2)^2} \left| \frac{d(m^2)}{d(s)} \right|^2 + \frac{R_0}{(s_0 - m^2)^2} \tag{52}$$

Notice that Eq. (51) can be satisfied by $s_0 \to m^2$. Then from Eq. (52)

$$R_0 = (s_0 - m^2)^2 \tag{53}$$

By inserting Eq. (53) into (47) and taking the limit $s_0 \to m^2$, one finds

$$g^2 = - \frac{n(m^2)}{d'(m^2)} \tag{54}$$

thus realizing the S-matrix conditions Eq. (43) for the particle being composite. Furthermore in this case we have Eq. (16) which gives[9]

$$-Z\delta m^2 = - \lim_{s \to \infty} sZ(s) = s_0 - m^2 = 0 \tag{55}$$

so that δm^2 is finite. If there is a zero in d(s) below m^2, it is when the $Z \to 0$ limit is already reached as was mentioned before. Hence it is clear that the zero $s_0 > m^2$ in d(s) approaches to m^2 in the $Z \to 0$, $Z_1 \to 0$ limit. Moreover in this case we obtain

$$T(s) \to [t(s)]_{elementary} = [t(s)]_{composite} = [T(s)]_{composite}$$

because not only $d(m^2)$ but also $\Gamma\Delta'\Gamma$ in Eq. (35) vanishes as $s_0 \to m^2$. Observe from Eq. (32) that[9] $\Delta'(s)$ blow up and one cannot determine the propagator function or the vertex function except at the pole position. The fact that $\Delta'(s)$ blows up is consistent with having Eq. (55) for the unsubtracted Lehman representation.

Notice that Eq. (51) can also be satisfied if the driving force is such that lim $d(s) = \infty$, say, like $\ln s$. From $Z = 0$, Eq. (18) holds and by using $d(s) \xrightarrow{s \to \infty} \ln s$ one gets $Z(s) \sim (s \ln s)^{-1}$ so that $\Delta'(s) \sim \ln s$. This means the propagator function would need a subtraction like Eq. (2) and have a zero below m^2. Thus a zero in d(s) exists. In this case n(s) will need a subtraction while $N(s)$ will not because $D(s) \sim 1/s$. We remark that in this case the position of the propagator zero below m^2 can be controlled by the subtraction constant in Eq. (2). As the subtraction constant increases, the position of the propagator zero below m^2 moves away toward minus infinity while another zero appears in the interval (m^2, s_t). In particular, if the subtraction constant in Eq. (2) is chosen arbitrarily large, we have the same situation as before where the propagator is arbitrarily large. But once the zero position in Eq. (2) is chosen, then the subtraction constant in Eq. (2) is fixed and the subtraction constant in $n(s)$ can be determined from Eq. (32). Since, however, the full amplitude in this case does not require any subtraction, the parameters g^2 and m^2 of the particle determined from Eq. (43) should be independent of the position of the zero below m^2 in Eq. (2). We have shown that the position of zero is actually a sort of hidden variable in the theory.

Finally, we remark that various asymptotic behaviors of the S-matrix and field-theoretic quantities can be considered for various different limits of Z_1 and Z. For example, if $Z_1 \neq 0$, $\lim_{s \to \infty} d(s)$ will be a constant. If $Z \neq 0$ in addition, then $\lim_{s \to \infty} D(s)$ is a constant too. The propagator will behave like $1/s$, thus there is no zero in $d(s)$ to be left of m^2. The right hand sides of Eq. (16) and (21) can never be made finite. For the driving force responsible for this, $n(s)$ as well as $N(s)$ will not need any subtraction. Similarly one can show that if the driving force is such that $Z_1 \neq 0$ and $Z = 0$, the propagator, $N(s)$ and $n(s)$ need no subtractions, while for $Z_1 = 0$ and $Z \neq 0$, both denominator functions of $T(s)$ and $t(s)$ will require a subyraction but the propagator function will not. The results are summarized in Table I. These results are fairly general in the sense that the left hand cut is specified only by Eq. (41). If the driving force is given, then one will be able to show these results along with the dynamical origin of the propagator zeros and their cancellation mechanism with the vertex poles explicitely in so far as Eq. (41) is not violated. Further investigation is under way for the case when the driving force is given by pole terms.

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	$T(s)$	$t(s)$	$\Delta'(s)$
$Z_1 = 0,\ Z = 0$	No	Yes	Yes
$Z_1 \neq 0,\ Z \neq 0$	No	No	No
$Z_1 \neq 0,\ Z = 0$	No	No	No
$Z_1 = 0,\ Z \neq 0$	Yes	Yes	No

Table I - Necessity (yes) and non-necessity (no) of subtractions in the amplitude $T(s)$, the irreducible part $t(s)$ and the propagator function $\Delta'(s)$.

FOOTNOTES AND REFERENCES

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