

# COUPLED-BUNCH LONGITUDINAL INSTABILITIES WITH A HARMONIC CAVITY \*

R. Lindberg, ANL, Argonne, IL, USA

## Abstract

We present a theory of coupled-bunch longitudinal instabilities for storage rings that employ a harmonic cavity to lengthen the bunch. We find growth rates associated with the  $m = 0$  and  $m = 1$  modes for both “optimally” and “overstretched” bunches; the former is a Robinson-like instability, while the latter corresponds to the “periodic transient beam loading” effect described in Ref. [1]. By self-consistently including longitudinal feedback, we then show that controlling the instability may require feedback damping rates that are higher than the growth rate. For parameters considered, we find that controlling the  $m = 0$  mode may require damping rates that are up to 2 times higher than the instability growth rate, while in many cases the  $m = 1$  mode cannot be stabilized with any feedback gain. We verify these predictions using particle tracking for APS-U like parameters.

## THEORY

We consider longitudinal motion in phase space spanned by the co-moving coordinate  $z = -ct + s$  and the negative energy deviation  $p_z = -(\gamma - \gamma_0)/\gamma_0$ . Here,  $t$  is the particle time at location  $s$  along the ring,  $c$  is the speed of light, while  $\gamma$  and  $\gamma_0$  are the particle and reference Lorentz factor, respectively. We assume that there are  $M$  bunches in the ring, and that the distribution function  $F$  of each bunch satisfies the Vlasov equation, so that for bunch  $n$  we have

$$0 = \frac{d}{ds} F_n(z, p_z; s) = \frac{\partial F_n}{\partial s} + \alpha_c p_z \frac{\partial F_n}{\partial z} + \left[ \mathcal{F}(z) + \frac{\Delta p_z}{cT_0} \right] \frac{\partial F_n}{\partial p_z}. \quad (1)$$

Here,  $\alpha_c$  is the momentum compaction,  $\mathcal{F}$  is total rf-force composed of that due to the applied (generator) fields and the collective, long-range wakefields, while  $\Delta p_z/cT_0$  denotes the energy correction  $\Delta p_z$  given every revolution period  $T_0$  by the feedback system.

The long-range wakefield is obtained by summing the effects of all other bunches over all previous turns. For a total current  $I_{\text{tot}}$  divided equally amongst  $M$  bunches we have

$$\mathcal{F}_w = \sum_{j=0}^{M-1} \frac{eI_{\text{tot}}/M}{\gamma mc^2} \int d\hat{z} d\hat{p}_z \int dk \frac{Z_{\parallel}(k)}{2\pi} e^{ik(z-\hat{z})} \times \sum_{\ell=0}^{\infty} F_j(\hat{z}, \hat{p}_z; s - \ell cT_0) e^{i\ell cT_0[\ell+(j-n)/M]}. \quad (2)$$

In particular, we integrate the contributions of  $T_0 I_{\text{tot}}/e$  electrons in bunch  $j$ , and sum  $\ell$  over previous turns. In (2) we

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have introduced the electron mass  $m$ , its charge  $e$ , the longitudinal impedance  $Z_{\parallel}(k)$ , and assumed  $\gamma \approx \gamma_0$ . Note that the exponential includes both mean distances between bunch  $j$  and  $n$  given by  $cT_0|j-n|/M$  and the accumulated distance from previous turns.

We assume that the feedback uses  $N$  previous measurements of the bunch displacement to infer the present energy and choose a corrective kick. Mathematically, we have

$$\Delta p_z = \frac{1}{cT_0} \sum_{\ell=0}^{N-1} \mathcal{E}_{\ell} \int d\hat{z} d\hat{p}_z \hat{z} F_j(\hat{z}, \hat{p}_z; s - \ell cT_0), \quad (3)$$

where  $\mathcal{E}_{\ell}$  are the finite impulse response (FIR) coefficients that convert position data from the previous  $N$  turns into a feedback kick. We will specify the  $\mathcal{E}_{\ell}$  later.

We assume that an equilibrium exists without feedback, and that at equilibrium each bunch has its own stationary distribution  $\bar{F}_n(z, p_z)$ . We then transform to action-angle variables of this stationary problem including wakes,  $\bar{F}_n(z, p_z) \rightarrow \bar{F}_n(\mathcal{J})$  with  $\mathcal{J}$  the particle action. To make our analysis tractable we next assume that the ring is uniformly filled, so that each bunch is separated by  $cT_0/M$  and has the same equilibrium  $\bar{F}_n(\mathcal{J}) = \bar{F}(\mathcal{J})$ . Then, we consider small deviations about the equilibrium,

$$F_n(\Phi, \mathcal{J}; s) = \bar{F}(\mathcal{J}) + f_n(\Phi, \mathcal{J}) e^{-i\Omega s/c}, \quad (4)$$

and linearize the resulting Vlasov system. We find that the linear stability is governed by

$$\begin{aligned} & \left[ -\frac{i\Omega}{c} + \frac{\omega(\mathcal{J})}{c} \frac{\partial}{\partial \Phi} \right] f_n \\ &= \frac{\partial z}{\partial \Phi} \frac{\partial \bar{F}}{\partial \mathcal{J}} \sum_{\ell=0}^{N-1} \frac{\mathcal{E}_{\ell}}{(cT_0)^2} e^{i\ell \Omega T_0} \int d\hat{\Phi} d\hat{\mathcal{J}} \hat{z} f_n(\hat{\Phi}, \hat{\mathcal{J}}) \\ &+ \frac{eI_{\text{tot}}}{\gamma mc^2} \frac{\partial z}{\partial \Phi} \frac{\partial \bar{F}}{\partial \mathcal{J}} \int dk \frac{Z_{\parallel}(k)}{2\pi M} \int d\hat{\Phi} d\hat{\mathcal{J}} e^{ik(z-\hat{z})} \\ &\times \sum_{j=0}^{M-1} f_j(\hat{\Phi}, \hat{\mathcal{J}}) \sum_{\ell=-\infty}^{\infty} e^{-i\ell cT_0[\ell+(j-n)/M]} e^{i\ell \Omega T_0}, \end{aligned} \quad (5)$$

where  $\omega(\mathcal{J})$  is the oscillation frequency of a particle with action  $\mathcal{J}$  and we introduced the short-hand  $z = z(\Phi, \mathcal{J})$  and  $\hat{z} = z(\hat{\Phi}, \hat{\mathcal{J}})$ . Next, we define the coupled-bunch mode distributions as

$$\tau_{\mu}(\Phi, \mathcal{J}) = \frac{1}{M} \sum_{j=0}^{M-1} e^{-2\pi i \mu j/M} e^{-ij\Omega T_0/M} f_j(\Phi, \mathcal{J}). \quad (6)$$

and then multiply (5) by  $e^{-in(2\pi\mu + \Omega T_0)/M}$  and sum over  $M$ . The sums in the last term can be collapsed into a single one by introducing the index  $q = j - n$ . We then apply the Poisson summation formula  $\sum_p e^{ip(\omega - \bar{\omega})T_0/M} = \sum_p (2\pi/M) \delta(\omega - \bar{\omega} - Mp\omega_0)$  and integrate over  $k = \omega/c$ .

Finally, we simplify notation by defining the frequencies  $\varpi_p = (Mp + \mu)\omega_0 + \Omega$  and the dipole and Fourier moments

$$\mathcal{D}_\mu = \frac{1}{cT_0} \int d\Phi d\mathcal{T} z(\Phi, \mathcal{T}) \tau_\mu(\Phi, \mathcal{T}), \quad (7)$$

$$\mathcal{D}_\mu(\bar{\omega}) = \frac{i}{\bar{\omega}T_0} \int d\Phi d\mathcal{T} e^{-i\bar{\omega}z(\Phi, \mathcal{T})/c} \tau_\mu(\Phi, \mathcal{T}). \quad (8)$$

If the bunch length is much less than the rf wavelength the two perturbations in (8) are equal, but this does not apply when the double rf system significantly lengthens the bunch. Collecting everything, we find that

$$\left[ \frac{i\Omega}{c} + \frac{\omega(\mathcal{T})}{c} \frac{\partial}{\partial \Phi} \right] \tau_\mu = \frac{\partial z}{\partial \Phi} \frac{\partial \bar{F}}{\partial \mathcal{T}} \sum_{\ell=0}^{N-1} \frac{\mathcal{E}_\ell}{cT_0} e^{i\ell\Omega T_0} \mathcal{D}_\mu \quad (9)$$

$$+ i \frac{eI_{\text{tot}}}{\gamma mc^3} \frac{\partial z}{\partial \Phi} \frac{\partial \bar{F}}{\partial \mathcal{T}} \sum_{p=-\infty}^{\infty} Z_{\parallel}(\varpi_p) e^{i\omega_p z/c} \varpi_p \mathcal{D}_\mu(\varpi_p).$$

We now assume that the equilibrium is Gaussian in  $p_z$  with rms width  $\sigma_\delta$ . Since the single particle Hamiltonian  $\mathcal{H}_0 = \alpha_c p_z^2/2 + V(z)$  we have  $\bar{F} \propto e^{-\mathcal{H}_0/\alpha_c \sigma_\delta^2}$  so that  $\partial \bar{F}/\partial \mathcal{T} = -\bar{F}(\partial \mathcal{H}_0/\partial \mathcal{T})/(\alpha_c \sigma_\delta^2) = -\omega(\mathcal{T})\bar{F}/(\alpha_c \sigma_\delta^2)$ . Next, we assume that the feedback samples the motion over a small number of turns such that  $NT_0$  is much shorter than a synchrotron period or growth time. Then,  $\sum_{\ell} \mathcal{E}_\ell e^{i\ell\Omega T_0} \approx -(2iG/\alpha_c)\Omega T_0$ , where  $G$  is the gain and when  $G = 1$  the feedback tries to completely correct the inferred error.

Now, we invert the differential operator to “solve” for  $\tau_\mu$  as shown in [2, 3]. We then close the system of linear equations for  $\mathcal{D}_\mu$  and  $\mathcal{D}_\mu$  by multiplying by  $z$  or  $e^{-i\bar{\omega}z/c}$  and integrating over phase space. The final results are

$$\mathcal{D}_\mu = -\mathcal{D}_\mu \frac{iG\Omega T_0}{2\alpha_c^2 \sigma_\delta^2} \int d\mathcal{T} \bar{F} \sum_{n=1}^{\infty} \frac{[n\zeta_n(\mathcal{T})/cT_0]^2}{n^2 - [\Omega/\omega(\mathcal{T})]^2} \quad (10)$$

$$- \frac{eI_{\text{tot}}}{\alpha_c \sigma_\delta^2 \gamma mc^2} \sum_p Z_{\parallel}(\varpi_p) \mathcal{D}_\mu(\varpi_p)$$

$$\times \int d\mathcal{T} \bar{F} \sum_{n=1}^{\infty} \frac{i^n n^2 \zeta_n(\mathcal{T})/cT_0}{n^2 - [\Omega/\omega(\mathcal{T})]^2} \mathcal{J}_n(\varpi_p, \mathcal{T})$$

and

$$\mathcal{D}_\mu(\bar{\omega}) =$$

$$- \mathcal{D}_\mu \frac{G\Omega}{\alpha_c^2 \sigma_\delta^2 \bar{\omega}} \int d\mathcal{T} \sum_{n=1}^{\infty} \frac{\bar{F}(\mathcal{T}) n^2 \zeta_n(\mathcal{T}) \mathcal{J}_n^*(\mathcal{T}, \bar{\omega})}{i^n cT_0 \{n^2 - [(\Omega/\omega(\mathcal{T}))^2]\}} \quad (11)$$

$$- \frac{2eI_{\text{tot}}}{\alpha_c \sigma_\delta^2 \gamma mc^2} \sum_p \frac{Z_{\parallel}(\varpi_p)}{\bar{\omega}T_0} \mathcal{D}_\mu(\varpi_p)$$

$$\times \int d\mathcal{T} \sum_{n=1}^{\infty} \bar{F}(\mathcal{T}) \frac{n^2 \mathcal{J}_n(\mathcal{T}, \varpi_p) \mathcal{J}_n^*(\mathcal{T}, \bar{\omega})}{n^2 - [(\Omega/\omega(\mathcal{T}))^2]}.$$

Here, the Fourier coefficients  $\zeta_n$  and  $\mathcal{J}_n$  are defined by

$$z = \sum_{n=0}^{\infty} \zeta_n \cos(n\Phi) \quad e^{i\bar{\omega}z/c} = \sum_{n=-\infty}^{\infty} i^n \mathcal{J}_n e^{in\Phi}. \quad (12)$$

## DISCUSSION AND SIMULATION

The set of equations (10) and (11) reflect the fact that the feedback acts only on the dipole moment, so that when the

bunch length approaches the rf period it typically will not damp as efficiently as expected. We will show this explicitly using both the theoretical solutions and elegant tracking [4] for APS-U type parameters listed in Table 1.

We model each rf cavity as a single resonator with [5]

$$Z_{\parallel}(\omega) = \frac{R_s}{1 + iQ_L \left( \frac{\omega_R}{\omega} - \frac{\omega}{\omega_R} \right)}, \quad (13)$$

where  $R_s$  is the (loaded) shunt impedance,  $Q_L$  is the loaded quality factor, and  $\omega_R$  is the resonance frequency. A typical cavity resonance is sufficiently narrow such that only two terms dominate the sums in (10) and (11). For example, the contribution for the APS-U main rf cavity is at  $pM\omega_0 = \pm\omega_{\text{rf}} = \pm h\omega_0$  with  $h = 1296$ , while at the fourth harmonic it is at  $\pm 4\omega_{\text{rf}}$ ; then, Eqs. (10) and (11) reduce to five linear equations (four when  $G = 0$ ), and  $\Omega$  is found by setting the matrix determinant of the system to zero.

Reference [6] identified two dominant instabilities in the double rf system. The first is essentially the usual  $\mu = 0$  Robinson instability, and we can understand its growth rate using the familiar expression [5]

$$\Im(\Omega) \approx \frac{\alpha_c eI_{\text{tot}}}{\gamma mc^2 T_0} \sum_{p=-\infty}^{\infty} \frac{Mp\omega_0 + \Omega}{2\Omega} \Re Z_{\parallel}(Mp\omega_0 + \Omega) \quad (14)$$

$$\approx \frac{\alpha_c eI_{\text{tot}}}{\gamma mc^2 T_0} \sum_{p=0}^{\infty} \left[ \omega \Re \frac{d}{d\omega} Z_{\parallel}(\omega) \right]_{\omega=Mp\omega_0}.$$

The main cavity has  $\omega_R < h\omega_0$ ,  $\Re(dZ_{\parallel}/d\omega) < 0$ , and it helps stabilize the dynamics. In contrast, bunch lengthening requires the HHC to be tuned slightly above a beam harmonic which contributes to instability. Since the HHC lengthening scales with  $\Im(Z_{\parallel})$  for  $z$ -symmetric bunches, we have

$$\Im Z_{\text{HHC}}(4\omega_{\text{rf}}) \approx \frac{R_s \omega_R}{2Q_L(\omega_R - 4\omega_{\text{rf}})} \approx C(\sigma_t) \quad (15)$$

with  $C$  a function only of the bunch length  $\sigma_t$ . Then, evaluating the HHC’s contribution to the growth rate shows that

$$\Im(\Omega) \approx 2\omega_R \frac{\alpha_c eI_{\text{tot}} C^3(\sigma_t) Q_L}{\pi \gamma mc^2 R_s^2}. \quad (16)$$

Hence, if the stabilizing effect of the main cavities is small we expect the mode 0 growth rate to scale as  $Q_L/R_s^2$ ; for a fixed cavity geometry (and  $R_s/Q_L$ ) the growth rate scales inversely with the HHC shunt impedance.

We illustrate this scaling for “optimal” stretching in Fig. 1, in which the HHC settings are chosen to flatten the potential such that it is approximately quartic, which for the APS-U leads to  $\sigma_t \approx 50$  ps. The top plot shows how the growth rate

Table 1: APS-U Ring and Main Cavity Parameters

Ring	Value	Main Cavity	Value
$\alpha_c$	$4.04 \times 10^{-5}$	$R_s$	10.1 M $\Omega$
$T_0$	3.628 $\mu$ s	$Q_L$	7380
$\sigma_\delta$	0.135 %	Total $V$	4.6 MV

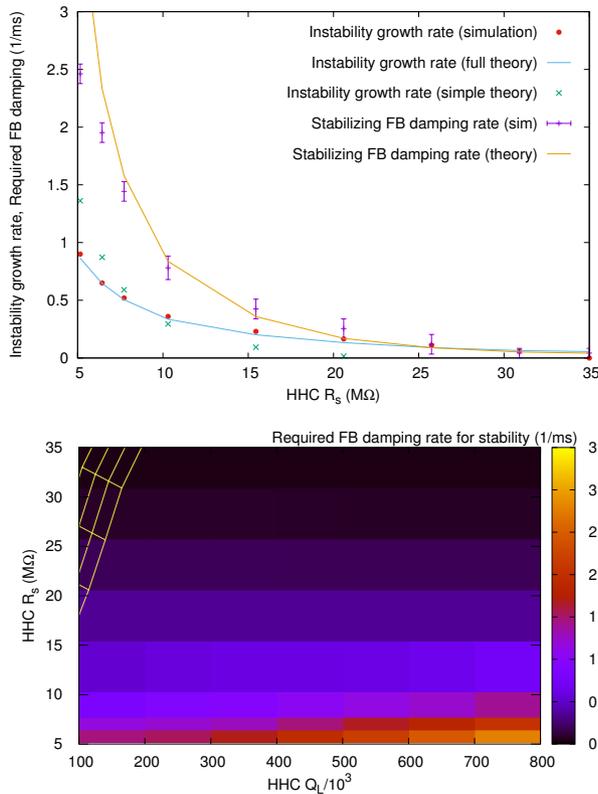


Figure 1: “Optimal” stretching with  $\sigma_t \approx 50$  ps. Top: instability growth rate and stabilizing feedback damping rate as a function of the HHC shunt impedance for  $Q_L = 600 \times 10^3$ . Both scale  $\propto 1/R_s^2$ , but stability can require a feedback significantly stronger than the instability. Bottom: Feedback damping required for stable simulations as a function of the HHC  $Q_L$  and  $R_s$ . In addition to the mode zero instability  $\propto Q_L/R_s^2$ , the hatched region with  $R_s/Q_L \geq 180 \Omega$  denotes a mode 1 instability that cannot be stabilized.

varies with  $R_s$  for fixed  $Q_L = 600 \times 10^3$ , and also the required feedback damping rate to stabilize the system. When the growth rates are small the two approximately agree, but they quickly diverge at small  $R_s$ .

Similar behavior is seen for the “overstretched” bunch in the top of Fig. 2, wherein the HHC is set to maximize Touschek lifetime, and the final distribution has two humps along  $z$  and  $\sigma_t \approx 90$  ps. The growth rates are approximately the same as for optimal stretching, but the required feedback is approximately twice as large. Interestingly, both theory and simulation indicate that this difference is largely due to a sudden “jump” in the required feedback near  $R_s \approx 18$  M $\Omega$ .

The bottom plots of both Figs. 1 and 2 show the required feedback as a function of HHC  $Q_L$  and  $R_s$ . The main part of the plot depicts the  $\propto Q_L/R_s^2$  behavior of the mode zero Robinson instability. The theoretical predictions agree to a similar level as seen in the top plots. In addition, the yellow hatched region indicates a mode 1 instability that appears to be immune to feedback, whose existence was previously described in Ref. [6]; furthermore, it appears

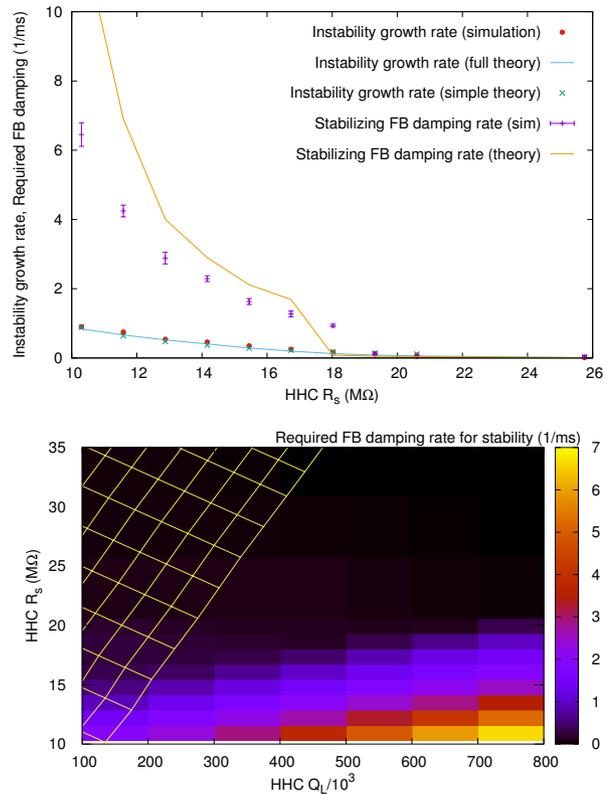


Figure 2: “Overstretched” bunch with  $\sigma_t \approx 90$  ps. Top: instability growth rate and stabilizing feedback damping rate as a function of the HHC  $R_s$  for  $Q_L = 600 \times 10^3$ . The required FB jumps significantly at  $R_s \approx 18$  M $\Omega$ . Bottom: Feedback damping required for stable simulations as a function of the HHC  $Q_L$  and  $R_s$ . The hatched region with  $R_s/Q_L \geq 75 \Omega$  exhibits a mode 1 instability that is immune to feedback.

to be the same as the “periodic transient beam loading” effect described in Ref. [1]. Its phenomenology including its threshold  $\propto R_s/Q_L$  and immunity to feedback was first described to me by M. Venturini [7].

Our theory predicts the  $R_s/Q_L \approx 180 \Omega$  mode 1 threshold to within 10% for optimal stretching. On the other hand, the theory overestimates the observed  $R_s/Q_L \approx 75 \Omega$  threshold for the overstretched case by about 60%. We do not understand this, nor can we explain how the simulated mode 1 threshold depends upon the number of bunches  $M$ . These odd findings warrant additional study.

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