EFFECTIVE GAUGE THEORIES FROM FUZZY EXTRA DIMENSIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

GÖNÜL ÜNAL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY
IN
PHYSICS

OCTOBER 2016

Approval of the thesis:

EFFECTIVE GAUGE THEORIES FROM FUZZY EXTRA DIMENSIONS

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ABSTRACT

EFFECTIVE GAUGE THEORIES FROM FUZZY EXTRA DIMENSIONS

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October 2016, 224 pages

In this thesis, we investigate the formulation and various aspects of gauge theories with fuzzy extra dimensions. In $SU(\mathcal{N})$ gauge theories coupled to a suitable number of adjoint scalar fields, we determine a family of fuzzy vacuum configurations dynamically emerging after the spontaneously symmetry breaking of the gauge symmetry. The emergent models are conjectured to be effective $U(n)\,(n < \mathcal{N})$ gauge theories with fuzzy extra dimensions. Making use of the equivariant parametrization technique and focusing on the simplest member of the family of fuzzy vacua, we obtain all the $SU(2) \times SU(2)$-equivariant gauge fields in a $U(4)$ model which characterize its low energy degrees of freedom. Low energy effective action of a $U(3)$ gauge theory on $\mathbb{R}^2 \times S_F^2$ is also determined and its vortex type solutions are investigated in detail. In this thesis, we also formulate the quantum Hall effect (QHE) on the complex Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$. We use the group theoretical techniques to solve the Landau problem and provide the energy spectrum and eigenstates of charged particles on this space under the influence of Abelian and non-Abelian background magnetic monopoles.

Keywords: Gauge Theory in Higher Dimensions, Fuzzy Spaces, Equivariant Parametrization, Dimensional Reduction, Quantum Hall Effect in Higher Dimensions

ÖZ

FUZZY EKSTRA BOYUTLARDAN EFEKTİF AYAR TEORİLERİ

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Ekim 2016 , 224 sayfa

Bu tez çalışmasında, fuzzy ekstra boyutlara sahip ayar teorilerinin formülasyonu ve çeşitli yönleri incelenmiştir. Uygun sayıda adjoint skaler alanla eşleşmiş $SU(\mathcal{N})$ ayar teorilerinin spontane simetri kırılımı ile dinamik olarak oluşan bir küme fuzzy vakum konfigürasyonları belirlenmiştir. Bu biçimde ortaya çıkan modellerin fuzzy ekstra boyutlu efektif $U(n)\,(n < \mathcal{N})$ ayar teorileri olarak yorumlanması üzerinde durulmuştur. Simetrik parametrizasyon tekniğiyle ve bahsi geçen fuzzy vakum kümesinin en basit üyesine odaklanarak, $U(4)$ modelinde düşük enerjili serbestlik derecelerini karakterize eden tüm $SU(2)\times SU(2)$-simetrik ayar alanları elde edilmiştir. $\mathbb{R}^2 \times S_F^2$ üstündeki $U(3)$ ayar teorisinin düşük enerjili eylemleri hesaplanıp, vorteks tipi çözümleri de incelenmiştir. Ayrıca, bu tezde kompleks Grassmann manifoldlarında, $\mathbf{Gr}_2(\mathbb{C}^N)$, kuantum Hall etkisi formüle edilmiştir. Grup teori teknikleri kullanarak, Landau problemini çözülüp, bu uzayda abelyen ve abelyen-olmayan manyetik monopollerin etkisi altındaki yüklü parçacıkların enerji spektrumları ve özdurumları belirlenmiştir.

Anahtar Kelimeler: Yüksek Boyutlarda Ayar Teorisi, Fuzzy Uzaylar, Simetrik Parametrizasyon, Boyutsal İndirgeme, Yüksek Boyutlarda Kuantum Hall Etkisi

To Murat

ACKNOWLEDGMENTS

First of all, I would like to express my deepest gratitude to my supervisor, Assoc. Prof. Dr. Seçkin Kürkçüoğlu, for his endless patience, support, and kind attitude all the time. His guidance greatly helped me not only in writing this thesis but also in many ways during my PhD research and education.

Besides my supervisor, I also would like to thank to the thesis monitoring committee (TIK) of this thesis: Assoc. Prof. Dr. İsmet Yurduşen and Assoc. Prof. Dr. İsmail Turan. Their critical comments and remarks helped to improve several aspects of the research results presented in this thesis.

Discussions in chapter 6 is based on the work in collaboration with Fatih Ballı, Alireza Behtash and my supervisor. I would like to thank Fatih Ballı and Alireza Behtash for their collaboration and providing perspectives on many different topics.

Next, I am so grateful to have the chance to thank my parents and my two sisters. They have been always supporting and encouraging me with their best wishes all through my research like they do in all parts of my life.

Finally, I would like to thank a lot to my husband, Murat Mesta, for his understanding during my most stressful times and his joy that has made my days enjoyable. This thesis would not been completed without his endless support and generosity of sharing his experiences with me.

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CHAPTER 1

INTRODUCTION

In the past few decades, non-abelian gauge theories with extra dimensions have been a continually appearing theme in theoretical physics studies attempting to explore physics beyond the Standard Model. One of the early motivations in this context was to explore Grand Unified Theories (GUTs) by formulating gauge theories whose extra dimension are symmetric spaces with the coset structure $K = G/H$ and subsequently dimensionally reducing the models to the Minkowski space in certain a manner that captures some new ingredients coming from extra dimensions. In the literature, this approach is known by the name of coset space dimensional reduction (CSDR) [1,2] (See also, [3] in this context) and we will have much more to say on it later on in this introduction, to make its connection to the research presented in this thesis as concrete as possible. Another motivation for their study, which is in fact not completely disconnected from the first, is the appearance of extra dimensions in (super)string theories and related supersymmetric Yang-Mills theories. Dimensional reduction of supersymmetric (SUSY) $N = 1$ Yang-Mills theory in $9 + 1$-dimensions to $N = 4$ SUSY Yang-Mills (SYM) in $3 + 1$-dimensional Minkowski space is an extremely well-known example, essentially due to a several numbers of physically appealing properties of the $N = 4$ SYM. As a quantum field theory (QFT), the latter has several appealing properties, among which its conformal invariance and UV finiteness, may be indicated at first glance. It is invariant under S-duality, interchanging the coupling constants g_{YM} and $\frac{4\pi}{g_{YM}}$ and it plays a central role in gauge/gravity duality as it is the most prominent example on the conformal field theory (CFT) side for AdS/CFT correspondence [4,5]. However, it is generally considered that this theory is not realistic as it has too much symmetry. One possible route for accessing phenomenologically more viable theories and its connection to our work will presented in the ensuing discussions.

New directions of research have opened up after it was shown by Arkani-Hamed et. al. [6] that extra dimensions may emerge dynamically in a four-dimensional renormalizable and asymptotically free gauge theory. As widely known, idea of extra dimensions goes back to the works of Kaluza and Klein [7,8] in which the product $\mathbb{M}^4 \times K_n$ of Minkowski space $\mathbb{M}^4$ and a compact space K_n is considered in an attempt to unify the theories gravitation and electromagnetism. It is important to stress that extra dimensions are input in Kaluza-Klein theory. However, in [6], extra dimensions appear spontaneously and it brings forth a new perspective in approaching field theories with extra dimensions. In the present literature, this phenomenon is frequently referred to deconstruction. One such new direction which was recognized by Aschieri et. al. [9] is the dynamical generation of the fuzzy sphere S_F^2 as an extra dimension in an $SU(\mathcal{N})$ gauge theory coupled to a triplet of scalar fields in the adjoint representation of the gauge group. Dynamical generation of product of two fuzzy spheres, $S_F^2 \times S_F^2$ was examined subsequently in a model which contains six scalar fields in the adjoint representation of the gauge group and which is essentially a deformation of the bosonic sector of $N = 4$ SYM containing quadratic and cubic interaction terms in addition to the usual quartic one. As these results have a crucial standing for the research conducted in this thesis, we will have more to say on them soon. However, before doing so we also would like to draw the attention of the reader to the connection of these developments to certain facts and results in string inspired matrix models.

A so-called BFSS matrix model due to Banks, Fischler, Shenker and Susskind [10] proposes to give a non-perturbative description of M-theory on flat backgrounds. This matrix model can be shown to emerge from the dimensional reduction of $N = 1$ supersymmetric Yang-Mills theory in $9+1$-dimensions down to the zero volume limit, i.e. to $0 + 1$ dimensions. In other words, one could say that the M-theory on flat backgrounds is described by “matrix quantum mechanics”. Massive deformations of the BFSS model have also been studied in the literature [11, 12]. One such model is the Berenstein-Maldacena-Nastase (BMN) matrix model and it proposes to give a non-perturbative description of the M-theory on maximally supersymmetric pp-wave backgrounds. Fuzzy sphere, S_F^2, or more generally direct sum of fuzzy spheres $\mathcal{S}_F^2 := \oplus S_F^2$ at different levels provide nontrivial vacuum configurations in this model. Another model providing a dual description of BFSS was developed by Ishibashi, Kawai, Kitazawa, and Tsuchiya (IKKT) [13]. It is obtained by reducing the SYM in 10-dimensions to a pure matrix model. Fuzzy spaces emerge from this model too [14–18]. For example, noncommutative $U(1)$ and $U(n)$ gauge theories on S_F^2 can be constructed from the IKKT matrix model supplemented by the Chern-Simons term [14]. One of the most interesting feature of fuzzy spaces in matrix models is that they are not an input in the model but they arise as brane-type solutions which are generically given as direct sums of fuzzy spheres $\mathcal{S}_F^2 := \oplus S_F^2$, that of products of fuzzy spheres $\mathcal{S}_F^2 \times \mathcal{S}_F^2 := \oplus S_F^2 \times S_F^2$, or higher dimensional fuzzy spaces such as fuzzy four sphere [19, 20]. Thus, the analogy to the models in [9, 21] and in general to the phenomenon of deconstruction is evident in this respect. In the present context, it is also useful to mention that fluctuations around such fuzzy vacua may be examined to yield gauge field excitations living on the world volume of the brane configurations.

After these cursory remarks indicating some of the connections between string theoretic matrix models and gauge field theories of interest in this thesis, we can now return to elaborate on the latter. As already mentioned, for the $SU(\mathcal{N})$ YM theory on Minkowski space $\mathbb{M}^4$ coupled to a triplet of adjoint scalar fields, fuzzy sphere S_F^2 vacuum was investigated in [9]. In this model, three matrices describing the S_F^2 are the vacuum expectation values of the scalar fields and the $SU(2)$ symmetry of S_F^2 is inherited from the global $SU(2)$ gauge symmetry of the YM model. Nonzero vacuum expectation values (VEVs) of the scalar fields imply that the $SU(\mathcal{N})$ gauge symmetry is spontaneously broken down to a $U(2\ell+1) \otimes U(n)$, where $\mathcal{N}$, n and the level ℓ of the fuzzy sphere are related as $\mathcal{N} = (2\ell+1)n$. Fluctuations around this vacuum configuration are found to have the structure of $U(n)$ gauge fields over S_F^2, which preliminarily indicates that the emerging model after symmetry breaking may be conjectured to be an effective $U(n)$

gauge theory over $\mathbb{M}^4 \times S_F^2$ in which fuzzy sphere appears as extra dimensions. The effective $U(n)$ gauge theory interpretation over $\mathbb{M}^4 \times S_F^2$ can be supported by two different approaches. A Kaluza-Klein (KK) type mode expansion of the gauge fields over the fuzzy extra dimensions can be considered and a detailed analysis of its low lying modes can be performed. This is already carried out in [9] and placed the effective gauge theory interpretation on firm grounds. A complementary treatment is given by the equivariant parametrization approach [22–26]. This involves imposing proper symmetry conditions on the fields of the model so that they transform covariantly under the action of the symmetry group of the extra dimensions up to the gauge transformations of the emergent model [1]. These conditions may be solved using the representation theory of Lie groups and explicit equivariant parametrizations of all the fields in the model can be obtained providing strong evidence for the interpretation of such models as effective gauge theories, since, subsequently, an effective low energy action (LEA) may be obtained by integrating out (i.e. tracing over) the fuzzy extra dimensions and dimensionally reducing the theory. Models with non-Abelian gauge symmetry groups, $U(2)$ and $U(3)$ for the case of $\mathcal{M} \times \mathcal{S}_F^2$, and a $U(4)$ model for $\mathcal{M} \times \mathcal{S}_F^2 \times \mathcal{S}_F^2$ have been investigated in [22, 24, 27] and LEAs were obtained. These LEAs are generalized Abelian Higgs type models with several $U(1)$ gauge fields and complex and real scalars with vortex solutions for $\mathcal{M} = \mathbb{R}^2$. We present the detailed results of two approaches [22, 24] in chapter 3 and our new results on $U(3)$ theory over $\mathcal{M} \times \mathcal{S}_F^2$ in chapter 5 [27].

Fuzzy vacua in the form of $S_F^2 \times S_F^2$ is spontaneously generated from a deformation of $N = 4$ SYM containing cubic and quadratic terms in the scalar fields. These deformation terms breaks supersymmetry completely and the $SO(6)$ R-symmetry down to a global $SU(2) \times SU(2)$. After breaking of the gauge symmetry, the latter serves as the isometry group of the $S_F^2 \times S_F^2$ vacua. That the emergent model behaves as an effective gauge theory on $\mathbb{M}^4 \times S_F^2 \times S_F^2$ was shown using equivariant parametrization techniques and computing the LEA for a $U(4)$ model [24]. These results are reviewed in some detail in chapter 3.

[1] In this aspect it is equivalent to the CSDR approach adapted to the present case of fuzzy extra dimensions.

This thesis is based on three articles [26–28], two of which are on the general framework described here while the last treats a separate problem. We conclude this introduction by giving a brief summary of our results.

In chapter 4, we further investigate the model which is a particular deformation of the $N = 4$ SYM theory with cubic SSB and mass deformation terms given in [21, 24]. We determine a family of fuzzy vacua which are expressed in terms of direct sums of product of two fuzzy spheres, i.e $S_F^{2\,Int} \times S_F^{2\,Int} := \oplus S_F^2 \times S_F^2$. Structure of these vacuum configurations is revealed by permitting splittings of the scalar fields that involve the introduction of $k_1 + k_2$ component multiplets transforming under the representation $(\frac{k_1-1}{2}, 0) \oplus (0, \frac{k_2-1}{2})$ of the global symmetry and it is found that all fuzzy monopole sectors over $S_F^2 \times S_F^2$ are systematically accessed thorough projections of these vacua. Focusing on the simplest member $S_F^{2\,Int} \times S_F^{2\,Int}$ of this family, we demonstrate that the fluctuations about this vacuum have precisely the form of gauge fields, which allow us to conjecture that the emerging model is an effective $U(n)$ $(n < \mathcal{N})$ gauge theory on $M^4 \times S_F^{2\,Int} \times S_F^{2\,Int}$. To support this interpretation, we study the $U(4)$ model and obtain all the $SU(2) \times SU(2)$-equivariant fields by equivariant parametrization technique, which characterize its low energy degrees of freedom and also examine the monopole sectors with winding numbers $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 1, \pm 1)$ in some detail. We note that spinorial modes that naturally come out of this analysis do not comprise independent degrees of freedom in the effective theory, but they may be used to find the "square roots" of the equivariant gauge field modes. Moreover, stability of our vacuum solutions is addressed by showing that they may be interpreted as mixed states with non-zero von Neumann entropy. Finally, we show that $S_F^{2\,Int} \times S_F^{2\,Int}$ identifies with the bosonic part of the product of two fuzzy superspheres with $OSP(2,2) \times OSP(2,2)$ supersymmetry and discuss how this comes about. Our results applies just as well to matrix models with the same type of vacua and methods are quite versatile to investigate other fuzzy vacuum configurations, which may be of physical interest.

In chapter 5, we explore the low energy structure of a $U(3)$ gauge theory over spaces with fuzzy sphere(s) as extra dimensions. In particular, we determine the equivariant parametrization of the gauge fields, which transform either invariantly or as vectors under the combined action of $SU(2)$ rotations of the fuzzy spheres and those $U(3)$ gauge transformations generated by $SU(2) \subset U(3)$ carrying the spin 1 irreducible representation of $SU(2)$. The cases of a single fuzzy sphere S_F^2 and a particular direct sum of concentric fuzzy spheres, $S_F^{2\,Int}$, covering the monopole bundle sectors with windings ± 1 are treated in full and the low energy degrees of freedom for the gauge fields are obtained. Employing the parametrizations of the fields in the former case, we determine a low energy action by tracing over the fuzzy sphere and show that the emerging model is abelian Higgs type with $U(1) \times U(1)$ gauge symmetry and possesses vortex solutions on $\mathbb{R}^2$, which we discuss in some detail. Generalization of our formulation to the equivariant parametrization of gauge fields in $U(n)$ theories is also briefly addressed.

In chapter 6, we formulate Quantum Hall Effects (QHEs) on the complex Grassmann manifolds $\mathbf{Gr}_2(\mathbb{C}^N)$ which are generalizations of complex projective spaces $\mathbb{C}P^N$. We set up the Landau problem in $\mathbf{Gr}_2(\mathbb{C}^N)$ and solve it using group theoretical techniques and provide the energy spectrum and the eigenstates in terms of the $SU(N)$ Wigner $\mathcal{D}$-functions for charged particles on $\mathbf{Gr}_2(\mathbb{C}^N)$ under the influence of abelian and non-abelian background magnetic monopoles or a combination of these thereof. In particular, for the simplest case of $\mathbf{Gr}_2(\mathbb{C}^4)$ we explicitly write down the $U(1)$ background gauge field as well as the single and many-particle eigenstates by introducing the Plücker coordinates and show by calculating the two-point correlation function that the Lowest Landau Level (LLL) at filling factor $\nu = 1$ forms an incompressible fluid. Our results are in agreement with the previous results in the literature for QHE on $\mathbb{C}P^N$ [29] and generalize them to all $\mathbf{Gr}_2(\mathbb{C}^N)$ in a suitable manner. At first sight, the discussions in this subject may look irrelevant from the rest of the thesis. However, there is an interesting connection between QHE and fuzzy spaces. The Landau problem on two- and higher-dimensional spaces has close connections to the physics of strings and D-branes in the matrix theory, and to fuzzy spaces such as S_F^2 and $\mathbb{C}P_F^N$. These connections were studied in the literature [30–32] where it was shown that construction of fuzzy spaces using geometric quantization methods yields that Hilbert spaces $\mathcal{H}_N$ of wave functions are holomorphic sections of $U(1)$ bundles over the commutative parent manifold, and the matrix algebras Mat_N of linear transformations on $\mathcal{H}_N$'s form the fuzzy spaces [32]. It has been observed that the LLL in Landau problems over S^2, $\mathbb{CP}^N$ in $U(1)$ backgrounds define Hilbert spaces that are identical to $\mathcal{H}_N$ as they are also holomorphic sections of $U(1)$ bundles over these spaces. Similar structural relations between S^4_F and the QHE on S^4 also exist [32]. Building upon this connection, observables of the QHE problem are also contemplated as linear transformations in Mat_N acting on $\mathcal{H}_N$. From this angle, we see that there appears almost an immediate connection of our findings for the QHE problem on $\mathbf{Gr}_2(\mathbb{C}^N)$ to fuzzy Grassmann spaces, which are discussed in some detail in the literature [33–35].

CHAPTER 2

FUZZY SPHERE

In this chapter, we introduce the basic formalism of noncommmutative spaces, in particular the noncommmutative or fuzzy two sphere, S_F^2, and $S_F^2 \times S_F^2$ and subsequently we give some essential features of the formulations of classical field theories over S_F^2 and $S_F^2 \times S_F^2$. Our discussion does not attempt to give a full review of the vast literature on the subject but focuses on a number of selected topics which provide the necessary background required to put the developments of ensuing chapters on a broader perspective, and make the thesis as self-contained as possible. There are several approaches to obtain the fuzzy spheres S_F^2, the product of two fuzzy spheres $S_F^2 \times S_F^2$, their supersymmetric generalization [36–44] and fuzzy complex projective space $\mathbb{C}P^N$ [45, 46]. Here, we will follow the practical and the transparent approach given in [38, 41] which is essentially based on quantizing the chain of manifolds such as $\mathbb{C}^2 \to S^3 \to S^2$ to obtain the fuzzy sphere S_F^2, or $\mathbb{C}^{N+1} \to S^{2N+1} \to \mathbb{C}P^N$ to obtain $\mathbb{C}P_F^N$. Another, somewhat more rigorous approach is given by the canonical (Dirac) quantization of Lagrangians composed of appropriate Wess-Zumino terms to achieve the desired form of the quantization of S^2 and several other compact manifolds [37], which fall into the broad class of coadjoint orbits of compact Lie groups.

In order to make this chapter self-contained, before discussing the quantization of S^2, we give a review of some basic properties of commutative manifolds S^3, S^2 and the descent chain $\mathbb{C}^2 \to S^3 \to S^2$ which will be necessary for the quantization process. In particular, we explain how S^3 form a fiber bundle of S^2

with a $U(1)$ fiber, i.e. the first Hopf fibration. Harmonic expansion of functions over S^2 is recalled and a simple description of a scalar field theory over S^2 is also provided in this chapter. Our goal is to give the relevant aspects of field theory on S^2, once the scalar field theory on the fuzzy sphere S_F^2 is constructed, this will allow us to observe its continuum limit in a transparent manner. We also describe the topologically nontrivial configurations of scalar fields over S^2 as sections of complex line bundles over S^2 with non-vanishing winding numbers following the exposition given in [38], which is amenable to obtain their fuzzy version. A discussion on the latter is also provided in the present chapter.

For quantization, our departure point is to obtain the noncommutative complex plane $\mathbb{C}^2$ by replacing its coordinates with the annihilation-creation operators. Making use of the first Hopf fibration provides the construction of noncommmutative three sphere, S_F^3 and the noncommmutative two sphere, S_F^2. We focus our attention on the latter and discuss its structure and properties in considerable detail. Subsequently, we investigate the formulation of scalar field theories on S_F^2 [37] including monopole sectors with nonvanishing winding numbers [38]. Finally, we turn our attention to gauge theories on S_F^2. Using the matrix model approach of [47], we first construct the $U(1)$ gauge theory on S_F^2 and provide also a brief description of $U(n)$ gauge theories over S_F^2. Monopole sectors of these gauge theories are also discussed in some detail.

2.1 Hopf Fibration

First Hopf fibration describes the three sphere S^3 as a fiber bundle over the base space S^2 via the $U(1)$ fibers. It is possible to say that every point on S^2 corresponds to a circle on S^3. In order to explain this relation, let us start with the embedding of S^3 in $\mathbb{R}^4$ as

$$\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 = 1 \,. \tag{2.1}$$

Using the coordinate transformation,

$$x_1 = 2(\omega_1\omega_3 + \omega_2\omega_4) \,, \quad x_2 = 2(\omega_2\omega_3 - \omega_1\omega_4) \,, \quad x_3 = \omega_1^2 + \omega_2^2 - \omega_3^2 - \omega_4^2 \,, \tag{2.2}$$

we can obtain the coordinates of S^2 in $\mathbb{R}^3$ since

$$x_1^2 + x_2^2 + x_3^2 = 1 \,. \tag{2.3}$$

There is a simple way to describe S^2 using the coordinates of a two-dimensional complex plane $\mathbb{C}^2$ [41]. This description will be very useful to obtain the non-commutative version of S^3 and S^2. Let us denote by $z \equiv (z_1\,, z_2)$ the coordinates of $\mathbb{C}^2$ and remove the origin $0 \equiv (0\,, 0)$, i.e. consider $\mathbb{C}^2 \backslash \{0\}$ so that we are able to define the coordinates of S^3 in the following form

$$\xi_i := \frac{z_i}{|z|}\,, \quad i = 1, 2\,, \quad |z| = \sqrt{|z_1|^2 + |z_2|^2}\,, \tag{2.4}$$

since $z \neq 0$. It can be easily seen that these coordinates are normalized to 1

$$\xi^\dagger \xi = 1\,, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} . \tag{2.5}$$

Now, we are ready to give the projection map π from S^3 to the base manifold S^2 as

$$\pi : S^3 \to S^2\,, \quad \xi_i \to x_i(\xi) = \xi^\dagger \tau_i \xi\,, \tag{2.6}$$

where τ_i are the Pauli matrices. This map can be written explicitly as follows

$$\begin{aligned} x_1 &= \xi_1^* \xi_2 + \xi_2^* \xi_1\,, \\ x_2 &= -i\xi_1^* \xi_2 + i\xi_2^* \xi_1\,, \\ x_3 &= \xi_1^* \xi_1 - \xi_2^* \xi_2\,. \end{aligned} \tag{2.7}$$

We observe that the coordinates x_i are left invariant under the $U(1)$ action $\xi \to e^{i\theta}\xi$ and also satisfy

$$\vec{x}(\xi)^* = \vec{x}(\xi), \quad \vec{x}(\xi) \cdot \vec{x}(\xi) = 1\,. \tag{2.8}$$

Hence, we have obtained the first Hopf fibration which may be denoted as

$$U(1) \to S^3 \to S^2\,. \tag{2.9}$$

Now, we would like to focus on S^2. In the next section, we will present some basic geometrical properties of S^2 and also discuss the Harmonic expansion of functions on it.

2.2 Two-sphere

Two-sphere S^2 is a two-dimensional compact, real manifold which may be simply described by embedding it into the three dimensional Euclidean space $\mathbb{R}^3$ by imposing the constraint

$$x_1^2 + x_2^2 + x_3^2 = r^2 \,, \tag{2.10}$$

where r is the radius of S^2. These coordinates may be given in terms of spherical angles $\theta\,,\varphi$ as follows

$$x_1 = r\cos\varphi\sin\theta\,, \quad x_2 = r\sin\varphi\sin\theta\,, \quad x_3 = r\cos\theta\,, \tag{2.11}$$

where $0 \leq \theta \leq \pi\,,\, 0 \leq \varphi \leq 2\pi$.

The coordinates x_i fulfilling (2.10) generate an infinite dimensional commutative algebra $\mathcal{A}$ of smooth functions on the two-sphere with the standard point-wise product $(fg)(x_i) = f(x_i)g(x_i)$. Any continuous function f on S^2 can be expanded in terms of the coordinate functions x_i as

$$f(\vec{x}) = \sum_{i_1,\cdots i_n} a_{i_1,\cdots i_n} x_{i_1}\cdots x_{i_n}\,. \tag{2.12}$$

It is possible to express the expansion of functions in terms of the spherical harmonics, $Y_{jm}(\theta\,,\phi)$ as

$$f(\vec{x}) = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} a_{jm} Y_{jm}(\vec{x})\,, \tag{2.13}$$

where the spherical harmonics fulfill the following orthogonality relation

$$\int \frac{d\Omega}{4\pi}\, Y_{jm}(\vec{x}) Y^*_{j'm'}(\vec{x}) = \delta_{j'j}\delta_{m'm}\,, \tag{2.14}$$

where $d\Omega = \sin\theta\, d\theta\, d\varphi$ is the solid angle. The derivations on S^2 are given by the "angular momentum" operator $L_i = -i(\vec{x}\wedge\nabla)_i = -i\epsilon_{ijk}x_k\partial_k$, which satisfy the $SU(2)$ commutation relations

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{2.15}$$

It is known that the eigenvectors of the square of "orbital angular momentum", L^2, and its third component, L_3, are the spherical harmonics with the following eigenvalue equations

$$
\begin{aligned}
L^2 Y_{jm}(\vec{x}) &= j(j+1) Y_{jm}(\vec{x}) \,, \quad j = 0, 1, \cdots \\
L_3 Y_{jm}(\vec{x}) &= m Y_{jm}(\vec{x}) \,, \quad m = -j, \cdots, j \,.
\end{aligned}
\tag{2.16}
$$

2.2.1 Scalar Fields on S^2

Using the properties of functions on S^2 given in the previous section, we can examine a few basic properties of a simple scalar field theory on S^2. Let us construct the complex scalar fields on S^2. For a massless complex scalar fields on S^2, it is possible to write the following action by using the Laplacian $-L^2 = -(-i\vec{x} \wedge \vec{\nabla})^2$ on S^2

$$
S = \int \frac{d\Omega}{4\pi} \phi^* L^2 \phi \,. \tag{2.17}
$$

For a real scalar field, we need to impose the reality condition $\phi^* = \phi$. As we mentioned earlier in (2.13), we can expand the complex scalar fields on S^2 in terms of spherical harmonics as follows

$$
\phi = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \phi_{jm} Y_{jm}(\vec{x}) \,. \tag{2.18}
$$

Using (2.14) and (2.16), the action (2.17) becomes

$$
\begin{aligned}
S = \int \frac{d\Omega}{4\pi} \sum_{jm} \sum_{kn} \phi^*_{jm} \phi_{kn} Y^*_{jm} L^2 Y_{kn} &= \sum_{jm} \sum_{kn} \phi^*_{jm} \phi_{kn} k(k+1) \int \frac{d\Omega}{4\pi} Y^*_{jm} Y_{kn} \\
&= \sum_{j=0}^{\infty} \sum_{m=-j}^{j} j(j+1) \phi^*_{jm} \phi_{jm} \,.
\end{aligned}
\tag{2.19}
$$

We note that it is possible to add a potential term in the form of $V(\phi^*\phi)$ in the action (2.17). The mode expansion in (2.19) is given here for future comparison that will be derived on S_F^2 later on subsection 2.5. It is certainly possible to study quantum field theory of scalar, spinor and gauge fields on S^2. For the developments along these lines existing literature may be consulted [48].

2.2.2 Monopole Sectors

In this subsection, we would like to discuss some features of the topologically nontrivial configurations of a complex scalar field on the two sphere. Following the exposition given in [38], we will demonstrate how complex line bundles characterized by a winding number can be given a form amenable to a quantization to obtain their counterparts over the fuzzy sphere S_F^2.

Let us start with the stereographic coordinates in order to describe S^2 in $\mathbb{R}^3$. It is well-known that all points on S^2, say U_S, except the north pole, can be defined by two coordinates (X, Y) where

$$(X, Y) = (\frac{x}{1-z}, \frac{y}{1-z}), \tag{2.20}$$

and similarly all points on S^2, say U_N, except the south pole, are well-defined by (U, V)

$$(U, V) = (\frac{x}{1+z}, -\frac{y}{1+z}). \tag{2.21}$$

Using the coordinates (θ, φ), they may be written as

$$(X, Y) = (\cot\frac{\theta}{2}\cos\varphi\,, \cot\frac{\theta}{2}\sin\varphi)\,, \quad (U, V) = (\tan\frac{\theta}{2}\cos\varphi\,, -\tan\frac{\theta}{2}\sin\varphi)\,. \tag{2.22}$$

Defining two complex number $Z = X + iY$ and $W = U + iV$, we obtain the following inhomogeneous coordinates of S^2 on U_S and U_N respectively

$$Z = \cot\frac{\theta}{2}e^{i\varphi}\,, \quad W = \tan\frac{\theta}{2}e^{-i\varphi}\,. \tag{2.23}$$

We may as well introduce the homogeneous coordinates on U_S, these may be given as

$$\chi_1 = \sin\frac{\theta}{2}\,, \quad \chi_2 = \cos\frac{\theta}{2}e^{i\varphi}\,, \tag{2.24}$$

while on U_N, we may take

$$\chi_1' = \sin\frac{\theta}{2}e^{-i\varphi}\,, \quad \chi_2' = \cos\frac{\theta}{2}\,, \tag{2.25}$$

Using these coordinate systems, we can expand the complex scalar fields on U_S as

$$\phi(\chi, \chi^*) = \sum c_{m_1 m_2 n_1 n_2} {\chi_1^*}^{m_1} {\chi_2^*}^{m_2} \chi_1^{n_1} \chi_2^{n_2}\,, \tag{2.26}$$

whereas on U_N as

$$\phi(\chi', \chi'^*) = \sum c'_{m_1 m_2 n_1 n_2} \chi_1'^{*m_1} \chi_2'^{*m_2} \chi_1'^{n_1} \chi_2'^{n_2}\,. \tag{2.27}$$

It is easily seen that on equator ($\theta = \frac{\pi}{2}$), the expansions (2.26) and (2.27) include the phases $e^{i(-m_2+n_2)\varphi}$ and $e^{i(m_1-n_1)\varphi}$, respectively and, therefore the complex scalars are related by

$$\phi'|_{\theta=\frac{\pi}{2}} = e^{i(m_1+m_2-n_1-n_2)\varphi}\phi|_{\theta=\frac{\pi}{2}} \tag{2.28}$$

Let us define the number $k = \frac{1}{2}(m_1+m_2-n_1-n_2)$, $2k \in \mathbb{Z}$. Let us take k fixed and consider the expansion of $\phi\,,\phi'$

$$\phi = \phi(\chi,\chi^*) = \sum c_{m_1m_2n_1n_2}\chi_1^{*\,m_1}\chi_2^{*\,m_2}\chi_1^{n_1}\chi_2^{n_2}\,, \tag{2.29}$$

$$\phi' = \phi(\chi',\chi'^*) = \sum c_{m_1m_2n_1n_2}\chi_1'^{*m_1}\chi_2'^{*m_2}\chi_1'^{n_1}\chi_2'^{n_2}\,, \tag{2.30}$$

with the same coefficients. Then we see that the scalar fields ϕ and ϕ' form the local sections of a $U(1)$ line bundle which we denote by $\tilde{\mathcal{G}}_k$. For $k=0$, the scalar fields ϕ and ϕ' are smooth functions in the algebra $A \equiv C^\infty(S^2) = \tilde{\mathcal{G}}_0$ while for $k \neq 0$, they form the modules over the algebra $\mathcal{A}$, i.e. $\mathcal{A}$-modules. This means that $\tilde{\mathcal{G}}_0\tilde{\mathcal{G}}_k = \tilde{\mathcal{G}}_k\tilde{\mathcal{G}}_0 = \tilde{\mathcal{G}}_k$. On $U_N \cap U_S$, we have the following transformation between ϕ and ϕ'

$$\phi' = e^{i\kappa\varphi}\phi\,, \tag{2.31}$$

where $\kappa = 2k \in \mathbb{Z}$ is the topological winding number.

The gauge transformation in (2.31) enables us to define the covariant derivatives on U_S and U_N, respectively as

$$D_\mu = i\partial_\mu + A_\mu\,, \quad D'_\mu = i\partial'_\mu + A'_\mu\,, \tag{2.32}$$

where A_μ and A'_μ are the topological gauge fields, whose explicitly form may be defined in terms of χ and χ'

$$A_\mu = i\kappa\chi^\dagger\partial_\mu\chi\,, \quad A'_\mu = i\kappa\chi'^\dagger\partial'_\mu\chi'\,. \tag{2.33}$$

We know that the transformation property of the covariant derivative should be same with complex scalar fields in order to obtain gauge invariant Lagrangian, i.e it must be of the form $D'_\mu\phi' = e^{i\kappa\varphi}D_\mu\phi$. Making use of this fact, we can construct the relation between A_μ and A'_μ as

$$A'_\mu = A_\mu - ig^{-1}\partial_\mu g\,, \quad g = e^{i\kappa\varphi}\,. \tag{2.34}$$

It is important to note that the gauge fields given in (2.33) steam from the nontrivial topological structure of complex scalar fields on S^2, can be called the topological κ-monopole gauge field. It is the same monopole gauge field encountered in discussion of Berry's phase and it has noting to do with the dynamical gauge fields.

The action for the complex scalar fields on S^2 with topological κ-monopole gauge field can be written as

$$\int \frac{d\Omega}{4\pi}\left(\phi^* D_\mu^2 \phi + V(\phi^*\phi)\right) , \tag{2.35}$$

for a suitably given potential $V(\phi^*\phi)$ satisfying the standard field theoretical requirements such as being bounded from below.

Now, we would like to describe this topological nontrivial field configuration without making any specific coordinate choice. It can be easily seen that the coordinate system (2.24) and (2.25) correspond to the coordinates of S^3 in the Hopf fibration map (2.7). To be more precise, setting $\xi = \chi$ or $\xi = \chi'$ in the map (2.7), we can obtain $x_1 = \cos\varphi \sin\theta\,, x_2 = \sin\varphi\sin\theta$ and $x_3 = \cos\theta$. Hence, we can write those complex scalar fields ϕ in S^3 with $k = \frac{1}{2}(m_1 + m_2 - n_1 - n_2)$ fixed as

$$\phi(\xi, \xi^*) = \sum c_{m_1 m_2 n_1 n_2} \xi_1^{*\,m_1} \xi_2^{*\,m_2} \xi_1^{n_1} \xi_2^{n_2} \,. \tag{2.36}$$

This corresponds to the set functions denoted by $\mathcal{G}_k$, $2k \in \mathbb{Z}$. As we mentioned before, under the $U(1)$ action, the coordinates of S^2 in (2.7) does not change. Under $U(1)$ action

$$\xi \to e^{\frac{i}{2}\psi}\xi\,, \quad \xi^\dagger \to e^{-\frac{i}{2}\psi}\xi^\dagger \tag{2.37}$$

we see that $\phi \in \mathcal{G}_k$ transforms as

$$\phi \to e^{-ik\psi}\phi\,. \tag{2.38}$$

Let us introduce the operator K_0 in the form

$$K_0 = \frac{1}{2}(\xi_\alpha^* \partial_{\xi_\alpha^*} - \xi_\alpha \partial_{\xi_\alpha})\,. \tag{2.39}$$

It can be easily seen that $\phi \in \mathcal{G}_k$ are the eigenvectors of this operator with the eigenvalue k;

$$\begin{aligned}
K_0\phi &= \frac{1}{2}(\xi_\alpha^* \partial_{\xi_\alpha^*} - \xi_\alpha \partial_{\xi_\alpha})\phi \\
&= \frac{1}{2}(\xi_1^* \partial_{\xi_1^*} + \xi_2^* \partial_{\xi_2^*} - \xi_1 \partial_{\xi_1} - \xi_2 \partial_{\xi_2}) \sum c_{m_1 m_2 n_1 n_2} {\xi_1^*}^{m_1} {\xi_2^*}^{m_2} \xi_1^{n_1} \xi_2^{n_2} \\
&= \frac{1}{2}(m_1 + m_2 - n_1 - n_2)\phi = k\phi \,. \qquad (2.40)
\end{aligned}$$

Here, we again stress that when $k = 0$, the complex scalar fields are just the scalar fields given in section 2.2.1 with the commutative algebra $\mathcal{A} = C^\infty(S^2)$, while $k \neq 0$, they are the element of $\mathcal{A}$-bimodules, $\mathcal{G}_k$. Furthermore, we see that $K_0\phi^* = -k\phi^*$ for $\phi \in \mathcal{G}_k$, therefore $\phi^* \in \mathcal{G}_k^* = \mathcal{G}_{-k}$ and also that $\mathcal{G}_k\mathcal{G}_l \subset \mathcal{G}_{k+l}$.

It is also possible to define the operators mapping $\mathcal{G}_k$ onto itself as follows

$$J_i = \frac{i}{2}\left(\xi_\alpha^* \tau_{\alpha\beta}^{i\,*} \partial_{\xi_\beta^*} - \xi_\alpha \tau_{\alpha\beta}^i \partial_{\xi_\beta}\right) \,. \qquad (2.41)$$

They form a differential realization of $SU(2)$ generators (in S^3 coordinates) satisfying the commutation relation $[J_i, J_j] = \epsilon_{ijk} J_k$.

Under the action of the generators J_i, ξ transform as spinors

$$J_i \xi_\beta = \frac{1}{2i}\tau_{\alpha\beta}^i \xi_\alpha \,, \quad J_i \xi_\beta^* = -\frac{1}{2i}\tau_{\alpha\beta}^i \xi_\alpha^* \qquad (2.42)$$

and x_i transform as vectors in $\mathbb{R}^3$. It can be easily seen that $x_i^2 = r^2$ is an invariant function under the action of J_i as expected. We can also define the operators which map $\mathcal{G}_k$ to $\mathcal{G}_{k+1}$ and $\mathcal{G}_k$ to $\mathcal{G}_{k-1}$, respectively as

$$K_+\phi = i\epsilon_{\alpha\beta}\xi_\alpha^*(\partial_{\xi_\beta}\phi) \,, \quad K_-\phi = i\epsilon_{\alpha\beta}(\partial_{\xi_\alpha^*}\phi)\xi_\beta \,, \qquad (2.43)$$

satisfying

$$[K_0, K_\pm] = \pm K_\pm \,, \quad [K_+, K_-] = 2K_0 \,. \qquad (2.44)$$

Let us explicitly show the first relation in (2.43)

$$\begin{aligned}
K_+\phi &= (i\xi_1^* \partial_{\xi_2} - i\xi_2^* \partial_{\xi_1}) \sum c_{m_1 m_2 n_1 n_2} {\xi_1^*}^{m_1} {\xi_2^*}^{m_2} \xi_1^{n_1} \xi_2^{n_2} \\
&= \sum c_{m_1 m_2 n_1 n_2} \left(i n_2 {\xi_1^*}^{m_1+1} {\xi_2^*}^{m_2} \xi_1^{n_1} \xi_2^{n_2-1} - i n_1 {\xi_1^*}^{m_1} {\xi_2^*}^{m_2+1} \xi_1^{n_1-1} \xi_2^{n_2} \right) \\
&:= i(n_2\phi' - n_1\phi'') \,, \qquad (2.45)
\end{aligned}$$

where ϕ' and ϕ'' have the topological number $k+1$. We note that although K_0, $K_\pm$ commute with the operators J_i, they depend on each other by the following relation

$$J_i^2 = K_0^2 + \frac{1}{2}(K_+K_- + K_-K_+) \,. \tag{2.46}$$

Now, we are ready to write the action for the complex scalar field with the topological monopole charge $\kappa = 2k$ in terms of the operators K_+ and K_- as follows

$$\int \frac{d\Omega}{4\pi}\left(\phi^* \frac{1}{2}(K_+K_- + K_-K_+)\phi + V(\phi^*\phi)\right) = \int \frac{d\Omega}{4\pi}\left(\phi^*(J^2 - k^2)\phi + V(\phi^*\phi)\right) \,, \tag{2.47}$$

where $\phi \in \mathcal{G}_k$ and we have used (2.40) and (2.46) to get the second line in (2.47). It can be easily seen that for $k = 0$, we obtain the action given in section (2.2.1). The noncommmutative version of this complex line bundles will be given in section (2.5.1).

2.3 Noncommutative Version of the Hopf Fibration

So far, we have given the necessary information which prepared us to take up task of describing the noncommutative or the fuzzy sphere S_F^2. In this section, we construct the noncommutative version of the first Hopf fibration [41] by using the quantization of $\mathbb{C}^2$. The quantized version of the first Hopf fibration enables us to construct the noncommmutative three sphere, S_F^3, and subsequently the noncommmutative two sphere, S_F^2, by exploring descent chain $\mathbb{C}^2 \to S^3 \to S^2$ given in section 2.1.

Let us first quantize $\mathbb{C}^2$. On $\mathbb{C}^2$, the Poisson bracket of two functions may be defined as

$$\{A, B\} = \sum_j \frac{\partial A}{\partial z_j}\frac{\partial B}{\partial \bar{z}_j} - \frac{\partial A}{\partial \bar{z}_j}\frac{\partial B}{\partial z_j} \,, \quad j = (1,2) \tag{2.48}$$

where

$$\{z_i, z_j\} = 0 \,, \quad \{\bar{z}_i, \bar{z}_j\} = 0 \,, \quad \{z_i, \bar{z}_j\} = \delta_{ij} \,. \tag{2.49}$$

We want to quantize the manifold $\mathbb{C}^2$ by replacing the Poisson bracket with an appropriate commutator $\{\hat{A},\hat{B}\} = \hat{A}\hat{B} - \hat{B}\hat{A}$ of operators $\hat{A}\,,\hat{B}$ acting on an infinite dimensional Hilbert space which may be conceived as the two-particle Fock space. This is equivalent to take the coordinates z_i as harmonic oscillator annihilation operators A_i and z_i^* as their adjoint $A_i^\dagger$. Then, we obtain the two-dimensional noncommmutative complex plane $\mathbb{C}_\theta^2$ with the commutation relations

$$[A_i, A_j] = 0\,, \quad [A_i^\dagger, A_j^\dagger] = 0\,, \quad [A_i, A_j^\dagger] = \theta\delta_{ij}\,, \tag{2.50}$$

where we have introduced the noncommmutative parameter θ with dimension length squared. Taking $\theta \to 0$, we obtain the classical manifold $\mathbb{C}^2$. With the scaling $A_i \to \frac{A_i}{\sqrt{\theta}}$, we can express non-trivial commutation relation simply as $[A_i, A_j^\dagger] = \delta_{ij}$. With the same method, we can achieve the quantization of the $N+1$-dimensional complex plane $\mathbb{C}^{N+1}$ for any N using suitable number of pair of annihilation-creation operators. The noncommmutative complex space $\mathbb{C}_\theta^{N+1}$ can be employed to obtain the fuzzy version of complex projective plane $\mathbb{C}P^N$ [41]. We will explain this relation later.

Now, using the definition (2.4) and these annihilation-creation operators, we can obtain the noncommmutative version of S^3 as

$$\begin{aligned}
\xi_i = \frac{z_i}{|z|} \to \hat{\xi}_i = A_i\frac{1}{\sqrt{\hat{N}}} = \frac{1}{\sqrt{\hat{N}+1}}A_i\,, \\
\xi_i^* = \frac{z_i^*}{|z|} \to \hat{\xi}_i^* = \frac{1}{\sqrt{\hat{N}}}A_i^\dagger = A_i^\dagger\frac{1}{\sqrt{\hat{N}+1}}\,,
\end{aligned} \tag{2.51}$$

where $\hat{N}$ is the number operator: $\hat{N} = \sum A_j^\dagger A_j$ with the condition $\hat{N} \neq 0$. This condition means that we omit the vacuum state from Hilbert space of states which is like removing the origin from $\mathbb{C}^2$ in section (2.1). To be more precise, S_F^3 is defined on the Hilbert space which is the orthogonal complement of the vacuum in the Fock space. We note that this construction of S_F^3 does not yield a truncated finite dimensional Hilbert space and it is ill-defined as acting on any state in this Hilbert space with some suitable power of the operator $A_i\frac{1}{\sqrt{\hat{N}}}$, the vacuum state will eventually be created. S_F^3 may be viewed as an auxiliary space in the construction of the fuzzy two sphere S_F^2 and this is the reason why it is introduced here. Let us also note that this space has nothing to do with the fuzzy three sphere construction given in [49].

Making use of the Hopf fibration map given in (2.7), the "coordinates" of the fuzzy sphere can be given by the operators

$$x_i(\xi) \to \hat{x}_i = \frac{1}{\sqrt{\hat{N}}} A^\dagger \tau_i A \frac{1}{\sqrt{\hat{N}}} = \frac{1}{\hat{N}} A^\dagger \tau_i A \,, \quad \hat{N} \neq 0 \,, \quad A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \,. \tag{2.52}$$

We investigate the properties of this noncommutative manifold in the next section.

2.4 Fuzzy Sphere

In the previous subsection, we have shown that we obtain the noncommmutative two sphere S_F^2 by replacing the commutative coordinates x_i of S^2 by the noncommmutative coordinates $\hat{x}_i$ and expressing them in terms of annihilation creation operators as

$$\hat{x}_i = \frac{1}{\hat{N}} A^\dagger \tau_i A \,, \quad \hat{N} \neq 0 \tag{2.53}$$

We point out that these coordinates commute with the number operator

$$[\hat{x}_i, \hat{N}] = 0 \,, \tag{2.54}$$

and this means that $\hat{x}_i$ can be restricted to the subspace $\mathcal{H}_n$ of the Fock space for $\hat{N} = n \neq 0$. $\mathcal{H}_n$ is a $(n+1)$-dimensional subspace of the Fock space which is spanned by the following orthogonal vectors

$$\frac{(A_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(A_2^\dagger)^{n_2}}{\sqrt{n_2!}} |0\,,0\rangle \equiv |n_1\,,n_2\rangle \,, \quad n_1 + n_2 = n \neq 0 \,. \tag{2.55}$$

The algebra of $(n+1) \times (n+1)$ matrices, $Mat(n+1)$ is completely generated by the polynomials in $\hat{x}_i$ restricted to the subspace $\mathcal{H}_n$.

We know that there exist a connection between the algebra of angular momentum and two independent harmonic oscillators, namely this is the Schwinger construction [50]. Consider the operators L_i defined in the form

$$L_i = \frac{1}{2} A^\dagger \tau_i A \,, \quad A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{2.56}$$

where A_α and $A_\alpha^\dagger$ fulfill (2.50). Then it is straightforward to verify that

$$[L_i, L_j] = i\epsilon_{ijk} L_k \,, \tag{2.57}$$

which is the commutation relation of angular momentum operator familiar to us from quantum mechanics. L_i generate the $su(2)$ Lie algebra and also the $SU(2)$ Lie group. Here, we can see that annihilation and creation operators carry the spin $\frac{1}{2}$ IRR of $SU(2)$ and thus they transform as spinors under the action of $SU(2)$

$$[L_i, A_\alpha] = -\frac{1}{2}(\tau_i)_{\alpha\beta} A_\beta \,, \quad [L_i, A_\alpha^\dagger] = \frac{1}{2}(\tau_i)_{\beta\alpha} A_\beta^\dagger \,. \tag{2.58}$$

This means that by the following n-fold symmetric product of these spinors

$$\underbrace{\frac{1}{2} \otimes_s \cdots \otimes_s \frac{1}{2}}_{n-\text{fold}} = \frac{n}{2} \,, \tag{2.59}$$

we can obtain the angular momentum $\ell = \frac{n}{2}$ irreducible representation of $SU(2)$ with the Casimir

$$L^2 |n_1\,, n_2\rangle = \frac{n}{2}(\frac{n}{2} + 1)|n_1\,, n_2\rangle \,. \tag{2.60}$$

Using (2.52) and (2.56), we have the relation between the coordinates of S_F^2 and the generators of $SU(2)$ as follows

$$\hat{x}_i |n_1\,, n_2\rangle = \frac{2}{n} L_i |n_1\,, n_2\rangle \,, \tag{2.61}$$

Hence, we obtain

$$[\hat{x}_i, \hat{x}_j] = \frac{2}{n} i\epsilon_{ijk} \hat{x}_k \,, \quad \sum_i \hat{x}_i^2 = (1 + \frac{2}{n}) \,, \tag{2.62}$$

where it is understood that these relations are given on the Hilbert space $\mathcal{H}_n$. As $n \to \infty$, we see from (2.62) that two-sphere S^2 is recovered.

We note that with the scaling $\hat{x}_i \to \frac{\hat{x}_i}{\sqrt{1+\frac{2}{n}}}$ and recalling that $n = 2\ell$, (2.61) and (2.62) can be rewritten in the following form

$$\hat{x}_i = \frac{L_i}{\sqrt{\ell(\ell+1)}} \,, \quad [\hat{x}_i, \hat{x}_j] = \frac{i}{\sqrt{\ell(\ell+1)}} \epsilon_{abc} \hat{x}_j \,, \quad \sum_i \hat{x}_i^2 = 1 \,. \tag{2.63}$$

These relations summarize the description of the fuzzy sphere and they will be frequently used throughout this thesis. If we need, we can make the scaling

$\hat{x}'_i = R\hat{x}_i$ to introduce the radius R for S_F^2 and work with the dimensionful quantities.

Any element $m \in Mat(2\ell+1)$ is an element of the fuzzy sphere S_F^2 and it is finitely generated by $\hat{x}_i$. We may express this fact by writing

$$m = \sum c_{i_1,\cdots,i_k} \hat{x}_{i_1} \cdots , \hat{x}_{i_k} \,. \tag{2.64}$$

A scalar product S_F^2 can be defined as

$$(m_1, m_2) = \mathrm{Tr}(m_1^\dagger m_2) = \frac{1}{n+1} Tr(m_1^\dagger m_2) \,, \quad m_i \in Mat(n+1) \,, \tag{2.65}$$

where $\mathrm{Tr} = \frac{1}{n+1} Tr$ is the normalized trace. Commutative limit of this product may be shown to correspond to integration over S^2. In particular, we see that $\mathrm{Tr}\mathbf{1} = 1$ corresponding to $\int \frac{d\Omega}{4\pi} = 1$.

Left acting and right acting linear operators may be defined on $Mat(2\ell+1)$. Let us consider the two linear operator α^L and α^R. We can write

$$\alpha^L m = \alpha m \,, \quad \alpha^R m = m\alpha \,, \quad \alpha^{L,R} \,, m \in Mat(2\ell+1) \,. \tag{2.66}$$

It is easy to see that α^L and α^R satisfy

$$(\alpha\beta)^L = \alpha^L \beta^L \,, \quad (\alpha\beta)^R = \beta^R \alpha^R \,, \quad [\alpha^L, \beta^R] = 0 \text{ for any } \alpha \,, \beta \in Mat(2\ell+1) \,. \tag{2.67}$$

Here, we have two commuting matrix algebras $Mat_L(2\ell+1)$ and $Mat_R(2\ell+1)$ generated by the left acting and right acting operators. As we mentioned earlier, the matrix algebra $Mat(2\ell+1)$ is generated by the coordinates $\hat{x}_i$ which contain the terms $A_i^\dagger A_j$ with the domain $\mathcal{H}_n$. Hence, the algebras $Mat_{L,R}(2\ell+1)$ are generated by the operators $(A_i^\dagger A_j)^{L,R}$.

We may as well write

$$A_i^L m = A_i m \,, \quad {A_i^\dagger}^L m = A_i^\dagger m \,, \quad A_i^R = m A_i \,, \quad {A_i^R}^\dagger = m A_i^\dagger \,. \tag{2.68}$$

However, the operators $A_i^L \,, A_i^R$ take $\mathcal{H}_n$ to $\mathcal{H}_{n-1}$ while $A_i^{L\dagger} \,, A_i^{R\dagger}$ take $\mathcal{H}_n$ to $\mathcal{H}_{n+1}$. Such operators play a role in the description of fiber bundles over S_F^2 as we will see in subsection 2.5.1.

Now, we are ready to define the derivations on S_F^2. It is known that since $su(2)$ act on $Mat(2\ell+1)$ by the adjoint action, the derivations on S_F^2 can be defined as

$$\mathcal{L}_i m := adL_i m = (L_i^L - L_i^R)m = [L_i, m]\,. \tag{2.69}$$

This provides a map from $Mat(2\ell+1)$ onto itself and it satisfies the Leibniz rule:

$$\mathcal{L}_i(m_1 m_2) = (\mathcal{L}_i m_1)m_2 + m_1(\mathcal{L}_i m_2)\,, \tag{2.70}$$

which means that it is indeed a derivation over the algebra $Mat(2\ell+1)$. It is also seen that the action of the operator $\mathcal{L}_i$ on the identity matrix $\mathbf{1}$ is zero. These facts are analogous to the chain rule in differentiation and the annihilation of constant functions by the continuum orbital angular momentum operator. From these facts, we can conjecture that the operator $\mathcal{L}_i$ is the fuzzy sphere "orbital angular momentum" which reduces to the usual derivative operator L_i on S^2 as $\ell \to \infty$;

$$\mathcal{L}_i \to L_i = -i(\vec{x}(\xi) \wedge \vec{\nabla})_i \equiv -i\epsilon_{ijk} x(\xi)_j \frac{\partial}{\partial x(\xi)_k}\,. \tag{2.71}$$

To make this correspondence precise, we determine the spectrum of the orbital angular momentum operator on S_F^2. Equation in (2.69) indicates that the operator $\mathcal{L}_i$ includes both the left and the right $SU(2)$ actions on $Mat(2\ell+1)$, L_i^L and L_i^R each carry the IRR ℓ of $SU(2)$. This is readily seen from the fact that

$$L_i^L L_i^L = \ell(\ell+1)\,, \quad L_i^R L_i^R = \ell(\ell+1)\,, \tag{2.72}$$

on $Mat(2\ell+1)$. Consequently its representation content is given by tensor product

$$\ell \otimes \ell \equiv 0 \oplus 1 \cdots \oplus 2\ell\,. \tag{2.73}$$

Spectrum of $\mathcal{L}^2$ is then given as

$$j(j+1)\,, \quad j = 0, 1 \cdots, 2\ell\,. \tag{2.74}$$

This is exactly the spectrum of L^2, but now truncated at $j = 2\ell$ and corresponds to it in the limit $\ell \to \infty$. Thus, it is justified to have $\mathcal{L}_i$ as the "orbital angular momentum" over S_F^2. What are the corresponding eigenvectors of $\mathcal{L}^2$ and $\mathcal{L}_3$ in the fuzzy case. These are given by so called the polarization tensors $T_{jm}(\ell)\,, j = 0, \cdots, 2\ell\,, m = -j, \cdots, j$ which are $(2\ell+1)\times(2\ell+1)$ matrices [51]. These matrices form a basis in $Mat(2\ell{+}1)$ since there are $(2\ell{+}1)^2$ linearly independent $T_{jm}(\ell)$s. This can easily be seen from the fact that since for a given j, there are $2j+1$ different values for m and therefore there are $\sum_{j=0}^{2\ell}(2j+1) = (2\ell+1)^2$, independent degrees of freedom in $T_{jm}(\ell)$. These tensors have the following eigenvalue equations and the orthogonality relation

$$\mathcal{L}^2 T_{jm} = j(j+1)T_{jm}\,, \quad \mathcal{L}_3 T_{jm} = [L_3, T_{jm}] = mT_{jm}\,, \quad (T_{j'm'}, T_{jm}) = \delta_{j'j}\delta_{m'm}\,. \tag{2.75}$$

$T_{jm}(\ell)$ carry the spin ℓ IRR of $SU(2)$ just as their continuum counter parts Y_{jm}. Under $SU(2)$ rotations, $T_{jm}(\ell)$ transform as

$$\tilde{T}_{jm'}(\ell) = \mathcal{D}(g)T_{jm'}(\ell)(\mathcal{D}(g))^{-1} = \sum_m \mathcal{D}(g)^j_{mm'}T_{jm}(\ell)\,, \tag{2.76}$$

where $\mathcal{D}(g)^j_{mm'}$ are Wigner functions for $SU(2)$, i.e. elements of the $SU(2)$ rotation matrices.

We note that S_F^2 preserves the rotational symmetry of two-sphere S^2. In other words, it is invariant under the $SU(2)$ action. This fact can be easily seen by the transformation property of commutation relation of the coordinates on S_F^2. A group element g of $SU(2)$ act on $\hat{x}_i$ adjointly: $Ad\hat{x}_i = g\hat{x}_i g^{-1}$ and the commutation relation in (2.63) becomes

$$[g\hat{x}_i g^{-1}, g\hat{x}_j g^{-1}] = \frac{i}{\sqrt{\ell(\ell+1)}}\epsilon_{abc}\, g\hat{x}_k g^{-1}\,, \tag{2.77}$$

which means that rotational symmetry is preserved by S_F^2. As S_F^2 is a truncation of S^2 with finite numbers of degrees of freedom and it preserves the rotational symmetry of S^2, it appears to be well suited setting for investigating classical and quantum field theories. This is the task we take up next.

2.5 Scalar Fields on S_F^2

Our aim is to adopt the formulations of scalar fields on S^2 given in section 2.2.1 to the fuzzy case. Let us first expand the complex scalar field $\Phi \in Mat(2\ell+1)$

in terms of the polarization tensors as

$$\Phi = \sum_{j=0}^{2\ell} \sum_{m=-j}^{j} \Phi_{jm} T_{jm}(\ell) \,. \tag{2.78}$$

Euclidean action for a massless field on S_F^2 may be written as

$$S = (\mathcal{L}_i \Phi, \mathcal{L}_i \Phi) = (\Phi, \mathcal{L}^2 \Phi) = \frac{1}{(2\ell+1)} Tr(\Phi^\dagger \mathcal{L}^2 \Phi) \,. \tag{2.79}$$

Using the expansion (2.78), this action becomes

$$\begin{aligned} S = \mathrm{Tr}(\Phi^\dagger \mathcal{L}^2 \Phi) &= \sum_{jm} \sum_{kn} k(k+1) \mathrm{Tr}(T_{jm}^\dagger T_{kn}) \Phi_{jm}^\dagger \Phi_{kn} \\ &= \sum_{j=0}^{2\ell} \sum_{m=-j}^{j} j(j+1) |\Phi_{jm}|^2 \,. \end{aligned} \tag{2.80}$$

This action has finite degrees of freedom and approaches to (2.19) in the limit $\ell \to \infty$. It is possible to add the potential term consisting of polynomials $P(\Phi)$ of Φ in this action

$$V(\Phi) = (1, P(\Phi)) = \frac{1}{(2\ell+1)} Tr(P(\Phi)) \,. \tag{2.81}$$

We note that the action (2.79) with or without the potential term $V(\Phi)$ is invariant under $SU(2)$ transformations. This can be checked both for finite $SU(2)$ transformation, taking $\Phi \to g^{-1} \Phi g \,,\ g \in SU(2)$ or by taking an infinitesimal $SU(2)$ transformation $\Phi \to \Phi + i\epsilon_i [L_i, \Phi]$, using $Adg = e^{i\epsilon_i adL_i} \approx 1 + i\epsilon_i adL_i$ for infinitesimal ϵ_i. It is instructive to check this for the infinitesimal case. Under the infinitesimal action $\Phi \to \Phi + i\epsilon_i [L_i, \Phi]$, the action (2.79) becomes

$$\begin{aligned} S &= \mathrm{Tr}\left((\Phi + i\epsilon_i [L_i, \Phi])^\dagger \, \mathcal{L}^2 \, (\Phi + i\epsilon_i [L_i, \Phi]) \right) \,, \\ &= \mathrm{Tr}\left(\Phi^\dagger \mathcal{L}^2 \Phi + i\epsilon_i [L_i, \Phi^\dagger] \mathcal{L}^2 \Phi + i\epsilon_i \Phi^\dagger \mathcal{L}^2 [L_i, \Phi] \right) + O(\epsilon_i^2) \,, \\ &= \mathrm{Tr}\Big(\Phi^\dagger \mathcal{L}^2 \Phi + i\epsilon_i \big(\Phi^\dagger L^2 L_i \Phi + \Phi^\dagger L_i \Phi L^2 - 2\Phi^\dagger L_j L_i \Phi L_j - \Phi^\dagger L^2 \Phi L_i \\ &\quad - \Phi^\dagger \Phi L_i L^2 + 2\Phi^\dagger L_j \Phi L_i L_j + L_i \Phi^\dagger (L^2 \Phi + \Phi L^2 - 2L_j \Phi L_j) \\ &\quad - \Phi^\dagger L_i (L^2 \Phi + \Phi L^2 - 2L_j \Phi L_j) \big) \Big) \,, \\ &= \mathrm{Tr}\left(\Phi^\dagger \mathcal{L}^2 \Phi \right) \end{aligned} \tag{2.82}$$

For finite $SU(2)$ transformation, the invariance is already observed due to cyclic property of the trace.

2.5.1 Monopole Sectors

In subsection 2.2.2 we have constructed the topologically nontrivial complex line bundles over S^2. Here, our goal is to carry that construction to the fuzzy setting and obtain the corresponding quantized version of line bundles over S_F^2 following the treatment given in [38]. As explained in section 2.3, in the quantized version of first Hopf fibration, the coordinates on S_F^3 may be defined in terms of annihilation-creation operator as follows

$$\hat{\xi}_i = A_i \frac{1}{\sqrt{\hat{N}}} , \quad \hat{\xi}_i^* = \frac{1}{\sqrt{\hat{N}}} A_i^\dagger . \tag{2.83}$$

Consider the monomial

$$\hat{\xi}_1^{*m_1} \hat{\xi}_2^{*m_2} \hat{\xi}_1^{n_1} \hat{\xi}_2^{n_2} \tag{2.84}$$

where $k = \frac{1}{2}(m_1 + m_2 - n_1 - n_2)$ is fixed. Let us denote the linear space which is spanned by monomials of this form as $\hat{\mathcal{G}}_k$. For $k = 0$, it collapses to the noncommutative algebra $\hat{\mathcal{G}}_0 = \hat{\mathcal{A}}$ generated by the operators L_i in (2.56) satisfying (2.57). For nonzero k, $\hat{\mathcal{G}}_k$ form the bimodules over $\hat{\mathcal{A}}$ and the operator L_i act adjointly on the elements of this space as $J_i \cdot := [L_i, \cdot]$. The quantization of $\mathcal{G}_k$ to $\hat{\mathcal{G}}_k$ is evident from these facts.

Now, just like the commutative case, the complex scalar fields are the elements of $\hat{\mathcal{A}}$-bimodules, $\hat{\mathcal{G}}_k$, in the form

$$\Phi = \sum a_{m_1 m_2 n_1 n_2} \hat{\xi}_1^{*m_1} \hat{\xi}_2^{*m_2} \hat{\xi}_1^{n_1} \hat{\xi}_2^{n_2} \tag{2.85}$$

with $k = \frac{1}{2}(m_1 + m_2 - n_1 - n_2)$ fixed. Let us define $m = m_1 + m_2$, $n = n_1 + n_2$ and $k = \frac{1}{2}(m - n)$. As we mentioned earlier, $(n+1)$-dimensional subspace $\mathcal{H}_n$ of the Fock space for the eigenvalue of the number operator $n \neq 0$ can be spanned by the annihilation-creation operators explicitly given in (2.55). Making use of this information, it can be seen that the complex scalars in (2.85) are the operators which map $\mathcal{H}_n$ to $\mathcal{H}_m$ since it contains n annihilation and m creation operators. Precisely, they are the $(m+1) \times (n+1)$ matrices.

Let us denote this linear mapping space as $\hat{\mathcal{G}}_{mn}$ instead of $\hat{\mathcal{G}}_k$. $\hat{\mathcal{G}}_{nn} = \hat{\mathcal{G}}_0$ is the $(n+1) \times (n+1)$ matrix algebra with the restriction $n \neq 0$. Hence, for this case, the complex scalar fields Φ are $(n+1)\times(n+1)$ matrices with $n=2\ell$ given in the previous section. On the other hand, $\hat{\mathcal{G}}_{mn}$ forms a left module over $\hat{\mathcal{G}}_{mm}$ and a right module over $\hat{\mathcal{G}}_{nn}$. The $su(2)$ rotation operators act on Φ as

$$J_j\Phi = L_j^m\Phi - \Phi L_j^n \,, \tag{2.86}$$

where L_j^m are $(m+1)\times(m+1)$-dimensional $su(2)$ generators whereas L_j^n are the $su(2)$ generators with $(n+1)\times(n+1)$ dimension. $su(2)$ IRR content of J_j can be seen from the following tensor product

$$\frac{m}{2}\otimes\frac{n}{2}\equiv |k|\oplus(|k|+1)\cdots\oplus J \,, \tag{2.87}$$

where $k=\frac{1}{2}(m-n)$ and $J=\frac{1}{2}(m+n)$. The eigenvectors of the operator J^2 and J_3 can be given by the generalized harmonics Φ^j_{Jks} with the eigenvalue equations

$$\begin{aligned} &J^2\Phi^j_{Jks} = j(j+1)\Phi^j_{Jks}\,, \quad J_3\Phi^j_{Jks} = s\Phi^j_{Jks}\,, \\ &j = |k|\,, |k|+1\,, \cdots\,, \quad s = -j\,, \cdots j\,, \end{aligned} \tag{2.88}$$

where there are $(m+1)(n+1)$ linearly independent Φ^j_{Jks}. Equation (2.88) may be compared with (2.75). Any element of $\hat{\mathcal{G}}_{mn}$ can be expanded in terms of the operators Φ^j_{Jks}.

Let us introduce the operator that gives the eigenvalue $k=\frac{1}{2}(m-n)$ under the action on the scalar fields $\Phi\in\hat{\mathcal{G}}_{mn}$ as

$$K_0\Phi = \frac{1}{2}[\hat{N},\Phi]\,, \quad \hat{N} = A_i^\dagger A_i\,. \tag{2.89}$$

In order to avoid any confusion, let us note that $\hat{N}|n_1\,,n_2\rangle = n|n_1\,,n_2\rangle$ and $\hat{N}|m_1\,,m_2\rangle = m|m_1\,,m_2\rangle$. Equation (2.89) can be checked as

$$\begin{aligned} K_0\Phi &= \frac{1}{2}[\hat{N},\Phi] = \frac{1}{2}(\hat{N}\Phi - \Phi\hat{N}) \\ &= \frac{1}{2}\Big((A_1^\dagger A_1 + A_2^\dagger A_2)\sum a_{m_1m_2n_1n_2}\hat{\xi}_1^{*m_1}\hat{\xi}_2^{*m_2}\hat{\xi}_1^{n_1}\hat{\xi}_2^{n_2}\Big) - \frac{1}{2}\Phi\hat{N} \\ &= \sum a_{m_1m_2n_1n_2}\hat{\xi}_1^{*m_1}\hat{\xi}_2^{*m_2}\hat{\xi}_1^{n_1}\hat{\xi}_2^{n_2}\Big(A_1^\dagger A_1 + A_2^\dagger A_2 \\ &+ \frac{1}{2}(m_1+m_2-n_1-n_2)\Big) - \frac{1}{2}\Phi\hat{N} \\ &= \frac{1}{2}(m_1+m_2-n_1-n_2)\Phi + \frac{1}{2}(\Phi\hat{N} - \Phi\hat{N}) \\ &= k\Phi\,, \end{aligned} \tag{2.90}$$

where $\kappa = 2k$ is the winding number. We can also give the operators K_+ and K_- which increase and decrease the topological number k by 1, respectively as

$$K_+\Phi = i\epsilon_{\alpha\beta}A^\dagger_\beta[\Phi, A^\dagger_\alpha]\,, \quad K_-\Phi = i\epsilon_{\alpha\beta}[A_\alpha, \Phi]A_\beta\,. \tag{2.91}$$

They can be readily compare with their commutative counterparts in (2.43). Finally, it remains to write down the action for the complex scalar fields with nonzero winding number κ. This is given as

$$S = \mathrm{Tr}(\Phi^\dagger \frac{1}{2}(K_+K_- + K_-K_+)\Phi + V(\Phi^\dagger\Phi)) = \mathrm{Tr}(\Phi^\dagger(J^2 - k^2)\Phi + V(\Phi^\dagger\Phi))\,. \tag{2.92}$$

A comparison with (2.47) reveals the analogy and summarized the result that we were set to achieve in this section.

2.6 $U(n)$ Gauge Theory on S^2_F

Scalar, spinor and gauge theories on S^2_F and their various aspects have been investigated in the recent past [36,38,52–62]. In this thesis, we will be essentially concerned with the aspects of gauge theories over S^2_F, $S^2_F \times S^2_F$, we therefore focus on the latter and refer the interested reader to the references [53, 57, 59] to find out more on scalar and spinor fields on S^2_F. Here, we investigate the gauge theory on S^2_F by using a matrix model. Following the approach in [47], we will first give the gauge theory on S^2_F which reduce to a $U(1)$ gauge theory on S^2 in the commutative limit $\ell \to \infty$ and then generalize this construction to $U(n)$ gauge theory on S^2_F.

2.6.1 $U(1)$ Gauge Theory

Let us consider the following matrix model with the action

$$S = Tr(V(\Lambda)) = \frac{1}{(2\ell+1)g^2} Tr\left((\Lambda^2 - (\ell + \frac{1}{2})^2 \mathbf{1}_{2(2\ell+1)})^2\right)\,, \tag{2.93}$$

where Λ is a $2(2\ell+1) \times 2(2\ell+1)$ Hermitian matrices, i.e. $\Lambda \in Mat(2(2\ell+1))$. It can be easily seen that this action is invariant under the adjoint action of

$U(2(2\ell+1))$

$$\Lambda \to U^{-1}\Lambda U\,, \quad U \in U(2(2\ell+1))\,. \tag{2.94}$$

The equation of motion can be obtained as

$$\Lambda(\Lambda^2 - (\ell+\frac{1}{2})^2) = 0\,. \tag{2.95}$$

This means that for the vacuum configuration, the eigenvalues of Λ are $\pm(\ell+\frac{1}{2}), 0$ with some multiplicities, say $n_\pm, n_0$. With $n_+ + n_- + n_0 = 2(2\ell+1)$, up to a unitary transformation, the matrix Λ satisfying (2.95) can be written explicitly as follows

$$\Lambda = \begin{pmatrix} (\ell+\frac{1}{2})\mathbf{1}_{n_+} & 0 & 0 \\ 0 & -(\ell+\frac{1}{2})\mathbf{1}_{n_-} & 0 \\ 0 & 0 & 0\mathbf{1}_{n_0} \end{pmatrix}\,. \tag{2.96}$$

Let us focus on the specific case by choosing $n_0 = 0\,, n_+ = 2\ell+2$ and $n_- = 2\ell$. We will show that by a unitary transformation of $U(2(2\ell+1))$, this matrix can be transformed in the following form

$$\Lambda = \frac{1}{2}\mathbf{1} + L_i \otimes \tau_i\,, \tag{2.97}$$

where τ_i are the Pauli matrices and L_i are the generators of spin ℓ IRR of $SU(2)$ which were used to define the fuzzy sphere as explained earlier. In order to see this, we note that the square of Λ and $Tr(\Lambda)$ are

$$\Lambda^2 = \frac{1}{4}\mathbf{1} + L_i\otimes\tau_i + L_iL_j(\delta_{ij} + i\epsilon_{ijk}\tau_k) = \frac{1}{4}\mathbf{1} + L_iL_i = (\ell+\frac{1}{2})^2\mathbf{1}\,, \tag{2.98}$$

$$Tr(\Lambda) = \frac{1}{2}2(2\ell+1) = (2\ell+1)\,. \tag{2.99}$$

Hence, the eigenvalues of Λ are $\pm(\ell+\frac{1}{2})$ and the multiplicities can be easily found by using

$$Tr(\Lambda) = (n_+ - n_-)(\ell+\frac{1}{2}) = 2\ell+1\,, \quad n_+ + n_- = 2(2\ell+1)\,, \tag{2.100}$$

from which we see that the $(\ell+\frac{1}{2})$ eigenvalue has the multiplicity $n_+ = 2\ell+2$ while the $-(\ell+\frac{1}{2})$ eigenvalue has multiplicity $n_- = 2\ell$. It is instructive to obtain these results in a slightly different manner as well. Let us introduce the operator $J_i = L_i\otimes\mathbf{1}_2 + \mathbf{1}_{2\ell+1}\otimes\frac{\tau_i}{2}$ which has the $SU(2)$ representation content

$$\ell\otimes\frac{1}{2} = (\ell-\frac{1}{2})\oplus(\ell+\frac{1}{2})\,. \tag{2.101}$$

Hence, the eigenvalues of J^2 are $(\ell - \frac{1}{2})(\ell + \frac{1}{2})$ and $(\ell + \frac{1}{2})(\ell + \frac{3}{2})$. Since $J^2 = L_iL_i + L_i\tau_i + \tau_i^2/4$, our matrix Λ can be written in terms of $J^2\,, L^2$ and τ_i^2 as follows

$$\Lambda = J^2 - L^2 - \tau_i^2/4 + 1/2 = J^2 - \ell(\ell+1) - 1/4\,. \tag{2.102}$$

Here, we can read the eigenvalues of Λ as $(\ell + \frac{1}{2})$ and $-(\ell + \frac{1}{2})$.

All these results prove that there exists a unitary transformation such that

$$diag((\ell + \frac{1}{2}), \cdots, -(\ell + \frac{1}{2})) = U(\frac{1}{2}\mathbf{1} + L_i \otimes \tau_i)U^{-1}\,, \tag{2.103}$$

and the equation of motion (2.95) is satisfied by Λ given in (2.97). In fact, we see that the equation of motion in (2.95) takes the form $L_iL_i = \ell(\ell+1)$ which is used to describe a fuzzy sphere at level ℓ.

Let us consider a general $2(2\ell+1) \times 2(2\ell+1)$ Hermitian matrix Λ;

$$\Lambda = \Lambda_\mu \otimes \tau_\mu = (\frac{1}{2} + \beta)\mathbf{1}_{2(2\ell+1)} + B_i \otimes \tau_i\,, \tag{2.104}$$

where B_i is a $(2\ell+1) \times (2\ell+1)$ matrix whereas β is a $2(2\ell+1) \times 2(2\ell+1)$ matrix. If we insert this matrix into the action (2.93), we obtain

$$\begin{aligned} S = Tr(V(\Lambda)) = &\frac{2}{g^2(2\ell+1)} Tr\Big((\ell+1/2)^4 - 2(\ell+1/2)^2((\frac{1}{2}+\beta)^2 + B_iB_i)) \\ &+ (\frac{1}{2}+\beta)^4 + B_iB_i(2\beta^2 + 6\beta + 3/2) + 4\beta B_i\beta B_i + 2i\epsilon_{ijk}B_kB_iB_j \\ &+ 4i\epsilon_{ijk}\beta B_kB_iB_j - \epsilon_{ijk}B_iB_j\epsilon_{lnk}B_lB_n + B_iB_iB_jB_j \Big) \end{aligned} \tag{2.105}$$

$$\begin{aligned} &= \frac{2}{g^2(2\ell+1)} Tr\Big((B_iB_i - L_iL_i)^2 + (B_i + i\epsilon_{ijk}B_jB_k)(B_i + i\epsilon_{ijk}B_jB_k) \\ &+ [B_i,\beta][B_i,\beta] + (2\ell+1)^2\beta^2 + 2\beta^2 + \beta^4 + 6\beta(\beta+1)(B_iB_i - L_iL_i) \\ &+ 4i\beta\epsilon_{ijk}(B_iB_jB_k - L_iL_jL_k)\Big)\,, \end{aligned} \tag{2.106}$$

where we have used $Tr_{2(2\ell+1)} = Tr_{(2\ell+1)} \otimes Tr_2$.

At this point, if we impose the constraint $\beta = 0$, we get $Tr(\Lambda) = 2\ell+1$ and equation (2.106) becomes

$$S = \frac{2}{g^2(2\ell+1)} Tr\Big((B_iB_i - \ell(\ell+1))^2 + (B_i + i\epsilon_{ijk}B_jB_k)(B_i + i\epsilon_{ijk}B_jB_k)\Big)\,. \tag{2.107}$$

We will show that this is indeed a possible form of a $U(1)$ gauge theory action on S_F^2. Let us first note that imposing the constraint $\beta = 0$ cause the breaking of $SU(2(2\ell+1))$ symmetry of (2.93) down to a smaller group $SU(2\ell+1)$. Indeed, it is easy to see that the action in (2.106) is invariant under the adjoint action of $SU(2\ell+1)$

$$B_i \to U^{-1} B_i U \,, \quad U \in SU(2\ell+1) \,. \tag{2.108}$$

Now, we can derive the equation of motion from the action (2.107) as

$$\begin{aligned} &\left(\frac{\partial}{\partial B_i}\right)_{lm} \left(\sum_{\alpha\beta} (B_i B_i - \ell(\ell+1))_{\alpha\beta} (B_i B_i - \ell(\ell+1))_{\beta\alpha} \right. \\ &\qquad \left. + \sum_{\alpha\beta} (B_i + i\epsilon_{ijk} B_j B_k)_{\alpha\beta} (B_i + i\epsilon_{ijk} B_j B_k)_{\beta\alpha} \right) = 0 \\ &\{B_i, B_j B_j - \ell(\ell+1)\} + (B_i + i\epsilon_{ijk} B_j B_k) + i\epsilon_{ijk} [B_j, B_k + i\epsilon_{klm} B_l B_m] = 0 \,. \end{aligned} \tag{2.109}$$

It is straightforward to see that (2.109) is satisfied if we take $B_i B_i - \ell(\ell+1) = 0$ and $B_i + i\epsilon_{ijk} B_j B_k = 0$ and this corresponds to taking $B_i = L_i$ up to a unitary transformation with $U \in U(2\ell+1)$. Since S in (2.107) is positive definite, we also see that with $B_i = L_i$, S is minimized, $S = 0$. Consider now the fluctuations A_i around this vacuum by writing

$$B_i = L_i + A_i \,, \tag{2.110}$$

then, we get

$$B_i B_i - \ell(\ell+1) = L_i A_i + A_i L_i + A_i A_i \,, \quad B_i + i\epsilon_{ijk} B_j B_k = \frac{1}{2} \epsilon_{ijk} F_{jk} \,, \tag{2.111}$$

where

$$F_{ij} = i[L_i, A_j] - i[L_j, A_i] + i[A_i, A_j] - \epsilon_{ijk} A_k \,. \tag{2.112}$$

This means that the fluctuation around vacuum solution provide the kinetic terms in F_{ij} automatically and (2.112) precisely has the form of the field strength term for a $U(1)$ gauge field A_i on S_F^2. Does this interpretation indeed hold? To see this, let us investigate the transformation property of A_i under the $U \in SU(2\ell+1)$ gauge transformations. Using (2.108) and (2.110), we can write

$$\begin{aligned} B_i' &= U^{-1} B_i U = U^{-1}(L_i + A_i) U = U^{-1} A_i U + U^{-1} L_i U \\ &= U^{-1} A_i U + U^{-1} [L_i, U] + L_i \,. \end{aligned} \tag{2.113}$$

Noting also that S is covariant under (2.108), we can write $B'_i = L_i + A'_i$ which yields from (2.113)

$$A'_i = U^{-1} A_i U + U^{-1}[L_i, U]\,. \tag{2.114}$$

This is indeed the correct form for the gauge transformation of a connection in a non-abelian theory. The latter is clearly due to the fuzzy structure of the theory. In the commutative limit with $U = e^{i\Lambda(x)}$, this takes the form of a usual abelian gauge transformation

$$\begin{aligned} A'_i &= A_i - i\epsilon_{ijk} x_j \partial_k \Lambda \\ &= A_i + L_i \Lambda\,. \end{aligned} \tag{2.115}$$

We note that $x_i L_i \Lambda = 0$ and therefore $L_i \Lambda$ is a vector field on S^2 for $\Lambda(x) \in C^\infty(S^2)$. Indeed on S^2, there are only two independent components of a gauge field. When S^2 is embedded in $\mathbb{R}^3$, this requirement on the gauge fields A_i may be satisfied by imposing the gauge invariant condition

$$x_i A_i = 0\,, \tag{2.116}$$

on A_i's. (From (2.115), we immediately see that $x_i A'_i = x_i A_i$). The condition (2.116) means that the component of A_i normal to S^2 is set to zero. It can also be read as being able to write any gauge field on S^2 as $A_i = \epsilon_{ijk} x_j \tilde{A}_k$ with $\tilde{A}_i$ on $\mathbb{R}^3$.

(2.116) can not be imposed on A_i on S_F^2 due to the fuzzy nature of the latter. However, a gauge invariant condition that approaches to (2.116) as $\ell \to \infty$ can be proposed as [41, 59, 63]

$$(L_i + A_i^L)^2 = L_i^2 = \ell(\ell+1)\mathbf{1}\,. \tag{2.117}$$

Gauge invariance of (2.117) may be checked as follows. Consider a scalar field $\Phi \in Mat(2\ell+1)$ on S_F^2 which is coupled to the gauge fields A_i in the fundamental representation of the gauge group. Thus we have [41, 59, 63]

$$\Phi \to U^{-1}\Phi\,, \quad U \in SU(2\ell+1)\,, \tag{2.118}$$

and

$$D_i(\Phi) = D_i \Phi - \Phi L_i\,, \quad D_i := L_i + A_i \tag{2.119}$$

We can write

$$\begin{aligned}
D_i'(U^{-1}\Phi) &= D_i'U^{-1}\Phi - U^{-1}\Phi L_i \\
&= L_iU^{-1}\Phi + A_i'U^{-1}\Phi - U^{-1}\Phi L_i \\
&= L_iU^{-1}\Phi - U^{-1}\Phi L_i + (U^{-1}A_iU + U^{-1}[L_i, U])U^{-1}\Phi \\
&= U^{-1}D_i(\Phi)\,,
\end{aligned} \tag{2.120}$$

which verifies that $D_i(\Phi)$ transform covariantly under the left action of the gauge group. We may write $D_i'(\cdot) = U^{-1}D_i(U\cdot)$ and also that $D_i'\cdot = U^{-1}D_iU\cdot$. This gives

$$\begin{aligned}
D_i'D_i'\cdot &= U^{-1}D_i'UU^{-1}D_iU \\
&= \ell(\ell+1)\mathbf{1}\,,
\end{aligned} \tag{2.121}$$

where we have used (2.117) in the second line. (2.121) explicitly shows that (2.117) is a gauge invariant condition.

An alternative way to handle this problem is to interpret the normal component of A_i on S_F^2 as a scalar field φ with a large mass, which can not be easily excited and becomes infinitely heavy in the $\ell \to \infty$ limit. In the latter case, this interpretation becomes equivalent to the initial solution offered for this problem. The term $B_iB_i - \ell(\ell+1)$ in action is used to suppress the scalar φ. More precisely, using (2.63), we get

$$\begin{aligned}
\frac{1}{\sqrt{\ell(\ell+1)}}(B_iB_i - \ell(\ell+1)) &= \frac{1}{\sqrt{\ell(\ell+1)}}(L_iA_i + A_iL_i + A_iA_i) \\
&= \{\hat{x}_i, A_i\} + \frac{1}{\sqrt{\ell(\ell+1)}}A_iA_i\,,
\end{aligned} \tag{2.122}$$

and as $\ell \to \infty$, it reduces to $2x_iA_i$ which is twice the radial component of gauge fields on S^2. Consequently as $\ell \to \infty$ the first term in (2.107) becomes

$$\begin{aligned}
\frac{2}{g^2(2\ell+1)}Tr(B_iB_i - \ell(\ell+1))^2 \xrightarrow[\ell\to\infty]{} &\int \frac{d\Omega}{4\pi}\frac{8}{g^2}\ell(\ell+1)(x_iA_i)^2 \\
= &\int \frac{d\Omega}{4\pi}\frac{8}{g^2}\ell(\ell+1)\varphi^2\,,
\end{aligned} \tag{2.123}$$

indicating that φ has the mass $m = \frac{2\sqrt{2}}{g}\sqrt{\ell(\ell+1)}$ for $\ell \to \infty$. Thus essentially φ decouples from the rest of theory as it achieves a large mass in this limit. In chapter 3, we will see the same phenomenon happening in the context of a model where the gauge theory on S_F^2 emerges dynamically from an $SU(\mathcal{N})$ gauge theory coupled to a triplet of adjoint scalar fields.

In the commutative limit $\ell \to \infty$, the action (2.107) becomes

$$S = \frac{1}{g^2} \int \frac{d\Omega}{4\pi} \tilde{F}_{ij} \tilde{F}^{ij} \,, \quad \tilde{F}_{ij} = iL_i A_j - iL_j A_i + \epsilon_{ijk} A_k \tag{2.124}$$

and it is the action of a $U(1)$ gauge theory on the unit sphere S^2. In the planar limit, two independent component of the gauge fields can be obtained by imposing the constraint (2.116) as follows. Consider for instance the plane obtained from the sphere at the north pole $(0,0,1)$ by taking the radius of the sphere to infinity. The constraint indicates that $A_3 = 0$ and $iL_1 = x_2\partial_3 - x_3\partial_2 = -\partial_2 \,, iL_2 = -x_1\partial_3 + x_3\partial_1 = \partial_1$. Hence, we have only $\tilde{F}_{12}$ nonvanishing and it is given as

$$\tilde{F}_{12} = -\partial_2 A_1 + \partial_1 A_2 + A_3 = \partial_1 A_2 - \partial_2 A_1 \,, \tag{2.125}$$

which is the more familiar form for the field strength in $U(1)$ theory on 2-dimensions.

2.6.1.1 Monopole Sectors

It is possible to obtain the monopole sectors over S_F^2 by using the matrix model whose essential ingredients were given in the previous section. We continue to follow discussion in [47]. Taking Λ as a $2(2\ell+1) \times 2(2\ell+1)$ matrix and considering the action (2.93) with

$$\Lambda^{\tilde{\ell}} = \frac{1}{2} \mathbf{1}_{2(2\tilde{\ell}+1)} + B_i^{\tilde{\ell}} \otimes \tau_i \,, \tag{2.126}$$

where $\tilde{\ell} \approx \ell$, we observe that the equation of motion (2.109) is going to be valid with B_i replace by $B_i^{\tilde{\ell}}$. Now, if we assume that $B_i^{\tilde{\ell}}$ contains only spin $\tilde{\ell}$ IRR of $SU(2)$ with the eigenvalue of Casimir $C_2 = \tilde{\ell}(\tilde{\ell}+1) \approx \ell(\ell+1)$, i.e. $B_i^{\tilde{\ell}} = \alpha L_i^{\tilde{\ell}}$ with α being a constant, the relevant equation of motion gives

$$\{\alpha L_i^{\tilde{\ell}}, \alpha^2 L_i^{\tilde{\ell}} L_i^{\tilde{\ell}}\} + \alpha L_i^{\tilde{\ell}} + i\epsilon_{ijk} L_j^{\tilde{\ell}} L_k^{\tilde{\ell}} + i\epsilon_{ijk}[\alpha L_j^{\tilde{\ell}}, \alpha L_k^{\tilde{\ell}} + i\alpha^2 \epsilon_{klm} L_l^{\tilde{\ell}} L_m^{\tilde{\ell}}] = 0 \,. \tag{2.127}$$

It is possible to simplify this equation as follows

$$\alpha L_i^{\tilde{\ell}}\left(2\alpha^2\tilde{\ell}(\tilde{\ell}+1)-2\ell(\ell+1)+1-\alpha-2\alpha(1-\alpha)\right)=0$$

$$(2\alpha-1)(\alpha-1)+\alpha^2 2\tilde{\ell}(\tilde{\ell}+1)-2\ell(2\ell+1)=0\,. \qquad (2.128)$$

The solution for the equation (2.128) can be found by expansion of α at the order $\frac{1}{\ell^2}$ and it gives

$$\alpha=1+\frac{m}{2\ell+1}+O(\frac{1}{\ell^2})\,,\quad m=2(\ell-\tilde{\ell})\,,\quad m\ll\ell\,. \qquad (2.129)$$

Hence, $\Lambda^{\tilde{\ell}}=\frac{1}{2}+\alpha L_i^{\tilde{\ell}}\otimes\tau_i$ is a vacuum solution for the matrix model (2.93) and we obtain in the large ℓ limit ($\ell\to\infty$)

$$\begin{aligned}
B_i^{\tilde{\ell}}+i\epsilon_{ijk}B_j^{\tilde{\ell}}B_k^{\tilde{\ell}} &= -(\alpha^2-\alpha)L_i^{\tilde{\ell}}=-(\frac{m}{2\ell+1}+\frac{m^2}{(2\ell+1)^2})\hat{x}_i\sqrt{\tilde{\ell}(\tilde{\ell}+1)}\\
&= -\frac{1}{2\ell}(1-\frac{1}{2\ell}+\frac{1}{4\ell^2})\sqrt{\ell^2+\ell-m\ell-\frac{m}{2}+\frac{m^2}{4}}\hat{x}_i=-\frac{m}{2}\hat{x}_i\,,\\
B_i^{\tilde{\ell}}B_i^{\tilde{\ell}}-\ell(\ell+1) &= \alpha^2\tilde{\ell}(\tilde{\ell}+1)-\ell(\ell+1)\\
&= (1+\frac{2m}{2\ell+1}+\frac{m^2}{(2\ell+1)^2})(\ell^2+\ell-m\ell-\frac{m}{2}+\frac{m^2}{4})-\ell(\ell+1)\\
&= O(m^2)\,.
\end{aligned} \qquad (2.130)$$

This means that in the commutative limit ($\ell\to\infty$), we obtain the classical action for the magnetic field strength with the magnetic monopole number m as

$$S=\frac{2}{(2\ell+1)}Tr\left((-\frac{m\hat{x}_i}{2})(-\frac{m\hat{x}_i}{2})\right)=\frac{m^2}{2g^2}\,. \qquad (2.131)$$

Here, it is possible to write down the magnetic field as $G_i=-\frac{1}{2}\epsilon_{ijk}F_{jk}=\frac{m}{2}\hat{x}_i$ and using (2.124), we get the same action as follows

$$S=\frac{1}{g^2}\int\frac{d\Omega}{4\pi}(-\epsilon_{ijk}G_k)(-\epsilon_{ijl}G_l)=\frac{m^2}{2g^2}\,. \qquad (2.132)$$

We note that since we use the dimensionless matrices $B_i^{\tilde{\ell}}$, G_i does not look like the magnetic field strength of the magnetic monopole which should have the inverse length square dimension expected from a monopole. However, we can easily restore the dimensions by taking $\tilde{B}_i^{\tilde{\ell}}=\frac{B_i^{\tilde{\ell}}}{r}$, then the magnetic field appears in the more familiar form $\tilde{G}_i=\frac{m\hat{x}_i}{2r^2}$ and the action reads

$$S=\frac{1}{g^2}\int\frac{d\Omega}{4\pi}r^2 2\tilde{G}_i\tilde{G}_i=\frac{m^2}{2g^2r^2}\,. \qquad (2.133)$$

In (2.133), g has the dimension of inverse length while it is dimensionless in (2.132).

2.6.2 Nonabelian Gauge over S_F^2

In order to construct a $U(n)$ gauge theory on S_F^2, we may start with a $2N \times 2N$ matrix Λ for the model described by the action given in (2.93) where $N > (2\ell+1)$. Assuming that $N = n(2\ell+1)$ and choosing $n_0 = 0\,, n_+ = n(2\ell+2)$ and $n_- = n(2\ell)$ in (2.96), the vacuum solution for this matrix model in a suitable basis can be written as

$$\Lambda = \frac{1}{2}\mathbf{1}_{2N} + L_i \otimes \tau_i \mathbf{1}_n\,, \tag{2.134}$$

where has the eigenvalue $\pm(\ell+\frac{1}{2})$ with multiplicities $N+n\,, N-n$, respectively. The latter follows from $Tr(\Lambda) = (n_+ - n_-)(\ell+\frac{1}{2}) = n(2\ell+1) = N$ and $n_+ + n_- = 2N$. Full set of the vacuum solutions are given by a unitary transformation of Λ in (2.134) as $U^{-1}\Lambda U$ with $U \in U(n(2\ell+1))$. We want to note that this block matrix contains n copies of solution given in section (2.6.1). Hence, we can define the fluctuation about this vacuum configuration by the following matrix (with the choice $\beta = 0$ again)

$$\Lambda = \frac{1}{2}\mathbf{1}_{2N} + B_i \otimes \tau_i\,, \tag{2.135}$$

where B_i are $N \times N$ matrices. This enables us to obtain the same action as (2.107) with $N \times N$ matrices as

$$S = \frac{2}{g^2(2\ell+1)} Tr\bigg((B_i B_i - \ell(\ell+1))^2 + (B_i + i\epsilon_{ijk} B_j B_k)(B_i + i\epsilon_{ijk} B_j B_k) \bigg)\,, \tag{2.136}$$

and the vacuum solution for this action is $B_i = L_i \mathbf{1}_n$. We stress that the action (2.136) is invariant under the adjoint action of $U(N)$;

$$B_i \rightarrow U^{-1} B_i U\,, \quad U \in U(N)\,. \tag{2.137}$$

We now want to show that this matrix model yields a $U(n)$ gauge theory over S_F^2. To do so, we first use $u(n(2\ell+1)) \cong u(n) \otimes u(2\ell+1)$ and see that the matrix B_i should naturally carry an additional index for $u(n)$. To be more precise, we need to write the fluctuations of B_i as

$$B_i = B_{i,\mu}\lambda^\mu = L_i \lambda^0 + A_{i,\mu}\lambda^\mu\,, \quad A_{i,\mu} = A_{i,0}\lambda^0 + A_{i,a}\lambda^a\,, \tag{2.138}$$

where $\lambda^0 = \mathbf{1}_n$ and λ_a are the Gell-man matrices of $su(n)$ which satisfy

$$\lambda_a \lambda_b = \frac{\delta_{ab}}{n}\mathbf{1}_n + \frac{1}{2}(d_{abc} + i f_{abc})\lambda_c \,, \tag{2.139}$$

where f_{abc} is anti-symmetric structure constant and d_{abc} is totally symmetric tensor.

With the fluctuation term A_i, we obtain the field strength tensor as

$$B_i + i\epsilon_{ijk}B_j B_k = \frac{1}{2}\epsilon_{ijk}F_{jk} \,, \quad F_{ij} = i[L_i, A_j] - i[L_j, A_i] + i[A_i, A_j] - \epsilon_{ijk}A_k \,. \tag{2.140}$$

This suggest that, then $A_i = A_{i,\mu}\lambda^\mu$ are the $su(n)$ valued gauge fields over S_F^2. For $\ell \to \infty$, we obtain a $U(n)$ gauge theory on S^2 with the action

$$S = \frac{1}{g^2}\int \frac{d\Omega}{4\pi} F_{ij}F^{ij} = \frac{1}{g^2}\int \frac{d\Omega}{4\pi}(F_{ij,0}F^{ij,0} + F_{ij,a}F^{ij,a}) \,, \tag{2.141}$$

where

$$\begin{aligned} F_{ij,0} &= iL_i A_{j,0} - iL_j A_{i,0} + \epsilon_{ijk}A_{k,0} \,, \\ F_{ij,a} &= iL_i A_{j,a} - iL_j A_{i,a} + iA_{i,b}A_{i,c}f_a^{bc} + \epsilon_{ijk}A_{k,a} \,. \end{aligned} \tag{2.142}$$

Just like we did in the previous section, let us assume that Λ is an $\tilde{N} \times \tilde{N}$ matrix with $\tilde{N} = n(2\tilde{\ell} + 1) = n(2\ell + 1) - m$ for small $m \in \mathbb{Z}$. We see that equation of motion in (2.109) can be fulfilled by a reducible representation of $SU(2)$ consisting of block matrices in the form of (2.126). Therefore, after a suitable unitary transformation Λ can be written

$$\Lambda^{(m_1, \cdots, m_n)} = \begin{pmatrix} \Lambda^{\tilde{\ell}_1} & 0 & \cdots & 0 \\ 0 & \Lambda^{\tilde{\ell}_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda^{\tilde{\ell}_n} \end{pmatrix} \,, \tag{2.143}$$

where $\Lambda^{\tilde{\ell}_i}$ is given by (2.126), $m_i = \frac{(\ell - \tilde{\ell}_i)}{2}$ and $\sum_{i=1}^n m_i = m$.

Following the similar steps of calculation given in the previous section, we obtain the action in the large ℓ limit as

$$S = \frac{1}{2g^2}\sum_{i=1}^n m_i^2 \,, \tag{2.144}$$

which is the action for the instantons with the topological numbers $m_i \neq 0 \,, i = 1 \,, \cdots \,, n$ on the fuzzy sphere S_F^2. This can be understood as the value of the classical action in certain instanton solutions of $U(n)$ Yang-Mills theories [64,65]. As the details of these works are not related to the main themes of this thesis, they will not be discussed in this thesis.

2.7 Higher Dimensional Fuzzy Spaces

So far, we have focused our attention on the fuzzy two-sphere and provided all the necessary background on fuzzy spaces that is going to be exploited in the rest of the thesis. However, it is also possible to construct the higher dimensional fuzzy spaces using the annihilation-creation operator method given in section 2.3. In this section, we would like to discuss briefly the fuzzy version of higher dimensional spaces, in particular the complex projective space $\mathbb{C}P^N$ and the product of two spheres $S^2 \times S^2$.

2.7.1 Fuzzy $\mathbb{C}P^N$

Here, we construct the N-dimensional fuzzy complex projective space by generalizing the technique given in section 2.3. For this purpose, we continue to follow the idea of [41]. $\mathbb{C}P^N$ is the N-dimensional complex projective space. It is possible to describe this space using the chain $\mathbb{C}^{N+1} \to S^{2N+1} \to \mathbb{C}P^N$ as we explain below. We have that S^{2N+1} is a $2N+1$-dimensional sphere described as

$$S^{2N+1} = \left\{ \xi = (\xi_1 \,, \cdots \,, \xi_{N+1}) \,,\ \xi \in \mathbb{C}^{N+1} \backslash \{0\} \,,\ \xi_i \xi_i = 1 \right\} . \tag{2.145}$$

S^{2N+1} forms a fiber bundle with $U(1)$ fibers since S^{2N+1} admit the $U(1)$ action $\xi \to e^{i\theta}\xi$ which may be defined as

$$U(1) \to S^{2N+1} \to \mathbb{C}P^N \,, \quad \text{or} \quad \mathbb{C}P^N = \frac{S^{2N+1}}{U(1)} \,. \tag{2.146}$$

The projection map of this fibration may be given as

$$\mathbf{X}_a(\xi) = \xi^\dagger \lambda_a \, \xi \,, \quad \xi \in S^{2N+1} \,, \tag{2.147}$$

where $\lambda_a \,, a = 1 \,, \cdots \,, N^2 - 1$ are the Gell-Mann matrices of $SU(N+1)$. We note that this map gives the embedding of $\mathbb{C}P^N$ into $\mathbb{R}^{N^2+2N}$ where we have the following constraints for the coordinates $\mathbf{X}_a$ [45]

$$\mathbf{X}_a\mathbf{X}_a = \frac{N}{N+1}\,, \quad d_{abc}\mathbf{X}_a\mathbf{X}_b = \frac{\sqrt{2}(N-1)}{N+1}\,, \tag{2.148}$$

where d_{abc} is totally symmetric tensor given in (2.139). Let us note that another realization of $\mathbb{C}P^N$ may be given by the coset space

$$\mathbb{C}P^N \equiv \frac{SU(N+1)}{U(N)} \approx \frac{SU(N+1)}{SU(N)\times U(1)}\,. \tag{2.149}$$

Now, we are ready to construct the fuzzy complex projective space $\mathbb{C}P^N_F$ by quantizing the chain $\mathbb{C}^{N+1} \to S^{2N+1} \to \mathbb{C}P^N$. To proceed, let us remove the origin of $N+1$-dimensional complex plane and so we can define the coordinates of S^{2N+1} as

$$\xi_i := \frac{z_i}{|z|}\,,\ z_i \in \mathbb{C}^{N+1}\,,\ |z| = \sqrt{|z_1|^2 + \cdots + |z_{N+1}|^2}\,. \tag{2.150}$$

Replacing the coordinates z_i and z_i^* by annihilation and creation operators A_i and $A_i^\dagger$, respectively, we obtain $N+1$ dimensional noncommutative complex plane $\mathbb{C}^{N+1}_\theta$ with $N+1$ sets of commutation relations

$$[A_i, A_j] = 0\,, \quad [A_i^\dagger, A_j^\dagger] = 0\,, \quad [A_i, A_j^\dagger] = \theta\delta_{ij}\,. \tag{2.151}$$

Using the definition in (2.150) and the fibration map (2.147), the coordinates of $\mathbb{C}P^N_F$ can be constructed as follows

$$\begin{aligned} \xi_i &:= \frac{z_i}{|z|} \to A_i\frac{1}{\hat{N}}\,, \quad \hat{N} = a_i^\dagger a_i \neq 0\,, \\ \mathbf{X}_a(\xi) &= \xi^\dagger\lambda_a\,\xi \to \hat{\mathbf{X}}_a = \frac{1}{\hat{N}}A^\dagger\lambda_a A\,. \end{aligned} \tag{2.152}$$

Just as before, $\mathbf{X}_a$ can be restricted to the subspace $\mathcal{H}_n$ of the Fock space for $\hat{N} = n$. In a similar manner, we can define the coordinate $\mathbf{X}_a$ of $\mathbb{C}P^N_F$ in terms of the $SU(N+1)$ angular momentum operator using the generalized Schwinger construction as

$$\hat{\mathbf{X}}_a = \frac{2}{n}\,, \quad [\hat{\mathbf{X}}_a, \hat{\mathbf{X}}_b] = \frac{2}{n}if_{abc}\hat{\mathbf{X}}_k \tag{2.153}$$

where f_{abc} is the structure constant of $SU(N+1)$. Since $\mathbb{C}P^N_F$ is not used in this thesis, we will not discuss the details of this space any further and refer the interested reader to the reference [41].

2.7.2 Fuzzy $S^2 \times S^2$

In this section, we give a brief review of the product of two fuzzy sphere $S^2_F \times S^2_F$ by following [24]. For detailed discussion on this space and QFT on it, the reader is referred to the references [43, 44]. The fuzzy version of $S^2 \times S^2$ can be obtained by modifying the results of quantization of sphere into the product of two spheres. As we have shown in section 2.4, the fuzzy sphere coordinates can be described by the generators of spin ℓ IRR of $SU(2)$ in (2.63) and $Mat(2\ell+1)$ is completely generated by polynomials in these coordinates acting on $(2\ell+1)$-dimensional Hilbert space. In a similar manner, $S^2_F \times S^2_F$ is the algebra $Mat[(2\ell_L+1)(2\ell_R+1)]$. This algebra is generated by the matrices $\mathbf{1}_{(2\ell_L+1)(2\ell_R+1)}$, $L^L_i := L^{(2\ell_L+1)}_i \otimes \mathbf{1}_{(2\ell_R+1)}$ and $L^R_i := \mathbf{1}_{(2\ell_L+1)} \otimes L^{(2\ell_R+1)}_i$ where (L^L_i, L^R_i) are the generators of (ℓ_L, ℓ_R) IRR of $SU(2)_L \times SU(2)_R$ with

$$[L^L_i, L^R_j] = i\epsilon_{ijk}L^L_k\,, \quad [L^R_i, L^R_j] = i\epsilon_{ijk}L^R_k\,, \quad [L^L_i, L^R_i] = 0\,,$$
$$L^L_i L^L_i = \ell_L(\ell_L+1)\mathbf{1}_{(2\ell_L+1)(2\ell_R+1)}\,, \quad L^R_i L^R_i = \ell_R(\ell_R+1)\mathbf{1}_{(2\ell_L+1)(2\ell_R+1)} \tag{2.154}$$

The coordinates of $S^2_F \times S^2_F$ can be described in terms of these six matrices as

$$\hat{x}^L_i = \frac{1}{\sqrt{\ell_L(\ell_L+1)}} L^{(2\ell_L+1)}_i \otimes \mathbf{1}_{(2\ell_R+1)}\,,$$
$$\hat{x}^R_i = \mathbf{1}_{(2\ell_L+1)} \otimes \frac{1}{\sqrt{\ell_R(\ell_R+1)}} L^{(2\ell_R+1)}_i\,, \quad i = 1,2,3 \tag{2.155}$$

acting on a $(2\ell_L+1)(2\ell_R+1)$-dimensional Hilbert space. They satisfy

$$[\hat{x}^L_i, \hat{x}^L_j] = \frac{i}{\sqrt{\ell_L(\ell_L+1)}}\epsilon_{ijk}\hat{x}^L_j\,, \quad [\hat{x}^R_i, \hat{x}^R_j] = \frac{i}{\sqrt{\ell_L(\ell_L+1)}}\epsilon_{ijk}\hat{x}^R_j\,,$$
$$[\hat{x}^L_i, \hat{x}^R_j] = 0\,, \quad \hat{x}^L_i\hat{x}^L_i = 1\,, \quad \hat{x}^R_i\hat{x}^R_i = 1\,. \tag{2.156}$$

In the commutative limit ($\ell_L, \ell_R \to \infty$), these coordinate become the standard coordinates of $S^2 \times S^2$ embedded in $\mathbb{R}^6$ and generate an infinite dimensional algebra of smooth functions $C^\infty(S^2 \times S^2)$ which can be expanded in terms of the product of two spherical harmonics $Y_{\ell_L m_L}(\theta, \phi) Y_{\ell_R m_R}(\theta', \phi')$.

Following a similar line of development as in the fuzzy sphere, the derivations on $S^2_F \times S^2_F$ can be obtained by the adjoint action of $su(2) \oplus su(2) = so(4)$ as

$$\mathcal{L}^L_i m := [L^{(2\ell_L+1)}_i \otimes \mathbf{1}_{2\ell_R+1}\,, m]\,, \quad \mathcal{L}^R_i m := [\mathbf{1}_{2\ell_L+1} \otimes L^{(2\ell_R+1)}_i\,, m] \tag{2.157}$$

where $m \in Mat(2\ell_L + 1)(2\ell_R + 1)$. In the commutative limit, these derivations reduce to the usual derivations $iL_i^L = \epsilon_{abc} x_c^L \partial_b^L$ and $iL_i^R = \epsilon_{abc} x_c^R \partial_b^R$ on $S^2 \times S^2$.

CHAPTER 3

DYNAMICAL GENERATION OF FUZZY SPHERE(S) AND EQUIVARIANT PARAMETRIZATION

So far, we have given the essential geometric structure of fuzzy sphere S_F^2 and also brief summary of field theory on S_F^2. Now, we focus our attention to the concept of dynamical generation of fuzzy sphere from an $SU(\mathcal{N})$ Yang-Mills theory coupled to a suitable number of scalar fields. We give the details of these results by following the work of Aschieri et. al. [9]. We start with a renormalizable $SU(\mathcal{N})$ gauge theory in 4-dimensional Minkowski space M^4 coupled to a triplet of scalar fields transforming adjointly under the action of $SU(\mathcal{N})$ and as vectors under the global action of $SO(3) \cong SU(2)$. Working on this model with the most general renormalizable potential term which spontaneously breaks $SU(\mathcal{N})$ symmetry down to a smaller group shows that the vacuum expectation value of scalar fields takes the form of fuzzy sphere S_F^2 and fluctuations around this vacuum have the structure of gauge fields over S_F^2. These results enables to interpret that after the spontaneously symmetry breaking, $SU(\mathcal{N})$ gauge theory on M^4 behaves as an effective $U(n)$ gauge theory on $M^4 \times S_F^2$ where $n(2\ell+1) = \mathcal{N}$ and ℓ is the level of fuzzy sphere. Here, the fuzzy sphere S_F^2 emerges dynamically as extra dimensions from 4-dimensional renormalizable gauge theory. In order to support this interpretation, we construct the Kaluza-Klein (KK) mode expansion of gauge fields on fuzzy extra dimensions and we investigate the low-energy effective action of $U(n)$ gauge theory on M^4 for the lowest lying KK modes.

Afterwards, we continue to develop the effective gauge theory interpretation by another complementing viewpoint approach which is so-called the equivariant parametrization technique. This technique impose the proper symmetry condition on the fields on $\mathcal{M} \times X$ where $\mathcal{M}$ is any physical space and X is some coset space so that the fields transform covariantly under the action of symmetry group of extra dimensions X up to a gauge transformation. This method enables us to construct the low-energy limits of the effective gauge theory on $\mathcal{M}$ since with equivariant modes of gauge fields, it is possible to integrating out (tracing over) the extra dimensions. We would like to mention that this technique is the application of the coset space dimensional reduction techniques (CSDR) [1, 2] (see also [3] in this context). In this chapter, we are particularly concern to the cases where X is the form of fuzzy sphere S_F^2 and the product of two fuzzy sphere $S_F^2 \times S_F^2$ and we will explain the detail of these concepts from the results of [22, 24], respectively. First, we focus on the equivariant parametrization of a $U(n)$ gauge theory on $\mathcal{M} \times S_F^2$ and for the minimal non-abelian gauge symmetry $n = 2$, we construct the $SU(2)$-equivariant modes of gauge fields on $\mathcal{M} \times S_F^2$ up to $U(2)$ gauge transformation. After tracing over S_F^2, we obtain the low-energy effective action on $\mathcal{M}$ which leads to Abelian Higgs type model and for $\mathcal{M} \equiv \mathbb{R}^2$ the vortex type solution can be found in certain limits [22].

In the last section of this chapter, we turn our attention to the models where extra dimensions take the form of the product of two fuzzy sphere $S_F^2 \times S_F^2$ by following [21, 24]. First, we would like to explain how the product of two fuzzy sphere dynamically emerges from a $SU(\mathcal{N})$ gauge theory coupled to suitable number of scalar fields. We consider the deformed $N = 4$ supersymmetric Yang-Mills theory with $SU(\mathcal{N})$ symmetry [21]. We work on the bosonic part of the $N = 4$ supersymmetric Yang-Mills theory where we have six scalar fields in the adjoint representation of $SU(\mathcal{N})$ transforming as vectors under the global action of $SO(6) \approx SU(4)$. We consider the cubic and quadratic interaction terms in the scalar fields in addition to usual quartic one. These deformation terms break both the supersymmetry and global $SO(6)$ symmetry of the model. Indeed, it breaks the global $SO(6)$ symmetry down to $SU(2) \times SU(2)$. Then, we show that the vacuum expectation value of scalar fields appear as the product of two fuzzy sphere and the fluctuations around this vacuum gives the gauge fields on $S_F^2 \times S_F^2$. Hence, it seems possible to interpret this model as an effective gauge theory on $\mathcal{M}\times S_F^2\times S_F^2$. Making use of the equivariant parametrization technique given in [24], we obtain the $SU(2)\times SU(2)$-equivariant modes of gauge fields on $\mathcal{M}\times S_F^2\times S_F^2$ and integrating out the extra dimensions, we construct low-energy effective action of this theory on $\mathcal{M}$. We show that this models leads to abelian Higgs type model with $U(1)\times U(1)\times U(1)$ symmetry and we find the vortex type solutions in this model.

3.1 Dynamical Generation of Fuzzy Extra Dimensions from an $SU(\mathcal{N})$ Gauge Theory

Let us consider an $SU(\mathcal{N})$ gauge theory on the Minkovski space M^4 and label the coordinates on M^4 by $y^\mu\,,(\mu=0,1,2,3)$. We have three anti-Hermitian scalar fields $\Phi_a\,,(a=1,2,3)$ coupled to the $su(\mathcal{N})$ valued anti-Hermitian gauge fields A_μ. Our scalar fields Φ_a are $\mathcal{N}\times\mathcal{N}$ matrices transforming adjointly under the action of $SU(\mathcal{N})$

$$\Phi_a\to U^\dagger\Phi_a U\,,\quad U\in SU(\mathcal{N})\,. \tag{3.1}$$

Let us consider the action [9]

$$S=\int d^4y\,\mathrm{Tr}\left(-\frac{1}{4g^2}F^\dagger_{\mu\nu}F^{\mu\nu}-(D_\mu\Phi_a)^\dagger(D^\mu\Phi_a)\right)-V(\Phi)\,, \tag{3.2}$$

where we have the covariant derivative in the form of $D_\mu=\partial_\mu+[A_\mu,\cdot]$ and the field strength tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$. We note that the kinetic and the gradient part of the action (3.2) have also a global $SO(3)$ symmetry under which the scalar fields transform in the vector representation of $SO(3)$.

The most general renormalizable potential which preserves both the $SO(3)$ and the $SU(\mathcal{N})$ symmetries can be written as

$$\begin{aligned}V(\Phi)=\mathrm{Tr}\,(g_1\Phi_a\Phi_a\Phi_b\Phi_b+g_2\Phi_a\Phi_b\Phi_a\Phi_b-g_3\epsilon_{abc}\Phi_a\Phi_b\Phi_c+g_4\Phi_a\Phi_a)\\+\frac{g_5}{\mathcal{N}}\mathrm{Tr}(\Phi_a\Phi_a)\mathrm{Tr}(\Phi_b\Phi_b)+\frac{g_6}{\mathcal{N}}\mathrm{Tr}(\Phi_a\Phi_b)\mathrm{Tr}(\Phi_a\Phi_b)+g_7\,,\end{aligned} \tag{3.3}$$

where $g_i\,,(i=1\,,\cdots,7)$ are coupling constants with appropriate dimensions. It is possible to obtain a more useful expression for the potential term (3.3). If we define the dimensionless scalar fields as $\Phi'_a=R\Phi_a$ where R has the dimension of length [1], we can rewrite the potential (3.3) in the following form for the suitable choice of the constants $R, \mathfrak{g}, b, c, d$;

$$V(\Phi) = \mathrm{Tr}\left(\frac{1}{\tilde{g}^2} F'^{\dagger}_{ab} F'_{ab} + \mathfrak{g}^2(\Phi'_a\Phi'_a + \tilde{b}'\mathbf{1})^2 + c\right) + \frac{h}{\mathcal{N}} g'_{ab} g'_{ab}\,, \tag{3.4}$$

where

$$F'_{ab} = [\Phi'_a, \Phi'_b] - \epsilon_{abc}\Phi'_c\,, \quad \tilde{b}' = b + \frac{d}{\mathcal{N}}\mathrm{Tr}(\Phi'_a\Phi'_a)\,, \quad g'_{ab} = \mathrm{Tr}(\Phi'_a\Phi'_a)\,. \tag{3.5}$$

The constants $b, c, d, h, \tilde{g}, \mathfrak{g}$ and R may be found in terms of $g_i\,, i = 1, \cdots, 7$ by solving the following equations;

$$\begin{aligned}
&\mathfrak{g}^2 R^4 + \frac{2R^4}{\tilde{g}^2} = g_1\,, \quad -\frac{2R^4}{\tilde{g}^2} = g_2\,, \quad \frac{4R^3}{\tilde{g}^2} = -g_3\,, \\
&2\mathfrak{g}^2 bR^2 + \frac{2\mathfrak{g}^2 bdR^2}{\mathcal{N}} - \frac{2R^2}{\tilde{g}^2} = g_4\,, \quad \frac{2R^4\mathfrak{g}^2 d}{\mathcal{N}} + \frac{d^2R^4\mathfrak{g}^2}{\mathcal{N}^2} = \frac{g_5}{\mathcal{N}}\,, \\
&\frac{R^4 h}{\mathcal{N}} = \frac{g_6}{\mathcal{N}}\,, \quad Tr(\mathfrak{g}^2 b^2) + Tr(c) = g_7\,.
\end{aligned} \tag{3.6}$$

Here, we can see that the suitable choice of R is

$$R = \frac{2g_2}{g_3}\,. \tag{3.7}$$

Now, taking $\mathfrak{g}' = R^2\mathfrak{g}\,, b' = \frac{b}{R^2}\,, \tilde{g}' = \frac{\tilde{g}}{R^2}$ and $h' = R^4 h$, we can suppress R and after dropping the primes and omitting c, we obtain the potential $V(\Phi)$;

$$\begin{aligned}
V(\Phi) &= \frac{1}{\tilde{g}^2}\mathrm{Tr}(F^{\dagger}_{ab}F_{ab}) + \mathfrak{g}^2\mathrm{Tr}(\Phi_a\Phi_a + \tilde{b}\mathbf{1})^2 + \frac{h}{\mathcal{N}} g_{ab} g_{ab}\,. \\
&\equiv V_1(\Phi) + V_2(\Phi) + V_3(\Phi)\,,
\end{aligned} \tag{3.8}$$

where

$$\mathfrak{g}^2 = g_1 + g_2\,, \quad \frac{2}{\tilde{g}^2} = -g_2\,, \quad h = g_6\,. \tag{3.9}$$

It is easy to see that the potential is positive definite if

$$\mathfrak{g}^2 > 0\,, \quad \frac{2}{\tilde{g}^2} > 0\,, \quad h \geq 0\,, \tag{3.10}$$

and from now on we assume that $\mathfrak{g}^2, \tilde{g}^2$ are positive and we are going eventually set $h = 0$, although V_3 term is kept in some of the formulas in order to work in

[1] Since Φ_a has dimension of mass in 4-dimension, $\Phi'_a = R\Phi_a$ are dimensionless.

a more general setting. For the minimum of the potential, we can see that the following conditions should be fulfilled

$$F_{ab} = [\Phi_a, \Phi_b] - \epsilon_{abc}\Phi_c = 0\,,$$
$$-\Phi_a\Phi_a = \tilde{b}\mathbf{1}_{\mathcal{N}}\,. \tag{3.11}$$

Here, we observe that the condition $F_{ab} = 0$ indicates that the scalar fields might be in any reducible representation of $SU(2)$ but the second condition $-\Phi_a\Phi_a = \tilde{b}\mathbf{1}_{\mathcal{N}}$ restrict the scalar fields to be in an IRR of $SU(2)$ according to the value of $\tilde{b}$. In order to find the solutions to these equation, the value of $\tilde{b}$ plays a significant role.

3.1.1 Vacuum Configuration: Type 1

If we take $\tilde{b}$ as the eigenvalue of quadratic Casimir of the $(2\ell+1)$-dimensional irreducible representation of $SU(2)$ labeled by ℓ

$$\tilde{b} = C_2 = \ell(\ell+1)\,, \quad 2\ell \in \mathbb{Z}\,, \tag{3.12}$$

and also assume that

$$\mathcal{N} = (2\ell+1)n\,, \tag{3.13}$$

then, up to gauge transformations $(U^{-1}\Phi_a U\,, U \in SU(\mathcal{N}))$, we can write the vacuum configuration as

$$\Phi_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_n\,, \tag{3.14}$$

where $X_a^{(2\ell+1)}$ are the anti-Hermitian generators of $(2\ell+1)$-dimensional IRRs of $SU(2)$ which are used to define the fuzzy sphere S_F^2 as explained earlier [2]. These generators satisfy

$$[X_a^{(2\ell+1)}, X_b^{(2\ell+1)}] = \epsilon_{abc}X_c^{(2\ell+1)}\,, \quad X_a^{(2\ell+1)}X_a^{(2\ell+1)} = -\ell(\ell+1)\,. \tag{3.15}$$

With the help of these representation properties, it is easy to check that the vacuum configuration (3.14) satisfy the conditions (3.11) minimizing the potential $V(\Phi)$ given in (3.8) with $h = 0$ as noted previously.

[2] In chapter 2, we have used the Hermitian generators L_i of $SU(2)$ IRR. Here, we switch our conventions and use anti-Hermitian generators. It is readily seen that with $X_i = -iL_i$, we can switch between those two.

We observe that the vacuum configuration in (3.14) spontaneously breaks $SU(\mathcal{N})$ symmetry down to $U(n)$. Here, the commutant of Φ_a is $U(n)$, i.e. the maximal commuting subgroup of $SU(\mathcal{N})$ with Φ_a is given by $U(n)$.

Let us consider the fluctuations about the vacuum (3.26)

$$\Phi_a = X_a + A_a \tag{3.16}$$

where $A_a \in u(2\ell+1) \otimes u(n)$ and we have introduced the short-hand notation $X_a^{(2\ell+1)} \otimes \mathbf{1}_n =: X_a$. With the fluctuation term A_a, F_{ab} becomes [3]

$$F_{ab} = [X_a\,, A_b] - [X_b\,, A_a] + [A_a\,, A_b] - \varepsilon_{abc}A_c. \tag{3.17}$$

This means that A_a $(a = 1, 2, 3)$ can be interpreted as the three components of a $U(n)$ gauge field on S_F^2. We can say that Φ_a are the "covariant coordinates" on S_F^2 with the associated curvature tensor F_{ab} .

At the beginning of this section, we have started with an $SU(\mathcal{N})$ gauge theory on M^4. Now, it can be conjectured that after spontaneously symmetry breaking, an effective $U(n)$ gauge theory emerges on $M^4 \times S_F^2$ with the gauge fields

$$A_M \equiv (A_\mu, A_a) \in u(n) \otimes u(2\ell+1)\,, \tag{3.18}$$

and field strength tensors F_{MN}

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,, \\ F_{\mu a} &= D_\mu \Phi_a = \partial_\mu \Phi_a + [A_\mu, \Phi_a]\,, \\ F_{ab} &= [X_a\,, A_b] - [X_b\,, A_a] + [A_a\,, A_b] - \varepsilon_{abc}A_c\,. \end{aligned} \tag{3.19}$$

Let us remark that A_M transform as a vector under the product group $SO(3,1)\times SU(2)$. It should be clear from our notation that, here A_μ transform as a vector under $SO(3,1)$ and trivially i.e. as scalar under $SU(2)$ and A_a transform as a vector under $SU(2)$ and trivially under $SO(3,1)$. This interpretation will be made manifest in section (3.2).

[3] In this chapter, we work with anti-Hermitian fields for convenience.

3.1.2 General Consideration for the Type 1

If we consider the solution for generic $\tilde{b}$, the conditions in (3.11) can not be fulfilled by the finite-dimensional matrices Φ_a. However, as the potential (3.8) is positive definite, the vacuum configuration for generic $\tilde{b}$ exist and it must be given by a solution of the equation $\frac{\partial V}{\partial \Phi_a} = 0$;

$$\frac{\partial \left(\mathrm{Tr} \left(\frac{1}{\tilde{g}^2} {F_{ab}}^{\dagger} F_{ab} + \mathfrak{g}^2 (\Phi_a \Phi_a + \tilde{b}\mathbf{1})^2 \right) + \frac{h}{\mathcal{N}} g_{ab} g_{ab} \right)}{\partial (\Phi_a)_{lm}} = 0 \,. \tag{3.20}$$

Let us calculate the derivative of each term in equation (3.20) separately; the first term gives

$$\begin{aligned}\frac{\partial \, \mathrm{Tr}(\Phi_a \Phi_a + \tilde{b}\mathbf{1})^2}{\partial (\Phi_a)_{lm}} = \frac{1}{\mathcal{N}} \Bigg(& 2(\Phi_a \Phi_b \Phi_b)_{ml} + 2(\Phi_b \Phi_b \Phi_a)_{ml} + 2\frac{d}{\mathcal{N}} (\Phi_a)_{ml} \mathrm{Tr}(\Phi_b \Phi_b) \\ & + 2\tilde{b}(\Phi_a)_{ml} + 2\frac{d}{\mathcal{N}} \mathrm{Tr}(\Phi_b \Phi_b)(\Phi_a)_{ml} + 2(\Phi_a)_{ml}\tilde{b} + 2\frac{bd}{\mathcal{N}}(\Phi_a)_{ml} \\ & + 2(\Phi_a)_{ml} \frac{bd}{\mathcal{N}} + 2\frac{d^2}{\mathcal{N}^2}(\Phi_a)_{ml} \mathrm{Tr}(\Phi_b \Phi_b) + 2\frac{d^2}{\mathcal{N}^2} \mathrm{Tr}(\Phi_b \Phi_b)(\Phi_a)_{ml} \Bigg) \,,\end{aligned} \tag{3.21}$$

and the second term can be calculated as

$$\begin{aligned}\frac{\partial \, \mathrm{Tr}(F_{bc}^{\dagger} F_{bc})}{\partial (\Phi_a)_{lm}} &= -\frac{1}{\mathcal{N}} \left(\frac{\partial}{\partial (\Phi_a)_{lm}} \sum_{ij} (F_{bc})_{ij} (F_{bc})_{ji} \right) \\ &= -\frac{1}{\mathcal{N}} \Bigg(\frac{\partial}{\partial (\Phi_a)_{lm}} \Big(\sum_{ijk} ((\Phi_b)_{ik} (\Phi_c)_{kj} - (\Phi_c)_{ik} (\Phi_b)_{kj}) \\ &\quad - \sum_{ij} \epsilon_{bcd} (\Phi_d)_{ij} \Big) + i \leftrightarrow j \Bigg) \\ &= -\frac{1}{\mathcal{N}} \Big(2(\Phi_c F_{ac})_{ml} - 2(F_{ac}\Phi_c)_{ml} + 2(F_{ba}\Phi_b)_{ml} - 2(\Phi_b F_{ba})_{ml} \\ &\quad - 2\epsilon_{bca}(F_{bc})_{ml} \Big) \\ &= \frac{1}{\mathcal{N}} (4[F_{ac}, \Phi_c] + 2\epsilon_{abc} F_{bc})_{ml} \,.\end{aligned} \tag{3.22}$$

For the last term, we have

$$\begin{aligned}\frac{\partial \, g_{bc} g_{bc}}{\partial (\Phi_a)_{lm}} &= \frac{\partial \, \mathrm{Tr}(\Phi_b \Phi_c) \mathrm{Tr}(\Phi_b \Phi_c)}{\partial (\Phi_a)_{lm}} \\ &= \frac{1}{\mathcal{N}^2} \left(\frac{\partial}{\partial (\Phi_a)_{lm}} \left(\sum_{ij} (\Phi_b)_{ij} (\Phi_c)_{ji} \right)^2 \right) = \frac{1}{\mathcal{N}} 4 g_{ab} (\Phi_b)_{lm} \,,\end{aligned} \tag{3.23}$$

where we have used the identity $g_{ab} = \frac{1}{3}\delta_{ab}\mathrm{Tr}(\Phi\cdot\Phi)$. Hence, the equation $\frac{\partial V}{\partial \Phi_a} = 0$ can be written as follows

$$\mathfrak{g}^2\{\Phi_a, \Phi\cdot\Phi + \tilde{b} + \frac{d}{\mathcal{N}}\mathrm{Tr}(\Phi\cdot\Phi + \tilde{b})\} + \frac{2h}{\mathcal{N}} g_{ab}\Phi_b + \frac{1}{\tilde{g}^2}\left(2[F_{ab}, \Phi_c] + \epsilon_{abc}F_{bc}\right) = 0\,. \tag{3.24}$$

Although the general solution to this equations is not known for generic $\tilde{b}$, a possible vacuum configuration (up to unitary transformation) can be proposed as follows

$$\Phi_a = diag\left(\alpha_1 X_a^{(2\ell_1+1)} \otimes \mathbf{1}_{n_1}, \cdots, \alpha_k X_a^{(2\ell_k+1)} \otimes \mathbf{1}_{n_k}\right)\,, \tag{3.25}$$

for suitable constants α_i. We aim to construct examples of vacuum configurations in the form of (3.25) which satisfy the minimal potential conditions (3.11) at least approximately.

If we assume that the vacuum configuration consist of only a single $SU(2)$ IRR with Casimirs $C_2 = \ell(\ell+1)$ and $\tilde{b}$ has the value which is very close to $C_2 = \ell(\ell+1)$, and then with the factorization $\mathcal{N} = (2\ell+1)n$, a vacuum configuration up to unitary transformation can be constructed as

$$\Phi_a = \alpha X_a^{(2\ell+1)} \otimes \mathbf{1}_n\,. \tag{3.26}$$

This vacuum configuration can be made to satisfy the minimum potential conditions (3.11) approximately as we demonstrate below and it may be taken as the general form of the type 1 vacuum configuration in (3.14).

In order to find α, we insert the solution (3.26) in the equation (3.24)

$$\begin{aligned} 2\mathfrak{g}^2\alpha X_a^{(2\ell+1)}\left(-\alpha^2 C_2 + \tilde{b} - \alpha^2 C_2 d + \frac{d}{\mathcal{N}}\left(b - \alpha^2 C_2 d\right)\mathcal{N}\right) - \frac{2h}{3}\alpha^3 C_2 X_a^{(2\ell+1)} \\ + \frac{1}{\tilde{g}^2}\left(-4X_a^{(2\ell+1)}\alpha(\alpha^2-\alpha) + 2X_a^{(2\ell+1)}(\alpha^2-\alpha)\right) = 0\,. \end{aligned} \tag{3.27}$$

Multiplying both side with the inverse of $X_a^{(2\ell+1)}$ and dividing 2α, we get

$$\mathfrak{g}^2(\alpha^2 C_2 - \tilde{b})(d+1) + \frac{h}{3}\alpha^2 C_2 - \frac{1}{\tilde{g}^2}(\alpha-1)(1-2\alpha) = 0\,. \tag{3.28}$$

It is possible to find the exact solutions to the equation (3.28), but we are interested in the expansion around $\alpha = 1$ solution. Let us assume that $d = h = 0$ and $\mathfrak{g}^2 \approx \frac{1}{\tilde{g}^2}$. Taking

$$\tilde{b} = \tilde{\ell}(\tilde{\ell} + 1)\,, \tag{3.29}$$

where $\tilde{\ell}$ is a real number (not necessarily integer or half integer), we obtain the solution for α to the order $\frac{1}{\ell^2}$ as

$$\alpha = 1 - \frac{m}{2\ell + 1} + O(\frac{1}{\ell^2})\,, \quad \text{for } m \ll \ell\,, \quad m = 2(\ell - \tilde{\ell})\,. \tag{3.30}$$

With the vacuum solution (3.26) and the expansion of α, the leading term of potential $V(\Phi)$ can be found

$$\begin{aligned} V_1(\Phi) &= \frac{1}{\tilde{g}^2}\mathrm{Tr}(F_{ab}^\dagger F_{ab}) = -\frac{1}{\tilde{g}^2}\mathrm{Tr}\left(\frac{m^2}{(2\ell+1)^2}\epsilon_{abc}X_c^{(2\ell+1)}\epsilon_{abd}X_d^{(2\ell+1)}\right) \\ &= \frac{2m^2}{(2\ell+1)^2\tilde{g}^2}\ell(\ell+1) = \frac{m^2}{2\tilde{g}^2} + O(\frac{1}{\ell^2})\,. \end{aligned} \tag{3.31}$$

$$V_2(\Phi) = \mathfrak{g}^2\mathrm{Tr}(\Phi_a\Phi_a + \tilde{b})^2 = \mathfrak{g}^2\mathrm{Tr}(-\alpha^2 C_2 + \tilde{b})^2 = \frac{\mathfrak{g}^2 m^2}{4(2\ell+1)^2} = O(\frac{1}{\ell^2})\,. \tag{3.32}$$

For the type 1 vacuum solution, $\tilde{\ell} = \ell$ in (3.31) and (3.32), both V_1 and V_2 tend to zero at large ℓ, which is the minimum of $V(\Phi)$. On the other hand, taking $\tilde{\ell} \approx \ell$ yields essentially another vacuum, V_1 still survives for large ℓ and has the value $\frac{m}{2\tilde{g}^2}$. As we mentioned in chapter 2, this is the value of classical action of field strength with magnetic monopole number m when $2\tilde{\ell}$ is an integer. Hence, it is possible to interpret that the vacuum configuration (3.26) as a fuzzy sphere carrying the magnetic monopole strength m [9].

3.1.3 Vacuum Configuration: Type 2

Let us again consider the vacuum configuration in (3.25) with n_i blocks of size $(2\ell_i + 1) = (2\tilde{\ell} + 1) + m_i$ which eventually will be a vacuum solution given as the direct sum of two fuzzy spheres. If we assume that $\tilde{\ell}$ is large and $\frac{m_i}{\tilde{\ell}} \ll 1$, the result of previous case can be generalized as

$$V(\Phi) = \mathrm{Tr}\left(\frac{1}{2\tilde{g}^2}\sum_i n_i m_i^2 \mathbf{1}_{(2\ell_i+1)}\right) \approx \frac{1}{2\tilde{g}^2 k}\sum_i n_i m_i^2\,, \tag{3.33}$$

with the constraint $\sum n_i m_i = \mathcal{N} - k(2\tilde{\ell}+1)$ where k is the total number of IRRs $k = \sum n_i$. This result can be interpreted that the solution (3.25) is the internal fuzzy sphere carrying the instantons with the action (3.33).

We note that with the approximation given above, the vacuum configuration (3.25) can be written in terms of only two distinct block diagonal as follows

$$\Phi_a = \begin{pmatrix} \alpha_1 X_a^{(2\ell_1+1)} \otimes \mathbf{1}_{n_1} & 0 \\ 0 & \alpha_2 X_a^{(2\ell_2+1)} \otimes \mathbf{1}_{n_2} \end{pmatrix}, \tag{3.34}$$

where $\mathcal{N} = (2\ell_1+1)n_1 + (2\ell_2+1)n_2$ and $\ell_2 = \ell_1 + \frac{1}{2}$. This can be seen from convexity of (3.33). The overall dimension of vacuum configuration (3.25) (with the assumption $2\ell_1 + 1 < 2\ell_2 + 1 < \cdots < 2\ell_k + 1$) does not change, if we lower n_1 and n_k by one and add two blocks with size $2\ell_1 + 2$ and $2\ell_k$ and the action in (3.33) becomes smaller due to convexity. It is possible to apply this method until we obtain two distinct blocks or one block with maximal size which means that we can get the vacuum configuration from (3.25) in the form of either (3.26) or (3.34).

3.2 Kaluza-Klein Modes

So far, we have shown that it is possible to interpret that, after the spontaneously symmetry breaking, $SU(\mathcal{N})$ Yang-Mills theory on M^4 coupled to a triplet of adjoint scalar fields behaves as an effective $U(n)$ gauge theory on $M^4 \times S_F^2$, where the extra dimensions are dynamically generated in the form of fuzzy a sphere. This interpretation can be supported with the construction of Kaluza-Klein mode expansion of the fields over the fuzzy extra dimensions. In this thesis, we will show that for the vacuum configurations (3.26) and (3.34), the Kaluza-Klein mode expansion yields the mass spectrum of the excitations with large mass gaps [9], corroborating with the effective gauge theory interpretation.

3.2.1 Kaluza-Klein Mode Expansion of Type 1 Vacuum

Let us start with the construction of Kaluza-Klein modes for the vacuum configuration given in (3.14) or in (3.26) with $\alpha = 1$. We will inspect $\alpha \neq 1$ in the next subsection. As we have noted in the equation (3.18), after the breaking of $SU(\mathcal{N})$ gauge symmetry down to $U(n)$, A_μ and A_a are interpreted as $u(n)$ valued gauge fields of the emerging model over $M^4 \times S_F^2$. Consider first the field $A_\mu \in u(n) \otimes u(2\ell+1)$, we can expand into their modes over S_F^2 using the polarization tensor T_{jm} on S_F^2 introduced in section 2.4 of chapter 2. Thus, we may write

$$A_\mu = \sum_{j=0}^{2\ell} \sum_{m=-j}^{j} T_{jm} \otimes A_{\mu,jm}(y) \quad j = 0, 1, \cdots, 2\ell\,, \tag{3.35}$$

where T_{jm} are $(2\ell+1) \times (2\ell+1)$ matrices, and $A_{\mu,lm}(y)$ are $u(n)$ valued gauge fields on M^4. Therefore, A_μ can be interpreted as $u(n)$ valued functions on $M^4 \times S_F^2$ with the Kaluza-Klein modes expansion on S_F^2.

We can consider the mode expansion of the fluctuations modes A_a around the vacuum configuration in a similar manner. Kaluza-Klein mode expansion of A_a can be written as

$$A_a = \sum_{j=0}^{2\ell} \sum_{m=-j}^{j} T_{jm} \otimes A_{a,jm}(y)\,, \tag{3.36}$$

where $A_{a,jm}(y)$ are $u(n)$ valued functions (scalar fields) on M^4. Equations (3.35) and (3.36) make our earlier remark following (3.19) manifest, as promised.

3.2.2 Mass Spectrum of the Gauge Sector

Now, let us determine the masses of the KK modes. To do so, let us focus on the term

$$\int \mathrm{Tr}(D_\mu \Phi_a)^\dagger D^\mu \Phi_a = \int \mathrm{Tr}\left(\partial_\mu \Phi_a^\dagger \partial^\mu \Phi_a + 2(\partial_\mu \Phi_a^\dagger)[A^\mu, \Phi_a] + [A_\mu, \Phi_a]^\dagger [A^\mu, \Phi_a]\right)\,, \tag{3.37}$$

in the action (3.2). First, we will show that the second term in (3.37) is of no relevance for the calculation of the masses since it includes only the cubic interaction terms. Using the cyclic property of the trace, this term can be written as

$$\int \mathrm{Tr}(\partial_\mu \Phi_a^\dagger [A^\mu, \Phi_a]) = \int \mathrm{Tr}(A^\mu [\Phi_a, \partial_\mu \Phi_a^\dagger]) = -\int \mathrm{Tr}(A^\mu [\Phi_a, \partial_\mu \Phi_a]) \,. \quad (3.38)$$

Now, let us consider y-dependent α which enables us to define the scalar fields in the most general form as follows

$$\Phi_a(y) = \alpha(y) X_a^{(2\ell+1)} \otimes \mathbf{1}_n + A_a \,. \quad (3.39)$$

In order to see the interaction terms, we insert (3.39) into (3.38) and we have

$$\begin{aligned} \int \mathrm{Tr}(A^\mu [\Phi_a, \partial_\mu \Phi_a^\dagger]) &= -\int \mathrm{Tr} A^\mu \left(\alpha [X_a, \partial_\mu A_a] + (\partial_\mu \alpha)[A_a, X_a] + [A_a, \partial_\mu A_a] \right) \,, \\ &= -\int \mathrm{Tr} A^\mu \left(\alpha \partial_\mu \left([X_a, A_a] \right) + (\partial_\mu \alpha)[A_a, X_a] + [A_a, \partial_\mu A_a] \right) \end{aligned} \quad (3.40)$$

Imposing the gauge condition [4] $[X_a, A_a] = 0$ we get

$$\int \mathrm{Tr}(A^\mu [\Phi_a, \partial_\mu \Phi_a^\dagger]) = -\int \mathrm{Tr} A^\mu [A_a, \partial_\mu A_a] \,. \quad (3.42)$$

Hence, no mass terms can be derived from this part of the action.

The last term in the equation (3.37) gives the mass for the gauge field A_μ and some higher order interaction terms. To see this, we use (3.39) and find

$$\int \mathrm{Tr}[A_\mu, \Phi_a]^\dagger [A^\mu, \Phi_a] = \int \mathrm{Tr}(\alpha^2 A_\mu^\dagger \mathcal{L}_a^\dagger \mathcal{L}_a A^\mu) + \text{higher order terms} \,, \quad (3.43)$$

where $\mathcal{L}_a \cdot = i[X_a, \cdot]$ are the derivations on fuzzy sphere in terms of anti-Hermitian generators of $SU(2)$ (2.69). Let us introduce the notation S_{int} for all higher order interaction terms coming from (3.37).

The relevant part for the mass can be calculated as

$$\begin{aligned} \int \mathrm{Tr}(\alpha^2 A_\mu^\dagger \mathcal{L}_a^\dagger \mathcal{L}_a A^\mu) &= \int \mathrm{Tr}(\alpha^2 \sum_{j=0}^{2\ell} \sum_{m=-j}^{j} \sum_{k=0}^{2\ell} \sum_{n=-k}^{k} k(k+1) A_{\mu,jm}^\dagger A^\mu{}_{kn} T_{jm}^\dagger T_{kn}) \\ &= \int \mathrm{Tr}_n (\sum_{j=0}^{2\ell} \sum_{m=-j}^{j} \alpha^2 j(j+1) A_{\mu,jm}^\dagger A^\mu{}_{jm}) \,. \end{aligned} \quad (3.44)$$

---

[4] In the commutative limit $\ell \to \infty$, the adjoint action $[X_a, \cdot]$ reduces to $-iL_a = -\epsilon_{abc} x_b \partial_c = \epsilon_{abc} x_c \partial_b$, then the gauge condition $[X_a, A_a] = 0$ becomes

$$[X_a, A_a] \xrightarrow[\ell \to \infty]{} \epsilon_{abc} x_c \partial_b A_a = \epsilon_{abc} (\partial_b (x_c A_a) - (\partial_b x_c) A_a) = \vec{\nabla} \cdot (\vec{x} \times \vec{A}) = \vec{\nabla} \cdot \vec{A}' = 0 \quad (3.41)$$

where $\vec{A}' = \vec{x} \times \vec{A}$ is on S^2 since $\vec{x}.\vec{A}' = 0$. This is the Lorenz gauge on S^2.

Here, in order to obtain the last line in (3.44), we have used the orthogonality properties of the polarization tensors given in (2.75). Now, it is easy to see that $u(n)$ gauge fields $A_{\mu,jm}(y)$ acquire the masses

$$m_j^2 = \frac{\alpha^2 g^2}{R^2} j(j+1)\,, \tag{3.45}$$

where we have inserted back the parameter R.

From (3.45), we immediately see that there is only one massless mode in KK spectrum and it is obviously given by the $j = 0\,, m = 0$ mode $A_{\mu,00}$. Other KK modes are separated from each other and $A_{\mu,00}$ with ever increasing mass gaps, with the model truncating at $j = 2\ell$.

Focusing only on the gauge sector of the effective theory and ignoring all the KK modes with the character mass scale $1/R$, we have the Low-energy effective action (LEA) written down as

$$S_{LEA} = -\int d^4y \frac{1}{4g^2} \mathrm{Tr}_n F^{0\dagger}_{\mu\nu} F^{0\mu\nu}\,, \tag{3.46}$$

where $F^0_{\mu\nu}$ is the fields strength associated with the massless mode $A_{\mu,00}$. We see that S_{LEA} corresponds to an $SU(n)$ gauge theory on M^4. Inclusion of $j \neq 0$ modes brings in the corresponding field strength terms as well as the mass terms as derived in (3.44).

Finally, let us note that we can predict the radius of internal fuzzy sphere as

$$r_{S^2} = \frac{\alpha R}{g}\,. \tag{3.47}$$

3.2.3 Mass Spectrum of the Scalar Sector

In order to inspect mass spectrum of scalar fields, we may proceed as follows. We consider the splitting of A_a into its radial and tangential components $A_a^r\,, A_a^t$ as

$$A_a = A_a^r + A_a^t = i\varphi(y) X_a + A_a^t\,, \tag{3.48}$$

where $\varphi(y) = -\frac{X_a A_a}{\ell(\ell+1)}$ are $u(n)$ valued and $X_a A_a^t = 0$. Then, we may expand the scalar fields Φ_a by writing out the radial component of fluctuations $\varphi(y)$

explicitly

$$\Phi_a(y) = X_a^{(2\ell+1)} \otimes (\alpha \mathbf{1}_n + \varphi(y)) + \sum_k A_{a,k}(x) \otimes \varphi_k(y)\,. \tag{3.49}$$

In this expression, $A_{a,k}(x)$ represent the fluctuation modes of gauge fields on the fuzzy sphere S_F^2 given in a suitable basis which is labeled by k. It is possible to find out how the expansion (3.36) and relevant part of (3.49) are related to each other but being unnecessary for our purpose, it will not be carried out here. In the light of the analysis of the previous subsection, one would expect that $A_{a,k}$ acquire masses with large gaps of the order of KK scale and therefore they will not contribute the LEA of the scalar sector. A detailed analysis proving this result is given in the appendix C of [9] and will not be discussed here. The same analysis also show that the lowest lying mode $j = 0$ yields the mass of the fluctuations $\varphi(y)$ in the radial component of A_a. We can find out the mass of this excitations as follows. If we insert the vacuum configuration $\Phi_a(y) = X_a^{(2\ell+1)} \otimes (\alpha \mathbf{1}_n + \varphi(y))$ into the potential term in (3.8), assuming, $\mathfrak{g}^2 \approx \frac{1}{\tilde{g}^2}$, we obtain

$$\begin{aligned}
V(\varphi(y)) &= \mathfrak{g}^2 \mathrm{Tr}\big((\Phi_a\Phi_a + \tilde{b})^2\big) \\
&= \mathfrak{g}^2 \mathrm{Tr}\left(\left(X_a^{(2\ell+1)} X_a^{(2\ell+1)}\big(\alpha^2 + 2\alpha\varphi(y) + \varphi^2(y)\big) + \tilde{b}\right)^2 + O(\varphi^3)\right) \\
&= \mathfrak{g}^2 \mathrm{Tr}\bigg(C_2^2\alpha^4 + (4C_2^2\alpha^2 + 2C_2^2\alpha^2 - 2C_2\tilde{b})\varphi^2 + (4C_2^2\alpha^3 - 4C_2\alpha\tilde{b})\varphi \\
&\qquad - 2C_2\tilde{b}\alpha^2 + \tilde{b}^2 + O(\varphi^3)\bigg) \\
&= \mathrm{Tr}_n\left(\mathfrak{g}^2 C_2^2 \varphi(y)^2 + O(\varphi^3)\right)\,,
\end{aligned} \tag{3.50}$$

where we have taken $\alpha^2 \approx 1$ and $\tilde{b} \approx C_2$. This means that after the spontaneously symmetry breaking of $SU(\mathcal{N})$ symmetry, the field φ acquire a mass (m_Φ) which can be interpreted as the order parameter of the Higgs mechanism and we will give its explicit form below. All these results indicate that the model emerging after the breaking of $SU(\mathcal{N})$ symmetry behaves as an effective $U(n)$ gauge theory over $M^4 \times S_F^2$; with KK modes on S_F^2 of the radius given in (3.47). Putting together the gauge and scalar sectors LEA action given at the lowest

KK modes reads

$$S_{LEA} = -\int d^4y \mathrm{Tr}_n \left(\frac{1}{4g^2} F^{0\dagger}_{\mu\nu} F^{0\mu\nu} + D_\mu \varphi(y) D^\mu \varphi(y) C_2 + \mathfrak{g}^2 C_2^2 \varphi(y)^2 \right) + S_{int} \tag{3.51}$$

where $\varphi(y)$ have the mass $m_\varphi^2 = \frac{\mathfrak{g}^2}{R^2} C_2$ reinserting R.

3.2.4 Kaluza-Klein Mode Expansion of Type 2 Vacuum

Now, let us consider the Kaluza-Klein mode expansion for the vacuum configuration (3.34). Here, the maximal subgroup of $SU(\mathcal{N})$ commuting with Φ_a is

$$K := SU(n_1) \times SU(n_2) \times U(1) \,, \tag{3.52}$$

and it will turn out to be the gauge group of the emerging effective model over $M^4 \times S_F^2$. Our purpose is to determine the mass spectrum of the KK modes of the gauge fields. The analysis outlined here is based on the extensive discussion given in [9]. To proceed we write the gauge field as follows

$$A_\mu = \begin{pmatrix} A_\mu^{(1)} & A_\mu^+ \\ A_\mu^- & A_\mu^{(2)} \end{pmatrix} \,, \tag{3.53}$$

where $A_\mu^{(1)}$ and $A_\mu^{(2)}$ are square matrices of size, $(2\ell_1+1)n_1 \times (2\ell_1+1)n_1$ and $(2\ell_2+1)n_2 \times (2\ell_2+1)n_2$, respectively, while A_μ^+ and A_μ^- are rectangular matrices of the size $(2\ell_1+1)n_1 \times (2\ell_2+1)n_2$ and $(2\ell_2+1)n_2 \times (2\ell_1+1)n_1$, respectively. Since A_μ is anti-Hermitian, we have for the blocks in (3.53)

$$(A_\mu^{(1)})^\dagger = -(A_\mu^{(1)}) \,, \quad (A_\mu^{(2)})^\dagger = -(A_\mu^{(2)}) \,, \quad (A_\mu^+)^\dagger = -(A_\mu^-) \,, \quad (A_\mu^-)^\dagger = -(A_\mu^+) \,. \tag{3.54}$$

The KK expansion of the gauge field component $A_\mu^{(1),(2)}$ can be given by introducing the polarization basis tensors as before

$$A_\mu^{(1)} = \sum_{j=0}^{2\ell_1} \sum_{m=-j}^{j} T_{jm}^{(1)} \otimes A_{\mu,jm}^{(1)}(y) \,, \quad A_\mu^{(2)} = \sum_{j=0}^{2\ell_2} \sum_{m=-j}^{j} T_{jm}^{(2)} \otimes A_{\mu,\ell m}^{(2)}(y) \,, \tag{3.55}$$

where $T_{jm}^{(1)}$ and $T_{jm}^{(2)}$ are $(2\ell_1+1)\times(2\ell_1+1)$ and $(2\ell_2+1)\times(2\ell_2+1)$ matrices respectively. As we mentioned earlier, the polarization tensors are the eigenstate of the operator $\mathcal{L}_i$ given in (2.69). Here, we have

$$\mathcal{L}^2 T_{jm}^{(1)} = j(j+1)T_{jm}^{(1)}\,, \quad \mathcal{L}^2 T_{jm}^{(2)} = j(j+1)T_{jm}^{(2)}\,. \tag{3.56}$$

We note that $A_{\mu,\ell m}^{(1)}(y)$ and $A_{\mu,\ell m}^{(2)}(y)$ are $u(n_1)$ and $u(n_2)$-valued gauge fields respectively and they transform as vectors on M^4.

Since A_μ^+ and A_μ^- are the rectangular matrices, we can not expand them in terms of these polarization tensors. However, it is possible to define the specific tensors which form a basis for the vector space including the rectangular matrices. These tensors are closely related to the spherical harmonics Y_{jm} on S^2 and encountered in discussion of spin orbit coupling in non-relativistic quantum mechanics [66]. In the subsection (2.5.1), we have shown that how $su(2)$ rotation generators act on the complex scalar fields in the form of rectangular matrices. Here, modifying this formalism for any gauge field A of the size $(2\ell_1+1)\times(2\ell_2+1)$, we see that the $su(2)$ rotation acting on A is defined by

$$\mathcal{L}_i A = L_i^{\ell_1} A - A L_i^{\ell_2}\,, \tag{3.57}$$

where $L_i^{\ell_1}$ are $(2\ell_1+1)\times(2\ell_1+1)$-dimensional while $L_i^{\ell_2}$ are $(2\ell_2+1)\times(2\ell_2+1)$-dimensional $su(2)$ generators, respectively and the $SU(2)$ representation content of the operator $\mathcal{L}_i$ in (3.57) is given by the Clebsch-Gordan decomposition

$$\ell_1 \otimes \ell_2 = |\ell_1 - \ell_2| \oplus \cdots (\ell_1+\ell_2)\,. \tag{3.58}$$

Hence, the spectrum of $\mathcal{L}^2$ is deduced from (3.58) to be

$$j'(j'+1)\,, \quad j' = |\ell_1-\ell_2|\,, \cdots, (\ell_1+\ell_2)\,. \tag{3.59}$$

The corresponding eigenstates of the operator $\mathcal{L}^2$ can be expressed in terms of the generalized harmonics given in (2.88). In this section, we prefer to use the following notation for these generalized harmonics in order to avoid any notational confusion

$$\tilde{T}_{j'm}\,, \quad j' = \frac{k}{2}\,, \cdots, \ell_1+\ell_2\,, \quad m = -j'\,, \cdots, j'\,, \tag{3.60}$$

where $k = 2(\ell_2 - \ell_1)$. For given (ℓ_1 , ℓ_2), there are $\sum_{j'=\frac{k}{2}}^{(\ell_1+\ell_2)}(2\ell+1) = (2\ell_1 + 1)(2\ell_2+1)$ linearly independent tensors $\tilde{T}_{j'm}$. Hence, they form a basis for the algebra of $(2\ell_1 + 1) \times (2\ell_2 + 1)$ matrices. We may summarize these results by writing

$$\mathcal{L}^2 \tilde{T}_{j'm} = j'(j'+1)\tilde{T}_{j'm} \,, \tag{3.61}$$

[38]. For completeness let us note that $\mathcal{L}^2$ is determined by applying the formula (3.57) twice. This gives

$$\mathcal{L}^2 A = (L_i^{\ell_1})^2 A - 2L_i^{\ell_1} A L_i^{\ell_2} + A(L_i^{\ell_2})^2 \,. \tag{3.62}$$

Using these information, it is possible to expand the gauge field components A_μ^+ and A_μ^- as

$$A_\mu^+ = \sum_{j'=\frac{k}{2}}^{\ell_1+\ell_2} \sum_{m=-j'}^{j'} \tilde{T}_{j'm}^+ \otimes A_{\mu,j'm}^+(y) \,, \quad A_\mu^- = \sum_{j'=\frac{k}{2}}^{\ell_1+\ell_2} \sum_{m=-j'}^{j'} \tilde{T}_{j'm}^- \otimes A_{\mu,j'm}^-(y) \,, \tag{3.63}$$

where $\tilde{T}_{j'm}^+$ and $\tilde{T}_{j'm}^-$ are $(2\ell_1+1)\times(2\ell_2+1)$ and $(2\ell_2+1)\times(2\ell_1+1)$ matrices, respectively and it is seen that $A_{\mu,\ell m}^+(y)$ is a vector field in the bifundamental representation $(n_1, \bar{n}_2)$ of $u(n_1)\times u(n_2)$ while $A_{\mu,\ell m}^-(y)$ is in the bifundamental representation $(n_2, \bar{n}_1)$ of $u(n_1)\times u(n_2)$.

The mass of spectrum of KK modes of A_μ can be determined from by the last term of (3.37). With both diagonal and off-diagonal contributions we have

$$\begin{aligned} \int \mathrm{Tr}([A_\mu, \Phi_a]^\dagger [A^\mu, \Phi_a]) = \mathrm{Tr}_n \Bigg(& \sum_{j=0}^{2\ell_1} m_{j,1}^2 A_{\mu,jm}^{(1)\dagger}(y) A_{jm}^{(1)\mu}(y) \\ & + \sum_{j=0}^{2\ell_2} m_{j,2}^2 A_{\mu,jm}^{(2)\dagger}(y) A_{jm}^{(2)\mu}(y) + \sum_{j'=\frac{k}{2}}^{\ell_1+\ell_2} 2m_{j';\pm}^2 (A_{\mu,j'm}^+(y))^\dagger A_{j'm}^{+\mu}(y) \Bigg) \\ & + \text{higher order terms} \,. \end{aligned} \tag{3.64}$$

The diagonal components of the commutator $[A_\mu, \Phi_a]$ gives the adjoint actions $-[X_a^{(2\ell_1+1)}, A_\mu^{(1)}]$ and $-[X_a^{(2\ell_2+1)}, A_\mu^{(2)}]$ and for the off-diagonal components, we obtain the following terms

$$\left(-\alpha_1 X_a^{(2\ell_1+1)} A_\mu^+ + \alpha_2 A_\mu^+ X_a^{(2\ell_2+1)}\right) \,, \quad \left(-\alpha_2 X_a^{(2\ell_2+1)} A_\mu^- + \alpha_1 A_\mu^- X_a^{(2\ell_1+1)}\right) \,. \tag{3.65}$$

Spectrum of the diagonal fluctuations is easily determined by straightforward calculations for each diagonal blocks, similar to the one performed in (3.43) and (3.44). We find that $A^{(1)}_{\mu,jm}$ and $A^{(2)}_{\mu,jm}$ acquire the masses

$$
\begin{aligned}
m^2_{j,1} &= \frac{\alpha_1^2 g^2}{R^2} j(j+1)\,, \quad j = 0\,, \cdots, 2\ell_1\,, \\
m^2_{j,2} &= \frac{\alpha_2^2 g^2}{R^2} j(j+1)\,, \quad j = 0\,, \cdots, 2\ell_2\,.
\end{aligned} \tag{3.66}
$$

For the off-diagonal contribution, it is better to use the following identity;

$$
\begin{aligned}
\mathrm{Tr}\left([A_\mu, \Phi_a]^\dagger [A^\mu, \Phi_a]\right) &= -\mathrm{Tr}\left([\Phi_a, A_\mu][\Phi_a, A^\mu]\right) \\
&= -2\mathrm{Tr}\left(\Phi_a A_\mu \Phi_a A^\mu - \Phi_a \Phi_a A_\mu A^\mu\right)\,,
\end{aligned} \tag{3.67}
$$

that follows from the cyclicality of the trace. Then, we obtain the following contribution from the off-diagonal terms

$$
\begin{aligned}
-2\mathrm{Tr}\Big(& \alpha_1\alpha_2 X_a^{(1)} A_\mu^+ X_a^{(2)} A_\mu^- + \alpha_1\alpha_2 X_a^{(2)} A_\mu^- X_a^{(1)} A_\mu^+ \\
& -\alpha_1^2 (X_a^{(1)})^2 A_\mu^+ A_\mu^- - \alpha_2^2 (X_a^{(2)})^2 A_\mu^- A_\mu^+ \Big)\,,
\end{aligned} \tag{3.68}
$$

where we have introduced the notations $X_a^{2\ell_1+1} =: X_a^{(1)}$ and $X_a^{2\ell_2+1} := X_a^{(2)}$. First two terms in (3.68) can be written as

$$
\begin{aligned}
& -2\alpha_1\alpha_2 \mathrm{Tr}(X_a^1 A_\mu^+ X_a^2 A_\mu^- + X_a^2 A_\mu^- X_a^1 A_\mu^+) \\
&= -2\alpha_1\alpha_2 \mathrm{Tr}\Big((X_a^1 A_\mu^+ - A_\mu^+ X_a^2)(X_a^2 A_\mu^- - A_\mu^- X_a^1) + (X_a^1)^2 A_\mu^+ A_\mu^- \\
&\quad + (X_a^2)^2 A_\mu^- A_\mu^+ \Big) \\
&= -2\alpha_1\alpha_2 \mathrm{Tr}\Big((\mathcal{L}_a A_\mu^+)(\mathcal{L}_a A_\mu^-) + (X_a^1)^2 A_\mu^+ A_\mu^- + (X_a^2)^2 A_\mu^- A_\mu^+ \Big) \\
&= -2\alpha_1\alpha_2 \mathrm{Tr}\Big(-(\mathcal{L}^2 A_\mu^+) A_\mu^- + (X_a^1)^2 A_\mu^+ A_\mu^- + (X_a^2)^2 A_\mu^- A_\mu^+ \Big)\,.
\end{aligned} \tag{3.69}
$$

It is instructive to give the calculation required for the first term in descending form. The second line to the last line of this equation, we have

$$
\begin{aligned}
\mathrm{Tr}\big((\mathcal{L}_a A^+_\mu)(\mathcal{L}_a A^-_\mu)\big) =& \,\mathrm{Tr}\bigg((L^{\ell_1}_a A^+_\mu - A^+_\mu L^{\ell_2}_a)(L^{\ell_2}_a A^-_\mu - A^-_\mu L^{\ell_1}_a) \bigg) \\
=&\mathrm{Tr}\bigg(L^{\ell_1}_a A^+_\mu L^{\ell_2}_a A^-_\mu - L^{\ell_1}_a A^+_\mu A^-_\mu L^{\ell_1}_a - A^+_\mu (L^{\ell_2}_a)^2 A^-_\mu + A^+_\mu L^{\ell_2}_a A^-_\mu L^{\ell_1}_a \bigg) \\
=& - \mathrm{Tr}\bigg(\big((L^{\ell_1}_a)^2 A^+_\mu + A^+_\mu (L^{\ell_1}_a)^2 - 2(L^{\ell_1}_a A^+_\mu L^{\ell_2}_a)\big) A^-_\mu \bigg) \\
=& - \mathrm{Tr}\big((\mathcal{L}^2 A^+_\mu) A^-_\mu\big) \,,
\end{aligned} \tag{3.70}
$$

where we have used $(X^{(1)}_a A^+_\mu - A^+_\mu X^{(2)}_a) = \mathcal{L}_a A^+_\mu$ and $(X^{(2)}_a A^-_\mu - A^-_\mu X^{(1)}_a) = \mathcal{L}_a A^-_\mu$ from (3.57). Substituting for the relevant terms in (3.68) and (3.69) and using the expansion (3.63) and noting that $\mathrm{Tr}\big(\tilde{T}^+_{j'm'} \tilde{T}^-_{j''m''}\big) = \delta_{j'j''}\delta_{m'm''}$, we have

$$
\begin{aligned}
\sum_{j'm} \bigg(& - 2\alpha_1^2 X_1^2 - 2\alpha_2^2 X_2^2 + 2\alpha_1\alpha_2 \big(j'(j'+1) \\
& + (X^1_a)^2 + (X^2_a)^2\big) \bigg) (A^+_{\mu,j'm}(y))^\dagger A^+_{\mu,j'm}(y) \,.
\end{aligned} \tag{3.71}
$$

The off-diagonal excitations have therefore the mass spectrum given by

$$
m^2_{j',\pm} = \frac{g^2}{R^2} \left(\alpha_1\alpha_2 j'(j'+1) + (\alpha_1 - \alpha_2)(X_2^2\alpha_2 - X_1^2\alpha_1)\right) \,. \tag{3.72}
$$

Using $(X^1_a)^2 = -\ell_1(\ell_1+1)\,, (X^2_a)^2 = -\ell_2(\ell_2+1)$ and the following approximation for α_1 and α_2 (as discussed in section 3.1.2)

$$
\begin{aligned}
\alpha_1 &= 1 - \frac{m_1}{2\ell_1+1} + O(\frac{1}{\ell_1^2}) \,, \quad m_1 = 2(\ell_1 - \tilde{\ell}) \,, \\
\alpha_2 &= 1 - \frac{m_2}{2\ell_2+1} + O(\frac{1}{\ell_2^2}) \,, \quad m_2 = 2(\ell_2 - \tilde{\ell}) \,,
\end{aligned} \tag{3.73}
$$

the mass spectrum of $m_{j',\pm}$ in (3.72) can be written as

$$
\begin{aligned}
m^2_{j',\pm} &\approx \frac{g^2}{R^2} \left(j'(j'+1) + \frac{1}{4}(m_2 - m_1)^2 + O(\frac{1}{\mathcal{N}}) \right) \,. \\
&\approx \frac{g^2}{R^2} \left(j'(j'+1) + (\ell_2 - \ell_1)^2 + O(\frac{1}{\mathcal{N}}) \right) \,.
\end{aligned} \tag{3.74}
$$

From (3.66), we see that zero mass KK modes are those given by the lowest lying modes $A^{(1)}_{\mu,00}$ and $A^{(2)}_{\mu,00}$ in the diagonal blocks. These are $su(n_1)$ and $su(n_2)$ valued gauge fields. From (3.74), it can be seen that there is no massless gauge fields from the off-diagonal bifundamental fluctuations fields, i.e. $A^{+,-}_{\mu,\ell m}$, since there is an additional term in (3.74) with $\ell_2 - \ell_1 = \frac{k}{2} \neq 0$. This indicates that because of nonzero magnetic monopole number $2k$, we can not obtain the massless KK modes from the off-diagonal fluctuations fields.

Considering only the lowest lying KK modes $A^{(1)}_{\mu,00}$ and $A^{(2)}_{\mu,00}$, we obtain the LEA in the form of (3.51) for each mode $A^{(1)}_{\mu,00}$ and $A^{(2)}_{\mu,00}$. Noting that $A^{(1)}_{\mu,00}$ and $A^{(2)}_{\mu,00}$ are $su(n_1)$ and $su(n_2)$ valued, respectively, we observe that our gauge theory on M^4 behaves like an the effective $SU(n_1) \times SU(n_2) \times U(1)$ gauge theory on $M^4 \times S^2_F$ after the symmetry breaking of $SU(\mathcal{N})$.

3.3 Equivariant Parametrization of Yang-Mills Theory on $\mathcal{M} \times S^2_F$

Let us consider the action in (3.2) with the same field content written on d-dimensional manifold $\mathcal{M}$

$$S = \int_{\mathcal{M}} d^d y \, \mathrm{Tr}\Big(-\frac{1}{4g^2} F^\dagger_{\mu\nu} F^{\mu\nu} - (D_\mu \Phi_a)^\dagger (D^\mu \Phi_a)\Big) - \frac{1}{\tilde{g}^2} \mathrm{Tr}\big(F^\dagger_{ab} F_{ab}\big) \\ - \mathfrak{g}^2 \mathrm{Tr}\big((\Phi_a \Phi_a + \tilde{b})^2\big) \qquad (3.75)$$

where $y^\mu = (y^1 , \cdots , y^d)$ stands for a set of local coordinates on $\mathcal{M}$, Φ_a are dimensionless and we have suppressed a scaling constant γ^{-2} in the covariant derivative term where γ has the dimensions $[m]^{d/2-1}$. We see that g has dimensions $[m]^{-d/2+2}$, while restoring the dimension of Φ_a, $\tilde{g}$ has the dimension of $[m]^{d/2-2}$. Without the constraint term $V_2(\Phi)$, it is possible to express the action (3.75) of emerging model on $\mathcal{M} \times S^2_F$ as the L^2-norm of F_{MN} by the scaling $\tilde{\Phi}_a = \sqrt{2} g \Phi_a$ and taking $g\tilde{g} = 1$. Then, we may have [26]

$$S = \int d^d \, y \, \frac{1}{4g^2} Tr_{n(2\ell+1)} F^\dagger_{MN} F^{MN} + \mathfrak{g}^2 V_2(\Phi) \, . \qquad (3.76)$$

We are primarily interested in the vacuum configuration (3.14) for the action (3.75). Clearly, the discussion in section 3.1 indicates that a $U(n)$ gauge theory on manifold $\mathcal{M} \times S^2_F$ can be conjectured to emerge after the spontaneous breaking of the $SU(\mathcal{N})$ gauge symmetry of (3.75) by the vacuum configuration (3.14). In this section, we are going to focus on the $U(2)$ gauge theory on $\mathcal{M} \times S^2_F$ and initially treat the problem of determining the $SU(2)$-equivariant modes of the gauge fields. Subsequently, this data is going to employed to obtain a low energy effective action (LEA) on $\mathcal{M}$ by integrating out (tracing over) the fuzzy sphere

S_F^2. Equivariant field modes and LEA obtained in this manner provides us with another viewpoint, complementing the KK-modes analysis of section 3.2, supporting effective gauge theory interpretation of the model on $\mathcal{M} \times S_F^2$. Our discussion in this section is based on the treatment given in [22].

3.3.1 Equivariant Fields on $\mathcal{M} \times S_F^2$

In order to proceed, we consider choosing the $SU(2)$-symmetry generators as

$$\omega_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_2 - \mathbf{1}_{2\ell+1} \otimes \frac{i\tau_a}{2}\,, \quad \omega_a \in u(2\ell+1) \otimes u(2)\,, \tag{3.77}$$

where τ_a are the Pauli matrices. It is easy to check that these generators satisfy the $SU(2)$ commutation relations

$$[\omega_a, \omega_b] = \epsilon_{abc}\omega_c\,. \tag{3.78}$$

ω_a carries the tensor product representation $\ell \otimes 1/2 \equiv (\ell + 1/2) \oplus (\ell - 1/2)$. What physics do ω_a's describe? We observe that adjoint action of ω_a imposes rotational symmetry up to a $SU(2)$ gauge transformation. This is seen, since adjoint action of X_a generates the infinitesimal rotations on S_F^2, while the adjoint action of the Pauli matrices generate the infinitesimal $SU(2)$ gauge transformation.

Let us impose the $SU(2)$-equivariance conditions on the gauge fields as follows

$$[\omega_a\,, A_\mu] = 0\,, \tag{3.79}$$

$$[\omega_a, A_b] = \epsilon_{abc}A_c\,. \tag{3.80}$$

First of these equations means that we require A_μ to transform as a scalar under combined action of rotations of S_F^2 and $SU(2)$ gauge transformation while the second equation indicates that A_a transform as a vector under the same action.

It is possible to derive the dimensions of set of solutions to A_μ and A_a from the $SU(2)$ IRR content of the adjoint action of ω_a. We can expand the representation content of the adjoint action of ω_a into Clebsch-Gordan series as

$$(\ell \otimes \frac{1}{2}) \otimes (\ell \otimes \frac{1}{2}) = \mathbf{2}\,\underline{0} \oplus \mathbf{4}\,\underline{1} \oplus \cdots\,, \tag{3.81}$$

where the coefficients in bold denote the multiplicities of respective IRR in front of which they appear. From this expansion , we observe that under the adjoint action of ω_a, there are two linearly independent objects transforming as scalars (spin 0) and four linearly independent objects transforming as vectors (spin 1). Hence, the solution space of A_μ is two-dimensional and that of A_a is four-dimensional.

In order to parametrise A_μ and A_a, let us first find the rotational invariants under the adjoint action of ω_a. $X_a^{(2\ell+1)} \otimes \tau_a$ and $\mathbf{1}_{(2\ell+1)2}$ are the rotational invariants under the adjoint action of ω_a since $\mathbf{1}_{(2\ell+1)2}$ trivially commutes with ω_a and we find

$$\begin{aligned}
[\omega_a, X_b^{(2\ell+1)} \otimes \tau_b] &= [X_a^{(2\ell+1)} \otimes \mathbf{1}_2 - \mathbf{1}_{2\ell+1} \otimes \frac{i\tau_a}{2}, X_b^{(2\ell+1)} \otimes \tau_b] \,, \\
&= \epsilon_{abc} X_c^{(2\ell+1)} \otimes \tau_b + \epsilon_{abc} X_b^{(2\ell+1)} \otimes \tau_c = 0 \,, \\
&= 0 \,, \qquad (3.82)
\end{aligned}$$

where anti-symmetry of the permutation symbol is used in the last line of the calculations.

Let us consider a linear combination of $X_a \otimes \tau_a$ and $\mathbf{1}_{2(2\ell+1)}$ which will allow us to express several formula in what follows in a compact manner and simplify the several of the ensuing calculations:

$$Q := \frac{X_a \otimes \tau_a - \frac{i}{2}\mathbf{1}_{(2\ell+1)2}}{\ell + \frac{1}{2}} \,. \qquad (3.83)$$

Here, Q is an anti-Hermitian "idempotent" since $Q^\dagger = -Q$ and $Q^2 = -\mathbf{1}_{2(2\ell+1)}$. Therefore, the parametrization of A_μ in terms of Q and identity matrix $\mathbf{1}$ may be written as

$$A_\mu = \frac{1}{2} a_\mu(y) Q + \frac{1}{2} i b_\mu(y) \mathbf{1}_{2(2\ell+1)} \,, \qquad (3.84)$$

where $a_\mu(y)$, $b_\mu(y)$ are Hermitian U(1) gauge fields on the manifold $\mathcal{M}$. We stress that imposing the symmetry constraint (3.79) causes the breaking of $U(2)$ gauge symmetry down to $U(1) \times U(1)$. Under the action of $U = e^{\frac{1}{2}\theta_1(y) Q} e^{\frac{i}{2}\theta_2(y)\mathbf{1}}$,$U \in U(1) \times U(1)$, A_μ remains covariant with $a'_\mu = a_\mu + \partial_\mu \theta_1$ and $b'_\mu = b_\mu + \partial_\mu \theta_2$. Let us explicitly show this invariance;

$$
\begin{aligned}
A_\mu \to U A_\mu U^{-1} + \partial_\mu U U^{-1} &= A_\mu + \frac{1}{2}(\partial_\mu \theta_1 Q + i\partial_\mu \theta_2 \mathbf{1}) \\
&= \frac{1}{2}(a_\mu + \partial_\mu \theta_1) Q + \frac{i}{2}(b_\mu + \partial_\mu \theta_2)\,, \\
&= \frac{1}{2} a'_\mu Q + \frac{i}{2} b'_\mu \mathbf{1} = A'_\mu\,.
\end{aligned} \tag{3.85}
$$

For the parametrization of A_a, we can write

$$
\begin{aligned}
A_a = \frac{1}{2}\varphi_1(y)[X_a, Q] + \frac{1}{2}(\varphi_2(y) - 1) Q[X_a, Q] + i\frac{1}{2}\varphi_3(y)\frac{1}{2(\ell + \frac{1}{2})}\{X_a, Q\} \\
+ \frac{1}{2(\ell + \frac{1}{2})}\varphi_4(y)\omega_a\,,
\end{aligned} \tag{3.86}
$$

where $\{\cdot, \cdot\}$ stands for anti-commutator and φ_i are Hermitian scalar fields over $\mathcal{M}$. It can be easily shown that the four basis $[X_a, Q]\,, Q[X_a, Q]\,, \{X_a, Q\}\,, \omega_a$ elements fulfill the vector condition in the equation (3.80); As an example, let us show the first one

$$
\begin{aligned}
[\omega_a, [X_b, Q]] &= [\omega_a, X_b] Q - Q[\omega_a, X_b] + X_b[\omega_a, Q] - [\omega_a, Q] X_b\,, \\
&= \epsilon_{abc}[X_c, Q]\,,
\end{aligned} \tag{3.87}
$$

where last two terms in the first line are zero because Q is rotational invariant. We would like to note that we have introduced the real scalar fields $\varphi_1\,, \varphi_2\,, \varphi_3$ and φ_4 on $\mathcal{M}$ as coefficients of these vectors. We will show that some of these scalar fields naturally combine to form complex scalars after we trace over the extra dimension S_F^2.

In the commutative limit $\ell \to \infty$, using 2.71 A_a becomes

$$
A_a \xrightarrow[\ell \to \infty]{} -\left(\frac{i}{2}\varphi_1(y)\mathcal{L}_a q + \frac{i}{2}\left(\varphi_2(y) - 1\right) q\mathcal{L}_a q + \frac{1}{2}\varphi_3(y) x_a q + \frac{1}{4}\varphi_4(y) x_a\right)\,. \tag{3.88}
$$

It can be seen from (3.83) and (2.63) that q is the commutative limit of Q given as $q := -i\tau \cdot \mathbf{x}$. In this limit, the fuzzy sphere S_F^2 reduces to ordinary two sphere S^2 and it is seen that we have three components of A_a. As we mentioned before in section 2.6, we can eliminate the normal component of gauge fields by imposing the constraint $x_a A_a = 0$. This constraint is fulfilled if and only if by taking $\varphi_3 = \varphi_4 = 0$. At this point, we can see that this theory have the same structure as the spherical symmetric gauge field over $\mathcal{M} \times S^2$ [67].

3.3.2 Obtaining the Low Energy Effective Action

Now, we are ready to substitute the equivariant gauge fields A_μ in (3.84) and A_a in (3.86) into the Yang-Mills action (3.75) and trace it over S_F^2. It is convenient to denote the reduced action in the form

$$S := \int d^d y \left(\mathcal{L}_F + \mathcal{L}_G + V_1 + V_2\right) , \tag{3.89}$$

where each term in the integrand in (3.89) is evaluated below. In the calculations leading to the result below, the identities

$$\begin{aligned} &\{Q, [X_a, Q]\} = 0\,, \quad \{X_a, [X_a, Q]\} = 0\,, \quad [Q, \{X_a, Q\}] = 0\,, \\ &[X_a, \{X_a, Q\}] = 0\,, \quad \text{sum over repeated } a \text{ is implied}\,, \end{aligned} \tag{3.90}$$

are repeatedly made use of. We obtain

$$\mathcal{L}_F = \frac{1}{4g^2} \mathrm{Tr}\left(F_{\mu\nu}^\dagger F^{\mu\nu}\right) = \frac{1}{16g^2}\left(f_{\mu\nu}f^{\mu\nu} + h_{\mu\nu}h^{\mu\nu} - \frac{2}{2\ell+1} f_{\mu\nu}h^{\mu\nu}\right) , \tag{3.91}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the field strength tensor of the $U(1)$ gauge field a_μ whereas $h_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$ is that of b_μ. With the equivariant gauge fields, the covariant derivative of scalar fields becomes

$$D_\mu \Phi_a = \frac{1}{2}\left(D_\mu\varphi_1 + QD_\mu\varphi_2\right)[X_a, Q] + \frac{i}{4(\ell+\frac{1}{2})}\partial_\mu\varphi_3\{X_a, Q\} + \frac{1}{2(\ell+\frac{1}{2})}\partial_\mu\varphi_4\omega_a\,, \tag{3.92}$$

where $D_\mu\varphi_i = \partial_\mu\varphi_i + \epsilon_{ji}a_\mu\varphi_j$. Then, the gradient term can be evaluated to be

$$\begin{aligned} \mathcal{L}_G &= \mathrm{Tr}\left((D_\mu\Phi_a)^\dagger(D^\mu\Phi_a)\right) \\ &= \frac{\ell(\ell+1)}{2(\ell+\frac{1}{2})^2}\left((D_\mu\varphi_1)^2 + (D_\mu\varphi_2)^2\right) + \frac{\ell(\ell+1)(\ell^2+\ell-1/4)}{4(\ell+\frac{1}{2})^4}(\partial_\mu\varphi_3)^2 \\ &\quad + \frac{\ell(\ell+1)}{2(\ell+\frac{1}{2})^3}\partial_\mu\varphi_3\partial^\mu\varphi_4 + \frac{(\ell^2+\ell+3/4)}{4(\ell+\frac{1}{2})^2}(\partial_\mu\varphi_4)^2\,. \end{aligned} \tag{3.93}$$

Here, it is possible to interpret $\varphi_1 + i\varphi_2$ as a complex scalar field carrying charge $+1$ under a_μ since $D_\mu\varphi = \partial_\mu\varphi + ia_\mu\varphi$. it can easily be seen that

$$\begin{aligned} (D_\mu\varphi_1)^2 + (D_\mu\varphi_2)^2 &= (\partial_\mu\varphi_1)^2 + (\partial_\mu\varphi_2)^2 + 2a_\mu\varphi_1\partial_\mu\varphi_2 - 2a_\mu\varphi_2\partial_\mu\varphi_1 \\ &= D_\mu\varphi\overline{D_\mu\varphi}\,. \end{aligned} \tag{3.94}$$

Proceeding to write the potential term V_1, let us first indicate that the dual of F_{ab} can be written in the compact form

$$\begin{aligned}\frac{1}{2}\epsilon_{abc}F_{ab} &= \frac{1}{2}\epsilon_{abc}[\Phi_a, \Phi_b] - \Phi_c \\ &= \frac{1}{2}\mathcal{W}_1(\varphi_1 + \varphi_2 Q)[X_a, Q] + \frac{i}{4}(|\varphi|^2 - \mathcal{W}_2)\frac{\{X_a, Q\}}{\ell + \frac{1}{2}} + \frac{\mathcal{W}_3}{4}\frac{\omega_c}{(\ell + \frac{1}{2})^2}\,,\end{aligned} \tag{3.95}$$

where we have introduced, for notational brevity, $\mathcal{W}_i\,, (i = 1, 2, 3)$ which are the certain combinations involving the scalar fields φ_3 and φ_4 are listed in Appendix A. Then, the potential term V_1 can be calculated to give

$$V_1 = \frac{1}{\tilde{g}^2}\mathrm{Tr}(F_{ab}^\dagger F_{ab}) = \frac{1}{\tilde{g}^2}\left(\mathcal{X}_1|\varphi|^4 + \mathcal{X}_2|\varphi|^2 + \mathcal{X}_3\right)\,, \tag{3.96}$$

where, one again $\mathcal{X}_i\,, (i = 1, 2, 3)$ are introduced for notational brevity and their explicit form as certain combinations involving the scalar fields φ_3 and φ_4 are also given in appendix A.

For the evaluation of the last term in (3.89), an intermediate step is given as

$$\Phi_a\Phi_a + \ell(\ell + 1) = \mathcal{Y}_1 + i\mathcal{Y}_2 Q\,, \tag{3.97}$$

where $\mathcal{Y}_1\,, \mathcal{Y}_2$ are given in appendix A. After another few steps of calculation, this yields the V_2 term as

$$\begin{aligned}V_2 &= \mathfrak{g}^2\mathrm{Tr}\big((\Phi_a\Phi_a + \tilde{b})^2\big) = \mathfrak{g}^2\mathrm{Tr}\big((\Phi_a\Phi_a + \ell(\ell + 1))^2\big) \\ &= \mathfrak{g}^2\left(\mathcal{Y}_1^2 + \mathcal{Y}_2^2 + \frac{1}{\ell + \frac{1}{2}}\mathcal{Y}_1\mathcal{Y}_2\right)\,.\end{aligned} \tag{3.98}$$

3.3.3 Vacua of the Reduced Potential $V_1 + V_2$

Another step forward is to examine the vacua of our potential terms V_1 and V_2. We recall that both V_1 and V_2 are positive definite. To get the field configuration minimizing V_1, this simply implies that we need to determine the zeros of F_{ab}. Since $[X_a, Q]\,, Q[X_a, Q]\,, \{X_a, Q\}$ and ω_a are linearly independent, it can be easily seen that in order to obtain $F_{ab} = 0$, the second and third terms in (3.95) imply $|\varphi|^2 - \mathcal{W}_2 = 0$ and $\mathcal{W}_3 = 0$, so we have the following conditions

$$|\varphi|^2 = (1-\varphi_3)\left(1 + \frac{1}{\ell+\frac{1}{2}}\varphi_4 - \frac{1}{2(\ell+\frac{1}{2})^2}\varphi_3\right), \tag{3.99}$$

$$0 = \frac{\ell(\ell+1)}{(\ell+\frac{1}{2})^2}(\varphi_3^2 - 2\varphi_3) + \varphi_4^2 + \frac{2(\ell^2+\ell-1/4)}{\ell+\frac{1}{2}}\varphi_4\,. \tag{3.100}$$

Multiplying the first part in (3.95) with $(\varphi_1 - \varphi_2 Q)$, we obtain

$$\mathcal{W}_1(\varphi_1 + \varphi_2 Q)(\varphi_1 - \varphi_2 Q) = \mathcal{W}_1|\varphi|^2 = 0\,, \tag{3.101}$$

which can be rewritten as

$$0 = |\varphi|\left(\frac{\ell^2+\ell-1/4}{(\ell+\frac{1}{2})^2}\varphi_3 + \frac{1}{\ell+\frac{1}{2}}\varphi_4\right). \tag{3.102}$$

Solving these three conditions (3.99), (3.100) and (3.102), we can obtain the vacua of potential V_1. There are five sets of solution for these equations given as follows

$$i)\ \ |\varphi| = 1\,,\quad \varphi_3 = 0\,,\quad \varphi_4 = 0\,, \tag{3.103a}$$

$$ii)\ \ |\varphi| = 1\,,\quad \varphi_3 = 2\,,\quad \varphi_4 = \frac{-2(\ell^2+\ell-1/4)}{\ell+\frac{1}{2}}\,, \tag{3.103b}$$

$$iii)\ \ |\varphi| = 0\,,\quad \varphi_3 = 1\,,\quad \varphi_4 = \frac{1}{2(\ell+\frac{1}{2})}\,, \tag{3.103c}$$

$$iv)\ \ |\varphi| = 0\,,\quad \varphi_3 = 1\,,\quad \varphi_4 = \frac{-2\ell(\ell+1)}{\ell+\frac{1}{2}}\,, \tag{3.103d}$$

$$v)\ \ |\varphi| = 0\,,\quad \varphi_3 = 1 \pm (\ell + \frac{1}{2})\,,\quad \varphi_4 = -\frac{\ell^2+\ell-1/4}{\ell+\frac{1}{2}} \pm \frac{1}{2}\,. \tag{3.103e}$$

When we also consider the zeros of potential term V_2 which are given by $\mathcal{Y}_1 = 0$ and $\mathcal{Y}_2 = 0$, we see that only (3.103a) fulfills this conditions so V_1 and V_2 has the vacua given by (3.103a) in (3.103).

In commutative limit ($\ell \to \infty$), the potentials V_1 and V_2 take the form

$$V_1(\Phi) \xrightarrow[\ell\to\infty]{} \frac{1}{\tilde{g}^2}\left(\frac{1}{2}(|\varphi|^2 + \varphi_3 - 1)^2 + |\varphi|^2\varphi_3^2 + \frac{1}{2}\varphi_4^2\right), \tag{3.104}$$

$$V_2(\Phi) \xrightarrow[\ell\to\infty]{} \mathfrak{g}^2\ell^2(\varphi_4^2 + \varphi_3^2)\,. \tag{3.105}$$

In (3.105), we consider in fact the limit $\ell \to \infty$, $\mathfrak{g} \to 0$ with $\mathfrak{g}\ell$ finite but small. The reason for this will be clear in the next section. It can be easily seen that (3.103a) is the vacuum configuration for V_1 and V_2 in the stated limit.

3.3.4 Vortex Type Solutions

We now examine the large ℓ limit of the low energy action in detail. Focusing on $\mathcal{M} = \mathbb{R}^2$, we determine the vortex type solutions for i) $\ell \to \infty$, $\mathfrak{g} \to 0$ with $\mathfrak{g}\ell$ is finite but small and ii) where ℓ is large but finite and $\mathfrak{g} \to \infty$. These two different limits governed by ℓ and $\mathfrak{g}$ enable us to handle the constraint term V_2 in effectively two extreme cases.

3.3.4.1 Case *i)*

In the commutative limit $\ell \to \infty$, we obtain the reduced LEA

$$S = \int_{\ell\to\infty} d^2y \Bigg(\frac{1}{16g^2}\left(f_{\mu\nu}f^{\mu\nu} + h_{\mu\nu}h^{\mu\nu}\right) + \frac{1}{2}|D_\mu\varphi|^2 + \frac{1}{4}\left((\partial_\mu\varphi_3)^2 + (\partial_\mu\varphi_4)^2\right) + \frac{1}{\tilde{g}^2}\left(\frac{1}{2}(|\varphi|^2+\varphi_3-1)^2 + |\varphi|^2\varphi_3^2 + \frac{1}{2}\varphi_4^2\right)\Bigg) , \tag{3.106}$$

where we have not written down the V_2 term with the proviso that $\mathfrak{g}\ell$ is small. Here, we can see that there are no interaction terms for both the real scalar field φ_4 and gauge field b_μ, so they are decoupled from the rest of action. Since they do not give any additional information in the equation of motions, from now on we can ignore these fields. The remaining fields are the $U(1)$ gauge field a_μ, the complex scalar field φ and the real scalar field φ_3. The vacuum configuration given in (3.103a) has the structure of S^1 with first homotopy group $\pi(S^1) = \mathbb{Z}$. This indicates that the vortex solution on $\mathbb{R}^2$ are characterized by an integer winding number. To search for the vortex solutions, let us consider the $SO(2) \approx U(1)$ rotationally symmetric ansatz [68] as

$$\varphi = \xi(r)\mathrm{e}^{iN\theta} , \quad \varphi_3 = \rho(r) , \quad a = a_\theta(r)d\theta , \quad a_r = 0 , \tag{3.107}$$

written in polar coordinate on $\mathbb{R}^2$ where $g_{rr} = 1\,, g_{\theta\theta} = r^2\,, g_{r\theta} = 0$ and N is winding number. The form of $f_{\mu\nu}f^{\mu\nu}$ and $|D_\mu\varphi|^2$ with this ansatz can be found as follows

$$
\begin{aligned}
f_{\mu\nu}f^{\mu\nu} &= \frac{1}{r^2}f_{r\theta}f_{r\theta} = \frac{1}{r^2}(\partial_r a_\theta)^2\,,\\
|D_\mu\varphi|^2 &= (\partial_r\varphi + ia_r\varphi)(\partial_r\bar{\varphi} - ia_r\bar{\varphi}) + \frac{1}{r^2}(\partial_\theta\varphi + ia_\theta\varphi)(\partial_\theta\bar{\varphi} - ia_\theta\bar{\varphi})\\
&= (\partial_r\xi)^2 + \frac{1}{r^2}\xi^2(N + a_\theta)^2\,,
\end{aligned}
\tag{3.108}
$$

then our action becomes

$$
\begin{aligned}
S = 2\pi\int_0^\infty dr\Bigg(&\frac{1}{8g^2 r}{a'_\theta}^2 + \frac{r}{2}\xi'^2 + \frac{1}{2r}(N + a_\theta)^2\xi^2 + \frac{r}{4}\rho'^2\\
&+\frac{r}{\tilde{g}^2}\left(\frac{1}{2}(\xi^2 + \rho - 1)^2 + \xi^2\rho^2\right)\Bigg)\,,
\end{aligned}
\tag{3.109}
$$

where primes are denoting the derivatives with respect to r.

The Euler-Lagrange equations can be derived from (3.109) to be

$$
\begin{aligned}
&\xi'' + \frac{1}{r}\xi' - \left(\frac{1}{r^2}(N + a_\theta)^2 + \frac{2}{\tilde{g}^2}(\xi^2 + \rho^2 + \rho - 1)\right)\xi = 0\,,\\
&a''_\theta - \frac{1}{r}a'_\theta - 4g^2(a_\theta + N)\xi^2 = 0\,,\\
&\rho'' + \frac{1}{r}\rho' - \frac{2}{\tilde{g}^2}(\xi^2 + 2\xi^2\rho + \rho - 1) = 0\,.
\end{aligned}
\tag{3.110}
$$

These are nonlinear coupled differential equations and we do not know their analytic solutions. However, it is possible to find their approximate solutions in the small r and large r regions. For $r \to 0$, using the Frobenius method, and power series expansion around $r = 0$, we obtain the solutions

$$
\xi = \xi_0 r^N + O(r^{N+2})\,, \quad a_\theta = a_0 r^2 + 0(r^4)\,, \quad \rho = \rho_0 + O(r^2)\,. \tag{3.111}
$$

For $r \to \infty$, finiteness of the action indicates some important features of our fields. It is easy to see that the action (3.109) is finite if $\xi(r) \to 1\,, a_\theta(r) = -N\,, \rho(r) \to 0$ as $r \to \infty$. Using these profiles, the asymptotic behavior of our fields can be determined by adding small fluctuations to these vacuum values and thus writing

$$
\xi = 1 - \delta\xi\,, \quad a_\theta = -N + \delta a\,, \quad \rho = \delta\rho\,. \tag{3.112}
$$

Now, it is possible to solve $\delta\chi\,, \delta a$ and $\delta\lambda$ by inserting our fields (3.112) in the Euler-Lagrange equations (3.110). Assuming that $(\frac{\delta a}{r})^2$ is subleading corresponding to $\delta\xi$ and $\delta\rho$, we obtain the following differential equations

$$\delta\xi'' + \frac{1}{r}\delta\xi' - \frac{2}{\tilde{g}^2}(-\delta\rho + 2\delta\xi) = 0\,, \tag{3.113}$$

$$\delta a'' - \frac{1}{r}\delta a' - 4g^2\delta a = 0\,, \tag{3.114}$$

$$\delta\rho'' + \frac{1}{2}\delta\rho' - \frac{2}{\tilde{g}^2}(3\delta\rho - 2\delta\xi) = 0\,. \tag{3.115}$$

Let us start with finding the solution to the differential equation (3.114). Here, setting $\delta a = r\delta\tilde{a}$, this equation becomes the modified Bessel equations at order 1

$$r^2\delta\tilde{a}'' + r\delta\tilde{a}' - \delta\tilde{a}(4g^2r^2 + 1) = 0\,. \tag{3.116}$$

Two linearly independent solutions to this modified Bessel equation are modified Bessel function of the first kind $I_1(2gr)$ and second kind $K_1(2gr)$. Here, since we are looking for solution for large r and $I_1(2gr)$ diverges in this limit, the solution to this equation is $K_1(2gr)$. Hence, the solution to (3.114) is constructed as

$$\delta a = \alpha_1 r K_1(2gr)\,. \tag{3.117}$$

For the solution to coupled differential equations (3.113) and (3.115), let us define a linear operator $\mathcal{D} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$, then they become

$$(\mathcal{D} - \frac{4}{\tilde{g}^2})\delta\xi + \frac{2}{\tilde{g}^2}\delta\rho = 0\,, \quad (\mathcal{D} - \frac{6}{\tilde{g}^2})\delta\rho + \frac{4}{\tilde{g}^2}\delta\xi = 0\,. \tag{3.118}$$

Multiplying first equation by $(\mathcal{D} - \frac{6}{\tilde{g}^2})$, we obtain $(\mathcal{D} - \frac{8}{\tilde{g}^2})(\mathcal{D} - \frac{2}{\tilde{g}^2})\delta\xi = 0$. This means that there are two second order differential equation for $\delta\xi$ in the following form

$$\delta\xi'' + \frac{1}{r}\delta\xi' + \frac{8}{\tilde{g}^2}\delta\xi = 0\,, \quad \delta\xi'' + \frac{1}{r}\delta\xi' + \frac{2}{\tilde{g}^2}\delta\xi = 0\,. \tag{3.119}$$

These are the modified Bessel equations at order 0. Taking the regular solutions at $r \to \infty$, we have

$$\delta\xi = \alpha_2 K_0(\sqrt{2}r/\tilde{g}) + \alpha_3 K_0(2\sqrt{2}r/\tilde{g})\,. \tag{3.120}$$

It is easy to see that we have the same differential equation (3.119) for $\delta\rho$ and after a straightforward calculation, its solution is found to be

$$\delta\rho = \alpha_2 K_0(\sqrt{2}r/\tilde{g}) - 2\alpha_3 K_0(2\sqrt{2}r/\tilde{g})\,. \tag{3.121}$$

Here, our assumption that $(\frac{\delta a}{r})^2$ is subleading to $\delta\xi\,,\delta\rho$ yields the condition $4g > \frac{\sqrt{2}}{\tilde{g}}$.

It is known from general consideration in Vortex dynamics that if field strength dominates the asymptotic profile than the vortices repel and they attract if scalars are dominant [68]. We can easily determine the leading order behavior of the field profiles in (3.117), (3.120) and (3.121). Since $K_0(2\sqrt{2}r/\tilde{g})$ is subleading to $K_0(\sqrt{2}r/\tilde{g})$, we should deal with the latter which indicates that scalar fields decay like $\frac{1}{\sqrt{r}}e^{-\frac{\sqrt{2}}{\tilde{g}}r}$ as $r \to \infty$. Whereas for the field strength, we have $B = f_{12} = \frac{1}{2}\partial_r a_\theta$ and it decays like $\frac{1}{\sqrt{r}}e^{-2gr}$. Hence, the attractive-repulsive nature of forces depends on the value of the coupling constants such that they are attractive for $g\tilde{g} > \frac{\sqrt{2}}{2}$ and repulsive for $\frac{\sqrt{2}}{4} < g\tilde{g} < \frac{\sqrt{2}}{2}$. Thus, vortices of the LEA corresponding to the standard Yang-Mills theory in (3.76) attract since $g\tilde{g} = 1$ in this case.

3.3.4.2 Case *ii)*

Here, we would like find the vortex solutions by taking large but finite ℓ and $\mathfrak{g} \to \infty$ which is equivalent to imposing the constraint $\Phi_a\Phi_a + \ell(\ell+1) = 0$ by hand. This constraint is fulfilled by setting $\mathcal{Y}_1 = 0\,,\mathcal{Y}_2 = 0$ in equation (3.97) and the solutions for φ_3 and φ_4 in leading powers of $\frac{1}{\ell}$ can be constructed in terms of $|\varphi|$ as

$$\varphi_3 = -\frac{1}{2\ell^2}(1-|\varphi|^2) + O(\frac{1}{\ell^3})\,, \quad \varphi_4 = \frac{1}{2\ell}(1-|\varphi|^2) + O(\frac{1}{\ell^2})\,. \tag{3.122}$$

Inserting these real scalar in the reduced action at the order $\frac{1}{\ell}$, we get

$$S = \int d^2y \frac{1}{16g^2}\left(f_{\mu\nu}f^{\mu\nu} + h_{\mu\nu}h^{\mu\nu} - \frac{1}{\ell}f_{\mu\nu}h^{\mu\nu}\right) + \frac{1}{2}\left(1 - \frac{1}{4\ell^2}\right)|D_\mu\varphi|^2 + \frac{1}{16\ell^2}(\partial_\mu|\varphi|^2)^2 + \frac{1}{2\tilde{g}^2}\left(1+\frac{1}{2\ell^2}\right)(1-|\varphi|^2)^2\,. \tag{3.123}$$

Using the equation of motion of the field b_μ enables us to find $h_{\mu\nu} = -\frac{f_{\mu\nu}}{2\ell}$. We substitute this in the action (3.123) to eliminate $h_{\mu\nu}$.

Making the same symmetric ansatz as in (3.107), we construct the Euler-Lagrange equations for $\xi(r)$ and $a_\theta(r)$

$$\Big(1-\frac{1}{4\ell^2}+\frac{\xi^2}{2\ell^2}\Big)\Big(\xi''+\frac{\xi'}{r}\Big)+\frac{1}{2\ell^2}\xi'^2-\Big(1-\frac{1}{4\ell^2}\Big)\frac{1}{r^2}(N+a_\theta)^2\xi \\ -\frac{2}{\tilde{g}^2}\Big(1+\frac{1}{2\ell^2}\Big)(\xi^2-1)\xi=0 \qquad (3.124)$$

$$a_\theta''-\frac{a_\theta'}{r}-4g^2(a_\theta+N)\Big(1-\frac{1}{\ell^2}\Big)\xi^2=0\,. \qquad (3.125)$$

The solutions to ξ and a_θ around $r=0$ are the same as in the previous case (3.111). For $r\to\infty$, the differential equations in (3.125) can be solved in terms of modified Bessel functions as

$$\delta\xi=\beta_1 K_0\Big(\frac{2}{\tilde{g}}\sqrt{(1+1/4\ell^2)}r\Big)\,,\quad \delta a=\beta_2 r K_1\left(2g\sqrt{\Big(1-\frac{1}{\ell^2}\Big)}r\right)\,, \qquad (3.126)$$

where $\xi=1-\delta\xi\,,a_\theta=N+\delta a$ and $\beta_1\,,\beta_2$ are constants. Here, the interval for the attractive and repulsive vortices can be determined by performing a similar calculation just as in the previous case. It is found as

$$g\tilde{g}>\sqrt{1+\frac{5}{4\ell^2}}\,,\quad \text{for attractive vortices}\,, \qquad (3.127)$$

$$\frac{1}{2}\sqrt{1+\frac{5}{4\ell^2}}<g\tilde{g}<\sqrt{1+\frac{5}{4\ell^2}}\,,\quad \text{for repulsive vortices}\,. \qquad (3.128)$$

For the standard Yang-Mills theory ($g\tilde{g}=1$), since scalar fields decay faster than the field strength, these vortices repel each other in this model. Finally, it is worth to note that $\ell\to\infty$ limit in (3.123) with $h_{\mu\nu}=-\frac{f_{\mu\nu}}{2\ell}\to 0$ yields the standard BPS vortex action [68].

3.4 Gauge Theory on $\mathcal{M}\times S_F^2\times S_F^2$

3.4.1 A Deformation of $N=4$ SYM

In this section, we focus our attention to the dynamical generation of the product of two fuzzy sphere $S_F^2\times S_F^2$ from an $SU(\mathcal{N})$ gauge theory coupled to suitable number of scalar fields. Let us consider an $SU(\mathcal{N})$ gauge theory coupled to six anti-Hermitian scalar fields Φ_i with the action [21, 24]

$$S = \int_{\mathcal{M}} d^d y \mathrm{Tr}\bigg(- \frac{1}{4g^2} F^\dagger_{\mu\nu} F^{\mu\nu} - (D_\mu \Phi_i)^\dagger (D^\mu \Phi_i) \bigg) - V(\Phi) \,, \quad i = 1, \cdots, 6 \,, \tag{3.129}$$

where Φ_i transform adjointly under the $SU(\mathcal{N})$ group action and also transform in the vector representation of a global $SO(6) \approx SU(4)$ symmetry and $V(\Phi)$ is such that it is invariant under these actions. If we consider the 4-dimensional Minkovski space and take the potential term $V(\Phi)$ as

$$V_{N=4}(\Phi) = -\frac{1}{4} g^2_{YM} \sum_{ij}^{6} [\Phi_i, \Phi_j]^2 \,, \tag{3.130}$$

the action (3.129) becomes the bosonic part of the $N = 4$ supersymmetric Yang Mills theory [4, 5]. The global $SO(6)$ symmetry of (3.129) is nothing but the R-symmetry of the super Yang-Mills theory. We will show that considering the action (3.129) with the potential terms

$$V(\Phi) = V_{N=4}(\Phi) + V_{\mathrm{break}}(\Phi) \,. \tag{3.131}$$

where $V_{\mathrm{break}}(\Phi)$ contains cubic soft symmetry breaking and quadratic mass deformation terms which breaks $N = 4$ supersymmetry completely and global $SU(4)$ symmetry down to a subgroup. The product of two fuzzy sphere $S^2_F \times S^2_F$ emerges as a vacuum configuration after the spontaneously symmetry breaking of $SU(\mathcal{N})$. Let us examine the action (3.129) with the potential term (3.131) which is written in the form of [21, 24]

$$V(\Phi) = \frac{1}{g^2_L} V_1(\Phi^L) + \frac{1}{g^2_R} V_1(\Phi^R) + \frac{1}{g^2_{LR}} V_1(\Phi^{L,R}) + \mathfrak{g}^2_L V_2(\Phi^L) + \mathfrak{g}^2_R V_2(\Phi^R) \,, \tag{3.132}$$

where we have divided six scalar fields Φ_i into two parts $\Phi^L_a = \Phi_a$, $\Phi^R_a = \Phi_{a+3}$, $(a = 1, 2, 3)$ and

$$\begin{aligned}
&V_1(\Phi^L) = \mathrm{Tr} F^{L\dagger}_{ab} F^L_{ab} \,, \quad F^L_{ab} = [\Phi^L_a, \Phi^L_b] - \epsilon_{abc} \Phi^L_c \\
&V_1(\Phi^R) = \mathrm{Tr} F^{R\dagger}_{ab} F^R_{ab} \,, \quad F^R_{ab} = [\Phi^R_a, \Phi^R_b] - \epsilon_{abc} \Phi^R_c \,, \\
&V_2(\Phi^L) = \mathrm{Tr}(\Phi^L_a \Phi^L_a + \tilde{b}^L)^2 \,, \quad V_2(\Phi^R) = \mathrm{Tr}(\Phi^R_a \Phi^R_a + \tilde{b}^R)^2 \,, \\
&V_1(\Phi^{L,R}) = \mathrm{Tr} F^{(L,R)\dagger}_{ab} F^{(L,R)}_{ab} \,, \quad F^{(L,R)}_{ab} = [\Phi^L_a, \Phi^R_b] \,.
\end{aligned} \tag{3.133}$$

Here, the covariant derivative in the action (3.129) can be rewritten in terms of Φ_a^L and Φ_a^R as follows

$$(D_\mu \Phi_i)^\dagger (D_\mu \Phi_i) = (D_\mu \Phi_a^L)^\dagger (D_\mu \Phi_a^L) + (D_\mu \Phi_a^R)^\dagger (D_\mu \Phi_a^R) \,. \tag{3.134}$$

We note that the model (3.129) with the potential (3.133) breaks the global $SU(4)$ R-symmetry down to a global $SU(2) \times SU(2)$. The scalar fields $\Phi_i \equiv (\Phi_a^L \,, \Phi_a^R)$ transform under the $(1,0) \oplus (0,1)$ representation of this global symmetry.

3.4.2 $S_F^2 \times S_F^2$ Vacuum

Let us focus on the construction of the vacuum configuration for this problem. It is easy to observe that the potential terms are positive definite and the minimization of these terms require

$$F_{ab}^L = 0 \,, \quad F_{ab}^R = 0 \,, \quad -\Phi_a^L \Phi_a^L = \tilde{b}^L \,, \quad -\Phi_a^R \Phi_a^R = \tilde{b}^R \,, \quad F_{ab}^{L,R} = 0 \,. \tag{3.135}$$

The most general solution to these equations is not known. However, by following a procedure similar to the one used in section (3.3), it is possible to construct a specific solution to these equations. Let us take $\tilde{b}^L$ as the eigenvalue of quadratic Casimir of an $SU(2)_L$ IRR labeled by ℓ_L and $\tilde{b}^R$ as the eigenvalue of quadratic Casimir of an $SU(2)_R$ IRR labeled by ℓ_R

$$\tilde{b}^R = \ell_L(\ell_L + 1) \,, \quad \tilde{b}^R = \ell_R(\ell_R + 1) \,, \quad 2\ell_L \,, 2\ell_R \in \mathbb{Z} \,. \tag{3.136}$$

With the assumption $\mathcal{N} = (2\ell_L + 1)(2\ell_R + 1)n$, we can choose the vacuum configuration

$$\begin{aligned} \Phi_a^L &= X_a^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_n \,, \\ \Phi_a^R &= \mathbf{1}^{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_n \,, \\ [\Phi_a^L, \Phi_b^R] &= 0 \,, \end{aligned} \tag{3.137}$$

where $(X_a^{(2\ell_L+1)}, X_a^{(2\ell_R+1)})$ are the anti-Hermitian generators of $SU(2)_L \times SU(2)_R$ in the irreducible representation (IRR) (ℓ_L, ℓ_R) satisfying the relations in (2.154). Clearly, this vacuum configuration spontaneously breaks the $SU(\mathcal{N})$ symmetry down to a $U(n)$. Here, the commutant of Φ_a^L and Φ_a^R in (3.137) is $U(n)$. In fact, this vacuum can be interpreted as the product of two fuzzy spheres $S_F^2 \times S_F^2$ generated by $\hat{x}_a^L$ and $\hat{x}_a^L$ where

$$\begin{aligned}\hat{x}_a^L &= \frac{i}{\sqrt{\ell_L(\ell_L+1)}} X_a^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)}\,, \\ \hat{x}_a^R &= \frac{i}{\sqrt{\ell_R(\ell_R+1)}} \mathbf{1}^{(2\ell_R+1)} \otimes X_a^{(2\ell_R+1)}\,,\end{aligned} \tag{3.138}$$

are given in subsection 2.7.2.

Fluctuations about the vacuum configuration in (3.137) can be written as

$$\Phi_a^L = X_a^L + A_a^L\,, \quad \Phi_a^R = X_a^R + A_a^R\,, \tag{3.139}$$

where A_a^L, $A_a^R \in u(2\ell_L+1) \otimes u(2\ell_R+1) \otimes u(n)$ and the notation $X_a^L = X_a^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_n$ and $X_a^R = \mathbf{1}^{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_n$ has been introduced. With the fluctuation terms A_a^L and A_a^R, F_{ab}^L, F_{ab}^R, $F_{ab}^{L,R}$ become

$$F_{ab}^L = [X_a^L, A_b^L] - [X_b^L, A_a^L] + [A_a^L, A_b^L] - \epsilon_{abc} A_c^L\,, \tag{3.140}$$

$$F_{ab}^R = [X_a^R, A_b^R] - [X_b^R, A_a^R] + [A_a^R, A_b^R] - \epsilon_{abc} A_c^R\,, \tag{3.141}$$

$$F_{ab}^{L,R} = [X_a^L, A_b^R] - [X_b^R, A_a^L] + [A_a^L, A_b^R]\,. \tag{3.142}$$

It can be observed that $(F_{ab}^L, F_{ab}^R, F_{ab}^{L,R})$ have the form of curvature tensor on the manifold $S_F^2 \times S_F^2$. Hence, we can interpret A_a^L, A_a^R, $(a = 1, 2, 3))$ as six components of a $U(n)$ gauge field over $S_F^2 \times S_F^2$ and Φ_a^L and Φ_a^R are "covariant coordinates" on $S_F^2 \times S_F^2$. In other words, after the spontaneously symmetry breaking the gauge theory on $\mathcal{M}$ can be interpreted as the gauge theory $\mathcal{M} \times S_F^2 \times S_F^2$ where the gauge fields are in the form of $A_M := (A_\mu, A_a^L, A_a^R)$ and the field strength tensor is $F_{MN} := (F_{\mu\nu}, F_{\mu a}^L, F_{\mu a}^R, F_{ab}^L, F_{ab}^R, F_{ab}^{L,R})$ with

$$\begin{aligned}F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,, \\ F_{\mu a}^L &= D_\mu \Phi_a^L = \partial_\mu A_a^L - [X_a^L, A_\mu] + [A_\mu, A_a^L]\,, \\ F_{\mu a}^R &= D_\mu \Phi_a^R = \partial_\mu A_a^R - [X_a^R, A_\mu] + [A_\mu, A_a^R]\,,\end{aligned} \tag{3.143}$$

and the rest given in (3.142) above. We note that by scaling $\tilde{\Phi}_a^L = \sqrt{2} g \Phi_a^L$, $\tilde{\Phi}_a^R = \sqrt{2} g \Phi_a^R$ and taking $g_L = g_R = \sqrt{2} g_{L,R} = \tilde{g}$ and $g\tilde{g} = 1$, this model can be expressed as follows

$$S = \int d^d y \, \frac{1}{4g^2} \mathrm{Tr}\big(F_{MN}^\dagger F^{MN}\big) + \mathfrak{g}_L^2 V_2(\Phi^L) + \mathfrak{g}_R^2 V_2(\Phi^R)\,. \tag{3.144}$$

This means that, apart from the constraint terms, it is possible to express the Lagrangian of the emerging model on $\mathcal{M}\times S_F^2\times S_F^2$ as the L^2-norm of F_{MN} with given relation between couplings .

3.4.3 Equivariant Parametrization

Let us consider a $U(4)$ gauge theory on $\mathcal{M}\times S_F^2\times S_F^2$ studied in [24]. In this case, the symmetry group of $S_F^2\times S_F^2$ is $SU(2)\times SU(2)$. Our aim is to construct the most general $SU(2)\times SU(2)$-equivariant gauge fields. Let us define the symmetry generators ω_a which generate $SU(2)\times SU(2)$ rotations up to a $U(4)$ gauge transformation as follows

$$\begin{aligned}\omega_a^L &= X_a^{(2\ell_L+1)}\otimes\mathbf{1}^{(2\ell_R+1)}\otimes\mathbf{1}_4-\mathbf{1}^{(2\ell_L+1)}\otimes\mathbf{1}^{(2\ell_R+1)}\otimes i\frac{L_a^L}{2}\,,\\ \omega_a^R &= \mathbf{1}^{(2\ell_L+1)}\otimes X_a^{(2\ell_R+1)}\otimes\mathbf{1}_4-\mathbf{1}^{(2\ell_L+1)}\otimes\mathbf{1}^{(2\ell_R+1)}\otimes i\frac{L_a^R}{2}\,,\end{aligned}\tag{3.145}$$

where ω_a^L and ω_a^R are required to satisfy $so(4)=su(2)\oplus su(2)$ commutation relations

$$[\omega_a^L,\omega_b^L]=\epsilon_{abc}\omega_c^L\,,\quad[\omega_a^R,\omega_b^R]=\epsilon_{abc}\omega_c^R\,,\quad[\omega_a^L,\omega_b^R]=0\,.\tag{3.146}$$

A suitable choice for $(L_a^L\,,L_a^R)$ is to take them to carry 4-dimensional $(\frac{1}{2}\,,\frac{1}{2})$ IRR of $SU(2)\times SU(2)$ [24] and satisfy the relations

$$\begin{aligned}&[L_a^L,L_b^L]=2i\epsilon_{abc}L_c^L\,,\quad[L_a^R,L_b^R]=2i\epsilon_{abc}L_c^R\,,\quad[L_a^L,L_b^R]=0\,,\\ &L_a^LL_b^L=i\epsilon_{abc}L_c^L+\delta_{ab}\mathbf{1}_4\,,\quad L_a^RL_b^R=i\epsilon_{abc}L_c^R+\delta_{ab}\mathbf{1}_4\,.\end{aligned}\tag{3.147}$$

For concreteness, explicit 4×4 matrix forms of L_a^L and L_a^R may be given as

$$\begin{aligned}&L_1^L=\begin{pmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{pmatrix},\ L_2^L=\begin{pmatrix}0&0&i&0\\0&0&0&-i\\-i&0&0&0\\0&i&0&0\end{pmatrix},\ L_3^L=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}\\ &L_1^R=\begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix},\ L_2^R=\begin{pmatrix}0&0&i&0\\0&0&0&i\\-i&0&0&0\\0&-i&0&0\end{pmatrix},\ L_3^R=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&i\\0&0&-i&0\end{pmatrix}\end{aligned}\tag{3.148}$$

To make this choice for $(L_a^L\,, L_a^R)$ clear, let us indicate a few facts regarding the IRRs of $U(4)$. This group has sixteen generators and its fundamental representation is four dimensional. This representation can be spanned by the sixteen 4×4 matrices $L_a^L\,, L_a^R\,, L_a^L L_b^R\,, \mathbf{1}$. $(L_a^L\,, L_a^R)$ generate the $SU(2)\times SU(2)$ subgroup of $U(4)$ as we see from (3.147) which is why they have been introduced in (3.145). Relation of these generators to the more familiar Gell-Mann type $\lambda_i\,, i = 1\,, \cdots\,, 15$ will not be worked out here as it is not necessary for our purpose. Since our aim is to construct the $SU(2)\times SU(2)$-equivariant gauge fields of $U(4)$ gauge theory, the choice for $(L_a^L\,, L_a^R)$ in (3.148) are suitable in order to define the generators $(\omega_a^L\,, \omega_a^R)$.

$SU(2)\times SU(2)$-equivariance on the $U(4)$ gauge theory may be imposed by requiring the constraints

$$
\begin{aligned}
&[\omega_a^L, A_\mu] = 0\,, \quad [\omega_a^L, A_b^L] = \epsilon_{abc} A_c^L\,,\\
&[\omega_a^R, A_\mu] = 0\,, \quad [\omega_a^R, A_b^R] = \epsilon_{abc} A_c^R\,,\\
&[\omega_a^L, A_b^R] = 0 = [\omega_a^R, A_b^L]\,.
\end{aligned} \tag{3.149}
$$

In order to find the explicit parametrization of A_μ, A_a^L and A_a^R, we use the Clebsch-Gordan series expansion of the adjoint action of $(\omega_a^L\,, \omega_a^R)$;

$$
[(\ell_L,\ell_R)\otimes(\frac{1}{2},\frac{1}{2})]\otimes[(\ell_L,\ell_R)\otimes(\frac{1}{2},\frac{1}{2})] \equiv \mathbf{4}(0,\,0)\oplus\mathbf{8}(1,\,0)\oplus\mathbf{8}(0,\,1)\oplus\cdots\,. \tag{3.150}
$$

This expansion implies that the solution of A_μ is 4-dimensional and each of the solution space of A_a^L and A_a^R is 8-dimensional. Here, we may define two "idempotents" Q_L and Q_R as

$$
\begin{aligned}
Q_L &= \frac{X_a^{(2\ell_L+1)}\otimes \mathbf{1}^{(2\ell_R+1)}\otimes L_a^L - \frac{i}{2}\mathbf{1}}{\ell_L+\frac{1}{2}}\,, \quad Q_L^\dagger = -Q_L\,, \quad Q_L^2 = -\mathbf{1}_{4(2\ell_L+2)(2\ell_R+1)}\,,\\
Q_R &= \frac{\mathbf{1}^{(2\ell_L+1)}\otimes X_a^{(2\ell_R+1)}\otimes L_a^R - \frac{i}{2}\mathbf{1}}{\ell_R+\frac{1}{2}}\,, \quad Q_R^\dagger = -Q_R\,, \quad Q_R^2 = -\mathbf{1}_{4(2\ell_L+2)(2\ell_R+1)}\,.
\end{aligned} \tag{3.151}
$$

The invariants under the adjoint action of $(\omega_a^L\,, \omega_a^R)$ can be expressed in terms of Q_L, Q_R, $Q_L Q_R$ and $\mathbf{1}$. Then, A_μ may be parametrized in terms of these invariants,

$$
A_\mu = \frac{1}{2}a_\mu^L(y) Q_L + \frac{1}{2}a_\mu^R(y) Q_R + \frac{1}{2} i a_\mu^{L,R} Q_L Q_R + \frac{i}{2} b_\mu \mathbf{1}\,, \tag{3.152}
$$

where $a^L_\mu(y)$, $a^R_\mu(y), a^{L,R}_\mu(y)$, $b_\mu(y)$ are all Hermitian $U(1)$ gauge fields. Here, we can also see that imposing the constraint (3.149) causes the breaking of $U(4)$ symmetry down to $U(1)^{\otimes 4}$. In a manner similar to the one given in section 3.3.1, it is possible to show that A_μ preserve the rotational symmetry under the gauge transformation generated by $U = e^{\frac{1}{2}Q_L\theta_1(y)}e^{\frac{1}{2}Q_R\theta_2(y)}e^{\frac{1}{2}Q_LQ_R\theta_3(y)}e^{\frac{i}{2}\mathbf{1}\theta_4(y)}$ with $a^{L\prime}_\mu = a^L_\mu + \partial_\mu\theta_1\,, a^{R\prime}_\mu = a^R_\mu + \partial_\mu\theta_2\,, a^{L,R\prime}_\mu = a^{L,R}_\mu + \partial_\mu\theta_3\,,$ and $b'_\mu = b_\mu + \partial_\mu\theta_4$.

With the parametrization (3.152), the field strength tensor takes the following form

$$F_{\mu\nu} = \frac{1}{2}f^L_{\mu\nu}Q_L + \frac{1}{2}f^R_{\mu\nu}Q_R + \frac{1}{2}f^{L,R}_{\mu\nu}Q_LQ_R + \frac{i}{2}h_{\mu\nu}\mathbf{1}\,, \tag{3.153}$$

where we have introduced

$$\begin{aligned} f^L_{\mu\nu} &= \partial_\mu a^L_\nu - \partial_\nu a^L_\mu\,, \quad f^R_{\mu\nu} = \partial_\mu a^R_\nu - \partial_\nu a^R_\mu\,, \\ f^{L,R}_{\mu\nu} &= \partial_\mu a^{L,R}_\nu - \partial_\nu a^{L,R}_\mu\,, \quad h_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu\,. \end{aligned} \tag{3.154}$$

For the parametrization of A^L_a and A^R_a, a suitable basis may be chosen as follows

$$\begin{aligned} A^L_a &= \frac{1}{2}(\varphi_1 + \tilde{\varphi}_1)[X^L_a, Q_L] + \frac{1}{2}(\varphi_2 + \tilde{\varphi}_2 - 1)Q_L[X^L_a, Q_L] + i\frac{\varphi_3}{4(\ell_L + \frac{1}{2})}\{X^L_a, Q_L\} \\ &+ \frac{\varphi_4}{2(\ell_L + \frac{1}{2})}\omega^L_a + iQ_R\bigg(\frac{1}{2}(\varphi_1 - \tilde{\varphi}_1)iQ_R[X^L_a, Q_L] + \frac{1}{2}(\varphi_2 - \tilde{\varphi}_2)Q_L[X^L_a, Q_L] \\ &+ i\frac{\tilde{\varphi}_3}{4(\ell_L + \frac{1}{2})}\{X^L_a, Q_L\} + \frac{\tilde{\varphi}_4}{2(\ell_L + \frac{1}{2})}\omega^L_a\bigg)\,, \end{aligned} \tag{3.155}$$

$$\begin{aligned} A^R_a &= \frac{1}{2}(\chi_1 + \tilde{\chi}_1)[X^R_a, Q_R] + \frac{1}{2}(\chi_2 + \tilde{\chi}_2 - 1)Q_R[X^R_a, Q_R] + i\frac{\chi_3}{4(\ell_R + \frac{1}{2})}\{X^R_a, Q_R\} \\ &+ \frac{\chi_4}{2(\ell_R + \frac{1}{2})}\omega^R_a + iQ_L\bigg(\frac{1}{2}(\chi_1 - \tilde{\chi}_1)[X^R_a, Q_R] + \frac{1}{2}(\chi_2 - \tilde{\chi}_2)Q_R[X^R_a, Q_R] \\ &+ i\frac{\tilde{\chi}_3}{4(\ell_R + \frac{1}{2})}\{X^R_a, Q_R\} + \frac{\tilde{\chi}_4}{2(\ell_R + \frac{1}{2})}\omega^R_a\bigg)\,, \end{aligned} \tag{3.156}$$

where $\varphi_i\,, \tilde{\varphi}_i\,, \chi_i\,, \tilde{\chi}_i\,,\ (i = 1,2,3,4)$ are Hermitian scalar fields over $\mathcal{M}$. A convenient notation for future use is to write $A^{L,R}_a =: \hat{A}^{L,R}_a + iQ^{L,R}\hat{A}'^{L,R}_a$.

Using the basis given above, the covariant derivatives can be constructed in terms of scalar fields as

$$
\begin{aligned}
D_\mu\Phi_a^L = \frac{1}{2}\Big(D_\mu(\varphi_1+\tilde\varphi_1)+Q^L D_\mu(\varphi_2+\tilde\varphi_2)\Big)[X_a^L,Q^L] + i\frac{\partial_\mu\varphi_3}{4(\ell_L+\frac{1}{2})}\{X_a^L,Q^L\} \\
+\frac{\partial_\mu\varphi_4}{2(\ell_L+\frac{1}{2})}\omega_a^L + iQ^R\Bigg(\frac{1}{2}\big(D_\mu(\varphi_1-\tilde\varphi_1)+Q^L D_\mu(\varphi_2-\tilde\varphi_2)\big)[X_a^L,Q^L] \\
+ i\frac{\partial_\mu\tilde\varphi_3}{4(\ell_L+\frac{1}{2})}\{X_a^L,Q^L\} + \frac{\partial_\mu\tilde\varphi_4}{2(\ell_L+\frac{1}{2})}\omega_a^L\Bigg)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
D_\mu\Phi_a^R = \frac{1}{2}\Big(D_\mu(\chi_1+\tilde\chi_1)+Q^R D_\mu(\chi_2+\tilde\chi_2)\Big)[X_a^R,Q^R] + i\frac{\partial_\mu\chi_3}{4(\ell_R+\frac{1}{2})}\{X_a^R,Q^R\} \\
+\frac{\partial_\mu\chi_4}{2(\ell_R+\frac{1}{2})}\omega_a^R + iQ^L\Bigg(\frac{1}{2}\big(D_\mu(\chi_1-\tilde\chi_1)+Q^R D_\mu(\chi_2-\tilde\chi_2)\big)[X_a^R,Q^R] \\
+ i\frac{\partial_\mu\tilde\chi_3}{4(\ell_R+\frac{1}{2})}\{X_a^R,Q^R\} + \frac{\partial_\mu\tilde\chi_4}{2(\ell_R+\frac{1}{2})}\omega_a^R\Bigg)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
D_\mu\varphi_i = \partial_\mu\varphi_i + \varepsilon_{ji}a_\mu^L\varphi_j + \varepsilon_{ji}a_\mu^{L,R}\varphi_j\,, \quad D_\mu\tilde\varphi_i = \partial_\mu\tilde\varphi_i + \varepsilon_{ji}a_\mu^L\tilde\varphi_j + \varepsilon_{ij}a_\mu^{L,R}\tilde\varphi_j \\
D_\mu\chi_i = \partial_\mu\chi_i + \varepsilon_{ji}a_\mu^R\chi_i + \varepsilon_{ji}a_\mu^{L,R}\chi_i\,, \quad D_\mu\tilde\chi_i = \partial_\mu\tilde\chi_i + \varepsilon_{ji}a_\mu^L\tilde\chi_i + \varepsilon_{ij}a_\mu^{L,R}\tilde\chi_j\,,
\end{aligned}
\tag{3.157}
$$

with $i = 1, 2$.

We are ready to find the reduced action from these parametrizations. Using the following notation for the reduced action

$$
S := \int \mathcal{L}_F + \mathcal{L}_G^L + \mathcal{L}_G^R + V_1^L + V_1^R + V_1^{L,R} + V_2^L + V_2^R\,, \tag{3.158}
$$

and the identities given in (3.90) for both Q_L and Q_R, it is possible to determine each summand in the integrand of (3.158). Since we are essentially interested in the exploring vortex type solutions to this low energy effective action in certain limits, we relegate explicit forms of (3.158) to Appendix A and consider the commutative limit ($\ell_L\,,\ell_R\to\infty$) of these terms. As $\ell_L\,,\ell_R\to\infty$, they read

$$
\begin{aligned}
\mathcal{L}_F &= \frac{1}{16g^2}\big(f^L_{\mu\nu}f^{L\mu\nu}+f^R_{\mu\nu}f^{R\mu\nu}+f^{L,R}_{\mu\nu}f^{L,R\mu\nu}+h_{\mu\nu}h^{\mu\nu}\big)\,,\\
\mathcal{L}^L_G &= |D_\mu\varphi|^2+|D_\mu\tilde\varphi|^2+\frac{1}{4}\left((\partial_\mu\varphi_3)^2+(\partial_\mu\tilde\varphi_3)^2+(\partial_\mu\varphi_4)^2+(\partial_\mu\tilde\varphi_4)^2\right)\,,\\
\mathcal{L}^R_G &= |D_\mu\chi|^2+|D_\mu\tilde\chi|^2+\frac{1}{4}\left((\partial_\mu\chi_3)^2+(\partial_\mu\tilde\chi_3)^2+(\partial_\mu\chi_4)^2+(\partial_\mu\tilde\chi_4)^2\right)\\
V^L_1 &= \frac{4}{{g_L}^2}\Bigg((|\varphi|^2+\frac{1}{4}(\varphi_3+\tilde\varphi_3-1))^2+(|\tilde\varphi|^2+\frac{1}{4}(\varphi_3-\tilde\varphi_3-1))^2\\
&\quad+\frac{1}{2}(\varphi_3+\tilde\varphi_3)^2|\varphi|^2+\frac{1}{2}(\varphi_3-\tilde\varphi_3)^2|\tilde\varphi|^2+\frac{1}{8}(\varphi_4^2+\tilde\varphi_4^2)\Bigg)\,,\\
V^R_1 &= \frac{4}{{g_R}^2}\Bigg((|\chi|^2+\frac{1}{4}(\chi_3+\tilde\chi_3-1))^2+(|\tilde\chi|^2+\frac{1}{4}(\chi_3-\tilde\chi_3-1))^2\\
&\quad+\frac{1}{2}(\chi_3+\tilde\chi_3)^2|\chi|^2+\frac{1}{2}(\chi_3-\tilde\chi_3)^2|\tilde\chi|^2+\frac{1}{8}(\chi_4^2+\tilde\chi_4)\Bigg)\,,\\
V^{L,R}_1 &= \frac{2}{g^2_{L,R}}\Bigg(|\varphi\tilde\chi-\tilde\varphi\chi|^2+|\bar\chi\varphi-\tilde\varphi\bar{\tilde\chi}|^2+\frac{1}{4}\big((|\varphi|^2+|\tilde\varphi|^2)(\tilde\chi_3^2+\tilde\chi_4^2)\\
&\quad+(|\chi|^2+|\tilde\chi|^2)(\tilde\varphi_3^2+\tilde\varphi_4^2)\big)\Bigg)\\
V^L_2 &= \mathfrak{g}_L^2\ell_L^2\left(\varphi_3^2+\varphi_4^2+\tilde\varphi_3^2+\tilde\varphi_4^2\right)\,,\\
V^R_2 &= \mathfrak{g}_R^2\ell_R^2\left(\chi_3^2+\chi_4^2+\tilde\chi_3^2+\tilde\chi_4^2\right)\,,
\end{aligned}
\tag{3.159}
$$

where in V^L_2 and V^R_2, we shall consider either $\mathfrak{g}_L\to 0\,,\mathfrak{g}_R\to 0$ such that $\mathfrak{g}_L\ell_L$ and $\mathfrak{g}_R\ell_R$ small but finite corresponding to the first limit case we explore in section 3.3.4 or $\mathfrak{g}_L\,,\mathfrak{g}_R\to\infty$ corresponding to the second case we explore in section 3.3.4. It can be easily seen that the vacuum configuration for the scalar fields is given by

$$|\varphi|=|\tilde\varphi|=|\chi|=|\tilde\chi|=\frac{1}{2}\,, \tag{3.160a}$$

$$\varphi_3=\tilde\varphi_3=\varphi_4=\tilde\varphi_4=0\,, \tag{3.160b}$$

$$\chi_3=\tilde\chi_3=\chi_4=\tilde\chi_4=0\,, \tag{3.160c}$$

$$\varphi\tilde\chi=\tilde\varphi\chi\,,\quad \bar\chi\varphi=\tilde\varphi\bar{\tilde\chi}\,. \tag{3.160d}$$

In fact, the two conditions given in the last line (3.160d) are not independent from each other. Using (3.160a), we can derive one from the other. This means that we have one constraint over our complex scalar fields $\varphi\,,\tilde\varphi\,,\chi\,,\tilde\chi$. In other words, we can write one of them in terms of other three. This implies that the vacuum configuration has the structure of a three torus, $T^3=S^1\times S^1\times S^1$, with first homotopy group $\pi_1(T^3) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. This result indicate that LEA has classical solution on $\mathbb{R}^2$ with nontrivial topology which are characterized by three integer winding numbers. Indeed, in the next section, we will give the vortex type solutions for this problem with the winding numbers (n_1, n_2, n_3).

3.4.4 Vortex Type Solutions

In subsection 3.3.4, we have constructed the vortex type solutions for the reduced action (3.89) on manifold $\mathcal{M} = \mathbb{R}^2$ after the integrating out the extra dimensions S_F^2. Here, we would like to examine the vortex type solutions for the reduced action (3.158) obtained by tracing over the extra dimensions $S_F^2 \times S_F^2$. In order to proceed, we define our limits as follows. First, we consider the reduced action in the commutative limit $\ell_L\,, \ell_R \rightarrow \infty$ and $\mathfrak{g}_L\,, \mathfrak{g}_R \rightarrow 0$ such that $\mathfrak{g}_L \ell_L$ and $\mathfrak{g}_R \ell_R$ are small but finite which is effectively equivalent to consider the commutative limit of the action (3.158) without the constraint terms V_2^L and V_2^R. In the second case, we take ℓ_L and ℓ_R large but finite and $\mathfrak{g}_L\,, \mathfrak{g}_R \rightarrow \infty$, which actually amounts to imposing the constraints by hand, i.e. taking $\Phi_a^L \Phi_a^L + \ell_L(\ell_L + 1) = 0$ and $\Phi_a^R \Phi_a^R + \ell_R(\ell_R + 1) = 0$.

3.4.4.1 Case *i*)

It can be easily seen from (3.159) that in the limit $\ell_L\,, \ell_R \rightarrow \infty$ and $\mathfrak{g}_L\,, \mathfrak{g}_R \rightarrow 0$, $b_\mu\,, \varphi_4$ and χ_4 are decoupled from the rest of the action. This means that their equations of motion do not affect the rest of the fields. Consequently, in this case we have $U(1)^3$ gauge theory. In order to proceed, let us make the rotationally symmetric ansatz

$$
\begin{aligned}
&a_r^L = 0\,, \quad a_r^R = 0\,, \quad a_r^{L,R} = 0\,, \\
&a^L = a_\theta^L(r) d\theta\,, \quad a^R = a_\theta^R(r) d\theta\,, \quad a^{L,R} = a_\theta^{L,R}(r) d\theta\,,
\end{aligned} \tag{3.161}
$$

and

$$
\begin{aligned}
&\varphi=\varphi(r)e^{in_1\theta}\,,\quad \tilde{\varphi}=\tilde{\varphi}(r)e^{in_2\theta}\,,\quad \chi=\chi(r)e^{im_1\theta}\,,\quad \tilde{\chi}=\tilde{\chi}(r)e^{im_2\theta}\,,\\
&\varphi_3=\varphi_3(r)\,,\quad \tilde{\varphi}_3=\tilde{\varphi}_3(r)\,,\quad \tilde{\varphi}_4=\tilde{\varphi}_4(r)\,,\quad \chi_3=\chi_3(r)\,,\quad \tilde{\chi}_3=\tilde{\chi}_3(r)\\
&\tilde{\chi}_4=\tilde{\chi}_4(r) \qquad (3.162)
\end{aligned}
$$

It is important to note that $n_1\,, n_2\,, n_3$ and n_4 are not linearly independent. This can be easily seen from (3.162) and (3.160d), they satisfy

$$
(n_1-n_2)-(n_3-n_4)=0\,. \qquad (3.163)
$$

Using this relation, we can eliminate n_4 as $n_4=n_3-(n_1-n_2)$. As mentioned before, the nontrivial structure of complex scalar fields can be defined by three winding numbers (n_1,n_2,n_3) in this case. Now, the reduced action becomes

$$
\begin{aligned}
S=2\pi\int_0^\infty dr\Big[&\frac{1}{8g^2}\Big(\frac{1}{r}a_\theta^{L\prime}a_\theta^{L\prime}+\frac{1}{r}a_\theta^{R\prime}a_\theta^{R\prime}+\frac{1}{r}a_\theta^{L,R\prime}a_\theta^{L,R\prime}\Big)+r\varphi'^2\\
&+\frac{1}{r}(n_1+a_\theta^L+a_\theta^{L,R})^2\varphi^2+r\tilde{\varphi}'^2+\frac{1}{r}(n_2+a_\theta^L-a_\theta^{L,R})^2\tilde{\varphi}^2+r\chi'^2\\
&+\frac{1}{r}(n_3+a_\theta^R+a_\theta^{L,R})^2\chi^2+r\tilde{\chi}'^2+\frac{1}{r}\big(n_3-(n_1-n_2)+a_\theta^R-a_\theta^{L,R}\big)^2\tilde{\chi}^2\\
&+\frac{r}{4}\Big(\varphi_3'^2+\tilde{\varphi}_3'^2+\tilde{\varphi}_4'^2+\chi_3'^2+\tilde{\chi}_3'^2+\tilde{\chi}_4'^2\Big)+\frac{r}{g_L^2}\Big(4\big(\varphi^2+\frac{1}{4}(\varphi_3+\tilde{\varphi}_3)-\frac{1}{4}\big)^2\\
&+4\big(\tilde{\varphi}^2+\frac{1}{4}(\varphi_3-\tilde{\varphi}_3)-\frac{1}{4}\big)^2+2(\varphi_3+\tilde{\varphi}_3)^2\varphi^2+2(\varphi_3-\tilde{\varphi}_3)^2\tilde{\varphi}^2+\frac{1}{2}\tilde{\varphi}_4^2\Big)\\
&+\frac{r}{g_R^2}\Big(4\big(\chi^2+\frac{1}{4}(\chi_3+\tilde{\chi}_3)-\frac{1}{4}\big)^2+4\big(\tilde{\chi}^2+\frac{1}{4}(\chi_3-\tilde{\chi}_3)-\frac{1}{4}\big)^2\\
&+2(\chi_3+\tilde{\chi}_3)^2\chi^2+2(\chi_3-\tilde{\chi}_3)^2\tilde{\chi}^2+\frac{1}{2}\tilde{\chi}_4^2\Big)+\frac{r}{g_{L,R}^2}\Big(2(\varphi\tilde{\chi}-\tilde{\varphi}\chi)^2\\
&+2(\chi\varphi-\tilde{\chi}\tilde{\varphi})^2+\frac{1}{2}\big(\varphi^2+\tilde{\varphi}^2\big)(\tilde{\chi}_3^2+\tilde{\chi}_4^2)+\frac{1}{2}\big(\chi^2+\tilde{\chi}^2\big)(\tilde{\varphi}_3^2+\tilde{\varphi}_4^2)\Big)\Big]\,, \qquad (3.164)
\end{aligned}
$$

With a straightforward calculation, we can construct Euler-Lagrange equations for our fields which are given in appendix A. These are nonlinear coupled differential equations and the analytic solutions to these equations are not known. However, like the previous section, it seems possible to obtain the approximate solutions by focusing on two regions; small r and large r. With the expansion around $r=0$, the profiles of our fields can be found as

$$
\begin{aligned}
&\varphi(r)=\varphi_0 r^{n_1}+O(r^{n_1+2})\,,\quad \chi(r)=\chi_0 r^{n_3}+O(r^{n_3+2})\,,\\
&\tilde{\varphi}(r)=\tilde{\varphi}_0 r^{n_2}+O(r^{n_2+2})\,,\quad \tilde{\chi}(r)=\tilde{\chi}_0 r^{n_3-(n_1-n_2)}+O(r^{n_3-(n_1-n_2)+2})\,,\\
&a_\theta^L(r)=a_0^L r^2+O(r^4)\,,\quad a_\theta^R(r)=a_0^R r^2+O(r^4)\,,\quad a_\theta^{L,R}(r)=a_0^{L,R} r^2+O(r^4)\,,\\
&\varphi_3=\varphi_0^3+O(r^2)\,,\quad \chi_3=\chi_0^3+O(r^2)\,,\quad \tilde{\varphi}_3=\tilde{\varphi}_0^3+O(r^2)\,,\quad \tilde{\chi}_3=\tilde{\chi}_0^3+O(r^2)\,,\\
&\tilde{\varphi}_4=\tilde{\varphi}_0^4+O(r^2)\,,\quad \tilde{\chi}_4=\tilde{\chi}_0^4+O(r^2)\,.
\end{aligned}
\tag{3.165}
$$

For large r, let us first notice the asymptotic behavior of our fields is defined by the finiteness of action (3.164). This implies that

$$
\begin{aligned}
&a_\theta^L(r)=-\frac{n_1+n_2}{2}\,,\quad a_\theta^R(r)=-\frac{2n_3-(n_1-n_2)}{2}\,,\quad a_\theta^{L,R}(r)=-\frac{n_1-n_2}{2}\,,\\
&\varphi(r)\to\frac{1}{2}\,,\quad \tilde{\varphi}(r)\to\frac{1}{2}\,,\quad \chi(r)\to\frac{1}{2}\,,\quad \tilde{\chi}(r)\to\frac{1}{2}\,,\quad \varphi_3(r)\to 0\,,\quad \chi_3\to 0\,,\\
&\tilde{\varphi}_3\to 0\,,\quad \tilde{\chi}_3\to 0\,,\quad \tilde{\varphi}_4\to\,,\quad \tilde{\chi}_4\to 0\,,\quad \text{as}\quad r\to\infty\,.
\end{aligned}
\tag{3.166}
$$

Now, considering the small fluctuations around these vacuums, we can obtain the asymptotic profile of our fields for large r. These fluctuations can be written as follows

$$
\begin{aligned}
&\varphi=\frac{1}{2}-\delta\varphi\,,\quad \tilde{\varphi}=\frac{1}{2}-\delta\tilde{\varphi}\,,\quad \chi=\frac{1}{2}-\delta\chi\,,\quad \tilde{\chi}=\frac{1}{2}-\delta\tilde{\chi}\,,\\
&a_\theta^L=-\frac{n_1+n_2}{2}+\delta a^L\,,\quad a_\theta^R=-\frac{2n_3-(n_1-n_2)}{2}+\delta a^R\,,\\
&a_\theta^{L,R}=-\frac{n_1-n_2}{2}+\delta a^{L,R}\,,
\end{aligned}
\tag{3.167}
$$

and

$$
\varphi_3=\delta\varphi_3\,,\quad \chi_3=\delta\chi_3\,,\quad \tilde{\varphi}_3=\delta\tilde{\varphi}_3\,,\quad \tilde{\chi}_3=\delta\tilde{\chi}_3\,,\quad \tilde{\varphi}_4=\delta\tilde{\varphi}_4\,,\quad \tilde{\chi}_4=\delta\tilde{\chi}_4\,.
\tag{3.168}
$$

Then, the equations of motion (A.20) for $\delta a^L\,,\delta a^{L,R}\,,\delta\varphi\,,\delta\tilde{\varphi}\,,\delta\varphi_3\,,\delta\tilde{\varphi}_3\,,\delta\tilde{\varphi}_4$ reduce to

$$\delta a^{L\prime\prime} - \frac{1}{r}\delta a^{L\prime} - 4g^2\delta a^L = 0\,, \tag{3.169}$$
$$\delta a^{L,R\prime\prime} - \frac{1}{r}\delta a^{L,R\prime} - 8g^2\delta a^{L,R} = 0\,, \tag{3.170}$$
$$\delta\varphi'' + \frac{1}{r}\delta\varphi' + \frac{4}{g_L^2}\left(-\delta\varphi + \frac{1}{4}(\delta\varphi_3 + \delta\tilde{\varphi}_3)\right) - \frac{1}{g_{L,R}^2}(\delta\varphi - \delta\tilde{\varphi}) = 0\,, \tag{3.171}$$
$$\delta\tilde{\varphi}'' + \frac{1}{r}\delta\tilde{\varphi}' + \frac{4}{g_L^2}\left(-\delta\tilde{\varphi} + \frac{1}{4}(\delta\varphi_3 - \delta\tilde{\varphi}_3)\right) - \frac{1}{g_{L,R}^2}(\delta\tilde{\varphi} - \delta\varphi) = 0\,, \tag{3.172}$$
$$\delta{\varphi_3}'' + \frac{1}{r}\delta{\varphi_3}' + \frac{4}{g_L^2}\left(\delta\varphi + \delta\tilde{\varphi} - \frac{3}{2}\delta\varphi_3\right) = 0\,, \tag{3.173}$$
$$\delta\tilde{\varphi}_3'' + \frac{1}{r}\delta\tilde{\varphi}_3' + \frac{4}{g_L^2}\left(\delta\varphi - \delta\tilde{\varphi} - \frac{3}{2}\delta\tilde{\varphi}_3\right) + \frac{1}{g_{L,R}^2}\delta\tilde{\varphi}_3 = 0\,, \tag{3.174}$$
$$\delta\tilde{\varphi}_4'' + \frac{1}{r}\delta\tilde{\varphi}_4' - \frac{2}{g_L^2}\delta\tilde{\varphi}_4 + \frac{1}{g_{L,R}^2}\delta\tilde{\varphi}_4 = 0\,, \tag{3.175}$$

and the equations of motion for $\delta a^R\,,\delta\chi\,,\delta\tilde{\chi}\,,\delta\chi_3\,,\delta\tilde{\chi}_3\,,\delta\tilde{\chi}_4$ can be obtain by replacing $g_L \to g_R\,,\delta a^L \to \delta a^R\,,\delta\varphi \to \delta\chi\,,\delta\tilde{\varphi} \to \delta\tilde{\chi}$ in the equations above. We note that we have assumed that $(\frac{\delta a^L}{r})^2\,,(\frac{\delta a^R}{r})^2$ and $(\frac{\delta a^{L,R}}{r})^2$ are subleading compared to the complex and real scalar fields in order to get the above equations.

Since the equations (3.114-3.175) are linear differential equations, we can find the exact solutions for each equations. For simplicity, let us consider the case $g_L = g_R = \sqrt{2}g_{L,R} = \tilde{g}$, for the solutions with general $g_L\,,g_R\,,g_{L,R}$, the reader is referred to the reference [24]. Equations (3.169) and (3.170) is in the same form of (3.114) and hence the gauge fields have the following asymptotic profile

$$\begin{aligned}
\delta a^L &= A^L r K_1(2gr)\\
\delta a^R &= A^R r K_1(2gr)\\
\delta a^{L,R} &= A^{L,R} r K_1(2\sqrt{2}gr)\,.
\end{aligned} \tag{3.176}$$

Following the linear operator method given in the previous section, we can construct the solutions for our scalar fields as follows

$$
\begin{aligned}
\delta\varphi &= B_1 K_0(\frac{\sqrt{2}r}{\tilde{g}}) + B_2 K_0(\frac{2\sqrt{2}r}{\tilde{g}}) + B_3 K_0(\frac{\sqrt{\alpha_1}r}{\tilde{g}}) + B_4 K_0(\frac{\sqrt{\alpha_2}r}{\tilde{g}})\,, \\
\delta\tilde{\varphi} &= B_1 K_0(\frac{\sqrt{2}r}{\tilde{g}}) + B_2 K_0(\frac{2\sqrt{2}r}{\tilde{g}}) - B_3 K_0(\frac{\sqrt{\alpha_1}r}{\tilde{g}}) - B_4 K_0(\frac{\sqrt{\alpha_2}r}{\tilde{g}})\,, \\
\delta\varphi_3 &= B_1 K_0(\frac{\sqrt{2}r}{\tilde{g}}) - 2B_2 K_0(\frac{2\sqrt{2}r}{\tilde{g}})\,, \\
\delta\tilde{\varphi}_3 &= B_3' K_0(\frac{\sqrt{\alpha_1}r}{\tilde{g}}) + B_4' K_0(\frac{\sqrt{\alpha_2}r}{\tilde{g}})\,, \\
\delta\chi &= C_1 K_0(\frac{\sqrt{2}r}{\tilde{g}}) + C_2 K_0(\frac{2\sqrt{2}r}{\tilde{g}}) + C_3 K_0(\frac{\sqrt{\alpha_1}r}{\tilde{g}}) + C_4 K_0(\frac{\sqrt{\alpha_2}r}{\tilde{g}})\,, \\
\delta\tilde{\chi} &= C_1 K_0(\frac{\sqrt{2}r}{\tilde{g}}) + C_2 K_0(\frac{2\sqrt{2}r}{\tilde{g}}) - C_3 K_0(\frac{\sqrt{\alpha_1}r}{\tilde{g}}) - C_4 K_0(\frac{\sqrt{\alpha_2}r}{\tilde{g}})\,, \\
\delta\chi_3 &= C_1 K_0(\frac{\sqrt{2}r}{\tilde{g}}) - 2C_2 K_0(\frac{2\sqrt{2}r}{\tilde{g}})\,, \\
\delta\tilde{\chi}_3 &= C_3' K_0(\frac{\sqrt{\alpha_1}r}{\tilde{g}}) + C_4' K_0(\frac{\sqrt{\alpha_2}r}{\tilde{g}})\,,
\end{aligned}
$$

where $\alpha_1 = 6 + 2\sqrt{3}$ and $\alpha_2 = 6 - 2\sqrt{3}$ and for the fields $\tilde{\varphi}_4$ and $\tilde{\chi}_4$, we have no fluctuations. Taking $(\frac{\delta a^L}{r})^2\,, (\frac{\delta a^R}{r})^2$ and $(\frac{\delta a^{L,R}}{r})^2$ are subleading to the complex and real scalars implies a condition between the coupling constant g and $\tilde{g}$ such that $4g > \frac{\sqrt{2}}{\tilde{g}}$. Since, the leading terms for the solutions both scalar fields and gauge fields are the same as in the case explained in the subsection (3.3.4.1), we have the same condition for the attractive and repulsive forces between vortices such that

$$
\begin{aligned}
&\text{attractive for} \quad g\tilde{g} > \frac{\sqrt{2}}{2}\,, \\
&\text{repulsive for} \quad \frac{\sqrt{2}}{4} < g\tilde{g} < \frac{\sqrt{2}}{2}\,.
\end{aligned} \tag{3.177}
$$

For the standard Yang-Mills theory $g\tilde{g} = 1$ in (3.144), we have attractive forces between vortices.

3.4.4.2 Case *ii*)

Now, let us examine the vortex type solutions for the reduced action with large but finite $\ell_L\,, \ell_R$ and $\mathfrak{g}_L\,, \mathfrak{g}_R \to \infty$. This corresponds to impose two constraints $\Phi_a^L \Phi_a^L + \ell_L(\ell_L + 1) = 0$ and $\Phi_a^R \Phi_a^R + \ell_R(\ell_R + 1) = 0$. It can be easily seen from

(A.13) that these two equations is satisfied if we have

$$
\begin{aligned}
&\mathcal{Y}_1^L = 0\,, \quad \mathcal{Y}_2^L = 0\,, \quad \tilde{\mathcal{Y}}_1^L = 0\,, \quad \tilde{\mathcal{Y}}_2^L = 0\,, \\
&\mathcal{Y}_1^R = 0\,, \quad \mathcal{Y}_2^R = 0\,, \quad \tilde{\mathcal{Y}}_1^R = 0\,, \quad \tilde{\mathcal{Y}}_2^R = 0\,.
\end{aligned} \tag{3.178}
$$

These equations enable us to solve the real scalar fields in terms of the complex scalars in powers of $\frac{1}{\ell_L}$ and $\frac{1}{\ell_R}$ as

$$
\begin{aligned}
\varphi_3 &= \frac{1}{\ell_L^2}(|\varphi|^2 + |\tilde{\varphi}|^2 - \frac{1}{2})\,, \quad \varphi_4 = -\frac{1}{\ell_L}(|\varphi|^2 + |\tilde{\varphi}|^2 - \frac{1}{2})\,, \\
\tilde{\varphi}_3 &= \frac{1}{\ell_L^2}(|\varphi|^2 - |\tilde{\varphi}|^2)\,, \quad \tilde{\varphi}_4 = -\frac{1}{\ell_L}(|\varphi|^2 - |\tilde{\varphi}|^2)\,,
\end{aligned} \tag{3.179}
$$

where $\chi_3\,, \chi_4\,, \tilde{\chi}_3\,, \tilde{\chi}_4$ can be obtained by $\varphi \to \chi$ and $\tilde{\varphi} \to \tilde{\chi}$. Let us expand the ℓ_L and ℓ_R dependent coefficients in the reduced action (3.158) to the order $\frac{1}{\ell_L^2}\,, \frac{1}{\ell_R^2}$ and substitute the approximate solutions (3.179) in it. Then, we obtain

$$
\begin{aligned}
\mathcal{L}_F &= \frac{1}{16g^2}\bigg(f^L_{\mu\nu} f^{L\mu\nu} + f^R_{\mu\nu} f^{R\mu\nu} + f^{L,R}_{\mu\nu} f^{L,R\mu\nu} + h_{\mu\nu}h^{\mu\nu} \\
&\quad + \frac{\big(f^L_{\mu\nu} f^{R\mu\nu} - h_{\mu\nu} f^{L,R\mu\nu}\big)}{2\ell_L\ell_R} + \big(f^R_{\mu\nu} f^{L,R\mu\nu} - f^L_{\mu\nu}h^{\mu\nu}\big)(\frac{1}{2\ell_L} - \frac{1}{4\ell_L^2}) \\
&\quad + \big(f^L_{\mu\nu} f^{L,R\mu\nu} - f^R_{\mu\nu}h^{\mu\nu}\big)(\frac{1}{2\ell_R} - \frac{1}{4\ell_R^2})\bigg)\,, \\
\mathcal{L}_G^L &= (1 + \frac{1}{2\ell_R} - \frac{1}{4\ell_R^2} - \frac{1}{4\ell_L^2})|D_\mu\varphi|^2 + (1 - \frac{1}{2\ell_R} + \frac{1}{4\ell_R^2} - \frac{1}{4\ell_L^2})|D_\mu\tilde{\varphi}|^2 \\
&\quad + \frac{1}{2\ell_L^2}\left((\partial_\mu|\varphi|^2)^2 + (\partial_\mu|\tilde{\varphi}|^2)^2\right)\,, \\
\mathcal{L}_G^R &= (1 + \frac{1}{2\ell_L} - \frac{1}{4\ell_L^2} - \frac{1}{4\ell_R^2})|D_\mu\chi|^2 + (1 - \frac{1}{2\ell_L} + \frac{1}{4\ell_L^2} - \frac{1}{4\ell_R^2})|D_\mu\tilde{\chi}|^2 \\
&\quad + \frac{1}{2\ell_R^2}\left((\partial_\mu|\chi|^2)^2 + (\partial_\mu|\tilde{\chi}|^2)^2\right)\,, \\
V_1^L &= \frac{4}{g_L^2}(1 + \frac{5}{4\ell_L^2})\left((|\varphi|^2 - \frac{1}{4})^2 + (|\tilde{\varphi}|^2 - \frac{1}{4})^2\right)\,, \\
V_1^R &= \frac{4}{g_R^2}(1 + \frac{5}{4\ell_R^2})\left((|\chi|^2 - \frac{1}{4})^2 + (|\tilde{\chi}|^2 - \frac{1}{4})^2\right)\,, \\
V_1^{L,R} &= \frac{1}{g_{L,R}^2}\bigg(2|\varphi\tilde{\chi} - \tilde{\varphi}\chi|^2 + 2|\bar{\chi}\varphi - \tilde{\varphi}\bar{\tilde{\chi}}|^2 + \frac{1}{2\ell_R^2}(|\varphi|^2 + |\tilde{\varphi}|^2)(|\chi|^2 - |\tilde{\chi}|^2)^2 \\
&\quad + \frac{1}{2\ell_L^2}(|\chi|^2 + |\tilde{\chi}|^2)(|\varphi|^2 - |\tilde{\varphi}|^2)^2\bigg)\,.
\end{aligned} \tag{3.180}
$$

The equation of motion for b_μ can be solved as follows

$$
h_{\mu\nu} = \left(\frac{1}{2\ell_L} - \frac{1}{4\ell_L^2}\right) f^L_{\mu\nu} + \left(\frac{1}{2\ell_R} - \frac{1}{4\ell_R^2}\right) f^R_{\mu\nu} + \frac{1}{4\ell_L\ell_R} f^{L,R}_{\mu\nu}\,. \tag{3.181}
$$

Inserting this solution for $h_{\mu\nu}$ into $\mathcal{L}_F$ and making the rotationally symmetric ansatz (3.166) and (3.167), our reduced action becomes

$$
\begin{aligned}
S =& 2\pi \int_0^\infty dr \Bigg[\frac{1}{8g^2} \Bigg(\frac{1}{r}(1-\frac{1}{4\ell_L^2}) a_\theta^{L'} a_\theta^{L'} + \frac{1}{r}(1-\frac{1}{4\ell_R^2}) a_\theta^{R'} a_\theta^{R'} + \frac{1}{r} a_\theta^{L,R'} a_\theta^{L,R'} \\
&+ \frac{1}{2r}\left(\frac{1}{\ell_R} - \frac{1}{2\ell_R^2}\right) a_\theta^{L'} a_\theta^{L,R'} + \frac{1}{2r}\left(\frac{1}{\ell_L} - \frac{1}{2\ell_L^2}\right) a_\theta^{R'} a_\theta^{L,R'} \Bigg) \\
&+ (1+\frac{1}{2\ell_R} - \frac{1}{4\ell_R^2} - \frac{1}{4\ell_L^2}) \left(r\varphi'^2 + \frac{1}{r}(n_1 + a_\theta^L + a_\theta^{L,R})^2 \varphi^2 \right) \\
&+ (1-\frac{1}{2\ell_R} + \frac{1}{4\ell_R^2} - \frac{1}{4\ell_L^2}) \left(r\tilde{\varphi}'^2 + \frac{1}{r}(n_2 + a_\theta^L - a_\theta^{L,R})^2 \tilde{\varphi}^2 \right) \\
&+ (1+\frac{1}{2\ell_L} - \frac{1}{4\ell_L^2} - \frac{1}{4\ell_R^2}) \left(r\chi'^2 + \frac{1}{r}(n_3 + a_\theta^R + a_\theta^{L,R})^2 \chi^2 \right) \\
&+ (1-\frac{1}{2\ell_L} + \frac{1}{4\ell_L^2} - \frac{1}{4\ell_R^2})) \left(r\tilde{\chi}'^2 + \frac{1}{r}(n_3 - (n_1 - n_2) + a_\theta^R - a_\theta^{L,R})^2 \tilde{\chi}^2 \right) \\
&+ \frac{2r}{\ell_L^2}\left(\varphi^2\varphi'^2 + \tilde{\varphi}^2\tilde{\varphi}'^2\right) + \frac{2r}{\ell_R^2}\left(\chi^2\chi'^2 + \tilde{\chi}^2\tilde{\chi}'^2\right) \\
&+ \frac{4r}{g_L^2}(1+\frac{5}{4\ell_L^2}) \left((\varphi^2 - \frac{1}{4})^2 + (\tilde{\varphi}^2 - \frac{1}{4})^2 \right) \\
&+ \frac{4r}{g_R^2}(1+\frac{5}{4\ell_R^2}) \left((\chi^2 - \frac{1}{4})^2 + (\tilde{\chi}^2 - \frac{1}{4})^2 \right) \\
&+ \frac{r}{g_{L,R}^2} \Bigg(2(\varphi\tilde{\chi} - \tilde{\varphi}\chi)^2 + 2(\chi\varphi - \tilde{\chi}\tilde{\varphi})^2 \\
&+ \frac{1}{2\ell_R^2}(\varphi^2 + \tilde{\varphi}^2)(\chi^2 - \tilde{\chi}^2)^2 + \frac{1}{2\ell_R^L}(\chi^2 + \tilde{\chi}^2)(\varphi^2 - \tilde{\varphi}^2)^2 \Bigg] .
\end{aligned}
\tag{3.182}
$$

Euler-Lagrange equations for this action can be found by straightforward calculations but these are nonlinear coupled differential equations. In a manner similar to the one given in section 3.3.4, we can obtain their approximate solutions by focusing on two regions. The profile of our fields around $r = 0$ are the same as (3.165). For large r, the solution for our fields can be found as follows

$$
\begin{aligned}
\delta a^L &= A_1^L r K_1(2gr) + A_2^L r K_1\left(2g\left(1 + \frac{1}{4}\left(\frac{1}{\ell_L^2} + \frac{1}{\ell_R^2}\right)\right) r\right) , \\
\delta a^R &= A_1^R r K_1(2gr) + A_2^R r K_1\left(2g\left(1 + \frac{1}{4}\left(\frac{1}{\ell_L^2} + \frac{1}{\ell_R^2}\right)\right) r\right) , \\
\delta a^{L,R} &= A^{L,R} r K_1\left(2\sqrt{2}g\left(1 - \frac{3}{8}\left(\frac{1}{\ell_L^2} + \frac{1}{\ell_R^2}\right)\right) r\right) ,
\end{aligned}
\tag{3.183}
$$

$$
\begin{aligned}
\delta\varphi &= E_1 K_0\left(\frac{\sqrt{\beta_1}}{\tilde{g}}r\right) + E_2 K_0\left(\frac{\sqrt{\beta_2}}{\tilde{g}}r\right) \, , \\
\delta\tilde{\varphi} &= E_3 K_0\left(\frac{\sqrt{\beta_1}}{\tilde{g}}r\right) + E_4 K_0\left(\frac{\sqrt{\beta_2}}{\tilde{g}}r\right) \, , \\
\delta\chi &= F_1 K_0\left(\frac{\sqrt{\gamma_1}}{\tilde{g}}r\right) + F_2 K_0\left(\frac{\sqrt{\gamma_2}}{\tilde{g}}r\right) \, , \\
\delta\tilde{\chi} &= F_3 K_0\left(\frac{\sqrt{\gamma_1}}{\tilde{g}}r\right) + F_4 K_0\left(\frac{\sqrt{\gamma_2}}{\tilde{g}}r\right) \, ,
\end{aligned}
\tag{3.184}
$$

where

$$
\begin{aligned}
\sqrt{\beta_1} &= 2\sqrt{2}\left(1+\frac{1}{4\ell_L^2}\right) \, , \quad \sqrt{\beta_2} = 2\left(1+\frac{3}{8}\left(\frac{1}{\ell_L^2}-\frac{1}{\ell_R^2}\right)\right) \, , \\
\sqrt{\gamma_1} &= 2\sqrt{2}\left(1+\frac{1}{4\ell_R^2}\right) \, , \quad \sqrt{\gamma_2} = 2\left(1+\frac{3}{8}\left(\frac{1}{\ell_R^2}-\frac{1}{\ell_L^2}\right)\right) \, ,
\end{aligned}
\tag{3.185}
$$

and $g_L = g_R = \sqrt{2} g_{L,R} := \tilde{g}$. The vortices are repulsive for the standard Yang-Mills theory in (3.144) with $g\tilde{g} = 1$.

CHAPTER 4

GAUGE THEORY OVER $\mathcal{M} \times S_F^{2\,INT} \times S_F^{2\,INT}$

In this chapter, we explore another interesting aspect of the particular deformed $N = 4$ supersymmetric Yang-Mills theory given in previous chapter. We have already given the detail on the dynamical generalization of the product of two fuzzy sphere $S_F^2 \times S_F^2$ from $SU(\mathcal{N})$ gauge theory coupled to six scalar fields in the adjoint representation of $SU(\mathcal{N})$. Here, we examine the new form of vacuum solutions of this deformed $N = 4$ SYM model which can be expressed in terms of a particular direct sum of product of fuzzy sphere, denoted by $S_F^{2\,Int} \times S_F^{2\,Int}$ [26][1]. To proceed, we suitably split the scalar fields in this model as $\Phi_a^L = \phi_a^L + \Gamma_a^L$, $\Phi_a^R = \phi_a^R + \Gamma_a^R$ where (Γ_a^L, Γ_a^R) are defined by determining the four scalar fields Ψ_α^L, Ψ_α^R $(\alpha = 1, 2)$ and their Hermitian conjugates, which are still in the adjoint of the $SU(\mathcal{N})$, but transforming under the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the global symmetry group $SO(6)$. We show that $(\Psi_\alpha^L, \Psi_\alpha^R)$ transform under the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of $SO(6)$ and (Γ_a^L, Γ_a^R) transform in the $(1, 0) \oplus (0, 1)$ representation of $SU(2) \times SU(2)$ by the suitable definition, so do (Φ_a^L, Φ_a^R). We would like to stress that the degrees of freedom is preserved for both sides of this redefinition of scalar fields and the equations of motion for $\phi_a^{L,R}$ and $\Psi_\alpha^{L\,\dagger}$ and $\Psi_\alpha^{R\,\dagger}$ simply reproduce from the variations of $\Phi_a^{L,R}$. Hence, there is no new degrees of freedom in this model. However, this redefinition provides to obtain the vacuum solution of the scalar fields which takes the form of direct sum of $S_F^2 \times S_F^2$.

To make this chapter self-contained, we would like to review of the original idea

[1] This chapter is based on the work that has been published: S. Kurkcuoglu and G. Unal "Equivariant fields in an $SU(\mathcal{N})$ gauge theory with new spontaneously generated fuzzy extra dimensions" Phys.Rev. D**93** (2016) 105019.

of the redefinition of scalar fields in terms of bilinears which was first introduced in [25]. This technique was applied to the $SU(\mathcal{N})$ gauge theory coupled to a triplet of scalar fields discussed in detail in chapter 3 and shown that it is possible to obtain the vacuum solution of scalar fields in terms of the direct sum of concentric fuzzy spheres. After a brief introduction of this technique, we are ready to apply this technique to our problem. First, with the redefinition given above, we show the vacuum configuration of scalar fields may be defined in terms of direct sum of $S_F^2 \times S_F^2$. Considering the fluctuations about this vacuum, we obtain the structure of gauge fields over $S_F^{2\,Int} \times S_F^{2\,Int}$ and enables us to conjecture that the spontaneous broken model is an effective gauge theory on the product manifold $\mathcal{M} \times S_F^{2\,Int} \times S_F^{2\,Int}$. In order to support this interpretation, we construct all of the $SU(2) \times SU(2)$-equivariant field modes by using equivariant parametrization technique. The redefinition of scalar fields in terms of bilinears (Γ_a^L, Γ_a^R) gives the interesting feature such that we find the equivariant spinor field modes which do not constitute independent dynamical degrees of freedom in the $U(4)$ effective gauge theory, but it is readily conceived that their suitable bilinears shall yield the equivariant gauge field modes on $S_F^{2\,Int} \times S_F^{2\,Int}$.

In addition, with this redefinition, the monopole sectors with non-vanishing winding number are accessed after certain projections of $\mathcal{M} \times S_F^{2\,Int} \times S_F^{2\,Int}$. We obtain the monopole sectors with winding numbers $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 1, \pm 1)$ from $S_F^{2\,Int} \times S_F^{2\,Int}$ and the equivariant fields in these sectors as a subset of those of the parent model. The latter characterizes the low energy modes of the theory and making contact with the results of [24] studied in detail in the section 3.4, we show that tracing over the fuzzy monopole sectors is bound to yield two decoupled Abelian Higgs-type models, each with a $U(1)^3$ gauge symmetry and static multivortex solutions characterized by three winding numbers. It seems possible to examine the splitting of the fields (Φ_a^L, Φ_a^R) with the composite part involving a $k_1 + k_2$ component multiplets transforming under the representation $(\frac{k_1-1}{2}, 0) \oplus (0, \frac{k_2-1}{2})$ of the global symmetry and determine a family of fuzzy vacuum solutions. It is manifestly seen from our results that suitable projections of these vacuum solutions yield all higher winding number monopole sectors.

An unexpected feature of the vacuum configuration $S_F^{2\,Int} \times S_F^{2\,Int}$ that it identifies with the bosonic part of the product of two fuzzy superspheres with $OSP(2,2) \times OSP(2,2)$ supersymmetry. We present it by examining the decomposition of typical superspin IRRs of $OSP(2,2) \times OSP(2,2)$ under $SU(2) \times SU(2)$ IRR and how a particular typical IRR of this group matches with the $SU(2) \times SU(2)$ IRR content of $S_F^{2\,Int} \times S_F^{2\,Int}$. In addition, we also give a construction of the generators of $OSP(2,2) \times OSP(2,2)$ in its nine-dimensional fundamental atypical representation, by projecting a relevant set of 16×16 matrices, which appear in our model as building blocks in the construction of the matrix algebra of the composite fields.

In the last section of this chapter, we discuss another vacuum solution to this model. Although, the structure we encounter looks superficially similar to the one obtained in section 4.2, we find that there is in fact a crucial difference; namely that the objects whose bilinears are Γ_a^L and Γ_a^R, do not transform as $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of $SU(2) \times SU(2)$. Nevertheless, treating this model as one in its own right we examine it in some detail.

4.1 Review of Gauge Theory over $M \times S_F^{2\,Int}$

In this section, let us briefly give the idea of obtaining the vacuum configuration as the direct sum of fuzzy sphere by splitting of scalar fields as

$$S_F^{2\,Int} := S_F^2(\ell) \oplus S_F^2(\ell) \oplus S_F^2\left(\ell + \frac{1}{2}\right) \oplus S_F^2\left(\ell - \frac{1}{2}\right) \,, \tag{4.1}$$

studied in [25]. Consider the $SU(\mathcal{N})$ gauge theory given in the section 3.1. Let us take the action (3.2) as

$$S = \int_{\mathcal{M}} d^d y \, \mathrm{Tr}\Big(- \frac{1}{4g^2} F_{\mu\nu}^\dagger F^{\mu\nu} - (D_\mu \Phi_a)^\dagger (D^\mu \Phi_a) \Big) - \frac{1}{\tilde{g}^2} \mathrm{Tr}\big(F_{ab}^\dagger F_{ab}\big) \,. \tag{4.2}$$

We note that in this case V_2 term is omitted from the action and the reason of its absence will be clear after the determining solution for the minimum potential. The structure of the vacuum (4.1) was revealed by performing the field redefinition

$$\Phi_a = \phi_a + \Gamma_a \,, \quad \Gamma_a = -\frac{i}{2} \Psi^\dagger \tilde{\tau}_a \Psi \,, \tag{4.3}$$

where

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} , \quad \Psi_\alpha \in \mathrm{Mat}(\mathcal{N}) , \quad \alpha = 1, 2 , \tag{4.4}$$

is a doublet of the global $SU(2)$ symmetry of the action (4.2). Here, ϕ_a, Ψ_α and Γ_a are all transforming adjointly under $SU(\mathcal{N})$ and $\tilde{\tau}_a = \tau_a \otimes 1_{\mathcal{N}}$.

We note that (4.3) is indeed a reparametrization of the fields Φ_a. Let us make this fact clear by determining degrees of freedom in (4.3). Noting that ϕ_a and Ψ_α are $\mathcal{N} \times \mathcal{N}$ matrices and they are anti-Hermitian, we can see that ϕ_a have $6\mathcal{N}^2$ real degrees of freedom and $3\mathcal{N} + 3(\mathcal{N}^2 - \mathcal{N})$ constraints, so they have $3\mathcal{N}^2$ real degrees of freedom while Ψ have $4\mathcal{N}^2$ real degrees of freedom in total. However, what enters into the definition of Γ_a are the equivalence classes $\Psi \sim U\Psi$, $U \in SU(\mathcal{N})$, as it can readily be observed that Γ_a are invariant under the left action $U\Psi$ of $SU(\mathcal{N})$ on Ψ. Since the unitary matrices $U \in SU(\mathcal{N})$ have $\mathcal{N}^2$ real degrees of freedom, it is thus clear that Γ_a have in total $4\mathcal{N}^2 - \mathcal{N}^2 = 3\mathcal{N}^2$ degrees of freedom [26].

As we mentioned earlier, since the potential term V_1 is positive definite, we have the minimum potential condition as $F_{ab} = 0$. This condition indicates that the vacuum configuration satisfying the minimum potential condition might carry any reducible representation of $SU(2)$. Hence, it is possible to obtain the vacuum configuration (4.1) by the suitable choice of ϕ_a and Γ_a. However, the presence of the potential term $V_2(\Phi)$ in (3.8) restrict the vacuum configuration in an irreducible representation of $SU(2)$ by the minimum potential condition $-\Phi_a\Phi_a = \tilde{b}$. Therefore, we omit the potential term V_2 in order to obtain the vacuum configuration consists of the direct sum of fuzzy sphere (4.1). Nevertheless, it is possible to impose it as a constraint as explained in the section 2.6.1 and given explicitly in (2.117).

With the assumption $\mathcal{N} = (2\ell + 1)4n$, we see that up to gauge transformations the vacuum configuration satisfying the minimum potential condition may be chosen as

$$\Phi_a = (X_a^{(2\ell+1)} \otimes \mathbf{1}_4 \otimes \mathbf{1}_n) + (\mathbf{1}_{2\ell+1} \otimes \Gamma_a^0 \otimes \mathbf{1}_n) , \tag{4.5}$$

where

$$\Gamma^0_a = -\frac{i}{2}\psi^\dagger \tau_a \psi\,, \tag{4.6}$$

are 4×4 matrices satisfying

$$[\Gamma^0_a, \Gamma^0_b] = \epsilon_{abc}\Gamma^0_c\,. \tag{4.7}$$

In order to determine Γ^0_a explicitly, let us define two sets of fermionic annihilation-creation operators $a_\alpha\,, a^\dagger_\alpha$

$$\{a_\alpha, a_\beta\} = 0\,, \quad \{a^\dagger_\alpha, a^\dagger_\beta\} = 0\,, \quad \{a_\alpha, a^\dagger_\beta\} = \delta_{\alpha\beta}\,. \tag{4.8}$$

These operators span the 4-dimensional Hilbert space with the basis vectors

$$|n_1\,, n_2\rangle \equiv (a^\dagger_1)^{n_1}(a^\dagger_2)^{n_2}|0\,,0\rangle\,, \quad n_1\,, n_2 = 0\,,1\,. \tag{4.9}$$

It can be seen that if the two-component spinor ψ is taken as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\,, \tag{4.10}$$

then the forms of Γ^0_a's become

$$\Gamma^0_1 = -\frac{i}{2}(a^\dagger_1 a_2 + a^\dagger_2 a_1)\,, \quad \Gamma^0_2 = -\frac{1}{2}(a^\dagger_1 a_2 - a^\dagger_2 a_1)\,, \quad \Gamma^0_3 = -\frac{i}{2}(a^\dagger_1 a_1 + a^\dagger_2 a_2) \tag{4.11}$$

and they satisfy $SU(2)$ commutation relations (4.7). Therefore, the vacuum configuration (4.5) fulfill the minimum potential condition $F_{ab} = 0$. It is possible to find $SU(2)$ IRR content of Γ^0_a by defining the ladder operators as

$$\Gamma^0_+ = \Gamma^0_1 + i\Gamma^0_2 = -ia^\dagger_1 a_2\,, \quad \Gamma^0_- = \Gamma^0_1 - i\Gamma^0_2 = -ia^\dagger_2 a_1\,, \tag{4.12}$$

then we have

$$\begin{aligned} &\Gamma^0_\pm|0\,,0\rangle = 0\,, \quad \Gamma^0_\pm|1\,,1\rangle = 0\,, \quad \Gamma^0_-|0\,,1\rangle = 0\,, \\ &\Gamma^0_+|0\,,1\rangle = -i|1\,,0\rangle\,, \quad \Gamma^0_-|1\,,0\rangle = -i|0\,,1\rangle\,, \quad \Gamma^0_+|0\,,1\rangle = 0\,. \end{aligned} \tag{4.13}$$

From these equations, it can be easily seen that $SU(2)$ IRR content of Γ^0_a includes two singlets $|0\,,0\rangle$, $|1\,,1\rangle$ which can be distinguished in terms of the eigenvalues of number operator $N = a^\dagger_\alpha a_\alpha$ and a doublet. Therefore, IRR content of Γ^0_a is

$$0_{\mathbf{0}} \oplus 0_{\mathbf{2}} \oplus \frac{1}{2}\,, \tag{4.14}$$

where $0_{\mathbf{0}}\,,0_{\mathbf{2}}$ stand for the eigenvalues of the number operator which take the values 0 and 2, respectively. It is easy to see that the projections to the singlet and doublet subspaces respectively may be found on these representations as

$$
\begin{aligned}
P_0 &= 1 - N + 2N_1N_2\,, \\
P_{0_{\mathbf{0}}} &= -\frac{1}{2}(N-2)P_0 = 1 - N + N_1N_2\,, \\
P_{0_{\mathbf{2}}} &= \frac{1}{2}NP_0 = N_1N_2 = -\frac{1}{2}N + \frac{1}{2}P_{\frac{1}{2}}\,, \\
P_{\frac{1}{2}} &= N - 2N_1N_2\,,
\end{aligned} \tag{4.15}
$$

where $N = N_1 + N_2\,, N_1 = a_1^\dagger a_1\,, N_2 = a_2^\dagger a_2$.

$SU(2)$ IRR content of vacuum configuration (4.5) can be derived from the Clebsch-Gordan decomposition as

$$
\ell \otimes \left(0_{\mathbf{0}} \oplus 0_{\mathbf{2}} \oplus \frac{1}{2}\right) \equiv \ell \oplus \ell \oplus \left(\ell + \frac{1}{2}\right) \oplus \left(\ell - \frac{1}{2}\right)\,, \quad \ell \neq 0\,. \tag{4.16}
$$

This indicated that the vacuum configuration (4.1) can be interpreted as a direct sum of four concentric fuzzy spheres as it has been already discussed and shown that it is also possible to conjecture that after the spontaneously symmetry breaking, the emergent model is an effective gauge theory on $\mathcal{M} \times S_F^2$ in [25]. In order to find the most general $SU(2)$-equivariant gauge field modes, the $SU(2)$ symmetry generators ω_a may be chosen as

$$
\begin{aligned}
\omega_a =& (X_a^{(2\ell+1)} \otimes \mathbf{1}_4 \otimes \mathbf{1}_2) + (\mathbf{1}_{2\ell+1} \otimes \Gamma_a^0 \otimes \mathbf{1}_2) - (\mathbf{1}_{2\ell+1} \otimes \mathbf{1}_4 \otimes \frac{i}{2}\tau_a) \\
=:& X_a + \Gamma_a^0 - \frac{i}{2}\tau_a \\
=:& D_a - \frac{i}{2}\tau_a\,, \qquad \omega_a \in u(2\ell+1) \otimes u(4) \otimes u(2)\,,
\end{aligned} \tag{4.17}
$$

and satisfying (3.78). ω_a carries a direct sum of IRRs of $SU(2)$, which is given as

$$
\begin{aligned}
\left(\ell \oplus \ell \oplus \left(\ell + \frac{1}{2}\right) \oplus \left(\ell - \frac{1}{2}\right)\right) \otimes \frac{1}{2} \equiv (\ell - 1) \oplus \mathbf{2}\left((\ell + \frac{1}{2}) \oplus (\ell - \frac{1}{2})\right) \\
+ \mathbf{2}\ell + (\ell + 1)\,.
\end{aligned} \tag{4.18}
$$

Projections to the representations appearing in the r.h.s of (4.18) is given in the table 4.1, where

Projector	Representation
$\Pi_{0_\mathbf{0}} = \mathbf{1}_{2\ell+1} \otimes P_{0_\mathbf{0}} \otimes \mathbf{1}_2$	$(\ell - \frac{1}{2}) \oplus (\ell + \frac{1}{2})$
$\Pi_{0_\mathbf{2}} = \mathbf{1}_{2\ell+1} \otimes P_{0_\mathbf{2}} \otimes \mathbf{1}_2$	$(\ell - \frac{1}{2}) \oplus (\ell + \frac{1}{2})$
$\Pi_{+} = \frac{1}{2}(iQ_I + \Pi_{\frac{1}{2}})$	$\ell \oplus (\ell + 1)$
$\Pi_{-} = \frac{1}{2}(-iQ_I + \Pi_{\frac{1}{2}})$	$\ell \oplus (\ell - 1)$
$\Pi_0 = \Pi_{0_\mathbf{0}} + \Pi_{0_\mathbf{2}} = \mathbf{1}_{2\ell+1} \otimes P_0 \otimes \mathbf{1}_2$	$\mathbf{2}\Big((\ell - \frac{1}{2}) \oplus (\ell + \frac{1}{2}) \Big)$
$\Pi_{\frac{1}{2}} = \Pi_+ + \Pi_- = \mathbf{1}_{2\ell+1} \otimes P_{\frac{1}{2}} \otimes \mathbf{1}_2$	$(\ell - 1) \oplus \mathbf{2}\ell \oplus (\ell + 1)$

Table4.1: Projections to the representations appearing in the r.h.s of (4.18).

$$Q_I = \frac{i}{\frac{1}{2}(\ell + \frac{1}{2})}(X_a \Gamma_a - \frac{1}{4}\Pi_{\frac{1}{2}}) \, , \quad Q_I^2 = -\Pi_{\frac{1}{2}} \, . \tag{4.19}$$

$SU(2)$-equivariant gauge fields can be obtained by imposing the symmetry constraints in (3.79), (3.80) and the additional constraint

$$[\omega_a \, , \Psi_\alpha] = \frac{i}{2}(\tilde{\tau}_a)_{\alpha\beta}\Psi_\beta \, . \tag{4.20}$$

The dimensions of solution spaces for $A_\mu \, , A_a$ and Ψ_α can be derived by the Clebsch-Gordan decomposition of the adjoint action of ω_a. The relevant part of this decomposition is

$$\Big[(\ell - 1) \oplus \mathbf{2}\Big((\ell + \frac{1}{2}) \oplus (\ell - \frac{1}{2}) \Big) + \mathbf{2}\ell + (\ell + 1) \Big]^{\otimes 2}$$
$$\equiv \mathbf{14}\, 0 \oplus \mathbf{24}\, \frac{1}{2} \oplus \mathbf{30}\, 1 \oplus \cdots \, . \tag{4.21}$$

This means that under the adjoint action of ω_a, there are 14 objects which transform as scalars. All these equivariant scalars can be constructed by using the projectors given in the table 4.1 and the suitable projection of the "idempotent"

$$Q_B = \frac{X_a \otimes \mathbf{1}_4 \otimes \tau_a - \frac{i}{2}\mathbf{1}_{(2\ell+1)8}}{\ell + \frac{1}{2}} \, , \tag{4.22}$$

and using the rotational invariants, it is possible to construct the equivariant vectors and spinors as well. We omit the explicit form of these equivariant gauge fields here and refer the interested reader to the original literature [25]. However, we would like to stress that in that article, it was shown that after suitable projections, the monopole sectors with winding number ± 1 can be accessed and the reduced model yields two decoupled Abelian Higgs-type models as found in subsection 3.3.1 and the vortex solutions determined in the subsection 3.3.4 are valid within each sector. Now, we are ready to apply this technique on the deformed $N = 4$ SYM theory given in the section 3.4.

4.2 New Fuzzy Extra Dimensions from $SU(\mathcal{N})$ Gauge Theory

Let us consider the deformed $N = 4$ SYM theory with $SU(\mathcal{N})$ gauge symmetry given in the section 3.4 and take the potential terms as follows

$$\begin{aligned}
&V(\Phi^L) = Tr_{\mathcal{N}} F^{L\dagger}_{ab} F^L_{ab} \,, \quad F^L_{ab} = [\Phi^L_a, \Phi^L_b] - \epsilon_{abc}\Phi^L_c \,, \\
&V(\Phi^R) = Tr_{\mathcal{N}} F^{R\dagger}_{ab} F^R_{ab} \,, \quad F^R_{ab} = [\Phi^R_a, \Phi^R_b] - \epsilon_{abc}\Phi^R_c \,, \\
&V(\Phi^{L,R}) = Tr_{\mathcal{N}} F^{(L,R)\dagger}_{ab} F^{(L,R)}_{ab} \,, \quad F^{(L,R)}_{ab} = [\Phi^L_a, \Phi^R_b] \,, \\
&\Phi^L_a = \Phi_a \,, \quad \Phi^R_a = \Phi_{a+3} \,, \quad (a = 1,2,3) \,,
\end{aligned} \tag{4.23}$$

where the potential terms V_2^L and V_2^R in (3.133) are again absent because of the same reason given in the previous section but we will impose these terms as constraints.

Following and generalizing the developments in [25], we are going to consider that Φ^L_a and Φ^R_a are split in the form

$$\Phi^L_a = \phi^L_a + \Gamma^L_a \,, \quad \Phi^R_a = \phi^R_a + \Gamma^R_a \,, \tag{4.24}$$

with the definitions

$$\Gamma^L_a = -\frac{i}{2} \Psi^{L\dagger} \tilde{\tau}_a \Psi^L \,, \quad \Gamma^R_a = -\frac{i}{2} \Psi^{R\dagger} \tilde{\tau}_a \Psi^R \,, \tag{4.25}$$

where the scalar fields Ψ^L and Ψ^R are doublets of the global $SU(2)_L \times SU(2)_R$, transforming under its IRRs $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively. Thus, we may form the 4-component multiplet

$$\Psi = \begin{pmatrix} \Psi^L \\ \Psi^R \end{pmatrix} = \begin{pmatrix} \Psi^L_1 \\ \Psi^L_2 \\ \Psi^R_1 \\ \Psi^R_2 \end{pmatrix} \,, \tag{4.26}$$

transforming under the representation $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ of the global symmetry group. We have that all the components $(\Psi^L_\alpha\,,\Psi^R_\alpha)$ $(\alpha = 1,2)$ of Ψ are scalar fields; they are $\mathcal{N} \times \mathcal{N}$ matrices, transforming adjointly $(\Psi^{L,R}_\alpha \to U^\dagger \Psi^{L,R}_\alpha U)$ under $SU(\mathcal{N})$. Clearly, then $(\Gamma^L_a\,,\Gamma^R_a)$ are bilinears of Ψ's transforming under the $(1,0) \oplus (0,1)$ of $SU(2)_L \times SU(2)_R$. Under the $SU(\mathcal{N})$ gauge symmetry $(\Gamma^L_a\,,\Gamma^R_a)$ transform adjointly $(\Gamma^{L,R}_a \to U^\dagger \Gamma^{L,R}_a U)$ as expected.

Note that there are $6\mathcal{N}^2$ real degrees of freedom in $(\Phi^L_a\,,\Phi^R_a)$ and the doublets Ψ^L and Ψ^R have $4\mathcal{N}^2$ real degrees of freedom each. Just like the previous case, here we also have equivalence classes for Ψ^L and Ψ^R. It is readily observed that, under the left action $\Psi^L \to U\Psi^L$, $\Psi^R \to V\Psi^R$, with $U, V \in SU(\mathcal{N})$, we have $(\Gamma^L_a\,,\Gamma^R_a)$ remaining invariant. Thus, what essentially enters into the definition of $(\Gamma^L_a\,,\Gamma^R_a)$ are the equivalence classes $(\Psi^L\,,\Psi^R) \sim (U\Psi^L\,,V\Psi^R)$. Since each of the unitary matrices $U\,,V \in SU(\mathcal{N})$ have $\mathcal{N}^2$ real degrees of freedom, this means that each of Γ^L_a and Γ^R_a has $4\mathcal{N}^2 - \mathcal{N}^2 = 3\mathcal{N}^2$ real degrees of freedom, which yields exactly the same $6\mathcal{N}^2$ real degrees of freedom in $(\Gamma^L_a\,,\Gamma^R_a)$ as in $(\Phi^L_a\,,\Phi^R_a)$.

In fact, it can also be shown in a straightforward manner that the variations with respect to $\phi^{L,R}_a$, $\Psi^{L\,\dagger}_\alpha$ and $\Psi^{R\,\dagger}_\alpha$ simply reproduce the same equations of motion as those that emerge from the variations[2] of $\Phi^{L,R}_a$ indicating that no new degrees of freedom are introduced into the model by (4.24). This splitting is rather premature as it lacks any physical motivation at the present stage, but our reasons will become clear as we move forward and show that the model spontaneously develops fuzzy extra dimensions, which may be written as direct sums of the products $S^2_F \times S^2_F$ as we shall now demonstrate.

4.2.1 The Vacuum Configuration

With the absence of V^L_2 and V^R_2, the conditions for minimum of the potential terms (3.135) reduces to

$$F^L_{ab} = 0\,, \quad F^R_{ab} = 0\,, \quad F^{L,R}_{ab} = 0\,. \tag{4.27}$$

[2] See Appendix B for details.

These equations indicate that the solution for minimum of potential are given by $\mathcal{N} \times \mathcal{N}$ matrices carrying reducible representations of $SU(2) \times SU(2)$ that decompose into direct sums of its IRRs. We want to consider such a solution to the equations (4.27) in which we can take advantage of the splitting of the fields indicated in (4.24) and (4.25) in its construction. Let us emphasize that, the particular vacuum solution we want to construct this way exists regardless of our use of relations given in (4.24) and (4.25) as it is clear from our initial remark. Keeping these in mind, we can proceed to observe that the requirements in (4.24) and (4.25) naturally restrict the possible $SU(2)_L \times SU(2)_R$ representation that (Γ_a^L, Γ_a^R) may carry to the one for which $(\Psi_\alpha^L, \Psi_\alpha^R)$ exists. In other words, (Γ_a^L, Γ_a^R) may not be in some arbitrary representation of $SU(2) \times SU(2)$, since then the corresponding $(\Psi_\alpha^L, \Psi_\alpha^R)$ will not exist in general. Here we consider the only possible solution for which both (ϕ_a^L, ϕ_a^R) and (Γ_a^L, Γ_a^R) are nonzero matrices.

We are going to show that the solution fulfilling the equations in (4.27) with the structure given in (4.24) and (4.25) may be written, assuming that $\mathcal{N}$ factors in the form $\mathcal{N} = (2\ell_L + 1) \times (2\ell_R + 1) \times 16 \times n$, as

$$\begin{aligned} \Phi_a^L &= (X_a^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_{16} \otimes \mathbf{1}_n) + (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \Gamma_a^{0\,L} \otimes \mathbf{1}_n)\,, \\ \Phi_a^R &= (\mathbf{1}^{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_{16} \otimes \mathbf{1}_n) + (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \Gamma_a^{0\,R} \otimes \mathbf{1}_n)\,, \end{aligned} \tag{4.28}$$

up to gauge transformations $\Phi_i \to U^\dagger \Phi_i U$.

$(\Gamma_a^{0\,L}, \Gamma_a^{0\,R})$ are conceived, for reasons that will become clear shortly, as 16×16 anti-Hermitian matrices which satisfy the $SU(2)_L \times SU(2)_R$ commutation relations

$$[\Gamma_a^{0\,L}, \Gamma_b^{0\,L}] = \epsilon_{abc} \Gamma_c^{0\,L}\,, \quad [\Gamma_a^{0\,R}, \Gamma_b^{0\,R}] = \epsilon_{abc} \Gamma_c^{0\,R}\,, \quad [\Gamma_a^{0\,L}, \Gamma_b^{0\,R}] = 0\,, \tag{4.29}$$

and form a reducible representation of $SU(2)_L \times SU(2)_R$.

We will now see that $\Gamma_a^{0\,L}$ and $\Gamma_a^{0\,R}$ can be written as bilinears of spinors carrying the IRR's $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively. For this purpose, let us introduce four sets of fermionic annihilation-creation operators $(b_\alpha\,, b_\alpha^\dagger\,, c_\alpha\,, c_\alpha^\dagger)$ with the anticommutation relations

$$\{b_\alpha, b^\dagger_\beta\} = \delta_{\alpha\beta}\,, \quad \{c_\alpha, c^\dagger_\beta\} = \delta_{\alpha\beta}\,, \tag{4.30}$$

and all other anticommutators vanishing. They span the sixteen-dimensional Hilbert space $\mathcal{H}$ with the basis vectors

$$|n_1,\, n_2,\, n_3,\, n_4\rangle \equiv (b^\dagger_1)^{n_1}(b^\dagger_2)^{n_2}(c^\dagger_1)^{n_3}(c^\dagger_2)^{n_4}|0,\, 0,\, 0,\, 0\rangle, \tag{4.31}$$

with $n_1, n_2, n_3, n_4 = 0, 1$.

We can now take

$$\Gamma^{0\,L}_a = -\frac{i}{2}\psi^{L\dagger}\tau_a\psi^L\,, \quad \Gamma^{0\,R}_a = -\frac{i}{2}\psi^{R\dagger}\tau_a\psi^R\,, \tag{4.32}$$

where

$$\psi^L := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\,, \quad \psi^R := \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\,. \tag{4.33}$$

It is easy to see that $(\Gamma^{0\,L}_a, \Gamma^{0\,R}_a)$ fulfill the $SU(2)_L \times SU(2)_R$ commutation relations in (4.29). We furthermore have that

$$\begin{aligned}
&[\psi^L_\alpha\,, \Gamma^{0\,L}_a] = -\frac{i}{2}(\tau_a)_{\alpha\beta}\psi^L_\beta\,, \quad [\psi^{\dagger\,L}_\alpha\,, \Gamma^{0\,L}_a] = \frac{i}{2}(\tau_a)_{\beta\alpha}\psi^{\dagger\,L}_\beta\,, \quad [\psi^L_\alpha\,, \Gamma^{0\,R}_a] = 0\,, \\
&[\psi^R_\alpha, \Gamma^{0\,R}_a] = -\frac{i}{2}(\tau_a)_{\alpha\beta}\psi^R_\beta\,, \quad [\psi^{\dagger\,R}_\alpha\,, \Gamma^{0\,R}_a] = \frac{i}{2}(\tau_a)_{\beta\alpha}\psi^{\dagger\,R}_\beta\,, \quad [\psi^R_\alpha\,, \Gamma^{0\,L}_a] = 0\,,
\end{aligned} \tag{4.34}$$

therefore ψ^L and ψ^R carry the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ IRRs of $SU(2)_L \times SU(2)_R$, respectively.

The quadratic Casimir of the representation spanned by $(\Gamma^{0\,L}_a, \Gamma^{0\,R}_a)$ may be straightforwardly calculated to give

$$C_2 = (\Gamma^{0\,L}_a)^2 + (\Gamma^{0\,R}_a)^2 = \begin{pmatrix} \mathbf{0}_4 & 0 & 0 \\ 0 & -\frac{3}{4}\mathbf{1}_8 & 0 \\ 0 & 0 & -\frac{3}{2}\mathbf{1}_4 \end{pmatrix}\,, \tag{4.35}$$

where we have used

$$(\Gamma^{0\,L}_a)^2 = -\frac{3}{4}N^L + \frac{3}{2}N^L_1 N^L_2\,, \quad (\Gamma^{0\,R}_a)^2 = -\frac{3}{4}N^R + \frac{3}{2}N^R_1 N^R_2\,, \tag{4.36}$$

with the number operators on the Hilbert space $\mathcal{H}$ given as

$$\begin{aligned} N_1^L &= b_1^\dagger b_1 \,, \quad N_2^L = b_2^\dagger b_2 \,, \quad N^L = N_1^L + N_2^L \,, \\ N_1^R &= c_1^\dagger c_1 \,, \quad N_2^R = c_2^\dagger c_2 \,, \quad N^R = N_1^R + N_2^R \,, \end{aligned} \tag{4.37}$$

and we have taken the basis vectors of $\mathcal{H}$ oriented in the order $|0000\rangle$, $|0011\rangle$ $|0001\rangle$, $|0010\rangle$, $|1100\rangle$, $|1111\rangle$, $|1101\rangle$, $|1110\rangle$, $|0100\rangle$, $|0111\rangle$, $|0101\rangle$, $|0110\rangle$, $|1000\rangle$, $|1011\rangle$, $|1001\rangle$, $|1010\rangle$.

We can construct the $SU(2)_L \times SU(2)_R$ representation content of $(\Gamma_a^{0\,L}, \Gamma_a^{0\,R})$ by the following ladder operators

$$\Gamma_+^{0\ L} = \Gamma_1^{0\,L} + i\Gamma_2^{0\,L} = -ib_1^\dagger b_2 \,, \quad \Gamma_+^{0\ R} = \Gamma_1^{0\,R} + i\Gamma_a^{0\,R} = -ic_1^\dagger c_2 \,, \tag{4.38}$$

$$\Gamma_-^{0\ L} = \Gamma_1^{0\,L} - i\Gamma_2^{0\,L} = -ib_2^\dagger b_1 \,, \quad \Gamma_-^{0\ R} = \Gamma_1^{0\,R} - i\Gamma_2^{0\,R} = -ic_2^\dagger c_1 \,. \tag{4.39}$$

Using these ladder operators and the eigenvalues of these ladder operators on the basis vectors in Hilbert space $\mathcal{H}$, we obtain

$$\begin{aligned} &|0000\rangle \,, |0011\rangle \,, |1100\rangle \,, |1111\rangle \longrightarrow (0,0) \,, \\ &|0010\rangle \,, |0001\rangle \longrightarrow (0, \tfrac{1}{2}) \,, \\ &|1110\rangle \,, |1101\rangle \longrightarrow (0, \tfrac{1}{2}) \,, \\ &|1000\rangle \,, |0100\rangle \longrightarrow (\tfrac{1}{2}, 0) \,, \\ &|1011\rangle \,, |0111\rangle \longrightarrow (\tfrac{1}{2}, 0) \,, \\ &|1010\rangle \,, |1001\rangle \,, |0110\rangle \,, |0101\rangle \longrightarrow (\tfrac{1}{2}, \tfrac{1}{2}) \,. \end{aligned} \tag{4.40}$$

Hence, $(\Gamma_a^{0\,L}, \Gamma_a^{0\,R})$ has the representation content expressed as the following direct sum of IRR's of $SU(2)_L \times SU(2)_R$:

$$\mathbf{4}(0,0) \oplus \mathbf{2}\big(\tfrac{1}{2}, 0\big) \oplus \mathbf{2}\big(0, \tfrac{1}{2}\big) \oplus \big(\tfrac{1}{2}, \tfrac{1}{2}\big) \,. \tag{4.41}$$

It is also possible to express $(\Gamma_a^{0\,L}, \Gamma_a^{0\,R})$ as

$$\Gamma_a^{0\,L} = \Gamma_a^0 \otimes \mathbf{1}_4 \,, \quad \Gamma_a^{0\,R} = \mathbf{1}_4 \otimes \Gamma_a^0 \,, \tag{4.42}$$

where Γ_a^0 given in (4.6) with the two-component spinor (4.10). Since Γ_a^0 fulfill the $SU(2)$ commutation relations, it is clear that $(\Gamma_a^{0\,L}, \Gamma_a^{0\,R})$ as defined in (4.42) fulfill the commutation relations in (4.29). It is easily observed that (4.32) and (4.42) describe unitarily equivalent representations and (4.42) indeed yields identically the same set of $({\Gamma^0_a}^L, {\Gamma^0_a}^R)$ as in Eq. (4.32) if the basis vectors of $\mathcal{H}_d$ are taken in the order $|0\,,0\rangle\,,|1\,,1\rangle\,,|0\,,1\rangle\,,|1\,,0\rangle$.

Let us consider the $SU(2)_L \times SU(2)_R$ IRR representation content of (4.28). Clebsch-Gordan decomposition gives

$$
\begin{aligned}
&(\ell_L, \ell_R) \otimes \left(\mathbf{4}(0,0) \oplus \mathbf{2}\big(\frac{1}{2},0\big) \oplus \mathbf{2}\big(0,\frac{1}{2}\big) \oplus \big(\frac{1}{2},\frac{1}{2}\big) \right) \\
&\equiv \mathbf{4}(\ell_L,\ell_R) \oplus \mathbf{2}\big(\ell_L - \frac{1}{2}, \ell_R\big) \oplus \mathbf{2}\big(\ell_L + \frac{1}{2}, \ell_R\big) \oplus \mathbf{2}\big(\ell_L, \ell_R - \frac{1}{2}\big) \oplus \mathbf{2}\big(\ell_L, \ell_R + \frac{1}{2}\big) \\
&\oplus \big(\ell_L - \frac{1}{2}, \ell_R - \frac{1}{2}\big) \oplus \big(\ell_L + \frac{1}{2}, \ell_R - \frac{1}{2}\big) \oplus \big(\ell_L - \frac{1}{2}, \ell_R + \frac{1}{2}\big) \oplus \big(\ell_L + \frac{1}{2}, \ell_R + \frac{1}{2}\big).
\end{aligned}
\tag{4.43}
$$

For convenience, we introduce the short-hand notation $D_a^L := X_a^L + {\Gamma^0_a}^L\,, D_a^R := X_a^R + {\Gamma^0_a}^R$ for the vacuum solutions (4.28)

In accordance with the decomposition in (4.43), a unitary transformation puts (D_a^L, D_a^R) into the block diagonal form $(\mathcal{D}_a^L, \mathcal{D}_a^R) \equiv (U^\dagger D_a^L U\,, U^\dagger D_a^R U)$ whose entries can be inferred from the casimir of IRR's appearing in (4.43) and their multiplicities (see Appendix B). Therefore, we may interpret the vacuum configuration of the gauge theory (3.129) with the redefinition of scalar fields (4.24) in terms of direct sums of $S_F^2 \times S_F^2$ given as

$$
\begin{aligned}
S_F^{2\,Int} \times S_F^{2\,Int} :\equiv\; & \mathbf{4}\left(S_F^2(\ell_L) \times S_F^2(\ell_R)\right) \oplus \mathbf{2}\left(S_F^2(\ell_L - \frac{1}{2}) \times S_F^2(\ell_R) \right) \\
& \oplus \mathbf{2}\left(S_F^2(\ell_L + \frac{1}{2}) \times S_F^2(\ell_R) \right) \oplus \mathbf{2}\left(S_F^2(\ell_L) \times S_F^2(\ell_R - \frac{1}{2}) \right) \\
& \oplus \mathbf{2}\left(S_F^2(\ell_L) \times S_F^2(\ell_R + \frac{1}{2}) \right) \oplus \left(S_F^2(\ell_L - \frac{1}{2}) \times S_F^2(\ell_R - \frac{1}{2}) \right) \\
& \oplus \left(S_F^2(\ell_L + \frac{1}{2}) \times S_F^2(\ell_R - \frac{1}{2}) \right) \oplus \left(S_F^2(\ell_L - \frac{1}{2}) \times S_F^2(\ell_R + \frac{1}{2}) \right) \\
& \oplus \left(S_F^2(\ell_L + \frac{1}{2}) \times S_F^2(\ell_R + \frac{1}{2}) \right) .
\end{aligned}
\tag{4.44}
$$

In order to obtain each summand occurring in (4.44), we have the corresponding projections given in the form

$$\Pi_{\alpha\beta} = \prod_{\gamma\neq\alpha,\,\delta\neq\beta} \frac{-(X_a^L + {\Gamma^0}_a^L)^2 - (X_a^R + {\Gamma^0}_a^R)^2 - \lambda_\gamma^L(\lambda_\gamma^L+1) - \lambda_\delta^R(\lambda_\delta^R+1)}{\lambda_\alpha^L(\lambda_\alpha^L+1) + \lambda_\beta^R(\lambda_\beta^R+1) - \lambda_\gamma^L(\lambda_\gamma^L+1) - \lambda_\delta^R(\lambda_\delta^R+1)}, \tag{4.45}$$

where $\alpha, \beta, \gamma,\, \delta \,=\, 0, +, -$ and $\lambda_\alpha^L\,, \lambda_\alpha^R$ take on the values $\ell_L\,, \ell_L \pm \frac{1}{2}\,, \ell_R\,, \ell_R \pm \frac{1}{2}$ respectively. This gives nine projectors. Note that $\Pi_{\alpha\beta}$ does not resolve the repeated summands in (4.44). For instance, Π_{00} projects to the sector $\mathbf{4}\,(S_F^2(\ell_L) \times S_F^2(\ell_R))$. We will see, how the projection to each repeated summand is accomplished as we proceed.

It is important to note that these projectors may be expressed, after a unitary transformation, in terms of the products of the projectors Π_α^L and Π_β^R, which are given as

$$\begin{aligned} \Pi_\alpha^L &= \prod_{\gamma\neq\alpha} \frac{-(X_a^L + {\Gamma^0}_a^L)^2 - \lambda_\gamma^L(\lambda_\gamma^L+1)}{\lambda_\alpha^L(\lambda_\alpha^L+1) - \lambda_\gamma^L(\lambda_\gamma^L+1)}\,, \\ \Pi_\beta^R &= \prod_{\delta\neq\beta} \frac{-(X_a^R + {\Gamma^0}_a^R)^2 - \lambda_\delta^R(\lambda_\delta^R+1)}{\lambda_\beta^R(\lambda_\beta^R+1) - \lambda_\delta^R(\lambda_\delta^R+1)}\,. \end{aligned} \tag{4.46}$$

From (4.46), we may find that Π_0^L, Π_0^R, $\Pi_\pm^L$, $\Pi_\pm^R$ take the form

$$\begin{aligned} \Pi_0^L &= \mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes P_0 \otimes \mathbf{1}_4 \otimes \mathbf{1}_n\,, \\ \Pi_0^R &= \mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_4 \otimes P_0 \otimes \mathbf{1}_n, \\ \Pi_\pm^L &= \frac{1}{2}(\pm i Q_I^L + \Pi_{\frac{1}{2}}^L)\,, \quad \Pi_\pm^R = \frac{1}{2}(\pm i Q_I^R + \Pi_{\frac{1}{2}}^R)\,, \end{aligned} \tag{4.47}$$

where

$$Q_I^L = i\frac{X_a^L {\Gamma^0}_a^L - \frac{1}{4}\Pi_{\frac{1}{2}}^L}{\frac{1}{2}(\ell_L + \frac{1}{2})}\,, \quad Q_I^R = i\frac{X_a^R {\Gamma^0}_a^R - \frac{1}{4}\Pi_{\frac{1}{2}}^R}{\frac{1}{2}(\ell_R + \frac{1}{2})}\,, \tag{4.48}$$

and $\Pi_{\frac{1}{2}}^L = \Pi_+^L + \Pi_-^L$, $\Pi_{\frac{1}{2}}^R = \Pi_+^R + \Pi_-^R$.

As $\Pi_{\alpha\beta}$ and $\Pi_\alpha^L \Pi_\beta^R$ project to the same subspaces, they are unitarily equivalent, $\Pi_{\alpha\beta} = U^\dagger \Pi_\alpha^L \Pi_\beta^R U$, for some unitary matrix U. Using the notation $\Pi_{\alpha\beta} \equiv \Pi_\alpha^L \Pi_\beta^R$ to denote this equivalence, we can list these nine projections onto the distinct IRRs in (4.43) as given in the table 4.2.

Projector	To the Representation
$\Pi_{00} \equiv \Pi_0^L \Pi_0^R$	$\mathbf{4}(\ell_L, \ell_R)$
$\Pi_{0\pm} \equiv \Pi_0^L \Pi_\pm^R$	$\mathbf{2}(\ell_L, \ell_R \pm \frac{1}{2})$
$\Pi_{\pm 0} \equiv \Pi_\pm^L \Pi_0^R$	$\mathbf{2}(\ell_L \pm \frac{1}{2}, \ell_R)$
$\Pi_{\pm\pm} \equiv \Pi_\pm^L \Pi_\pm^R$	$(\ell_L \pm \frac{1}{2}, \ell_R \pm \frac{1}{2})$
$\Pi_{\pm\mp} \equiv \Pi_\pm^L \Pi_\mp^R$	$(\ell_L \pm \frac{1}{2}, \ell_R \mp \frac{1}{2})$

Table4.2: Projections $\Pi_{\alpha\beta}$

Projector	To the Representation
$\Pi_{0_\mathbf{0}}^L \Pi_{0_\mathbf{0}}^R$	(ℓ_L, ℓ_R)
$\Pi_{0_\mathbf{0}}^L \Pi_{0_\mathbf{2}}^R$	(ℓ_L, ℓ_R)
$\Pi_{0_\mathbf{2}}^L \Pi_{0_\mathbf{0}}^R$	(ℓ_L, ℓ_R)
$\Pi_{0_\mathbf{2}}^L \Pi_{0_\mathbf{2}}^R$	(ℓ_L, ℓ_R)
$\Pi_{0_\mathbf{0}}^L \Pi_\pm^R$	$(\ell_L, \ell_R \pm \frac{1}{2})$
$\Pi_{0_\mathbf{2}}^L \Pi_\pm^R$	$(\ell_L, \ell_R \pm \frac{1}{2})$
$\Pi_\pm^L \Pi_{0_\mathbf{0}}^R$	$(\ell_L \pm \frac{1}{2}, \ell_R)$
$\Pi_\pm^L \Pi_{0_\mathbf{2}}^R$	$(\ell_L \pm \frac{1}{2}, \ell_R)$
$\Pi_\pm^L \Pi_\pm^R$	$(\ell_L \pm \frac{1}{2}, \ell_R \pm \frac{1}{2})$
$\Pi_\pm^L \Pi_\mp^R$	$(\ell_L \pm \frac{1}{2}, \ell_R \mp \frac{1}{2})$

Table4.3: Projections to all fuzzy subspaces in r.h.s. of (4.44).

It is possible to split Π_0^L to the projectors $\Pi_{0_\mathbf{0}}^L, \Pi_{0_\mathbf{2}}^L$ and Π_0^R to $\Pi_{0_\mathbf{0}}^R, \Pi_{0_\mathbf{2}}^R$, as

$$
\begin{aligned}
\Pi_{0_\mathbf{0}}^L &= \mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes P_{0_\mathbf{0}} \otimes \mathbf{1}_4 \otimes \mathbf{1}_n \,, \\
\Pi_{0_\mathbf{2}}^L &= \mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes P_{0_\mathbf{2}} \otimes \mathbf{1}_4 \otimes \mathbf{1}_n \,, \\
\Pi_{0_\mathbf{0}}^R &= \mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_4 \otimes P_{0_\mathbf{0}} \otimes \mathbf{1}_n \,, \\
\Pi_{0_\mathbf{2}}^R &= \mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_4 \otimes P_{0_\mathbf{2}} \otimes \mathbf{1}_n \,,
\end{aligned}
\tag{4.49}
$$

where $P_{0_\mathbf{0}}\,, P_{0_\mathbf{2}}$ are given in (4.15). Taking the above splitting of Π_0^L and Π_0^R into account, we can resolve $\Pi_{00}\,, \Pi_{0\pm}\,, \Pi_{\pm 0}$ into the projections, which project onto subspaces carrying a single IRR as given in table 4.3. These constitute the 16 projectors onto the fuzzy subspaces appearing in the right hand side of equation (4.44).

4.2.2 Gauge Theory over $M^4 \times S_F^{2\,Int} \times S_F^{2\,Int}$

We may now turn our attention back to the vacuum configuration (4.28). The latter breaks the $SU(\mathcal{N})$ gauge symmetry to a $U(n)$. Clearly, this is the commutant of (Φ_a^L, Φ_a^R) given in (4.28). In addition, the global symmetry is totally broken by the vacuum. However, we note that, it is still possible to combine a global rotation with a gauge transformation which leaves the vacuum invariant.

We may introduce the fluctuations (A_a^L, A_a^R) about the vacuum as

$$\begin{aligned}\Phi_a^L &= X_a^L + \Gamma_a^{0\,L} + A_a^L = D_a^L + A_a^L\,,\\ \Phi_a^R &= X_a^R + \Gamma_a^{0\,R} + A_a^R = D_a^R + A_a^R\,,\end{aligned} \tag{4.50}$$

where A_a^L, $A_a^R \in u(2\ell_L+1)\otimes u(2\ell_R+1)\otimes u(4)\otimes u(4)\otimes u(n)$.

Evaluating F_{ab}^L, F_{ab}^R, $F_{ab}^{L,R}$, we find

$$\begin{aligned}F_{ab}^L &= [D_a^L, A_b^L] - [D_b^L, A_a^L] + [A_a^L, A_b^L] - \epsilon_{abc}A_c^L\,,\\ F_{ab}^R &= [D_a^R, A_b^R] - [D_b^R, A_a^R] + [A_a^R, A_b^R] - \epsilon_{abc}A_c^R\,,\\ F_{ab}^{L,R} &= [D_a^L, A_b^R] - [D_b^R, A_a^L] + [A_a^L, A_b^R]\,.\end{aligned} \tag{4.51}$$

This suggests that we can think of A_a^L and A_a^R as the six components of a $U(n)$ gauge field living on $S_F^{2\,Int} \times S_F^{2\,Int}$ including the two normal components. As we mentioned earlier, we can eliminate these two normal components of gauge fields by imposing gauge invariant conditions on the fields in the commutative limit, ℓ_L, $\ell_R \to \infty$. Following the approaches in [41, 59, 63], we introduce the conditions

$$\begin{aligned}(X_a^L + \Gamma_a^{0\,L} + A_a^L)^2 &= (X_a^L + \Gamma_a^{0\,L})^2\\ &= -(\ell_L+\gamma)(\ell_L+\gamma+1)\mathbf{1}_{(2(\ell_L+\gamma)+1)(4(2\ell_R+1)n)}\,,\\ (X_a^R + \Gamma_a^{0\,R} + A_a^R)^2 &= (X_a^R + \Gamma_a^{0\,R})^2\\ &= -(\ell_R+\gamma)(\ell_R+\gamma+1)\mathbf{1}_{(2(\ell_R+\gamma)+1)(4(2\ell_L+1)n)}\,,\end{aligned} \tag{4.52}$$

where $\gamma = 0, \pm\frac{1}{2}$. In the commutative limit, ℓ_L, $\ell_R \to \infty$, (4.52) yields the transversality condition on $\Gamma_a^{0\,L} + A_a^L$ and $\Gamma_a^{0\,R} + A_a^R$ to be

$$\hat{x}_a^L(\Gamma_a^{0\,L} + A_a^L) \to -\gamma\,, \quad \hat{x}_a^R(\Gamma_a^{0\,R} + A_a^R) \to -\gamma\,, \tag{4.53}$$

as long as $A_a^{L,R}$ are smooth and bounded for ℓ_L, $\ell_R \to \infty$ and converge to $A_a^L(x)\,, A_a^R(x)$ in this limit. Here we have $i\frac{X_a^{L,R}}{\ell} \to \hat{x}_a^{L,R}$ as ℓ_L, $\ell_R \to \infty$, with $(\hat{x}_a^L\,, \hat{x}_a^R)$ being the coordinates of $S^2 \times S^2$.

To summarize, we have a $U(n)$ gauge theory on $\mathcal{M} \times S_F^{2\,Int} \times S_F^{2\,Int}$. Writing $A_M := (A_\mu, A_a)$, the field strength tensor takes the form $F_{MN} = (F_{\mu\nu},\, F_{\mu a}^L\,, F_{\mu a}^R,\, F_{ab}^L\,, F_{ab}^R\,, F_{ab}^{L,R})$ with

$$
\begin{aligned}
&F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,, \\
&F_{\mu a}^L := D_\mu \Phi_a^L = \partial_\mu A_a^L - [X_a^L + \Gamma_a^{0\,L}, A_\mu] + [A_\mu, A_a^L]\,, \\
&F_{\mu a}^R := D_\mu \Phi_a^R = \partial_\mu A_a^R - [X_a^R + \Gamma_a^{0\,R}, A_\mu] + [A_\mu, A_a^R]\,. \\
&F_{ab}^L = [X_a^L + \Gamma_a^{0\,L}, A_b^L] - [X_b^L + \Gamma_b^{0\,L}, A_a^L] + [A_a^L, A_b^L] - \epsilon_{abc} A_c^L\,, \\
&F_{ab}^R = [X_a^R + \Gamma_a^{0\,R}, A_b^R] - [X_b^R + \Gamma_b^{0\,R}, A_a^R] + [A_a^R, A_b^R] - \epsilon_{abc} A_c^R\,, \\
&F_{ab}^{L,R} = [X_a^L + \Gamma_a^{0\,L}, A_b^R] - [X_b^R + \Gamma_b^{0\,R}, A_a^L] + [A_a^L, A_b^R]\,.
\end{aligned}
$$

4.3 The $SU(2) \times SU(2)$-equivariant $U(4)$ gauge theory

In this section, we investigate the $U(4)$ gauge theory on $M^4 \times S_F^{2\,Int} \times S_F^{2\,Int}$. Following a similar line of development as in the subsection 3.4.3, we introduce $SU(2) \times SU(2) \approx SO(4)$ symmetry generators in order to construct the $SU(2) \times SU(2)$-equivariant gauge fields. Our anti-Hermitian symmetry generators are

$$
\begin{aligned}
\omega_a^L &= (X_a^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_{16} \otimes \mathbf{1}_4) + (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \Gamma_a^{0\,L} \otimes \mathbf{1}_4) \\
&\quad - (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_{16} \otimes i\frac{L_a^L}{2})\,, \\
\omega_a^R &= (\mathbf{1}^{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_{16} \otimes \mathbf{1}_4) + (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \Gamma_a^{0\,R} \otimes \mathbf{1}_4) \\
&\quad - (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_{16} \otimes i\frac{L_a^R}{2})\,,
\end{aligned}
\tag{4.54}
$$

and $\omega_a^L = X_a^L + \Gamma_a^{0\,L} + \frac{i}{2} L_a^L\,, \omega_a^R = X_a^L + \Gamma_a^{0\,R} + \frac{i}{2} L_a^R$ for short. L_a^L and L_a^R are chosen as in (3.148) so that ω_a^L and ω_a^R satisfy (3.146).

Since (L_a^L, L_a^R) carry the $(\frac{1}{2}, \frac{1}{2})$ IRR of $SU(2) \times SU(2)$, it is readily seen that the

Projector	To the Representation
$\Pi^L_{\mathbf{0_0}}\Pi^R_{\mathbf{0_0}}$	$(\ell_L+\frac{1}{2},\ell_R+\frac{1}{2})\oplus(\ell_L+\frac{1}{2},\ell_R-\frac{1}{2})\oplus(\ell_L-\frac{1}{2},\ell_R+\frac{1}{2})\oplus(\ell_L-\frac{1}{2},\ell_R-\frac{1}{2})$
$\Pi^L_{\mathbf{0_0}}\Pi^R_{\mathbf{0_2}}$	$(\ell_L+\frac{1}{2},\ell_R+\frac{1}{2})\oplus(\ell_L+\frac{1}{2},\ell_R-\frac{1}{2})\oplus(\ell_L-\frac{1}{2},\ell_R+\frac{1}{2})\oplus(\ell_L-\frac{1}{2},\ell_R-\frac{1}{2})$
$\Pi^L_{\mathbf{0_2}}\Pi^R_{\mathbf{0_0}}$	$(\ell_L+\frac{1}{2},\ell_R+\frac{1}{2})\oplus(\ell_L+\frac{1}{2},\ell_R-\frac{1}{2})\oplus(\ell_L-\frac{1}{2},\ell_R+\frac{1}{2})\oplus(\ell_L-\frac{1}{2},\ell_R-\frac{1}{2})$
$\Pi^L_{\mathbf{0_2}}\Pi^R_{\mathbf{0_2}}$	$(\ell_L+\frac{1}{2},\ell_R+\frac{1}{2})\oplus(\ell_L+\frac{1}{2},\ell_R-\frac{1}{2})\oplus(\ell_L-\frac{1}{2},\ell_R+\frac{1}{2})\oplus(\ell_L-\frac{1}{2},\ell_R-\frac{1}{2})$
$\Pi^L_{\mathbf{0_0}}\Pi^R_{\pm}$	$(\ell_L+\frac{1}{2},\ell_R\pm 1)\oplus(\ell_L-\frac{1}{2},\ell_R\pm 1)\oplus(\ell_L+\frac{1}{2},\ell_R)\oplus(\ell_L-\frac{1}{2},\ell_R)$
$\Pi^L_{\mathbf{0_2}}\Pi^R_{\pm}$	$(\ell_L+\frac{1}{2},\ell_R\pm 1)\oplus(\ell_L-\frac{1}{2},\ell_R\pm 1)\oplus(\ell_L+\frac{1}{2},\ell_R)\oplus(\ell_L-\frac{1}{2},\ell_R)$
$\Pi^L_{\pm}\Pi^R_{\mathbf{0_0}}$	$(\ell_L\pm 1,\ell_R+\frac{1}{2})\oplus(\ell_L\pm 1,\ell_R-\frac{1}{2})\oplus(\ell_L,\ell_R+\frac{1}{2})\oplus(\ell_L,\ell_R-\frac{1}{2})$
$\Pi^L_{\pm}\Pi^R_{\mathbf{0_2}}$	$(\ell_L\pm 1,\ell_R+\frac{1}{2})\oplus(\ell_L\pm 1,\ell_R-\frac{1}{2})\oplus(\ell_L,\ell_R+\frac{1}{2})\oplus(\ell_L,\ell_R-\frac{1}{2})$
$\Pi^L_{\pm}\Pi^R_{\pm}$	$(\ell_L\pm 1,\ell_R\pm 1)\oplus(\ell_L\pm 1,\ell_R)\oplus(\ell_L,\ell_R\pm 1)\oplus(\ell_L,\ell_R)$
$\Pi^L_{\pm}\Pi^R_{\mp}$	$(\ell_L\pm 1,\ell_R)\oplus(\ell_L\pm 1,\ell_R\mp 1)\oplus(\ell_L,\ell_R)\oplus(\ell_L,\ell_R\mp 1)$

Table4.4: Projections to the representations appearing in the r.h.s of (4.55).

symmetry generators (ω^L_a,ω^R_a) have the $SU(2)\times SU(2)$ representation content

$$
\begin{aligned}
&(\ell_L,\ell_R)\otimes\left(\mathbf{4}(0,0)\oplus\mathbf{2}(\frac{1}{2},0)\oplus\mathbf{2}(0,\frac{1}{2})\oplus(\frac{1}{2},\frac{1}{2})\right)\otimes(\frac{1}{2},\frac{1}{2})\\
&\equiv\mathbf{4}[(\ell_L+\frac{1}{2},\ell_R+\frac{1}{2})\oplus(\ell_L+\frac{1}{2},\ell_R-\frac{1}{2})\oplus(\ell_L-\frac{1}{2},\ell_R+\frac{1}{2})\oplus(\ell_L-\frac{1}{2},\ell_R-\frac{1}{2})]\\
&\oplus\mathbf{2}[(\ell_L-1,\ell_R-\frac{1}{2})\oplus(\ell_L-1,\ell_R+\frac{1}{2})]\oplus\mathbf{4}[(\ell_L,\ell_R+\frac{1}{2})\oplus(\ell_L,\ell_R-\frac{1}{2})]\\
&\oplus\mathbf{2}[(\ell_L+1,\ell_R-\frac{1}{2})\oplus(\ell_L+1,\ell_R+\frac{1}{2})]\oplus\mathbf{2}[(\ell_L-\frac{1}{2},\ell_R-1)\oplus(\ell_L+\frac{1}{2},\ell_R-1)]\\
&\oplus\mathbf{4}[(\ell_L-\frac{1}{2},\ell_R)\oplus(\ell_L+\frac{1}{2},\ell_R)]\oplus\mathbf{2}[(\ell_L-\frac{1}{2},\ell_R+1)\oplus(\ell_L+\frac{1}{2},\ell_R+1)]\\
&\oplus(\ell_L-1,\ell_R-1)\oplus\mathbf{2}(\ell_L-1,\ell_R)\oplus\mathbf{2}(\ell_L,\ell_R-1)\oplus\mathbf{4}(\ell_L,\ell_R)\oplus(\ell_L+1,\ell_R-1)\\
&\oplus\mathbf{2}(\ell_L+1,\ell_R)\oplus(\ell_L-1,\ell_R+1)\oplus\mathbf{2}(\ell_L,\ell_R+1)\oplus(\ell_L+1,\ell_R+1)\equiv I\,.
\end{aligned}
\tag{4.55}
$$

Adjoint action of (ω^L_a,ω^R_a) implies the $SU(2)\times SU(2)$-equivariance conditions in (3.149) and

$$
\begin{aligned}
&[\omega^L_a,\Psi^L_\alpha]=\frac{i}{2}(\tau_a)_{\alpha\beta}\Psi^L_\beta\,,\quad[\omega^R_a,\Psi^R_\alpha]=\frac{i}{2}(\tau_a)_{\alpha\beta}\Psi^R_\beta\,,\\
&[\omega^L_a,\Psi^R_\alpha]=0=[\omega^R_a,\Psi^L_\alpha]\,.
\end{aligned}
\tag{4.56}
$$

For the $U(4)$ theory under investigation, we list the projectors and the subspaces to which they project in the equation (4.55) in the table (4.4). In order to avoid the possibility of any notational confusion, we note that the representation content of (ω^L_a,ω^R_a) includes the tensor product with the IRR $(\frac{1}{2},\frac{1}{2})$ as seen in the l.h.s. of (4.55) and $\Pi^L_\alpha \Pi^R_\beta$ project to the subspaces as listed in the table 4.4, while in the absence of the gauge symmetry generators (L^L_a, L^R_a), $\Pi^L_\alpha \Pi^R_\beta$ project to the subspaces as listed in table 4.3.

We can find the dimension of solution space for A_μ, A^L_a, A^R_a and Ψ^L_α, Ψ^R_α using the Clebsch-Gordan decomposition of the adjoint action of (ω^L_a, ω^R_a) whose representation content can be found by $I \otimes I$. The relevant part of this decomposition gives

$$\mathbf{196}(0,0) \oplus \mathbf{336}(\frac{1}{2},0) \oplus \mathbf{336}(0,\frac{1}{2}) \oplus \mathbf{420}(1,0) \oplus \mathbf{420}(0,1) \cdots . \tag{4.57}$$

This means that there are 196 equivariant scalars (i.e rotational invariants under (ω^L_a, ω^R_a)), 336 equivariant spinors in each of the IRRs $(\frac{1}{2}\,,0)$ and $(0\,,\frac{1}{2})$ and 420 vectors in each of the IRRs $(1\,,0)$ and $(0\,,1)$. Employing the matrices

$$S^L_i = \mathbf{1}_{2\ell_L+1} \otimes \mathbf{1}_{2\ell_R+1} \otimes s_i \otimes \mathbf{1}_4 \otimes \mathbf{1}_4\,, \quad S^R_i = \mathbf{1}_{2\ell_L+1} \otimes \mathbf{1}_{2\ell_R+1} \otimes \mathbf{1}_4 \otimes s_i \otimes \mathbf{1}_4\,,$$
$$s_i = \begin{pmatrix} \sigma_i & 0_2 \\ 0_2 & 0_2 \end{pmatrix}, \quad i = 1\,,2\,, \tag{4.58}$$

$$\begin{aligned}
Q^L_B &= \frac{X^L_a L^L_a - \frac{i}{2}\mathbf{1}}{\ell_L + \frac{1}{2}}\,, \quad Q^L_{0_\mathbf{0}} = \Pi^L_{0_\mathbf{0}} Q^L_B\,, \quad Q^L_{0_\mathbf{2}} = \Pi^L_{0_\mathbf{2}} Q^L_B\,,\\
Q^L_+ &= \frac{1}{4\ell_L(\ell_L+1)}\Pi^L_+\left((2\ell_L+1)^2 Q^L_B + i\right)\Pi^L_+\,,\\
Q^L_- &= \frac{1}{4\ell_L(\ell_L+1)}\Pi^L_-\left((2\ell_L+1)^2 Q^L_B - i\right)\Pi^L_-\,,\\
Q^L_F &= \Gamma^{0\,L}_a L^L_a - i\frac{1}{2}\Pi^L_{\frac{1}{2}}\,, \quad Q^L_H = -i\frac{\epsilon_{abc} X^L_a \Gamma^{0\,L}_b L^L_c}{\sqrt{\ell_L(\ell_L+1)}} - \frac{1}{2}Q^L_{BI} + i\frac{1}{2}\Pi^L_{\frac{1}{2}}\,,\\
Q^L_{BI} &= i\frac{(\ell_L+\frac{1}{2})^2\{Q^L_B, Q^L_I\} + \frac{1}{2}\Pi^L_{\frac{1}{2}}}{2\ell_L(\ell_L+1)}\,, \quad Q^L_{S_i} = \frac{X^L_a S^L_i L^L_a - \frac{i}{2}S^L_i}{\ell_L + \frac{1}{2}}\,,
\end{aligned} \tag{4.59}$$

and $L \to R$ in (4.59) for the right constituents, a judicious choice of a basis for the equivariant scalars can be made so that they are "idempotents" in the subspace they live in, and they can be listed as

$$\begin{gathered}
\Pi^L_i \Pi^R_i, \quad \Pi^L_i S^R_k, \quad \Pi^L_i Q^R_j, \quad \Pi^L_i Q^R_{S_k}, \quad Q^L_j \Pi^R_i, \quad Q^L_j S^R_k, \quad Q^L_j Q^R_j, \quad Q^L_j Q^R_{S_k},\\
Q^L_{S_k}\Pi^R_i, \quad Q^L_{S_k} S^R_k, \quad Q^L_{S_k} Q^R_j, \quad Q^L_{S_k} Q^R_{S_k}, \quad S^L_k \Pi^R_i, \quad S^L_k S^R_k, \quad S^L_k Q^R_j, \quad S^L_k Q^R_{S_k},
\end{gathered} \tag{4.60}$$

where i runs over $0_\mathbf{0}, 0_\mathbf{2}, +, -$, j runs over $0_\mathbf{0}, 0_\mathbf{2}, +, -, H, F$ and k takes on the values $1, 2$ and no sum over repeated indices is implied. Full lists of the equivariant spinors and vectors are not our immediate corcern in what follows and therefore they are relegated to Appendix B.

We note that that the index α ($\alpha = 1, 2$) of Ψ^L_α and Ψ^R_α implying the transformation properties of these fields under the global symmetry $SU(2)_L \times SU(2)_R$, becomes, after symmetry breaking, the spinor index on $S_F^{2\,Int} \times S_F^{2\,Int}$, just like the index a ($a = 1, 2, 3$) of (Φ^L_a, Φ^R_a) becomes the vector index. We stress that the pure group theoretical result in equation (4.57) predicts the presence of equivariant spinor fields in the IRRs $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the symmetry group $SU(2)_L \times SU(2)_R$ of the fuzzy extra dimensions $S_F^{2\,Int} \times S_F^{2\,Int}$. Their explicit construction, as listed in (B.6), is only facilitated by the splittings of Φ^L an Φ^R in (4.24) and (4.25). As it should be already clear from our discussions in subsections 4.1 and 4.2, these spinorial modes do not constitute independent dynamical degrees of freedom in the $U(4)$ effective gauge theory. Taking suitable bilinears of these spinors, we may construct all the equivariant gauge field modes on $S_F^{2\,Int} \times S_F^{2\,Int}$. In other words, it is in principle possible to express the "square roots" of the equivariant gauge field modes through these equivariant spinorial modes.

4.3.1 Projection to the Monopole Sectors

In this section, we derive the monopole sectors with the winding numbers $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 1, \pm 1)$ from the suitable projections of $S_F^{2\,Int} \times S_F^{2\,Int}$ and we gain much insight on the structure of the model by examining projections to its subsectors. We will see how to systematically access all higher winding number monopole sectors in the next subsection.

We observe that ${S_F}^{2\,Int} \times {S_F}^{2\,Int}$ may be projected down to the monopole sectors

$$S_F^{2\,\pm} \times S_F^2 = \Big(S_F^2(\ell_L) \times S_F^2(\ell_R) \Big) \oplus \Big(S_F^2(\ell_L \pm \frac{1}{2}) \times S_F^2(\ell_R) \Big) \,, \tag{4.61}$$

$$S_F^2 \times S_F^{2\,\pm} = \Big(S_F^2(\ell_L) \times S_F^2(\ell_R) \Big) \oplus \Big(S_F^2(\ell_L) \times S_F^2(\ell_R \pm \frac{1}{2}) \Big) \,, \tag{4.62}$$

$$S_F^{2\,\pm} \times {S_F}^{2\,\pm} = \Big(S_F^2(\ell_L) \times S_F^2(\ell_R) \Big) \oplus \Big(S_F^2(\ell_L \pm \frac{1}{2}) \times S_F^2(\ell_R \pm \frac{1}{2}) \Big) \,, \tag{4.63}$$

with the winding numbers $(\pm 1, 0)$, $(0, \pm 1), (\pm 1, \pm 1)$, respectively. We can now probe the low energy structure of the $U(4)$ model in these monopole sectors by writing down their equivariant gauge field modes.

Let us inspect each of the sectors briefly.

i. $\underline{S_F^{2\,\pm} \times S_F^2}$:

We may consider, for instance, the projection

$$\Pi^L_{0_\mathbf{0}} \Pi^R_{0_\mathbf{0}} + \Pi^L_{\pm} \Pi^R_{0_\mathbf{0}} \,. \tag{4.64}$$

We note that the projection (4.64) to this sector is not unique, in the sense that there is in fact a set of projections which give the same monopole sector. We infer from (4.61) to which IRRs the projection (4.64) restricts the direct sum given in the r.h.s. of (4.43). After this projection, the number of equivariant fields are greatly reduced and they can be most easily found by working out the adjoint action of (ω_a^L, ω_a^R), which in this subspace takes the simple form

$$\left[\left((\ell_L, \ell_R) \oplus (\ell_L \pm \frac{1}{2}, \ell_R) \right) \otimes (\frac{1}{2}, \frac{1}{2}) \right]^{\otimes 2} \equiv \mathbf{8}(0,0) \oplus \mathbf{12}(\frac{1}{2}, 0) \oplus \mathbf{16}(1,0) \\ \oplus \mathbf{16}(0,1) \cdots \,.$$

Thus, there are 8 invariants which we read from (4.60) as

$$\begin{gathered} \Pi^L_{0_\mathbf{0}} \Pi^R_{0_\mathbf{0}} \,, \quad \Pi^L_{\pm} \Pi^R_{0_\mathbf{0}} \,, \quad \Pi^L_{0_\mathbf{0}} Q^R_{0_\mathbf{0}} \,, \quad \Pi^L_{\pm} Q^R_{0_\mathbf{0}} \,, \\ Q^L_{0_\mathbf{0}} \Pi^R_{0_\mathbf{0}} \,, \quad Q^L_{\pm} \Pi^R_{0_\mathbf{0}} \,, \quad Q^L_{0_\mathbf{0}} Q^R_{0_\mathbf{0}} \,, \quad Q^L_{\pm} Q^R_{0_\mathbf{0}} \,, \end{gathered} \tag{4.65}$$

16 vectors carrying the $(1,0)$ IRR

$$\begin{gathered} \Pi^R_{0_\mathbf{0}} [D^L_a, Q^L_{0_\mathbf{0}}] \,, \quad \Pi^R_{0_\mathbf{0}} Q^L_{0_\mathbf{0}} [D^L_a, Q^L_{0_\mathbf{0}}] \,, \quad \Pi^R_{0_\mathbf{0}} \{D^L_a, Q^L_{0_\mathbf{0}}\} \,, \\ Q^R_{0_\mathbf{0}} [D^L_a, Q^L_{0_\mathbf{0}}] \,, \quad Q^R_{0_\mathbf{0}} Q^L_{0_\mathbf{0}} [D^L_a, Q^L_{0_\mathbf{0}}] \,, \quad Q^R_{0_\mathbf{0}} \{D^L_a, Q^L_{0_\mathbf{0}}\} \,, \\ \Pi^R_{0_\mathbf{0}} [D^L_a, Q^L_{\pm}] \,, \quad \Pi^R_{0_\mathbf{0}} Q^L_{\pm} [D^L_a, Q^L_{\pm}] \,, \quad \Pi^R_{0_\mathbf{0}} \{D^L_a, Q^L_{\pm}\} \,, \\ Q^R_{0_\mathbf{0}} [D^L_a, Q^L_{\pm}] \,, \quad Q^R_{0_\mathbf{0}} Q^L_{\pm} [D^L_a, Q^L_{\pm}] \,, \quad Q^R_{0_\mathbf{0}} \{D^L_a, Q^L_{\pm}\} \,, \\ \Pi^R_{0_\mathbf{0}} \Pi^L_{0_\mathbf{0}} \omega^L_a \,, \quad \Pi^R_{0_\mathbf{0}} \Pi^L_{\pm} \omega^L_a \,, \quad Q^R_{0_\mathbf{0}} \Pi^L_{0_\mathbf{0}} \omega^L_a \,, \quad Q^R_{0_\mathbf{0}} \Pi^L_{\pm} \omega^L_a \,, \end{gathered} \tag{4.66}$$

and 16 vectors in the $(0,1)$ IRR are

$$\begin{aligned}
&\Pi^L_{0_{\mathbf{0}}}[D^R_a, Q^R_{0_{\mathbf{0}}}],\ \ \Pi^L_{0_{\mathbf{0}}} Q^R_{0_{\mathbf{0}}}[D^R_a, Q^R_{0_{\mathbf{0}}}],\ \ \Pi^L_{0_{\mathbf{0}}}\{D^R_a, Q^R_{0_{\mathbf{0}}}\},\\
&Q^L_{0_{\mathbf{0}}}[D^R_a, Q^R_{0_{\mathbf{0}}}],\ \ Q^L_{0_{\mathbf{0}}} Q^R_{0_{\mathbf{0}}}[D^R_a, Q^R_{0_{\mathbf{0}}}],\ \ Q^L_{0_{\mathbf{0}}}\{D^R_a, Q^R_{0_{\mathbf{0}}}\},\\
&\Pi^L_{\mp}[D^R_a, Q^R_{0_{\mathbf{0}}}],\ \ \Pi^L_{\mp} Q^R_{0_{\mathbf{0}}}[D^R_a, Q^R_{0_{\mathbf{0}}}],\ \ \Pi^L_{\mp}\{D^R_a, Q^R_{0_{\mathbf{0}}}\},\\
&Q^L_{\mp}[D^R_a, Q^R_{0_{\mathbf{0}}}],\ \ Q^L_{\mp} Q^R_{0_{\mathbf{0}}}[D^R_a, Q^R_{0_{\mathbf{0}}}],\ \ Q^L_{\mp}\{D^R_a, Q^R_{0_{\mathbf{0}}}\},\\
&\Pi^L_{0_{\mathbf{0}}}\Pi^R_{0_{\mathbf{0}}}\omega^R_a,\ \ \Pi^L_{\mp}\Pi^R_{0_{\mathbf{0}}}\omega^R_a,\ \ Q^L_{0_{\mathbf{0}}}\Pi^R_{0_{\mathbf{0}}}\omega^R_a,\ \ Q^L_{\mp}\Pi^R_{0_{\mathbf{0}}}\omega^R_a\,.
\end{aligned} \tag{4.67}$$

We see that there are 12 equivariant spinor in the IRR $(\frac{1}{2},0)$

$$\begin{aligned}
&\Pi^R_{0_{\mathbf{0}}}\Pi^L_{0_{\mathbf{0}}}\beta^L_\alpha Q^L_\pm\,,\ \Pi^R_{0_{\mathbf{0}}}Q^L_{0_{\mathbf{0}}}\beta^L_\alpha \Pi^L_\pm\,,\ \Pi^R_{0_{\mathbf{0}}}Q^L_{0_{\mathbf{0}}}\beta^L_\alpha Q^L_\pm\,,\ Q^R_{0_{\mathbf{0}}}\Pi^L_{0_{\mathbf{0}}}\beta^L_\alpha Q^L_\pm\,,\ Q^R_{0_{\mathbf{0}}}Q^L_{0_{\mathbf{0}}}\beta^L_\alpha \Pi^L_\pm\,,\\
&Q^R_{0_{\mathbf{0}}}Q^L_{0_{\mathbf{0}}}\beta^L_\alpha Q^L_\pm\,,\ \Pi^R_{0_{\mathbf{0}}}\Pi^L_\pm\beta^L_\alpha S^L_2\,,\ \Pi^R_{0_{\mathbf{0}}}\Pi^L_\pm\beta^L_\alpha Q^L_{s2}\,,\ \Pi^R_{0_{\mathbf{0}}}Q^L_\pm\beta^L_\alpha Q^L_{s2}\,,\ Q^R_{0_{\mathbf{0}}}\Pi^L_\pm\beta^L_\alpha S^L_2\,,\\
&Q^R_{0_{\mathbf{0}}}\Pi^L_\pm\beta^L_\alpha Q^L_{s2}\,,\ Q^R_{0_{\mathbf{0}}}Q^L_\pm\beta^L_\alpha Q^L_{s2}\,,
\end{aligned} \tag{4.68}$$

where $\beta^L_\alpha = \mathbf{1}^{2\ell_L+1}\otimes \mathbf{1}^{2\ell_R+1}\otimes b_\alpha\otimes \mathbf{1}_4$ and due to the form of this monopole sector, we find no equivariant spinors in the IRR $(0,\frac{1}{2})$.

One, rather trivial alternative to (4.64) is to change $\Pi^R_{0_{\mathbf{0}}}$ with $\Pi^R_{0_{\mathbf{2}}}$ in (4.64), this simply amounts to taking $\Pi^R_{0_{\mathbf{0}}}\to \Pi^R_{0_{\mathbf{2}}}\,, Q^R_{0_{\mathbf{0}}}\to Q^R_{0_{\mathbf{2}}}$ in (4.65),(4.66),(4.67) and (4.68). Another choice is the projector

$$\Pi^L_{0_{\mathbf{0}}}\Pi^R_{0_{\mathbf{0}}} + \Pi^L_\pm \Pi^R_{0_{\mathbf{2}}}\,. \tag{4.69}$$

Equivariant fields in this case can be obtained in a similar fashion.

i. $\underline{S^2_F\times S^{2\,\pm}_F}$:

We observe that the only change in (4.65) is the replacement of $(\frac{1}{2},0)$ with $(0,\frac{1}{2})$. Bearing this fact in mind, results in (4.65) to (4.69) apply with the exchange $L\leftrightarrow R$.

i. $\underline{S^{2\,\pm}_F\times S^{2\,\pm}_F}$:

To obtain this monopole sector we can use any one of the projections

$$\Pi^L_i\Pi^R_j + \Pi^L_\pm\Pi^R_\pm\,,\quad i\,,j = 0_{\mathbf{0}}, 0_{\mathbf{2}}\,. \tag{4.70}$$

In this case, the adjoint action of (ω_a^L, ω_a^R) yields the representation content

$$\Big[\big((\ell_L, \ell_R) \oplus (\ell_L \mp \frac{1}{2}, \ell_R \mp \frac{1}{2})\big) \otimes (\frac{1}{2}, \frac{1}{2})\Big]^{\otimes 2} \equiv \mathbf{8}(0,0) \oplus \mathbf{16}(1,0) \oplus \mathbf{16}(0,1) \oplus \cdots \,. \tag{4.71}$$

We immediately observe that equivariant spinors are completely absent in this sector. Taking, for instance, $i\,, j = 0_{\mathbf{0}}$ we find that 8 scalars can be written as

$$\begin{aligned}
&\Pi^L_{0_{\mathbf{0}}}\Pi^R_{0_{\mathbf{0}}}\,, \quad \Pi^L_{\pm}\Pi^R_{\pm}\,, \quad \Pi^L_{0_{\mathbf{0}}}Q^R_{0_{\mathbf{0}}}\,, \quad \Pi^L_{\pm}Q^R_{\pm}\,, \\
&Q^L_{0_{\mathbf{0}}}\Pi^R_{0_{\mathbf{0}}}\,, \quad Q^L_{\pm}\Pi^R_{\pm}\,, \quad Q^L_{0_{\mathbf{0}}}Q^R_{0_{\mathbf{0}}}\,, \quad Q^L_{\pm}Q^R_{\pm}\,,
\end{aligned} \tag{4.72}$$

and 16 vectors carrying the $(1,0)$ IRR are

$$\begin{aligned}
&\Pi^R_{0_{\mathbf{0}}}[D^L_a, Q^L_{0_{\mathbf{0}}}]\,, \quad \Pi^R_{0_{\mathbf{0}}}Q^L_{0_{\mathbf{0}}}[D^L_a, Q^L_{0_{\mathbf{0}}}]\,, \quad \Pi^R_{0_{\mathbf{0}}}\{D^L_a, Q^L_{0_{\mathbf{0}}}\}\,, \\
&Q^R_{0_{\mathbf{0}}}[D^L_a, Q^L_{0_{\mathbf{0}}}]\,, \quad Q^R_{0_{\mathbf{0}}}Q^L_{0_{\mathbf{0}}}[D^L_a, Q^L_{0_{\mathbf{0}}}]\,, \quad Q^R_{0_{\mathbf{0}}}\{D^L_a, Q^L_{0_{\mathbf{0}}}\}\,, \\
&\Pi^R_{\pm}[D^L_a, Q^L_{\pm}]\,, \quad \Pi^R_{\pm}Q^L_{\pm}[D^L_a, Q^L_{\pm}]\,, \quad \Pi^R_{\pm}\{D^L_a, Q^L_{\pm}\}\,, \\
&Q^R_{\pm}[D^L_a, Q^L_{\pm}]\,, \quad Q^R_{\pm}Q^L_{\pm}[D^L_a, Q^L_{\pm}]\,, \quad Q^R_{\pm}\{D^L_a, Q^L_{\pm}\}\,, \\
&\Pi^R_{0_{\mathbf{0}}}\Pi^L_{0_{\mathbf{0}}}\omega^L_a\,, \quad \Pi^R_{\pm}\Pi^L_{\pm}\omega^L_a\,, \quad Q^R_{0_{\mathbf{0}}}\Pi^L_{0_{\mathbf{0}}}\omega^L_a\,, \quad Q^R_{\pm}\Pi^L_{\pm}\omega^L_a\,,
\end{aligned} \tag{4.73}$$

while the vectors carrying the $(0,1)$ representation follow from (4.73) by the exchange $L \leftrightarrow R$.

In all cases that we have discussed in this subsection, each summand of the projectors (given in (4.64), (4.69), (4.70), etc.) splits the equivariant fields into mutually orthogonal subsectors under matrix product. For concreteness, let us briefly discuss the consequences of this fact for the sector given by the projection in (4.64). We have found the rotational invariants under the symmetry generators (ω_a^L, ω_a^R) given in (4.65), so the parametrization of the fields A_μ can be defined in terms of two mutually orthogonal sets $(\Pi^R_{0_{\mathbf{0}}}Q^L_{0_{\mathbf{0}}}\,, \Pi^L_{0_{\mathbf{0}}}Q^R_{0_{\mathbf{0}}}\,, Q^L_{0_{\mathbf{0}}}Q^R_{0_{\mathbf{0}}},$ $\Pi^L_{0_{\mathbf{0}}}\Pi^R_{0_{\mathbf{0}}})$ and $(\Pi^R_{0_{\mathbf{0}}}Q^L_{\pm}\,, \Pi^L_{\pm}Q^R_{0_{\mathbf{0}}}\,, Q^L_{\pm}Q^R_{0_{\mathbf{0}}}\,, \Pi^L_{\pm}\Pi^R_{0_{\mathbf{0}}})$. Comparing these two sets to the paramet- rization of gauge field constructed in the section 3.4, we observe that each set in the subspace it lives is equivalent to the basis for parametrization of gauge field (3.152). Hence, the low energy effective action of this model consists of two decoupled set of Abelian Higgs-type models with $U(1)^3$ gauge symmetry and each set possesses static multivortex solutions characterized by three winding numbers as given in the subsection 3.4.4 [24].

4.4 Generalization of the Model with k-component Multiplets

It is possible to search for other vacuum solutions for the deformed $N = 4$ SYM theory given in the section 4.2. We may generalize the construction of this section by replacing the doublets Ψ^L and Ψ^R in equation (4.25) with k_1-, k_2-component multiplets of the global $SU(2) \times SU(2)$ as

$$\Psi^L = \begin{pmatrix} \Psi_1^L \\ \Psi_2^L \\ \vdots \\ \Psi_{k_1}^L \end{pmatrix} , \quad \Psi^R = \begin{pmatrix} \Psi_1^R \\ \Psi_2^R \\ \vdots \\ \Psi_{k_2}^R \end{pmatrix} , \quad \Psi = \begin{pmatrix} \Psi^L \\ \Psi^R \end{pmatrix} , \tag{4.74}$$

transforming in its $(\frac{k_1-1}{2}, 0)$ and $(0, \frac{k_2-1}{2})$ IRR, respectively. Then, Ψ is the $k_1 + k_2$-component multiplet in the representation $(\frac{k_1-1}{2}, 0) \oplus (0, \frac{k_2-1}{2})$. Components $\Psi_\alpha^L\,, \Psi_\beta^R \in Mat(\mathcal{N})\,, (\alpha = 1, \cdots, k_1)\,, (\beta = 1 \cdots, k_2)$ of Ψ are scalar fields transforming in the adjoint representation of $SU(\mathcal{N})$ as $\Psi_\alpha^{L,R} \to U^\dagger \Psi_\alpha^{L,R} U$. Bilinears Γ_a^L and Γ_a^R in Ψ^L and Ψ^R are defined similarly as before in the form

$$\Gamma_a^L = -\frac{i}{2} \Psi^{L\dagger} \tilde{\lambda}_a^L \Psi^L\,, \quad \Gamma_a^R = -\frac{i}{2} \Psi^{R\dagger} \tilde{\lambda}_a^R \Psi^R\,, \quad \tilde{\lambda}_a^{L,R} = \lambda_a^{L,R} \otimes \mathbf{1}_{\mathcal{N}}\,, \tag{4.75}$$

where now $\lambda_a^{L,R}$ are the generators of spin $(\frac{k_{L,R}-1}{2})$ representation of $SU(2)$.

In subsection 4.2.1, we have seen that the vacuum configuration of our model can be written as the direct sum of products of fuzzy spheres whose structure is determined by the representation content of $(\Gamma_a^{0L}, \Gamma_a^{0R})$ with the corresponding doublet scalar fields taking the form given in (4.33). In order to generalize the latter, we need $k = k_1 + k_2$ sets of annihilation-creation operators which satisfy

$$\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}\,, \quad \alpha, \beta = 1, \cdots, k_1\,, \quad \{c_\rho, c_\sigma^\dagger\} = \delta_{\rho\sigma}\,, \quad \rho, \sigma = 1, \cdots, k_2\,, \tag{4.76}$$

with all other anticommutators vanishing. Thus, these operators span the $2^{k_1+k_2}$-dimensional Hilbert space with the basis vectors

$$|n_1, \cdots, n_{k_1}, m_1, \cdots, m_{k_2}\rangle = (b_1^\dagger)^{n_1} \cdots (b_{k_1}^\dagger)^{n_{k_1}} (c_1^\dagger)^{m_1} \cdots (c_{k_2}^\dagger)^{m_{k_2}} |0, 0 \cdots, 0\rangle\,, \tag{4.77}$$

where $n_i, m_j = 0, 1,\ (i = 1, \cdots, k_1,\ j = 1, \cdots, k_2)$. For $\Psi^L = \psi^L$ and $\Psi^R = \psi^R$

with

$$\psi^L := \begin{pmatrix} b_1 \\ \vdots \\ b_{k_1} \end{pmatrix}, \quad \psi^R := \begin{pmatrix} c_1 \\ \vdots \\ c_{k_2} \end{pmatrix}. \tag{4.78}$$

It is straightforward to show that ${\Gamma^0_a}^L = -\frac{i}{2}\psi^{L\dagger}\lambda_a\psi^L$ and ${\Gamma^0_a}^R = -\frac{i}{2}\psi^{R\dagger}\lambda_a\psi^R$ satisfy the $SU(2)\times SU(2)$ commutation relations and in addition fulfill

$$\begin{aligned} [\psi^L_\alpha\,, {\Gamma^0_a}^L] &= -\frac{i}{2}(\lambda_a)_{\alpha\beta}\psi^L_\beta\,, \quad [\psi^L_\alpha\,, {\Gamma^0_a}^R] = 0\,, \\ [\psi^R_\alpha, {\Gamma^0_a}^R] &= -\frac{i}{2}(\lambda_a)_{\alpha\beta}\psi^R_\beta\,, \quad [\psi^R_\alpha\,, {\Gamma^0_a}^L] = 0\,, \end{aligned} \tag{4.79}$$

implying that ψ^L_α and ψ^R_α indeed carry the $(\frac{k_1-1}{2}, 0)$ and $(0, \frac{k_2-1}{2})$ IRRs, respectively.

In order to obtain the vacuum configuration in the present case, first we have to find out the $SU(2)\times SU(2)$ IRR content of $({\Gamma^0_a}^L, {\Gamma^0_a}^R)$. Number operators $N^L = b^\dagger_\alpha b_\alpha$ and $N^R = c^\dagger_\alpha c_\alpha$ commute with ${\Gamma^0_a}^L$ and ${\Gamma^0_a}^R$. This means that, the number of states in a given sector with eigenvalues (n^L, n^R) $(n^L = (0, \cdots, k_1)\,, n^R = (0, \cdots, k_2))$ of N^L and N^R is equal to the dimension of one of the $SU(2)\times SU(2)$ IRR sectors occurring in the decomposition of the representation of $({\Gamma^0_a}^L, {\Gamma^0_a}^R)$ into the irreducibles of $SU(2)\times SU(2)$. Therefore, the IRRs of $SU(2)\times SU(2)$ that appear in $({\Gamma^0_a}^L, {\Gamma^0_a}^R)$ may be labeled as

$$(\ell^{k_1}_n, \ell^{k_2}_m) = \left(\frac{\binom{k_1}{n} - 1}{2}, \frac{\binom{k_2}{m} - 1}{2} \right), \tag{4.80}$$

and the reducible representation carried by $({\Gamma^0_a}^L, {\Gamma^0_a}^R)$ decomposes into the direct sum

$$L^{k_1\,k_2} := \sum_{n=0}^{k_1}\sum_{m=0}^{k_2} \oplus(\ell^{k_1}_n, \ell^{k_2}_m)\,. \tag{4.81}$$

Since $\binom{k_i}{n} = \binom{k_i}{k-n}$, we see that $\ell^{k_i}_n = \ell^{k_i}_{k_i-n}$. As a consequence, not all the summands in (4.81) are distinct IRRs. Noting also that $\ell^{k_i}_{\frac{k_i}{2}}$ occurs only once for k_i even, we may rewrite (4.81) as the direct sum of distinct IRRs together with its multiplicities as

$$\begin{aligned}
L^{k_{1even}\,k_{2even}} =& (\ell^{k_1}_{\frac{k_1}{2}}, \ell^{k_2}_{\frac{k_2}{2}}) \oplus \mathbf{2} \sum_{n=0}^{\frac{k_1}{2}-1} \sum_{m=0}^{\frac{k_2}{2}} (\ell^{k_1}_n, \ell^{k_2}_m) \oplus \mathbf{2} \sum_{n=0}^{\frac{k_1}{2}} \sum_{m=0}^{\frac{k_2}{2}-1} (\ell^{k_1}_n, \ell^{k_2}_m)\,, \\
=& (\ell^{k_1}_{\frac{k_1}{2}}, \ell^{k_2}_{\frac{k_2}{2}}) \oplus \mathbf{4} \sum_{n=0}^{\frac{k_1}{2}-1} \sum_{m=0}^{\frac{k_2}{2}-1} (\ell^{k_1}_n, \ell^{k_2}_m) \oplus \mathbf{2} \sum_{n=0}^{\frac{k_1}{2}-1} (\ell^{k_1}_n, \ell^{k_2}_{\frac{k_2}{2}}) \\
& \oplus \mathbf{2} \sum_{m=0}^{\frac{k_2}{2}-1} (\ell^{k_1}_{\frac{k_1}{2}}, \ell^{k_2}_m)\,, \qquad (4.82) \\
L^{k_{1odd}\,k_{2odd}} =& \mathbf{4} \sum_{n=0}^{\frac{k_1-1}{2}} \sum_{m=0}^{\frac{k_2-1}{2}} (\ell^{k_1}_n, \ell^{k_2}_m)\,, \qquad (4.83) \\
L^{k_{1even}\,k_{2odd}} =& \mathbf{4} \sum_{n=0}^{\frac{k_1}{2}-1} \sum_{m=0}^{\frac{k_2-1}{2}} (\ell^{k_1}_n, \ell^{k_2}_m) \oplus \mathbf{2} \sum_{m=0}^{\frac{k_2-1}{2}} (\ell^{k_1}_{\frac{k_1}{2}}, \ell^{k_2}_m)\,. \qquad (4.84)
\end{aligned}$$

$L^{k_{1odd}\,k_{2even}}$ can be obtained by taking $k_1 \leftrightarrow k_2$ in equation (4.84).

With the assumption $\mathcal{N} = 2^{k_1+k_2}(2\ell_L+1)(2\ell_R+1)n$, the vacuum configuration of our $SU(\mathcal{N})$ gauge theory can be written as

$$\begin{aligned}
\Phi^L_a &= (X^{(2\ell_L+1)}_a \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_{2^{k_1+k_2}} \otimes \mathbf{1}_n) + (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \Gamma^{0\,L}_a \otimes \mathbf{1}_n) \\
\Phi^R_a &= (\mathbf{1}^{(2\ell_L+1)} \otimes X^{(2\ell_R+1)}_a \otimes \mathbf{1}_{2^{k_1+k_2}} \otimes \mathbf{1}_n) + (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \Gamma^{0\,R}_a \otimes \mathbf{1}_n)\,,
\end{aligned} \qquad (4.85)$$

up to $SU(\mathcal{N})$ gauge transformations.

Clebsch-Gordan decomposition of the tensor products $(\ell_L, \ell_R) \otimes L^{k_{1even}k_{2odd}}$ and $(\ell_L, \ell_R) \otimes L^{k_{1even}k_{2even}}$ and $(\ell_L, \ell_R) \otimes L^{k_{1odd}k_{2odd}}$ reveal the vacuum configurations in terms of direct sums of $S^2_F \times S^2_F$. For instance, we have

$$\begin{aligned}
& {S^2_F}^{Int}_{k_1\,odd} \times {S^2_F}^{Int}_{k_2\,odd} := \\
& \mathbf{4} \sum_{n=0}^{\frac{k_1-1}{2}} \sum_{m=0}^{\frac{k_2-1}{2}} \Big[S^2_F(\ell_L + \ell^{k_1}_n) \times S^2_F(\ell_R + \ell^{k_2}_m) \oplus \cdots \oplus S^2_F(\ell_L + \ell^{k_1}_n) \times S^2_F(|\ell_R - \ell^{k_2}_m|) \\
& \quad \oplus S^2_F(\ell_L + \ell^{k_1}_n - 1) \times S^2_F(\ell_R + \ell^{k_2}_m) \oplus \cdots \oplus S^2_F(\ell_L + \ell^{k_1}_n - 1) \times S^2_F(|\ell_R - \ell^{k_2}_m|) \\
& \quad \oplus \vdots \\
& \quad \oplus S^2_F(|\ell_L - \ell^{k_1}_n|) \times S^2_F(\ell_R + \ell^{k_2}_m) \oplus \cdots \oplus S^2_F(|\ell_L - \ell^{k_1}_n|) \times S^2_F(|\ell_R - \ell^{k_2}_m|) \Big]\,.
\end{aligned} \qquad (4.86)$$

Remaining two cases are worked out explicitly in Appendix B.

We easily see from (4.86) and (B.7), (B.8) that, all higher winding number monopole sectors may be obtained from suitable projections of ${S_F^{2\,Int}}_{k_1} \times {S_F^{2\,Int}}_{k_2}$ in a systematic manner. As a quick example, let us consider the case with $k_1 = k_2 = 3$. Then, $({\Gamma^0_a}^L, {\Gamma^0_a}^R)$ has the representation content

$$\mathbf{4}[(0,0) \oplus (0,1) \oplus (1,0) \oplus (1,1)]\,, \tag{4.87}$$

and the vacuum configuration takes the form

$$\begin{aligned} {S_F^{2\,Int}}_{k_1=3} \times {S_F^{2\,Int}}_{k_2=3} = \mathbf{4}\Big[& \mathbf{4} S_F^2(\ell_L) \times S_F^2(\ell_R) \oplus \mathbf{2} S_F^2(\ell_L) \times S_F^2(\ell_R - 1) \\ & \oplus \mathbf{2} S_F^2(\ell_L) \times S_F^2(\ell_R + 1) \oplus \mathbf{2} S_F^2(\ell_L - 1) \times S_F^2(\ell_R) \\ & \oplus \mathbf{2} S_F^2(\ell_L + 1) \times S_F^2(\ell_R) \oplus \mathbf{2} S_F^2(\ell_L - 1) \times S_F^2(\ell_R - 1) \\ & \oplus \mathbf{2} S_F^2(\ell_L - 1) \times S_F^2(\ell_R + 1) \oplus \mathbf{2} S_F^2(\ell_L + 1) \times S_F^2(\ell_R - 1) \\ & \oplus \mathbf{2} S_F^2(\ell_L + 1) \times S_F^2(\ell_R + 1) \Big]\,. \end{aligned} \tag{4.88}$$

Monopole sectors with winding numbers $(0, \pm 2), (\pm 2, 0), (\pm 2\,, \pm 2), (\pm 2\,, \mp 2)$ are all available through projections of ${S_F^{2\,Int}}_{k_1=3} \times {S_F^{2\,Int}}_{k_2=3}$. Sectors with winding numbers, such as $(n, n-1)$, appear through projections of ${S_F^{2\,Int}}_{k_1} \times {S_F^{2\,Int}}_{k_2}$ for $k_1 \neq k_2$.

Before closing this section, let us also remark that for the $U(4)$ gauge theory over ${S_F^{2\,Int}}_{k_1=3} \times {S_F^{2\,Int}}_{k_2=3}$ there are no equivariant spinors. This is quiet expected, since, for $k_1 = k_2 = 3$, Ψ^L and Ψ^R transform under the IRRs $(1,0)$ and $(0,1)$ respectively and under the adjoint action of the symmetry generators we have

$$[\omega_a^L, \Psi_b^L] = \frac{i}{2} (\tilde{\lambda}_a)_{bc} \Psi_c^L = \epsilon_{abc} \Psi_c^L\,, \quad [\omega_a^R, \Psi_b^R] = \frac{i}{2} (\tilde{\lambda}_a)_{bc} \Psi_c^R = \epsilon_{abc} \Psi_c^R\,, \tag{4.89}$$

since $(\tilde{\lambda}_a)_{bc} = -2i\epsilon_{abc}$ in the adjoint representation of $SU(2)$. Thus these equivariant field modes are one and the same as those obtained from the equivariance conditions on Φ_a^L and Φ_a^R. From our results, we infer that the equivariant spinor fields over left and right fuzzy extra dimensions do exist only for both k_1 and k_2 even integers, while only left(right) spinor modes exist for $k_1(k_2)$ even only, and these modes do not exist at all for k_1 and k_2 both odd.

4.5 Relation to Fuzzy Superspace $S_F^{(2,2)} \times S_F^{(2,2)}$

It is possible to identify the vacuum configuration given in equation (4.44) as the bosonic (even) part of the fuzzy space $S_F^{(2,2)} \times S_F^{(2,2)}$ with $OSP(2,2) \times OSP(2,2)$ symmetry. This observation makes the vacuum configuration $S_F^{(2,2)} \times S_F^{(2,2)}$ especially interesting since, it simply comes out naturally and we have in no way intended for it to emerge.

In order to reveal this relation, we have to write down the decomposition of IRRs of $OSP(2,2) \times OSP(2,2)$ under the $SU(2) \times SU(2)$ IRRs. Irreducible representations of $OSP(2,1) \times OSP(2,1)$ are characterized by two integer or half-integer numbers
$(\mathcal{J}_1, \mathcal{J}_2)_{OSP(2,1)\times OSP(2,1)}$ and it has the decomposition under the $SU(2) \times SU(2)$ IRRs as

$$
\begin{aligned}
(\mathcal{J}_1, \mathcal{J}_2) = \Big[& (\mathcal{J}_1, \mathcal{J}_2) \oplus (\mathcal{J}_1 - \frac{1}{2}, \mathcal{J}_2) \oplus (\mathcal{J}_1, \mathcal{J}_2 - \frac{1}{2}) \\
& \oplus (\mathcal{J}_1 - \frac{1}{2}, \mathcal{J}_2 - \frac{1}{2}) \Big]_{SU(2)\times SU(2)} .
\end{aligned} \tag{4.90}
$$

Irreducible representations of $OSP(2,2) \times OSP(2,2)$ can be divided into two parts. These are the typical $(\mathcal{J}_1, \mathcal{J}_2)_T$, and the atypical $(\mathcal{J}_1, \mathcal{J}_2)_A$ representations. Typical representations $(\mathcal{J}_1, \mathcal{J}_2)_T$ are reducible under the $OSP(2,1) \times OSP(2,1)$ IRRs as

$$
(\mathcal{J}_1, \mathcal{J}_2)_T = (\mathcal{J}_1, \mathcal{J}_2) \oplus (\mathcal{J}_1 - \frac{1}{2}, \mathcal{J}_2) \oplus (\mathcal{J}_1, \mathcal{J}_2 - \frac{1}{2}) \oplus (\mathcal{J}_1 - \frac{1}{2}, \mathcal{J}_2 - \frac{1}{2}) \,, \tag{4.91}
$$

whereas the atypical ones are irreducible with respect to the group $OSP(2,1) \times OSP(2,1)$ and in fact $(\mathcal{J}_1, \mathcal{J}_2)_A$ is equivalent to the IRR $(\mathcal{J}_1, \mathcal{J}_2)$ of $OSP(2,1) \times OSP(2,1)$. All these facts follow from the generalization of the representation theory of $OSP(2,2)$ and $OSP(2,1)$, which is extensively discussed in [41, 42, 69]. With the help of equations (4.90) and (4.91), we see that $(\mathcal{J}_1, \mathcal{J}_2)_T$ of $OSP(2,2) \times OSP(2,2)$ has the decomposition in terms of the IRRs of $SU(2) \times$

$SU(2)$ as

$$(\mathcal{J}_1,\mathcal{J}_2)_T = \Big[(\mathcal{J}_1,\mathcal{J}_2)\oplus \mathbf{2}(\mathcal{J}_1,\mathcal{J}_2-\frac{1}{2})\oplus \mathbf{2}(\mathcal{J}_1-\frac{1}{2},\mathcal{J}_2)\oplus \mathbf{4}(\mathcal{J}_1-\frac{1}{2},\mathcal{J}_2-\frac{1}{2}) \\ \oplus(\mathcal{J}_1-1,\mathcal{J}_2)\oplus \mathbf{2}(\mathcal{J}_1-1,\mathcal{J}_2-\frac{1}{2})\oplus \mathbf{2}(\mathcal{J}_1-\frac{1}{2},\mathcal{J}_2-1) \\ \oplus(\mathcal{J}_1,\mathcal{J}_2-1)\oplus(\mathcal{J}_1-1,\mathcal{J}_2-1)\Big]_{SU(2)\times SU(2)}\,, \qquad \mathcal{J}_1\,,\mathcal{J}_2\geq 1\,, \tag{4.92}$$

while the representation $(\frac{1}{2},\frac{1}{2})_T$ decomposes as

$$(\frac{1}{2},\frac{1}{2})_T \equiv (\frac{1}{2},\frac{1}{2})\oplus(0,\frac{1}{2})\oplus(\frac{1}{2},0)\oplus(0,0) \\ \equiv \Big[(\frac{1}{2},\frac{1}{2})+\mathbf{2}(0,\frac{1}{2})\oplus \mathbf{2}(\frac{1}{2},0)\oplus \mathbf{4}(0,0)\Big]_{SU(2)\times SU(2)}\,. \tag{4.93}$$

It is now easy to see that, for $(\mathcal{J}_1,\mathcal{J}_2)_T\equiv(\ell_L+\frac{1}{2},\ell_R+\frac{1}{2})_T$, we obtain precisely the same IRR content from (4.92) as the one that appears for the vacuum configuration given in (4.43). This means that $S_F^{2\,Int}\times S_F^{2\,Int}$ can be identified with the bosonic part of the $OSP(2,2)\times OSP(2,2)$ fuzzy space $S_F^{(2,2)}\times S_F^{(2,2)}$ at the level $(\ell_L+\frac{1}{2},\ell_R+\frac{1}{2})_T$.

We further observe that $(\mathcal{J}_1,\mathcal{J}_2)\equiv(\ell_L+\frac{1}{2},\ell_R+\frac{1}{2})$ IRR of $OSP(2,1)\times OSP(2,1)$ matches with a particular sector of the representation given in (4.43) and allows us to identify

$$\Big(S_F^2(\ell_L+\frac{1}{2})\times S_F^2(\ell_R+\frac{1}{2})\Big)\oplus\Big(S_F^2(\ell_L)\times S_F^2(\ell_R+\frac{1}{2})\Big) \\ \oplus\Big(S_F^2(\ell_L+\frac{1}{2}\times S_F^2(\ell_R)\Big)\oplus\big(S_F^2(\ell_L)\times S_F^2(\ell_R)\big)\,, \tag{4.94}$$

with the bosonic part of $OSP(2,1)\times OSP(2,1)$ fuzzy space $S_F^{(2,1)}\times S_F^{(2,1)}$. The subsector given in (4.94) may be seen as the direct sum of two winding number $(1,0)$ monopole sectors as in (4.61) where one monopole sector differs from the other by the level of the right fuzzy spheres.

The superalgebra $osp(2,2)\times osp(2,2)$ has 16 generators $\Lambda_M^i:=(\Lambda_a^i,\Lambda_\mu^i,\Lambda_8^i)\,, i=L\,,R$ which satisfy the graded commutation relations

$$[\Lambda_a^i\,,\Lambda_b^i]=i\varepsilon_{abc}\Lambda_c^i\,,\quad [\Lambda_a^i\,,\Lambda_\mu^i]=\frac{1}{2}(\Sigma_a)_{\nu\mu}\Lambda_\nu^i\,,\quad [\Lambda_a^i\,,\Lambda_8^i]=0\,, \\ [\Lambda_8^i\,,\Lambda_\mu^i]=\Xi_{\mu\nu}\Lambda_\nu^i\,,\quad \{\Lambda_\mu^i\,,\Lambda_\nu^i\}=\frac{1}{2}(\mathcal{C}\Sigma_a)_{\mu\nu}\Lambda_a^i+\frac{1}{4}(\Xi\mathcal{C})_{\mu\nu}\Lambda_8^i\,, \tag{4.95}$$

where

$$\Sigma_a = \begin{pmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}, \quad \Xi = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{4.96}$$

and C is the two-dimensional Levi-Civita symbol and all the other graded commutation are zero. Reality condition implemented by the graded dagger operation on the generators reads

$$\Lambda_a^\ddagger = \Lambda_a^\dagger = \Lambda_a\,, \quad \Lambda_\mu^\ddagger = -\mathcal{C}_{\mu\nu}\Lambda_\nu\,, \quad \Lambda_8^\ddagger = \Lambda_8^\dagger = \Lambda_8\,, \tag{4.97}$$

for both the left and the right generators.

Using the representation theory of $osp(2,1)$ and $osp(2,2)$, it is rather straightforward to construct the nine-dimensional fundamental representation $(\frac{1}{2},\frac{1}{2})_A$ of $osp(2,2)\times osp(2,2)$ which is at the same time the $(\frac{1}{2},\frac{1}{2})$ IRR of $osp(2,1)\times osp(2,1)$. Generators of the three-dimensional representation of $osp(2,2)$ may be written as

$$\lambda_a := \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\sigma_a \end{pmatrix}, \quad \lambda_4 := \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 := \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{4.98}$$

$$\lambda_6 := \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_7 := \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_8 := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Construction of these generators and a detailed exposition of the properties of the $osp(2,2)$ and $osp(2,1)$ superalgebras can be found in [25, 41]. 16 generators $(\Lambda_M^L\,,\Lambda_M^R)$ in the IRR $(\frac{1}{2},\frac{1}{2})_A$ can be given as

$$\begin{aligned} &\Lambda_M^L \equiv \lambda_M\otimes \mathbf{1}_3\,, \quad \Lambda_a^R = \mathbf{1}_3\otimes\lambda_a\,, \quad \Lambda_{4,5}^R = \alpha\otimes\lambda_{4,5}\,, \quad \Lambda_{6,7}^R = -\alpha\otimes\lambda_{6,7}\,, \\ &\Lambda_8^R = -\mathbf{1}_3\otimes\lambda_8\,, \end{aligned} \tag{4.99}$$

where $\alpha = 3\mathbf{1}_3 - 2\lambda_8$.

The matrices $\Gamma_a^{0\,L}\,,\Gamma_a^{0\,R}\,,b_\alpha\,,c_\alpha\,,b_\alpha^\dagger\,,c_\alpha^\dagger\,,N^L\,,N^R$ constitute a basis for the 16×16 matrices acting on the sixteen-dimensional module corresponding to the representation space in (4.41), and coincides with that of (4.93). We can make use of these matrices to construct generators of the representation $(\frac{1}{2},\frac{1}{2})_A$ given in (4.99). To do so, we should restrict to one of the nine-dimensional submodules with the representation content $(\frac{1}{2},\frac{1}{2})\oplus(0,\frac{1}{2})\oplus(\frac{1}{2},0)\oplus(0,0)$. Clearly, there exists a set of projectors which yield the same representation, and a particular projector from this set is

$$\mathcal{P} := \mathcal{P}^L_{0_{\mathbf{2}}}\mathcal{P}^R_{0_{\mathbf{2}}} + \mathcal{P}^L_{0_{\mathbf{2}}}\mathcal{P}^R_{\frac{1}{2}} + \mathcal{P}^R_{0_{\mathbf{2}}}\mathcal{P}^L_{\frac{1}{2}} + \mathcal{P}^L_{\frac{1}{2}}\mathcal{P}^R_{\frac{1}{2}}\,, \tag{4.100}$$

where we have $\mathcal{P}^L_{0_{\mathbf{2}}} = \mathbf{1}_4\otimes P_{0_{\mathbf{2}}}$, $\mathcal{P}^L_{\frac{1}{2}} = \mathbf{1}_4\otimes P_{\frac{1}{2}}$, $\mathcal{P}^R_{0_{\mathbf{2}}} = P_{0_{\mathbf{2}}}\otimes\mathbf{1}_4$, $\mathcal{P}^R_{\frac{1}{2}} = P_{\frac{1}{2}}\otimes\mathbf{1}_4$. Using $\mathcal{P}$, we can restrict to the nine-dimensional submodule and subsequently get

$$\begin{aligned}
&\Lambda^L_1 := -i\mathcal{P}{\Gamma^0_1}^L\,,\quad \Lambda^L_2 := i\mathcal{P}{\Gamma^0_2}^L\,,\quad \Lambda^L_3 := -i\mathcal{P}{\Gamma^0_3}^L\,,\quad \Lambda^L_4 := -\frac{1}{2}(\tilde{b}_1+\tilde{b}^\dagger_2)\,,\\
&\Lambda^L_5 := \frac{1}{2}(\tilde{b}^\dagger_1-\tilde{b}_2)\,,\quad \Lambda^L_6 := \frac{1}{2}(\tilde{b}_1-\tilde{b}^\dagger_2)\,,\quad \Lambda^L_7 := \frac{1}{2}(\tilde{b}^\dagger_1+\tilde{b}_2)\,,\quad \Lambda^L_8 := \mathcal{P}N\,,
\end{aligned} \tag{4.101}$$

and

$$\begin{aligned}
&\Lambda^R_1 := -i\mathcal{P}{\Gamma^0_1}^R\,,\quad \Lambda^R_2 := i\mathcal{P}{\Gamma^0_2}^R\,,\quad \Lambda^R_3 := i\mathcal{P}{\Gamma^0_3}^R\,,\quad \Lambda^R_4 := \frac{1}{2}(\tilde{c}_1+\tilde{c}^\dagger_2)\,,\\
&\Lambda^R_5 := -\frac{1}{2}(\tilde{c}^\dagger_1-\tilde{c}_2)\,,\quad \Lambda^R_6 := \frac{1}{2}(\tilde{c}_1-\tilde{c}^\dagger_2)\,,\quad \Lambda^R_7 := \frac{1}{2}(\tilde{c}^\dagger_1+\tilde{c}_2)\,,\quad \Lambda^R_8 := -\mathcal{P}M\,,
\end{aligned} \tag{4.102}$$

where

$$\tilde{b}_\alpha = \mathcal{P}b_\alpha\mathcal{P},\quad \tilde{b}^\dagger_\alpha = \mathcal{P}b^\dagger_\alpha\mathcal{P},\quad \tilde{c}_\alpha = \mathcal{P}c_\alpha\mathcal{P},\quad \tilde{c}^\dagger_\alpha = \mathcal{P}c^\dagger_\alpha\mathcal{P}\,. \tag{4.103}$$

We note in passing that the graded dagger operation on the matrices given in (4.103) reads

$$\tilde{b}^\ddagger_\alpha = \tilde{b}^\dagger_\alpha\,,\quad (\tilde{b}^\dagger_\alpha)^\ddagger = -\tilde{b}_\alpha\quad \tilde{c}^\ddagger_\alpha = \tilde{c}^\dagger_\alpha\,,\quad (\tilde{c}^\dagger_\alpha)^\ddagger = -\tilde{c}_\alpha\,. \tag{4.104}$$

Finally, in (4.101) and (4.102), it is understood that the columns and rows of zero are deleted after the projection and therefore, we have 9×9 matrices $(\Lambda^L_M,\Lambda^R_M)$ as intended.

4.6 Stability of the Vacuum Solutions

The recent new approach introduced in [70] has already been successfully employed in [25] to argue the stability of vacuum solutions in the form of direct sums of fuzzy spheres for an $SU(\mathcal{N})$ gauge theory coupled to scalar fields in the vector and spinor representation of a global $SU(2)$ symmetry. In this section, we follow the ideas of [70] which were adapted to the present context in [25] to investigate and demonstrate the stability of our vacuum configuration $S_F^{2\,Int} \times S_F^{2\,Int}$. As we have emphasized earlier we are working on a gauge theory matrix model, which is a massive deformation of the $N = 4$ supersymmetric Yang-Mills theory. For matrix models stemming from low energy limit of string theories, vacuum configurations for the potentials are usually described either by a single fuzzy sphere or its direct sums, or, as in the present model, in terms of the product $S_F^2 \times S_F^2$ or its direct sums. The critical observation that was made in [70] is that, such direct sum of fuzzy spheres form mixed states if one or several of the fuzzy sphere(s) at a given level occur more than once in the direct sum, whereas the solutions given by a single fuzzy sphere are pure states. The stability of the former type vacuum configurations are guaranteed due to the fact that mixed states cannot unitarily evolve to pure states.

In order to understand the structure of the vacuum solutions in this work, we now apply the ideas in [70] following [25]. Let us think of a state ω on the matrix algebra $\mathcal{A} = Mat(\mathcal{N})$. All the matrices spanning the vacuum configuration are in this matrix algebra. ω is a linear map $\omega : \mathcal{A} \to \mathbb{C}$ which satisfies

$$\omega(\Phi^*\Phi) \geq 0\,, \quad \forall \Phi \in \mathcal{A}\,, \quad \omega(\mathbf{1}) = 1\,. \tag{4.105}$$

To describe a single fuzzy sphere S_F^2 at the level L in this setting, we demand the following condition

$$\omega(X_a X_a) = L(L+1)\omega(\mathbf{1}) = -L(L+1)\,, \tag{4.106}$$

is satisfied. In a similar manner $S_F^2(\ell_L) \times S_F^2(\ell_R)$ is described by imposing the condition

$$\omega(X_a^L X_a^L + X_a^R X_a^R) = -L(L+1) - R(R+1)\,. \tag{4.107}$$

We have the vacuum configuration given in (4.44) as :

$$
\begin{aligned}
S_F^{2\,Int} \times S_F^{2\,Int} = & \; \mathbf{4}\left(S_F^2(\ell_L) \times S_F^2(\ell_R)\right) \oplus \mathbf{2}\left(S_F^2(\ell_L - \frac{1}{2}) \times S_F^2(\ell_R)\right) \\
& \oplus \mathbf{2}\left(S_F^2(\ell_L + \frac{1}{2}) \times S_F^2(\ell_R)\right) \oplus \mathbf{2}\left(S_F^2(\ell_L) \times S_F^2(\ell_R - \frac{1}{2})\right) \\
& \oplus \mathbf{2}\left(S_F^2(\ell_L) \times S_F^2(\ell_R + \frac{1}{2})\right) \oplus \left(S_F^2(\ell_L - \frac{1}{2}) \times S_F^2(\ell_R - \frac{1}{2})\right) \\
& \oplus \left(S_F^2(\ell_L + \frac{1}{2}) \times S_F^2(\ell_R - \frac{1}{2})\right) \oplus \left(S_F^2(\ell_L - \frac{1}{2}) \times S_F^2(\ell_R + \frac{1}{2})\right) \\
& \oplus \left(S_F^2(\ell_L + \frac{1}{2}) \times S_F^2(\ell_R + \frac{1}{2})\right) . \qquad (4.108)
\end{aligned}
$$

We list the projections to each summand in this expression and their rank in the table below.

Projector	Rank
$\Pi^L_{0_{\mathbf{0}}}\Pi^R_{0_{\mathbf{0}}}$, $\Pi^L_{0_{\mathbf{0}}}\Pi^R_{0_{\mathbf{2}}}$, $\Pi^L_{0_{\mathbf{2}}}\Pi^R_{0_{\mathbf{0}}}$, $\Pi^L_{0_{\mathbf{2}}}\Pi^R_{0_{\mathbf{2}}}$	$(2\ell_L+1)(2\ell_R+1)n$
$\Pi^L_{-}\Pi^R_{0_{\mathbf{0}}}$, $\Pi^L_{-}\Pi^R_{0_{\mathbf{2}}}$	$(2\ell_L)(2\ell_R+1)n$
$\Pi^L_{+}\Pi^R_{0_{\mathbf{0}}}$, $\Pi^L_{+}\Pi^R_{0_{\mathbf{2}}}$	$(2\ell_L+2)(2\ell_R+1)n$
$\Pi^L_{0_{\mathbf{0}}}\Pi^R_{-}$, $\Pi^L_{0_{\mathbf{2}}}\Pi^R_{-}$	$(2\ell_L+1)(2\ell_R)n$
$\Pi^L_{0_{\mathbf{0}}}\Pi^R_{+}$, $\Pi^L_{0_{\mathbf{2}}}\Pi^R_{+}$	$(2\ell_L+1)(2\ell_R+2)n$
$\Pi^L_{-}\Pi^R_{-}$	$(2\ell_L)(2\ell_R)n$
$\Pi^L_{-}\Pi^R_{+}$	$(2\ell_L)(2\ell_R+2)n$
$\Pi^L_{+}\Pi^R_{-}$	$(2\ell_L+2)(2\ell_R)n$
$\Pi^L_{+}\Pi^R_{+}$	$(2\ell_L+2)(2\ell_R+2)n$

Let us define the states $\omega_{\alpha\beta}$ by the requirement

$$
\omega_{\alpha\beta}\big(\Pi^L_\alpha \Pi^R_\beta (\mathcal{D}^L_a \mathcal{D}^L_a + \mathcal{D}^R_a \mathcal{D}^R_a)\Pi^L_\alpha \Pi^R_\beta\big) = -L_\alpha(L_\alpha+1) - R_\beta(R_\beta+1)\,, \qquad (4.109)
$$

where the indices α, β take on the values $(0_{\mathbf{0}},\, 0_{\mathbf{2}},\, +,\, -)$, L_α, R_β take on the values $(\ell_L,\, \ell_L,\, \ell_L + \frac{1}{2}, \ell_L - \frac{1}{2})$ and $(\ell_R,\, \ell_R,\, \ell_R + \frac{1}{2}, \ell_R - \frac{1}{2})$, respectively, and no sum over α, β is implied. In (4.109) we have used $\mathcal{D}^L_a$ and $\mathcal{D}^R_a$ introduced earlier in (B.3). With the condition given in (4.109), the matrix algebra $\mathcal{A}$ is divided into the direct sum of the matrix algebras

$$
\begin{aligned}
\mathcal{A}_\Pi := &\, \mathbf{4}\, Mat((2\ell_L+1)(2\ell_R+1)n) \oplus \mathbf{2}\, Mat((2\ell_L)(2\ell_R+1)n) \\
&\oplus \mathbf{2}\, Mat((2\ell_L+2)(2\ell_R+1)n) \oplus \mathbf{2}\, Mat((2\ell_L+1)(2\ell_R)n) \\
&\oplus \mathbf{2}\, Mat((2\ell_L+1)(2\ell_R+2)n) \oplus Mat((2\ell_L)(2\ell_R)n) \\
&\oplus Mat((2\ell_L+1)(2\ell_R+2)n) \oplus Mat((2\ell_L+2)(2\ell_R)n) \\
&\oplus Mat((2\ell_L+2)(2\ell_R+2)n). \qquad (4.110)
\end{aligned}
$$

We can conventionally label the basis kets of the module on which $\mathcal{A}_\Pi$ acts as $|L_\alpha\,, L_3;\, R_\beta\,, R_3\rangle$. For brevity and clarity of notation, we suppress the labels L and R, write the α, β subscripts separately and hence write these kets as $|L_3,\, R_3; [\alpha, \beta]\rangle$. Projections $\Pi^L_\alpha \Pi^R_\beta$ can be expressed as

$$
\Pi^L_\alpha \Pi^R_\beta = \sum_{R_3=-R}^{R} \sum_{L_3=-L}^{L} |L_3,\, R_3; [\alpha, \beta]\rangle\langle L_3,\, R_3; [\alpha, \beta]| \,, \quad \Pi^L_\alpha \Pi^R_\beta \in \mathcal{A}_\Pi \,. \qquad (4.111)
$$

Although the projections to subsectors that appear only once in (4.108) are unique up to unitary transformations, this is not the case for the sectors of (4.108) that occur with multiplicities. In fact we observe that under the unitary transformation u belonging to the group $U(4) \otimes \mathbf{4}\, U(2) \otimes \mathbf{4}\, U(1) \equiv \mathcal{U}$, basis kets become

$$
|L_3,\, R_3; [\alpha, \beta]\rangle = \sum_{\sigma\rho} u_{[\alpha\beta]\,[\sigma\rho]} |L_3,\, R_3; [\sigma, \rho]\rangle \,. \qquad (4.112)
$$

From (4.112), we find that the projectors (4.111) transform as follows $\Pi^L_\alpha \Pi^R_\beta \to \mathcal{U}^\dagger \Pi^L_\alpha \Pi^R_\beta \mathcal{U}$, which gives

$$
\Pi^L_\alpha \Pi^R_\beta[u] = \sum_{L_3=-L}^{L} \sum_{R_3=-R}^{R} \sum_{[\sigma\rho],\,[rs]} u^\dagger_{[rs][\alpha\beta]} u_{[\alpha\beta][\sigma\rho]} |L_3,\, R_3; [\sigma, \rho]\rangle\langle L_3,\, R_3; [r, s]| \,. \qquad (4.113)
$$

Thus, after this unitary transformation, $\Pi^L_\alpha \Pi^R_\alpha[u]$ are still projectors, since they satisfy

$$
(\Pi^L_\alpha \Pi^R_\beta[u])^2 = \Pi^L_\alpha \Pi^R_\beta[u] \,, \quad (\Pi^L_\alpha \Pi^R_\beta[u])^\dagger = \Pi^L_\alpha \Pi^R_\beta[u] \,. \qquad (4.114)
$$

Here, it is important to note that we have $u_{[\alpha\beta][\sigma\rho]} = \delta_{[\alpha\beta][\sigma\rho]}$ for $\alpha, \beta = +, -$, which gives $\Pi^L_\pm \Pi^R_\pm[u] = \Pi^L_\pm \Pi^R_\pm$; this however does not hold for the remaining 12 projectors. For instance, the projections to the four subsectors carrying the IRR (ℓ_L, ℓ_R) get mixed by the $U(4)$ subgroup of $\mathcal{U}$. Likewise, there are four distinct subsectors, in each of which two projectors get mixed by separate $U(2)$ subgroups of $\mathcal{U}$. All this means is that, in general not all the transformed $\Pi^L_\pm \Pi^R_\pm[u]$ belong to the algebra of observables $\mathcal{A}_\Pi$.

Let us consider the expectation value of an element $\mathcal{O}$ of $\mathcal{A}_\Pi$ in the state ω

$$\omega(\mathcal{O}) = \sum_{[\alpha\beta]} \lambda_{[\alpha\beta]} \omega_{[\alpha\beta]}(\mathcal{O}) \,, \tag{4.115}$$

where

$$\omega_{[\alpha\beta]}(\mathcal{O}) = \frac{1}{2L_\alpha+1}\frac{1}{2R_\beta+1} \sum_{L_3,R_3}\sum_{L'_3,R'_3} \big\langle L_3,\, R_3; [\alpha\beta] \big| \mathcal{O} \big| L'_3,\, R'_3; [\alpha\beta] \big\rangle \,, \tag{4.116}$$

and $\lambda_{[\alpha\beta]}$ is a probability vector satisfying

$$0 \leq \lambda_{[\alpha\beta]} \leq 1, \quad \sum_{\alpha\beta} \lambda_{[\alpha\beta]} = 1 \,. \tag{4.117}$$

It is obvious that the state $\omega_{[\alpha\beta]}(\mathcal{O})$ is invariant under the unitary transform (4.112) and therefore it has the unitary symmetry $U(4) \otimes \mathbf{4}\, U(2) \otimes \mathbf{4}\, U(1)$. This fact indicates that under $\mathcal{U}$, $\lambda_{[\alpha\beta]}$ transforms to

$$\lambda_{[\sigma\rho]}(u) = \sum_{[\alpha\beta]} \lambda_{[\alpha\beta]} u^\dagger_{[\sigma\rho][\alpha\beta]} u_{[\alpha\beta][\sigma\rho]} = \sum_{[\alpha\beta]} \lambda_{[\alpha\beta]} |u_{[\alpha\beta][\rho\sigma]}|^2 \,,$$
$$\text{no sum over } [\sigma\rho] \text{ in the r.h.s.} \,. \tag{4.118}$$

Alluding to our remark after (4.114), we note that we have $\lambda_{\pm\pm}(u) = \lambda_{\pm\pm}$, $\lambda_{\pm\mp}(u) = \lambda_{\pm\mp}$ while in general $\lambda_{[\alpha\beta]}(u) \neq \lambda_{[\alpha\beta]}$ for $\alpha,\, \beta \neq +,\, -$. Thus, the decomposition of $\omega(\mathcal{O})$ into $\omega_{[\alpha\beta]}(\mathcal{O})$ is not unique, Consequences of this fact may be most easily recognized in the density matrix language.

We first express $\omega_{[\alpha\beta]}(\mathcal{O})$ by introducing a density matrix $\rho_{[\alpha\beta]}$.This is the density matrix of the pure state

$$\rho_{[\alpha\beta]} = |\psi_{[\alpha\beta]}\rangle\langle\psi_{[\alpha\beta]}| = \sum_{L_3,\, R_3\,, L'_3,\, R'_3} C^*_{L'_3 R'_3} C_{L_3 R_3} |L_3,\, R_3; [\alpha\beta]\rangle\langle L'_3,\, R'_3; [\alpha\beta]| \,, \tag{4.119}$$

where

$$|\psi_{[\alpha\beta]}\rangle = \sum_{L_3,R_3} C_{L_3R_3}|L_3,\ R_3;[\alpha\beta]\rangle\,,\ \sum_{L_3,R_3}|C_{L_3R_3}|^2 = 1\,,\ 0\leq |C^*_{L_3'R_3'}C_{L_3R_3}|\leq 1\,. \tag{4.120}$$

To construct the same state as in (4.115) in this language, we introduce the density matrix ρ

$$\rho = \sum_{[\alpha\beta]}\lambda_{[\alpha\beta]}(u)\rho_{[\alpha\beta]}\,,\quad 0<\lambda_{[\alpha\beta]}<1\,,\quad \sum_{[\alpha\beta]}\lambda_{[\alpha\beta]}=1\,. \tag{4.121}$$

Expectation values of $\mathcal{O}$ in the state $\omega_{[\alpha\beta]}$ and ω are expressed as

$$\omega_{[\alpha\beta]}(\mathcal{O}) = Tr(\rho_{[\alpha\beta]}\mathcal{O})\,,\quad \omega(\mathcal{O}) = Tr(\rho\mathcal{O})\,. \tag{4.122}$$

Noting that $\rho_{[\alpha\beta]}\Pi^L_\alpha\Pi^R_\beta = \rho_{[\alpha\beta]}$, we can easily check that $\omega_{[\alpha\beta]}(\mathcal{O})$ in (4.122) is consistent with the condition give in (4.109) and matches with the form given in (4.116) and therefore $\omega(\mathcal{O})$ agrees with the expression given in equation (4.115). Due to the unitary symmetry $\mathcal{U}$ transforming $\lambda_{[\sigma\rho]}(u)$'s as given in (4.118), the decomposition of ρ into $\rho_{[\alpha\beta]}$ is not unique. This means that ρ characterizes a mixed state. The latter is also evident from the fact that

$$Tr(\rho^2) = \sum_{[\alpha\beta]}|\lambda_{[\alpha\beta]}(u)|^2 < 1\,. \tag{4.123}$$

Since $S^{2\,Int}_F \times S^{2\,Int}_F$ is characterized by the density matrix ρ, we arrive at the conclusion that our vacuum solution forms a mixed state. As mixed states cannot unitarily evolve into pure states in time, $S^{2\,Int}_F \times S^{2\,Int}_F$ cannot decay to $S^2_F \times S^2_F$, a pure state, and hence the $S^{2\,Int}_F \times S^{2\,Int}_F$ vacuum is stable. From the same reasoning, it follows that the generalized vacuum solution obtained in section 4 are also stable as they form mixed states too.

We can compute the von Neumann entropy of $S^{2\,Int}_F \times S^{2\,Int}_F$. This is given as

$$\begin{aligned} S(\rho) &= -Tr(\rho\log\rho)\\ &= -\sum_{[\alpha\beta]}\lambda_{[\alpha\beta]}(u)\log\lambda_{[\alpha\beta]}(u) + \sum_{[\alpha\beta]}\lambda_{[\alpha\beta]}(u)S(\rho_{[\alpha\beta]})\\ &= -\sum_{[\alpha\beta]}\lambda_{[\alpha\beta]}(u)\log\lambda_{[\alpha\beta]}(u)\,, \end{aligned} \tag{4.124}$$

where we have used the entropy theorem [71] in writing the second line and the fact that $S(\rho_{[\alpha\beta]}) = 0$ in writing the last line. Dependence of $\lambda_{[\alpha\beta]}(u)$ on u as given in (4.118) indicates a Markovian process which is doubly stochastic since

$$\sum_{\alpha\beta} |u_{[\alpha\beta][\rho\sigma]}|^2 = \sum_{\rho\sigma} |u_{[\alpha\beta][\rho\sigma]}|^2 = 1\,. \tag{4.125}$$

This process will increase the entropy of $S_F^{2\,Int} \times S_F^{2\,Int}$, since it is irreversible. We see that $S(\rho)$ has its maximal value $S^{max}(\rho) = 4\log 2$ which is attained if $\lambda_{[\alpha\beta]} = \frac{1}{16}$ for all $[\alpha\beta]$. We see that this maximal value may only be reached if and only if the system starts with the probabilities $\lambda_{\pm\pm} = \lambda_{\pm\mp} = \frac{1}{16}$ since $\lambda_{\pm\pm}(u) = \lambda_{\pm\pm}\,, \lambda_{\pm\mp}(u) = \lambda_{\pm\mp}$. If the latter is not the case, the quantum entropy still increases but cannot reach the value $4\log 2$.

4.7 Another Vacuum Solution

It is worthwhile to ask whether it is possible to find solutions to equations given in (4.27) in the form

$$\begin{aligned}
\Phi_a^L &= (X_a^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_4 \otimes \mathbf{1}_n) + (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \tilde{\Gamma_a^0}^L \otimes \mathbf{1}_n)\,,\\
\Phi_a^R &= (\mathbf{1}^{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_4 \otimes \mathbf{1}_n) + (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \tilde{\Gamma_a^0}^R \otimes \mathbf{1}_n)\,,
\end{aligned} \tag{4.126}$$

with the factorization $\mathcal{N} = (2\ell_L + 1) \times (2\ell_R + 1) \times 4 \times n$ and where $\tilde{\Gamma_a^0}^L$ and $\tilde{\Gamma_a^0}^R$ are 4×4 matrices instead of the 16×16 matrices determined in section 4.2.1, satisfying the relations in (4.29). The answer to this question is only superficially affirmative as such $\tilde{\Gamma_a^0}^L$ and $\tilde{\Gamma_a^0}^R$ exist, but against the very premise of our initial requirement that $\tilde{\Gamma_a^0}^L$ and $\tilde{\Gamma_a^0}^R$ are bilinears of the doublets Ψ^L and Ψ^R of $SU(2) \times SU(2)$ transforming under its $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ IRR's. To be more concrete, it turns out that it is possible to express $\tilde{\Gamma_a^0}^L$ and $\tilde{\Gamma_a^0}^R$ in terms of bilinears of some matrices χ^L and χ^R, which, however, do not transform as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ under $SU(2)\times SU(2)$. This fact suggests that, we should expect to find no equivariant spinor field modes at all for the emerging effective $U(4)$ gauge theory. It appears instructive to examine this case in some detail.

If we start with two sets of fermionic annihilation-creation operators $a_\alpha, a_\alpha^\dagger$ given in (4.8) and we choose

$$\chi^L = \begin{pmatrix} \chi_1^L \\ \chi_2^L \end{pmatrix} := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} , \quad \chi^R = \begin{pmatrix} \chi_1^R \\ \chi_2^R \end{pmatrix} := \begin{pmatrix} a_1^\dagger \\ a_2 \end{pmatrix} , \tag{4.127}$$

then, $\tilde{\Gamma^0_a}^L = -\frac{i}{2}{\chi^L}^\dagger \tau_a \chi^L$, $\Gamma^0{}_a^R = -\frac{i}{2}{\chi^R}^\dagger \tau_a \chi^R$ satisfy

$$[\tilde{\Gamma^0_a}^L, \tilde{\Gamma^0_b}^L] = \epsilon_{abc}\tilde{\Gamma^0_c}^L , \quad [\tilde{\Gamma^0_a}^R, \tilde{\Gamma^0_b}^R] = \epsilon_{abc}\tilde{\Gamma^0_c}^R , \quad [\tilde{\Gamma^0_a}^L, \tilde{\Gamma^0_b}^R] = 0 . \tag{4.128}$$

However, we find that

$$\begin{aligned} &[\chi_\alpha^L, \tilde{\Gamma^0_a}^L] = -\frac{i}{2}(\tau_a)_{\alpha\beta}\chi_\beta^L , \quad [\chi_\alpha^L, \tilde{\Gamma^0_a}^R] \neq 0 , \\ &[\chi_\alpha^R, \tilde{\Gamma^0_a}^R] = -\frac{i}{2}(\tau_a)_{\alpha\beta}\chi_\beta^R , \quad [\chi_\alpha^R, \tilde{\Gamma^0_a}^L] \neq 0 . \end{aligned} \tag{4.129}$$

Thus, due to the two nonvanishing commutators in (4.129), χ^L and χ^R are *not* transforming in the IRRs $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of $SU(2) \times SU(2)$, respectively. Bearing this fact in mind, we can nevertheless continue to work with the matrices $\tilde{\Gamma^0_a}^L$ and $\tilde{\Gamma^0_a}^R$ satisfying (4.128), and investigate the structure of the emerging model in its own right.

Using the identities

$$(\tilde{\Gamma^0_a}^L)^2 = -\frac{3}{4}N + \frac{3}{2}N_1N_2 , \quad (\tilde{\Gamma^0_a}^R)^2 = \frac{3}{4}N - \frac{3}{2}N_1N_2 - \frac{3}{4} , \tag{4.130}$$

where $N = N_1 + N_2$, $N_1 = a_1^\dagger a_1$, $N_2 = a_2^\dagger a_2$, the quadratic Casimir operator can be evaluated and we simply find

$$C_2 = (\tilde{\Gamma^0_a}^L)^2 + (\tilde{\Gamma^0_a}^R)^2 = -\frac{3}{4}\mathbf{1}_4 . \tag{4.131}$$

This means that $(\tilde{\Gamma^0_a}^L, \tilde{\Gamma^0_a}^R)$ carry the direct sum representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.

As we mentioned earlier in (4.9), these annihilation-creation operators span the 4-dimensional Hilbert space which has four states: $|0,\, 0\rangle$, $|0,\, 1\rangle$, $|1,\, 0\rangle$, $|1,\, 1\rangle$. With the choice (4.127), $\tilde{\Gamma^0_a}^L$ is reducible with respect to $SU(2)_L$ and has two inequivalent singlets, $|0,\, 0\rangle$, $|1,\, 1\rangle$ and a doublet, spanned by $|0,\, 1\rangle$, $|1,\, 0\rangle$. Similarly, $\tilde{\Gamma^0_a}^R$ is reducible with respect to $SU(2)_R$ and has two inequivalent singlets, $|0,\, 1\rangle$, $|1,\, 0\rangle$, and a doublet, spanned by $|0,\, 0\rangle$, $|1,\, 1\rangle$:

$$\begin{aligned} &\tilde{\Gamma^0_a}^L \to (0_{\mathbf{0}}, 0) \oplus (0_{\mathbf{2}}, 0) \oplus (\frac{1}{2}, 0) , \\ &\tilde{\Gamma^0_a}^R \to (0, 0_{\mathbf{0}}) \oplus (0, 0_{\mathbf{2}}) \oplus (0, \frac{1}{2}) . \end{aligned} \tag{4.132}$$

Two inequivalent singlets of $\tilde{\Gamma_a^0}^L$ can be distinguished by the eigenvalues $0, 2$ of N, since $[\tilde{\Gamma_a^0}^L, N] = 0$. Likewise, the eigenvalues $0, 2$ of the operator $(\mathbf{1}_4 - (N_1 - N_2))$ distinguishes the two inequivalent singlets of $\tilde{\Gamma_a^0}^R$ since $[\tilde{\Gamma_a^0}^R, \mathbf{1}_4 - (N_1 - N_2)] = 0$.

Let us define the two projectors

$$
\begin{aligned}
P_0 &= \frac{(\tilde{\Gamma_a^0}^L)^2 + \frac{3}{4}}{\frac{3}{4}} = -\frac{(\tilde{\Gamma_a^0}^R)^2}{\frac{3}{4}} = 1 - N + 2N_1N_2\,,\\
P_{\frac{1}{2}} &= -\frac{(\tilde{\Gamma_a^0}^L)^2}{\frac{3}{4}} = \frac{(\tilde{\Gamma_a^0}^R)^2 + \frac{3}{4}}{\frac{3}{4}} = N - 2N_1N_2\,,
\end{aligned}
\tag{4.133}
$$

where P_0 projects to the singlets of $\tilde{\Gamma_a^0}^L$ and to the doublet of $\tilde{\Gamma_a^0}^R$, and $P_{\frac{1}{2}}$ projects to the doublet of $\tilde{\Gamma_a^0}^L$ and to the singlet of $\tilde{\Gamma_a^0}^R$. Projections to the inequivalent singlets and spin up and down components of doublets read

$$
\begin{aligned}
&P^L_{0_\mathbf{0}} = -\frac{1}{2}(N-2)P_0 = 1 - N + N_1N_2\,, \quad P^L_{0_\mathbf{2}} = \frac{1}{2}NP_0 = N_1N_2\,,\\
&P^L_{\frac{1}{2}+} = P_{\frac{1}{2}}N_1 = N_1 - N_1N_2\,, \quad P^L_{\frac{1}{2}-} = P_{\frac{1}{2}}N_2 = N_2 - N_1N_2\,,\\
&P^R_{0_\mathbf{0}} = P^L_{\frac{1}{2}+}\,, \quad P^R_{0_\mathbf{2}} = P^L_{\frac{1}{2}-}\,, \quad P^R_{\frac{1}{2}+} = P^L_{0_\mathbf{0}} \quad P^R_{\frac{1}{2}-} = P^L_{0_\mathbf{2}}\,.
\end{aligned}
\tag{4.134}
$$

The Clebsch-Gordan decomposition of the vacuum configuration proposed in equation (4.126) is determined as

$$
\begin{aligned}
(\ell_L, \ell_R) \otimes \left((\frac{1}{2}, 0) \oplus (\frac{1}{2}, 0)\right) \equiv &(\ell_L + \frac{1}{2}, \ell_R) \oplus (\ell_L - \frac{1}{2}, \ell_R) \oplus (\ell_L, \ell_R + \frac{1}{2})\\
&\oplus (\ell_L, \ell_R - \frac{1}{2})\,.
\end{aligned}
\tag{4.135}
$$

This means that the vacuum configuration can be written as the direct sum

$$
\begin{aligned}
S_F^{2\,Int} \times S_F^{2\,Int} \equiv &\left(S_F^2(\ell_L + \frac{1}{2}) \times S_F^2(\ell_R)\right) \oplus \left(S_F^2(\ell_L - \frac{1}{2}) \times S_F^2(\ell_R)\right)\\
&\oplus \left(S_F^2(\ell_L) \times S_F^2(\ell_R + \frac{1}{2})\right) \oplus \left(S_F^2(\ell_L) \times S_F^2(\ell_R - \frac{1}{2})\right)\,.
\end{aligned}
\tag{4.136}
$$

Projections to each summand in (4.136) can be obtained by adapting the formula in (4.45) to the present case. This yields the projectors $\Pi_{\alpha\beta} \equiv \{\Pi_{+0}, \Pi_{-0}\,, \Pi_{0+}\,, \Pi_{0-}\}$ (see, equation (4.139) below) which, upon using the suitably adapted version of (4.46), are unitarily equivalent to the product $\Pi^L_\alpha \Pi^R_\beta$, which we write as $\Pi_{\alpha\beta} \equiv \Pi^L_\alpha \Pi^R_\beta$.

For the projectors $\Pi_0^L\,, \Pi_0^R\,, \Pi_\pm^L\,, \Pi_\pm^R$, we have the explicit forms

$$\begin{aligned} &\Pi_0^L = \mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes P_0 \otimes \mathbf{1}_n\,, \quad \Pi_0^R = \mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes P_{\frac{1}{2}} \otimes \mathbf{1}_n\,, \\ &\Pi_\pm^L = \frac{1}{2}(\pm i Q_I^L + \Pi_{\frac{1}{2}}^L)\,, \quad \Pi_\pm^R = \frac{1}{2}(\pm i Q_I^R + \Pi_{\frac{1}{2}}^R)\,, \end{aligned} \tag{4.137}$$

where

$$Q_I^L = i \frac{X_a^L \tilde{\Gamma_a^0}^L - \frac{1}{4}\Pi_{\frac{1}{2}}^L}{\frac{1}{2}(\ell_L + \frac{1}{2})}\,, \quad Q_I^R = i \frac{X_a^R \tilde{\Gamma_a^0}^R - \frac{1}{4}\Pi_{\frac{1}{2}}^R}{\frac{1}{2}(\ell_R + \frac{1}{2})}\,. \tag{4.138}$$

In observation of the relations given in (4.134), we see that

$$\Pi_{\pm 0} \equiv \Pi_\pm^L \Pi_0^R = \Pi_\pm^L\,, \quad \Pi_{0\pm} \equiv \Pi_0^L \Pi_\pm^R = \Pi_\pm^L\,, \tag{4.139}$$

while all other products vanish. Therefore, $\Pi_\pm^R\,, \Pi_\pm^L$ are simply the required four projectors. For convenience, we list them in the table below.

Projector	To the Representation
$\Pi_\pm^L = \frac{1}{2}(\pm i Q_I^L + \Pi_{\frac{1}{2}}^L)$	$(\ell_L \pm \frac{1}{2}, \ell_R)$
$\Pi_\pm^R = \frac{1}{2}(\pm i Q_I^R + \Pi_{\frac{1}{2}}^R)$	$(\ell_L, \ell_R \pm \frac{1}{2})$

At this stage we can consider the fluctuations about the vacuum configuration (4.126)

$$\begin{aligned} \Phi_a^L &= X_a^L + \tilde{\Gamma_a^0} + A_a^L := D_a^L + A_a^L\,, \\ \Phi_a^R &= X_a^R + \tilde{\Gamma_a^0} + A_a^R := D_a^R + A_a^R\,, \end{aligned} \tag{4.140}$$

where A_a^L, $A_a^R \in u(2\ell_L+1) \otimes u(2\ell_R+1) \otimes u(4) \otimes u(n)$.

We can view A_a^L and A_a^R $(a = 1,2,3)$ as the six components of a $U(n)$ gauge field on $S_F^{2\,Int} \times S_F^{2\,Int}$ since F_{ab}^L, F_{ab}^R, $F_{ab}^{L,R}$ take the form of the curvature tensor

$$\begin{aligned} F_{ab}^L &= [D_a^L, A_b^L] - [D_b^L, A_a^L] + [A_a^L, A_b^L] - \epsilon_{abc} A_c^L\,, \\ F_{ab}^R &= [D_a^R, A_b^R] - [D_b^R, A_a^R] + [A_a^R, A_b^R] - \epsilon_{abc} A_c^R\,, \\ F_{ab}^{L,R} &= [D_a^L, A_b^R] - [D_b^R, A_a^L] + [A_a^L, A_b^R]\,. \end{aligned} \tag{4.141}$$

Adapting the discussion, starting with equation (4.52), it can be seen that only four of these six gauge fields constitute independent degrees of freedom in the commutative limit, ℓ_L, $\ell_R \to \infty$.

The emerging model has the structure of a $U(n)$ gauge theory on $\mathcal{M} \times S_F^{2\,Int} \times S_F^{2\,Int}$ with the gauge fields $A_M = (A_\mu\,, A_a)$ and corresponding field strength tensor $F_{MN} = (F_{\mu\nu}\,, F^L_{\mu a}\,, F^R_{\mu a},\, F^L_{ab}\,, F^R_{ab}\,, F^{L,R}_{ab})$. We can quickly glance over some of the essential features of the $U(4)$ gauge theory on $\mathcal{M} \times S_F^{2\,Int} \times S_F^{2\,Int}$.

For the $U(4)$ theory, taking the symmetry generators ω_a^L and ω_a^R

$$\begin{aligned}\omega_a^L = (X_a^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_4 \otimes \mathbf{1}_4) + (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \tilde{\Gamma^0_a}^L \otimes \mathbf{1}_4) \\ - (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_4 \otimes i\frac{L_a^L}{2})\,,\end{aligned} \tag{4.142}$$

$$\begin{aligned}\omega_a^R = (\mathbf{1}^{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_4 \otimes \mathbf{1}_4) + (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \tilde{\Gamma^0_a}^R \otimes \mathbf{1}_4) \\ - (\mathbf{1}^{(2\ell_L+1)} \otimes \mathbf{1}^{(2\ell_R+1)} \otimes \mathbf{1}_4 \otimes i\frac{L_a^R}{2})\,,\end{aligned} \tag{4.143}$$

with (L_a^L, L_a^R) same as before, we can construct the $SU(2) \times SU(2)$-equivariant fields. $SU(2) \times SU(2)$ representation content of (ω_a^L, ω_a^R) follows from the Clebsch-Gordan expansion

$$\begin{aligned}(\ell_L, \ell_R) \otimes \Big((\frac{1}{2}, 0) \oplus (\frac{1}{2}, 0) \Big) \otimes (\frac{1}{2}, \frac{1}{2}) &\equiv \mathbf{2}(\ell_L, \ell_R + \frac{1}{2}) \oplus \mathbf{2}(\ell_L, \ell_R - \frac{1}{2}) \\ &\oplus \mathbf{2}(\ell_L + \frac{1}{2}, \ell_R) \oplus \mathbf{2}(\ell_L - \frac{1}{2}, \ell_R) \oplus (\ell + 1, \ell_R - \frac{1}{2}) \oplus (\ell + 1, \ell_R + \frac{1}{2}) \\ &\oplus (\ell - 1, \ell_R - \frac{1}{2}) \oplus (\ell - 1, \ell_R + \frac{1}{2}) \oplus (\ell_L - \frac{1}{2}, \ell_R - 1) \\ &\oplus (\ell_L + \frac{1}{2}, \ell_R - 1) \oplus (\ell_L + \frac{1}{2}, \ell_R + 1) \\ &\oplus (\ell_L - \frac{1}{2}, \ell_R + 1) \\ &:= I\,.\end{aligned} \tag{4.144}$$

$\Pi^L_\pm\,, \Pi^R_\pm \in Mat((2\ell_L + 1) \times (2\ell_R + 1) \times 4 \times 4)$ project to the representations in the decomposition (4.144) as given in the table below.

Projector	To the Representation
$\Pi^L_\pm = \frac{1}{2}(\pm i Q^L_I + \Pi^L_{\frac{1}{2}})$	$(\ell_L, \ell_R + \frac{1}{2}) \oplus (\ell_L, \ell_R - \frac{1}{2}) \oplus (\ell_L \pm 1, \ell_R + \frac{1}{2})$ $\oplus(\ell_L \pm 1, \ell_R - \frac{1}{2})$
$\Pi^R_\pm = \frac{1}{2}(\pm i Q^R_I + \Pi^R_{\frac{1}{2}})$	$(\ell_L + \frac{1}{2}, \ell_R) \oplus (\ell_L + \frac{1}{2}, \ell_R \pm 1) \oplus (\ell_L - \frac{1}{2}, \ell_R)$ $\oplus(\ell_L - \frac{1}{2}, \ell_R \pm 1)$

The $SU(2) \times SU(2)$-equivariance conditions indicate that $A_\mu\,, A_a^L\,, A_b^R$ satisfy the relevant adapted version of (3.149) and (4.56). As before, we can determine the dimensions of solution spaces for A_μ, A_a^L and A_a^R using the Clebsch-Gordan decomposition of the adjoint action of (ω_a^L, ω_a^R). We find

$$I \otimes I \equiv \mathbf{24}(0,0) \oplus \mathbf{52}(1,0) \oplus \mathbf{52}(0,1) \oplus \cdots\,. \tag{4.145}$$

This means that there are 24-invariants. The solution space for each of A_a^L, A_a^R is 52-dimensional. We further see that there are no spinor representations $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ occurring in (4.145). This corroborates perfectly with our initial expectations, in view of the fact that $({\Gamma^0_a}^L, {\Gamma^0_a}^R)$ cannot be expressed through a bilinear of fields with the desired symmetry properties. If the latter was possible, it would have contradicted the absence of the equivariant spinor field modes and vice versa.

A suitable set of 24 invariants is given by the following matrices

$$\begin{aligned}
&\Pi_+^L\,,\ Q_+^L\,,\ \Pi_-^L\,,\ Q_-^L\,,\ \Pi_+^R\,,\ Q_+^R\,,\ \Pi_-^R\,,\ Q_-^R\,,\ Q_F^L\,,\ Q_H^L\,,\ Q_F^R\,,\ Q_H^R\,,\\
&\Pi_+^L Q_B^R\,,\ \Pi_-^L Q_B^R\,,\ \Pi_+^R Q_B^L\,,\ \Pi_-^R Q_B^L\,,\ Q_+^L Q_B^R\,,\ Q_-^L Q_B^R\,,\ Q_F^L Q_B^R\,,\ Q_H^L Q_B^R\,,\\
&Q_+^R Q_B^L\,,\ Q_-^R Q_B^L\,,\ Q_F^R Q_B^L\,,\ Q_H^R Q_B^L\,,
\end{aligned} \tag{4.146}$$

where $Q_\pm^L$, Q_F^L, Q_H^L, Q_{BI}^L are in same formal form as (4.59) and likewise for the set of matrices Q^R.

A set of 52 linearly matrices transforming under the $(1,0)$ representation may be provided as

$$\begin{aligned}
&[D_a^L, Q_+^L]\,,\ Q_+^L[D_a^L, Q_+^L]\,,\ \{D_a^L, Q_+^L\}\,,\ Q_B^R[D_a^L, Q_+^L]\,,\ Q_B^R Q_+^L[D_a^L, Q_+^L]\,,\ Q_B^R\{D_a^L, Q_+^L\}\,,\\
&[D_a^L, Q_-^L]\,,\ Q_-^L[D_a^L, Q_-^L]\,,\ \{D_a^L, Q_-^L\}\,,\ Q_B^R[D_a^L, Q_-^L]\,,\ Q_B^R Q_-^L[D_a^L, Q_-^L]\,,\ Q_B^R\{D_a^L, Q_-^L\}\,,\\
&[D_a^L, Q_F^L]\,,\ Q_F^L[D_a^L, Q_F^L]\,,\ \{D_a^L, Q_F^L\}\,,\ Q_B^R[D_a^L, Q_F^L]\,,\ Q_B^R Q_F^L[D_a^L, Q_F^L]\,,\ Q_B^R\{D_a^L, Q_F^L\}\,,\\
&[D_a^L, Q_H^L]\,,\ Q_H^L[D_a^L, Q_H^L]\,,\ \{D_a^L, Q_H^L\}\,,\ Q_B^R[D_a^L, Q_H^L]\,,\ Q_B^R Q_H^L[D_a^L, Q_H^L]\,,\ Q_B^R\{D_a^L, Q_H^L\}\,,\\
&\Pi_+^R[D_a^L, Q_B^L]\,,\ \Pi_+^R Q_B^L[D_a^L, Q_B^L]\,,\ \Pi_+^R\{D_a^L, Q_B^L\}\,,\ Q_+^R[D_a^L, Q_B^L]\,,\ Q_+^R Q_B^L[D_a^L, Q_B^L]\,,\\
&Q_+^R\{D_a^L, Q_B^L\}\,,\ \Pi_-^R[D_a^L, Q_B^L]\,,\ \Pi_-^R Q_B^L[D_a^L, Q_B^L]\,,\ \Pi_-^R\{D_a^L, Q_B^L\}\,,\ Q_-^R[D_a^L, Q_B^L]\,,\\
&Q_-^R Q_B^L[D_a^L, Q_B^L]\,,\ Q_-^R\{D_a^L, Q_B^L\}\,,\ Q_F^R[D_a^L, Q_B^L]\,,\ Q_F^R Q_B^L[D_a^L, Q_B^L]\,,\ Q_F^R\{D_a^L, Q_B^L\}\,,\\
&Q_H^R[D_a^L, Q_B^L]\,,\ Q_H^R Q_B^L[D_a^L, Q_B^L]\,,\ Q_H^R\{D_a^L, Q_B^L\}\,,\ \Pi_+^L\omega_a^L\,,\ \Pi_-^L\omega_a^L\,,\ Q_B^R\Pi_+^L\omega_a^L\,,\\
&Q_B^R\Pi_-^L\omega_a^L\,,\ \Pi_+^R\omega_a^L\,,\ \Pi_-^R\omega_a^L\,,\ Q_+^R\omega_a^L\,,\ Q_-^R\omega_a^L\,,\quad Q_F^R\omega_a^L\,,\quad Q_H^R\omega_a^L
\end{aligned} \tag{4.147}$$

while a linearly independent set transforming as $(0,1)$ is obtained from (4.147) by taking $L \leftrightarrow R$.

Monopole sectors exist in this case too and they can be accessed by projecting from $S_F^{2\,Int} \times S_F^{2\,Int}$. We have, for instance

$$S_F{}^{2^{L\pm}} \times S_F{}^{2^{R\pm}} = \Big(S_F^2(\ell_L) \times S_F^2(\ell_R \pm \frac{1}{2})\Big) \oplus \Big(S_F^2(\ell_L \pm \frac{1}{2}) \times S_F^2(\ell_R)\Big), \tag{4.148}$$

$$S_F{}^{2^{L,2}} \times S_F{}^{2^{R,0}} = \Big(S_F^2(\ell_L + \frac{1}{2}) \times S_F^2(\ell_R)\Big) \oplus \Big(S_F^2(\ell_L - \frac{1}{2}) \times S_F^2(\ell_R)\Big), \tag{4.149}$$

$$S_F{}^{2^{L,0}} \times S_F{}^{2^{R,\,2}} = \Big(S_F^2(\ell_L) \times S_F^2(\ell_R + \frac{1}{2})\Big) \oplus \Big(S_F^2(\ell_L) \times S_F^2(\ell_R - \frac{1}{2})\Big), \tag{4.150}$$

with the winding numbers $(\pm 1, \pm 1)$, $(2,0)$, $(0,2)$, respectively.

We can project to the $(\pm 1, \pm 1)$ sector using

$$(1 - \Pi^L_{\mp})(1 - \Pi^R_{\mp}). \tag{4.151}$$

This projection leaves us with 8 equivariant scalars

$$\Pi^L_{\pm}\,, \quad \Pi^R_{\pm}\,, \quad Q^L_{\pm}\,, \quad Q^R_{\pm}\,, \quad Q^R_B \Pi^L_{\pm}\,, \quad Q^R_B Q^L_{\pm}\,, \quad Q^L_B \Pi^R_{\pm}\,, \quad Q^L_B Q^R_{\pm}\,, \tag{4.152}$$

and 16 vectors carrying the $(1,0)$ representation,

$$\begin{aligned}&[D^L_a, Q^L_{\pm}]\,,\ Q^L_{\pm}[D^L_a, Q^L_{\pm}]\,,\ \{D^L_a, Q^L_{\pm}\}\,,\ Q^R_B[D^L_a, Q^L_{\pm}]\,,\ Q^R_B Q^L_{\pm}[D^L_a, Q^L_{\pm}]\,,\\ &Q^R_B\{D^L_a, Q^L_{\pm}\}\,,\ \Pi^R_{\pm}[D^L_a, Q^L_B]\,,\ \Pi^R_{\pm} Q^L_B[D^L_a, Q^L_B]\,,\ \Pi^R_{\pm}\{D^L_a, Q^L_B\}\,,\ Q^R_{\pm}[D^L_a, Q^L_B]\,,\\ &Q^R_{\pm} Q^L_B[D^L_a, Q^L_B]\,,\ Q^R_{\pm}\{D^L_a, Q^L_B\}\,,\ \Pi^L_{\pm}\omega^L_a\,,\ Q^R_B \Pi^L_{\pm}\omega^L_a\,,\ \Pi^R_{\pm}\omega^L_a\,,\ Q^R_{\pm}\omega^L_a\,,\end{aligned} \tag{4.153}$$

and another 16 carrying the $(0,1)$ IRR which are obtained from (4.153) by $L \leftrightarrow R$.

For the winding number sector $(2,0)$ in the equation (4.149), we can use the projection operator

$$(1 - \Pi^R_{+})(1 - \Pi^R_{-})\,. \tag{4.154}$$

In this case, the relevant part of the Clebsch-Gordan expansion gives the result $\mathbf{12}(0,0) \oplus \mathbf{28}(1,0) \oplus \mathbf{24}(0,1)$. Equivariant scalars may be given as the following subset of those in (4.146)

$$\Pi^L_+\,,\quad \Pi^L_-\,,\quad Q^L_+\,,\quad Q^L_-\,,\quad Q^L_F\,,\quad Q^L_H\,,\quad Q^R_B\Pi^L_+\,,\quad Q^R_B\Pi^L_-\,,\quad Q^R_BQ^L_+\,,\quad Q^R_BQ^L_-\,,$$
$$Q^R_BQ^L_F \quad Q^R_BQ^L_H. \tag{4.155}$$

28 vectors which carry the $(1,0)$ IRR can be given as

$$[D^L_a, Q^L_+]\,,\, Q^L_+[D^L_a, Q^L_+]\,,\, \{D^L_a, Q^L_+\}\,,\, Q^R_B[D^L_a, Q^L_+]\,,\, Q^R_BQ^L_+[D^L_a, Q^L_+]\,,\, Q^R_B\{D^L_a, Q^L_+\}\,,$$
$$[D^L_a, Q^L_-]\,,\, Q^L_-[D^L_a, Q^L_-]\,,\, \{D^L_a, Q^L_-\}\,,\, Q^R_B[D^L_a, Q^L_-]\,,\, Q^R_BQ^L_-[D^L_a, Q^L_-]\,,\, Q^R_B\{D^L_a, Q^L_-\}\,,$$
$$[D^L_a, Q^L_F]\,,\, Q^L_F[D^L_a, Q^L_F]\,,\, \{D^L_a, Q^L_F\}\,,\, Q^R_B[D^L_a, Q^L_F]\,,\, Q^R_BQ^L_F[D^L_a, Q^L_F]\,,\, Q^R_B\{D^L_a, Q^L_F\}\,,$$
$$[D^L_a, Q^L_H]\,,\, Q^L_H[D^L_a, Q^L_H]\,,\, \{D^L_a, Q^L_H\}\,,\, Q^R_B[D^L_a, Q^L_H]\,,\, Q^R_BQ^L_H[D^L_a, Q^L_H],$$
$$Q^R_B\{D^L_a, Q^L_H\}\,,\; \Pi^L_+\omega^L_a\,,\, \Pi^L_-\omega^L_a\,,\, Q^R_B\Pi^L_+\omega^L_a\,,\, Q^R_B\Pi^L_-\omega^L_a\,. \tag{4.156}$$

while there are 24 matrices which carry the $(0,1)$ IRR and they may be listed as

$$\Pi^L_+[D^R_a, Q^R_B]\,,\, \Pi^L_+Q^R_B[D^R_a, Q^R_B]\,,\, \Pi^L_+\{D^R_a, Q^R_B\}\,,\, Q^L_+[D^R_a, Q^R_B]\,,\, Q^L_+Q^R_B[D^R_a, Q^R_B]\,,$$
$$Q^L_+\{D^R_a, Q^R_B\}\,,\, \Pi^L_-[D^R_a, Q^R_B]\,,\, \Pi^L_-Q^R_B[D^R_a, Q^R_B]\,,\, \Pi^L_-\{D^R_a, Q^R_B\}\,,\, Q^L_-[D^R_a, Q^R_B]\,,$$
$$Q^L_-Q^R_B[D^R_a, Q^R_B]\,,\, Q^L_-\{D^R_a, Q^R_B\}\,,\, Q^L_F[D^R_a, Q^R_B]\,,\, Q^L_FQ^R_B[D^R_a, Q^R_B]\,,\, Q^L_F\{D^R_a, Q^R_B\}\,,$$
$$Q^L_H[D^R_a, Q^R_B]\,,\, Q^L_HQ^R_B[D^R_a, Q^R_B]\,,\, Q^L_H\{D^R_a, Q^R_B\}\,,\, \Pi^L_+\omega^R_a\,,\, \Pi^L_-\omega^R_a\,,\, Q^L_+\omega^R_a\,,$$
$$Q^L_-\omega^R_a\,,\, Q^L_F\omega^R_a\,,\, Q^L_H\omega^R_a. \tag{4.157}$$

To describe the monopole sectors with the winding number $(0,2)$, it is sufficient to make the exchange $L \leftrightarrow R$.

CHAPTER 5

$U(3)$ GAUGE THEORY OVER $\mathcal{M} \times S_F^2$

In chapter 3, we have explained that a $U(n)$ gauge theory over $\mathcal{M} \times S_F^2$ can be interpreted as an effective gauge theory obtained from an $SU(\mathcal{N})$ gauge theory on $\mathcal{M}$ coupled to a triplet of scalar fields in the adjoint representation of $SU(\mathcal{N})$ with $\mathcal{N} = n(2\ell + 1)$ after the spontaneous symmetry breaking. This interpretation has been supported by the construction of the Kaluza-Klein mode expansion of gauge fields over the fuzzy extra dimensions. We have examined the equivariant parametrization of this model as a complementing aspects of developing the effective gauge theory interpretation. For concreteness, we have focused on the specific emergent model; $U(2)$ gauge theory on $\mathcal{M} \times S_F^2$ and making use of the equivariant parametrization technique, we have shown that the commutative limit of equivariant modes reduces to the gauge fields of spherical symmetric gauge theory on $\mathcal{M} \times S^2$ [67] and the low energy limit of this model yields abelian Higgs-type model [22]. In this chapter, we would like to concentrate on this model. Taking a step forward, we investigate the low energy structure of this model with larger gauge groups [27][1].

Our initial attention is to determine in full detail the equivariant field modes of a $U(3)$ gauge theory over $\mathcal{M} \times S_F^2$ and obtain the corresponding LEA by tracing over S_F^2. In order to obtain the equivariant modes of gauge fields, we impose the proper symmetry conditions on the fields of the model so that they transform either invariantly or as vectors under the combined action of $SU(2)$ rotations of the fuzzy spheres and those $U(3)$ gauge transformations. We find that equivari-

[1] This chapter is based on the work that has been published: S. Kurkcuoglu and G. Unal "$U(3)$ gauge theory on fuzzy extra dimensions" Phys.Rev. D**94** (2016) 036003.

ant scalars may be constructed by taking advantage of the dipole and quadrupole terms, which appear in the branching of the adjoint representation of $SU(3)$ as $\underline{8} \to \underline{5} \oplus \underline{3}$ when the $SU(2)$ subgroup is maximally embedded in $SU(3)$. Using these considerations and other group theoretical input coming from the equivariance conditions, we find the invariants as "idempotents" involving intertwiners combining spin ℓ irreducible representation of $SU(2)$ generating the rotations of S_F^2 and those $U(3)$ gauge transformations generated by $SU(2) \subset U(3)$ carrying the spin 1 IRR of $SU(2)$. There is also another invariant proportional to the $\mathcal{N}$-dimensional identity matrix, which essentially appears due to a $U(1)$ subgroup of $U(3) \approx SU(3) \times U(1)$. Equivariant vectors are built using these invariants and the generators of S_F^2. We show that how the commutative limit of our equivariant field modes relate to the cylindrically symmetric gauge fields of $SU(3)$ Yang-Mills theory of Bais and Weldon [72].

Integrating out the extra dimension S_F^2, we obtain the LEA in which there are three abelian gauge fields and two complex scalars each coupling to only one of the gauge fields and three real scalars interacting with the complex fields and with each other through a quartic potential. From this LEA, we derive the vortex solutions on $\mathcal{M} \equiv \mathbb{R}^2$ in two different limits governed together by ℓ and the coupling constant of the constraint term in the potential and for both two limits, we need two winding numbers in order to express vortex solutions. In particular, we point out the connection between the BPS vortices that we obtain in a certain commutative limit in section 5.3 and the instanton solution in [72]

Next, we briefly outline the generalization of equivariant parametrization (EP) of gauge fields to $U(n)$ theories over $\mathcal{M} \times S_F^2$, and show that equivariant scalar are obtained by employing the $n-1$ multipole terms, that appear in the branching of the adjoint representation of $SU(n)$ under $SU(2)$, when the latter is maximally embedded in $SU(n)$.

Adapting the approach given in the previous chapter, we study the $U(3)$-equivariant fields over $\mathcal{M} \times S_F^{2\,Int}$, where $S_F^{2\,Int} := S_F^2(\ell) \oplus S_F^2(\ell) \oplus S_F^2\left(\ell + \frac{1}{2}\right) \oplus S_F^2\left(\ell - \frac{1}{2}\right)$ was revealed in [25] via a certain field redefinition of the triplet of scalars for the $SU(\mathcal{N})$ Yang-Mills theory. The reason of interest on this vacuum is two fold. Firstly, through its certain projections it gives us access to fuzzy monopole bundles with winding numbers ± 1 and secondly it naturally identifies with the bosonic part of the $N = 2$ fuzzy supersphere with $OSP(2,2)$ supersymmetry as discussed in [25]. We express all the equivariant field modes characterizing the low energy behaviour of the effective $U(3)$ theory on $\mathcal{M} \times S_F^{2\,Int}$ in terms of suitable "idempotents" and projection operators.

5.1 $SU(2)$-equivariant Gauge Fields

In this section, we focus our attention on the equivariant parametrization of $U(3)$ gauge theory $\mathcal{M} \times S_F^2$. All necessary and sufficient information on the dynamical generation of an effective $U(n)$ gauge theory on $\mathcal{M} \times S_F^2$ from an $SU(\mathcal{N})$ gauge theory on $\mathcal{M}$ coupled to suitably number of scalar fields has been given in the section 3.1 and the gauge field configuration of this model defined in (3.18). Focusing on the effective $U(2)$ gauge theory, the equivariant parametrization technique has been applied on this model and its the low energy limit has been constructed in the section 3.3. Here, adapting the formulas and the approach in the section 3.3, we construct the explicit form of $SU(2)$-equivariant gauge fields in $U(3)$ gauge theory $\mathcal{M} \times S_F^2$. To be more precise, we impose the symmetry condition on the gauge fields in (3.18) so that they transform as scalars and vectors under rotations of S_F^2 up to $U(3)$ gauge transformation. For this purpose, we introduce the infinitesimal symmetry generators ω_a as

$$\omega_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_3 - \mathbf{1}_{(2\ell+1)} \otimes i\Sigma_a \,, \tag{5.1}$$

where Σ_a are the spin 1 irreducible representation of $SU(2) \subset SU(3)$: $(\Sigma_a)_{ij} = i\epsilon_{iaj}$ and ω_a satisfy the condition (3.78). Clearly, the adjoint action $ad\,\omega_a \cdot = [\omega_a, \cdot]$, is composed of infinitesimal rotations over S_F^2 combined with those infinitesimal $SU(3)$ transformations, which are generated by Σ_a. The adjoint representation of $SU(3)$ decomposes to $SU(2)$ IRR's as

$$\underline{8} \rightarrow \underline{5} \oplus \underline{3} \,. \tag{5.2}$$

In this branching, Σ_a generate the $\underline{3}$ (spin 1) IRR of $SU(2)$, while the remaining five generators of $SU(3)$ may be given in the form of the quadrupole tensor

$$Q_{ab} = \frac{1}{2}\{\Sigma_a, \Sigma_b\} - \frac{2}{3}\delta_{ab} \,, \tag{5.3}$$

$$(Q_{ab})_{ij} = \delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi} - \frac{2}{3}\delta_{ab}\delta_{ij} \,, \tag{5.4}$$

carrying the spin 2 (i.e $\underline{5}$) IRR of $SU(2)$. For each IRR of $SU(2)$ in the branching (5.2), we may expect to construct one rotational invariant under $ad\,\omega_a$ in addition to the identity matrix $\mathbf{1}_{(2\ell+1)3}$ and we will at once proceed to see that this is indeed so[2]. These invariants may be simply taken as $X_a\Sigma_a$ and $X_aX_bQ_{ab}$, however we again prefer to express them as "idempotent" matrices for simplify our future formulas and the explicit form of them will be given later in this section.

In order to find the $SU(2)$-equivariant gauge fields, we impose the symmetry constraints (3.79) and (3.80) which simply imply that, under the adjoint action of ω_a, A_μ are rotational invariants and A_a transform as vectors.

$SU(2)$ IRR content of ω_a may be found by the following tensor product

$$\ell \otimes 1 = (\ell - 1) \oplus \ell \oplus (\ell + 1) \,, \tag{5.5}$$

and therefore IRR decomposition of the adjoint action of ω_a is

$$[(\ell - 1) \oplus \ell \oplus (\ell + 1)] \otimes [(\ell - 1) \oplus \ell \oplus (\ell + 1)] = \mathbf{3}\mathbf{0} \oplus \mathbf{7}\mathbf{1} \oplus \cdots \,. \tag{5.6}$$

From this Clebsch-Gordan expansion, it can be seen that the set of solutions for A_μ is 3-dimensional. We span this space by the invariants Q_1, Q_2, as defined below and $\mathbf{1}_{(2\ell+1)3}$ and introduce the following explicit parametrization of A_μ:

$$A_\mu = -\frac{1}{2}a_\mu^{(1)}(y)Q_1 + \frac{1}{2}a_\mu^{(2)}(y)Q_2 + \frac{i}{2}\bigg(\frac{a_\mu^{(1)}(y) - a_\mu^{(2)}(y)}{3} + b_\mu(y)\bigg)\mathbf{1} \,, \tag{5.7}$$

where $a_\mu^{(1)}, a_\mu^{(2)}, b_\mu$ are Hermitian $U(1)$ gauge fields[3] on $\mathcal{M}$ and Q_1, Q_2 are anti-

[2] Generalization of this construction to all $U(n)$ gauge theories on $\mathcal{M} \times S_F^2$ is discussed in section 5.4.

[3] The reason for this particular form of the coefficients of $Q_1\,, Q_2$ and $\mathbf{1}$ in (5.7) will become clear as we proceed to perform the dimensional reduction over S_F^2 in the next section.

Hermitian idempotents given as [73][4]

$$Q_1 = \frac{2(iX_a\Sigma_a + \ell + 1)(iX_b\Sigma_b + 1) - (\ell + 1)(2\ell + 1)\mathbf{1}}{i(\ell + 1)(2\ell + 1)} \,,$$
$$Q_2 = \frac{2(iX_a\Sigma_a - \ell)(iX_b\Sigma_b + 1) - \ell(2\ell + 1)\mathbf{1}}{i\ell(2\ell + 1)} \,, \tag{5.8}$$

where

$$Q_1^\dagger = -Q_1 \,, \quad Q_1^2 = -\mathbf{1}_{3(2\ell+1)} \,, \quad Q_2^\dagger = -Q_2 \,, \quad Q_2^2 = -\mathbf{1}_{3(2\ell+1)} \,. \tag{5.9}$$

Here, we see that imposing the symmetry conditions (3.79) and (3.80) cause the breaking of $U(3)$ gauge symmetry down to $U(1) \times U(1) \times U(1)$. Just like we explained in the chapter 3, it is possible to show that under the gauge transformation generated by $U = e^{-\frac{1}{2}\theta_1(y)Q_1} e^{\frac{1}{2}\theta_2(y)Q_2} e^{i\left(\frac{1}{6}\theta_1(y) - \frac{1}{6}\theta_2(y) + \frac{1}{2}\theta_3(y)\right)\mathbf{1}}$, $A_\mu \to A'_\mu$ with $a_\mu^{(i)\prime} = a_\mu^{(i)} + \partial_\mu \theta_i$ and $b'_\mu = b_\mu + \partial_\mu\theta_3$, hence the rotationally symmetry of A_μ is preserved.

Equation (5.6) shows that the dimension of the set of solutions for A_a is seven and its parametrization may be chosen as follows

$$\begin{aligned} A_a = &\frac{1}{2}\varphi_1(y)[X_a, Q_1] + \frac{1}{2}\chi_1(y)[X_a, Q_2] - \frac{1}{2}(\varphi_2(y) + 1)Q_1[X_a, Q_1] \\ &+ \frac{1}{2}(\chi_2(y) - 1)Q_2[X_a, Q_2] + \frac{i}{2}\frac{\varphi_3(y)}{2(\ell + 1/2)}\Big(\{X_a, Q_1\} - iQ_2[X_a, Q_2]\Big) \\ &+ \frac{i}{2}\frac{\chi_3(y)}{2(\ell + 1/2)}\Big(\{X_a, Q_2\} - iQ_1[X_a, Q_1]\Big) + \frac{1}{2}\psi(y)\frac{\omega_a}{\ell + 1/2} \,, \end{aligned} \tag{5.10}$$

where we have introduced the real scalar fields $\varphi_1 \,, \varphi_2 \,, \varphi_3 \,, \chi_1 \,, \chi_2 \,, \chi_3$ and ψ on $\mathcal{M}$ and some of these naturally combine to form complex scalars when the model is dimensionally reduced over S_F^2.

In the commutative limit, $\ell \to \infty$, we have

$$iQ_1 = q_1 = (\Sigma_a \hat{x}_a)^2 + (\Sigma_a \hat{x}_a) - 1 \,,$$
$$iQ_2 = q_2 = (\Sigma_a \hat{x}_a)^2 - (\Sigma_a \hat{x}_a) - 1 \,, \tag{5.11}$$

where $q_1^2 = q_2^2 = \mathbf{1}_3$. Another idempotent may be given as a linear combination of q_1 and q_2 and $\mathbf{1}_3$ as $q_3 = -(q_1 + q_2) - \mathbf{1}_3$ [73]. Using (5.11), we find that the

[4] In [73], these idempotents were introduced for the purpose of constructing the spin 1 Dirac operator on the fuzzy sphere.

commutative limit of A_a in (5.10) takes the form

$$A_a \xrightarrow[\ell\to\infty]{} -\frac{\varphi_1(y)}{2}\mathcal{L}_a q_1 - \frac{\chi_1(y)}{2}\mathcal{L}_a q_2 - i\frac{(\varphi_2(y)+1)}{2} q_1 \mathcal{L}_a q_1 + i\frac{(\chi_2(y)-1)}{2} q_2 \mathcal{L}_a q_2 + \frac{\varphi_3(y)}{2}\hat{x}_a q_1 + \frac{\chi_3(y)}{2}\hat{x}_a q_2 + \frac{\psi(y)}{2}\hat{x}_a\,. \tag{5.12}$$

Imposing the constraint $x_a A_a = 0$ eliminates the radial component of the gauge field. We see from (5.12) that this condition is satisfied if and only if we set $\varphi_3 = \chi_3 = \psi = 0$. The remaining terms of A_a in (5.12) and the commutative limit of A_μ (apart from a b_μ-field due to the $U(1)$ subgroup of $U(3)$, which decouples from the rest in the commutative limit, or eliminated by solving its equation of motion in powers of $\frac{1}{\ell}$, as we shall see later on in section 5.3) are in agreement with the cylindrical symmetric ansatz for the $SU(3)$ Yang-Mills theory of Bais and Weldon [72].

5.2 Dimensional Reduction of the Yang-Mills Action

In this section, we pursue the dimensional reduction of $U(3)$ gauge theory on $\mathcal{M}\times S_F^2$ over S_F^2. We can substitute our equivariant gauge fields A_μ in (5.7) and A_a in (5.10) into the action (3.75), and then by tracing over the fuzzy sphere S_F^2, we obtain the reduced action on $\mathcal{M}$. In the present case, the identities in (3.90) take the form of

$$[X_a,\{X_a,Q_i\}]=0\,,\ [Q_i,\{X_a,Q_i\}]=0\,,\ \{X_a,[X_a,Q_i]\}=0\,,\ \{Q_i,[X_a,Q_i]\}=0\,, \tag{5.13}$$

where $i=1,2$ and sum over only the repeated index "a" is implied.

Now, we start to calculate each term in (3.89) separately. For the field strength term, the curvature $F_{\mu\nu}$ can be expressed in terms of the rotational invariants $Q_1\,, Q_2$ and $\mathbf{1}$ as

$$F_{\mu\nu} = -\frac{1}{2}f_{\mu\nu}^{(1)}Q_1 + \frac{1}{2}f_{\mu\nu}^{(2)}Q_2 + i\frac{1}{2}\left(\frac{f_{\mu\nu}^{(1)}-f_{\mu\nu}^{(2)}}{3} + h_{\mu\nu}\right)\mathbf{1} \tag{5.14}$$

where we have introduced

$$f_{\mu\nu}^{(1)} := \partial_\mu a_\nu^{(1)} - \partial_\nu a_\mu^{(1)}\,,\quad f_{\mu\nu}^{(2)} := \partial_\mu a_\nu^{(2)} - \partial_\nu a_\mu^{(2)}\,,\quad h_{\mu\nu} := \partial_\mu b_\nu - \partial_\nu b_\mu\,. \tag{5.15}$$

Then, $\mathcal{L}_F$ takes the form

$$\begin{aligned}\mathcal{L}_F &:= \frac{1}{4g^2}\mathrm{Tr}(F^\dagger_{\mu\nu}F^{\mu\nu})\\ &= \frac{1}{g^2}\Bigg(\frac{\ell+1}{9(2\ell+1)}f^{(1)}_{\mu\nu}f^{(1)\mu\nu} + \frac{\ell}{9(2\ell+1)}f^{(2)}_{\mu\nu}f^{(2)\mu\nu} + \frac{1}{18}f^{(1)}_{\mu\nu}f^{(2)\mu\nu} + \frac{1}{16}h_{\mu\nu}h^{\mu\nu}\\ &\qquad + \frac{1}{6(2\ell+1)}f^{(1)}_{\mu\nu}h^{\mu\nu} + \frac{1}{6(2\ell+1)}f^{(2)}_{\mu\nu}h^{\mu\nu}\Bigg)\,. \end{aligned}\tag{5.16}$$

The covariant derivative term $D_\mu\Phi_a$ is calculated to be

$$\begin{aligned}D_\mu\Phi_a &= \frac{1}{2}\left(D_\mu\varphi_1\right)[X_a,Q_1] + \frac{1}{2}\left(D_\mu\chi_1\right)[X_a,Q_2] - \frac{1}{2}\left(D_\mu\varphi_2\right)Q_1[X_a,Q_1]\\ &\quad + \frac{1}{2}\left(D_\mu\chi_2\right)Q_2[X_a,Q_2] + \frac{i}{4}\frac{\partial_\mu\varphi_3}{(\ell+1/2)}(\{X_a,Q_1\} - iQ_2[X_a,Q_2])\\ &\quad + \frac{i}{4}\frac{\partial_\mu\chi_3}{(\ell+1/2)}(\{X_a,Q_2\} - iQ_1[X_a,Q_1]) + \frac{1}{2(\ell+1/2)}\left(\partial_\mu\psi\right)\omega_a\,, \end{aligned}\tag{5.17}$$

where $D_\mu\varphi_i = \partial_\mu\varphi_i + \epsilon_{ji}a^{(1)}_\mu\varphi_j$ and $D_\mu\chi_i = \partial_\mu\chi_i + \epsilon_{ji}a^{(2)}_\mu\chi_j$. After tracing, the gradient term $\mathcal{L}_G$ reads

$$\mathcal{L}_G = \mathrm{Tr}((D_\mu\Phi_a)^\dagger D_\mu\Phi_a)\tag{5.18}$$

$$\begin{aligned}=&\frac{2\ell(2\ell+3)}{3(\ell+1)(2\ell+1)}((D_\mu\varphi_1)^2 + (D_\mu\varphi_2)^2) + \frac{2(2\ell-1)(\ell+1)}{3\ell(2\ell+1)}((D_\mu\chi_1)^2 + (D_\mu\chi_2)^2)\\ &+ \frac{6\ell^5+15\ell^4+4\ell^3-9\ell^2+2}{3\ell(\ell+1)(2\ell+1)^3}((\partial_\mu\varphi_3)^2 + (\partial_\mu\chi_3)^2) + \frac{\ell^2+\ell+2}{(2\ell+1)^2}(\partial_\mu\psi)^2\\ &- \frac{2\ell(\ell+1)}{3(2\ell+1)^2}\partial_\mu\varphi_3\partial_\mu\chi_3 - \frac{2\ell(2\ell^2-5\ell-9)}{3(2\ell+1)^3}\partial_\mu\psi\partial_\mu\varphi_3\\ &- \frac{2(2\ell^3+11\ell^2+7\ell-2)}{3(2\ell+1)^3}\partial_\mu\chi_3\partial_\mu\psi\,. \end{aligned}\tag{5.19}$$

We note that $\varphi_1\,,\varphi_2$ and $\chi_1\,,\chi_2$ naturally combine to two complex scalar fields $\varphi := \varphi_1 + i\varphi_2\,, \chi := \chi_1 + i\chi_2$, with $D_\mu\varphi = (\partial_\mu + ia^{(1)}_\mu)\varphi$ and $D_\mu\chi = (\partial_\mu + ia^{(2)}_\mu)\chi$, which we will make use of in the next section.

In order to calculate the potential term V_1, it is useful to work with the dual of the curvature F_{ab}. We find

$$\begin{aligned}\frac{1}{2}\epsilon_{abc}F_{ab} &= \Lambda_1 + \Lambda_2|\varphi|^2 + \Lambda_3|\chi|^2 + \Lambda_4(\varphi_3^2+\chi_3^2) + \Lambda_5\varphi_3 + \Lambda_6\chi_3 + \Lambda_7\varphi_3\chi_3\\ &\quad + \Lambda_8\varphi_3\psi + \Lambda_9\chi_3\psi + \Lambda_{10}(\varphi_1 + \varphi_2Q_1)[X_a,Q_1]\\ &\quad + \Lambda_{11}(\chi_1 + \chi_2Q_2)[X_a,Q_2] + \Lambda_{12}\psi + \Lambda_{13}\psi^2\,, \end{aligned}\tag{5.20}$$

where $\Lambda_i\,, i = 1, \cdots, 11$ are the $3(2\ell+1) \times 3(2\ell+1)$ dimensional matrices which are listed in the appendix C. Using (5.20), the potential term V_1 may be determined as

$$\begin{aligned} V_1 = \frac{1}{\tilde{g}^2}\mathrm{Tr}(F_{ab}^\dagger F_{ab}) = \frac{1}{\tilde{g}^2}\Big(& \alpha_1 - \alpha_2|\varphi|^2 - \alpha_3|\chi|^2 - \alpha_4\varphi_3^2 - \alpha_5\chi_3^2 - \alpha_6\varphi_3 + \alpha_7\chi_3 \\ & - \alpha_8\varphi_3\chi_3 - \alpha_9\varphi_3\psi - \alpha_{10}\chi_3\psi + \alpha_{11}\psi^2 + \beta_1|\varphi|^4 - \beta_2|\varphi|^2|\chi|^2 \\ & + \beta_3|\varphi|^2\varphi_3^2 + \beta_4|\varphi|^2\chi_3^2 - \beta_5|\varphi|^2\varphi_3 + \beta_6|\varphi|^2\chi_3 - \beta_7|\varphi|^2\varphi_3\chi_3 \\ & + \beta_8|\varphi|^2\varphi_3\psi - \beta_9|\varphi|^2\chi_3\psi + \beta_{10}|\varphi|^2\psi^2 + \gamma_1|\chi|^4 - \gamma_2|\chi|^2\varphi_3^2 \\ & + \gamma_3|\chi|^2\chi_3^2 + \gamma_4|\chi|^2\varphi_3 - \gamma_5|\chi|^2\chi_3 + \gamma_6|\chi|^2\varphi_3\chi_3 - \gamma_7|\chi|^2\varphi_3\psi \\ & - \gamma_8|\chi|^2\chi_3\psi + \gamma_9|\chi|^2\psi^2 - \delta_1(\varphi_3^4 + \chi_3^4 + 6\varphi_3^2\chi_3^2) \\ & - \delta_2(\varphi_3^3 + 3\varphi_3\chi_3^2) - \delta_3(\chi_3^3 + 3\chi_3\varphi_3^2) - \delta_4(\varphi_3^3\chi_3 + \chi_3^3\varphi_3) \\ & - \delta_5(\varphi_3^3\psi + 3\varphi_3\chi_3^2\psi) - \delta_6(\chi_3^3\psi + 3\chi_3\varphi_3^2\psi) + \delta_7(\varphi_3^2\psi + \chi_3^2\psi) \\ & + \delta_8(\varphi_3^2\psi^2 + \chi_3^2\psi^2) - \delta_9\varphi_3\chi_3\psi - \delta_{10}\varphi_3\psi^2 - \delta_{11}\chi_3\psi^2 \\ & - \delta_{12}\varphi_3\chi_3\psi^2 - \delta_{13}\varphi_3\psi^2 - \delta_{14}\chi_3\psi^3 - \delta_{15}\psi^3 - \delta_{16}\psi^4\Big)\,, \end{aligned} \tag{5.21}$$

where all the ℓ-dependent constants: $\alpha\,,\beta\,,\gamma\,,\delta$ are given in the appendix C.

In the $\ell \to \infty$ limit we find

$$\begin{aligned} V_1(\Phi)\Big|_{\ell\to\infty} = \frac{1}{\tilde{g}^2}\Big(& \frac{2}{3}(|\varphi|^2+\varphi_3-1)^2+\frac{2}{3}(|\chi|^2-\chi_3-1)^2+\frac{2}{3}(|\varphi|^2-|\chi|^2)^2+\frac{4}{3}|\varphi|^2\varphi_3^2 \\ & + \frac{4}{3}|\chi|^2\chi_3^2 - \frac{1}{6}\varphi_3^2 - \frac{1}{6}\chi_3^2 + \frac{1}{2}\psi^2 - \frac{1}{3}(\varphi_3\chi_3 + \varphi_3\psi + \chi_3\psi)\Big)\,. \end{aligned} \tag{5.22}$$

The potential $V_1(\Phi) = \frac{1}{\tilde{g}^2}\mathrm{Tr}(F_{ab}^\dagger F_{ab})$ is positive definite, although the r.h.s of (5.21) and (5.22) are not manifestly so. For the limiting case (5.22) we have determined that minima occurs at the following configurations

$$i)\quad |\varphi|^2 = 1\,, \quad |\chi|^2 = 1\,, \quad \varphi_3 = \chi_3 = \psi = 0\,, \tag{5.23}$$

$$ii)\quad |\varphi|^2 = 0\,, \quad |\chi|^2 = 0\,, \quad \varphi_3 = 1\,, \chi_3 = -1\,, \psi = 0\,, \tag{5.24}$$

$$iii)\quad |\varphi|^2 = \frac{1}{\sqrt{2}}\,, \quad |\chi|^2 = 0\,, \quad \varphi_3 = 0\,, \quad \chi_3 = -\frac{3}{2}\,, \quad \psi = -\frac{1}{2}\,, \tag{5.25}$$

$$iv)\quad |\varphi|^2 = 0\,, \quad |\chi|^2 = \frac{1}{\sqrt{2}}\,, \quad \varphi_3 = \frac{3}{2}\,, \quad \chi_3 = 0\,, \quad \psi = \frac{1}{2}\,. \tag{5.26}$$

For the computation of the last term in (3.89), we first obtain the expression

$$\Phi_a \Phi_a + \ell(\ell+1) = R_1 + R_2 i Q_1 + R_3 i Q_2 \,, \tag{5.27}$$

where $R_1 \,, R_2$ and R_3 are listed in the appendix C. Then, the potential term V_2 is determined to be

$$V_2(\Phi) = \mathfrak{g}^2 \left(R_1^2 + R_2^2 + R_3^2 - \frac{2(2\ell-3)}{3(2\ell+1)} R_1 R_2 - \frac{2(2\ell+5)}{3(2\ell+1)} R_1 R_3 - \frac{2}{3} R_2 R_3 \right) . \tag{5.28}$$

In the large ℓ limit, we find

$$\begin{aligned} V_2(\Phi)\Big|_{\ell\to\infty} &= \frac{1}{3}\mathfrak{g}^2 \Big((R_1 - R_2 - R_3)^2 + (-R_1 + R_2 - R_3)^2 \\ &\quad + (-R_1 - R_2 + R_3)^2 \Big)\Big|_{\ell\to\infty} \,, \\ &= \frac{1}{3}\mathfrak{g}^2 \ell^2 \left((-\psi + \varphi_3 + \chi_3)^2 + (\psi - \varphi_3 + \chi_3)^2 + (\psi + \varphi_3 - \chi_3)^2 \right) . \end{aligned} \tag{5.29}$$

In the next section we will first consider the scaling limit $\mathfrak{g} \to 0$, $\ell \to \infty$, with $\mathfrak{g}\ell$ kept finite but small. Then, among the minima of the potential $V_1(\Phi)$ listed above, only (5.23) minimizes (5.29) as can easily be observed.

5.3 Vortices

In this section, we would like to investigate the vortex type solutions for the reduced action on $\mathcal{M} \equiv \mathbb{R}^2$ which is obtained from the dimensional reduction of $U(3)$ gauge theory on $\mathcal{M} \times S_F^2$. As we mentioned earlier in the subsection 3.3.4, we focus on the exploring these vortex type solution in two different limits, *i)* $\ell \to \infty \,, \mathfrak{g} \to 0$ with $\mathfrak{g}\ell$ remaining finite but small and *ii)* $\mathfrak{g} \to \infty$ and ℓ is large but finite.

5.3.1 Case i)

In this case the reduced action becomes

$$S = \int d^2y \left(\frac{1}{18g^2}(f^{(1)}_{\mu\nu}f^{(1)\mu\nu} + f^{(2)}_{\mu\nu}f^{(2)\mu\nu} + f^{(1)}_{\mu\nu}f^{(2)\mu\nu}) + \frac{1}{16g^2}h_{\mu\nu}h^{\mu\nu} \right.$$
$$+ \frac{2}{3}(|D_\mu\varphi|^2 + |D_\mu\chi|^2) + \frac{1}{4}((\partial_\mu\varphi_3)^2 + (\partial_\mu\chi_3)^2 + (\partial_\mu\psi)^2)$$
$$\left. - \frac{1}{6}(\partial_\mu\varphi_3\partial_\mu\chi_3 + \partial_\mu\varphi_3\partial_\mu\psi + \partial_\mu\chi_3\partial_\mu\psi) + \frac{1}{\tilde{g}^2}V_1(\Phi)\Big|_{\ell\to\infty} \right) . \qquad (5.30)$$

We observe that, the gauge field b_μ decouples from the rest of the action, and does not play any role in the rest of this subsection. Thus we essentially have a abelian Higgs type model with $U(1) \times U(1)$ gauge symmetry. The vacuum configuration is given by (5.23) and has the structure of $T^2 = S^1 \times S^1$, with $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, indicating that the vortex solutions constructed below are characterized by two winding numbers, say (N, M).

To search for vortex solutions, we again work with the usual rotationally symmetric ansatz [68], which in this case may be written out as

$$a^{(1)}_r = a^{(2)}_r = 0\,, \quad a^1_\theta := a^{(1)}_\theta(r)\,, \quad a^2_\theta := a^{(2)}_\theta(r)\,,$$
$$\varphi = \zeta(r)e^{iN\theta}\,, \;\; \chi = \eta(r)e^{iM\theta}\,, \quad \varphi_3 = \rho(r)\,, \quad \chi_3 = \sigma(r)\,, \quad \psi = \tau(r)\,, \quad (5.31)$$

where the cartesian coordinates (y_1, y_2) are replaced by the polar variables $(r\,, \theta)$. With this ansatz the action reads

$$S = 2\pi \int dr \left(\frac{1}{9g^2r}(a^{1\prime}_\theta a^{1\prime}_\theta + a^{2\prime}_\theta a^{2\prime}_\theta + a^{1\prime}_\theta a^{2\prime}_\theta) + \frac{2r}{3}(\zeta'^2 + \eta'^2) + \frac{2}{3r}(N + a^1_\theta)^2\zeta^2 \right.$$
$$+ \frac{2}{3r}(M + a^2_\theta)^2\eta^2 + \frac{r}{4}(\rho'^2 + \sigma'^2 + \tau'^2) - \frac{r}{6}(\rho'\sigma' + \rho'\tau' + \sigma'\tau')$$
$$+ \frac{4r}{3\tilde{g}^2}\Big((1 - \zeta^2 - \eta^2) + \frac{3}{8}(\rho^2 + \sigma^2 + \tau^2) - \rho + \sigma - \frac{1}{4}(\rho\sigma + \rho\tau + \sigma\tau)$$
$$\left. + \zeta^4 + \eta^4 - \zeta^2\eta^2 + \zeta^2(\rho^2 + \rho) + \eta^2(\sigma^2 - \sigma) \Big) \right) .$$
$$(5.32)$$

Euler-Lagrange equations for the fields are

$$\begin{aligned}
&\zeta'' + \frac{\zeta'}{r} - \left(\frac{1}{r^2}(N + a^1_\theta)^2 + \frac{2}{\tilde{g}^2}(-1 + 2\zeta^2 - \eta^2 + \rho^2 + \rho)\right)\zeta = 0\,,\\
&\eta'' + \frac{\eta'}{r} - \left(\frac{1}{r^2}(M + a^2_\theta)^2 + \frac{2}{\tilde{g}^2}(-1 + 2\eta^2 - \zeta^2 + \sigma^2 - \sigma)\right)\eta = 0\,,\\
&a^{1''}_\theta - \frac{a^{1'}_\theta}{r} + \frac{1}{2}a^{2''}_\theta - \frac{a^{2'}_\theta}{2r} - 6g^2(N + a^1_\theta)\zeta^2 = 0\,,\\
&a^{2''}_\theta - \frac{a^{2'}_\theta}{r} + \frac{1}{2}a^{1''}_\theta - \frac{a^{1'}_\theta}{2r} - 6g^2(M + a^2_\theta)\eta^2 = 0\,,\\
&\rho'' + \frac{\rho'}{r} - \frac{\sigma' + \tau'}{3r} - \frac{\sigma'' + \tau''}{3} - \frac{2\rho}{\tilde{g}^2} + \frac{8}{3\tilde{g}^2} + \frac{2}{3\tilde{g}^2}(\sigma + \tau) - \frac{8}{3\tilde{g}^2}\zeta^2(2\rho + 1) = 0\,,\\
&\sigma'' + \frac{\sigma'}{r} - \frac{\rho' + \tau'}{3r} - \frac{\rho'' + \tau''}{3} - \frac{2\sigma}{\tilde{g}^2} - \frac{8}{3\tilde{g}^2} + \frac{2}{3\tilde{g}^2}(\rho + \tau) - \frac{8}{3\tilde{g}^2}\eta^2(2\sigma - 1) = 0\,,\\
&\tau'' + \frac{\tau'}{r} - \frac{\rho' + \sigma'}{3r} - \frac{\rho'' + \sigma''}{3} - \frac{2\tau}{\tilde{g}^2} + \frac{2}{3\tilde{g}^2}(\rho + \sigma) = 0\,.
\end{aligned} \tag{5.33}$$

We do not know any analytic solutions to these coupled non-linear differential equations. However, as pointed out earlier, we can construct the solutions profiles for small and large r. For $r \to 0$, the series solutions give

$$\begin{aligned}
&\zeta = \zeta_0 r^N + O(r^{N+2})\,, \quad \eta = \eta_0 r^M + O(r^{M+2})\,, \quad a^1_\theta = a_0^{(1)} r^2 + O(r^4)\,,\\
&a^2_\theta = a_0^{(2)} r^2 + O(r^4)\,, \quad \rho = \rho_0 + O(r^2)\,, \quad \sigma = \sigma_0 + O(r^2)\,, \quad \tau = \tau_0 + O(r^2)\,,
\end{aligned} \tag{5.34}$$

where $\zeta_0\,, \eta_0\,, a_0^{(1)}\,, a_0^{(2)}\,, \rho_0\,, \sigma_0\,, \tau_0$ are constants.

For large r, as we mentioned earlier, the asymptotic behavior of fields are enforced by the requirement of the finiteness of the action for the vortex type solutions. We have $\zeta(r) \to 1\,, \eta(r) \to 1\,, a^1_\theta(r) \to N\,, a^2_\theta(r) \to M\,, \rho(r) \to 0\,, \sigma(r) \to 0\,, \tau(r) \to 0$ as $r \to \infty$, where the integers N and M are the winding numbers of the vortex configuration. In order to obtain the profiles for large ℓ, we can consider the small fluctuations about these limiting values and write $\zeta = 1 - \delta\zeta\,, \eta = 1 - \delta\eta\,, a^1_\theta = -N + \delta a^1\,, a^2_\theta = -M + \delta a^2$. Assuming that $(\frac{\delta a^1_\theta}{r})^2$ and $(\frac{\delta a^2_\theta}{r})^2$ are subleading compared to $\delta\zeta\,, \delta\eta\,, \rho\,, \sigma\,, \tau$, the Euler-Lagrange equations

(5.33) become

$$\begin{aligned}
&\delta\zeta'' + \frac{\delta\zeta'}{r} - \frac{2}{\tilde{g}^2}(4\delta\zeta - \rho - 2\delta\eta) = 0\,, \quad \delta\eta'' + \frac{\delta\eta'}{r} - \frac{2}{\tilde{g}^2}(4\delta\eta + \sigma - 2\delta\zeta) = 0 \\
&\delta a^{1\prime\prime} - \frac{\delta a^{1\prime}}{r} + 4g^2\delta a^2 - 8g^2\delta a^1 = 0\,, \quad \delta a^{2\prime\prime} - \frac{\delta a^{2\prime}}{r} + 4g^2\delta a^1 - 8g^2\delta a^2 = 0\,, \\
&\rho'' + \frac{\rho'}{r} - \frac{10}{\tilde{g}^2}\rho - \frac{4}{\tilde{g}^2}\sigma + \frac{8}{\tilde{g}^2}\delta\zeta - \frac{4}{\tilde{g}^2}\delta\eta = 0\,, \\
&\sigma'' + \frac{\sigma'}{r} - \frac{10}{\tilde{g}^2}\sigma - \frac{4}{\tilde{g}^2}\rho - \frac{8}{\tilde{g}^2}\delta\eta + \frac{4}{\tilde{g}^2}\delta\zeta = 0\,, \\
&\tau'' + \frac{\tau'}{r} - \frac{2}{\tilde{g}^2}\tau - \frac{4}{\tilde{g}^2}\rho - \frac{4}{\tilde{g}^2}\sigma - \frac{4}{\tilde{g}^2}\delta\eta + \frac{4}{\tilde{g}^2}\delta\zeta = 0\,.
\end{aligned} \tag{5.35}$$

We can solve these coupled linear differential equations in terms of the modified Bessel functions K_α and find

$$\begin{aligned}
\delta\zeta &= A_1K_0(\frac{2\sqrt{2}r}{\tilde{g}}) + A_2K_0(\frac{\sqrt{2}r}{\tilde{g}}) - A_3K_0(\frac{3\sqrt{2}r}{\tilde{g}})\,, \\
\delta\eta &= A_2K_0(\frac{\sqrt{2}r}{\tilde{g}}) + A_3K_0(\frac{3\sqrt{2}r}{\tilde{g}}) + A_4K_0(\frac{2\sqrt{2}r}{\tilde{g}})\,, \\
\rho &= A_2K_0(\frac{\sqrt{2}r}{\tilde{g}}) + 3A_3K_0(\frac{3\sqrt{2}r}{\tilde{g}}) - 2A_4K_0(\frac{2\sqrt{2}r}{\tilde{g}})\,, \\
\sigma &= 2A_1K_0(\frac{2\sqrt{2}r}{\tilde{g}}) - A_2K_0(\frac{\sqrt{2}r}{\tilde{g}}) + 3A_3K_0(\frac{3\sqrt{2}r}{\tilde{g}})\,, \\
\tau &= \frac{2}{3}(A_1 - A_4)K_0(\frac{2\sqrt{2}r}{\tilde{g}}) + 2A_3K_0(\frac{3\sqrt{2}r}{\tilde{g}}) + A_5K_0(\frac{\sqrt{2}r}{\tilde{g}})\,, \\
\delta a^1 &= C_1rK_1(2gr) + C_2rK_1(2\sqrt{3}gr)\,, \\
\delta a^2 &= C_1rK_1(2gr) - C_2rK_1(2\sqrt{3}gr)\,,
\end{aligned} \tag{5.36}$$

where $A_i\,, i = 1\cdots,5$ and $C_j\,, j = 1,2$ are constants, which can only be determined numerically. It is easy to see that our assumption that $(\frac{\delta a^1_\theta}{r})^2$ and $(\frac{\delta a^2_\theta}{r})^2$ are subleading to $\delta\zeta\,, \delta\eta\,, \rho\,, \sigma\,, \tau$ gives the same condition as in the subsection 3.3.4.1, namely $4g > \sqrt{2}/\tilde{g}$. We find from (5.36) that, the field strengths $B^1 := f^1_{12} = \frac{1}{r}f^1_{r\theta} = \frac{1}{r}\partial_r a^1_\theta$ and $B^2 := f^2_{12} = \frac{1}{r}f^2_{r\theta} = \frac{1}{r}\partial_r a^2_\theta$ are proportional to $\propto \frac{1}{\sqrt{r}}e^{-2gr}$ while the scalar fields $\delta\zeta\,, \delta\eta\,, \rho\,, \sigma$ and τ decay like $\frac{1}{\sqrt{r}}e^{-\frac{\sqrt{2}}{\tilde{g}}r}$ asymptotically. Thus, we obtain the same interval as in the subsection 3.3.4.1 for the repulsive and attractive forces on this model such that these vortices attract for $g\tilde{g} > \frac{\sqrt{2}}{2}$ and they repel in the parameter interval $\frac{\sqrt{2}}{4} < g\tilde{g} < \frac{\sqrt{2}}{2}$. Particularly, for the case $g\tilde{g} = 1$ needed for the standard Yang-Mills (3.76), we have attractive vortices.

5.3.2 Case *ii*)

Taking the limit $\mathfrak{g} \to \infty$ is equivalent to enforcing the constraint $\Phi_a\Phi_a+\ell(\ell+1) = 0$. It can be easily seen from (5.27) that this constraint can only be fulfilled by setting $R_1 = 0\,, R_2 = 0$ and $R_3 = 0$. Using these three conditions, we can solve $\varphi_3\,, \chi_3$ and ψ in terms of $|\varphi|$ and $|\chi|$ in powers of $\frac{1}{\ell}$. Substituting back into the action should then give us an action with only two complex scalars φ and χ. To leading non-vanishing order in powers of $\frac{1}{\ell}$, we find that

$$\begin{aligned} \psi &= \frac{1}{2\ell}(1-|\varphi|^2) + \frac{1}{2\ell}(1-|\chi|^2) + O(\frac{1}{\ell^2})\,, \\ \varphi_3 &= -\frac{3}{4\ell^2}(1-|\varphi|^2) - \frac{2\ell+1}{4\ell^2}(1-|\chi|^2) + O(\frac{1}{\ell^3})\,, \\ \chi_3 &= \frac{1}{4\ell^2}(1-|\chi|^2) - \frac{2\ell+1}{4\ell^2}(1-|\varphi|^2) + O(\frac{1}{\ell^3})\,. \end{aligned} \tag{5.37}$$

Substituting from (5.37) for $\varphi_3\,, \chi_3\,, \psi$, expanding ℓ dependent coefficients to order $\frac{1}{\ell^2}$, we obtain the reduced action as follows

$$\begin{aligned} S = \int d^2y \Bigg(& \frac{1}{18g^2}(1+\frac{1}{2\ell}-\frac{3}{4\ell^2})f^{(1)}_{\mu\nu}f^{(1)\mu\nu} + \frac{1}{18g^2}(1-\frac{1}{2\ell}-\frac{1}{4\ell^2})f^{(2)}_{\mu\nu}f^{(2)\mu\nu} \\ & + \frac{1}{18g^2}(1-\frac{1}{\ell^2})f^{(1)}_{\mu\nu}f^{(2)\mu\nu} + \frac{2}{3}(1-\frac{1}{2\ell^2})(|D_\mu\varphi|^2+|D_\mu\chi|^2) \\ & + \frac{1}{6\ell^2}\big((\partial_\mu|\varphi|^2)^2 + (\partial_\mu|\chi|^2)^2 + \partial_\mu|\varphi|^2\partial_\mu|\chi|^2\big) + \frac{1}{\tilde{g}^2}\Bigg(\frac{4}{3}(1+\frac{1}{4\ell^2}) \\ & - \frac{4}{3}(1-\frac{1}{\ell}+\frac{1}{\ell^2})|\varphi|^2 - \frac{4}{3}(1+\frac{1}{\ell}-\frac{1}{\ell^2})|\chi|^2 - \frac{4}{3}(1+\frac{3}{4\ell^2})|\varphi|^2|\chi|^2 \\ & + \frac{4}{3}(1-\frac{1}{2\ell}+\frac{1}{2\ell^2})|\varphi|^4 + \frac{4}{3}(1+\frac{1}{2\ell}-\frac{1}{2\ell^2})|\chi|^4 \\ & + \frac{1}{3\ell^2}(|\varphi|^4|\chi|^2+|\chi|^4|\varphi|^2)\Bigg)\Bigg)\,, \end{aligned} \tag{5.38}$$

where we wrote

$$h_{\mu\nu} = -\frac{2}{3}(\frac{1}{\ell}-\frac{1}{2\ell^2})(f^{(1)}_{\mu\nu}+f^{(2)}_{\mu\nu})\,, \tag{5.39}$$

which follows from the equation of motion of b_μ at the $\frac{1}{\ell^2}$ order.

For this case too, we make the rotationally symmetric vortex solution ansatz

(5.31) and find the action to take the form

$$S = 2\pi \int dr \Bigg(\frac{1}{9g^2 r}(1+\frac{1}{2\ell}-\frac{3}{4\ell^2}) a_\theta^{1\prime} a_\theta^{1\prime} + \frac{1}{9g^2 r}(1-\frac{1}{2\ell}-\frac{1}{4\ell^2}) a_\theta^{2\prime} a_\theta^{2\prime}$$
$$+ \frac{1}{9g^2 r}(1-\frac{1}{\ell^2}) a_\theta^{1\prime} a_\theta^{2\prime} + \frac{2}{3}(1-\frac{1}{2\ell^2})(r\zeta'^2 + \frac{(N+a_\theta^1)^2}{r}\zeta^2 + r\eta'^2 + \frac{(M+a_\theta^2)^2}{r}\eta^2)$$
$$+ \frac{r}{6\ell^2}\big(4\zeta'^2\zeta^2 + 4\eta'^2\eta^2 + 4\zeta'\zeta\eta'\eta\big) + \frac{1}{\tilde{g}^2}\Bigg(\frac{4r}{3}(1+\frac{1}{4\ell^2})$$
$$- \frac{4r}{3}(1-\frac{1}{\ell}+\frac{1}{\ell^2})\zeta^2 - \frac{4r}{3}(1+\frac{1}{\ell}-\frac{1}{\ell^2})\eta^2 - \frac{4r}{3}(1+\frac{3}{4\ell^2})\zeta^2\eta^2$$
$$+ \frac{4r}{3}(1-\frac{1}{2\ell}+\frac{1}{2\ell^2})\zeta^4 + \frac{4r}{3}(1+\frac{1}{2\ell}-\frac{1}{2\ell^2})\eta^4 + \frac{r}{3\ell^2}(\zeta^4\eta^2 + \eta^4\zeta^2)\Bigg)\Bigg). \quad (5.40)$$

Equations of motion for the fields $\zeta\,,\eta\,,a_\theta^1\,,a_\theta^2$ after a straightforward calculation are given in the appendix C. Profiles of these fields around $r=0$ are the same as in the previous case (5.34).

For large r, it is easy to find the linearized equations for the fluctuations about the vacuum values. We write as before $\zeta = 1-\delta\zeta\,,\eta = 1-\delta\eta\,,a_\theta^1 = -N + \delta a^1\,,a_\theta^2 = -M + \delta a^2$ and we obtain the equations

$$\delta\zeta'' + \frac{\delta\zeta'}{r} - \frac{2}{\tilde{g}^2}(4-\frac{2}{\ell}+\frac{2}{\ell^2})\zeta + \frac{2}{\tilde{g}^2}(2+\frac{1}{2\ell^2})\eta = 0\,,$$
$$\delta\eta'' + \frac{\delta\eta'}{r} - \frac{2}{\tilde{g}^2}(4+\frac{2}{\ell}-\frac{2}{\ell^2})\eta + \frac{2}{\tilde{g}^2}(2+\frac{1}{2\ell^2})\zeta = 0\,,$$
$$\delta a^{1\prime\prime} - \frac{\delta a^{1\prime}}{r} - 2g^2(4-\frac{2}{\ell}+\frac{1}{\ell^2})\delta a^1 + 2g^2(2-\frac{1}{\ell^2})\delta a^2 = 0\,,$$
$$\delta a^{2\prime\prime} - \frac{\delta a^{2\prime}}{r} - 2g^2(4+\frac{2}{\ell}-\frac{1}{\ell^2})\delta a^2 + 2g^2(2-\frac{1}{\ell^2})\delta a^1 = 0\,. \quad (5.41)$$

Solutions for these equations are given in terms of modified Bessel functions K_n:

$$\delta\zeta = E_1(-1+\frac{1}{\ell}+\frac{3}{2\ell^2})K_0(\frac{\sqrt{12+3/\ell^2}\,r}{\tilde{g}}) + E_2(1+\frac{1}{\ell}-\frac{1}{2\ell^2})K_0(\frac{\sqrt{4-3/\ell^2}\,r}{\tilde{g}})\,,$$
$$\delta\eta = E_1 K_0(\frac{\sqrt{12+3/\ell^2}\,r}{\tilde{g}}) + E_2 K_0(\frac{\sqrt{4-3/\ell^2}\,r}{\tilde{g}})\,,$$
$$\delta a^1 = F_1(-1+\frac{1}{\ell}-\frac{1}{\ell^2})rK_1(2\sqrt{3}gr) + F_2(1+\frac{1}{\ell})rK_1(2gr)\,,$$
$$\delta a^2 = F_1 rK_1(2\sqrt{3}gr) + F_2 rK_1(2gr)\,, \quad (5.42)$$

where $E_1\,,E_2\,,F_1\,,F_2$ are constants. Here, we can also define the parameter intervals for the attractive and repulsive behaviour of forces between the vortices. It is easy to see that for $g\tilde{g} > \frac{\sqrt{4-3/\ell^2}}{2}$, the field strengths decay faster than the scalar fields, so we have attractive vortices. On the other hand, for $\frac{\sqrt{4-3/\ell^2}}{4} < g\tilde{g} < \frac{\sqrt{4-3/\ell^2}}{2}$ we have repulsive forces between the vortices.

As $\ell \to \infty$ the action (5.38) at the critical point $g\tilde{g} = 1$ becomes

$$S = \int d^2y \; \frac{1}{18g^2}\big(f^{(1)}_{\mu\nu}f^{(1)\mu\nu} + f^{(2)}_{\mu\nu}f^{(2)\mu\nu} + f^{(1)}_{\mu\nu}f^{(2)\mu\nu}\big) + \frac{2}{3}(|D_\mu\varphi|^2 + |D_\mu\chi|^2) + \frac{2}{3}g^2\Big((|\varphi|^2-1)^2 + (|\chi|^2-1)^2 + (|\varphi|^2-|\chi|^2)^2\Big)\,, \tag{5.43}$$

In this case, we may express the action in the form

$$\begin{aligned} S = \int d^2y \; & \frac{1}{18g^2}\big(B^1 + 2g^2(2|\varphi|^2 - |\chi|^2 - 1)\big)^2 \\ & + \frac{1}{18g^2}\big(B^2 + 2g^2(2|\chi|^2 - |\varphi|^2 - 1)\big)^2 + \frac{1}{18g^2}\big(B^1 + B^2 + 2g^2(|\varphi|^2 + |\chi|^2 - 2)\big)^2 \\ & + \frac{2}{3}\big(\overline{D_1\varphi} - i\overline{D_2\varphi}\big)\big(D_1\varphi + iD_2\varphi\big) + \frac{2}{3}\big(\overline{D_1\chi} - i\overline{D_2\chi}\big)\big(D_1\chi + iD_2\chi\big) \\ & + \frac{2}{3}(B^1 + B^2) - \frac{2i}{3}\big(\partial_1(\overline{\varphi}D_2\varphi) - \partial_2(\overline{\varphi}D_1\varphi)\big) - \frac{2i}{3}\big(\partial_1(\overline{\chi}D_2\chi) - \partial_2(\overline{\chi}D_1\chi)\big)\,, \end{aligned} \tag{5.44}$$

where $B^1 = f^1_{12}\,, B^2 = f^2_{12}$ as we have noted previously. The last two terms in (5.44) vanish as they can be expressed as line integrals around a circle at infinity. Noting that the fluxes of B^1 and B^2 are $2\pi N$ and $2\pi M$ respectively, N, M being the winding numbers of the vortex configuration, we see that the action is bounded from below with $S \geq \frac{4}{3}\pi(N+M)$. This bound is saturated, when the fields satisfy the BPS equations:

$$\begin{aligned} D_1\varphi + iD_2\varphi = 0\,, \quad & B^1 + 2g^2(2|\varphi|^2 - |\chi|^2 - 1) = 0\,, \\ D_1\chi + iD_2\chi = 0\,, \quad & B^2 + 2g^2(2|\chi|^2 - |\varphi|^2 - 1) = 0\,. \end{aligned} \tag{5.45}$$

These equations give a particular generalization of the BPS equations for the abelian Higgs model [68]. In fact, these equations appear to be formally the same as the self dual instanton equations for the $SU(3)$ Yang-Mills theory with cylindrical symmetry studied by Bais and Weldon [72]. There is a clear distinction between the two however; the latter are in the context of Yang-Mills theories over $\mathbb{R}^4$ and the cylindrically symmetric ansatz essentially dimensionally reduces that theory to an abelian Higgs type model over $\mathbb{H}^2$, with the $SU(3)$

instanton solutions being characterized by a Pontryagin index, which is given as the sum of the two winding numbers of the abelian Higgs type model over $\mathbb{H}^2$ with $U(1) \times U(1)$ gauge symmetry, while our BPS equations are obtained for $U(1) \times U(1)$ abelian Higgs type model over $\mathbb{R}^2$.

5.4 $SU(2)$-equivariant Gauge Fields for $U(n)$ Gauge Theory

Now, we briefly indicate how the results of section 5.1 generalizes to $U(n)$ gauge theories over $\mathcal{M} \times S_F^2$. For this purpose we write the symmetry generators ω_a

$$\omega_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_n - \mathbf{1}^{(2\ell+1)} \otimes i\tilde{\Sigma}_a^k \,, \tag{5.46}$$

where $\tilde{\Sigma}_a^k$ are spin k irreducible representation of $SU(2)$ with $n = 2k+1$. Thus, the $SU(2)$ IRR content of ω_a is

$$\ell \otimes k = (\ell + k) \oplus (\ell + k - 1) \oplus \cdots \oplus |\ell - k| \,, \tag{5.47}$$

and the IRR content of the adjoint action of ω_a can be found to be

$$[\ell \otimes k]^{\otimes 2} = (\mathbf{2k+1})0 \oplus (\mathbf{6k+1})1 \oplus \cdots \,. \tag{5.48}$$

This decomposition means that under the adjoint action of ω_a, there are $(2k+1)$ scalars and $(6k+1)$ vectors. It indicates that with our symmetry constraints (3.79) and (3.80), the set of solutions to A_μ should be $(2k+1)$-dimensional while the set of the solutions to A_a should be $(6k+1)$-dimensional. It is possible to find the parametrization of A_μ by using the following rotational invariants

$$\mathbf{1}_{(2\ell+1)(2k+1)} \,, \quad \tilde{\Sigma}_a^k X_a \,, \quad (\tilde{\Sigma}_a^k X_a)^2 \,, \quad (\tilde{\Sigma}_a^k X_a)^3 \,, \quad \cdots \,, \quad (\tilde{\Sigma}_a^k X_a)^{2k} \,. \tag{5.49}$$

We may recall that the adjoint representation of $SU(n)$ is $n^2 - 1$ dimensional and decomposes under the $SU(2)$ IRRs as

$$n^2 - 1 = \oplus \sum_{j=1}^{n-1} (2j+1) \,. \tag{5.50}$$

This is a multipole expansion starting with the dipole term and going up to the $(n-1)^{\text{th}}$-pole term. Thus, considering that we may construct one rotational invariant per multipole term, together with the identity we have $n = 2k+1$

rotational invariants as we have already inferred from (5.48). The invariants listed in (5.49) may be expressed in terms of the appropriate multipole tensors and can further be combined into idempotents as we given in (5.8) for the case of $k = 1$ and the vectors can be obtained subsequently.

5.5 Other Vacuum Configurations

In this section, we turn our attention to the treatment of the structure of equivariant fields over other fuzzy vacuum configurations. In the section 4.1 of chapter 4, we have investigated the vacuum configuration of scalar fields in the $U(n)$ gauge theory on $\mathcal{M} \times S_F^{2\,Int}$ and shown that it can be expressed in terms of the direct sum of fuzzy sphere as

$$S_F^{2\,Int} := S_F^2(\ell) \oplus S_F^2(\ell) \oplus S_F^2\left(\ell + \frac{1}{2}\right) \oplus S_F^2\left(\ell - \frac{1}{2}\right) , \qquad (5.51)$$

by performing the field redefinition (4.3). As we mentioned before, the structure of equivariant gauge fields and the low energy of $U(2)$ gauge theory over $\mathcal{M} \times S_F^{2\,Int}$ was investigated in detail [25]. Here, our aim is to consider the $U(3)$ gauge theory over $\mathcal{M} \times S_F^{2\,Int}$ and construct the $SU(2)$ equivariant gauge fields characterizing its low energy behavior [5]. In order to determine the latter, we choose the $SU(2)$ symmetry generators ω_a as

$$\begin{aligned}
\omega_a =& (X_a^{(2\ell+1)} \otimes \mathbf{1}_4 \otimes \mathbf{1}_3) + (\mathbf{1}_{2\ell+1} \otimes \Gamma_a^0 \otimes \mathbf{1}_3) - (\mathbf{1}_{2\ell+1} \otimes \mathbf{1}_4 \otimes i\Sigma_a) \\
=:& X_a + \Gamma_a^0 - i\Sigma_a \\
=:& D_a - i\Sigma_a \,, \qquad \omega_a \in u(2\ell+1) \otimes u(4) \otimes u(3) \,,
\end{aligned} \qquad (5.52)$$

and they satisfy (3.78). ω_a carries a direct sum of IRRs of $SU(2)$, which is given as

$$\begin{aligned}
\left(\ell \oplus \ell \oplus \left(\ell + \frac{1}{2}\right) \oplus \left(\ell - \frac{1}{2}\right)\right) \otimes 1 \equiv& \, \mathbf{2}\Big((\ell - 1) \oplus \ell \oplus (\ell + 1)\Big) \\
&\oplus \mathbf{2}\Big((\ell + \frac{1}{2}) \oplus (\ell - \frac{1}{2})\Big) \oplus (\ell - \frac{3}{2}) \oplus (\ell + \frac{3}{2}) \,. \quad (5.53)
\end{aligned}$$

Using the definitions in (4.15), the projections to the representations appearing in the r.h.s of (5.53) can be constructed and listed in the table 5.1 where

[5] Note that we omit $V_2(\Phi)$ term from the action (3.75) and we will impose it as a constraint as discussed in [25].

Projector	Representation
$\Pi_{0_{\mathbf{0}}} = \mathbf{1}_{2\ell+1} \otimes P_{0_{\mathbf{0}}} \otimes \mathbf{1}_3$	$(\ell-1) \oplus \ell \oplus (\ell+1)$
$\Pi_{0_{\mathbf{2}}} = \mathbf{1}_{2\ell+1} \otimes P_{0_{\mathbf{2}}} \otimes \mathbf{1}_3$	$(\ell-1) \oplus \ell \oplus (\ell+1)$
$\Pi_+ = \frac{1}{2}(iQ_I + \Pi_{\frac{1}{2}})$	$(\ell-\frac{1}{2}) \oplus (\ell+\frac{1}{2}) \oplus (\ell+\frac{3}{2})$
$\Pi_- = \frac{1}{2}(-iQ_I + \Pi_{\frac{1}{2}})$	$(\ell-\frac{3}{2}) \oplus (\ell-\frac{1}{2}) \oplus (\ell+\frac{1}{2})$
$\Pi_0 = \Pi_{0_{\mathbf{0}}} + \Pi_{0_{\mathbf{2}}} = \mathbf{1}_{2\ell+1} \otimes P_0 \otimes \mathbf{1}_3$	$\mathbf{2}\Big((\ell-1) \oplus \ell \oplus (\ell+1)\Big)$
$\Pi_{\frac{1}{2}} = \Pi_+ + \Pi_- = \mathbf{1}_{2\ell+1} \otimes P_{\frac{1}{2}} \otimes \mathbf{1}_3$	$\mathbf{2}\Big(\big(\ell+\frac{1}{2}\big) \oplus \big(\ell-\frac{1}{2}\big)\Big) \oplus (\ell-\frac{3}{2}) \oplus (\ell+\frac{3}{2})$

Table5.1: Projections to the representations appearing in the r.h.s of (5.53).

$$Q_I = \frac{i}{\frac{1}{2}(\ell+\frac{1}{2})}(X_a\Gamma_a - \frac{1}{4}\Pi_{\frac{1}{2}})\,, \quad Q_I^2 = -\Pi_{\frac{1}{2}}\,. \tag{5.54}$$

$SU(2)$-equivariant gauge fields can be obtained by imposing the symmetry constraints in (3.79), (3.80) and (4.20). The dimensions of solution spaces for A_μ , A_a and Ψ_α can be derived by the Clebsch-Gordan decomposition of the adjoint action of ω_a. The relevant part of this decomposition is

$$\left[\mathbf{2}\Big((\ell-1) \oplus \ell \oplus (\ell+1)\Big) \oplus \mathbf{2}\Big(\big(\ell+\frac{1}{2}\big) \oplus \big(\ell-\frac{1}{2}\big)\Big) \oplus (\ell-\frac{3}{2}) \oplus (\ell+\frac{3}{2})\right]^{\otimes 2} \equiv \mathbf{22}\,0 \oplus \mathbf{40}\,\frac{1}{2} \oplus \mathbf{54}\,1 \oplus \cdots\,. \tag{5.55}$$

This simply means that there are 22 rotationally invariants and A_μ may be parametrized by these invariants. A suitable set may be listed as the following projectors and “idempotents" (in the subspace they belong to)

$$\begin{aligned}
&\Pi_{0_{\mathbf{0}}}\,, \quad \Pi_{0_{\mathbf{2}}} \quad \Pi_+\,, \quad \Pi_- \quad iS_1\,, \quad iS_2\,, \quad Q^1_{0_{\mathbf{0}}} = \Pi_{0_{\mathbf{0}}} Q_1\,, \quad Q^2_{0_{\mathbf{0}}} = \Pi_{0_{\mathbf{0}}} Q_2\,,\\
&Q^1_{0_{\mathbf{2}}} = \Pi_{0_{\mathbf{2}}} Q_1\,, \quad Q^2_{0_{\mathbf{2}}} = \Pi_{0_{\mathbf{2}}} Q_2\,, \quad Q^1_-\,, \quad Q^2_-\,, \quad Q^1_+\,, \quad Q^2_+\,, \quad Q^1_{+-}\,, \quad Q^2_{-+}\,,\\
&Q_{S11} = S_1 Q_1\,, \quad Q_{S12} = S_1 Q_2\,, \quad Q_{S21} = S_2 Q_1\,, \quad Q_{S22} = S_2 Q_2\,, \quad Q_F\,, \quad Q_H\,,
\end{aligned} \tag{5.56}$$

where

$$
\begin{aligned}
Q^1_- &= \frac{1}{\ell(2\ell+3)}\Big((2\ell+1)(\ell+1)\Pi_- Q_1 \Pi_- - i\Pi_-\Big)\,, \\
Q^2_- &= \frac{\ell(2\ell+1)}{(\ell+1)(2\ell-1)}\Pi_- Q_2 \Pi_- + \frac{(2\ell+1)}{\ell(2\ell-1)(2\ell+3)}\Pi_- Q_1 \Pi_- \\
&\quad - \frac{i}{\ell(\ell+1)(2\ell-1)(2\ell+3)}\Pi_-\,, \\
Q^1_+ &= \frac{(2\ell+1)(\ell+1)}{\ell(2\ell+3)}\Pi_+ Q_1 \Pi_+ + \frac{(2\ell+1)^2}{(2\ell-1)(2\ell+3)}\Pi_+ Q_2 \Pi_+ \\
&\quad - i\frac{(4\ell^3+4\ell^2-\ell+1)}{\ell(2\ell-1)(2\ell+3)}\Pi_+\,, \\
Q^2_+ &= \frac{1}{(\ell+1)(2\ell-1)}\Big(\ell(2\ell+1)\Pi_+ Q_2 \Pi_+ - i\Pi_+\Big)\,, \\
Q^1_{+-} &= \Pi_+ Q_1 \Pi_- - i\Pi_{\frac{1}{2}} + 2i\Pi_+\,, \quad Q^2_{-+} = \Pi_- Q_2 \Pi_+ - i\Pi_{\frac{1}{2}} + 2i\Pi_-\,, \\
S_i &= \mathbf{1}_{2\ell+1} \otimes s_i \otimes \mathbf{1}_2\,, \quad s_i = \begin{pmatrix} \sigma_i & 0_2 \\ 0_2 & 0_2 \end{pmatrix}\,, \quad i = 1\,,2\,,
\end{aligned} \tag{5.57}
$$

and

$$
\begin{aligned}
Q_F &= \frac{1}{3}\Gamma_a \Sigma_a - 2i(\Gamma_a\Sigma_a)^2 - i\frac{4}{3}\Pi_{\frac{1}{2}}\,, \\
Q_H &= \frac{4(2\ell+1)}{6\ell^2+11\ell+1}Q' - \frac{4(2\ell^2+3\ell)}{6\ell^2+11\ell+1}Q'' - i\frac{(2\ell-1)(\ell+1)}{6\ell^2+11\ell+1}\Pi_+ \\
&\quad - i\frac{3(2\ell-1)(\ell+1)}{6\ell^2+11\ell+1}\Pi_- + i\frac{4\sqrt{4\ell^2+10\ell+2}}{6\ell^2+11\ell+1}\epsilon_{abc}X_a\Gamma_b\Sigma_c \\
&\quad + i\frac{16}{6\ell^2+11\ell+1}(\epsilon_{abc}X_a\Gamma_b\Sigma_c)^2\,, \\
Q' &= \frac{\ell(2\ell+1)}{(\ell+1)(2\ell-1)}\Pi_- Q_2 \Pi_- + \frac{(2\ell+1)^2}{(2\ell-1)(2\ell+3)}\Pi_- Q_1 \Pi_- \\
&\quad - i\frac{4\ell^3+8\ell^2+3\ell-2}{(\ell+1)(2\ell-1)(2\ell+3)}\Pi_-\,, \\
Q'' &= \frac{(2\ell+1)}{(\ell+1)(2\ell-1)(2\ell+3)}\Pi_+ Q_2 \Pi_+ + \frac{(2\ell+1)(\ell+1)}{\ell(2\ell+3)}\Pi_+ Q_1 \Pi_+ \\
&\quad - i\frac{1}{\ell(\ell+1)(2\ell-1)(2\ell+3)}\Pi_+\,.
\end{aligned} \tag{5.58}
$$

Using Mathematica it is easy to verify that

$$
\begin{aligned}
&(iS_i)^2 = -\Pi_0\,, \quad (Q^i_{0_\mathbf{0}})^2 = -\Pi^i_{0_\mathbf{0}}\,, \quad (Q^i_{0_\mathbf{2}})^2 = -\Pi^i_{0_\mathbf{2}}\,, \quad (Q^i_\pm)^2 = -\Pi_\pm\,, \\
&(Q^1_{+-})^2 = -\Pi_{\frac{1}{2}}\,, \quad (Q^2_{-+})^2 = -\Pi_{\frac{1}{2}}\,, \quad (Q_{S_{ij}})^2 = -\Pi_0\,, \quad Q_F^2 = -\Pi_{\frac{1}{2}}\,, \\
&Q_H^2 = -\Pi_{\frac{1}{2}}\,, \quad Q'^2 = -\Pi_-\,, \quad Q'^2 = -\Pi_+\,.
\end{aligned} \tag{5.59}
$$

In the equation (5.55), it is seen that under the adjoint action of ω_a, there are 54 objects which transform as vectors. Using the rotational invariant in (5.56), we can construct these as follows

$$
\begin{aligned}
&[D_a\,,Q^i_{0_\mathbf{0}}]\,, \quad Q^i_{0_\mathbf{0}}[D_a\,,Q^i_{0_\mathbf{0}}]\,, \quad \{D_a\,,Q^i_{0_\mathbf{0}}\}\,,\\
&[D_a\,,Q^i_{0_\mathbf{2}}]\,, \quad Q^i_{0_\mathbf{2}}[D_a\,,Q^i_{0_\mathbf{2}}]\,, \quad \{D_a\,,Q^i_{0_\mathbf{2}}\}\,,\\
&[D_a\,,Q^i_{-}]\,, \quad Q^i_{-}[D_a\,,Q^i_{-}]\,, \quad \{D_a\,,Q^i_{-}\}\,,\\
&[D_a\,,Q^i_{+}]\,, \quad Q^i_{+}[D_a\,,Q^i_{+}]\,, \quad \{D_a\,,Q^i_{+}\}\,,\\
&[D_a\,,Q_H]\,, \quad Q_H[D_a\,,Q_H]\,, \quad \{D_a\,,Q_H\}\,,\\
&[D_a\,,Q_F]\,, \quad Q_F[D_a\,,Q_F]\,, \quad \{D_a\,,Q_F\}\,,\\
&[D_a\,,Q_{S11}]\,, \quad Q^1_0[D_a\,,Q_{S11}]\,, \quad \{D_a\,,Q_{S11}\}\,,\\
&[D_a\,,Q_{S12}]\,, \quad Q^2_0[D_a\,,Q_{S12}]\,, \quad \{D_a\,,Q_{S12}\}\,,\\
&[D_a\,,Q_{S21}]\,, \quad Q^1_0[D_a\,,Q_{S21}]\,, \quad \{D_a\,,Q_{S21}\}\,,\\
&[D_a\,,Q_{S22}]\,, \quad Q^2_0[D_a\,,Q_{S22}]\,, \quad \{D_a\,,Q_{S22}\}\,,\\
&[D_a\,,Q^1_{+-}]\,, \quad Q^1_{\frac{1}{2}}[D_a\,,Q^1_{+-}]\,, \quad \{D_a\,,Q^1_{+-}\}\,,\\
&[D_a\,,Q^2_{-+}]\,, \quad Q^2_{\frac{1}{2}}[D_a\,,Q^2_{-+}]\,, \quad \{D_a\,,Q^2_{-+}\}\,,\\
&\Pi_{0_\mathbf{0}}\omega_a\,, \quad \Pi_{02}\omega_a\,, \quad \Pi_-\omega_a\,, \quad \Pi_+\omega_a\,, \quad S_1\omega_a\,, \quad S_2\omega_a\,.
\end{aligned}
\tag{5.60}
$$

Here $Q^1_0 = \Pi_0 Q_1\,, Q^2_0 = \Pi_0 Q_2\,, Q^1_{\frac{1}{2}} = \Pi_{\frac{1}{2}} Q_1\,, Q^2_{\frac{1}{2}} = \Pi_{\frac{1}{2}} Q_2\,,$ and no sum over repeated indices is implied. It is possible to parametrize A_a in terms of these 54-objects. For the 40 objects which transform as spinors under the adjoint action of ω_a, we can, for instance, take

$$
\begin{aligned}
&\Pi_{0_\mathbf{0}}\beta_\alpha Q_{-+}\,, \quad Q^1_{0_\mathbf{0}}\beta_\alpha\Pi_-\,, \quad Q^2_{0_\mathbf{0}}\beta_\alpha\Pi_-\,, \quad \Pi_{0_\mathbf{0}}\beta_\alpha Q_{+-}\,, \quad Q^1_{0_\mathbf{0}}\beta_\alpha\Pi_+\,, \quad Q^2_{0_\mathbf{0}}\beta_\alpha\Pi_+\,,\\
&Q^1_{0_\mathbf{0}}\beta_\alpha Q_{+-}\,, \quad Q^2_{0_\mathbf{0}}\beta_\alpha Q_{-+}\,, \quad \Pi_-\beta_\alpha Q^1_{0_\mathbf{2}}\,, \quad \Pi_-\beta_\alpha Q^2_{0_\mathbf{2}}\,, \quad \Pi_+\beta_\alpha Q^1_{0_\mathbf{2}}\,, \quad \Pi_+\beta_\alpha Q^2_{0_\mathbf{2}}\,,\\
&Q^1_{+-}\beta_\alpha\Pi_{0_\mathbf{2}}\,, \quad Q^2_{-+}\beta_\alpha\Pi_{0_\mathbf{2}}\,, \quad Q^1_{+-}\beta_\alpha Q^1_{0_\mathbf{2}}\,, \quad Q^2_{-+}\beta_\alpha Q^2_{0_\mathbf{2}}\,, \quad S_1\beta_\alpha\Pi_+\,, \quad S_1\beta_\alpha\Pi_-\,,
\end{aligned}
\tag{5.61}
$$

$$\Pi_-\beta_\alpha S_2\,, \quad \Pi_+\beta_\alpha S_2\,, \quad Q_{S11}\beta_\alpha\Pi_+\,, \quad Q_{S11}\beta_\alpha\Pi_-\,, \quad Q_{S12}\beta_\alpha\Pi_+\,, \quad Q_{S12}\beta_\alpha\Pi_-\,,$$
$$\Pi_-\beta_\alpha Q_{S21}\,, \quad \Pi_-\beta_\alpha Q_{S22}\,, \quad \Pi_+\beta_\alpha Q_{S12}\,, \quad \Pi_+\beta_\alpha Q_{S22}\,, \quad Q_{S11}\beta_\alpha Q^1_{+-}\,,$$
$$Q_{S12}\beta_\alpha Q^2_{-+}\,, \quad Q^1_{+-}\beta_\alpha Q_{S21}\,, \quad Q^2_{-+}\beta_\alpha Q_{S22}\,, \quad \Pi_{0_{\mathbf{0}}}\beta_\alpha Q^1_+\,, \quad \Pi_{0_{\mathbf{0}}}\beta_\alpha Q^2_-\,,$$
$$Q_{S11}\beta_\alpha Q^1_+\,, \quad Q_{S12}\beta_\alpha Q^2_-\,, \quad Q^1_+\beta_\alpha Q_{S21}\,, \quad Q^2_-\beta_\alpha Q_{S22}\,, \quad Q^1_+\beta_\alpha \Pi_{0_{\mathbf{2}}}\,, \quad Q^2_-\beta_\alpha \Pi_{0_{\mathbf{2}}}\,. \tag{5.62}$$

Thus, we have determined all the equivariant low energy degrees of freedom for the $U(3)$ gauge theory over $\mathcal{M} \times S_F^{2\,Int}$. A few remarks are now in order. Firstly, we wish to emphasize once again that, from a geometrical point of view the vacuum $S_F^{2\,Int}$ may be interpreted as stacks of concentric D2-branes with magnetic monopole fluxes and due to this fact it is possible to think of the equivariant gauge field modes that we have found as the modes of the gauge fields living on the world-volume of these D-branes. Let us also stress that the equivariant spinors given above, do not constitute independent degrees of freedom in the $U(3)$ effective gauge theory over $\mathcal{M} \times S_F^{2\,Int}$. Their bilinears, however, may be constructed to yield the equivariant scalars and vectors. In other words, it is possible to use these equivariant spinor modes to express the "square roots" of the equivariant gauge field modes.

It is possible to explore the dimensional reduction of the $U(3)$ gauge theory over $S_F^{2\,Int}$ or over its projections, such as the monopole bundles $S_F^{2\pm} = S_F^2(\ell) \oplus S_F^2(\ell \pm \frac{1}{2})$ with winding numbers ± 1. In this latter case, it easy to observe that the reduced model will yield two decoupled abelian Higgs type model, each carrying $U(1)^{\otimes 3}$ as found in section 5.2 and the vortex solutions determined in section 5.3 will be valid within each sector. Dimensional reduction over $S_F^{2\,Int}$ is quite tedious calculation-wise and will not be considered in this thesis.

CHAPTER 6

QUANTUM HALL EFFECT ON $\mathbf{GR}_2(\mathbb{C}^N)$

This chapter is oriented to formulate the quantum Hall effect (QHE) problem on the complex Grassmann manifolds $\mathbf{Gr}_2(\mathbb{C}^N)$ [1]. Although the subject of this chapter does not seem related to the previous parts of the thesis at first sight, there is an intimate connection between QHE problem and fuzzy spaces. We may briefly explain this connection as follows: A well known fact about Landau problems over compact manifolds such as $S^2\,,\mathbb{C}P^N$ is that the degenerate states at the lowest Landau level (LLL) and for that matter at any Landau level is finite. The finite dimensional Hilbert space of states $\mathcal{H}_N$ at the LLL correspond to holomorphic sections of complex line bundles for the QHE problem with abelian background gauge fields. Construction of fuzzy spaces via geometric quantization methods also yield Hilbert spaces which are holomorphic sections of complex line bundles over the commutative parent manifold. Thus there is a one to one correspondence between the Hilbert spaces for LLL states on $S^2\,,\mathbb{C}P^N$, $\mathbf{Gr}_2(\mathbb{C}^N)$ and the Hilbert spaces for the fuzzy manifolds $S_F^2\,,\mathbb{C}P_F^N$, $\mathbf{Gr}_2(\mathbb{C}^N)_F$. Similar structural relationship between fuzzy even spheres S_F^{2k} and QHE on S^{2k} also exists [74]. The matrix algebras describing fuzzy spaces such as $S_F^2\,,\mathbb{C}P_F^N$ act on this Hilbert space as linear transformations. Observables on fuzzy spaces belong to these matrix algebras. Therefore it is possible to conceive the observables of QHE on such spaces at LLL as linear transformations in the corresponding matrix algebras.

In early 80s, with strong motivation emerging from condensed matter physics,

[1] This chapter is based on the work that has been published: F. Ballı, A. Behtash, S. Kürkçüoğlu, and G. Ünal, "Quantum Hall effect on the Grassmannians $\mathbf{Gr}_k(\mathbb{C}^N)$" Phys.Rev. D**89** (2014) 105031.

Haldane has solved the problem of spherical cloud of electrons under the influence of Dirac monopole background fields [75]. With motivations from high energy physics and string theory, Hu and Zhang [76] have introduced a 4-dimensional version of QHE on 4-sphere, S^4 by generalizing the ideas of Haldane. They have formulated and solved the Landau problem on S^4 for fermions carrying an additional $SU(2)$ degree of freedom and under the influence of an $SU(2)$ background gauge field. For the multi-particle problem in the lowest Landau level (LLL) with filling factor $\nu = 1$, it turns out that in the thermodynamic limit, a finite spatial density is achieved only if the particles are in an infinitely large irreducible representations of $SU(2)$ (i.e. they carry infinitely large number of $SU(2)$ internal degrees of freedom). In this limit, two-point density correlation function immediately indicates incompressibility property of this 4-dimensional quantum Hall liquid. Appearance of massless chiral bosons at the edge of a 2-dimensional quantum Hall droplet [77–80] also generalizes to this setting. Nevertheless, it is found that among the edge excitations of this 4-dimensional quantum Hall droplet not only photons and gravitons but also other massless higher spin states occur. The latter is essentially due to the presence of a large number of $SU(2)$ degrees of freedom attached to each particle and, as such, it is not a desirable feature of the model.

Other developments ensued the work of Hu and Zhang. Several authors have addressed other higher-dimensional generalizations of QHE to a variety of manifolds including complex projective spaces $\mathbb{C}P^N$, S^8, S^3, the Flag manifold $\frac{SU(3)}{U(1)\times U(1)}$, as well as quantum Hall systems based on higher dimensional fuzzy spheres [29, 81–84]. Of particular interest to us is the work of Nair and Karabali on the formulation of QHE problem on $\mathbb{C}P^N$ [29]. These authors solve the Landau problem on $\mathbb{C}P^N$ by appealing to the coset realization of $\mathbb{C}P^N$ over $SU(N+1)$ and performing a suitable restriction of the Wigner $\mathcal{D}$-functions on the latter. In this manner, wave functions for charged particles under the influence of both $U(1)$ abelian and/or non abelian $SU(N)$ gauge backgrounds are obtained as sections of $U(1)$ and/or $SU(N)$ bundles over $\mathbb{C}P^N$. This formulation simultaneously permits the authors to give the energy spectrum of the LL, where the degeneracy in each LL is identified with the dimension of IRR to which the aforementioned restricted Wigner $\mathcal{D}$-functions belong. An important feature of these results is that the spatial density of particles remains finite without the need for infinitely large internal $SU(N)$ degrees of freedom, contrary to the situation encountered for the Hall effect on S^4. It also turns out that there is a close connection between the Hall effects on $\mathbb{C}P^3$ and $\mathbb{C}P^7$ with abelian backgrounds and those on the spheres S^4 and S^8 with $SU(2)$ and $SO(8)$ backgrounds, respectively [29, 81, 84].

In this chapter, we focus on the formulation of the QHE on the complex Grassmannians $\mathbf{Gr}_k(\mathbb{C}^N)$ [28], which are generalizations of complex projective spaces $\mathbb{C}P^N$ and share many of their nice features, such as being a Kähler manifold. Several of these features are effectively captured by their so-called the Plücker embedding into $\mathbb{C}P^{\binom{N}{k}-1}$. For the case $k = 2$, to which we will be restricting ourselves in this chapter, the Plücker embedding describes $\mathbf{Gr}_k(\mathbb{C}^N)$ as a projective algebraic hypersurface in $\mathbb{C}P^N$. For $\mathbf{Gr}_k(\mathbb{C}^N)$ this is the well-known Klein Quadric in $\mathbb{C}P^5$ [85]. The developments summarized above and the intriguing geometry of Grassmannian manifolds motivates us to take up the formulation of the QHE problem on the Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$. Using group theoretical techniques, we solve the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^N)$ and provide the energy spectrum and the eigenfunctions in terms of $SU(N)$ Wigner $\mathcal{D}$-functions for charged particles on $\mathbf{Gr}_2(\mathbb{C}^N)$ under the influence of abelian and/or non-abelian background magnetic monopoles, where the latter are obtained as sections of bundles over $\mathbf{Gr}_2(\mathbb{C}^N)$.

The organization of this chapter is as follows: we first provide a short account of the formulation of quantum Hall problem on $\mathbb{C}P^1$ and $\mathbb{C}P^2$ in section 6.1 for the purposes of orienting the developments in the subsequent sections and making the exposition self-contained. In section 6.2, we focus our attention to QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$ which is the simplest and perhaps the more interesting case and the solution for the most general case of non-zero $U(1)$ and $SU(2) \times SU(2)$ backgrounds are given. In particular, we show that at the LLL with $\nu = 1$, finite spatial densities are obtained at finite $SU(2) \times SU(2)$ internal degrees of freedom in agreement with the results of [29]. In section 6.3, we generalize these results to all $\mathbf{Gr}_2(\mathbb{C}^N)$. The local structure of the solutions on $\mathbf{Gr}_2(\mathbb{C}^4)$ in the presence of $U(1)$ background gauge field is presented in section 6.4. There we give the single and multi-particle wave functions by introducing the Plücker coordinates and show by calculating the two-point correlation function that the LLL at filling factor $\nu = 1$ forms an incompressible fluid. The $U(1)$ gauge field, its associated field strength and their properties are illustrated using the differential geometry on $\mathbf{Gr}_2(\mathbb{C}^4)$. We also briefly comment on the generalization of this local formulation to all $\mathbf{Gr}_2(\mathbb{C}^N)$.

6.1 Review of QHE on $\mathbb{C}P^1$ and $\mathbb{C}P^2$

The formulation of the QHE on $\mathbb{C}P^1 \equiv S^2$ is originally due to Haldane [75]. Karabali and Nair [29] have provided a reformulation of QHE on $\mathbb{C}P^1$ in a manner that is adaptable to formulate QHE on $\mathbb{C}P^N$. Here, we closely follow the discussion of [29] and while at it we provide the Young diagram techniques for handling the QHE problem on $\mathbb{C}P^2$. In section 6.2 and 6.3 we employ the latter to transparently handle the branching of the IRR of $SU(N)$ under the relevant subgroups appearing in the coset realizations of $\mathbf{Gr}_2(\mathbb{C}^N)$.

Landau problem on $\mathbb{C}P^1$ can be viewed as electrons on a two-sphere under the influence of a Dirac monopole sitting at the center. Our task is to construct the Hamiltonian for a single electron under the influence of a monopole field. To this end, let us first point out that by the Peter-Weyl theorem the functions on the group manifold of $SU(2) \equiv S^3$ may be expanded in terms of the Wigner-$\mathcal{D}$ functions $\mathcal{D}^{(j)}_{L_3 R_3}(g)$ where g is an $SU(2)$ group element and j is an integral or a half-odd integral number labeling the IRR of $SU(2)$. The subscripts L_3 and R_3 are the eigenvalues of the third component of the left- and right-invariant vector fields on $SU(2)$[2]. The left- and right-invariant vector fields on $SU(2)$ satisfy

$$[L_i \,, L_j] = -\varepsilon_{ijk} L_k \,, \quad [R_i \,, R_j] = \varepsilon_{ijk} R_k \,, \quad [L_i \,, R_j] = 0 \,. \tag{6.1}$$

The harmonics as well as sections of bundles over $\mathbb{C}P^1$ may be obtained from the Wigner-$\mathcal{D}$ functions on $SU(2)$ by a suitable restriction of the latter. The

[2] Throughout this chapter we sometimes denote the left and right invariant vector fields of $SU(N)$ and their eigenvalues by L_i and R_i, respectively, which one is meant will be clear from the context.

coset realization of $\mathbb{C}P^1$ is

$$\mathbb{C}P^1 \equiv S^2 = \frac{SU(2)}{U(1)}. \tag{6.2}$$

This implies that the sections of $U(1)$ bundle over $\mathbb{C}P^1$ should fulfill

$$\mathcal{D}(g e^{iR_3\theta}) = e^{i\frac{n}{2}\theta}\mathcal{D}(g)\,, \tag{6.3}$$

where n is an integer. This condition is solved by the functions of the form $\mathcal{D}^{(j)}_{L_3\frac{n}{2}}(g)$. In fact, the eigenvalue $\frac{n}{2}$ of R_3 corresponds to the strength of the Dirac monopole at the center of the sphere and $\mathcal{D}^{(j)}_{L_3\frac{n}{2}}(g)$ are the desired wavefunctions as will be made clear shortly. In particular, $\mathcal{D}^{(j)}_{L_3 0}(g)$ correspond to the spherical harmonics on S^2, which are the wavefunctions for electrons on a sphere with zero magnetic monopole background.

In the presence of a magnetic monopole field B, the Hamiltonian must involve covariant derivatives whose commutator is proportional to the magnetic field. Let us take this commutator as $[D_+\,, D_-] = B$. It is now observed that the covariant derivatives $D_\pm$ may be identified by the right invariant vector fields $R_\pm = R_1 \pm iR_2$, as

$$D_\pm = \frac{1}{\sqrt{2}\ell}R_\pm\,, \tag{6.4}$$

where ℓ denotes the radius of the sphere. Noting that $[R_+\,, R_-] = 2R_3$, for the eigenvalue $\frac{n}{2}$ of R_3 we have

$$B = \frac{n}{2\ell^2}\,, \tag{6.5}$$

for the magnetic monopole with the strength $\frac{n}{2}$ in accordance with the Dirac quantization condition. The associated magnetic flux through the sphere is $2\pi n$.

The Hamiltonian may be expressed as

$$\begin{aligned} H &= \frac{1}{2M}(D_+D_- + D_-D_+) \\ &= \frac{1}{2M\ell^2}(\sum_{i=1}^{3} R_i^2 - R_3^2)\,, \end{aligned} \tag{6.6}$$

where M is the mass of the particle. We have that $\sum_{i=1}^{3} R_i^2 = \sum_{i=1}^{3} L_i^2 = j(j+1)$. In order to guarantee that $\frac{n}{2}$ occurs as one of the possible eigenvalues of R_3, we need to have $j = \frac{1}{2}n + q$ where q is an integer. The spectrum of the Hamiltonian reads

$$\begin{aligned} E_{q,n} &= \frac{1}{2M\ell^2}\left((\frac{n}{2}+q)(\frac{n}{2}+q+1) - \frac{n^2}{4} \right) \\ &= \frac{B}{2M}(2q+1) + \frac{q(q+1)}{2M\ell^2} \,. \end{aligned} \tag{6.7}$$

The associated eigenfunctions are $\mathcal{D}^{(j)}_{L_3 \frac{n}{2}}(g)$ as noted earlier. In (6.7), q is readily interpreted as the Landau level (LL) index. The ground state, that is the Lowest Landau Level (LLL), is at $q = 0$ and has the energy $\frac{B}{2M}$. The LLL is separated from the higher LL by finite energy gaps.

The degeneracy of the LL are controlled by the left invariant vector fields L_i since they commute with the covariant derivatives $[L_i \, , D_j] = 0$. Each LL is $(2j + 1 = n + 1 + 2q)$-fold degenerate. In other words, there are this many wavefunctions $\mathcal{D}^{(j)}_{L_3 \frac{n}{2}}(g)$ at a given LL with L_3 eigenvalues ranging from $-j$ to j.

Local form of the wavefunctions may be written down by picking a suitable coordinate system. We omit this here and refer the reader to the original literature [29] where this is done in detail. In particular, it is shown in [29] that the LLL form an incompressible liquid by computing the two-point correlation function for the wave-function density. We will address this crucial property of the LLL for our case in section 6.4.

Let us now briefly turn our attention to the formulation of Landau problem on $\mathbb{C}P^2$. This and its generalization to $\mathbb{C}P^N$ is given in [29]. The coset realization of $\mathbb{C}P^2$ may be written as

$$\mathbb{C}P^2 \equiv \frac{SU(3)}{U(2)} \sim \frac{SU(3)}{SU(2) \times U(1)} \,. \tag{6.8}$$

Following a similar line of development as in the previous case, we can obtain the harmonics and local sections of bundles over $\mathbb{C}P^2$ from a suitable restriction of the Wigner-$\mathcal{D}$ functions on $SU(3)$. Let $g \in SU(3)$ and let us denote the left- and the right-invariant vector fields on $SU(3)$ by L_α and R_α ($\alpha : 1 \, , \cdots , 8$); they fulfill the Lie algebra commutation relations for $SU(3)$. We can introduce the Wigner-$\mathcal{D}$ functions on $SU(3)$ as

$$\mathcal{D}^{(p,q)}_{L,L_3,L_8;R,R_3,R_8}(g) \,, \tag{6.9}$$

where (p,q) label the irreducible representations of $SU(3)$, and the subscripts denote the relevant quantum numbers for the left- and right- rotations. In particular, the left and right generators of the $SU(2)$ subgroup are labeled by L_i and R_i $(i: 1\,,2\,,3)$ and $L_iL_i = L(L+1)$, $R_iR_i = R(R+1)$.

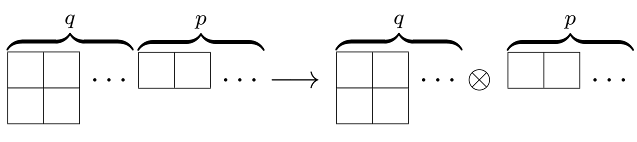
We note that the tangents along $\mathbb{C}P^2$ may be parametrized by the right invariant fields, R_α, $(\alpha: 4,5,6,7)$. Consequently, the Hamiltonian on $\mathbb{C}P^2$ may be written down as

$$\begin{aligned} H &= \frac{1}{2M\ell^2}\sum_{\alpha=4}^{7} R_\alpha^2 \\ &= \frac{1}{2M\ell^2}\left(\mathcal{C}_2(p,q) - R(R+1) - R_8^2\right) \,, \end{aligned} \tag{6.10}$$

where $\mathcal{C}_2(p,q)$ is the quadratic Casimir of $SU(3)$.

The coset realization of $\mathbb{C}P^2$ implies that there can be both abelian and non-abelian background gauge fields corresponding to the gauging of the $U(1)$ and $SU(2)$ subgroups, respectively.

Let us first obtain the wave functions with the $U(1)$ background gauge field. This means that our desired $\mathcal{D}^{(p,q)}$ should transform trivially under the $SU(2)$, and carry a $U(1)$ charge under the right actions of these groups. In other words, these wave functions must be singlets under $SU(2)$ with $R = 0\,, R_3 = 0$ and a non-zero R_8 eigenvalue. We can utilize the Young tableaux to see the branching of the $SU(3)$ IRR satisfying this requirement. The $SU(3)$ IRR labeled by (p,q) may be assigned to a Young tableau with p columns with one box each and q columns with two boxes on each. The branching $SU(3) \supset SU(2) \times U(1)$, which keeps the $SU(2)$ in the singlet representation, is therefore



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where the diagram on l.h.s. of the arrow represents the generic (p,q) IRR of $SU(3)$ and the first diagram on the r.h.s. of the arrow represent the $SU(2)$ IRR, which is singlet in this case. A general formula exits [86] for expressing the $U(1)$ charge of the branching $SU(3) \supset SU(2) \times U(1)$ (see equation 6.32 for a more general case):

$$n = \frac{1}{2}(J_1 - 2J_2)\,, \quad n \in \mathbb{Z}\,,$$

where J_1 is the number of boxes in the tableau of $SU(2)$ and J_2 is the number of boxes in the rightmost tableau in the branching. Thus for the tableaux given above, we conclude that $n = q - p$. In order to fix the relation between R_8 eigenvalues and the integer n, we use the fundamental representation $(1,0)$ with the generators λ_a fulfilling the normalization condition $\mathrm{Tr}(\lambda_a \lambda_b) = \frac{1}{2}\delta_{ab}$, and $\lambda_8 = \frac{1}{2\sqrt{3}}\mathrm{diag}(1,1,-2)$, so that

$$R_8 = -\frac{n}{\sqrt{3}} = -\frac{p-q}{\sqrt{3}}. \tag{6.11}$$

It is useful to note that the flux of the $U(1)$ field strength corresponding to the background gauge field is proportional to the number n. We omit the details of this here and refer the reader to [29].

The spectrum of the Hamiltonian (6.10) may be given as

$$E_{q,n} = \frac{1}{2M\ell^2}\left(q(q+n+2)+n\right)\,, \tag{6.12}$$

where we have used the eigenvalue of the quadratic Casimir $\mathcal{C}_2(p,q)$ of the IRR (p,q), which is

$$\mathcal{C}_2(p,q) = \frac{1}{3}\left(p(p+3)+q(q+3)+pq\right)\,. \tag{6.13}$$

and expressed the energy levels in terms of q and n only. In (6.12), q appears as the Landau level index; the ground state energy may be obtained by setting $q = 0$ and that gives LLL energy $E_{LLL} = \frac{n}{2M\ell^2}$.

The wave functions corresponding to this energy spectrum can be written in terms of the Wigner-$\mathcal{D}$ functions as

$$\mathcal{D}^{(p,q)}_{L,L_3,L_8;0,0,-\frac{n}{\sqrt{3}}}(g)\,. \tag{6.14}$$

The degeneracy of each Landau level q is given by the dimension of the IRR (p,q), which is

$$\dim(p,q) = \frac{(p+1)(q+1)(p+q+2)}{2}\,. \tag{6.15}$$

This means that the set of quantum numbers L, L_3 and L_8 can take $\dim(p,q)$ different values.

It is also useful to note that the case $n = 0$ simply reduces the Wigner-$\mathcal{D}$ functions to the harmonics on $\mathbb{C}P^2$, corresponding to the wave functions of a particle on $\mathbb{C}P^2$ with vanishing monopole background.

Consider the case of filling factor $\nu = 1$, i.e. each of the LL states is occupied by one fermion. We therefore have that $p = n$, $q = 0$ and the number of fermions $\mathcal{N}$ is equal to $\dim(n, 0) = (n+1)(n+2)/2$. The density of particles ρ is given by

$$\rho = \frac{\mathcal{N}}{\mathrm{vol}(\mathbb{C}P^2)}\,, \tag{6.16}$$

where $\mathrm{vol}(\mathbb{C}P^2) = 8\pi^2\ell^4$. In the thermodynamic limit $\ell \to \infty$ and $\mathcal{N} \to \infty$, this yields the finite result

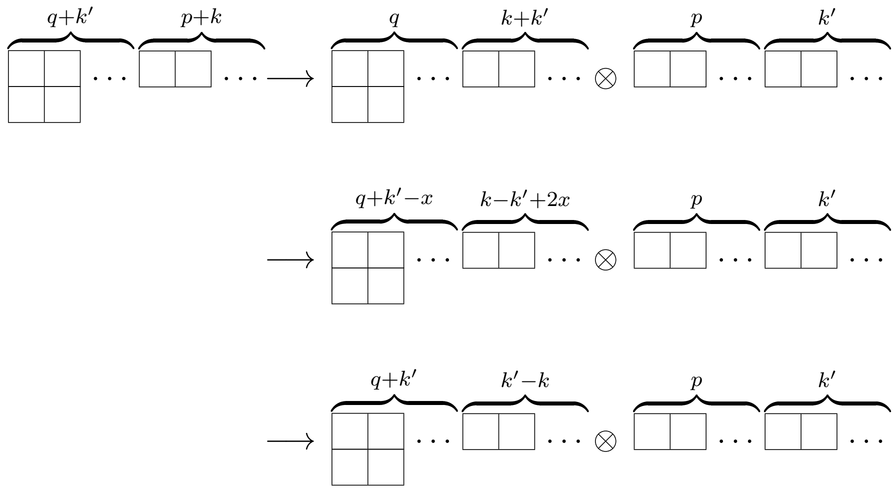
$$\rho = \frac{\mathcal{N}}{8\pi^2\ell^4} \underset{\ell\to\infty,\mathcal{N}\to\infty}{\longrightarrow} \frac{n^2}{16\pi^2\ell^4} = \left(\frac{B}{2\pi}\right)^2, \tag{6.17}$$

as first discussed in [29].

The wave functions can be expressed in suitable local coordinates and taking advantage of these functions, the multi-particle wave-function for the filling factor $\nu = 1$ state can immediately be constructed. A straightforward calculation for the two-point correlation function for the wave-function density may be given which signals the incompressibility of the LLL. We refer the reader to [29] for details.

The case of $SU(2)$ and $U(1)$ background gauge fields may be handled as follows. In this case we allow for all possible right $SU(2)$ IRR labeled by spin R. It is possible to label $SU(3)$ representations in the form $(p+k, q+k')$. The branching

$SU(3) \supset SU(2) \times U(1)$ may be represented by the Young tableaux



These tableaux represent the maximum, generic and minimum spin R-value configurations that can result from the branching, and we have assumed without loss of generality that $k' > k$ and $k \geq x \geq 0$. Here x is an integer introduced to conveniently represent the generic case. From the tableaux, the range of the spin R and R_8 eigenvalues may be easily obtained as follows:

$$R = \frac{|k-k'|}{2}, \cdots, \frac{k+k'}{2} \tag{6.18}$$

$$R_8 = \frac{1}{2\sqrt{3}}\left(-2(p-q)+(k-k')\right) = -\frac{n}{\sqrt{3}}\,. \tag{6.19}$$

Noting that n is an integer restricts the spin R to integer values. Spectrum of the Hamiltonian (6.10) is now

$$\begin{aligned} E &= \frac{1}{2M\ell^2}(\mathcal{C}_2(p+k, q+k') - R(R+1) - R_8^2) \\ &= \frac{1}{2M\ell^2}\Big(q^2 + q(2k-m+n+2) + n(k+1) + k^2 + 2k + m^2 \\ &\quad - m(k+1) - R(R+1)\Big)\,, \end{aligned} \tag{6.20}$$

where $k' = k - 2m$ and m is an integer. As indicated in (6.18), there is an interval for the values of R. The LLL is obtained when we choose the maximum value for R,

$$R_{max} = \frac{k+k'}{2} = k - m\,, \tag{6.21}$$

where m should take only integer values within the interval $m = 0, \cdots, \frac{k}{2}$ if k is even, and $m = 0, \cdots, \frac{k-1}{2}$ if k is odd. Using (6.21) in (6.20), the energy spectrum is expressed as

$$E = \frac{1}{2M\ell^2}\left(q^2 + q(2R+n+m+2) + n(R+m+1) + (R+m)(m+1)\right) . \tag{6.22}$$

For fixed n,R we observe from this expression that the LL are controlled by the two integers q and m. The LLL is obtained for $q=0$ and $m=0$.

As discussed in [29], for pure $SU(2)$ background, to ensure the finiteness of energy eigenvalues R should scale like $R \sim \ell^2$ in the thermodynamic limit. For $\nu = 1$ we have $\mathcal{N} = \dim(R,R) = \frac{1}{2}(R+1)(R+1)(2R+2)$ and this results in a finite density of particles

$$\rho \sim \frac{\mathcal{N}}{(2R+1)\ell^4} \underset{\ell\to\infty,\mathcal{N}\to\infty}{\longrightarrow} \frac{R^3}{2R\ell^4} \, . \tag{6.23}$$

As for the case of both $U(1)$ and $SU(2)$ backgrounds, it is possible to pick either n or R to scale like ℓ^2. Taking $n \sim \ell^2$ and R to be finite as $\ell \to \infty$, gives again a finite spatial density

$$\rho \sim \frac{\dim(R+n,R)}{(2R+1)\ell^4} \underset{\ell\to\infty,\mathcal{N}\to\infty}{\longrightarrow} \frac{n^2}{4\ell^4} \, , \tag{6.24}$$

for $\nu = 1$ with $\dim(R+n,R) = \frac{1}{2}(n+R+1)(R+1)(n+2R+1)$.

6.2 Landau Problem on the Grassmannian $\mathbf{Gr}_2(\mathbb{C}^4)$

Starting in this section we will consider the quantum Hall problem on the complex Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$. In order to set up the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^N)$, it is necessary to list a few facts about the Grassmannians and their geometry.

The complex Grassmannians $\mathbf{Gr}_k(\mathbb{C}^N)$ are the set of all k-dimensional linear subspaces of the vector space $\mathbb{C}^N$ with the complex dimension $k(N-k)$. They are smooth and compact complex manifolds and admit Kähler structures. Grassmannians are homogeneous spaces and can therefore be realized as the cosets of $SU(N)$ as

$$\mathbf{Gr}_k(\mathbb{C}^N) = \frac{SU(N)}{S[U(N-k)\times U(k)]} \sim \frac{SU(N)}{SU(N-k)\times SU(k)\times U(1)} . \tag{6.25}$$

It is clear from this realization that $\mathbf{Gr}_1(\mathbb{C}^N) \equiv \mathbb{C}P^N$. $\mathbf{Gr}_2(\mathbb{C}^4)$ is therefore the simplest Grassmannian that is not a projective space. The coset space realization of the Grassmannians is the most suitable setting for group theoretical techniques that we will employ to formulate and solve the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^4)$ first and subsequently on all $\mathbf{Gr}_2(\mathbb{C}^N)$.

In order to set up and solve the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^4)$, we contemplate, following the ideas reviewed in the previous section, that $SU(4)$ Wigner $\mathcal{D}$-functions may be suitably restricted to obtain the harmonics and local sections of bundles over $\mathbf{Gr}_2(\mathbb{C}^4)$. Let $g \in SU(4)$ and let us denote the left- and the right-invariant vector fields on $SU(4)$ by L_α and R_α ($\alpha : 1, \cdots, 15$); they fulfill the Lie algebra commutation relations for $SU(4)$. We can introduce the Wigner-$\mathcal{D}$ functions on $SU(4)$ as

$$g \to \mathcal{D}^{(p,q,r)}_{L^{(1)} L_3^{(1)} L^{(2)} L_3^{(2)} L_{15}; R^{(1)} R_3^{(1)} R^{(2)} R_3^{(2)} R_{15}}(g) \,, \tag{6.26}$$

where (p, q, r) are three integers labeling the irreducible representations of $SU(4)$, and the subscripts denote the relevant quantum numbers for the left- and right-rotations. In particular, the left and right generators of $SU(2) \times SU(2)$ subgroup are labeled by $L_\alpha \equiv (L_i^{(1)}, L_i^{(2)})$ and $R_\alpha \equiv (R_i^{(1)}, R_i^{(2)})$ ($i : 1, 2, 3$, $\alpha : 1, \cdots, 6$) with corresponding $SU(2) \times SU(2)$ quadratic Casimirs $\mathcal{C}_2^L = L^{(1)}(L^{(1)} + 1) + L^{(2)}(L^{(2)} + 1)$, $\mathcal{C}_2^R = R^{(1)}(R^{(1)} + 1) + R^{(2)}(R^{(2)} + 1)$.

The real dimension of $\mathbf{Gr}_2(\mathbb{C}^4)$ is 8 and tangents along $\mathbf{Gr}_2(\mathbb{C}^4)$ may be parametrized by the 8 right invariant fields R_α ($\alpha : 7, \cdots, 14$). Consequently, the Hamiltonian on $\mathbf{Gr}_2(\mathbb{C}^4)$ may be written down as

$$\begin{aligned} H &= \frac{1}{2M\ell^2} \sum_{\alpha=7}^{14} R_\alpha^2 \\ &= \frac{1}{2M\ell^2} \left(\mathcal{C}_2(p,q,r) - \mathcal{C}_2^R - R_{15}^2 \right) \,, \end{aligned} \tag{6.27}$$

where $\mathcal{C}_2(p,q,r)$ is the quadratic Casimir of $SU(4)$ in the IRR (p,q,r) with the eigenvalue

$$\mathcal{C}_2(p,q,r) = \frac{3}{8}(r^2 + p^2) + \frac{1}{2}q^2 + \frac{1}{8}(2pr + 4pq + 4qr + 12p + 16q + 12r). \tag{6.28}$$

The dimension of the IRR (p, q, r) is

$$\dim(p, q, r) = \frac{1}{12}(p+q+2)(p+q+r+3)(q+r+2)(p+1)(q+1)(r+1). \tag{6.29}$$

The coset realization of $\mathbf{Gr}_2(\mathbb{C}^4)$ implies that, there can be both abelian and non-abelian background gauge fields corresponding to the gauging of the $U(1)$ and one or both of the $SU(2)$ subgroups. We list these as three distinct cases:

i. $U(1)$ background gauge fields only

ii. $U(1)$ background gauge field and a single $SU(2)$ background gauge field,

iii. $U(1)$ background gauge field and $SU(2) \times SU(2)$ background gauge field.

It is useful to remark that the second case may be viewed as a certain restriction of the third. We will discuss these matters in detail in what follows.

Following [33, 87], it is useful to list a few facts regarding the branching

$$SU(N_1 + N_2) \supset SU(N_1) \times SU(N_2) \times U(1)\,. \tag{6.30}$$

We can embed $SU(N_1) \times SU(N_2) \times U(1)$ into $SU(N_1 + N_2)$ as

$$\begin{pmatrix} e^{iN_2\phi}U_1 & 0 \\ 0 & e^{-iN_1\phi}U_2 \end{pmatrix}\,, \tag{6.31}$$

where $U_1 \in SU(N_1)$ and $U_2 \in SU(N_2)$. Let us denote the IRR of $SU(N_1)$ and $SU(N_2)$ with $\mathcal{J}_1$ and $\mathcal{J}_2$. We also let J_a be the total number of boxes in the Young tableaux of $SU(N_a)$ $(a : 1, 2)$. The $U(1)$ charge may thus be expressed as

$$n = \frac{1}{N_1 N_2}(N_2 J_1 - N_1 J_2)\,. \tag{6.32}$$

Clearly, the IRR of $U(1)$ is fixed by those of the $SU(N_a)$ factors and the IRR content of the subgroup $SU(N_1) \times SU(N_2) \times U(1)$ may be denoted as $(\mathcal{J}_1, \mathcal{J}_2)_n$. The decomposition of a given IRR $\mathcal{J}$ of $SU(N_1 + N_2)$ under this subgroup is expressed as

$$\mathcal{J} = \bigoplus_{\mathcal{J}_1, \mathcal{J}_2} m^{\mathcal{J}}_{\mathcal{J}_1, \mathcal{J}_2} (\mathcal{J}_1, \mathcal{J}_2)_n\,, \tag{6.33}$$

where $m^{\mathcal{J}}_{\mathcal{J}_1, \mathcal{J}_2}$ are the multiplicities of the IRR $(\mathcal{J}_1, \mathcal{J}_2)_n$ occurring in the direct sum. Further details may be found in the references [33, 87] and in the original article of Hagen and Macfarlane [88].

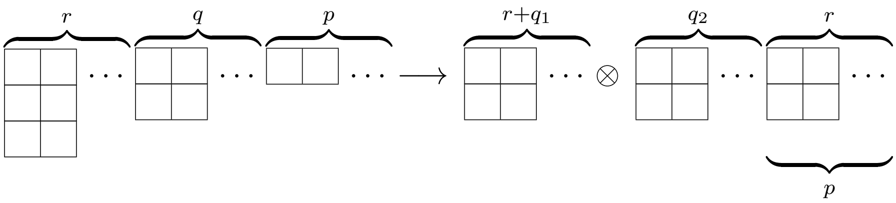
6.2.1 $U(1)$ Gauge Field Background

For the QHE problem on $\mathbf{Gr}_2(\mathbb{C}^4)$ we are concerned with the branching

$$SU(4) \supset SU(2) \times SU(2) \times U(1)\,. \tag{6.34}$$

Obtaining the wave functions with the $U(1)$ background gauge field, requires us to restrict $\mathcal{D}^{(p,q,r)}$ in such a way that they transform trivially under the right action of $SU(2) \times SU(2)$, and carry a right $U(1)$ charge, that is, they should be singlets under $SU(2) \times SU(2)$ with $\mathcal{C}_2^R = 0$ eigenvalue and a non-zero R_{15} eigenvalue.

We can utilize the Young tableaux to see the branching of the $SU(4)$ IRR fulfilling this requirement. The $SU(4)$ IRR labeled by (p, q, r) may be denoted as a Young tableau with p columns with one box on each, q columns with two boxes on each, and r columns with three boxes on each. The branching (6.34), which keeps the $SU(2) \times SU(2)$ in the singlet representation, is therefore



(6.35)

where we have introduced the splitting $q = q_1 + q_2$ in the representation in order to handle the partition of columns labeled by q in the branching. It is important to realize that in the last row of the $SU(4)$ representation there are r (fully symmetrized) boxes, which are moved as a whole under this branching to the second slot in the r.h.s. and the trivial representation of $SU(2) \times SU(2)$ is obtained if and only if p is equal to r. Otherwise, we have a nontrivial representation for the second $SU(2)$ in the branching (6.34).

Using the formula (6.32) we compute the $U(1)$ charge as

$$n = \frac{1}{2}\left((2r + 2q_1) - (p + r + 2q_2)\right) = q_1 - q_2\,, \tag{6.36}$$

where we have used $p = r$.

In order to fix the relation between the eigenvalues of R_{15} and the $U(1)$ charge n, we need to use the 6-dimensional fundamental representation $(0, 1, 0)$ (Young tableaux: $\begin{array}{c}\square\\[-4pt]\square\end{array}$) of $SU(4)$. As opposed to $\mathbb{C}P^3 \approx SU(4)/SU(3)\times U(1)$ where the branching of the 4-dimensional representations (i.e. $(1,0,0)$ and $(0,0,1)$) of $SU(4)$ contain singlets of $SU(3)$, in the present case however, the smallest $SU(4)$ IRR containing the singlet of $SU(2)\times SU(2)$ is $(0,1,0)$ and it has the branching

$$\begin{array}{c}\square\\[-4pt]\square\end{array} \longrightarrow \left(\cdot \otimes \begin{array}{c}\square\\[-4pt]\square\end{array}\right)_{-1} \oplus \left(\begin{array}{c}\square\\[-4pt]\square\end{array} \otimes \cdot\right)_{1} \oplus \left(\square \otimes \square\right)_{0}, \tag{6.37}$$

where subscripts show the charge (6.36). Taking the generators λ_a of $SU(4)$ fulfilling the normalization condition $\mathrm{Tr}(\lambda_a\lambda_b) = \frac{1}{2}\delta_{ab}$, in one of the 4-dimensional IRR ($(1,0,0)$ or $(0,0,1)$), it is possible to show that in the 6-dimensional IRR[3] $(0,1,0)$

$$R_{15} = \frac{1}{\sqrt{2}}\mathrm{diag}(0,0,0,0,-1,1), \tag{6.38}$$

and therefore we in general have

$$R_{15} = \frac{n}{\sqrt{2}} = \frac{q_1 - q_2}{\sqrt{2}}. \tag{6.39}$$

It is now easy to give the energy spectrum corresponding to the Hamiltonian (6.27), using (6.28), $p = r$, R_{15} taking the value in (6.39) and $\mathcal{C}_2^R = 0$:

$$E = \frac{1}{2M\ell^2}\left(p^2 + 3p + np + 2q_2^2 + 4q_2 + 2pq_2 + 2n(1+q_2)\right) . \tag{6.40}$$

The LLL energy at a fixed monopole background n is obtained for $q_2 = p = 0$ and it is

$$E_{LLL} = \frac{n}{M\ell^2} = \frac{2B}{M} , \tag{6.41}$$

with the degeneracy $\dim(0\,,n\,,0) = \frac{1}{12}(n+1)(n+2)^2(n+3)$. In (6.41), $B = \frac{n}{2\ell^2}$ is the field strength of the $U(1)$ magnetic monopole. The gauge field associated to B and related matters will be discussed in section 6.4.

The wave functions corresponding to this energy spectrum can be written in terms of the Wigner-$\mathcal{D}$ functions as

$$\mathcal{D}^{(p\,,q_1+q_2\,,p)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}\,;0\,,0\,,0\,,0\,,\frac{n}{\sqrt{2}}}(g) \equiv \mathcal{D}^{(p\,,[\frac{q+n}{2}]+[\frac{q-n}{2}]\,,p)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}\,;0\,,0\,,0\,,0\,,\frac{n}{\sqrt{2}}}(g)\,. \tag{6.42}$$

[3] Generalizing this result to the $\frac{N(N-1)}{2}$-dimensional representations of $SU(N)$ is used in the subsequent sections. A proof is provided in appendix D.

The degeneracy of each Landau level is given by the dimension of the IRR $(p\,,q\,,p)$ in equation (6.29). This means that the set of left quantum numbers $\{L^{(1)}\,,L_3^{(1)}\,,L^{(2)}\,,L_3^{(2)}\,,L_{15}\}$ can take on $\dim(p\,,q_1+q_2\,,p)$ different values as a set.

For the many-body fermion problem in which all the states of LLL are filled with the filling factor $\nu=1$, in the thermodynamic limit $\ell\to\infty,\mathcal{N}\to\infty$ we obtain a finite spatial density of particles

$$\rho=\frac{\mathcal{N}}{\frac{\pi^4\ell^8}{12}}\underset{\ell\to\infty\,,\mathcal{N}\to\infty}{\longrightarrow}\frac{n^4}{\pi^4\ell^8}=\left(\frac{2B}{\pi}\right)^4\,, \tag{6.43}$$

where we have used $\mathcal{N}=\dim(0\,,n\,,0)=\frac{1}{12}(n+1)(n+2)^2(n+3)$ for the number of fermions in the LLL with $\nu=1$, and[4] $\mathrm{vol}(\mathbf{Gr}_2(\mathbb{C}^4))=\frac{\pi^4\ell^8}{12}$.

We note that the case $n=0$ simply reduces the Wigner-$\mathcal{D}$ functions to the harmonics on $\mathbf{Gr}_2(\mathbb{C}^4)$ corresponding to the wave functions of a particle on $\mathbf{Gr}_2(\mathbb{C}^4)$ with vanishing monopole background.

It is possible to interchange the Young tableaux of the two $SU(2)$'s in (6.35). This flips the sign of the $U(1)$ charge, $n\to-n$; in the formulas for the energy and degeneracy, etc, this fact can be compensated by substituting $|n|$ for n.

In section 6.4 we give the single and many-particle wave functions (for the filling factor $\nu=1$ state) in terms of the Plücker coordinates for $\mathbf{Gr}_2(\mathbb{C}^4)$ and use the latter to obtain the two-point correlation function for the wave-function density signaling the incompressibility of the LLL. An account of the $U(1)$ gauge field

[4] It may be useful to state that this volume is computed with the help of the repeated iteration of (special) unitary group manifolds in terms of the odd dimensional spheres,

$$\begin{aligned} SU(N) &\approx \frac{SU(N)}{SU(N-1)}\times\frac{SU(N-1)}{SU(N-2)}\times\cdots\times\frac{SU(3)}{SU(2)}\times SU(2) \\ &\cong S^{2N-1}\times S^{2N-3}\times\cdots\times S^5\times S^3, \end{aligned} \tag{6.44}$$

(for $N\geq3$) where $\approx$ means "locally equal to" and $\cong$ indicates isomorphism. Considering this local expression we can expand all the special unitary groups in (6.25) and employ the volume formula for spheres to obtain an approximation for the volume of the Grassmannians [89], namely,

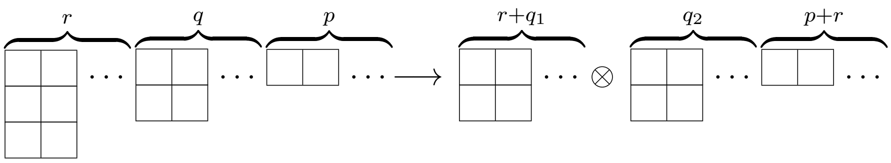
$$\mathrm{vol}(\mathbf{Gr}_k(\mathbb{C}^N))=\frac{1!2!\cdots(k-1)!}{(N-1)!(N-2)!\ldots(N-k)!}(\pi\ell^2)^{k(N-k)}, \tag{6.45}$$

which produces the factor $\frac{1}{12}$ for $k=2$ and $N=4$. This factor is in general subject to change upon using other methods. Since this is immaterial for our purposes, we will stick to the approximation (6.45) throughout this chapter.

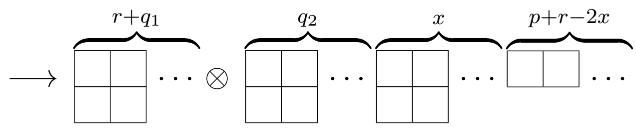
is also provided for illustrative purposes.

6.2.2 Single $SU(2)$ Gauge Field and $U(1)$ Gauge Field Background

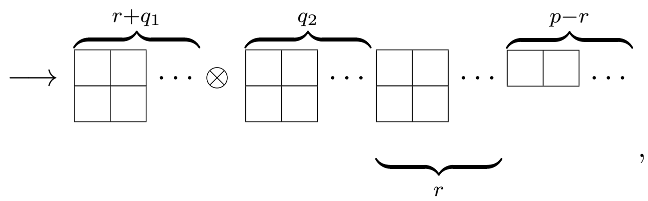
In this case, we need to restrict to $\mathcal{D}^{(p,q,r)}$, which transform as a singlet under one or the other $SU(2)$ in the right action of $SU(2) \times SU(2)$, and carry a $U(1)$ charge. Therefore, we have a range of possibilities within the branching (6.34) as given in the following Young tableaux decomposition:



(6.46)



(6.47)



(6.48)

We have assumed that $p > r$ and split $q_1 + q_2 = q$. We have introduced the integer x $(0 \leq x \leq r)$ to conveniently represent the generic case. From the tableaux, R_{15} eigenvalues may be easily obtained as:

$$n = \frac{1}{2}\left(2(q_1 - q_2) - (p - r)\right) , \tag{6.49}$$

and we observe that the first $SU(2)$ in the branching remains a singlet while the second may take on values over a range;

$$R^{(1)} = 0 \,, \quad R^{(2)} = \frac{p-r}{2}, \cdots, \frac{r+p}{2} \,. \tag{6.50}$$

Since n is an integer, we must have that $p - r$ is an even integer. This condition restricts the spin $R^{(2)}$ to integer values.

Using $\mathcal{C}_2^R = R^{(2)}(R^{(2)}+1)$, energy spectrum corresponding to the Hamiltonian (6.27) is given as

$$E = \frac{1}{2M\ell^2}\left(\mathcal{C}_2(p,q,r) - R^{(2)}(R^{(2)}+1) - \frac{n^2}{2}\right) . \tag{6.51}$$

This can be rewritten in terms of $q_2\,,n\,,p$ using (6.28), (6.49), assuming $p > r$ and introducing m via $r = p-2m$ ($m = 0\,,\cdots\,,\frac{p}{2}$ if p is even and $m = 0\,,\cdots\,,\frac{p-1}{2}$ if p is odd) as

$$E = \frac{1}{2M\ell^2}\Big(2q_2^2 + 2q_2(n+p+2) + n(p+2) + p^2 + 3p + m^2 - m(p+1) \\ - R^{(2)}(R^{(2)}+1)\Big) . \tag{6.52}$$

In order to obtain the lowest energy we have to take the maximum value of the spin $R^{(2)}_{max} = \frac{r+p}{2} = p - m$. Then, the energy spectrum becomes

$$E = \frac{1}{2M\ell^2}\Big(2q_2^2 + 2q_2(n + R^{(2)} + m + 2) + n(R^{(2)} + m + 2) \\ + (R^{(2)} + m)(2 + m)\Big) . \tag{6.53}$$

The LLL energy at fixed background fields $R^{(2)}$ and n, is obtained for $q_2 = m = 0$ as follows:

$$E_{LLL} = \frac{1}{2M\ell^2}\left(n(R^{(2)}+2) + 2R^{(2)}\right) . \tag{6.54}$$

The wave functions in the present case can be written in the form

$$\mathcal{D}^{(p\,,q_1+q_2\,,r)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}\,;\,0\,,0\,,R^{(2)}\,,R_3^{(2)}\,,\frac{n}{\sqrt{2}}}(g) \,, \tag{6.55}$$

where $R^{(2)}$ is given in (6.50).

In order to have finite energy eigenvalues in the thermodynamic limit $\ell \to \infty\,,\mathcal{N} \to \infty$, the scales of n and $R^{(2)}$ in terms of the powers of ℓ have to be determined. For a pure $SU(2)$ background ($n = 0\,, R^{(1)} = 0\,, R^{(2)} \neq 0$), $R^{(2)}$ should scale in the thermodynamic limit as $R^{(2)} \sim \ell^2$. The number of fermions in the LLL with $\nu = 1$ is

$$\mathcal{N} = \dim(R^{(2)}\,,0\,,R^{(2)}) = \frac{1}{12}(R^{(2)}+2)^2(2R^{(2)}+3)(R^{(2)}+1)^2 \underset{R^{(2)}\to\infty}{\longrightarrow} \frac{\left(R^{(2)}\right)^5}{6} \,, \tag{6.56}$$

and the corresponding spatial density is

$$\rho \sim \frac{\mathcal{N}}{\frac{\pi^4 \ell^8}{12}(2R^{(2)}+1)} \underset{\ell\to\infty\,,\mathcal{N}\to\infty}{\longrightarrow} \frac{\left(R^{(2)}\right)^4}{\pi^4 \ell^8}\,, \tag{6.57}$$

which is finite.

When both $U(1)$ and $SU(2)$ backgrounds are present (i.e. $n \neq 0, R^{(1)} = 0\,, R^{(2)} \neq 0$), just like the case of $\mathbb{C}P^2$ reviewed in the previous section, we may choose either one of n or $R^{(2)}$ to scale like ℓ^2. Taking $n \sim \ell^2$ and $R^{(2)}$ to be finite in thermodynamic limit, we again get a finite spatial density

$$\rho \sim \frac{\mathcal{N}}{\frac{\pi^4 \ell^8}{12}(2R^{(2)}+1)} \underset{\ell\to\infty\,,\mathcal{N}\to\infty}{\longrightarrow} \frac{n^4}{2\pi^4 \ell^8 R^{(2)}}\,, \tag{6.58}$$

where we have the number of fermions $\mathcal{N}$ in the LLL with $\nu = 1$ given in this case as

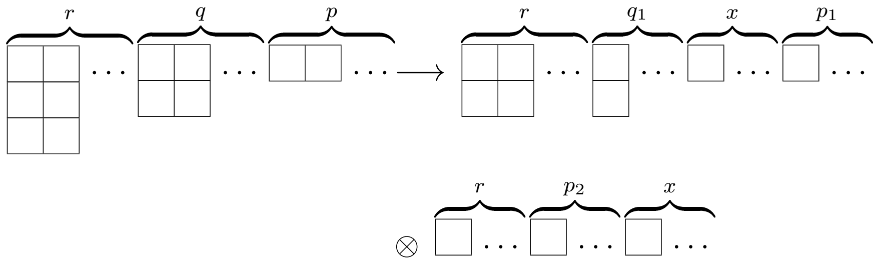
$$\begin{aligned} \dim(R^{(2)}\,, n\,, R^{(2)}) &= \frac{1}{12}(R^{(2)}+n+2)^2(2R^{(2)}+n+3)(R^{(2)}+1)^2(n+1) \\ &\underset{n\to\infty\,,R^{(2)}\to \text{finite}}{\longrightarrow} \frac{n^4}{12}\,. \end{aligned} \tag{6.59}$$

Before closing this subsection, we note that interchanging the Young tableaux of two $SU(2)$'s amounts to interchanging $R^{(1)}$ and $R^{(2)}$ in (6.50), and also a flip in the sign of the $U(1)$ charge. In the relevant formulas above, one can compensate for these changes by replacing $R^{(2)}$ with $R^{(1)}$ and substituting $|n|$ for n.

6.2.3 $SU(2) \times SU(2)$ Gauge Field Background

Now we need to restrict $\mathcal{D}^{(p,q,r)}$ to those wave functions that transform as an IRR $(R^{(1)}\,, R^{(2)})$ of $SU(2) \times SU(2)$, and carry a $U(1)$ charge. It is useful to partition IRR of $SU(4)$ as $(p_1 + p_2\,, q_1 + q_2 + x\,, r)$. There are now two classes of branchings which differ in their $U(1)$ charge as given in terms of p_1, p_2, q_1, q_2 and r below.

If $q_2 = 0$, the branching with maximal $R^{(2)}$ value is

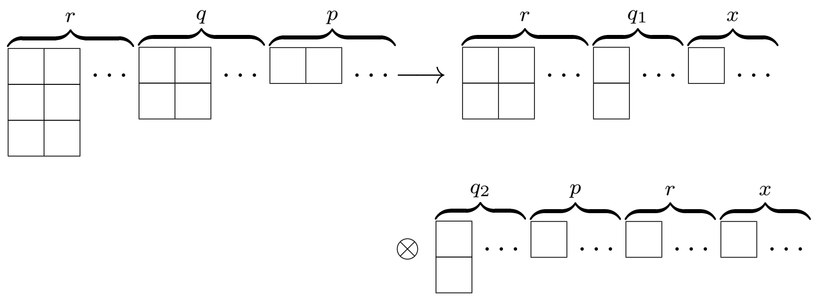


$$. \tag{6.60}$$

As $R^{(2)}$ decreases down from its maximal value $R^{(2)} = \frac{r+p_2+x}{2}$ in increments of 1, the total number of boxes in each $SU(2)$ does not vary so we have, with $q = q_1 + x$,

$$n = \frac{1}{2}(2q_1 - (p_2 - p_1 - r)) \,. \tag{6.61}$$

Suppose now that $q_2 \neq 0$. This may happen only if all p-boxes are already in the tableaux of the second $SU(2)$ in the branching; thus we must have that $p_1 = 0$. Once again, we have the branching with the maximal $R^{(2)}$ value as



$$\tag{6.62}$$

and the $U(1)$ charge is now (with $p = p_2$)

$$n = \frac{1}{2}\left(2(q_1 - q_2) - (p_2 - r)\right) \,. \tag{6.63}$$

Using both of the tableaux we observe that the first $SU(2)$ in the branching takes the value

$$R^{(1)} = \frac{p_1 + x}{2} \,, \quad 0 \leq x \leq q \,, \quad 0 \leq p_1 \leq p. \tag{6.64}$$

For this value of $R^{(1)}$ the second $SU(2)$ takes on values between $R^{(2)}_{max} = \frac{S}{2}$ and $R^{(2)}_{min} = \frac{|2\mathcal{M}-S|}{2}$

$$\frac{|2\mathcal{M} - S|}{2} \leq R^{(2)} \leq \frac{S}{2} \,, \quad S = p_2 + x + r \,, \tag{6.65}$$

where $\mathcal{M}$ is defined as the largest among the integers $p_2\,, x$ and r.

We consider the cases $q_2 = 0$ and $q_2 \neq 0$ with the $U(1)$ charges given in (6.61) and (6.63) separately to determine the energy spectrum corresponding to the Hamiltonian (6.81). We have that

$$E = \frac{1}{2M\ell^2}\left(\mathcal{C}_2(p,q,r) - R^{(1)}(R^{(1)}+1) - R^{(2)}(R^{(2)}+1) - \frac{n^2}{2}\right) . \tag{6.66}$$

For the case $q_2 = 0$, we have the condition that

$$m := \frac{p_2 - p_1 - r}{2} \tag{6.67}$$

is an integer to ensure that n is so. Let us assume that $p_2 > p_1 + r$ so that m is positive.

In order to obtain the lowest energy eigenvalues we use (6.64) together with the maximum value of $R^{(2)}$ as given in (6.65). Next, we eliminate $p_2\,, q_1\,, x$ and r in favor of $n\,, R^{(1)}\,, R^{(2)}\,, p_1$, and m (explicitly we have $p_2 = R^{(2)} - R^{(1)} + p_1 + m$, $q_1 = n + m$, $x = 2R^{(1)} - p_1$ and $r = R^{(2)} - R^{(1)} - 2m$) to get

$$\begin{aligned}
E =& \frac{1}{2M\ell^2}\Big(\mathcal{C}_2\left(R^{(2)} - R^{(1)} + 2p_1 + m\,, n + m + 2R^{(1)} - p_1\,, R^{(2)} - R^{(1)} - m\right) \\
& - R^{(1)}(R^{(1)}+1) - R^{(2)}(R^{(2)}+1) - \frac{n^2}{2}\Big) \\
=& \frac{1}{2M\ell^2}\big(p_1^2 + p_1(m + R^{(2)} - R^{(1)} + 1) + m^2 + m(R^{(1)} + R^{(2)} + n + 2) \\
& + n(R^{(1)} + R^{(2)} + 2) + 2R^{(2)}\big)\,,
\end{aligned} \tag{6.68}$$

where $R^{(2)} > R^{(1)}$ due to the assumption $p_2 > p_1 + r$. For fixed $R^{(1)}, R^{(2)}$, and n Landau levels are controlled by the two integers p_1 and m. Taking $p_1 = m = 0$ results in the LLL energy

$$E_{LLL} = \frac{1}{2M\ell^2}\left(n(R^{(1)} + R^{(2)} + 2) + 2R^{(2)}\right) . \tag{6.69}$$

We note that assuming $p_2 < p_1 + r$ flips the sign of m and in (6.68) $m \to -m$.[5]

It is also important to remark that for $R^{(1)} = R^{(2)} = R$, we have $p_1 = p_2 + r$ and thus

$$\tilde{m} := \frac{p_1 + r - p_2}{2} = r\,, \tag{6.70}$$

[5] The energy levels are still, of course, positive as can easily be checked.

and the energy levels are given by

$$\begin{aligned} E &= \frac{1}{2M\ell^2}\left(\mathcal{C}_2\left(2p_1 - r\,, n - r + 2R - p_1\,, r\right) - 2R(R+1) - \frac{n^2}{2}\right) \\ &= \frac{1}{2M\ell^2}\left(2R + p_1(1 + p_1 - \tilde{m}) + (n - \tilde{m})(2 + 2R - \tilde{m})\right) . \end{aligned} \tag{6.71}$$

The energy values here are positive since $p_1 \geq \tilde{m}$, $n \geq \tilde{m}$, and $2R - \tilde{m} \geq 0$ by construction. The LLL energy is given by $p_1 = \tilde{m} = 0$, which is indeed the same as the one obtained from (6.69) when $R := R^{(1)} = R^{(2)}$.

The case $p_1 = 0$ may be treated along similar lines. We have that

$$m := \frac{p - r}{2} \tag{6.72}$$

is an integer for the same reason that n is so. Let us assume $p > r$ so that m is positive. In this case we can write $p\,, q_1\,, x$ and r in terms of $n\,, R^{(1)}\,, R^{(2)}\,, q_2$ and m. Hence we find for the lowest energy eigenvalues

$$\begin{aligned} E = \frac{1}{2M\ell^2}\Big(&\mathcal{C}_2\left(R^{(2)} - R^{(1)} + m\,, 2q_2 + 2R^{(1)} + n + m\,, R^{(2)} - R^{(1)} - m\right) \\ &- R^{(1)}(R^{(1)} + 1) - R^{(2)}(R^{(2)} + 1) - \frac{n^2}{2}\Big) \\ = \frac{1}{2M\ell^2}\big(&2q_2^2 + 2q_2(n + R^{(1)} + R^{(2)} + m + 2) + n(R^{(1)} + R^{(2)} + 2) + m^2 \\ &+ m(R^{(1)} + R^{(2)} + n + 2) + 2R^{(2)}\big). \end{aligned} \tag{6.73}$$

We note that here we do have the condition $R^{(2)} > R^{(1)}$ as well. In this case q_2 and m specify the Landau levels. We take $q_2 = m = 0$ in (6.73) to obtain the LLL energy and this yields the same result given in (6.69) as expected.

LLL energy for $R^{(2)} < R^{(1)}$ can be found by interchanging $R^{(1)}$ and $R^{(2)}$ in (6.69) and taking n to $-n$ where now $n < 0$. This gives

$$E_{LLL} = \frac{1}{2M\ell^2}\left(-n(R^{(2)} + R^{(1)} + 2) + 2R^{(1)}\right) . \tag{6.74}$$

We do have two distinct cases to consider in the thermodynamic limit. For a pure $SU(2) \times SU(2)$ background: $n = 0, R^{(1)} \neq 0, R^{(2)} \neq 0$, both $R^{(1)}$ and $R^{(2)}$ should scale in the thermodynamic limit as ℓ^2. The number of fermions in the LLL with $\nu = 1$ is

$$\dim(R^{(2)} - R^{(1)}\,, 2R^{(1)}\,, R^{(2)} - R^{(1)}) \sim 4{R^{(1)}}^5 R^{(2)}\,, \tag{6.75}$$

and the corresponding spatial density in this limit is

$$\rho \sim \frac{4R^{(1)^5}R^{(2)}}{\pi^4\ell^8(2R^{(1)}+1)(2R^{(2)}+1)} \underset{\ell\to\infty\,,\mathcal{N}\to\infty}{\longrightarrow} \text{finite}\,. \tag{6.76}$$

For the nonzero background $n \neq 0, R^{(1)} \neq 0, R^{(2)} \neq 0$ we have three parameters n, $R^{(1)}$ and $R^{(2)}$. We can choose, say n to scale like ℓ^2 and the others to remain finite in thermodynamic limit. For $\nu = 1$ we get

$$\dim(R^{(2)} - R^{(1)}\,, 2R^{(1)} + n\,, R^{(2)} - R^{(1)}) \longrightarrow n^4 \tag{6.77}$$

and the spatial density is

$$\rho \sim \frac{n^4}{\pi^4\ell^8(2R^{(1)}+1)(2R^{(2)}+1)} \longrightarrow \text{finite}. \tag{6.78}$$

6.3 Landau Problem on $\mathbf{Gr}_2(\mathbb{C}^N)$

We are now ready to generalize the results of the previous section to all Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$. It is useful to write down the coset realization

$$\mathbf{Gr}_2(\mathbb{C}^N) = \frac{SU(N)}{S[U(N-2)\times U(2)]} \sim \frac{SU(N)}{SU(N-2)\times SU(2)\times U(1)}\,. \tag{6.79}$$

The $SU(N)$ Wigner $\mathcal{D}$-functions for $g \in SU(N)$,

$$\mathcal{D}^{(P_1,P_2,P_3,\ldots,P_{N-2},P_{N-1})}_{L^{SU(N-2)}\,,L\,,L_3\,,L_{N^2-1}\,,R^{SU(N-2)}\,,R\,,R_3\,,R_{N^2-1}}(g)\,, \tag{6.80}$$

carrying the IRR $(P_1\,, P_2\,, P_3\,, \cdots\,, P_{N-2}\,, P_{N-1})$ labeled by $N-1$ non-negative integers, may be appropriately restricted to obtain the harmonics and local sections of bundles over $\mathbf{Gr}_2(\mathbb{C}^N)$. Let us denote the left- and the right-invariant vector fields on $SU(N)$ by L_α and R_α $(\alpha : 1\,, \cdots\,, N^2-1)$; they satisfy the Lie algebra commutation relations for $SU(N)$. In (6.80), $L^{SU(N-2)}$ and $R^{SU(N-2)}$ stand for the suitable sets of left and right quantum numbers, which we will not need in what follows.

The real dimension of $\mathbf{Gr}_2(\mathbb{C}^N)$ is $4N-8$ and tangents along $\mathbf{Gr}_2(\mathbb{C}^N)$ may be parametrized by the $4N-8$ right-invariant fields, R_α, $(\alpha : N^2-4N+7, \cdots, N^2-$

2). Consequently, the Hamiltonian may be written as

$$
\begin{aligned}
H &= \frac{1}{2m\ell^2} \sum_{\alpha=N^2-4N+7}^{N^2-2} R_\alpha^2 \\
&= \frac{1}{2m\ell^2} \left(\mathcal{C}_2^{SU(N)} - \mathcal{C}_2^{SU(N-2)} - \mathcal{C}_2^{SU(2)} - R_{N^2-1}^2 \right) . \qquad (6.81)
\end{aligned}
$$

Here for future use we give the eigenvalue of $\mathcal{C}_2^{SU(N)}$ in the IRR $(P_1\,, P_2\,, 0\,, \cdots\,, 0\,, P_{N-2}\,, P_{N-1})$, which reads

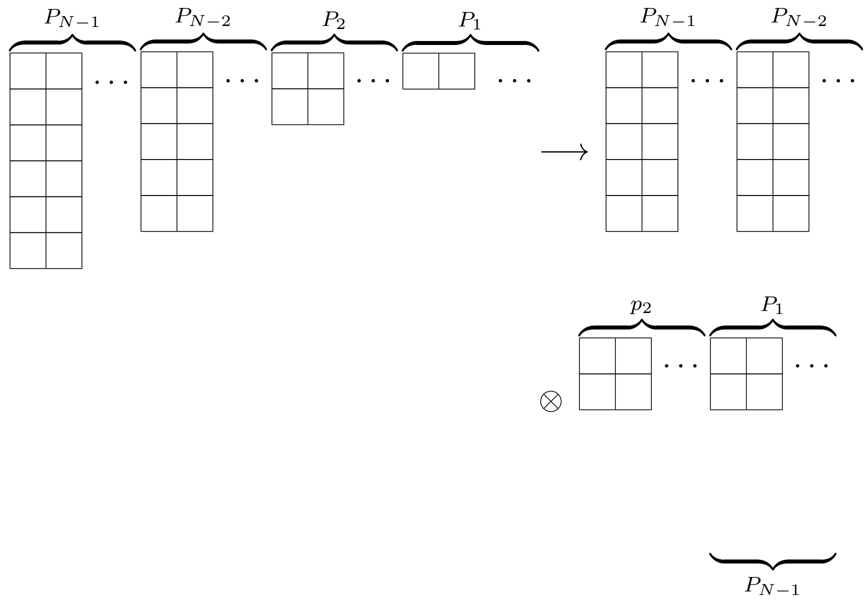
$$
\begin{aligned}
\mathcal{C}_2(P_1, P_2, 0, \ldots, 0, P_{N-2}, P_{N-1}) &= (\frac{N-1}{2N})P_1^2 + (\frac{N-2}{N})P_2^2 + (\frac{N-2}{N})P_{N-2}^2 \\
&+ (\frac{N-1}{2N})P_{N-1}^2 + (\frac{N-2}{N})P_1P_2 + \frac{2}{N}P_1P_{N-2} \\
&+ \frac{1}{N}P_1P_{N-1} + \frac{4}{N}P_2P_{N-2} + \frac{2}{N}P_2P_{N-1} \\
&+ (\frac{N-2}{N})P_{N-2}P_{N-1} + (\frac{N-1}{2})P_1 + (N-2)P_2 \\
&+ (N-2)P_{N-2} + (\frac{N-1}{2})P_{N-1}, \qquad (6.82)
\end{aligned}
$$

and the dimension of this representation is given in the appendix D.

In order to obtain the wave functions with only a $U(1)$ background gauge field, we consider those $\mathcal{D}$-functions that transform trivially under the right action of $SU(N-2)$ and $SU(2)$, and carry a right $U(1)$ charge. This means these wave functions remain singlets under $SU(N-2)$ and $SU(2)$ with non-zero $\mathcal{C}_2^{SU(N-2)}, \mathcal{C}_2^{SU(2)}$ eigenvalues and a non-zero R_{N^2-1} eigenvalue.

The branching $SU(N) \supset SU(N-2) \times SU(2) \times U(1)$ may be utilized for this purpose. In order to have both $SU(N-2)$ and $SU(2)$ as singlets in the branching we must require all P_i except $P_1\,, P_2\,, P_{N-2}\,, P_{N-1}$ to vanish, and also $P_{N-1} = P_1$.

In terms of Young tableaux, this branching can be shown by



where the tableaux on the l.h.s represent the IRR $(P_1\,, P_2\,, 0\,, \cdots\,, 0\,, P_{N-2}\,, P_1)$ of $SU(N)$. The tableaux on the r.h.s are those of $SU(N-2)$ and $SU(2)$, respectively, and both are singlets in this case.

From (6.32) we compute the $U(1)$ charge as

$$\begin{aligned} n &= \frac{1}{2(N-2)}(2J_1 - (N-2)J_2) \\ &= P_{N-2} - P_2\,. \end{aligned} \tag{6.83}$$

The relation between eigenvalues of R_{N^2-1} and n is found to be (see Appendix D)

$$R_{N^2-1} = -\sqrt{1-\frac{2}{N}}n\,. \tag{6.84}$$

The energy spectrum of the Hamiltonian is

$$\begin{aligned} E &= \frac{1}{2M\ell^2}\left(\mathcal{C}_2^{SU(N)} - (1-\frac{2}{N})n^2\right) \\ &= \frac{1}{2M\ell^2}\Bigg(P_1^2 + (2-\frac{4}{N})P_2^2 + (N-1+2n)P_1 + 2(n+N-2+\frac{2}{N})P_2 \\ &\qquad + 4P_1P_2 + n(N-2)\Bigg)\,, \end{aligned}$$

where we have used (6.82) with $P_{N-1} = P_1$ and $P_{N-2} = P_2 + n$. The integers P_1 and P_2 are in fact considered to be the Landau level indices. The LLL energy can be obtained by setting $P_1 = P_2 = 0$, which is

$$E_{LLL} = \frac{Nn - 2n}{2M\ell^2}\,. \tag{6.85}$$

The corresponding wave functions may be expressed by

$$\mathcal{D}^{(P_1\,,P_2\,,0\,,\cdots\,,0\,,P_{n-2}=P_2+n\,,P_{n-1}=P_1)}_{L^{SU(N-2)}\,,L\,,L_3\,,L_{N^2-1}\,,0\,,0\,,0\,,-\sqrt{1-\frac{2}{N}}n}(g)\,. \tag{6.86}$$

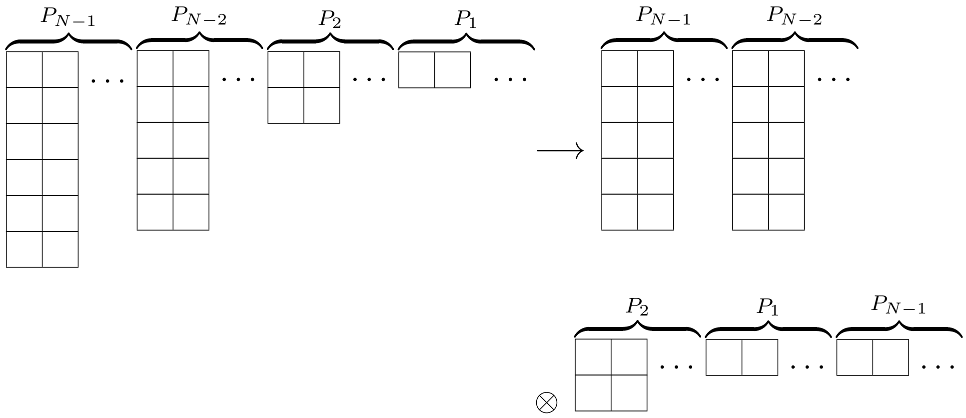
Spatial density of particles in the thermodynamic limit is computed in a manner analogous to those for the case of $\mathbf{Gr}_2(\mathbb{C}^4)$. We have $\mathrm{vol}(\mathbf{Gr}_2(\mathbb{C}^N)) = \frac{\pi^{2(N-2)}}{(N-2)!(N-1)!}\ell^{4N-8}$ from (6.45) and in the LLL, $P_{N-2} = n$, $P_1 = P_2 = P_{N-1} = 0$, with $\nu = 1$. Correspondingly, the dimension formula (D.1) for the LLL with $\nu = 1$ reduces to

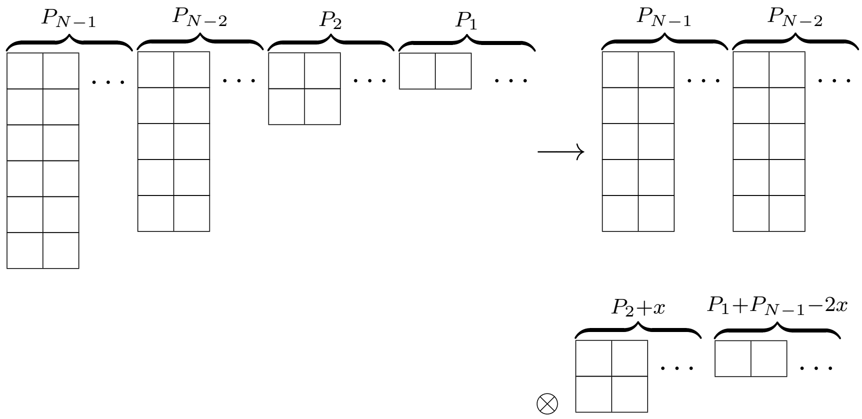
$$\begin{aligned}\mathcal{N} = \dim(0\,,0\,,\cdots\,,n\,,0) =& \frac{(n+N-3)!(n+N-4)!(n+N-2)^2}{(N+1)!(N-2)!} \\ & \times \frac{(n+N-1)(n+N-3)}{n!(n+1)!}\end{aligned} \tag{6.87}$$

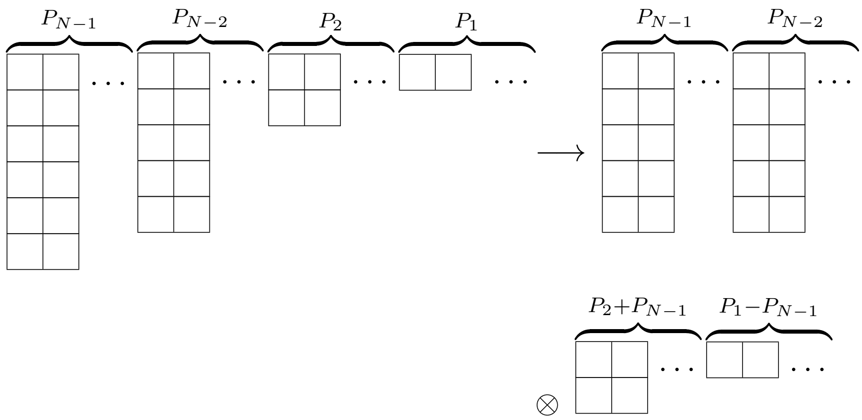
In the thermodynamic limit ($\ell \to \infty$ and $\mathcal{N} \to \infty$), the density of the states takes the form

$$\rho = \frac{\mathcal{N}}{\frac{\pi^{2(N-2)}}{(N-2)!(N-1)!}\ell^{4N-8}} \longrightarrow \frac{n^{2N-4}}{\ell^{4N-8}} = \left(\frac{B}{2\pi}\right)^{2N-4}\,. \tag{6.88}$$

For the case of both $SU(2)$ and $U(1)$ background gauge fields, the spectrum of the Hamiltonian and the wave functions are obtained in a similar manner. We still have to demand all P_i except $P_1\,, P_2\,, P_{N-2}\,, P_{N-1}$ to vanish, but no longer impose the condition $P_{N-1} = P_1$. The relevant branching of $SU(N)$ is now given by the Young tableaux below:







where the branching rule for maximum, generic and minimum $SU(2)$ spin are given, respectively and $0 \leq x \leq P_{N-1}$. We have assumed that $P_1 \geq P_{N-1}$. The $SU(2)$ spin interval is then

$$R = \frac{P_1 - P_{N-1}}{2}, \cdots, \frac{P_{N-1} + P_1}{2}, \tag{6.89}$$

and the $U(1)$ charge is given by

$$n = \frac{1}{2}\left(P_{N-1} + 2(P_{N-2} - P_2) - P_1\right). \tag{6.90}$$

By the Dirac quantization condition n should be an integer so we must have that

$$m := \frac{P_1 - P_{N-1}}{2}, \tag{6.91}$$

is an integer taking values within the interval $m = 0, \cdots, \frac{P_1}{2}$ if P_1 is even and $m = 0, \cdots, \frac{P_1 - 1}{2}$ if P_1 is odd. The energy spectrum corresponding to the Hamiltonian (6.81) reads

$$E = \frac{1}{2M\ell^2}\left(\mathcal{C}_2^{SU(N)} - R(R+1) - (1 - \frac{2}{N})n^2\right). \tag{6.92}$$

This equation can be re-written in terms of P_2, P_1, m and n by using (6.92), (6.90) and (6.91):

$$\begin{aligned} E = \frac{1}{2M\ell^2}\Bigg(& (\frac{N-1}{2N})P_1^2 + (\frac{N-2}{N})P_2^2 + (\frac{N-2}{N})(n^2 + m^2 + 2nm + P_2^2 \\ & + 2nP_2 + 2mP_2) + (\frac{N-1}{2N})(4m^2 + P_1^2 - 4mP_1) + (\frac{N-2}{N})P_1P_2 \\ & + \frac{2}{N}P_1(n + m + P_2) - \frac{1}{N}(2mP_1 - P_1^2) + \frac{4}{N}P_2(n + m + P_2) \\ & - \frac{2}{N}P_2(2m - P_1) - (\frac{N-2}{N})(2m - P_1)(n + m + P_2) + (\frac{N-1}{2})P_1 \\ & + (N-2)P_2 + (N-2)P_{N-2} + (\frac{N-1}{2})(-2m + P_1) - (\frac{N-2}{N})n^2 \\ & - R(R+1)\Bigg). \end{aligned} \tag{6.93}$$

Taking the maximum value of the spin R,

$$R = \frac{P_{N-1} + P_1}{2} = P_1 - m, \tag{6.94}$$

the lowest energy becomes

$$\begin{aligned}
E &= \frac{1}{2M\ell^2}\Bigg((\frac{N-1}{2N})(2R^2+2m^2) \\
&+ \frac{N-2}{N}(2P_2^2+mn+2nP_2+2RP_2+2mP_2+Rn+Rm) \\
&+ \frac{1}{N}(2Rn+4RP_2+2mn+R^2+m^2+2Rm+4P_2n+4P_2m+4P_2^2) \\
&+ (\frac{N-1}{2})(2R)+(N-2)(2P_2+n+m)-R(R+1)\Bigg)\,. \qquad (6.95)
\end{aligned}$$

Once again, the LLL at fixed background charges n and R are controlled by two integers, m and P_2. The LLL is found by putting $P_2 = m = 0$. This gives the energy eigenvalue

$$E_{LLL} = \frac{1}{2M\ell^2}\left(nR+(N-2)(n+R)\right)\,, \qquad (6.96)$$

which collapses to (6.54) for $N = 4$ as expected. More generally, to match the formulas of this section to those for $N = 4$, we note that the correspondence for the IRR labels is determined to be

$$(p\,,q=q_1+q_2\,,r) \longrightarrow (P_1\,,P_2=q_2\,,0\,,\cdots\,,P_{N-2}=q_1\,,P_{N-1})\,, \qquad (6.97)$$

For a pure $SU(2)$ background $n = 0, R \neq 0$, R should scale in the thermodynamic limit as $R^{(2)} \sim \ell^2$. The number of fermions in the LLL with $\nu = 1$ is $\mathcal{N} = \dim(R, 0, \cdots, 0, R)$ where

$$\begin{aligned}
\dim(R,0,\cdots,0,R) &= \frac{1}{(N-1)!(N-2)!(N-3)!(R+1)!R!}((R+N-3)! \\
&\times (N-4)!(R+N-3)!(R+N-2)(R+1)(2R+N-1) \\
&\times (N-3)(R+N-2)), \qquad (6.98)
\end{aligned}$$

and the corresponding spatial density is

$$\rho \sim \frac{\mathcal{N}}{\ell^{4N-8}(2R+1)} \longrightarrow \frac{R^{2N-3}}{k\ell^{4N-8}(2R+1)} \longrightarrow \text{finite}\,. \qquad (6.99)$$

For both $U(1)$ and $SU(2)$ backgrounds $n \neq 0, R \neq 0$, we can choose the scaling $n \sim \ell^2$ and keep R finite in thermodynamic limit. The $\mathcal{N}$ in the LLL with $\nu = 1$ is

$$\mathcal{N} = \dim(R\,,0\,,\cdots\,,n\,,R) \longrightarrow n^{2N-4}\,, \qquad (6.100)$$

and the spatial density reads

$$\rho \sim \frac{\mathcal{N}}{\ell^{4N-8}(2R+1)} \longrightarrow \frac{n^{2N-4}}{k\ell^{4N-8}(2R+1)} \longrightarrow \text{finite}\,. \qquad (6.101)$$

Before ending this section, let us briefly list a few of the results of our analysis for the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^5)$. Labeling the IRR of $SU(5)$ with $(p\,,q\,,r\,,s)$, we find that the energy spectrum due to only an abelian monopole background is

$$\begin{aligned} E &= \frac{1}{2M\ell^2}\left(\mathcal{C}_2^{SU(5)} - \frac{3}{5}n^2\right) \\ &= \frac{1}{2M\ell^2}\left(p^2 + 2q^2 + 2nq + 2qp + pn + 4p + 6q + 3n\right)\,, \end{aligned} \qquad (6.102)$$

where we have used $p = s$ and $r = n+q$ in $\mathcal{C}_2^{SU(5)}$. The numbers p and q play the role of Landau level indices. So the ground state energy is obtained by letting $p = q = 0$, which yields

$$E_{LLL} = \frac{3n}{2M\ell^2}\,, \qquad (6.103)$$

and wave functions take the form

$$\mathcal{D}^{(p\,,q\,,n+q\,,p)}_{L^{SU(3)}\,,L\,,L_3\,,L_{24}\,,0\,,0\,,0\,,-\sqrt{\frac{3}{5}}n}(g)\,. \qquad (6.104)$$

With reference to (D.1) the dimension of the $(0\,,0\,,n\,,0)$ representation gives the degeneracy of the LLL as follows:

$$\dim(0\,,0\,,n\,,0) = \frac{(n+2)!(n+1)!(n+3)^2(n+4)(n+2)}{4!3!n!(n+1)!}. \qquad (6.105)$$

Finally, the spatial density of fermions is readily computed to be

$$\rho \longrightarrow \frac{n^6}{\ell^{12}} = \left(\frac{B}{2\pi}\right)^6\,. \qquad (6.106)$$

For $SU(2)$ and $U(1)$ backgrounds together, the energy spectrum reads

$$E = \frac{1}{2M\ell^2}\left(\mathcal{C}_2^{SU(5)} - R(R+1) - \frac{3}{5}n^2\right)\,, \qquad (6.107)$$

where $SU(2)$ has the spin range

$$R = \frac{p-s}{2}, \cdots, \frac{s+p}{2}\,, \qquad (6.108)$$

assuming that $p > s$. The $U(1)$ charge now reads $n = \frac{1}{2}(s + 2(r-q) - p)$. Setting $s = p - 2m$, the maximal $SU(2)$ charge $R = p - m$ gives the energy eigenvalues

$$E = \frac{1}{2M\ell^2}(m^2 + 2q^2 + mn + 2qn + 2Rq + 2mq + Rn + Rm + 3R + 6q + 3n + 3m)\,. \tag{6.109}$$

Here applying the LLL condition gives the lowest energy as

$$E_{LLL} = \frac{1}{2M\ell^2}(n(R+3) + 3R)\,. \tag{6.110}$$

6.4 Local Form of the Wave Functions and the Gauge Fields

In this section, we first provide the local form of the wave functions for solutions of the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^4)$. For this purpose, we will utilize the well-known Plücker coordinates for $\mathbf{Gr}_2(\mathbb{C}^4)$.

The Plücker coordinates for $\mathbf{Gr}_k(\mathbb{C}^N)$ are constructed out of a projective embedding, the so-called Plücker embedding $\mathbf{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathbf{P}\left(\bigwedge^k \mathbb{C}^n\right)$, which provides a one-to-one map between the set of k-dimensional subspaces of $\mathbb{C}^n$ (i.e. the Grassmannian $\mathbf{Gr}_k(\mathbb{C}^N)$) and a subset of the projective space of the k^{th} exterior power of the vector space $\mathbb{C}^n$, where the latter is denoted as $\mathbf{P}\left(\bigwedge^k \mathbb{C}^n\right)$. This subset of $\mathbf{P}\left(\bigwedge^k \mathbb{C}^n\right)$ is a projective variety characterized by the intersection of quadrics induced by all possible relations between generalized Plücker coordinates. In what follows, we focus on the Plücker embedding of $\mathbf{Gr}_2(\mathbb{C}^4)$; more details and general discussions could be found in [85, 90].

For $\mathbf{Gr}_2(\mathbb{C}^4)$ this construction entails the projective space $\mathbf{P}\left(\mathbb{C}^4 \wedge \mathbb{C}^4\right) \equiv \mathbb{C}P^5$. Introducing two sets of complex coordinates $v_\alpha\,, w_\alpha$ $(\alpha = 1\,, \cdots, 4)$, that is one set for each $\mathbb{C}^4$, a fully antisymmetric basis for the exterior product space $\mathbb{C}^4 \wedge \mathbb{C}^4$ would be given in the form of

$$P_{\alpha\beta} = \frac{1}{\sqrt{2}}(v_\alpha w_\beta - v_\beta w_\alpha)\,. \tag{6.111}$$

$P_{\alpha\beta}$ may be contemplated as the homogenous coordinates on $\mathbb{C}P^5$ with the identification $P_{\alpha\beta} \sim \lambda P_{\alpha\beta}$ where $\lambda \in U(1)$ and $\sum_{\alpha,\beta}^4 |P_{\alpha\beta}|^2 = 1$.

The Plücker embedding of $\mathbf{Gr}_2(\mathbb{C}^4)$ in $\mathbb{C}P^5$ is given by the homogeneous condition

$$\varepsilon_{\alpha\beta\gamma\delta} P_{\alpha\beta} P_{\gamma\delta} = P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0\,, \tag{6.112}$$

defining the Klein quadric Q_4 in $\mathbb{C}P^5$, which is complex analytically equivalent to $\mathbf{Gr}_2(\mathbb{C}^4)$. The homogeneous equation $\varepsilon_{\alpha\beta\gamma\delta} P_{\alpha\beta} P_{\gamma\delta} = 0$ is nothing but the restriction to a projective hypersurface of degree two, which is the quadric Q_4.

It is possible to employ $P_{\alpha\beta}$ to parametrize the columns of $g \in SU(4)$ in the IRR $(0,1,0)$; we choose a parametrization of the form

$$g := \left(\begin{array}{c|c|c|c|cc} & & & & P^*_{34} & P_{12} \\ & & & & -P^*_{24} & P_{13} \\ \vdots & \vdots & \vdots & \vdots & P^*_{23} & P_{14} \\ & & & & P^*_{14} & P_{23} \\ & & & & -P^*_{13} & P_{24} \\ & & & & P^*_{12} & P_{34} \end{array} \right), \tag{6.113}$$

where the orthogonality of the columns follow from the Plücker relation in (6.112). For a short-hand notation, we will employ $g_{N6} = P_N := P_{\alpha\beta}$, $g_{N5} = \varepsilon_{NM} P^*_M = \varepsilon_{\alpha\beta\gamma\delta} P^*_{\gamma\delta}$ with $N \equiv [\alpha\beta]$, $N = 1\,, \cdots\,, 6$ and $\alpha\beta = (12, 13, 14, 23, 24, 34)$.

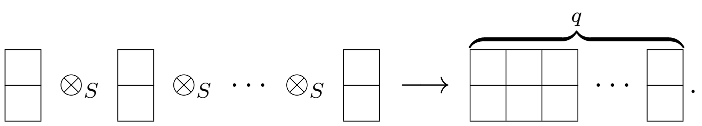
$\mathcal{D}^{(0\,,q_1+q_2\,,0)}_{L^{(1)} L^{(1)}_3 L^{(2)} L^{(2)}_3 L_{15}\,;\,0\,,0\,,0\,,0\,,\frac{n}{\sqrt{2}}}(g)$ are the wave functions in the $U(1)$ background gauge field. They are the sections of $U(1)$ bundle over $\mathbf{Gr}_2(\mathbb{C}^4)$, which fulfill the gauge transformation property

$$\mathcal{D}^{(0\,,q_1+q_2\,,0)}(gh) = \mathcal{D}^{(0\,,q_1+q_2\,,0)}(g e^{i\lambda_{15}\theta}) = \mathcal{D}^{(0\,,q_1+q_2\,,0)}(g) e^{i\frac{n}{\sqrt{2}}\theta}\,. \tag{6.114}$$

Using (6.38) for λ_{15} and (6.113), this yields immediately

$$\mathcal{D}^{(0\,,1\,,0)}(g) \sim P_{\alpha\beta}\,. \tag{6.115}$$

We point out that the $(0,q,0)$ IRR is the q-fold symmetric tensor product of the $(0,1,0)$ representation; to wit, $(0,q,0) \equiv \prod_{\otimes q}(0,1,0)$. This can be shown by the symmetric tensor product $(\otimes_S)$ of $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ tableaux as



We infer that

$$\mathcal{D}^{(0\,,q_1+q_2\,,0)}(g) \sim P_{\alpha_1\beta_1}P_{\alpha_2\beta_2}\cdots P_{\alpha_{q_1}\beta_{q_1}}P^*_{\gamma_1\delta_1}P^*_{\gamma_2\delta_2}\cdots P^*_{\gamma_{q_2}\delta_{q_2}}\,. \tag{6.116}$$

So the LLL wave functions are those with $q_2 = 0$:

$$\mathcal{D}^{(0\,,q_1\,,0)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}\,;0\,,0\,,0\,,0\,,\frac{n}{\sqrt{2}}}(g) \sim P_{\alpha_1\beta_1}P_{\alpha_2\beta_2}\cdots P_{\alpha_{q_1}\beta_{q_1}}\,, \tag{6.117}$$

which are holomorphic in the Plücker coordinates.

Another useful point to mention here is that, although the right-invariant vector fields on $SU(4)$ cannot be easily written down, the left-invariant vector fields can be easily given as [35]

$$L_k = -v_j(\lambda_k)_{ij}\frac{\partial}{\partial v_i} - w_j(\lambda_k)_{ij}\frac{\partial}{\partial w_i} + v_i^*(\lambda_k)_{ij}\frac{\partial}{\partial v_j^*} + w_i^*(\lambda_k)_{ij}\frac{\partial}{\partial w_j^*}\,, \tag{6.118}$$

where λ_k $(k = 1\,,\dots\,,15)$ are the Gell-Mann matrices for $SU(4)$. Choosing complex vectors $\mathbf{v}$ and $\mathbf{w}$ to satisfy the orthonormality conditions

$$v_i w_i^* = 0\,, \quad |\mathbf{v}|^2 = |\mathbf{w}|^2 = 1\,, \tag{6.119}$$

and using the identity

$$\sum_{k=1}^{N^2-1}\lambda^k_{ij}\lambda^k_{mn} = \frac{1}{2}\delta_{in}\delta_{jm} - \frac{1}{2N}\delta_{ij}\delta_{mn}\,, \tag{6.120}$$

for $N = 4$, the Casimir $\mathcal{C}_2^{SU(4)}$ may be realized as the differential operator:

$$\begin{aligned}
\mathcal{C}_2^{SU(4)} = {} & \frac{15}{8}\left(v_i\frac{\partial}{\partial v_i} + w_i\frac{\partial}{\partial w_i} + v_i^*\frac{\partial}{\partial v_i^*} + w_i^*\frac{\partial}{\partial w_i^*}\right) + \frac{3}{8}\bigg(v_iv_j\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j} \\
& + w_iw_j\frac{\partial}{\partial w_i}\frac{\partial}{\partial w_j} + \text{c.c.}\bigg) - \frac{2}{8}\left(v_iw_j\frac{\partial}{\partial v_i}\frac{\partial}{\partial w_j} - v_iw_j^*\frac{\partial}{\partial v_i}\frac{\partial}{\partial w_j^*} + \text{c.c.}\right) \\
& + \frac{1}{8}\left(v_iv_j^*\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j^*} + w_iw_j^*\frac{\partial}{\partial w_i}\frac{\partial}{\partial w_j^*} + \text{c.c.}\right) \\
& + v_iw_j\frac{\partial}{\partial v_j}\frac{\partial}{\partial w_i} + v_i^*w_j^*\frac{\partial}{\partial v_j^*}\frac{\partial}{\partial w_i^*} - \frac{\partial}{\partial v_j}\frac{\partial}{\partial v_j^*} - \frac{\partial}{\partial w_j}\frac{\partial}{\partial w_j^*}\,,
\end{aligned} \tag{6.121}$$

which clearly generates the eigenvalues $\frac{q^2}{2}+2q$ when applied to the wave functions (6.116).

The LLL with filling factor $\nu = 1$ has $\mathcal{N} = \dim(0,1,0) = \frac{1}{12}(n+1)(n+2)^2(n+3)$ number of particles. Its multi-particle wave-function is given in terms of the

Slater determinant as

$$\begin{aligned}
\Psi_{MP} &= \frac{1}{\sqrt{\mathcal{N}!}} \det \begin{pmatrix} \Psi_{\Lambda_1}(P^{\mathbf{1}}) & \cdots & \Psi_{\Lambda_{\mathcal{N}}}(P^{\mathbf{1}}) \\ \Psi_{\Lambda_1}(P^{\mathbf{2}}) & \cdots & \Psi_{\Lambda_{\mathcal{N}}}(P^{\mathbf{2}}) \\ \vdots & \ddots & \vdots \\ \Psi_{\Lambda_1}(P^{\mathcal{N}}) & \cdots & \Psi_{\Lambda_{\mathcal{N}}}(P^{\mathcal{N}}) \end{pmatrix} \\
&= \frac{1}{\sqrt{\mathcal{N}!}} \varepsilon^{\Lambda_1\Lambda_2\cdots\Lambda_n} \Psi_{\Lambda_1}(P^{(1)})\Psi_{\Lambda_2}(P^{(2)})\cdots\Psi_{\Lambda_N}(P^{(N)}) \,. \qquad (6.122)
\end{aligned}$$

Here $P^{\mathbf{i}}$ denotes the i^{th} position fixed in the Hall fluid and correspondingly $\Psi_{\Lambda_j}(P^{\mathbf{i}})$ refers to the wave function of the j^{th} particle located at the position $P^{\mathbf{i}}$. Now let us calculate the two-point correlation function in this fluid in the presence of only a $U(1)$ background. For a one-particle wave function in (6.115) (with $n = 1$) our notation transcribes as

$$\Psi_{\Lambda_i}(P^{\mathbf{i}}) \equiv \Psi^{\mathbf{i}}_{\alpha\beta} \sim P^{\mathbf{i}}_{\alpha\beta} \,. \qquad (6.123)$$

The LLL wave function given in (6.117) may now be denoted by

$$\Psi_{\Lambda_i}(P^{\mathbf{i}}) \equiv \Psi^{\mathbf{i}}_{\Lambda_i} \sim (P^{\mathbf{i}}_{\alpha\beta})^n \,. \qquad (6.124)$$

The general form of the correlation function between a pair of particles, say 1 and 2, on a manifold $\mathcal{M}$ is given by

$$\Omega(1,2) = \int_{\mathcal{M}} |\Psi_{MP}|^2 d\mu(3) d\mu(4) \cdots d\mu(\mathcal{N}) \,, \qquad (6.125)$$

with $d\mu(i)$ being the measure of integration on $\mathcal{M}$ in the coordinates of the i^{th} particle and Ψ_{MP} represents the multi-particle wave function of the Hall fluid on the manifold $\mathcal{M}$. Expanding the determinant formula (6.122) and using some algebra one can show that $\Omega(1,2)$ can be simplified as

$$\Omega(1,2) = \int_{\mathcal{M}} |\Psi_{MP}|^2 d\mu(3) d\mu(4) \cdots d\mu(\mathcal{N}) = |\Psi^{\mathbf{1}}|^2 |\Psi^{\mathbf{2}}|^2 - |\Psi^{*\mathbf{1}}_{\Lambda} \Psi^{\mathbf{2}}_{\Lambda}|^2 . \qquad (6.126)$$

In order to compute (6.126) for our case, we take the normalized coordinate chart $\gamma_i := \frac{P_{\alpha\beta}}{P_{12}}$ where $P_{12} \neq 0$

$$\mathcal{P} = \frac{1}{\sqrt{1+|\gamma_a|^2}} (1, \gamma_1, \ldots, \gamma_5)^T := \frac{1}{\sqrt{1+|\gamma_a|^2}} (1\,, \vec{\gamma}) \,, \qquad (6.127)$$

on the Grassmannian $\mathbf{Gr}_2(\mathbb{C}^4)$. In this coordinate patch (6.124) becomes $\Psi^{\mathbf{i}}_{\alpha} \sim (\mathcal{P}^{\mathbf{i}}_{\alpha})^n$. Inserting this into (6.126) yields

$$\begin{aligned}
\Omega(1,2) &= 1 - |\mathcal{P}^{*\mathbf{1}}_{\Lambda}\mathcal{P}^{\mathbf{2}}_{\Lambda}|^n \\
&= 1 - \left[\frac{\gamma^{*\mathbf{1}}_a\gamma^{\mathbf{2}}_a\gamma^{\mathbf{1}}_b\gamma^{*\mathbf{2}}_b}{1+|\gamma^{\mathbf{1}}_a|^2+|\gamma^{\mathbf{2}}_a|^2+|\gamma^{\mathbf{1}}_a|^2\,|\gamma^{\mathbf{2}}_a|^2}\right]^n \\
&= 1 - \left[1 - \frac{|\vec{\gamma}^{\mathbf{1}}-\vec{\gamma}^{\mathbf{2}}|^2}{1+|\gamma^{\mathbf{1}}_a|^2+|\gamma^{\mathbf{2}}_a|^2+|\gamma^{\mathbf{1}}_a|^2\,|\gamma^{\mathbf{2}}_a|^2}\right]^n .
\end{aligned} \tag{6.128}$$

Let us set $\vec{X} = \vec{\gamma}\ell$. In the thermodynamic limit $\mathcal{N} \to \infty$ and $n \to \infty$, (6.128) takes the form

$$\begin{aligned}
\Omega(1,2) &= 1 - \left[1 - \left|\vec{X}^{\mathbf{1}} - \vec{X}^{\mathbf{2}}\right|^2 \left[\ell^2 + \left|\vec{X}^{\mathbf{1}}\right|^2 + \left|\vec{X}^{\mathbf{2}}\right|^2 + \ell^{-2}\left|\vec{X}^{1}\right|^2\left|\vec{X}^{\mathbf{2}}\right|^2\right]^{-1}\right]^n \\
&\to 1 - \left[1 - \tfrac{2B}{n}\left|\vec{X}^{\mathbf{1}} - \vec{X}^{\mathbf{2}}\right|^2\right]^n \\
&\to 1 - e^{-2B\left|\vec{X}^{\mathbf{1}}-\vec{X}^{\mathbf{2}}\right|^2} \\
&= 1 - e^{-2B\left(\vec{x}^{\mathbf{1}}-\vec{x}^{\mathbf{2}}\right)^2} e^{-2B\ell^2\left(\det\Gamma^{\mathbf{1}}-\det\Gamma^{\mathbf{2}}\right)^2},
\end{aligned} \tag{6.129}$$

where we have used $n = 2B\ell^2$ and that $\Gamma^{\mathbf{i}} := \begin{pmatrix} \gamma^{\mathbf{i}}_2 & \gamma^{\mathbf{i}}_1 \\ \gamma^{\mathbf{i}}_4 & \gamma^{\mathbf{i}}_3 \end{pmatrix}$. Note that the last equality shows the two-point function of the particles located at the positions $\vec{x}^{\mathbf{1}}, \vec{x}^{\mathbf{2}}$ on $\mathbf{Gr}_2(\mathbb{C}^4)$, is extracted from that of the particles on $\mathbb{C}P^5$ at the positions $\vec{X}^{\mathbf{1}}, \vec{X}^{\mathbf{2}}$ by a restriction of these particles to the algebraic variety determined by $X^{\mathbf{i}}_5 \equiv \ell\det\Gamma^{\mathbf{i}}$, as expected. It is apparent from this function that the probability of finding two particles at the same point goes to zero. This result indicates the incompressibility of the Hall fluid.

Turning our attention to the $U(1)$ gauge field we may write

$$A = -\frac{in}{\sqrt{2}}\mathrm{Tr}\left(\lambda^{15}_{(6)}g^{-1}dg\right) . \tag{6.130}$$

With the help of (6.113) and (6.39), one can express A in terms of the Plücker coordinates as

$$
\begin{aligned}
A &= -\frac{in}{\sqrt{2}}\left(\lambda^{15}_{(6)}\right)_{LM}\left(g^{-1}\right)_{MN}(dg)_{NL} \\
&= -\frac{in}{2}\left(-\left(g^{-1}\right)_{5N}(dg)_{N5} + \left(g^{-1}\right)_{6N}(dg)_{N6}\right) \\
&= -\frac{in}{2}\left(-g^{*}_{N5}(dg)_{N5} + g^{*}_{N6}(dg)_{N6}\right) \\
&= -\frac{in}{2}\left(-P_N dP^{*}_N + P^{*}_N dP_N\right) \\
&= -inP^{*}_N dP_N\,, \qquad (6.131)
\end{aligned}
$$

where use has been made of the notational conventions stated below equation (6.113), and the fact that $d(P^{*}_N P_N) = 0$ due to (6.112). Under $U(1)$ gauge transformations A transforms to $A + d\left(\frac{n\theta}{\sqrt{2}}\right)$, which is consistent with the transformation of the wave functions given in (6.114).

Let us introduce the notation $\tilde{\mathcal{P}} \equiv (P_1, \ldots P_6)^T$ where T stands for transpose and define a non-homogeneous coordinate chart $\mathcal{Q} \equiv \frac{\tilde{\mathcal{P}}}{P_1}$ with $P_1 \neq 0$ on $\mathbf{Gr}_2(\mathbb{C}^4)$ as

$$
\mathcal{Q} \equiv (1\,, \gamma_1\,, \cdots\,, \gamma_5)^T\,, \qquad (6.132)
$$

subject to the Plücker relation (6.112) which in terms of the (affine coordinates) γ_i takes the form

$$
\gamma_5 = \gamma_2\gamma_3 - \gamma_1\gamma_4\,. \qquad (6.133)
$$

Without (6.133), $\mathcal{Q}$ is a non-homogenous coordinate chart in $\mathbb{C}P^5$. We can express our gauge potential as

$$
\begin{aligned}
A &= -in\mathcal{P}^{\dagger}d\mathcal{P} \\
&= -in|P_1|^2\mathcal{Q}^{\dagger}d\mathcal{Q} - inP^{*}_1|\mathcal{Q}|^2 dP_1 \\
&= -in|\mathcal{Q}|^{-2}\mathcal{Q}^{\dagger}d\mathcal{Q} - inP^{*}_1|P_1|^{-2}dP_1 \\
&= -in|\mathcal{Q}|^{-2}\mathcal{Q}^{\dagger}d\mathcal{Q} - inP^{-1}_1 dP_1 \\
&= -in\partial\ln(|\mathcal{Q}|^2) - ind\ln(P_1) \\
&= -in\partial K - ind\ln(P_1). \qquad (6.134)
\end{aligned}
$$

where K is the $\mathbb{C}P^5$ Kähler potential given by

$$
K = \ln|\mathcal{Q}|^2 \equiv \ln(1 + |\gamma_i|^2)\,, \qquad (6.135)
$$

and subject to the condition (6.133).

The field strength is calculated via

$$
\begin{aligned}
F = dA &= -\frac{in}{\sqrt{2}} \mathrm{Tr}\left(\lambda^{15}_{(6)} g^{-1} dg \wedge g^{-1} dg\right) \\
&= -ind P_N^* \wedge dP_N \,.
\end{aligned}
\tag{6.136}
$$

We note that F is an antisymmetic, gauge invariant, and closed two-form on $\mathbf{Gr}_2(\mathbb{C}^4)$ and as such it is proportional to the Kähler two-form Ω over $\mathbf{Gr}_2(\mathbb{C}^4)$. This fact can be readily verified using (6.134) and writing

$$
F = dA = in\partial\partial^* K = n\Omega \,, \tag{6.137}
$$

where ∂, ∂^* are the Dolbeault operators in the coordinates γ_i and γ_i^*, respectively, and $d = \partial + \partial^*$. The relation (6.137) with (6.135) leads to the following form of the field strength [90]:

$$
F = -in\left(\frac{d\gamma_i^* \wedge d\gamma_i}{1+|\gamma|^2} - \frac{\gamma_i d\gamma_i^* \wedge \gamma_j^* d\gamma_j}{(1+|\gamma|^2)^2}\right) \,, \tag{6.138}
$$

being subject to the Plücker relation (6.133). Let us associate with each index i a dual index $\hat{\imath}$ in the sense that i is dual to $\hat{\imath}$ if $\gamma_i\gamma_{\hat{\imath}}$ appears in the Plücker relation. Hence $1, 4$ and $2, 3$ are dual to one another. Expanding γ_5 in (6.133) results in the Hermitian components for the Kähler form Ω as

$$
\begin{aligned}
\Omega_{ii^*} &= iN_\gamma\left(1 + \prod_{\alpha=1,\alpha\neq i,\hat{\imath}}^{4} |\gamma_\alpha|^2 + (1+|\gamma_{\hat{\imath}}|^2)\sum_{\alpha=1,\alpha\neq i}^{4}|\gamma_\alpha|^2\right) \,, \\
\Omega_{ij^*} &= -iN_\gamma\left(1+|\gamma_{\hat{\imath}}|^2+|\gamma_{\hat{\jmath}}|^2\right)\left(\gamma_i^*\gamma_j + \gamma_{\hat{\imath}}\gamma_{\hat{\jmath}}^*\right) \,, \quad i<j\,, \quad j\neq\hat{\imath}\,, \\
\Omega_{i\hat{\imath}^*} &= -iN_\gamma\left(\gamma_i^*\gamma_{\hat{\imath}}\left(\sum_{\alpha=1}^{4}|\gamma_\alpha|^2 - |\gamma_i|^2 - |\gamma_{\hat{\imath}}|^2\right) - \frac{1}{2}(\gamma_i^*)^2\prod_{j\neq i,\hat{\imath}}\gamma_j\gamma_{\hat{\jmath}}\right. \\
&\quad \left. - \frac{1}{2}(\gamma_{\hat{\imath}})^2\prod_{j\neq i,\hat{\imath}}\gamma_j^*\gamma_{\hat{\jmath}}^*\right) \,, \quad i<\hat{\imath}\,,
\end{aligned}
\tag{6.139}
$$

where $N_\gamma = \left(1+\sum_{a=1}^5 |\gamma_a|^2\right)^{-2}$. In these formulas Einstein summation convention is not in use.

It is known from very general considerations [91] that the integral of F over a non-contractable two surface Σ in $\mathbf{Gr}_2(\mathbb{C}^4)$ is an integral multiple of 2π:

$$
\frac{1}{2\pi}\int_\Sigma F = n \,. \tag{6.140}
$$

In the present context, this result signals an analogue of the Dirac quantization condition with $\frac{n}{2}$ identified as the magnetic monopole charge. Therefore, we do have that the magnetic field is $B = \frac{n}{2\ell^2}$.

A number of remarks are in order. The generalization of our results to all higher dimensional Grassmannians is fairly straightforward. Taking $\mathbf{Gr}_2(\mathbb{C}^N)$, the only difference is that now both the vector potential A and field strength F are subject to the Plücker relations

$$\gamma_{ik}\gamma_{jl} = \gamma_{ij}\gamma_{kl} - \gamma_{il}\gamma_{kj}, \quad 1 \leq i < k < j < l \leq 2(N-2)\,, \tag{6.141}$$

in terms of the non-homogeneous coordinates $\gamma_{ij} := P_{ij}/P_{12}$ in the patch where $P_{12} \neq 0$. The parametrization in (6.113) can be generalized to $N(N-1)/2$-dimensional fundamental representations of the $SU(N)$ group by means of these Plücker relations. Let us also note that the Grassmannians have a non-trivial algebraic topological structure that, for the best of our purposes here, is reflected in their second cohomology group which is non-zero, or more precisely $H^2(\mathbf{Gr}_k(\mathbb{C}^N)) = \mathbb{Z}$ [92]. This is the reason why the integral of the first Chern character in (6.140) is an integer. Similarly, one may consider the integral of the d^{th} ($d = 2(N-2)$) order Chern character for the Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$ [93]:

$$\frac{1}{d!(2\pi)^d \mathrm{vol}(\mathbf{Gr}_2(\mathbb{C}^N))} \int_{\mathbf{Gr}_2(\mathbb{C}^N)} F \wedge \Omega \cdots \wedge \Omega = n\,, \tag{6.142}$$

for $F = n\Omega$.

CHAPTER 7

CONCLUSIONS

In this thesis, we have focused on the investigation of various aspects of fuzzy vacuum configurations arising in the context of $SU(\mathcal{N})$ gauge theories coupled to a multiplet of adjoint scalar fields. In chapter 3, we discussed the results of [9, 21, 22, 24] which demonstrated how $SU(\mathcal{N})$ gauge theories coupled to a suitable number of scalar fields develop fuzzy vacuum configurations in the form of fuzzy sphere, S^2_F, or the product of two fuzzy sphere, $S^2_F \times S^2_F$, after the spontaneously breaking of the $SU(\mathcal{N})$ symmetry. We showed that the fluctuations around these fuzzy vacua have the gauge field structure on fuzzy sphere(s) which made it possible to conjecture the emergent theories as effective gauge theories with fuzzy extra dimensions. KK-type mode expansion of the gauge fields and their equivariant parametrization provided two complementary approaches in understanding and interpreting these models and allowed us to compute their low energy limits. As concrete examples, we have examined the low energy limit of effective $U(n)$ gauge theory $\mathcal{M} \times S^2_F$ by Kaluza-Klein mode expansion of gauge fields and the low energy effective actions of $U(2)$ gauge theory on $\mathcal{M} \times S^2_F$ and that of $U(4)$ gauge theory on $\mathcal{M} \times S^2_F \times S^2_F$ were constructed by using the equivariant parametrization of gauge fields. These two review chapters are followed by three chapters covering the original results of research conducted for this thesis.

In chapter 4, we considered a model [21,24] in which an $SU(\mathcal{N})$ gauge theory coupled to six adjoint scalar fields which is in fact a particular deformation of $N = 4$ supersymmetric Yang-Mills theory with cubic soft supersymmetry breaking and mass deformation terms in scalar fields. We found new spontaneously generated fuzzy extra dimensions emerging from this model which are expressed in terms of a direct sum of product of two fuzzy spheres, $S_F^{2\,Int} \times S_F^{2\,Int}$. The direct sum structure of the vacuum was clearly revealed by a suitable splitting of the scalar fields in the model in a manner that generalizes the approach of [25]. Fluctuations around this vacuum have the structure of gauge fields over $S_F^{2\,Int} \times S_F^{2\,Int}$, and this enabled us to conjecture the spontaneous broken model as an effective $U(n)$ gauge theory on the product manifold $M^4 \times S_F^{2\,Int} \times S_F^{2\,Int}$. We supported this interpretation by the equivariant parametrization technique. We examined the $U(4)$ theory and determined all of the $SU(2) \times SU(2)$ equivariant fields in the model, characterizing its low energy degrees of freedom. Monopole sectors with winding numbers $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 1, \pm 1)$ were accessed from $S_F^{2\,Int} \times S_F^{2\,Int}$ after suitable projections and subsequently equivariant fields in these sectors were obtained. We indicated how Abelian Higgs type models with vortex solutions emerge after dimensionally reducing over the fuzzy monopole sectors as well. A family of fuzzy vacua was determined by giving a systematic treatment for the splitting of the scalar fields and it was made manifest that suitable projections of these vacuum solutions yield all higher winding number fuzzy monopole sectors. We observed that the vacuum configuration $S_F^{2\,Int} \times S_F^{2\,Int}$ identifies with the bosonic part of the product of two fuzzy superspheres with $OSP(2,2) \times OSP(2,2)$ supersymmetry and elaborate on this unexpected and intriguing feature. Finally, stability of our vacuum solutions was addressed by showing that they may be interpreted as mixed state with non-zero von Neumann entropy.

As mentioned before, the low energy limit of effective $U(2)$ gauge theory on $\mathcal{M} \times S_F^2$ was studied in [22] and reviewed in chapter 3 of this thesis. In chapter 5, we took a step forward and investigated the low energy structure of models with larger gauge group and obtained a new family of fuzzy vacua $S_F^{2\,Int}$ by suitable splitting of scalar fields as well. We analyzed the low energy structure of the $U(3)$ gauge theory on $\mathcal{M} \times S_F^2$ by using equivariant parametrization technique and subsequently determined the equivariant fields transforming invariantly and as vectors under the combined adjoint action of $SU(2)$ rotations over the fuzzy sphere and those $U(3)$ gauge transformations generated by $SU(2) \subset U(3)$ carrying the spin 1 IRR of $SU(2)$, when the $SU(2)$ subgroup is maximally embedded in $SU(3)$. Our results revealed that the dipole and quadrupole terms, which appear in the branching of the adjoint representation of $SU(3)$ as $\underline{8} \rightarrow \underline{5} \oplus \underline{3}$ under $SU(2)$ are the useful objects in constructing the equivariant scalars and we have shown how this generalizes to $U(n)$ theories over $\mathcal{M} \times S_F^2$ via employing the $n-1$ multipole terms which appear in the branching of the adjoint representation of $SU(n)$ under $SU(2)$. The equivariance conditions that we have imposed on the fields broke the $U(3)$ gauge symmetry down $U(1) \times U(1) \times U(1)$. Subsequently, we determined the LEA emanating from the equivariant parametrization of the fields and found that it consists of two complex scalars, each coupling to one of the gauge fields a_μ^i, $(i = 1, 2)$ only, and three real scalars coupling to the complex fields and to each other through a quartic potential. We have seen that in the $\ell \rightarrow \infty$ limit gauge field b_μ either decouples completely from the rest of the LEA or it is eliminated by solving its equation of motion in powers of $\frac{1}{\ell}$. Determining the vacuum structure of the effective potential for the scalars, we were able to give two different vortex solutions for the LEA on $\mathbb{R}^2$, both of which are characterized by two winding numbers in each case. We also made clear, how the commutative limit of our results relate to the instanton solutions in self-dual $SU(3)$ Yang-Mills theory for cylindrically symmetric gauge fields of Bais and Weldon [72] and indicated the apparent connection between the BPS vortices that we obtained in a certain commutative limit in section 5.3 and the instanton solution in [72]. Adapting the ideas in section 4.1 to this model, we have provided a complete analysis of the $U(3)$-equivariant fields over $\mathcal{M} \times S_F^{2\,Int}$ and determined the equivariant field modes characterizing the low energy behaviour of the effective $U(3)$ theory on $\mathcal{M} \times S_F^{2\,Int}$ in terms of suitable "idempotents" and projection operators. We noted that $S_F^{2\,Int}$ may be seen as stacks of concentric fuzzy D-branes carrying magnetic monopole fluxes from a stringy viewpoint, and consequently equivariant gauge field modes found in section 5.5 may be interpreted as those living on the world-volume of these D-branes, and may potentially be useful in bridging the effective gauge theory and the string theoretic perspectives.

In chapter 6, we gave a formulation of the quantum hall effects on the complex Grassmann manifolds $\mathbf{Gr}_2(\mathbb{C}^N)$. We first focused on the simplest case $\mathbf{Gr}_2(\mathbb{C}^4)$ in order to solve the Landau problem. We constructed the solution for the most general case of non-zero $U(1)$ and $SU(2) \times SU(2)$ backgrounds and showed that at the LLL with $\nu = 1$, finite spatial densities are obtained at finite $SU(2) \times SU(2)$ internal degrees of freedom in agreement with the results of [29]. Subsequently, we generalized these results to all $\mathbf{Gr}_2(\mathbb{C}^N)$. Moreover, the local structure of the solutions on $\mathbf{Gr}_2(\mathbb{C}^4)$ in the presence of $U(1)$ background gauge field was presented in this chapter, where we have computed the single and multi-particle wave functions in terms of the Plücker coordinates and showed that the LLL at filling factor $\nu = 1$ forms an incompressible fluid by calculating the two-point correlation function. The $U(1)$ gauge field, its associated field strength and their properties were illustrated using the differential geometry on $\mathbf{Gr}_2(\mathbb{C}^4)$. We have also briefly commented on the generalization of this local formulation to all $\mathbf{Gr}_2(\mathbb{C}^N)$.

APPENDIX A

SOME DETAILS FOR CHAPTER 3

$$\mathcal{W}_1 = \frac{(\ell^2+\ell-1/4)}{(\ell+\frac{1}{2})^2}\varphi_3 + \frac{1}{\ell+\frac{1}{2}}\varphi_4\,, \tag{A.1}$$

$$\mathcal{W}_2 = (1-\varphi_3)\left(1+\frac{1}{\ell+\frac{1}{2}}\varphi_4 - \frac{1}{2(\ell+\frac{1}{2})^2}\varphi_3\right)\,, \tag{A.2}$$

$$\mathcal{W}_3 = \frac{\ell(\ell+1)}{(\ell+\frac{1}{2})^2}\varphi_3(\varphi_3-2) + 2\frac{(\ell^2+\ell-1/4)}{(\ell+\frac{1}{2})}\varphi_4 + \varphi_4^2\,. \tag{A.3}$$

$$\begin{aligned}
\mathcal{X}_1 =& \frac{\ell(\ell+1)(\ell^2+\ell-1/4)}{2(\ell+\frac{1}{2})^4}\,,\\
\mathcal{X}_2 =& \frac{\ell(\ell+1)}{(\ell+\frac{1}{2})^2}\left(\mathcal{W}_1^2 - \frac{(\ell^2+\ell-\frac{1}{4})}{(\ell+\frac{1}{2})^2}\mathcal{W}_2 + \frac{\mathcal{W}_3}{2(\ell+\frac{1}{2})^2}\right)\,,\\
\mathcal{X}_3 =& \frac{\ell(\ell+1)}{2(\ell+\frac{1}{2})^4}\big((\ell^2+\ell-1/4)\mathcal{W}_2^2 - \mathcal{W}_2\mathcal{W}_3\big) + \frac{\ell^2+\ell+\frac{3}{4}}{8(\ell+\frac{1}{2})^4}\mathcal{W}_3^2\,.
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\mathcal{Y}_1 =& \frac{1}{2}(1-|\varphi|^2) - \frac{\varphi_3}{4(\ell+\frac{1}{2})^2} - \frac{(\ell^2+\ell-\frac{1}{4})}{(\ell+\frac{1}{2})}\varphi_4 - \frac{(\ell^2+\ell-\frac{1}{2})}{4(\ell+\frac{1}{2})^2}\varphi_3^2 - \frac{1}{4}\varphi_4^2 \qquad \text{(A.5)}\\
&- \frac{\varphi_3\varphi_4}{4(\ell+\frac{1}{2})}\,,\\
\mathcal{Y}_2 =& -\frac{1}{4(\ell+\frac{1}{2})}(1-|\varphi|^2) - \frac{(\ell^2+\ell-\frac{1}{2})}{(\ell+\frac{1}{2})}\varphi_3 - \frac{1}{2}\varphi_4 - \frac{\varphi_3^2}{16(\ell+\frac{1}{2})^3}\\
&- \frac{(\ell^2+\ell-\frac{1}{4})}{2(\ell+\frac{1}{2})^2}\varphi_3\varphi_4 - \frac{\varphi_4^2}{4(\ell+\frac{1}{2})}
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
\mathcal{L}_F =& \frac{1}{4g^2}\mathrm{Tr}(F_{\mu\nu}^\dagger F^{\mu\nu}) = \frac{1}{16g^2}\big(f_{\mu\nu}^L f^{L\mu\nu} + f_{\mu\nu}^R f^{R\mu\nu} + f_{\mu\nu}^{L,R} f^{L,R\mu\nu} + h_{\mu\nu}h^{\mu\nu}\\
&+ \frac{\big(f_{\mu\nu}^L f^{R\mu\nu} - h_{\mu\nu}f^{L,R\mu\nu}\big)}{2(\ell_L+\frac{1}{2})(\ell_R+\frac{1}{2})} + f_{\mu\nu}^R\bigg(\frac{f^{L,R\mu\nu}}{(\ell_L+\frac{1}{2})} - \frac{h^{\mu\nu}}{(\ell_R+\frac{1}{2})}\bigg)\\
&+ f_{\mu\nu}^L\bigg(\frac{f^{L,R\mu\nu}}{(\ell_R+\frac{1}{2})} - \frac{h^{\mu\nu}}{(\ell_L+\frac{1}{2})}\bigg)
\end{aligned} \tag{A.7}$$

$$\mathcal{L}_G^L = \mathrm{Tr}(D_\mu \Phi_a^L)^\dagger (D_\mu \Phi_a^L) = \frac{\ell_L(\ell_L+1)}{2(\ell_R+1)(\ell_L+\frac{1}{2})^2}\Big((2\ell_R+3)|D_\mu\varphi|^2$$
$$+(2\ell_R-1)|D_\mu\tilde{\varphi}|^2\Big) + \frac{1}{4}\frac{\ell_L(\ell_L+1)(\ell_L^2+\ell_L-\frac{1}{4})}{(\ell_L+\frac{1}{2})^4}\left((\partial_\mu\varphi_3)^2+(\partial_\mu\tilde{\varphi}_3)^2\right)$$
$$+\frac{1}{4}\frac{\ell_L^2+\ell_L+\frac{3}{4}}{(\ell_L+\frac{1}{2})^2}\left((\partial_\mu\varphi_4)^2+(\partial_\mu\tilde{\varphi}_4)^2\right)$$
$$+\frac{\ell_L(\ell_L+1)}{4(\ell_R+\frac{1}{2})(\ell_L+\frac{1}{2})^3}\left(\frac{(\ell_L^2+\ell_L-\frac{1}{4})}{(\ell_L+\frac{1}{2})}\partial_\mu\varphi_3\partial_\mu\tilde{\varphi}_3+\partial_\mu\varphi_3\partial_\mu\tilde{\varphi}_4+\partial_\mu\tilde{\varphi}_3\partial_\mu\varphi_4\right)$$
$$+\frac{1}{4}\frac{\ell_L^2+\ell_L+\frac{3}{4}}{(\ell_R+\frac{1}{2})(\ell_L+\frac{1}{2})^2}\left(\partial_\mu\varphi_4\partial_\mu\tilde{\varphi}_4\right)+\frac{1}{2}\frac{\ell_L(\ell_L+1)}{(\ell_L+\frac{1}{2})^3}\left(\partial_\mu\varphi_3\partial_\mu\varphi_4+\partial_\mu\tilde{\varphi}_3\partial_\mu\tilde{\varphi}_4\right),\tag{A.8}$$

where we have defined the complex scalar as $\varphi=\varphi_1+i\varphi_2$ and $\tilde{\varphi}=\tilde{\varphi}_1+i\tilde{\varphi}_2$. For $\mathcal{L}_G^R=\mathrm{Tr}(D_\mu\Phi_a^R)^\dagger(D_\mu\Phi_a^R)$, it is enough to replace $\ell_L\to\ell_R$ and $\varphi_i\to\chi_i\,,\tilde{\varphi}\to\tilde{\chi}_i\,,i=1,\cdots,4$ where $\chi=\chi_1+i\chi_2\,,\tilde{\chi}=\tilde{\chi}_1+i\tilde{\chi}_2$ in (A.8). For the potential terms $V_1^L\,,V_1^R$, we have

$$V_1^L=\frac{1}{g_L^2}\mathrm{Tr}F_{ab}^{L\dagger}F_{ab}^L=\mathcal{X}_1^L(|\varphi|^4+|\tilde{\varphi}|^4)+\mathcal{X}_{2+}^L|\varphi|^2+\mathcal{X}_{2-}^L|\tilde{\varphi}|^2+\mathcal{X}_3^L\,,$$
$$V_1^R=\frac{1}{g_R^2}\mathrm{Tr}F_{ab}^{R\dagger}F_{ab}^R=\mathcal{X}_1^R(|\chi|^4+|\chi'|^4)+\mathcal{X}_{2,+}^R|\chi|^2+\mathcal{X}_{2-}^R|\chi'|^2+\mathcal{X}_3^R\,,$$

where

$$\mathcal{X}_1^L=4\frac{\ell_L(\ell_L+1)(\ell_L^2+\ell_L-\frac{1}{4})}{(\ell_L+\frac{1}{2})^4}\,,$$
$$\mathcal{X}_{2\pm}^L=2\frac{\ell_L(\ell_L+1)}{(\ell_L+\frac{1}{2})^2}\Bigg((\mathcal{W}_{1,\pm}^L)^2-\frac{\ell_L^2+\ell_L-\frac{1}{4}}{(\ell_L+\frac{1}{2})^2}(\mathcal{W}_2^L\pm\tilde{\mathcal{W}}_2^L)$$
$$+\frac{1}{2(\ell_L+\frac{1}{2})^2}(\mathcal{W}_3^L\pm\tilde{\mathcal{W}}_3^L)\Bigg)\pm\frac{1}{(\ell_R+\frac{1}{2})}\frac{1}{(\ell_L+\frac{1}{2})^2}\Bigg(\ell_L(\ell_L+1)(\mathcal{W}_{1,\pm}^L)^2$$
$$+\frac{2\ell_L(\ell_L+1)(\ell_L^2+\ell_L-\frac{1}{4})}{(\ell_L+\frac{1}{2})^2}\left(1\mp\frac{1}{2}(\mathcal{W}_2^L\pm\tilde{\mathcal{W}}_2^L)\right)+\frac{1}{2(\ell_L+\frac{1}{2})}(\mathcal{W}_3^L\pm\tilde{\mathcal{W}}_3^L)\Bigg),$$
$$\mathcal{X}_3^L=\frac{1}{2(\ell_L+\frac{1}{2})^4}\Bigg(\ell_L(\ell_L+1)(\ell_L^2+\ell_L-\frac{1}{4})\left((\mathcal{W}_2^L)^2+(\tilde{\mathcal{W}}_2^L)^2\right)$$
$$+\frac{1}{4}(\ell_L^2+\ell_L+\frac{3}{4})\left((\mathcal{W}_3^L)^2+(\tilde{\mathcal{W}}_3^L)^2\right)-\ell_L(\ell_L+1)(\mathcal{W}_2^L\mathcal{W}_3^L+\tilde{\mathcal{W}}_2^L\tilde{\mathcal{W}}_3^L)\Bigg)$$
$$+\frac{1}{2}\frac{1}{(\ell_R+\frac{1}{2})}\frac{1}{(\ell_L+\frac{1}{2})^3}\Bigg(\frac{\ell_L(\ell_L+1)(\ell_L^2+\ell_L-\frac{1}{4})}{(\ell_L+\frac{1}{2})}\mathcal{W}_2^L\tilde{\mathcal{W}}_2^L$$
$$+\frac{1}{4}\frac{(\ell_L^2+\ell_L+\frac{3}{4})}{(\ell_L+\frac{1}{2})}\mathcal{W}_3^L\tilde{\mathcal{W}}_3^L-\frac{1}{2}(\mathcal{W}_2^L\tilde{\mathcal{W}}_3^L+\tilde{\mathcal{W}}_2^L\mathcal{W}_3^L)\Bigg),\tag{A.9}$$

where $\mathcal{X}_1^R\,,\mathcal{X}_{2\pm}^R\,,\mathcal{X}_3^R$ can be obtained by replacing $L\to R$ and $\ell_L\to\ell_R$. Here, we have introduced the variable $\mathcal{W}$ for simplicity as

$$
\begin{aligned}
\mathcal{W}^L_{1,\pm} &= \frac{\ell_L^2+\ell_L-1/4}{(\ell_L+1/2)^2}(\varphi_3\pm\tilde\varphi_3)+\frac{1}{\ell_L+1/2}(\varphi_4\pm\tilde\varphi_4)\,,\\
\mathcal{W}^L_2 &= (1-\varphi_3)\left(1+\frac{\varphi_4}{\ell_L+1/2}-\frac{\varphi_3}{2(\ell_L+1/2)^2}\right)\\
&\quad-\tilde\varphi_3\left(\frac{\tilde\varphi_4}{\ell_L+1/2}-\frac{\tilde\varphi_3}{2(\ell_L+1/2)^2}\right)\,,\\
\tilde{\mathcal{W}}^L_2 &= -\left(1+\frac{1}{2(\ell_L+1/2)^2}\right)\tilde\varphi_3+\frac{\tilde\varphi_4}{(\ell_L+1/2)}+\frac{\varphi_3\tilde\varphi_3}{(\ell_L+1/2)^2}\\
&\quad-\frac{1}{(\ell_L+1/2)}\left(\varphi_3\tilde\varphi_4+\tilde\varphi_3\varphi_4\right)\,,\\
\mathcal{W}^L_3 &= \frac{\ell_L(\ell_L+1)}{(\ell_L+1/2)^2}\left(\varphi_3^2-2\varphi_3\right)+\varphi_4^2+2\frac{\ell_L^2+\ell_L-1/4}{\ell_L+1/2}\varphi_4\\
&\quad+\frac{\ell_L(\ell_L+1)}{(\ell_L+1/2)^2}\tilde\varphi_3^2+\tilde\varphi_4^2\,,\\
\tilde{\mathcal{W}}^L_3 &= \frac{2(\ell_L^2+\ell_L-\frac{1}{4})}{(\ell_L+1/2)}\tilde\varphi_4+\frac{2\ell_L(\ell_L+1)}{(\ell_L+1/2)^2}(\varphi_3-1)\tilde\varphi_3+2\varphi_4\tilde\varphi_4\,,
\end{aligned}
\tag{A.10}
$$

and for $\mathcal{W}^R_{1,\pm}\,,\mathcal{W}^R_2\,,\tilde{\mathcal{W}}^R_2\,,\mathcal{W}^R_3\,,\tilde{\mathcal{W}}^R_3$ replace $\ell_L\to\ell_R$ and the scalars $(\tilde\varphi_3\,,\tilde\varphi_4)$ by the set $(\tilde\chi_3\,,\tilde\chi_4)$. The term $V_1^{L,R}$ are constructed as

$$
\begin{aligned}
V_1^{L,R} &= \frac{1}{g^2_{L,R}}\Bigg(2\alpha^1\left(|\varphi\tilde\chi-\tilde\varphi\chi|^2+|\bar\chi\varphi-\tilde\varphi\bar{\tilde\chi}|^2\right)+(|\varphi|^2+|\tilde\varphi|^2)\Big(\alpha_2^L\tilde\chi_3^2+\alpha_3^L\tilde\chi_4^2\\
&\quad+\alpha_4^L\tilde\chi_3\tilde\chi_4\Big)+(|\chi|^2+|\tilde\chi|^2)\left(\alpha_2^R\tilde\varphi_3^2+\alpha_3^R\tilde\varphi_4^2+\alpha_4^R\tilde\varphi_3\tilde\varphi_4\right)\Bigg)\,,
\end{aligned}
\tag{A.11}
$$

where

$$
\begin{aligned}
\alpha_1 &= \frac{\ell_L(\ell_L+1)\ell_R(\ell_R+1)}{(\ell_L+\frac{1}{2})^2(\ell_R+\frac{1}{2})^2}\,,\quad \alpha_2^L=\frac{1}{2}\frac{\ell_L(\ell_L+1)\ell_R(\ell_R+1)(\ell_R^2+\ell_R-\frac{1}{4})}{(\ell_L+\frac{1}{2})^2(\ell_R+\frac{1}{2})^4}\,,\\
\alpha_3^L &= \frac{1}{2}\frac{\ell_L(\ell_L+1)(\ell_R^2+\ell_R+\frac{3}{4})}{(\ell_L+\frac{1}{2})^2(\ell_R+\frac{1}{2})^2}\,,\quad \alpha_4^L=\frac{1}{2}\frac{\ell_L(\ell_L+1)\ell_R(\ell_R+1)}{(\ell_L+\frac{1}{2})^2(\ell_R+\frac{1}{2})^3}\,,
\end{aligned}
\tag{A.12}
$$

and $\alpha_2^R\,,\alpha_3^R\,,\alpha_4^R$ can be obtained by exchanging $\ell_R\leftrightarrow\ell_R$ in (A.12). For the potential term V_2^L and V_2^R, we find that

$$
\begin{aligned}
\Phi_a^L\Phi_a^L+\ell_L(\ell_L+1) &= \mathcal{Y}_1^L+iQ^L\mathcal{Y}_2^L+iQ^R(\tilde{\mathcal{Y}}_1^L+iQ^L\tilde{\mathcal{Y}}_2^L)\,,\\
\Phi_a^R\Phi_a^R+\ell_R(\ell_R+1) &= \mathcal{Y}_1^R+iQ^R\mathcal{Y}_2^R+iQ^L(\tilde{\mathcal{Y}}_1^R+iQ^L\tilde{\mathcal{Y}}_2^R)
\end{aligned}
\tag{A.13}
$$

where

$$\mathcal{Y}_1^L = -\frac{1}{2}(2|\varphi|^2 + 2|\tilde{\varphi}|^2 - 1) - \frac{\varphi_3}{4(\ell_L + \frac{1}{2})^2} - \frac{(\ell_L^2 + \ell_L - \frac{1}{4})}{\ell_L + \frac{1}{2}}\varphi_4$$
$$- \frac{(\ell_L^2 + \ell_L - \frac{1}{2})}{4(\ell_L + \frac{1}{2})^2}(\varphi_3^2 + \tilde{\varphi}_3^2) - \frac{1}{4}(\varphi_4^2 + \tilde{\varphi}_4^2) - \frac{1}{4(\ell_L + \frac{1}{2})}(\varphi_3\varphi_4 + \tilde{\varphi}_3\tilde{\varphi}_4)\,, \tag{A.14}$$

$$\mathcal{Y}_2^L = \frac{1}{4(\ell_L + \frac{1}{2})}(2|\varphi|^2 + 2|\tilde{\varphi}|^2 - 1) - \frac{(\ell_L^2 + \ell_L - \frac{1}{2})}{(\ell_L + \frac{1}{2})}\varphi_3 - \frac{1}{2}\varphi_4 - \frac{(\varphi_3^2 + \tilde{\varphi}_3^2)}{16(\ell_L + \frac{1}{2})^3}$$
$$- \frac{(\ell_L^2 + \ell_L - \frac{1}{4})}{2(\ell_L + \frac{1}{2})^2}(\varphi_3\varphi_4 + \tilde{\varphi}_3\tilde{\varphi}_4) - \frac{(\varphi_4^2 + \tilde{\varphi}_4^2)}{4(\ell_L + \frac{1}{2})}\,, \tag{A.15}$$

$$\tilde{\mathcal{Y}}_1^L = -\frac{1}{2}(2|\varphi|^2 + 2|\tilde{\varphi}|^2) - \frac{\tilde{\varphi}_3}{4(\ell_L + \frac{1}{2})^2} - \frac{(\ell_L^2 + \ell_L - \frac{1}{4})}{\ell_L + \frac{1}{2}}\tilde{\varphi}_4 - \frac{(\ell_L^2 + \ell_L - \frac{1}{2})}{2(\ell_L + \frac{1}{2})^2}\varphi_3\tilde{\varphi}_3$$
$$- \frac{1}{2}\varphi_4\tilde{\varphi}_4 - \frac{1}{4(\ell_L + \frac{1}{2})}(\varphi_3\tilde{\varphi}_4 + \tilde{\varphi}_3\varphi_4)\,, \tag{A.16}$$

$$\tilde{\mathcal{Y}}_2^L = \frac{1}{4(\ell_L + \frac{1}{2})}(2|\varphi|^2 + 2|\tilde{\varphi}|^2) - \frac{(\ell_L^2 + \ell_L - \frac{1}{2})}{(\ell_L + \frac{1}{2})}\tilde{\varphi}_3 - \frac{1}{2}\tilde{\varphi}_4 - \frac{\varphi_3\tilde{\varphi}_3}{8(\ell_L + \frac{1}{2})^3}$$
$$- \frac{(\ell_L^2 + \ell_L - \frac{1}{4})}{2(\ell_L + \frac{1}{2})^2}(\varphi_3\tilde{\varphi}_4 + \tilde{\varphi}_3\varphi_4) - \frac{\varphi_4\tilde{\varphi}_4}{2(\ell_L + \frac{1}{2})}\,, \tag{A.17}$$

and for $\mathcal{Y}_i^R$ and $\tilde{\mathcal{Y}}_i^R\,, (i = 1, 2)$, it is enough to change $\ell_L \to \ell_R$ and $\varphi \to \chi$ in the equations above. Then, we obtain

$$V_2^L = a_L^2\Bigg((\mathcal{Y}_1^L)^2 + (\mathcal{Y}_2^L)^2 + (\tilde{\mathcal{Y}}_1^L)^2 + (\tilde{\mathcal{Y}}_2^L)^2 + \frac{1}{(\ell_L + \frac{1}{2})}\left(\mathcal{Y}_1^L\mathcal{Y}_2^L + \tilde{\mathcal{Y}}_1^L\tilde{\mathcal{Y}}_2^L\right)$$
$$+ \frac{1}{(\ell_R + \frac{1}{2})}\left(\mathcal{Y}_1^L\tilde{\mathcal{Y}}_1^L + \mathcal{Y}_2^L\tilde{\mathcal{Y}}_2^L\right) + \frac{1}{2(\ell_L + \frac{1}{2})(\ell_R + \frac{1}{2})}\left(\mathcal{Y}_1^L\tilde{\mathcal{Y}}_2^L + \tilde{\mathcal{Y}}_1^L\mathcal{Y}_2^L\right)\Bigg) \tag{A.18}$$

and

$$V_2^R = a_R^2\Bigg((\mathcal{Y}_1^R)^2 + (\mathcal{Y}_2^R)^2 + (\tilde{\mathcal{Y}}_1^R)^2 + (\tilde{\mathcal{Y}}_2^R)^2 + \frac{1}{(\ell_R + \frac{1}{2})}\left(\mathcal{Y}_1^R\mathcal{Y}_2^R + \tilde{\mathcal{Y}}_1^R\tilde{\mathcal{Y}}_2^R\right)$$
$$+ \frac{1}{(\ell_L + \frac{1}{2})}\left(\mathcal{Y}_1^R\tilde{\mathcal{Y}}_1^R + \mathcal{Y}_2^R\tilde{\mathcal{Y}}_2^R\right) + \frac{1}{2(\ell_R + \frac{1}{2})(\ell_L + \frac{1}{2})}\left(\mathcal{Y}_1^R\tilde{\mathcal{Y}}_2^R + \tilde{\mathcal{Y}}_1^R\mathcal{Y}_2^R\right)\Bigg) \tag{A.19}$$

Euler-Lagrange equations

$$
\begin{aligned}
& a_\theta^{L\prime\prime} - \frac{1}{r} a_\theta^{L\prime} - 8g^2 \Big((a_\theta^L + a_\theta^{L,R} + n_1)\varphi^2 + (a_\theta^L - a_\theta^{L,R} + n_2)\tilde{\varphi}^2 \Big) = 0\,, \\
& a_\theta^{R\prime\prime} - \frac{1}{r} a_\theta^{R\prime} - 8g^2 \Bigg((a_\theta^R + a_\theta^{L,R} + n_3)\chi^2 + (a_\theta^R - a_\theta^{L,R} + n_3 \\
& \qquad - (n_1 - n_2))\tilde{\chi}^2 \Bigg) = 0\,, \\
& a_\theta^{L,R\prime\prime} - \frac{1}{r} a_\theta^{L,R\prime} - 8g^2 \Bigg((a_\theta^L + a_\theta^{L,R} + n_1)\varphi^2 - (a_\theta^L - a_\theta^{L,R} + n_2)\tilde{\varphi}^2 \\
& \qquad + (a_\theta^R + a_\theta^{L,R} + n_3)\chi^2 - (a_\theta^R - a_\theta^{L,R} + n_3 - (n_1 - n_2))\tilde{\chi}^2 \Bigg) = 0\,, \\
& \varphi'' + \frac{1}{r}\varphi' - \left(\frac{1}{r^2}(a_\theta^L + a_\theta^{L,R} + n_1)^2 + \frac{1}{g_L^2}\big(8(\varphi^2 + \frac{1}{4}(\varphi_3 + \tilde{\varphi}_3 - 1)) + 2(\varphi_3 + \tilde{\varphi}_3)^2\big) \right)\varphi \\
& \qquad - \frac{1}{g_{L,R}^2}\left(2\varphi(\tilde{\chi}^2 + \chi^2) - 4\chi\tilde{\chi}\tilde{\varphi} - \frac{1}{2}(\tilde{\chi}_3^2 + \tilde{\chi}_4^2)\varphi \right) = 0\,, \\
& \chi'' + \frac{1}{r}\chi' - \left(\frac{1}{r^2}(a_\theta^R + a_\theta^{L,R} + n_3)^2 + \frac{1}{g_R^2}\big(8(\chi^2 + \frac{1}{4}(\chi_3 + \tilde{\chi}_3 - 1)) + 2(\chi_3 + \tilde{\chi}_3)^2\big) \right)\chi \\
& \qquad - \frac{1}{g_{L,R}^2}\left(2\chi(\varphi^2 + \tilde{\varphi}^2) - 4\varphi\tilde{\varphi}\tilde{\chi} - \frac{1}{2}(\tilde{\varphi}_3^2 + \tilde{\varphi}_4^2)\chi \right) = 0\,, \\
& \tilde{\varphi}'' + \frac{1}{r}\tilde{\varphi}' - \left(\frac{1}{r^2}(a_\theta^L - a_\theta^{L,R} + n_2)^2 + \frac{1}{g_L^2}\big(8(\tilde{\varphi}^2 + \frac{1}{4}(\varphi_3 - \tilde{\varphi}_3 - 1)) + 2(\varphi_3 - \tilde{\varphi}_3)^2\big) \right)\tilde{\varphi} \\
& \qquad - \frac{1}{g_{L,R}^2}\left(2\tilde{\varphi}(\chi^2 + \tilde{\chi}^2) - 4\varphi\chi\tilde{\chi} - \frac{1}{2}(\tilde{\chi}_3^2 + \tilde{\chi}_4^2)\tilde{\varphi} \right) = 0\,, \\
& \tilde{\chi}'' + \frac{1}{r}\tilde{\chi}' - \Bigg(\frac{1}{r^2}(a_\theta^R - a_\theta^{L,R} + n_3 - (n_1 - n_2))^2 + \frac{1}{g_R^2}\big(8(\tilde{\chi}^2 + \frac{1}{4}(\chi_3 - \tilde{\chi}_3 - 1)) \\
& \qquad + 2(\chi_3 - \tilde{\chi}_3)^2\big) \Bigg)\tilde{\chi} - \frac{1}{g_{L,R}^2}\left(2\tilde{\chi}(\varphi^2 + \tilde{\varphi}^2) - 4\chi\varphi\tilde{\varphi} - \frac{1}{2}(\tilde{\varphi}_3^2 + \tilde{\varphi}_4^2)\tilde{\chi} \right) = 0\,, \\
& \varphi_3'' + \frac{1}{r}\varphi_3' - \left(\frac{4}{g_L^2}\big(\varphi^2 + \tilde{\varphi}^2 + \frac{1}{2}(\varphi_3 - 1)\big) + \frac{8}{g_L^2}\left((\varphi_3 + \tilde{\varphi}_3)\varphi^2 + (\varphi_3 - \tilde{\varphi}_3)\tilde{\varphi}^2 \right) \right) = 0\,, \\
& \chi_3'' + \frac{1}{r}\chi_3' - \left(\frac{4}{g_R^2}\big(\chi^2 + \tilde{\chi}^2 + \frac{1}{2}(\chi_3 - 1)\big) + \frac{8}{g_R^2}\left((\chi_3 + \tilde{\chi}_3)\chi^2 + (\chi_3 - \tilde{\chi}_3)\tilde{\chi}^2 \right) \right) = 0\,, \\
& \tilde{\varphi}_3'' + \frac{1}{r}\tilde{\varphi}_3' - \left(\frac{4}{g_L^2}\big(\varphi^2 - \tilde{\varphi}^2 + \frac{1}{2}\tilde{\varphi}_3\big) + \frac{8}{g_L^2}\left((\varphi_3 + \tilde{\varphi}_3)\varphi^2 - (\varphi_3 - \tilde{\varphi}_3)\tilde{\varphi}^2 \right) \right) \\
& \qquad - \frac{2}{g_{L,R}^2}(\chi^2 + \tilde{\chi}^2)\tilde{\varphi}_3 = 0\,, \\
& \tilde{\chi}_3'' + \frac{1}{r}\tilde{\chi}_3' - \left(\frac{4}{g_R^2}\big(\chi^2 - \tilde{\chi}^2 + \frac{1}{2}\tilde{\chi}_3\big) + \frac{8}{g_R^2}\left((\chi_3 + \tilde{\chi}_3)\chi^2 - (\chi_3 - \tilde{\chi}_3)\tilde{\chi}^2 \right) \right) \\
& \qquad - \frac{2}{g_{L,R}^2}(\varphi^2 + \tilde{\varphi}^2)\tilde{\chi}_3 = 0\,, \\
& \tilde{\varphi}_4'' + \frac{1}{r}\tilde{\varphi}_4' - \left(\frac{2}{g_L^2}\tilde{\varphi}_4 - \frac{2}{g_{L,R}^2}(\chi^2 + \tilde{\chi}^2)\tilde{\varphi}_4 \right) = 0\,, \\
& \tilde{\chi}_4'' + \frac{1}{r}\tilde{\chi}_4' - \left(\frac{2}{g_R^2}\tilde{\chi}_4 - \frac{2}{g_{L,R}^2}(\varphi^2 + \tilde{\varphi}^2)\tilde{\chi}_4 \right) = 0\,,
\end{aligned}
\tag{A.20}
$$

APPENDIX B

SOME DETAILS FOR CHAPTER 4

Variation of the action (3.75) with respect to $\Phi_a^{i\,L}$ gives

$$D_\mu D^\mu \Phi_a^{i\,L} + \frac{1}{g_L^2}(2 f_{ijk}\Phi_b^{j\,L} F_{ab}^{k\,L} - \varepsilon_{abc} F_{bc}^{i\,L}) = 0\,, \tag{B.1}$$

while the variation with respect to $\Psi_\alpha^{l\,L\,\dagger}$ yields

$$\left(D_\mu D^\mu \Phi_a^{i\,L} + \frac{1}{g_L^2}(2 f_{ijk}\Phi_b^{j\,L} F_{ab}^{k\,L} - \varepsilon_{abc} F_{bc}^{i\,L}) \right) \gamma_{lmi} (\tilde{\tau}_a \Psi^{m\,L})_\alpha = 0\,, \tag{B.2}$$

where $\Phi_a^L = \Phi_a^{i\,L}\lambda_i$, $\Psi_\alpha^L = \Psi_\alpha^{i\,L}\lambda_i$ with the anti-hermitian $SU(\mathcal{N})$ generators λ_i $(i = 1 \cdots, \mathcal{N}^2 - 1)$ fulfilling $\lambda_i\lambda_j = -\frac{2}{\mathcal{N}}\delta_{ij} + (d_{ijk} + f_{ijk})\lambda_k$ and $\gamma_{ijk} := d_{ijk} + f_{ijk}$ for short. Clearly, these equations imply each other. Variation with respect to $\Phi_a^{i\,R}$ and $\Psi_\alpha^{l\,R\,\dagger}$ yield analogous expressions with $L \to R$.

The block diagonal form $(\mathcal{D}_a^L, \mathcal{D}_a^R)$ indicated in the subsection 4.2.1 is given as

$$\begin{aligned}
\mathcal{D}_a^L\mathcal{D}_a^L + \mathcal{D}_a^R\mathcal{D}_a^R = \Bigg(& -\big(\ell_L(\ell_L+1) + \ell_R(\ell_R+1)\big)\mathbf{1}_{(2\ell_L+1)(2\ell_R+1)4n}\,, \\
& -\big((\ell_L - \tfrac{1}{2})(\ell_L + \tfrac{1}{2}) + \ell_R(\ell_R+1)\big)\mathbf{1}_{(2\ell_L)(2\ell_R+1)2n}\,, \\
& -\big((\ell_L + \tfrac{1}{2})(\ell_L + \tfrac{3}{2}) + \ell_R(\ell_R+1)\big)\mathbf{1}_{(2\ell_L+2)(2\ell_R+1)2n}\,, \\
& -\big(\ell_L(\ell_L+1) + (\ell_R - \tfrac{1}{2})(\ell_R + \tfrac{1}{2})\big)\mathbf{1}_{(2\ell_L+1)(2\ell_R)2n}\,, \\
& -\big(\ell_L(\ell_L+1) + (\ell_R + \tfrac{1}{2})(\ell_R + \tfrac{3}{2})\big)\mathbf{1}_{(2\ell_L+1)(2\ell_R+2)2n}\,, \\
& -\big((\ell_L - \tfrac{1}{2})(\ell_L + \tfrac{1}{2}) + (\ell_R - \tfrac{1}{2})(\ell_R + \tfrac{1}{2})\big)\mathbf{1}_{(2\ell_L)(2\ell_R)n}\,, \\
& -\big((\ell_L + \tfrac{1}{2})(\ell_L + \tfrac{3}{2}) + (\ell_R - \tfrac{1}{2})(\ell_R + \tfrac{1}{2})\big)\mathbf{1}_{(2\ell_L+2)(2\ell_R)n}\,, \\
& -\big((\ell_L - \tfrac{1}{2})(\ell_L + \tfrac{1}{2}) + (\ell_R + \tfrac{1}{2})(\ell_R + \tfrac{3}{2})\big)\mathbf{1}_{(2\ell_L)(2\ell_R+2)n}\,, \\
& -\big((\ell_L + \tfrac{1}{2})(\ell_L + \tfrac{3}{2}) + (\ell_R + \tfrac{1}{2})(\ell_R + \tfrac{3}{2})\big)\mathbf{1}_{(2\ell_L+2)(2\ell_R+2)n} \Bigg)\,.
\end{aligned} \tag{B.3}$$

The matrices in (4.58) and (4.59) square as

$$
\begin{aligned}
&(Q_B^L)^2 = -\mathbf{1}_{(2\ell_L+1)(2\ell_R+1)64}\,, \quad (Q_B^R)^2 = -\mathbf{1}_{(2\ell_L+1)(2\ell_R+1)64}\,, \quad (Q_\pm^L)^2 = -\Pi_\pm^L\,, \\
&(Q_\pm^R)^2 = -\Pi_\pm^R\,, \quad \left(Q_{0_\mathbf{0}}^L\right)^2 = -\Pi_{0_\mathbf{0}}^L\,, \quad \left(Q_{0_\mathbf{0}}^R\right)^2 = -\Pi_{0_\mathbf{0}}^R\,, \quad \left(Q_{0_\mathbf{2}}^L\right)^2 = -\Pi_{0_\mathbf{2}}^L\,, \,, \\
&\left(Q_{0_\mathbf{2}}^R\right)^2 = -\Pi_{0_\mathbf{2}}^R\,, \quad \left(iS_i^L\right)^2 = -\Pi_0^L\,, \quad \left(iS_i^R\right)^2 = -\Pi_0^R\,, \quad (Q_{S_i}^L)^2 = -\Pi_0^L\,, \\
&(Q_{S_i}^R)^2 = -\Pi_0^R\,, \quad (Q_F^L)^2 = -\Pi_{\frac{1}{2}}^L \quad (Q_F^R)^2 = -\Pi_{\frac{1}{2}}^R\,, \quad (Q_H^L)^2 = -\Pi_{\frac{1}{2}}^L\,, \\
&(Q_H^R)^2 = -\Pi_{\frac{1}{2}}^R\,, \quad (Q_{BI}^L)^2 = -\Pi_{\frac{1}{2}}^L\,, \quad (Q_{BI}^R)^2 = -\Pi_{\frac{1}{2}}^R\,, \quad (Q_I^L)^2 = -\Pi_{\frac{1}{2}}^L \\
&(Q_I^R)^2 = -\Pi_{\frac{1}{2}}^R\,,
\end{aligned}
\tag{B.4}
$$

justifying that they are "idempotent"s in the subspace they belong to.

Using the equivariant invariants in (4.60), vectors in the $(1,0)$ IRR may be listed as

$$
\begin{aligned}
&\Pi_i^R[D_a^L, Q_j^L], \quad \Pi_i^R Q_j^L[D_a^L, Q_j^L], \quad \Pi_i^R\{D_a^L, Q_j^L\}, \quad S_k^R[D_a^L, Q_j^L], \quad S_k^R Q_j^L[D_a^L, Q_j^L], \\
&S_k^R\{D_a^L, Q_j^L\} \quad Q_j^R[D_a^L, Q_j^L], \quad Q_j^R Q_j^L[D_a^L, Q_j^L], \quad Q_j^R\{D_a^L, Q_j^L\} \quad Q_{S_k}^R[D_a^L, Q_j^L], \\
&Q_{S_k}^R Q_j^L[D_a^L, Q_j^L], \; Q_{S_k}^R\{D_a^L, Q_j^L\}, \; \Pi_i^R[D_a^L, Q_{S_k}^L], \; \Pi_i^R Q_0^L[D_a^L, Q_{S_k}^L], \; \Pi_i^R\{D_a^L, Q_{S_k}^L\}, \\
&S_k^R[D_a^L, Q_{S_k}^L], \; S_k^R Q_0^L[D_a^L, Q_{S_k}^L], \; S_k^R\{D_a^L, Q_{S_k}^L\}, \; Q_j^R[D_a^L, Q_{S_k}^L], \; Q_j^R Q_0^L[D_a^L, Q_{S_k}^L], \\
&Q_j^R\{D_a^L, Q_{S_k}^L\}, \quad Q_{S_k}^R[D_a^L, Q_{S_k}^L], \quad Q_{S_k}^R Q_0^L[D_a^L, Q_{S_k}^L], \quad Q_{S_k}^R\{D_a^L, Q_{S_k}^L\}, \quad \Pi_i^R\Pi_i^L\omega_a^L, \\
&S_k^R\Pi_i^L\omega_a^L \quad Q_j^R\Pi_i^L\omega_a^L \quad Q_{S_k}^R\Pi_i^L\omega_a^L, \quad \Pi_i^R S_k^L\omega_a^L, \quad S_k^R S_k^L\omega_a^L \quad Q_j^R S_k^L\omega_a^L \quad Q_{S_k}^R S_k^L\omega_a^L,
\end{aligned}
\tag{B.5}
$$

where $Q_0^L = Q_{0_\mathbf{0}}^L + Q_{0_\mathbf{2}}^L$. Equivariant vectors in the $(0,1)$ IRR is obtained from (B.5) simply by the exchange $L \leftrightarrow R$.

336 equivariant spinors in the IRR $(\frac{1}{2}, 0)$ parametrized as

$$
\begin{aligned}
&\Pi_i^R\Pi_\mu^L\beta_\alpha^L Q_\nu^L, \; \Pi_i^R\Pi_\nu^L\beta_\alpha^L Q_\mu^L, \; \Pi_i^R Q_\mu^L\beta_\alpha^L\Pi_\nu^L, \; \Pi_i^R Q_\nu^L\beta_\alpha^L\Pi_\mu^L, \; \Pi_i^R Q_\mu^L\beta_\alpha^L Q_\nu^L, \; \Pi_i^R Q_\nu^L\beta_\alpha^L Q_\mu^L \\
&\Pi_i^R S_\rho^L\beta_\alpha^L\Pi_\nu^L, \; \Pi_i^R\Pi_\nu^L\beta_\alpha^L S_\rho^L, \; \Pi_i^R Q_{S_\rho}^L\beta_\alpha^L\Pi_\nu^L, \; \Pi_i^R\Pi_\nu^L\beta_\alpha^L Q_{S_\rho}^L, \; \Pi_i^R Q_{S_\rho}^L\beta_\alpha^L Q_\nu^L, \; \Pi_i^R Q_\nu^L\beta_\alpha^L Q_{S_\rho}^L \\
&S_k^R\Pi_\mu^L\beta_\alpha^L Q_\nu^L, \; S_k^R\Pi_\nu^L\beta_\alpha^L Q_\mu^L, \; S_k^R Q_\mu^L\beta_\alpha^L\Pi_\nu^L, \; S_k^R Q_\nu^L\beta_\alpha^L\Pi_\mu^L, \; S_k^R Q_\mu^L\beta_\alpha^L Q_\nu^L, \; S_k^R Q_\nu^L\beta_\alpha^L Q_\mu^L
\end{aligned}
$$

$$S_k^R S_\rho^L \beta_\alpha^L \Pi_\nu^L,\ S_k^R \Pi_\nu^L \beta_\alpha^L S_\rho^L,\ S_k^R Q_{S_\rho}^L \beta_\alpha^L \Pi_\nu^L,\ S_k^R \Pi_\nu^L \beta_\alpha^L Q_{S_\rho}^L,\ S_k^R Q_{S_\rho}^L \beta_\alpha^L Q_\nu^L,$$
$$S_k^R Q_\nu^L \beta_\alpha^L Q_{S_\rho}^L,\ Q_j^R \Pi_\mu^L \beta_\alpha^L Q_\nu^L,\ Q_j^R \Pi_\nu^L \beta_\alpha^L Q_\mu^L,\ Q_j^R Q_\mu^L \beta_\alpha^L \Pi_\nu^L,\ Q_j^R Q_\nu^L \beta_\alpha^L \Pi_\mu^L,$$
$$Q_j^R Q_\mu^L \beta_\alpha^L Q_\nu^L,\ Q_j^R Q_\nu^L \beta_\alpha^L Q_\mu^L,\ Q_j^R S_\rho^L \beta_\alpha^L \Pi_\nu^L,\ Q_j^R \Pi_\nu^L \beta_\alpha^L S_\rho^L,\ Q_j^R Q_{S_\rho}^L \beta_\alpha^L \Pi_\nu^L,$$
$$Q_j^R \Pi_\nu^L \beta_\alpha^L Q_{S_\rho}^L,\ Q_j^R Q_{S_\rho}^L \beta_\alpha^L Q_\nu^L,\ Q_j^R Q_\nu^L \beta_\alpha^L Q_{S_\rho}^L,\ Q_{S_k}^R \Pi_\mu^L \beta_\alpha^L Q_\nu^L,\ Q_{S_k}^R \Pi_\nu^L \beta_\alpha^L Q_\mu^L,$$
$$Q_{S_k}^R Q_\mu^L \beta_\alpha^L \Pi_\nu^L,\ Q_{S_k}^R Q_\nu^L \beta_\alpha^L \Pi_\mu^L,\ Q_{S_k}^R Q_\mu^L \beta_\alpha^L Q_\nu^L,\ Q_{S_k}^R Q_\nu^L \beta_\alpha^L Q_\mu^L,\ Q_{S_k}^R S_\rho^L \beta_\alpha^L \Pi_\nu^L$$
$$Q_{S_k}^R \Pi_\nu^L \beta_\alpha^L S_\rho^L,\ Q_{S_k}^R Q_{S_\rho}^L \beta_\alpha^L \Pi_\nu^L,\ Q_{S_k}^R \Pi_\nu^L \beta_\alpha^L Q_{S_\rho}^L,\ Q_{S_k}^R Q_{S_\rho}^L \beta_\alpha^L Q_\nu^L,\ Q_j^R Q_\nu^L \beta_\alpha^L Q_{S_\rho}^L\,, \tag{B.6}$$

where $\beta_\alpha^L = \mathbf{1}^{2\ell_L+1} \otimes \mathbf{1}^{2\ell_R+1} \otimes b_\alpha \otimes \mathbf{1}_4$, $\beta_\alpha^R = \mathbf{1}^{2\ell_L+1} \otimes \mathbf{1}^{2\ell_R+1} \otimes c_\alpha \otimes \mathbf{1}_4$, $\mu = 0_\mathbf{0}, 0_\mathbf{2}$, $\nu = +, -$, $\rho = 1, 2$ and where $\Pi_{0_\mathbf{0}}^L, Q_{0_\mathbf{0}}^L, S_1^L, Q_{S_1}^L$ on the left most and $\Pi_{0_\mathbf{2}}^L, Q_{0_\mathbf{2}}^L, S_2^L, Q_{S_2}^L$ on the right most side in any of these expressions are excluded. For the equivariant spinors carrying $(0, \frac{1}{2})$ representation, it is enough to take $L \leftrightarrow R$ in (B.6).

The vacuum configuration with $(k_1\,, k_2)$ component multiplets can be calculated for the cases $k_1 = \text{even}\,, k_2 = \text{even}$ and $k_1 = \text{even}\,, k_2 = \text{odd}$ as follows

$$\begin{aligned}
&{S_F^{2\,Int}}_{k_1\,even} \times {S_F^{2\,Int}}_{k_2\,even} := \\
&\quad S_F^2(\ell_L + \ell^{k_1}_{\frac{k_1}{2}}) \times S_F^2(\ell_R + \ell^{k_2}_{\frac{k_2}{2}}) \oplus \cdots \oplus S_F^2(\ell_L + \ell^{k_1}_{\frac{k_1}{2}}) \times S_F^2(|\ell_R - \ell^{k_2}_{\frac{k_2}{2}}|) \\
&\quad \oplus \vdots \\
&\quad \oplus S_F^2(|\ell_L - \ell^{k_1}_{\frac{k_1}{2}}|) \times S_F^2(\ell_R + \ell^{k_2}_{\frac{k_2}{2}}) \oplus \cdots \oplus S_F^2(|\ell_L - \ell^{k_1}_{\frac{k_1}{2}}|) \times S_F^2(|\ell_R - \ell^{k_2}_{\frac{k_2}{2}}|) \\
&\quad \oplus \mathbf{4} \sum_{n=0}^{\frac{k_1}{2}-1} \sum_{m=0}^{\frac{k_2}{2}-1} \Big[S_F^2(\ell_L + \ell^{k_1}_n) \times S_F^2(\ell_R + \ell^{k_2}_m) \oplus \cdots \oplus S_F^2(\ell_L + \ell^{k_1}_n) \times S_F^2(|\ell_R - \ell^{k_2}_m|) \\
&\quad \oplus \vdots \\
&\quad \oplus S_F^2(|\ell_L - \ell^{k_1}_n|) \times S_F^2(\ell_R + \ell^{k_2}_m) \oplus \cdots \oplus S_F^2(|\ell_L - \ell^{k_1}_n|) \times S_F^2(|\ell_R - \ell^{k_2}_m|) \Big] \\
&\quad \oplus \mathbf{2} \sum_{n=0}^{\frac{k_1}{2}-1} \Big[S_F^2(\ell_L + \ell^{k_1}_n) \times S_F^2(\ell_R + \ell^{k_2}_{\frac{k_2}{2}}) \oplus \cdots \oplus S_F^2(\ell_L + \ell^{k_1}_n) \times S_F^2(|\ell_R - \ell^{k_2}_{\frac{k_2}{2}}|) \\
&\quad \oplus \vdots \\
&\quad \oplus S_F^2(|\ell_L - \ell^{k_1}_n|) \times S_F^2(\ell_R + \ell^{k_2}_{\frac{k_2}{2}}) \oplus \cdots \oplus S_F^2(|\ell_L - \ell^{k_1}_n|) \times S_F^2(|\ell_R - \ell^{k_2}_{\frac{k_2}{2}}|) \Big]
\end{aligned}$$

$$\oplus\mathbf{2}\sum_{m=0}^{\frac{k_2}{2}-1}\Bigg[S_F^2(\ell_L+\ell_{\frac{k_1}{2}}^{k_1})\times S_F^2(\ell_R+\ell_m^{k_2})\oplus\cdots\oplus S_F^2(\ell_L+\ell_{\frac{k_1}{2}}^{k_1})\times S_F^2(|\ell_R-\ell_m^{k_2}|)$$

$$\oplus\vdots$$

$$\oplus S_F^2(|\ell_L-\ell_{\frac{k_1}{2}}^{k_1}|)\times S_F^2(\ell_R+\ell_m^{k_2})\oplus\cdots\oplus S_F^2(|\ell_L-\ell_{\frac{k_1}{2}}^{k_1}|)\times S_F^2(|\ell_R-\ell_m^{k_2}|)\Bigg]. \tag{B.7}$$

$$S_F^{2\,Int}{}_{k_1\,even}\times S_F^{2\,Int}{}_{k_2\,odd}:=$$

$$\mathbf{4}\sum_{n=0}^{\frac{k_1}{2}-1}\sum_{m=0}^{\frac{k_2-1}{2}}\Bigg[S_F^2(\ell_L+\ell_n^{k_1})\times S_F^2(\ell_R+\ell_m^{k_2})\oplus\cdots\oplus S_F^2(\ell_L+\ell_n^{k_1})\times S_F^2(|\ell_R-\ell_m^{k_2}|)$$

$$\oplus S_F^2(\ell_L+\ell_n^{k_1}-1)\times S_F^2(\ell_R+\ell_m^{k_2})\oplus\cdots\oplus S_F^2(\ell_L+\ell_n^{k_1}-1)\times S_F^2(|\ell_R-\ell_m^{k_2}|)$$

$$\oplus\vdots$$

$$\oplus S_F^2(|\ell_L-\ell_n^{k_1}|)\times S_F^2(\ell_R+\ell_m^{k_2})\oplus\cdots\oplus S_F^2(|\ell_L-\ell_n^{k_1}|)\times S_F^2(|\ell_R-\ell_m^{k_2}|)\Bigg]$$

$$\oplus\mathbf{2}\sum_{m=0}^{\frac{k_2-1}{2}}\Bigg[S_F^2(\ell_L+\ell_{\frac{k_1}{2}}^{k_1})\times S_F^2(\ell_R+\ell_m^{k_2})\oplus\cdots\oplus S_F^2(\ell_L+\ell_{\frac{k_1}{2}}^{k_1})\times S_F^2(|\ell_R-\ell_m^{k_2}|)$$

$$\oplus\vdots$$

$$\oplus S_F^2(|\ell_L-\ell_{\frac{k_1}{2}}^{k_1}|)\times S_F^2(\ell_R+\ell_m^{k_2})\oplus\cdots\oplus S_F^2(|\ell_L-\ell_{\frac{k_1}{2}}^{k_1}|)\times S_F^2(|\ell_R-\ell_m^{k_2}|)\Bigg]. \tag{B.8}$$

APPENDIX C

SOME DETAILS FOR CHAPTER 5

$$
\begin{aligned}
\Lambda_1 :=& -\frac{2\ell^4+6\ell^3+4\ell^2-\ell-2}{4\ell(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_1 \\
&+\frac{2\ell^4+2\ell^3-2\ell^2-\ell-1}{4(\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2+\frac{\omega_c}{2\ell^2+2\ell+1}\,, \\
\Lambda_2 :=& -\frac{4\ell^4+8\ell^3+5\ell^2}{4(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_1 \\
&-\frac{8\ell^5+18\ell^4+11\ell^3+3\ell^2}{4(\ell+1)(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2+\frac{\ell\omega_c}{2\ell^2+2\ell+1}\,, \\
\Lambda_3 :=& \frac{(\ell+1)(8\ell^4+14\ell^3+5\ell^2-3\ell-2)}{4\ell(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_1 \\
&+\frac{(\ell+1)(4\ell^3+4\ell^2+\ell+1)}{4(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2-\frac{(\ell+1)\omega_c}{2\ell^2+2\ell+1}\,, \\
\Lambda_4 :=& -\frac{4\ell^4+10\ell^3+4\ell^2-\ell-2}{4\ell(2\ell+1)^2(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_1 \\
&+\frac{4\ell^4+6\ell^3-2\ell^2-5\ell-3}{4(2\ell+1)^2(\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2+\frac{\omega_c}{(2\ell+1)^2}\,, \\
\Lambda_5 :=& \frac{2\ell^5+10\ell^4+14\ell^3+3\ell^2-3\ell-2}{2\ell(\ell+1)(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_1 \\
&-\frac{2\ell^4+2\ell^3-\ell^2-\ell-2}{2(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2-\frac{2\omega_c}{2\ell^2+2\ell+1}\,, \\
\Lambda_6 :=& -\frac{2\ell^4+6\ell^3+5\ell^2+\ell-2}{2(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_1 \\
&-\frac{2\ell^5-6\ell^3-\ell^2+3\ell+2}{2\ell(\ell+1)(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2+\frac{2\omega_c}{2\ell^2+2\ell+1}\,, \\
\Lambda_7 :=& \frac{2\ell^3+6\ell^2+3\ell-3}{2(\ell+1)(2\ell+1)^2(\ell^2+\ell-1)}P_1 \\
&+\frac{2\ell^3-3\ell+2}{2\ell(2\ell+1)^2(\ell^2+\ell-1)}P_2-\frac{2\omega_c}{(2\ell+1)^2}\,,
\end{aligned}
$$

$$\begin{aligned}
\Lambda_8 :&= \Lambda_9 := -\Lambda_{13} := -\frac{1}{(2\ell+1)^2}\,,\\
\Lambda_{10} :&= \frac{2\ell^2+3\ell-1}{2(\ell+1)(2\ell+1)}\varphi_3 - \frac{1}{2(2\ell+1)}\chi_3 + \frac{1}{2\ell+1}\psi\,,\\
\Lambda_{11} :&= -\frac{2\ell^2+\ell-2}{2\ell(2\ell+1)}\chi_3 - \frac{1}{2(2\ell+1)}\varphi_3 + \frac{1}{2\ell+1}\psi\,,\\
\Lambda_{12} :&= \frac{1}{2\ell+1}(-Q_1[X_c,Q_1] - Q_2[X_c,Q_2] - \omega_c + 2X_c)
\end{aligned} \tag{C.1}$$

where $P_1 := -Q_1[X_c,Q_1] - i\{X_c,Q_2\}$ and $P_2 := -Q_2[X_c,Q_2] - i\{X_c,Q_1\}$.

$$\begin{aligned}
\alpha_1 &= \frac{4(\ell^2+\ell-1)^2(\ell^2+\ell+1)}{3\ell^3(\ell+1)^3}\,, \quad \alpha_2 = \frac{4(2\ell^4+5\ell^3+\ell^2-\ell+3)}{3(\ell+1)^3(2\ell+1)}\,,\\
\alpha_3 &= \frac{4(2\ell^4+3\ell^3-2\ell^2-4\ell+2)}{3\ell^3(2\ell+1)}\,, \quad \alpha_4 = \alpha_5\\
\alpha_5 &= \frac{2(-3\ell^8-12\ell^7-14\ell^6+13\ell^4+12\ell^3+16\ell^2+12\ell-12)}{3\ell^3(\ell+1)^3(2\ell+1)^2}\,,\\
\alpha_6 &= \frac{4(4\ell^7+10\ell^6+2\ell^5-2\ell^4-3\ell^3-15\ell^2+4)}{3\ell^3(\ell+1)^2(2\ell+1)^2}\,,\\
\alpha_7 &= \frac{4(4\ell^7+18\ell^6+26\ell^5+2\ell^4-35\ell^3-28\ell^2+7\ell+6)}{3\ell^2(\ell+1)^3(2\ell+1)^2}\,,\\
\alpha_8 &= \frac{4(\ell^6+3\ell^5+15\ell^4+25\ell^3-30\ell^2-42\ell+24)}{3\ell^2(\ell+1)^2(2\ell+1)^2}\,,\\
\alpha_9 &= \frac{4(2\ell^6+23\ell^5+43\ell^4-11\ell^3-45\ell^2+6\ell+6)}{3\ell(\ell+1)^2(2\ell+1)^3}\,,\\
\alpha_{10} &= \frac{4(2\ell^6-11\ell^5-42\ell^4-7\ell^3+46\ell^2+6\ell-12)}{3\ell^2(\ell+1)(2\ell+1)^3}\,,\\
\alpha_{11} &= \frac{2(\ell^4+2\ell^3-5\ell^2-6\ell+4)}{\ell(\ell+1)(2\ell+1)^2}
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
\beta_1 &= \frac{4\ell^2(4\ell^3+14\ell^2+14\ell+3)}{3(\ell+1)^3(2\ell+1)^2}\,, \quad \beta_2 = \frac{4(4\ell^2+4\ell-3)}{3(2\ell+1)^2}\,,\\
\beta_3 &= \frac{4(8\ell^6+36\ell^5+46\ell^4+5\ell^3-9\ell^2+7\ell-3)}{3(\ell+1)^3(2\ell+1)^3}\,,\\
\beta_4 &= \frac{4(2\ell^4+9\ell^3+15\ell^2+7\ell-3)}{3(\ell+1)^3(2\ell+1)^3}\,, \quad \beta_5 = \frac{4(-4\ell^4-8\ell^3+7\ell^2+11\ell-6)}{3(\ell+1)^2(2\ell+1)^2}\,,\\
\beta_6 &= \frac{4(\ell-1)^2(2\ell^2+7\ell+6)}{3(\ell+1)^3(2\ell+1)^2}\,, \quad \beta_7 = \frac{8(8\ell^4+22\ell^3+7\ell^2-10\ell+3)}{3(\ell+1)^2(2\ell+1)^3}\\
\beta_8 &= \frac{8\ell(4\ell^3+12\ell^2+7\ell-3)}{(\ell+1)^2(2\ell+1)^3}\,, \quad \beta_9 = \beta_{10} = \frac{8\ell(2\ell+3)}{(\ell+1)(2\ell+1)^3}\,,
\end{aligned} \tag{C.3}$$

$$
\begin{aligned}
\gamma_1 &= \frac{4(\ell+1)^2(4\ell^3-2\ell^2-2\ell+1)}{3\ell^3(2\ell+1)^2}\,, \quad \gamma_2 = \frac{2(-4\ell^4+2\ell^3-8\ell+4)}{3\ell^3(2\ell+1)^3}\,,\\
\gamma_3 &= \frac{4(8\ell^6+12\ell^5-14\ell^4-21\ell^3+12\ell^2+12\ell-6)}{3\ell^3(2\ell+1)^3}\,,\\
\gamma_4 &= \frac{4(\ell+2)^2(2\ell^2-3\ell+1)}{3\ell^3(2\ell+1)^2}\,, \quad \gamma_5 = \frac{4(4\ell^4+8\ell^3-7\ell^2-11\ell+6)}{3\ell^2(2\ell+1)^2}\,,\\
\gamma_6 &= \frac{8(8\ell^4+10\ell^3-11\ell^2-10\ell+6)}{3\ell^2(2\ell+1)^3}\,, \quad \gamma_7 = \gamma_9 = \frac{8(\ell+1)(2\ell-1)}{\ell(2\ell+1)^3}\,,\\
\gamma_8 &= \frac{8(4\ell^4+4\ell^3-5\ell^2-3\ell+2)}{\ell^2(2\ell+1)^3}\,,
\end{aligned} \tag{C.4}
$$

$$
\begin{aligned}
\delta_1 &= \frac{2(-3\ell^8-12\ell^7-12\ell^6+6\ell^5+13\ell^4+2\ell^3+2\ell-2)}{3\ell^3(\ell+1)^3(2\ell+1)^4}\,,\\
\delta_2 &= \frac{4(2\ell^8+15\ell^7+23\ell^6-11\ell^5-23\ell^4+\ell^3-11\ell^2+4)}{3\ell^3(\ell+1)^2(2\ell+1)^4}\,,\\
\delta_3 &= \frac{4(2\ell^8+\ell^7-26\ell^6-54\ell^5-8\ell^4+64\ell^3+44\ell^2-13\ell-10)}{3\ell^2(\ell+1)^3(2\ell+1)^4}\,,\\
\delta_4 &= \frac{8(\ell^6+3\ell^5+5\ell^4+5\ell^3-8\ell^2-10\ell+6)}{3\ell^2(\ell+1)^2(2\ell+1)^4}\,,\\
\delta_5 &= \frac{8(2\ell^6+7\ell^5+3\ell^4-15\ell^3-15\ell^2+3\ell+3)}{3\ell(\ell+1)^2(2\ell+1)^5}\\
\delta_6 &= \frac{8(2\ell^6+5\ell^5-2\ell^4-3\ell^3+8\ell^2+\ell-2)}{3\ell^2(\ell+1)(2\ell+1)^5}\,,\\
\delta_7 &= \frac{4(3\ell^4+6\ell^3-5\ell^2-8\ell+4)}{\ell(\ell+1)(2\ell+1)^3}\,, \quad \delta_8 = \frac{4(3\ell^4+6\ell^3-\ell^2-4\ell+2)}{\ell(\ell+1)(2\ell+1)^4}\,,\\
\delta_9 &= \frac{8(\ell^6+3\ell^5+3\ell^4+\ell^3-6\ell^2-6\ell+4)}{\ell^2(\ell+1)^2(2\ell+1)^3}\,,\\
\delta_{10} &= \frac{4(2\ell^4+3\ell^3-5\ell^2-4\ell+4)}{\ell(2\ell+1)^4}\,, \quad \delta_{11} = \frac{4(2\ell^4+5\ell^3-2\ell^2-7\ell+2)}{(\ell+1)(2\ell+1)^4}\,,\\
\delta_{12} &= \frac{8\ell(\ell+1)}{(2\ell+1)^4}\,, \quad \delta_{13} = \frac{8\ell(2\ell^2-5\ell-9)}{3(2\ell+1)^5}\,, \quad \delta_{14} = \frac{8(2\ell^3+11\ell^2+7\ell-2)}{3(2\ell+1)^5}\,,\\
\delta_{15} &= \frac{4(-\ell^2-\ell+2)}{(2\ell+1)^3}\,, \quad \delta_{16} = \frac{2(-\ell^2-\ell-2)}{(2\ell+1)^4}
\end{aligned} \tag{C.5}
$$

$$
\begin{aligned}
R_1 = &-\frac{\ell}{2(\ell+1)}(|\varphi|^2-1) - \frac{\ell+1}{2\ell}(|\chi|^2-1) + \frac{1}{\ell^2+\ell}(\chi_3-\varphi_3)\\
&- \frac{2\ell^4+4\ell^3-2\ell-1}{2(2\ell+1)^2(\ell^2+\ell)}(\chi_3-\varphi_3)^2 - \frac{2\ell^2+2\ell-1}{2\ell+1}\psi + \frac{1}{2\ell+1}(\chi_3-\varphi_3)\psi\\
&- \frac{\ell^2+\ell+1}{(2\ell+1)^2}\psi^2\,, \qquad (C.6)\\
R_2 = &\frac{\ell}{2\ell^2+3\ell+1}(|\varphi|^2-1) + \frac{2\ell^2+\ell-1}{2(2\ell^2+1)}(|\chi|^2-1)\\
&+ \frac{\ell^2+2\ell-1}{(2\ell+1)(\ell^2+\ell)}(\chi_3 - \frac{\chi_3^2+\varphi_3^2}{2(2\ell+1)}) - \frac{2\ell^3+2\ell^2-3\ell+1}{\ell(2\ell+1)}(\varphi_3 - \frac{\varphi_3\chi_3}{2\ell+1})\\
&- \frac{\ell+1}{2\ell+1}(\psi + \frac{\psi^2}{2\ell+1}) - \frac{2\ell^2+3\ell-1}{(2\ell+1)^2}\varphi_3\psi + \frac{\ell+1}{(2\ell+1)^2}\chi_3\psi\,, \qquad (C.7)\\
R_3 = &\frac{2\ell^2+3\ell}{2(2\ell^2+3\ell+1)}(|\varphi|^2-1) - \frac{\ell+1}{2\ell^2+\ell}(|\chi|^2-1)\\
&+ \frac{\ell^2-2}{(2\ell+1)(\ell^2+\ell)}(\varphi_3 - \frac{\varphi_3^2+\chi_3^2}{2(2\ell+1)}) - \frac{2\ell^3+4\ell^2-\ell-4}{(2\ell+1)(\ell+1)}(\chi_3 - \frac{\chi_3\varphi_3}{2\ell+1})\\
&- \frac{\ell}{2\ell+1}(\psi + \frac{\psi^2}{2\ell+1}) - \frac{\ell}{(2\ell+1)^2}\varphi_3\psi - \frac{2\ell^2+\ell-2}{(2\ell+1)^2}\chi_3\psi\,, \qquad (C.8)
\end{aligned}
$$

Equations of motion that follow from the variations of the action (5.40) are

$$
\begin{aligned}
&\big(1-\frac{1}{\ell^2}+\frac{1}{\ell^2}(\zeta^2+\eta^2)\big)(\zeta''+\frac{\zeta'}{r}) - \Bigg(-\eta^2\big(1+\frac{3}{4\ell^2}+\frac{(M+a_\theta^2)^2}{2\ell^2r^2}\\
&-\frac{(N+a_\theta^1)^2}{\ell^2r^2}\big) + \frac{3}{\ell^2}\zeta^2\eta^2 - \frac{7}{4\ell^2}\eta^4 + (1-\frac{1}{\ell^2})\frac{(N+a_\theta^1)^2}{r^2} - \frac{1}{\ell^2}\zeta'^2 - \frac{1}{2\ell^2}\eta'^2\\
&\qquad - (1-\frac{1}{\ell}+\frac{1}{2\ell^2}) + (2-\frac{1}{\ell})\zeta^2\Bigg)\zeta = 0\,,\\
&\big(1-\frac{1}{\ell^2}+\frac{1}{\ell^2}(\zeta^2+\eta^2)\big)(\eta''+\frac{\eta'}{r}) - \Bigg(-\zeta^2\big(1+\frac{3}{4\ell^2}+\frac{(N+a_\theta^1)^2}{2\ell^2r^2}\\
&-\frac{(M+a_\theta^2)^2}{\ell^2r^2}\big) + \frac{3}{\ell^2}\zeta^2\eta^2 - \frac{7}{4\ell^2}\zeta^4 + (1-\frac{1}{\ell^2})\frac{(M+a_\theta^2)^2}{r^2} - \frac{1}{\ell^2}\eta'^2 - \frac{1}{2\ell^2}\zeta'^2\\
&\qquad - (1+\frac{1}{\ell}-\frac{3}{2\ell^2}) + (2-\frac{1}{\ell}-\frac{2}{\ell^2})\eta^2\Bigg)\eta = 0\,,\\
&a_\theta^{1\prime\prime} - \frac{a_\theta^{1\prime}}{r} + (2-\frac{1}{\ell^2})(M+a_\theta^2)\eta^2 - (4-\frac{2}{\ell}+\frac{1}{\ell^2})(N+a_\theta^1)\zeta^2 = 0\,,\\
&a_\theta^{2\prime\prime} - \frac{a_\theta^{2\prime}}{r} + (2-\frac{1}{\ell^2})(N+a_\theta^1)\zeta^2 - (4+\frac{2}{\ell}-\frac{1}{\ell^2})(M+a_\theta^2)\eta^2 = 0\,, \qquad (C.9)
\end{aligned}
$$

APPENDIX D

SOME DETAILS FOR CHAPTER 6

In this short appendix, we provide a derivation of the normalization coefficient of R_{N^2-1} in the $\frac{N(N-1)}{2}$-dimensional IRR of $SU(N)$ for $N \geq 3$. Let $T_a^{(D)}$ label the N^2-1 generators of $SU(N)$ in the defining N-dimensional representation. Let us choose their trace normalization to be

$$\mathrm{Tr}(T_a^{(D)} T_b^{(D)}) = \frac{1}{2}\delta_{ab} \,. \tag{D.1}$$

It is a well-known fact in the representation theory of Lie groups that such a choice fixes the trace normalization of the generators in all the IRR [94]. We can proceed to write the trace normalization in an IRR R of $SU(N)$ as

$$\mathrm{Tr}(T_a^{(R)} T_b^{(R)}) = \kappa_{ab} \,, \tag{D.2}$$

where κ_{ab} is a rank-2 tensor invariant under $SU(N)$ transformations. Since the only rank-2 invariant $SU(N)$ tensor is Kronecker delta, δ_{ab}, we have

$$\kappa_{ab} = X_{(R)}\delta_{ab} \,, \tag{D.3}$$

where $X_{(R)}$, commonly known as the *Dynkin index* of the representation R of the group $SU(N)$, is given by [94]

$$X_{(R)} = \frac{\dim(R)}{\dim(SU(N))}\mathcal{C}_2(R) \,. \tag{D.4}$$

We have that $\dim(SU(N))$ is equal to N^2-1 and $\mathcal{C}_2^R$ is the quadratic Casimir of the IRR R. For either of the $\frac{N(N-1)}{2}$-dimensional IRR, $(0\,,1\,,0\,,\cdots\,,0\,,0)$ or $(0\,,0\,,\cdots\,,1\,,0)$ of $SU(N)$, this gives, using (6.82),

$$X_{(R)} = \frac{N-2}{N} \,, \tag{D.5}$$

and the trace formula (D.2) then reads

$$\mathrm{Tr}(T_a T_b) = \frac{N-2}{N}\delta_{ab}, \tag{D.6}$$

in either of the $\frac{N(N-1)}{2}$-dimensional IRR. Our aim is to find the coefficient of R_{N^2-1} in these representations. In terms of the Young diagrams, the branching of, say, $(0\,,1\,,0\,,\cdots\,,0\,,0)$ representation under $SU(N-2)\times SU(2)\times U(1)$ gives

$$\yng(1,1) \;=\; \left(\,\cdot\,\otimes\,\yng(1,1)\right)_{-1} \oplus \left(\yng(1,1)\,\otimes\,\cdot\,\right)_{\frac{2}{N-2}} \oplus \left(\yng(1)\,\otimes\,\yng(1)\right)_{\frac{4-N}{2(N-2)}}\,, \tag{D.7}$$

where the subscripts give the $U(1)$ charge (6.32). Considering the dimension of each representation in this branching, we find

$$R_{N^2-1} = \zeta\,\mathrm{diag}\Bigg(\underbrace{\frac{N-4}{2(N-2)},\ldots,\frac{N-4}{2(N-2)}}_{2(N-2)},\underbrace{\frac{-2}{N-2},\ldots,\frac{-2}{N-2}}_{\frac{(N-2)(N-3)}{2}},1\Bigg)\,, \tag{D.8}$$

where ζ represents the coefficient of R_{N^2-1} and the dimensions of the IRR in the branching (D.7) are given in the underbraces. Finally, using (D.8) in (D.6) gives

$$\zeta = \sqrt{\frac{N-2}{N}}\,. \tag{D.9}$$

The dimension of the $(P_1\,,P_2\,,P_3\,,\cdots\,,P_{N-2}\,,P_{N-1})$ representation may be written as

$$\begin{aligned}
\dim(P_1,P_2,0,\ldots,0,P_{N-2},P_{N-1}) \;=\;& \frac{1}{j}((P_{N-2}+P_{N-1}+N-3)!(P_{N-2}+N-4)!\\
\times\;& (P_2+N-4)!(P_1+P_2+N-3)!\\
\times\;& (P_{N-2}+P_{N-1}+P_2+N-2)(P_{N-1}+1)\\
\times\;& (P_1+P_2+P_{N-2}+P_{N-1}+N-1)(P_1+1)\\
\times\;& (P_{N-2}+P_2+N-3)(P_1+P_2\\
\times\;& +P_{N-2}+N-2)),
\end{aligned} \tag{D.1}$$

where j is

$$j = (N-1)!(N-2)!(N-3)!(N-4)!P_2!P_{N-2}!(P_{N-2}+P_{N-1}+1)!(P_1+P_2+1)!. \tag{D.2}$$

REFERENCES

[1] P. Forgacs and N. S. Manton. Space-time symmetries in gauge theories. *Commun. Math. Phys.*, 72:15, 1980.

[2] D. Kapetanakis and G. Zoupanos. Coset space dimensional reduction of gauge theories. *Phys. Rep.*, 219:4, 1992.

[3] P. Aschieri, J. Madore, P. Manousselis, and G. Zoupanos. Dimensional reduction over fuzzy coset spaces. *J. High Energy Phys.*, 04:034, 2004.

[4] E. D'Hoker and D. Z. Freedman. Supersymmetric gauge theories and the AdS / CFT correspondence. Lecture Notes, 2002. hep-th/0201253.

[5] A. V. Ramallo. Introduction to the AdS/CFT correspondence. *Springer Proc. Phys.*, 161:411, 2015.

[6] N. Arkani-Hamed, A. G. Cohen, and H. Georgi. (De)Constructing Dimensions. *Phys. Rev. Lett.*, 86:4757–4761, 2001.

[7] T. Kaluza. On the problem of unity in physics. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, K1:966–972, 1921.

[8] O. Klein. Quantum theory and five-dimensional theory of relativity. *Z. Phys*, 37:895–906, 1926.

[9] P. Aschieri, T. Grammatikopoulos, H. Steinacker, and G. Zoupanos. Dynamical generation of fuzzy extra dimensions, dimensional reduction and symmetry breaking. *JHEP*, 0609:026, 2006.

[10] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind. M theory as a matrix model: A Conjecture. *Phys. Rev. D*, 55:5112, 1997.

[11] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase. Strings in flat space and pp waves from N=4 superYang-Mills. *JHEP*, 0204:013, 2002.

[12] K. Dasgupta, M. M. Sheikh-Jabbari, and M. Van Raamsdonk. Matrix perturbation theory for M theory on a PP wave. *JHEP*, 0205:056, 2002.

[13] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya. A Large-N Reduced Model as Superstring. *Nucl. Phys. B*, 498:467, 1997.

[14] S. Iso, Y. Kimura, K. Tanaka, and K. Wakatsuki. Noncommutative Gauge Theory on Fuzzy Sphere from Matrix Model. *Nucl. Phys. B*, 604:121, 2001.

[15] Y. Kitazawa. Matrix Models in Homogeneous Spaces. *Nucl. Phys. B*, 642:210, 2002.

[16] T. Azuma, K. Nagao, and J. Nishimura. Perturbative dynamics of fuzzy spheres at large-N. *J. High Energy Phys.*, 06:081, 2005.

[17] T. Azuma, S. Bal, and J. Nishimura. Dynamical generation of gauge groups in the massive Yang-Mills-Chern-Simons matrix model. *Phys. Rev. D*, 72:066005, 2005.

[18] T. Azuma, S. Bal, K. Nagao, and J. Nishimura. Nonperturbative studies of fuzzy spheres in a matrix model with the Chern-Simons term. *J. High Energy Phys.*, 05:005, 2004.

[19] J. Castelino, S-M. Lee, and W. Taylor. Longitudinal 5-branes as 4-spheres in Matrix theory. *Nucl. Phys. B*, 526:334, 1998.

[20] Y. Kimura. Noncommutative Gauge Theory on Fuzzy Four-Sphere and Matrix Model. *Nucl. Phys. B*, 637:177–198, 2002.

[21] A. Chatzistavrakidis, H. Steinacker, and G. Zoupanos. On the fermion spectrum of spontaneously generated fuzzy extra dimensions with fluxes. *Fortsch. Phys.*, 58:537, 2010.

[22] D. Harland and S. Kurkcuoglu. Equivariant reduction of Yang-Mills theory over the fuzzy sphere and the emergent vortices. *Nucl. Phys. B*, 821:380, 2009.

[23] S. Kurkcuoglu. Noncommutative Vortices and Flux-Tubes from Yang-Mills Theories with Spontaneously Generated Fuzzy Extra Dimensions. *Phys. Rev. D*, 82:105010, 2010.

[24] S. Kurkcuoglu. Equivariant reduction of $U(4)$ gauge theory over $S_F^2 \times S_F^2$ and the emergent vortices. *Phys. Rev. D*, 85:105004, 2012.

[25] S. Kurkcuoglu. New fuzzy extra dimensions from $SU(\mathcal{N})$ gauge theories. *Phys. Rev. D*, 92:025022, 2015.

[26] S. Kurkcuoglu and G. Unal. Equivariant fields in an $SU(\mathcal{N})$ gauge theory with new spontaneously generated fuzzy extra dimensions. *Phys. Rev. D*, 93:105019, 2016.

[27] S. Kurkcuoglu and G. Unal. $U(3)$ gauge theory on fuzzy extra dimensions. *Phys. Rev. D*, 94:036003, 2016.

[28] F. Balli, A. Behtash, S. Kurkcuoglu, and G. Unal. Quantum Hall Effect on the Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$. *Phys. Rev. D*, 89:105031, 2014.

[29] D. Karabali and V. P. Nair. Quantum Hall Effect in Higher Dimensions. *Nucl. Phys. B*, 641:533, 2002.

[30] B. A. Bernevig, J. H. Brodie, L. Susskind, and N. Toumbas. How Bob Laughlin tamed the giant graviton from Taub-NUT space. *J. High Energy Phys.*, 02:003, 2001.

[31] M. Fabinger. Higher-Dimensional Quantum Hall Effect in String Theory. *J. High Energy Phys.*, 05:037, 2002.

[32] D. Karabali, V. P. Nair, and S. Randjbar-Daemi. *Fields to Strings: Circumnavigating Theoretical Physics.* edited by M. Shifman, World Scientific, 2005.

[33] B. P. Dolan and O. Jahn. Fuzzy complex Grassmannian spaces and their star products. *Int. J. Mod. Phys. A*, 18:1935, 2003.

[34] Idrish Huet and Sean Murray. Polarization Tensors for the fuzzy Grassmannians $Gr^F_{2:n}$ and Grassmannian harmonics on $Gr_{2:n}$, 2012. hep-th/0201253.

[35] S. Murray and C. Sämann. Quantization of Flag Manifolds and their Supersymmetric Extensions. *Adv. Theor. Math. Phys.*, 12:641, 2008.

[36] J. Madore. The fuzzy sphere. *Class. Quant. Grav.*, 9:69, 1992.

[37] A. P. Balachandran, G. Marmo, B. S. Skagerstam, and A. Stern. *Classical Topology and Quantum States.* World Scientific, 1991.

[38] H. Grosse, C. Klimčík, and P. Prešnajder. Topologically nontrivial field configurations in non-commutative geometry. *Commun. Math. Phys.*, 178:507, 1996.

[39] H. Grosse and P. Prešnajder. The Construction of non-commutative manifolds using coherent states. *Lett. Math. Phys.*, 28:239, 1993.

[40] H. Grosse and G. Reiter. The Fuzzy Supersphere. *J. Geom. and Phys.*, 28:349–383, 1998.

[41] A. P. Balachandran, S. Kürkçüoğlu, and S. Vaidya. *Lectures on fuzzy and fuzzy SUSY physics.* World Scientific, 2007.

[42] A. P. Balachandran, S. Kurkcuoglu, and E. Rojas. The Star product on the fuzzy supersphere. *JHEP*, 0207:056, 2002.

[43] S. Vaidya and B. Ydri. On the origin of the UV-IR mixing in non-commutative matrix geometry. *Nucl. Phys. B*, 671:401, 2003.

[44] S. Vaidya and B. Ydri. New scaling limit for fuzzy spheres, 2002. hep-th/0201253.

[45] A. P. Balachandran, B. P. Dolan, J. H. Lee, X. Martin, and D. O'Connor. Fuzzy complex projective spaces and their star-products. *J. Geom. Phys.*, 43:184, 2002.

[46] G. Alexanian, A. P. B alachandran, G. Immirzi, and B. Ydri. Fuzzy CP(2). *J. Geom. Phys.*, 42:28, 2002.

[47] H. Steinacker. Quantized Gauge Theory on the Fuzzy Sphere as Random Matrix Model. *Nuclear Physics B*, 679:66–98, 2004.

[48] C. Jayawardena. Schwinger Model on S^2. *Helvetica Physica Acta*, 61:636–711, 1988.

[49] S. Ramgoolam. On spherical harmonics for fuzzy spheres in diverse dimensions. *Nucl. Phys. B*, 610:461–488, 2001.

[50] J. J. Sakurai. *Modern Quantum Mechanics.* Addison-Wesley Publishing Company, 1994.

[51] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonsky. *Quantum Theory of Angular Momentum.* World Scientific, 1998.

[52] J. Madore. *An Introduction to Non-commutative Differential Geometry and its Physical Applications.* Cambridge University Press, 1995.

[53] H. Grosse, C. Klimčík, and P. Prešnajder. Field Theory on a Supersymmetric Lattice. *Commun. Math. Phys.*, 185:155–175, 1997.

[54] H. Grosse, C. Klimčík, and P. Prešnajder. N=2 Superalgebra and Non-Commutative Geometry. *Preprint ESI*, 321, 1996.

[55] S. Baez, A. P. Balachandran, B. Ydri, and S. Vaidya. Monopoles and solitons in fuzzy physics. *Commun. Math. Phys.*, 208:797, 2000.

[56] A. P. Balachandran and S. Vaidya. Instantons and chiral anomaly in fuzzy physics. *Int. J. Mod. Phys. A*, 16:17, 2001.

[57] A. P. Balachandran, T. R. Govindarajan, and B. Ydri. The fermion doubling problem and noncommutative geometry. *Mod. Phys. Lett. A*, 15:1279, 2000.

[58] A. P. Balachandran and G. Immirzi. Fuzzy Nambu-Goldstone physics. *Int. J. Mod. Phys. A*, 18:5981, 2003.

[59] A. P. Balachandran and G. Immirzi. The Fuzzy Ginsparg-Wilson algebra: A Solution of the fermion doubling problem. *Phys. Rev. D*, 68:065023, 2003.

[60] S. Vaidya. Perturbative dynamics on fuzzy $S(2)$ and $RP(2)$. *Phys. Lett. B*, 512:403, 2001.

[61] C. S. Chu, J. Madore, and H. Steinacker. Scaling limits of the fuzzy sphere at one loop. *JHEP*, 0108:038, 2001.

[62] B. P. Dolan, D. O'Conner, and P. Prešnajder. Matrix ϕ^4 models on the fuzzy sphere and their continuum limits. *JHEP*, 0203:013, 2002.

[63] D. Karabali, V. P. Nair, and A. P. Polychronakos. Spectrum of Schrodinger field in a noncommutative magnetic monopole. *Nuclear Physics B*, 627:565, 2002.

[64] E. Witten. Two-dimensional gauge theories revisited. *J. Geom. Phys.*, 9:303, 1992.

[65] D. Gross and A. Matytsin. Instanton induced large N phase transitions in two-dimensional and four-dimensional QCD. *Nucl. Phys. B*, 429:50, 1994.

[66] S. Gasiorowicz. *Quantum Physics*. John Wiley & Sons Inc., 2003.

[67] Edward Witten. Some exact multipseudoparticle solutions of classical yang-mills theory. *Phys. Rev. Lett.*, 38:121–124, 1977.

[68] N. Manton and P. Sutcliffe. *Topological Solitons*. Cambridge University Press, 2004.

[69] S. Kurkcuoglu. Nonlinear sigma models on the fuzzy supersphere. *JHEP*, 0403:062, 2004.

[70] N. Acharyya, N. Chandra, and S. Vaidya. Quantum Entropy for the Fuzzy Sphere and its Monopoles. *JHEP*, 1411:078, 2014.

[71] R. Bertlmann. Decoherence and the physics of open quantum systems. Lecture Notes. https://homepage.univie.ac.at/reinhold.bertlmann/pdfs/decoscript_v2.pdf.

[72] F. A. Bais and H. A. Weldon. $SU(3)$ instantons with cylindrical symmetry. *Phys. Rev. D*, 08:561, 1978.

[73] A. P. Balachandran and P. Padmanabhan. Spin j Dirac Operators on the Fuzzy 2-Sphere. *JHEP*, 09:120, 2009.

[74] K. Hasebe. Higher Dimensional Quantum Hall Effect as A-Class Topological Insulator. *Nucl. Phys. B*, 886:952, 2014.

[75] F. D. M. Haldane. Fractional Quantization of the Hall Effect: A Hierarchy of Incompressible Quantum Fluid States. *Phys. Rev. Lett.*, 51:605, 1983.

[76] J. Hu and S. C. Zhang. A Four Dimensional Generalization of the Quantum Hall Effect. *Science*, 294:823, 2001.

[77] Z. F. Ezawa. *Quantum Hall Effects: Field Theoretical Approach and Related Topics.* World Scientific, 2003.

[78] B. I. Halperin. Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential. *Phys. Rev. B*, 25:2185, 1982.

[79] X. G. Wen. Electrodynamical Properties of Gapless Edge Excitations in the Fractional Quantum Hall States. *Phys. Rev. Lett.*, 64:2206, 1990.

[80] M. Stone. Schur functions, chiral bosons, and the quantum-Hall-effect edge states. *Phys. Rev. B*, 42:8399, 1990.

[81] B. A. Bernevig, C. H. Chern, J. P. Hu, N. Toumbas, and S. C. Zhang. The Eight Dimensional Quantum Hall Effect and the Octonions. *Phys. Rev. Lett.*, 91:236803, 2003.

[82] V.P. Nair and S. Randjbar-Daemi. Quantum Hall effect on S^3, edge states and fuzzy S^3/Z_2. *Nucl. Phys. B*, 679:447, 2004.

[83] M. Daoud and A. Jellal. Quantum Hall Effect on the Flag Manifold $\mathbf{F}_2$. *Int. J. Mod. Phys. A*, 23:3129, 2008.

[84] K. Hasebe and Y. Kimura. Dimensional Hierarchy in Quantum Hall Effects on Fuzzy Spheres. *Phys. Lett. B*, 602:255, 2004.

[85] R. S. Ward and R. O. Wells. *Twistor Geometry and Field Theory.* Cambridge Univ. Press, 1991.

[86] I. Bars. *Supergroups and their representations, In: Introduction to Supersymmetry in Particle and Nuclear Physics.* Plenum Press, 1981.

[87] A. B. Balantekin and I. Bars. Branching Rules for the Supergroup $SU(N/M)$ from those of $SU(N+M)$. *J. Math. Phys.*, 23:1239, 1982.

[88] C. R. Hagen and A. J. Macfarlane. Reduction of Representations of SU_{mn} with respect to the Subgroup $SU_m \otimes SU_n$. *J. Math. Phys.*, 6:1355, 1965.

[89] K. Fuji. Introduction to Grassmann Manifolds and Quantum Computation. *J. Appl. Math.*, 2:371, 2002.

[90] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* John Wiley & Sons Inc., 1994.

[91] V. P. Nair. *Quantum Field Theory: A Modern Perspective.* Springer, 2005.

[92] Y. Karshon and S. Tolman. The Gromov width of complex Grassmannians. *Algebr. Geom. Topol.*, 5:9110, 2005.

[93] A. Gray. *Tubes.* Birkhäuser Verlag, 2004.

[94] J. Fuchs and C. Schweigert. *Symmetries, Lie algebras and Representations: A Graduate Course for Physicists*. Cambridge Univ. Press, 2003.

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PUBLICATIONS

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S. Kurkcuoglu and G. Unal "$U(3)$ gauge theory on fuzzy extra dimensions" Phys.Rev. D94 (2016) 036003

F. Ballı, A. Behtash, S. Kürkçüoğlu,and G. Ünal, "Quantum Hall effect on the Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$" Phys.Rev. D89 (2014) 105031