Lepton Flavour Violation and $K \to \pi\nu\overline{\nu}$ Decays in the Standard Model (Effective Field Theory)

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by

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Declaration

I hereby declare that the work presented in this thesis is the result of my own research activities unless reference is given to others. None of this material has been previously submitted to this or any other university. All work presented here was carried out in the Department of Mathematical Sciences at the University of Liverpool, Liverpool, U.K. during the period from September 2014 to August 2018.

Contributions to this work have previously been published or are awaiting publication elsewhere in:

- S. Davidson, M. Gorbahn and **M. Leak**, *Majorana neutrino masses in the renormalization group equations for lepton flavor violation,* Phys. Rev. D **98**, no. 9, 095014 (2018).
- M. Cerdà Sevilla, M. Gorbahn and **M. Leak**, *Three Loop QCD Corrections to $K \to \pi\nu\bar{\nu}$,* paper in progress.

Abstract

A major goal of modern physics is the detection of physics beyond the Standard Model. Promising areas to search for such signals are processes that are either forbidden or highly suppressed within the Standard Model, including Lepton Flavour Violation (LFV) and rare K decays. While LFV is forbidden in the renormalisable Standard Model, the rare K decays $K^+ \to \pi^+\nu\overline{\nu}$ and $K_L \to \pi^0\nu\overline{\nu}$ are not, and so precision theoretical predictions are required to distinguish any signal of new physics from the Standard Model background.

We discuss the contribution of the dimension-five lepton number violating Weinberg operator to the running of dimension-six LFV operators in the Standard Model Effective Field Theory. We also consider contributions from a hypothetical second Higgs doublet, which gives rise to Wilson coefficients that cannot all be constrained by small neutrino masses. We then consider bounds on the magnitudes of these Wilson coefficients, by using phenomenological bounds on the low-energy effective theory.

Finally, we discuss the calculation of the rare decays $K^+ \to \pi^+\nu\overline{\nu}$ and $K_L \to \pi^0\nu\overline{\nu}$ to Next-to-Next-to-Leading Order in QCD. These decays are theoretically very clean, providing a promising arena for the detection of new physics. We discuss in detail the matching of box diagrams in the full Standard Model to diagrams of the effective theory below the weak scale, to $\mathcal{O}(\alpha_s^2)$. This involves the evaluation of three-loop integrals, and we provide the Wilson coefficients obtained from the matching.

Contents

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In memory of Grandma, who is sorely missed,
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Chapter 1

Introduction

The Standard Model (SM) of particle interactions describes nearly all phenomena of fundamental particles to an extremely high level of precision, at the highest energies we can currently probe. However, it is known to be an incomplete theory. The SM only describes visible matter, which comprises only a few percent of the contents of the universe [1], and has nothing to say about dark matter or dark energy, which make up the remainder. In addition, the SM is unable to describe gravity, the only one of the fundamental forces not yet to be quantised [2]. The renormalisable SM also does not allow neutrino masses, in contradiction to experimental observations [3]. Even when considering the fundamental properties of known particles, the SM is not infallible. While the agreement between the SM prediction of the electron's magnetic moment and its experimentally measured value is a major success for the SM [4, 5], it fares less well when considering the magnetic moment of the muon. Here there is a significant disagreement between the SM and experiment, which although is not currently at the level of a discovery, only becomes more significant with the passage of time [6, 7]. This discrepancy is all the more interesting since the heavy mass of the muon compared to the electron highly increases the sensitivity of the muon to the effects of new physics from heavy new particles.

The SM is a Quantum Field Theory (QFT), which is defined by its invariance under the symmetry group $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$, and by its particle content. Due to its shortcomings, the SM can be considered to be a low energy effective theory for a more complete theory valid at high energies [8, 9]. From this perspective, the SM can be divided into two sectors, the first containing marginal and relevant operators, and the second containing only irrelevant operators. Marginal and relevant operators may be renormalised to all orders in perturbation theory using a finite number of counterterms. The second sector, comprising of an infinite number of higher-dimension irrelevant operators, can be renormalised order-by-order by a finite number of counterterms. Therefore this theory is fully predictive once an expansion order is specified in the irrelevant operators. Higher-dimension operators are generated in the SM by heavy particles that have masses well above the electroweak scale, and the effect of these operators may be observed by experiments.

Higher-dimension operators generated from heavy particles are suppressed by the mass of the heavy particle. This natural suppression typically makes the signal of effective operators very small when compared to the signal from dimension-four operators. However, experimental searches for processes that are forbidden or highly suppressed when only considering dimension-four operators in the SM are sensitive to the effects of effective operators, and as such are an active field in the search for new physics. Current experiments that aim to probe these processes include LHCb and NA62 at CERN, KOTO at JPARC, Belle II at SuperKEKB, and Mu3e at the Paul Scherrer Institute. The first four of these experiments are probing processes that are highly suppressed within the dimension-four SM, specifically the properties of K and B mesons, while Mu3e is searching for signals of the decay $\mu \to ee\bar{e}$, which is forbidden in the dimension-four SM. These experiments promise to deliver unprecedented sensitivity to new physics, and as such it is necessary to improve theoretical predictions in parallel.

In the leptonic sector of the SM, the lowest-order effective operator is the Weinberg operator [10], which generates Majorana masses for left-handed neutrinos. The Weinberg operator is Lepton Number Violating, and also generates Lepton Flavour Violation through loop processes involving double-insertions of the Weinberg operator [11]. Since neutrinos are known to have non-zero masses, this process is the leading order contribution to Lepton Flavour Violating processes such as $\mu \to ee\bar{e}$, and we calculate this contribution in this thesis. We also consider an extension to this model, by considering the effects arising from the presence of a second Higgs doublet. The mixing of double-insertions of dimension-five operators into dimension-six operators of SMEFT generates contributions to the renormalisation group equations of the dimension-six processes, which we calculate and use to place bounds on the Wilson coefficients of the two Higgs doublet model.

We also calculate the contribution to the rare decays $K \to \pi\nu\bar{\nu}$ from the dimension-four operators of the SM to $\mathcal{O}(\alpha_s^2)$. We only calculate the contribution from box diagrams, but note that the additional penguin diagrams have already been evaluated in a previous work [12]. This precision calculation is necessary to be able to differentiate any contribution to the decay from higher-dimension operators generated by heavy new physics.

This work is organised as follows:

- In Chapter 2, we describe the relevant formalism of QFTs, discussing equations of motion, gauge theories, and quantisation. We also discuss the unbroken SM, which is used heavily in Chapter 5, followed by the Higgs mechanism, and the associated generation of particle masses, and charged- and neutral-currents in the weak sector. This is followed by a discussion of the CKM matrix, which is relevant for Chapter 6. Finally, we discuss Majorana masses and the PMNS matrix, which is relevant for left-handed neutrino masses, discussed in Chapter 5.

- In Chapter 3, we discuss the evaluation of Feynman integrals that arise in loop diagrams. We discuss techniques to reduce integrals to vacuum integrals, the symmetry properties of vacuum integrals, and their evaluation through the work of [13]. We also describe the reduction of integrals to a basis of master integrals using the technique of integration by parts. We then discuss dimensional regularisation and renormalisation in the $\overline{\text{MS}}$ scheme, as well as the extraction of UV divergences using infrared rearrangement. The relation between renormalisation constants and renormalisation group equations is also considered, which is used in Chapters 5 and 6. We finish the chapter by discussing a calculation concerning the decomposition of tensor integrals, providing an exact combinatorial relation for the decomposition of a class of tensor integrals into scalar integrals.

- In Chapter 4, we introduce effective field theories, starting with the decoupling of heavy particles from low-energy processes. This is followed by a categorisation of the types of effective operators that arise in this work, including evanescent and Equation-of-Motion-vanishing operators. We go on to discuss the renormalisation of effective field theories, with a focus on the renormalisation of loop diagrams that mix dimension-five operators into dimension-six operators. We then describe the process of matching in order to determine Wilson coefficients, as well as the importance of threshold corrections. Finally, we give a description of the two effective theories that arise in this work, namely the Standard Model Effective Field Theory (SMEFT), and the formalism of the weak Hamiltonian.

- In Chapter 5, we discuss the mixing of dimension-five operators into the dimension-six SMEFT operators of the Warsaw basis. These diagrams are the leading contribution in the SM to Lepton Flavour Violating processes, and have not previously been evaluated in the Warsaw basis. We also consider the addition of a second Higgs doublet, and how that affects the renormalisation group equations of the dimension-six operators of the Warsaw basis. Once the anomalous dimensions are determined, a phenomenological discussion is given, in which the Warsaw basis Wilson coefficients are matched onto the Wilson coefficients of the low energy effective theory. Experimental bounds on these low-energy coefficients are then converted to bounds on the SMEFT Wilson coefficients, which allow bounds to be placed on the Wilson coefficients of the dimension-five operators. We also discuss a previous calculation [11] that considered the mixing of the Weinberg operator into dimension-six operators of the Buchmuller-Wyler [14] basis, and show that we obtain results in disagreement with this earlier calculation, which we find to be due to a mistake in the projection of their results.

- Chapter 6 gives the contribution of the box diagrams of the dimension-four SM to the process $K \to \pi\nu\overline{\nu}$ to $\mathcal{O}(\alpha_s^2)$. We start with a discussion of the experimental status of searches for these rare decays, and a discussion of theoretical branching ratios for these decays. Since we match the full theory onto the effective theory, we discuss the renormalisation of the effective theory, including the role of a required evanescent operator, before moving on to the matching calculation. After giving details of the matching equation, we give the Wilson coefficients up to $\mathcal{O}(\alpha_s^2)$, and discuss the μ-independence of these coefficients as a check on the calculation.

Chapter 2

Background

Modern particle physics is formulated in terms of Quantum Field Theory (QFT), which is the product of the union of quantum mechanics and special relativity. In writing down Poincaré-invariant theories for quantum mechanical fields, non-linear terms arise which lead to interactions between different particle species. It is these interactions that make QFTs of such interest, and natural candidates for a theory of fundamental particles.

The richness of QFTs is enhanced further when considering their symmetry properties. Many QFTs have a Lagrangian density $\mathcal{L}$ (henceforth referred to as a Lagrangian), which displays a symmetry under some global transformation of the fields. By instead requiring that these global symmetries be respected locally, new gauge fields arise, which interact with the matter content of the theory. Quantum Electrodynamics (QED) was the first quantum gauge theory to be formulated [15, 16], describing the coupling of fermions with photons, and the theory yields theoretical predictions that have been experimentally verified to an astonishing level of precision [4, 5]. Quantum Chromodynamics (QCD), which describes the strong force, is also a gauge theory, but unlike QED has a non-Abelian group structure. Consequently, QCD exhibits interesting new phenomena including gluon self-coupling, which have profound implications for hadronisation.

The Standard Model (SM) [17–19] of particle physics is also a gauge theory, which at energies below the electroweak scale breaks into a simpler group structure containing both QCD and QED. The breaking from a large group structure to a smaller group structure is due to the Higgs mechanism [20–22], where a scalar particle obtains a non-zero vacuum expectation value (VEV). A candidate for the Higgs boson was discovered at the LHC in 2012 [23, 24], and there are large experimental and theoretical efforts to gain a greater understanding of the Higgs sector.

We shall here provide a brief outline of QFTs, their generalisation to gauge theories, and a discussion of the breaking of the SM gauge group via the Higgs mechanism.

2.1 Quantum Field Theory

To describe high energy processes at a fundamental level, it is necessary to unify special relativity and quantum mechanics, which yields a QFT. One way to do this is by taking a classical field theory that is Lorentz invariant, promoting the fields of the Lagrangian to operator-valued fields, and imposing commutation relations upon those fields. The requirements made of the Lagrangian are that it:

- is invariant under Poincaré transformations,
- contains only fields and their derivatives up to first order (as well as coupling constants),
- is renormalisable (has a mass dimension equal to four).

The requirement that a Lagrangian contains derivatives only up to first order ensures that the theory doesn't suffer from problems such as non-locality and spectra that are unbounded from below [25], and the requirement that the mass dimension of the Lagrangian must be equal to four follows from dimensional analysis. The fundamental object of interest is the action S, which contains all of the information of the theory. The action is a dimensionless quantity defined as the integral of the Lagrangian over spacetime,

$$S = \int \mathrm{d}^4x \, \mathcal{L}(\varphi_i, \partial\varphi_i). \tag{2.1}$$

Since the integral measure has mass dimension of minus four, the vanishing dimension of the action requires the Lagrangian to have mass dimension of four. We have here denoted the fields present in the Lagrangian by φ, which can be bosonic or fermionic. For the SM, matter is described in terms of fermions, while interactions are mediated by bosons (gauge vectors and a Higgs scalar). We shall see that these requirements on mass dimension can be relaxed when talking about effective field theories [25].

The equations of motion (EoM) of the classical theory can be found using the variational principle on the action, and are given by the Euler-Lagrange equations,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi_i} = 0. \tag{2.2}$$

2.2 Abelian Gauge Theories

Gauge theories are theories in which the Lagrangian (and therefore action) is invariant under local continuous symmetries. Gauge theories can be constructed by starting with some Lagrangian that displays a global (non-local) continuous symmetry, and then demanding that the theory should be locally invariant under the same symmetry.

Consider the Dirac Lagrangian,

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi, \tag{2.3}$$

where $\psi = \psi(x)$ are fermionic fields with mass m. Under the global continuous U(1) transformations

$$\psi(x) \to e^{i\alpha}\psi(x), \qquad \bar{\psi}(x) \to e^{-i\alpha}\bar{\psi}(x), \tag{2.4}$$

where α is some arbitrary constant, $\mathcal{L}_{\text{Dirac}}$ remains invariant. In order to make this a gauge theory, we demand that the Lagrangian remains invariant under the local U(1) symmetry

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \qquad \bar{\psi}(x) \to e^{-i\alpha(x)}\bar{\psi}(x). \tag{2.5}$$

Transforming the fields in this way causes the Lagrangian to transform as

$$\mathcal{L}_{\text{Dirac}} \to \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + i\bar{\psi}e^{-i\alpha(x)}\gamma^{\mu}\left(\partial_{\mu}e^{i\alpha(x)}\right)\psi \tag{2.6}$$

$$= \mathcal{L}_{\text{Dirac}} - \bar{\psi}\gamma^{\mu}\left(\partial_{\mu}\alpha(x)\right)\psi, \tag{2.7}$$

and so the Dirac Lagrangian is not invariant under the local U(1) transformation. While the mass term is invariant under the gauge transformation, the derivative term is not, and so the insistence of gauge symmetry corresponds to a redefinition of derivatives. The gauge-covariant derivative for QED is

$$D_{\mu}\psi(x) = \partial_{\mu}\psi(x) + ieA_{\mu}(x)\psi(x)\,, \tag{2.8}$$

where $A_{\mu}(x)$ is a gauge field that corresponds to the photon, and e is the QED coupling constant. This gauge field transforms under U(1) transformations as

$$A_{\mu}(x) \to A_{\mu}(x) - \frac{1}{e}\partial_{\mu}\alpha(x). \tag{2.9}$$

Since $A_{\mu}(x)$ is a new field that appears in the Lagrangian, it requires a kinetic term of its own that is built out of $A_{\mu}(x)$ and its derivatives, and is invariant under U(1) transformations. This is given in terms of the field strength tensor $F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, and the kinetic term is

$$\delta\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,.$$

It is seen that promoting a global symmetry of the Dirac Lagrangian to a local symmetry necessarily introduces a gauge field that can be identified as the photon. A Lagrangian for fermions that is Lorentz invariant, gauge invariant under the group U(1), charge- and parity-conserving, and only contains terms of mass-dimension four is

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left(iD\!\!\!\!/ - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^{\mu}\psi A_{\mu}. \tag{2.10}$$

This is the Lagrangian of QED. Imposing a gauge symmetry on a free theory causes to the field of the free theory to interact with a gauge particle.

QED is an Abelian gauge theory, which means that gauge transformations commute, which is clearly the case for U(1) transformations,

$$e^{i\alpha(x)}e^{i\beta(x)} = e^{i\beta(x)}e^{i\alpha(x)}. \tag{2.11}$$

2.3 Non-Abelian Gauge Theories

Lagrangians can also have global symmetries under continuous transformations whose group members do not commute. These are non-Abelian theories. Consider the example of QCD, where there are three copies of every quark, each with a different colour index, r, b or g, and we can construct triplets as

$$\Psi(x) = \begin{pmatrix} \psi_r(x) \\ \psi_b(x) \\ \psi_g(x) \end{pmatrix}. \tag{2.12}$$

Apart from colour, all ψ_i $(i \in \{r, b, g\})$ have the same quantum numbers. The free Lagrangian is again the Dirac Lagrangian, but with one copy for each colour,

$$\begin{aligned} \mathcal{L}_0 &= \bar{\psi}_r \left(i\partial\!\!\!/ - m\right) \psi_r + \bar{\psi}_b \left(i\partial\!\!\!/ - m\right) \psi_b + \bar{\psi}_g \left(i\partial\!\!\!/ - m\right) \psi_g && (2.13) \\ &= \bar{\Psi} \left[\left(i\partial\!\!\!/ - m\right) \mathbb{1}_{3\times 3}\right] \Psi. && (2.14) \end{aligned}$$

The Lagrangian $\mathcal{L}_0$ is invariant under the global SU(3) transformation

$$\Psi(x) \to V\Psi(x) \qquad \text{where} \qquad V = \exp\left(i\alpha^A T^A\right), \tag{2.15}$$

where $T^A = \frac{1}{2}\lambda^A$ are the generators of SU(3) $(A = 1, \ldots, 8)$, and λ^A are the Gell-Mann matrices [26]. To make this a gauge symmetry, the global symmetry is promoted to a local symmetry,

$$\Psi(x) \to V(x)\Psi(x) \qquad \text{where} \qquad V(x) = \exp\left(i\alpha^A(x) T^A\right), \tag{2.16}$$

and a covariant derivative $D_\mu\Psi(x)$ must be constructed that transforms in the same way as the triplet $\Psi(x)$. The covariant derivative for QCD is

$$D_\mu = \partial_\mu + ig_s G^A_\mu(x) T^A\,, \tag{2.17}$$

where g_s is the strong coupling of QCD. Since there are eight generators of SU(3), there are eight gauge fields $G^A_\mu(x)$, called gluons. The gluon field transforms under SU(3) as

$$G^A_\mu(x) \to G^A_\mu(x) - \frac{1}{g_s}\partial_\mu\alpha^A(x) - f^{ABC}G^B_\mu(x)\alpha^C(x), \tag{2.18}$$

where f^{ABC} are the structure constants of QCD.

The kinetic term of the gluon is written in terms of the QCD field strength tensor,

$$G^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - g_s f^{ABC} G^B_\mu G^C_\nu\,, \tag{2.19}$$

where the kinetic term is

$$\delta\mathcal{L} = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu}. \tag{2.20}$$

The QCD field strength tensor $G^A_{\mu\nu}$ contains an extra term compared to the QED tensor $F_{\mu\nu}$, which arises from the commutator of QCD generators. This term does not appear in the QED field strength tensor since the QED generators commute. The generators satisfy the commutation relations

$$\left[T^A, T^B\right] = i f^{ABC} T^C, \tag{2.21}$$

where f^{ABC} are the totally antisymmetric structure constants of SU(3).

Invariants of SU(n) Lie groups are called Casimirs. Two which will appear frequently in Chapter 6 are the quadratic Casimirs C_F and C_A, of the fundamental and adjoint representations respectively. These are given by

$$C_F(n) = \frac{n^2-1}{2n}, \quad \text{and} \quad C_A(n) = n\,, \tag{2.22}$$

which for SU(3) take the values $C_F = 4/3$ and $C_A = n_c = 3$.

A gauge-invariant Lagrangian for a locally SU(3) symmetric theory containing dynamical fermions and gluons can be written as

$$\mathcal{L}_{\text{QCD}} = \bar{\Psi}\left(i\not{D} - m\right)\Psi - \frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu}. \tag{2.23}$$

This is the classical Lagrangian of QCD. Due to the non-Abelian structure of the field strength tensor, this Lagrangian contains gluon cubic and quartic self-interactions, which are not present in Abelian theories such as QED. These self-couplings change the phenomenology of the theories drastically, and lead to confinement and asymptotic freedom in QCD [27, 28].

Since mass terms for gauge fields, such as $\frac{1}{2} m_A A^\mu A_\mu$ are not gauge invariant, such mass terms are forbidden, and gauge particles are massless.[1]

[1]However, if the gauge symmetry is spontaneously broken, then the gauge fields acquire a vacuum expectation value (VEV), and the gauge particles become massive. This happens in the electroweak sector via the Higgs mechanism, discussed in Section 2.5.

2.4 Quantisation of Gauge Theories

The fundamental quantities in QFTs are Green's functions,

$$G^{(n)}(x_1, \ldots, x_n) = \langle 0 | \, T(\varphi(x_1) \ldots \varphi(x_n)) \, |0\rangle \,, \tag{2.24}$$

which can be computed in the path-integral formalism by

$$G^{(n)}(x_1, \ldots, x_n) = \frac{\int \mathcal{D}\varphi \, \varphi(x_1) \ldots \varphi(x_n) e^{iS[\varphi(x), \partial_\mu \varphi(x)]}}{\int \mathcal{D}\varphi \, e^{iS[\varphi(x), \partial_\mu(x)]}} \,, \tag{2.25}$$

where $\mathcal{D}\varphi$ indicates that the integral is over all configurations of the field $\varphi(x)$, and $S\left[\varphi(x), \partial_\mu(x)\right]$ is the classical action for the field $\varphi(x)$.

Introducing the generating functional for a source $j(x)$,

$$Z[j] = N \int \mathcal{D}\varphi \, e^{iS[\varphi(x), \partial_\mu \varphi(x)] + i \int \mathrm{d}^4 x \, j(x) \varphi(x)} \,, \tag{2.26}$$

where N is a normalisation factor, Green's functions can be obtained by

$$G^{(n)}(x_1, \ldots, x_n) = (-i)^n \frac{1}{Z[0]} \frac{\delta^n}{\delta j(x_1) \ldots \delta j(x_n)} Z[j] \bigg|_{j=0} \,. \tag{2.27}$$

The full Green's function contains both connected and disconnected parts. Since only the connected pieces are of interest in scattering amplitudes, they can be isolated by defining

$$Z[j] = e^{iW[j]} \,, \tag{2.28}$$

where $W[j]$ is the generating functional for connected Green's functions,

$$G^{(n)}_{\mathrm{conn}}(x_1, \ldots, x_n) = (-i)^{n-1} \frac{\delta^n}{\delta j(x_1) \ldots \delta j(x_n)} W[j] \bigg|_{j=0} \,. \tag{2.29}$$

A subclass of connected Green's functions are one-particle-irreducible (1PI) Green's functions. When represented as Feynman diagrams, these correspond to diagrams that cannot be cut into two disconnected pieces by cutting a single propagator, and contain the entire quantum structure of the theory, since one-particle-reducible diagrams may be systematically built from 1PI diagrams. 1PI Green's functions may be directly obtained by considering the quantum effective action $\Gamma[\varphi]$, defined as the Legendre transform of $W[j]$,

$$\Gamma[\varphi] = -W[j] - \int \mathrm{d}^4 y \, j(y) \varphi(y) \,, \tag{2.30}$$

where

$$\frac{\delta}{\delta \varphi(x)} \Gamma[\varphi] = -j(x) \,. \tag{2.31}$$

A further class of important Green's functions for low-energy physics are the one-light-particle-irreducible Green's functions. These Green's functions correspond to Feynman diagrams that cannot be cut into two disconnected pieces by cutting a single propagator of a light particle. These Green's functions contain the dynamics of light particles, and contain the physics of effective theories obtained after integrating out heavy particles. This is relevant for Chapter 4.

2.4.1 Faddeev-Popov Quantisation and Ghosts

Naively using the construction of the previous section to calculate Green's functions in gauge theories involves taking the path integral over an infinite number of gauge-equivalent field configurations, leading to unphysical divergences. It is possible to avoid this issue by only integrating over field configurations that are gauge-inequivalent, using the Faddeev-Popov prescription [29]. In this prescription, the generating functional for an arbitrary gauge field in the presence of no sources is given by [30]

$$Z[0] = \prod_a \int \mathcal{D}A^a_\mu \prod_{b=1}^{n} \delta(G_b(A^a_\mu)) \det \left| \frac{\delta G_b}{\delta \alpha_a} \right| e^{iS[A^a_\mu]}, \tag{2.32}$$

where $\alpha_a(x)$ are the gauge parameters of Equation (2.16), and $G_b(A^a_\mu)$ are gauge-fixing terms that are vanishing for certain values of $A^a_\mu(x)$. This fixes the gauge, making sure that the path integral is only taken over gauge-inequivalent field configurations.

The general equation for a gauge-fixed generating functional given in Equation (2.32) contains a determinant of the form $\det |\delta G_b / \delta \alpha_a|$. This determinant can be removed by using the identity

$$\det M = \int \mathcal{D}c\, \mathcal{D}\bar{c}\, e^{i\bar{c}Mc}, \tag{2.33}$$

where c and $\bar{c}$ are anti-commuting Grassmann fields. For a non-Abelian theory such as QCD, the generating functional is [30]

$$Z[j^a_\mu] = \int \prod_{a,b,d} \mathcal{D}A^a_\mu\, \mathcal{D}c^b\, \mathcal{D}\bar{c}^d\, e^{i \int \mathrm{d}^4x \left[\mathcal{L}[A^a_\mu] + j^a_\mu A^\mu_a + \bar{c}^b M_{be} c^e - \frac{1}{2\xi} \sum_b F^2_b(A^a_\mu) \right]}. \tag{2.34}$$

Note that the gauge-fixing condition for non-Abelian theories has been denoted by $F_b(a^a_\mu)$. The quantities

$$M_{be} \equiv \frac{\delta F_b(A^a_\mu)}{\delta \alpha^e} \tag{2.35}$$

are in general functions of the gauge fields A^a_μ, and consequently the generating functional couples the gauge fields with the scalar Grassmann fields c and $\bar{c}$. Since the Grassmann fields have no source terms, they cannot exist as external particles and only couple to gauge fields within closed loops. Due to this behaviour, they are known as Faddeev-Popov ghosts, and they gauge transform under the fundamental representation of the gauge group.

Considering the case of QCD, the gauge-fixing condition can be taken to be $F_b = \partial_\mu G_b^\mu$, which transforms as (using Equation (2.18))

$$\frac{\delta F_b(G_\mu^a)}{\delta \alpha^e} = -\frac{1}{g_s}\partial^2 \delta_{be} - f^{abe}\partial_\mu G_a^\mu. \tag{2.36}$$

Therefore, the ghost-Lagrangian is

$$\mathcal{L} = \bar{c}^b M_{be} c^e = -\bar{c}^b \partial_\mu \left(\delta_{be}\partial^\mu + g_s f^{abe} G_a^\mu\right) c^e, \tag{2.37}$$

where the ghost fields have been rescaled as $\frac{\bar{c}^b c^e}{g_s} \to \bar{c}^b c^e$.

2.5 The Standard Model and the Higgs Mechanism

The Standard Model is a unified description of the electromagnetic, weak, and strong forces. It has a gauge symmetry of $\mathrm{SU(3)_C \times SU(2)_L \times U(1)_Y}$, where the SU(2) symmetry acts on doublets of left-handed fermions, and the charge of the U(1) group is hypercharge, Y. The SM Lagrangian is [8]

$$\begin{aligned}\mathcal{L}_{\mathrm{SM}} = &-\frac{1}{4}G_{\mu\nu}^A G^{A\mu\nu} - \frac{1}{4}W_{\mu\nu}^I W^{I\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) + m^2\varphi^\dagger\varphi - \frac{1}{2}\lambda\left(\varphi^\dagger\varphi\right)^2 \\ &+ i\left(\bar{\ell}\not{D}\ell + \bar{e}\not{D}e + \bar{q}\not{D}q + \bar{u}\not{D}u + \bar{d}\not{D}d\right) - \left(\bar{\ell}\Gamma_e e\varphi + \bar{q}\Gamma_u u\tilde{\varphi} + \bar{q}\Gamma_d d\varphi + \text{ H.c.}\right),\end{aligned} \tag{2.38}$$

where the Yukawa couplings $\Gamma_i,\ i \in \{e, u, d\}$ are matrices in generation space. The matter content (with group representations and charges listed in Table 2.1) is

- left-handed lepton doublets ℓ,
- right-handed lepton singlets e (no sterile right-handed neutrinos are included in the SM, since they are unobservable),
- left-handed quark doublets q,
- right-handed up-type quark singlets u,
- right-handed down-type quarks d,

and φ denotes the Higgs doublet with hypercharge $Y_\varphi = \frac{1}{2}$. The quadratic Higgs term is a mass term, but with the "wrong" sign, which leads to spontaneous symmetry breaking, discussed below. $\tilde{\varphi}$ is defined by

$$\tilde{\varphi}^i \equiv \varepsilon_{ij}(\varphi^*)^j, \tag{2.39}$$

where i and j are SU(2) indices of the fundamental representation, and ε_{ij} is the antisymmetric Levi-Civita symbol, with $\varepsilon_{12} = +1$. With this definition, $\tilde{\varphi}$ transforms under the fundamental representation of SU(2), and has hypercharge $Y_{\tilde{\varphi}} = -\frac{1}{2}$. $G_{\mu\nu}^A, W_{\mu\nu}^I$ and

Field	$\mathrm{SU}(3)_\mathrm{c}$	$\mathrm{SU}(2)_\mathrm{L}$	Y
ℓ	-	2	$-1/2$
e	-	1	-1
q	3	2	$1/6$
u	3	1	$2/3$
d	3	1	$-1/3$
φ	-	2	$1/2$

Table 2.1: Fermions and Higgs doublet of the SM, with their hypercharges Y, and their representation under the constituent gauge groups of the SM.

$B_{\mu\nu}$ are the field strength tensors of the gauge groups $\mathrm{SU}(3)_\mathrm{C}, \mathrm{SU}(2)_\mathrm{L}$, and $\mathrm{U}(1)_\mathrm{Y}$ respectively, given by

$$\begin{aligned} G^A_{\mu\nu} &= \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - g_s f^{ABC} G^B_\mu G^C_\nu, \\ W^I_{\mu\nu} &= \partial_\mu W^I_\nu - \partial_\nu W^I_\mu - g_2 \epsilon^{IJK} W^J_\mu W^K_\nu, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \end{aligned} \tag{2.40}$$

where f^{ABC} and ϵ^{IJK} are structure constants, and g_2 is the SU(2) coupling constant. The sign conventions for covariant derivatives are illustrated by the covariant derivative of q, which is charged under the entire SM gauge group:

$$D_\mu q = \left(\partial_\mu + i g_s T^A G^A_\mu + i g_2 S^I W^I_\mu + i g_1 Y_q B_\mu\right) q, \tag{2.41}$$

where T^A and S^I are respectively the generators of $\mathrm{SU}(3)_\mathrm{C}$ and $\mathrm{SU}(2)_\mathrm{L}$, and g_1 is the $\mathrm{U}(1)_\mathrm{Y}$ coupling.

It is important to note that $\mathcal{L}_\mathrm{SM}$ contains no mass terms, and so all fields (before electroweak symmetry breaking) are massless.

The $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ gauge group of the SM is broken by the Higgs doublet φ acquiring a VEV. The relevant part of the Lagrangian is

$$\mathcal{L}_\mathrm{SM}\Big|_{\mathrm{SU}(2)_\mathrm{L}\times\mathrm{U}(1)_\mathrm{Y}} = -\frac{1}{4}\left(W^I_{\mu\nu}\right)^2 - \frac{1}{4}\left(B_{\mu\nu}\right)^2 + \left(D_\mu\varphi\right)^\dagger\left(D^\mu\varphi\right) + m^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2, \tag{2.42}$$

which is invariant under gauge transformations

$$\varphi \to e^{i\alpha^I(x) S^I} e^{i\beta(x) Y_\varphi}\varphi. \tag{2.43}$$

The Higgs doublet is a weak isospin doublet of complex scalar fields

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}, \tag{2.44}$$

which has a potential $V(\varphi) = -m^2\varphi^\dagger\varphi + \lambda\left(\varphi^\dagger\varphi\right)^2$. The potential is minimised for

$$|\varphi|^2 = \frac{m^2}{2\lambda}, \tag{2.45}$$

which means φ acquires a VEV. Since Equation (2.45) only specifies the magnitude of the Higgs doublet that minimises the potential, the VEV of the Higgs doublet can be freely chosen to be

$$\langle\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}, \tag{2.46}$$

where $v = \sqrt{m^2/\lambda}$ is real. Fluctuations around this minimum (which can be treated perturbatively) can then be parameterised by a scalar field $h(x)$, called the Higgs field, allowing the Higgs doublet to be written as

$$\varphi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+h(x)\end{pmatrix}. \tag{2.47}$$

This choice is called the unitary gauge.

When the Higgs doublet acquires a non-zero VEV, it gives masses to otherwise massless gauge bosons and fermions of the unbroken SM Lagrangian, given in Equation (2.38). For the gauge bosons, the relevant part of the Lagrangian is in the covariant derivative of the Higgs doublet, and upon inserting the VEV of the Higgs doublet this becomes

$$\delta\mathcal{L} = \frac{1}{2}\begin{pmatrix}0 & v\end{pmatrix}\left[g_2 W^I_\mu S^I + g_1 Y_\varphi B_\mu\right]\left[g_2 W^{J\mu} S^J + g_1 Y_\varphi B^\mu\right]\begin{pmatrix}0\\ v\end{pmatrix}. \tag{2.48}$$

Explicitly entering the SU(2) generators as $S^I = \tau^I/2$, where τ^I are the Pauli matrices,

$$\tau^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \tau^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \tau^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{2.49}$$

this part of the Lagrangian becomes

$$\delta\mathcal{L} = \frac{v^2}{8}\left[g_2^2\left(\left(W^1_\mu\right)^2 + \left(W^2_\mu\right)^2\right) + \left(-g_2 W^3_\mu + g_1 B_\mu\right)^2\right]. \tag{2.50}$$

Consequently the fields W^1_μ and W^2_μ acquire a mass of $M_W = g_2 v/2$. The final term involving W^3_μ and B_μ can be rewritten as

$$\begin{aligned}\left(-g_2 W^3_\mu + g_1 B_\mu\right)^2 &= \begin{pmatrix}W^3_\mu & B_\mu\end{pmatrix}\begin{pmatrix}g_2^2 & -g_1 g_2\\ -g_1 g_2 & g_1^2\end{pmatrix}\begin{pmatrix}W^3_\mu\\ B_\mu\end{pmatrix}\\ &= \begin{pmatrix}W^3_\mu & B_\mu\end{pmatrix}\mathbf{M}\begin{pmatrix}W^3_\mu\\ B_\mu\end{pmatrix},\end{aligned} \tag{2.51}$$

where $\mathbf{M}$ is a mass matrix. To extract the mass eigenstates and their masses, it is necessary to diagonalise the system, by finding the eigenvalues and eigenvectors of $\mathbf{M}$. The eigenvalues are

$$\lambda_1 = 0, \qquad \lambda_2 = g_1^2 + g_2^2, \tag{2.52}$$

with eigenvectors

$$\vec{x}_1 = \frac{1}{\sqrt{g_1^2+g_2^2}}\begin{pmatrix}g_1\\ g_2\end{pmatrix} \quad \text{and} \quad \vec{x}_2 = \frac{1}{\sqrt{g_1^2+g_2^2}}\begin{pmatrix}-g_2\\ g_1\end{pmatrix} \tag{2.53}$$

respectively. The eigenvectors are normalised to respect the canonical normalisation of the W_μ^I and B_μ kinetic terms. Therefore in the propagating mass basis there are two bosons

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \frac{1}{\sqrt{g_1^2 + g_2^2}} \begin{pmatrix} g_1 & -g_2 \\ g_2 & g_1 \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}, \tag{2.54}$$

with masses

$$M_A = 0, \qquad M_Z = \frac{v}{2}\sqrt{g_1^2 + g_2^2}. \tag{2.55}$$

The leptonic gauge sector of the SM Lagrangian contains the terms

$$\begin{aligned}
\sum_r i\bar{\ell}_r \not{D} \ell_r \supset & -g_2 \sum_r \bar{\ell}_r \gamma^\mu \left(S^1 W_\mu^1 + S^2 W_\mu^2\right) P_L \ell_r \\
&= -\frac{g_2}{2} \sum_r \begin{pmatrix} \overline{\nu_{Lr}} & \overline{e_{Lr}} \end{pmatrix} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W_\mu^1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} W_\mu^2 \right] \gamma^\mu P_L \begin{pmatrix} \nu_{Lr} \\ e_{Lr} \end{pmatrix} \\
&= -\frac{g_2}{2} \sum_r \left[\overline{\nu_{Lr}} \left(W_\mu^1 - iW_\mu^2\right) \gamma^\mu P_L e_{Lr} + \overline{e_L} r \left(W_\mu^1 + iW_\mu^2\right) \gamma^\mu P_L \nu_{Lr} \right] \\
&\equiv -\frac{g_2}{\sqrt{2}} \sum_r \left[\overline{\nu_{Lr}} W_\mu^+ \gamma^\mu P_L e_{Lr} + \overline{e_L} r W_\mu^- \gamma^\mu P_L \nu_{Lr} \right],
\end{aligned} \tag{2.56}$$

where the fields $W_\mu^\pm(x)$ are defined as

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \mp iW_\mu^2\right). \tag{2.57}$$

These are the charged mediators of the weak force, which have mass $M_{W^\pm} = M_W$, since they are a linear combination of the fields W_μ^1 and W_μ^2.

To summarise the $\mathrm{SU(2)_L \times U(1)_Y}$ sector of the SM, the fields W_μ^1 and W_μ^2 mediate charged-current interactions due to the non-diagonal structure of the SU(2) generators S^1 and S^2, mixing to form fields of electric charge $W_\mu^\pm$. The W_μ^3 and B_μ mediate neutral-current electroweak interactions due to the diagonal structure of S^3 and $Y\mathbb{1}_{2\times 2}$, and mix to give the Z-boson and the photon. After electroweak symmetry breaking (EWSB), the Higgs mechanism gives mass to the $W_\mu^\pm$ and Z_μ fields, while leaving the photon A_μ massless.

Fermions also gain mass from the Higgs mechanism. Dirac mass terms are given by

$$m\bar{\psi}\psi = m\left(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L\right), \tag{2.58}$$

where

$$\psi_{L/R} = P_{L/R}\psi\,, \qquad \text{and} \qquad \overline{\psi_{L/R}} = \overline{\psi} P_{R/L}\,. \tag{2.59}$$

P_L and P_R are chirality projectors defined by

$$P_L = \frac{1-\gamma_5}{2}\,, \qquad P_R = \frac{1+\gamma_5}{2}\,. \tag{2.60}$$

Dirac mass terms cannot be written in the SM Lagrangian since the left- and right-handed fields transform under different representations of SU(2), and as such Dirac mass terms are not gauge invariant. However, it is instead possible to write Yukawa terms,

$$\mathcal{L}_{\text{SM,Yuk}} = -\left(\bar{\ell}\Gamma_e e\varphi + \bar{q}\Gamma_u u\tilde{\varphi} + \bar{q}\Gamma_d d\varphi + \text{ H.c.}\right) \tag{2.61}$$

which are gauge invariant and couple left- and right-handed fermions. Here, Γ_e, Γ_u, and Γ_d are respectively the Yukawa matrices for charged leptons, up-type quarks, and down-type quarks. Upon EWSB, the Higgs doublet can be replaced by its VEV to yield

$$\mathcal{L}_{\text{SM,Yuk}} = -\left(\frac{v}{\sqrt{2}}\overline{e_L}\Gamma_e e_R + \frac{v}{\sqrt{2}}\overline{u_L}\Gamma_u u_R + \frac{v}{\sqrt{2}}\overline{d_L}\Gamma_d d_R + \text{H.c.}\right), \tag{2.62}$$

resulting in the generation of Dirac mass terms for the fermions, proportional to the VEV of the Higgs doublet.

The SM is described by a QFT with quarks and leptons as the matter content, with an $\text{SU}(3)_\text{C} \times \text{SU}(2)_\text{L} \times \text{U}(1)_\text{Y}$ gauge symmetry that is spontaneously broken by the Higgs doublet acquiring a VEV. This spontaneous symmetry breaking produces a new physical massive Higgs boson, gives mass to the SU(2) gauge bosons, and gives rise to weak charged-current processes.

2.6 The CKM Matrix

The quark Yukawa sector of the unbroken SM is

$$\mathcal{L}_{\text{Yuk,q}} = -\bar{q}_p\Gamma^u_{pr}\tilde{\varphi}u_r - \bar{q}_p\Gamma^d_{pr}\varphi d_r + \text{ H.c.}, \tag{2.63}$$

which after EWSB becomes

$$\mathcal{L}_{\text{mass,q}} = -\frac{v}{\sqrt{2}}\left(\bar{d}^p_L\Gamma^d_{pr}d^r_R + \bar{u}^p_L\Gamma^u_{pr}u^r_R\right) + \text{ H.c.}, \tag{2.64}$$

where p, r are generation indices and the subscripts L/R denote a left-/right-handed Dirac fermion ψ. The Yukawa matrices Γ are non-diagonal, implying that the flavour states $u^p, d^p \ldots$ appearing in the SM Lagrangian are not propagating particles. To find the propagating states, it is necessary to diagonalise the mass terms into the canonical form. This requires the diagonalisation of the Yukawa matrices.

Yukawa matrices are non-Hermitian complex 3×3 matrices in flavour space. The products $\Gamma_u\Gamma_u^\dagger$ and $\Gamma_u^\dagger\Gamma_u$ are Hermitian, with equivalent Hermitian products for the down-type Yukawa matrix. These combinations can be diagonalised by unitary matrices as

$$\Gamma_u\Gamma_u^\dagger = U_u M_u^2 U_u^\dagger, \qquad \Gamma_u^\dagger\Gamma_u = K_u M_u^2 K_u^\dagger, \tag{2.65}$$

where M_u^2 is a diagonal matrix with real eigenvalues. An equivalent set of expressions holds for d-type Yukawa matrices. Both of the above relations are satisfied by the decompositions

$$\Gamma_u = U_u M_u K_u^\dagger \qquad \text{and} \qquad \Gamma_d = U_d M_d K_d^\dagger. \tag{2.66}$$

Inserting these decompositions into the quark mass Lagrangian gives

$$\mathcal{L}_{\text{mass,q}} = -\frac{v}{\sqrt{2}}\left[\bar{d}_L^p \left(U_d M_d K_d^\dagger\right)_{pr} d_R^r + \bar{u}_L^p \left(U_u M_u K_u^\dagger\right)_{pr} u_R^r\right] + \text{H.c.} \tag{2.67}$$

Rotating the right- and left-handed quarks independently in generation space as

$$d_R^p \to K_{pr}^d d_R^r, \qquad u_R^p \to K_{pr}^u u_R^r \tag{2.68}$$

and

$$d_L^p \to U_{pr}^d d_L^r, \qquad u_L^p \to U_{pr}^u u_L^r, \tag{2.69}$$

defines the quarks in the mass basis, in which the mass Lagrangian is diagonalised,

$$\mathcal{L}_{\text{mass,q}} = -\frac{v}{\sqrt{2}}\left[\bar{d}_L^p M_{pr}^d d_R^r + \bar{u}_L^p M_{pr}^u u_R^r\right] + \text{H.c.} \tag{2.70}$$

Applying this flavour rotation to the entire quark Lagrangian leaves the Lagrangian mostly unchanged, since terms that do not mix up- and down-type quarks are diagonal in flavour space. However, mixing does occur in the $i\bar{q}\not{D}q$ term,

$$\begin{aligned} i\bar{q}^p \not{D} q^p &\supset -\bar{q}^p \left(g_1 Y_q \not{B} \mathbb{1}_{2\times 2} + g_2 \not{W}^I S^I\right) q^p \\ &= \begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix}^p \begin{pmatrix} g_1 Y_q \not{B} + \frac{g_2}{2}\not{W}^3 & \frac{g_2}{\sqrt{2}}\not{W}^+ \\ \frac{g_2}{\sqrt{2}}\not{W}^- & g_1 Y_q \not{B} - \frac{g_2}{2}\not{W}^3 \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix}^p \\ &\supset \frac{g_2}{\sqrt{2}} \bar{u}_L^p \not{W}^+ d_L^p + \frac{g_2}{\sqrt{2}} \bar{d}_L^p \not{W}^- u_L^p. \end{aligned} \tag{2.71}$$

Rotating to the mass basis using Equation (2.69), these terms transform to

$$\begin{aligned} \frac{g_2}{\sqrt{2}}\left(\bar{u}_L^p \not{W}^+ d_L^p + \bar{d}_L^p \not{W}^- u_L^p\right) &\to \frac{g_2}{\sqrt{2}}\left(\bar{u}_L^m U_{mp}^{u\dagger} \not{W}^+ U_{pn}^d d_L^n + \bar{d}_L^m U_{mp}^{d\dagger} \not{W}^- U_{pn}^u u_L^n\right) \\ &= \frac{g_2}{\sqrt{2}}\left(\bar{u}_L^m \not{W}^+ V_{mn} d_L^n + \bar{d}_L^m \not{W}^- V_{mn}^\dagger u_L^n\right), \end{aligned} \tag{2.72}$$

where the unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix V is defined as

$$V \equiv U_u^\dagger U_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{2.73}$$

A general 3×3 complex matrix has 18 real parameters. Unitarity provides nine constraints, which can be parameterised by three angles and six phases. However, since the mass terms of the Lagrangian are invariant under the rephasing of the quark fields ($\bar{u}_L^m \to \bar{u}_L^p D_{pm}^{u\dagger}$, $d_R^n \to D_{nr}^d d_L^r$, where D^u and D^d are diagonal matrices of phases), the quark fields can also be rephased in the charged-current sector of the Lagrangian. This transforms the charged-current term to

$$\frac{g_2}{\sqrt{2}}\left(\bar{u}_L^m \not{W}^+ V_{mn} d_L^n + \text{H.c.}\right) \to \frac{g_2}{\sqrt{2}}\left(\bar{u}_L^p \not{W}^+ D_{pm}^{u\dagger} V_{mn} D_{nr}^d d_L^r + \text{H.c.}\right), \tag{2.74}$$

where

$$D^{u\dagger} V D^d = \begin{pmatrix} D_{11}^{u*} D_{11}^d V_{11} & D_{11}^{u*} D_{22}^d V_{12} & D_{11}^{u*} D_{33}^d V_{13} \\ D_{22}^{u*} D_{11}^d V_{21} & D_{22}^{u*} D_{22}^d V_{22} & D_{22}^{u*} D_{33}^d V_{23} \\ D_{33}^{u*} D_{11}^d V_{31} & D_{33}^{u*} D_{22}^d V_{32} & D_{33}^{u*} D_{33}^d V_{33} \end{pmatrix}. \tag{2.75}$$

Therefore, there are six quark phases that may be chosen in order to eliminate the six phases of the CKM matrix. However, if all of the rotations are the same, then the phase shifts will cancel and the CKM matrix will not be changed. Therefore, it is only possible to eliminate five phases from the CKM matrix by quark field redefinitions [31].

Since the free parameters of the CKM matrix may be expressed as three angles and a phase, the standard parameterisation of the CKM matrix is given by the product of three rotation matrices in orthogonal planes, with a phase. It is given by [32]

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{2.76}$$

where $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$, and δ is a phase which is responsible for all CP violation in the SM.

2.7 Majorana Masses and the PMNS Matrix

In the SM it is not possible to write down a mass term for neutrinos at dimension-four. However, Majorana masses for left-handed neutrinos can be generated from the dimension-five Weinberg operator [10]. In this section we discuss Dirac and Majorana masses in general, which will be useful for Chapter 5.

A 4-component Dirac spinor can be represented in the Weyl basis as

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{2.77}$$

where ψ_L and ψ_R are 2-component Weyl spinors that transform under Lorentz transformations as

$$\begin{aligned} \psi_L &\to \exp\left[\frac{1}{2}\left(i\theta_j\tau^j - \beta_j\tau^j\right)\right]\psi_L \\ \psi_R &\to \exp\left[\frac{1}{2}\left(i\theta_j\tau^j + \beta_j\tau^j\right)\right]\psi_R, \end{aligned} \tag{2.78}$$

where τ^j are the Pauli matrices. A Lorentz invariant mass term can therefore be written as (working in the Weyl basis of the gamma matrices)

$$m\overline{\Psi}\Psi = m\left(\psi_L^\dagger \psi_R^\dagger\right)\gamma^0 \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = m\left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L\right). \tag{2.79}$$

Lorentz invariance is confirmed infinitesimally since

$$\begin{aligned}\delta\left(\psi_L^\dagger \psi_R\right) &= \delta\left(\psi_L^\dagger\right)\psi_R + \psi_L^\dagger \delta\left(\psi_R\right) \\ &= \psi_L^\dagger\left(\frac{1}{2}\left(-i\theta_j\tau^j - \beta_j\tau^j\right)\right)\psi_R + \psi_L^\dagger\left(\frac{1}{2}\left(i\theta_j\tau^j + \beta_j\tau^j\right)\right)\psi_R = 0. \end{aligned}\tag{2.80}$$

The Dirac mass term is also often written in terms of chiral Dirac spinors as

$$m\overline{\Psi}\Psi = m\left(\overline{\Psi_L}\Psi_R + \overline{\Psi_R}\Psi_L\right), \tag{2.81}$$

where $\Psi_{L/R} = P_{L/R}\Psi$. Yukawa interactions in the SM produce Dirac masses after EWSB.

Another Lorentz invariant mass term can be written as

$$iM\psi_L^T \tau_2 \psi_L, \tag{2.82}$$

with another similar term in terms of right-handed Weyl spinors. This is a Majorana mass term. This mass term can be seen to be Lorentz invariant in the same way as the above Dirac mass, additionally using the relation $\tau_j^T \tau_2 = -\tau_2\tau_j$.

It is possible to write a Majorana mass term in terms of Dirac fermions. First it is necessary to define the *charge conjugate* Ψ^c of a fermion field Ψ, where Ψ and Ψ^c have opposite quantum numbers, but equal mass. Ψ^c is defined as [33],

$$\Psi^c \equiv C\left(\overline{\Psi}\right)^T = C\Psi^*, \tag{2.83}$$

where C is the charge conjugation matrix, $C = -i\gamma_2$ in the Weyl basis. C satisfies the relations [33]

$$C^\dagger = C^{-1}, \qquad C^T = -C, \qquad C\Gamma_i^T C^{-1} = \eta_i \Gamma_i, \tag{2.84}$$

where there is no summation over i, and

$$\eta_i = \begin{cases} 1 & \text{for } \Gamma_i = 1, i\gamma_5, \gamma_\mu\gamma_5 \\ -1 & \text{for } \Gamma_i = \gamma_\mu, \sigma_{\mu\nu}. \end{cases} \tag{2.85}$$

These relations imply that

$$\overline{(\Psi_L)^c} = (\Psi_L)^{c\dagger}\gamma^0 = \left(-\gamma^0 C\Psi_L^*\right)^\dagger \gamma^0 = -\Psi_L^T C^{-1}. \tag{2.86}$$

It is useful to write Ψ_L^c explicitly in terms of Weyl fermions:

$$\Psi_L^c = (\Psi_L)^c = -i\gamma_2\Psi_L^* = -i\begin{pmatrix}0 & \tau_2 \\ -\tau_2 & 0\end{pmatrix}\begin{pmatrix}\psi_L^* \\ 0\end{pmatrix} = \begin{pmatrix}0 \\ i\tau_2\psi_L^*\end{pmatrix}. \tag{2.87}$$

Similarly, using Equation (2.86),

$$\overline{\Psi_L^c} = \left[(\Psi_L)^c\right]^\dagger \gamma^0 = \begin{pmatrix} -i\psi_L^T\tau_2 & 0 \end{pmatrix}. \tag{2.88}$$

The Lagrangian term (written in terms of Dirac spinors)

$$-M\overline{\Psi_L^c}\Psi_L, \tag{2.89}$$

can be seen to be a Majorana mass term since

$$-M\overline{\Psi_L^c}\Psi_L = -M\begin{pmatrix} -i\psi_L^T\tau_2 & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ 0 \end{pmatrix} = iM\psi_L^T\tau_2\psi_L, \tag{2.90}$$

which is exactly the Lorentz invariant Majorana mass term of Equation (2.82).

The SM does not include right-handed neutrinos, since they are completely uncharged and therefore impossible to observe directly. Additionally, although the renormalisable SM does not contain mass terms for left-handed neutrinos, experimental observation of neutrino oscillations imply that the left-handed neutrinos must be massive, with the mass eigenvalues being non-degenerate [3]. Therefore, a Lagrangian that describes massive left-handed neutrinos without right-handed neutrinos after EWSB is

$$\mathcal{L} = i\bar{\ell}_p \not{D}\ell_p - \frac{v}{\sqrt{2}}\overline{e_L}_p\Gamma_e^{pr}e_{R,r} - \frac{1}{2}M_\nu^{pr}\overline{\nu_L^c}_p\nu_{L,r}, \tag{2.91}$$

where Γ_e is the lepton Yukawa matrix that gives masses to the charged leptons, and M_ν is the complex symmetric Majorana mass matrix for left-handed neutrinos [34]. Note that we have conventionally normalised the Majorana mass term by a factor of 1/2. As for the CKM matrix, the lepton Yukawa matrix can be diagonalised by a bi-unitary transformation,

$$\Gamma_e = U_e M_e K_e^\dagger, \tag{2.92}$$

where M_e is a diagonal matrix of masses, and the Lagrangian can be transformed into the charged-lepton mass basis by rotating in generation space as

$$\begin{aligned} e_{R,r} &\to K_e^{rm}e_{R,m} \\ \left.\begin{aligned} e_{L,p} &\to U_e^{pn}e_{L,n} \\ \nu_{L,p} &\to U_e^{pn}\nu_{L,n} \end{aligned}\right\} &\implies \ell_p \to U_e^{pn}\ell_n, \end{aligned} \tag{2.93}$$

where the individual rotations of the left-handed charged leptons and left-handed neutrinos imply that the left-handed doublet ℓ transforms as a single entity. Then, the Lagrangian becomes (including only the charged-current part of the covariant derivative, since this is where effects of generation rotations are manifest)

$$\mathcal{L} = -\frac{g_2}{\sqrt{2}}\left(\overline{\nu_L}_p\gamma^\mu e_{L,p}W_\mu^+ + \overline{e_L}_p\gamma^\mu\nu_{L,p}W_\mu^-\right) - \frac{vM_e^p}{\sqrt{2}}\overline{e_L}_p e_{R,p} - \frac{1}{2}\overline{\nu_L^c}_p C_\nu^{pr}\nu_{L,n}, \tag{2.94}$$

where

$$C_\nu \equiv U_e^T M_\nu U_e. \tag{2.95}$$

Since the Majorana mass matrix M_ν is complex symmetric and the matrix U_e is unitary, the matrix C_ν is also complex symmetric (equal to its transpose). C_ν can therefore be diagonalised using a congruence transformation [35],

$$D_\nu = U_{\mathrm{PMNS}}^T C_\nu U_{\mathrm{PMNS}}, \tag{2.96}$$

where D_ν is a diagonal matrix of neutrino masses and U_{PMNS} is a unitary matrix. Rotating the neutrinos in generation space as

$$\nu_{L,p} \to U_{\mathrm{PMNS}}^{pr} \nu_{L,r} \tag{2.97}$$

brings the lepton Lagrangian fully into the mass basis,

$$\mathcal{L} = -\frac{g_2}{\sqrt{2}} \left(\overline{\nu_{L}}_r U_{\mathrm{PMNS}}^{\dagger rp} \gamma^\mu e_{L,p} W_\mu^+ + \overline{e_L}_p \gamma^\mu U_{\mathrm{PMNS}}^{pr} \nu_{L,r} W_\mu^- \right) - \frac{v M_e^p}{\sqrt{2}} \overline{e_L}_p e_{R,p} - \frac{1}{2} \overline{\nu_L^c}_p D_\nu^p \nu_{L,p}. \tag{2.98}$$

The Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix U_{PMNS} is the neutrino equivalent of the CKM matrix, and relates flavour eigenstates to mass eigenstates. Since it is a unitary 3×3 matrix, it has in general nine degrees of freedom, like the CKM matrix. However, since Majorana mass terms are not invariant under the rephasing of fermion fields, there are only three degrees of freedom that may be removed from the PMNS matrix by field rephasings. Therefore the six degrees of freedom of the PMNS matrix may be parameterised by three angles and three phases analogously to the CKM matrix [32],

$$U_{\mathrm{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{pmatrix},$$

where α_{21} and α_{31} are called Majorana phases. If ignoring the Majorana phases, then the PMNS appears identical to the CKM matrix. However, the experimentally determined angles of the CKM matrix and PMNS matrix are very different [32], leading to different mixing behaviour in the quark and lepton sectors. Whereas the CKM matrix is nearly diagonal, the PMNS matrix has large non-diagonal terms, allowing considerable inter-generational mixing. It is not possible to measure the Majorana phases in neutrino oscillation experiments, since the phases cancel in the transition amplitude between neutrinos of different flavour [36], and consequently the Majorana phases are currently unknown.

Chapter 3

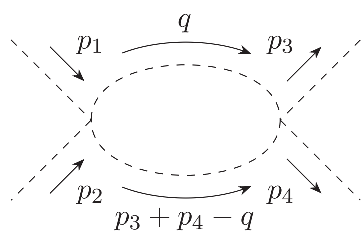
Renormalisation

Calculations in quantum field theories contain unphysical ultraviolet (UV) divergences, which suggest that all physical quantities should be infinite, which is physically nonsensical. This feature plagued the early years of QED, and was only resolved by the efforts of Tomonaga, Schwinger, and Feynman, for which they were awarded the Nobel Prize for physics in 1965 [37]. In perturbation theory, UV divergences appear when evaluating loop diagrams, which encode the quantum effects of the theory. Loop diagrams are typically divergent since they include virtual particles with indeterminate momenta, and so the diagrams generate momentum integrals that formally extend to infinite momenta. To extract physically meaningful finite quantities from divergent integrals, it is first necessary to parameterise the divergences, a process known as regularisation. Once the integrals have been regularised, the large-momentum divergences are systematically subtracted in a procedure known as renormalisation.

3.1 Feynman Integrals

Perturbation theory relies on the evaluation of Feynman diagrams, which can contain loops. These loops involve the exchange of virtual particles with unspecified 'loop' momenta, and consequently it is necessary to integrate the expressions from loop diagrams over all such loop momenta. Since loop momenta enter expressions in the propagators of the virtual particles, the result of a loop diagram contains a Feynman integral, whose integrand is a function of those propagators.

For example, in φ^4 theory, there exists the diagram



$$= \frac{\lambda^2}{2} \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi^4)} \frac{1}{q^2 - m^2 + i\epsilon} \frac{1}{(p_3 + p_4 - q)^2 - m^2 + i\epsilon}, \tag{3.1}$$

which contains an integral over all possible values of the loop momentum q. This integral can be formally evaluated by performing a Wick rotation to Euclidean momenta (i.e. by taking $q^0 \to iq^0_E$), and then transforming to spherical coordinates such that the loop momentum can take values between zero and infinity. The expression thus becomes

$$\frac{\lambda^2}{2} \frac{i}{(2\pi)^2} \int_0^{\infty} \mathrm{d}q_E \frac{q_E^3}{(q_E^2 + m^2)((p_{3,E} + p_{4,E} - q_E)^2 + m^2)}, \tag{3.2}$$

which is infrared (IR) safe, but in the UV limit $q_E \to \infty$

$$\frac{\lambda^2}{2} \frac{i}{(2\pi)^2} \int_0^{\infty} \mathrm{d}q_E \frac{q_E^3}{(q_E^2 + m^2)((p_{3,E} + p_{4,E} - q_E)^2 + m^2)} \xrightarrow{q_E \to \infty} \frac{\lambda^2}{2} \frac{i}{(2\pi)^2} \int \frac{\mathrm{d}q_E}{q_E}, \tag{3.3}$$

which is logarithmically divergent.

Feynman integrals can have a rich structure, and analysis of such integrals is a major area of research. With each loop that a diagram contains, an additional integral is required, and the presence of fermions as virtual particles gives Feynman integrals a tensor structure. However, all Feynman integrals are functions of the masses and momenta of the virtual particles of the diagram.

A general 1-loop Feynman integral may be written as (where from now on $i\epsilon$ terms that specify the contour prescription for propagators are omitted)

$$\mathcal{I}(q; k_1, \ldots, k_n; m_1, \ldots, m_n) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{f(q)}{\prod D(q, k_i, m_i)}, \tag{3.4}$$

where q is the loop momentum, k_i are external momenta, and m_i are the propagator masses. $\mathcal{I}$ and f can be Lorentz tensors, and propagators are denoted by

$$D(q, k_i, m_i) = \left(q - \sum_i k_i\right)^2 - m_i^2. \tag{3.5}$$

Feynman integrals can be simplified in a number of ways. The integrand may be Taylor expanded in external momenta [38], which are typically small relative to the masses of the internal particles in weak processes. This removes external momenta from the integrand, resulting in integrals that only depend on propagator masses and the loop momentum. Such integrals are called vacuum integrals. A 1-loop vacuum integral may then be written as

$$\mathcal{I}_{\text{vac}}(q; m_1, \ldots, m_n; \nu_1, \ldots, \nu_n) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{f(q^\mu, q^2, q^2 q^\mu, \ldots)}{(q^2 - m_1^2)^{\nu_1}(q^2 - m_2)^{\nu_2} \ldots (q^2 - m_n^2)^{\nu_n}}, \tag{3.6}$$

where ν_i denote the powers that each propagator is raised to. Integrals of this form may be simplified further simply by using partial fraction decomposition. For example,

$$\frac{1}{(q^2 - m_1^2)(q^2 - m_2^2)} = \frac{1}{m_1^2 - m_2^2} \frac{1}{q^2 - m_1^2} + \frac{1}{m_2^2 - m_1^2} \frac{1}{q^2 - m_2^2}, \tag{3.7}$$

and similarly, any integral of the form of Equation (3.6) may be reduced to a sum of 1-loop vacuum (tensor) integrals, each containing only a single mass. Once this has been done, the tensor integrals may be decomposed into a sum of scalar integrals, using relations such as

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{q^\mu}{(q^2 - m^2)^\nu} = 0 \tag{3.8}$$

and

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{q^\mu q^\nu}{(q^2 - m^2)^\nu} = \frac{g^{\mu\nu}}{4} \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{q^2}{(q^2 - m^2)^\nu}. \tag{3.9}$$

The first of these identities follows from integrating an odd function over an even domain, and holds for any odd power of q^μ in the numerator. The second follows from the fact that a Feynman integral must be Lorentz invariant, and therefore a rank-2 tensor integral must be proportional to $g^{\mu\nu}$, which is the only rank-2 Lorentz invariant tensor. The constant of proportionality is found by contracting both sides with $g_{\mu\nu}$. Similar relations hold for higher-rank tensor integrals.

Once tensor integrals have been decomposed into scalar integrals, the numerators can be set to one, simply by adding and subtracting the propagator mass. For example,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{q^2}{(q^2 - m^2)^\nu} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{(q^2 - m^2) + m^2}{(q^2 - m^2)^\nu} \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 - m^2)^{\nu-1}} + m^2 \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 - m^2)^\nu}. \end{aligned} \tag{3.10}$$

Therefore, general Feynman integrals may be expanded and simplified to vacuum integrals that only depend on a single mass m and a single propagator power ν. These are the types of integral that have been used throughout this work. They can be written as

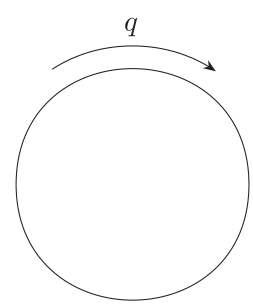
$$\mathcal{I}_{\rm vac}^{(1)}(m^2;\nu) = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2-m^2)^\nu}, \tag{3.11}$$

$$\mathcal{I}_{\rm vac}^{(2)}(m_1^2,m_2^2,m_3^2;\nu_1,\nu_2,\nu_3) = \\ \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{1}{(q_1^2-m_1^2)^{\nu_1}(q_2^2-m_2^2)^{\nu_2}((q_1-q_2)^2-m_3^2)^{\nu_3}}, \tag{3.12}$$

$$\mathcal{I}_{\rm vac}^{(3)}(m_1^2,m_2^2,m_3^2,m_4^2,m_5^2,m_6^2;\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6) = \\ \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_3}{(2\pi)^4} \frac{1}{(q_1^2-m_1^2)^{\nu_1}(q_2^2-m_2^2)^{\nu_2}(q_3^2-m_3^2)^{\nu_3}} \\ \times \frac{1}{((q_1-q_2)^2-m_4^2)^{\nu_4}((q_1-q_3)^2-m_5^2)^{\nu_5}((q_2-q_3)^2-m_6^2)^{\nu_6}}, \tag{3.13}$$

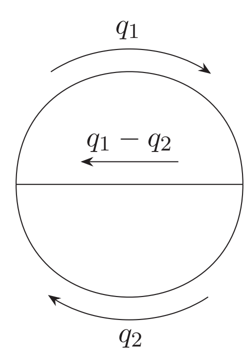
which are respectively the 1-loop, 2-loop, and 3-loop scalar vacuum integrals. Higher loop orders are not encountered in this thesis. The topologies of the above diagrams are shown below (where masses and propagator powers are suppressed in the diagrams).

$\mathcal{I}_{\rm vac}^{(1)}(m^2;\nu) =$



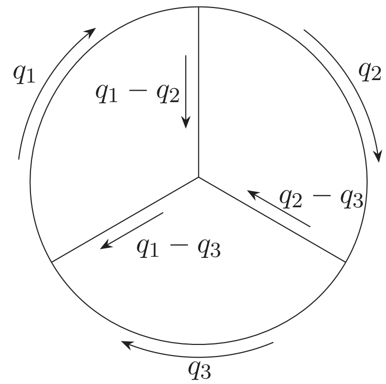
, (3.14)

$\mathcal{I}_{\rm vac}^{(2)}(m_1^2,m_2^2,m_3^2;\nu_1,\nu_2,\nu_3) =$



, (3.15)

$$\mathcal{I}^{(3)}_{\rm vac}(m_1^2,\ldots,m_6^2;\nu_1,\ldots,\nu_6) =$$



. (3.16)

These are the most general topologies for these loop levels. Different topologies may be generated for 3-loop vacuum integrals by having some propagator powers equal to zero. An extreme example is a 3-loop integral with $\nu_4 = \nu_5 = \nu_6 = 0$, which is simply the product of three 1-loop vacuum integrals.

3.2 Symmetries of Vacuum Integrals

Vacuum integrals have a high degree of symmetry, and in a calculation, many integrals may be generated which are equal under some symmetry transformation. Making use of these symmetries reduces the number of independent integrals that need to be solved.

A vacuum integral is symmetric if it is invariant under a simultaneous reordering of propagator masses m_i and propagator powers ν_i. The 1-loop vacuum integral has a trivial topology, and therefore no symmetries. However, the 2-loop vacuum integral has a non-trivial topology, and is in fact totally symmetric. By this, it is meant that any of the propagator masses and powers may be interchanged with any other, with the integral remaining unchanged. To demonstrate the total symmetry of the 2-loop vacuum diagram, it is sufficient to show that

$$\mathcal{I}^{(2)}_{\rm vac}(m_1^2,m_2^2,m_3^2;\nu_1,\nu_2,\nu_3) = \mathcal{I}^{(2)}_{\rm vac}(m_2^2,m_1^2,m_3^2;\nu_2,\nu_1,\nu_3)\,, \tag{3.17}$$

and

$$\mathcal{I}^{(2)}_{\rm vac}(m_1^2,m_2^2,m_3^2;\nu_1,\nu_2,\nu_3) = \mathcal{I}^{(2)}_{\rm vac}(m_3^2,m_1^2,m_2^2;\nu_3,\nu_1,\nu_2)\,, \tag{3.18}$$

since these permutations are a complete set of generators for the possible structures of 2-loop vacuum integrals. Starting with Equation (3.17),

$$
\begin{aligned}
&\mathcal{I}^{(2)}_{\text{vac}}(m_2^2, m_1^2, m_3^2; \nu_2, \nu_1, \nu_3) \\
&\qquad = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{1}{(q_1^2 - m_2^2)^{\nu_2}(q_2^2 - m_1^2)^{\nu_1}((q_1-q_2)^2 - m_3^2)^{\nu_3}} \\
&\qquad \overset{q_1 \leftrightarrow q_2}{=} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \frac{1}{(q_2^2 - m_2^2)^{\nu_2}(q_1^2 - m_1^2)^{\nu_1}((q_1-q_2)^2 - m_3^2)^{\nu_3}} \\
&\qquad = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{1}{(q_1^2 - m_1^2)^{\nu_1}(q_2^2 - m_2^2)^{\nu_2}((q_1-q_2)^2 - m_3^2)^{\nu_3}} \\
&\qquad = \mathcal{I}^{(2)}_{\text{vac}}(m_1^2, m_2^2, m_3^2; \nu_1, \nu_2, \nu_3).
\end{aligned} \tag{3.19}
$$

Equation (3.18) can be shown to hold in a similar manner. However, rather than the simple integral transform $q_1 \leftrightarrow q_2$ that was used for Equation (3.17), it is necessary to use the integral transform

$$
q_1 \to q_1' = q_1 - q_2, \qquad q_2 \to q_2' = q_1. \tag{3.20}
$$

Formally, it is necessary to compute the Jacobian to understand how the integral measures transform, since

$$
\mathrm{d}^4 q_1 \mathrm{d}^4 q_2 = J \mathrm{d}^4 q_1' \mathrm{d}^4 q_2', \quad \text{where} \quad J = \begin{vmatrix} \frac{\partial q_1}{\partial q_1'} & \frac{\partial q_1}{\partial q_2'} \\ \frac{\partial q_2}{\partial q_1'} & \frac{\partial q_2}{\partial q_2'} \end{vmatrix}. \tag{3.21}
$$

Since $q_1 = q_2'$ and $q_2 = q_2' - q_1'$, the Jacobian simply evaluates to $J = 1$. Therefore,

$$
\begin{aligned}
&\mathcal{I}^{(2)}_{\text{vac}}(m_3^2, m_1^2, m_2^2; \nu_3, \nu_1, \nu_2) \\
&\qquad = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{1}{(q_1^2 - m_3^2)^{\nu_3}(q_2^2 - m_1^2)^{\nu_1}((q_1-q_2)^2 - m_2^2)^{\nu_2}} \\
&\qquad = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{1}{((q_1-q_2)^2 - m_3^2)^{\nu_3}(q_1^2 - m_1^2)^{\nu_1}((-q_2)^2 - m_2^2)^{\nu_2}} \\
&\qquad = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \frac{1}{(q_1^2 - m_1^2)^{\nu_1}(q_2^2 - m_2^2)^{\nu_2}((q_1-q_2)^2 - m_3^2)^{\nu_3}} \\
&\qquad = \mathcal{I}^{(2)}_{\text{vac}}(m_1^2, m_2^2, m_3^2; \nu_1, \nu_2, \nu_3)
\end{aligned} \tag{3.22}
$$

It is thus demonstrated that the 2-loop vacuum integral is totally symmetric.

The 3-loop vacuum integral is not totally symmetric, but displays the symmetries of the tetrahedral group S_4, which has two generators [39], which are shown diagrammatically in Figure 3.1. These two generators generate 24 group elements.

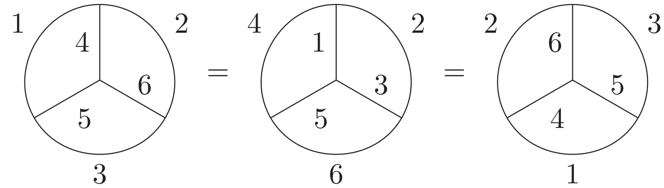


Fig. 3.1: Generators of the tetrahedral group S_4.

In integral form, the generators imply

$$\begin{aligned} &\mathcal{I}^{(3)}_{\text{vac}}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6) \\ &= \mathcal{I}^{(3)}_{\text{vac}}(m_4^2, m_2^2, m_6^2, m_1^2, m_5^2, m_3^2; \nu_4, \nu_2, \nu_6, \nu_1, \nu_5, \nu_3) \\ &= \mathcal{I}^{(3)}_{\text{vac}}(m_2^2, m_3^2, m_1^2, m_6^2, m_4^2, m_5^2; \nu_2, \nu_3, \nu_1, \nu_6, \nu_4, \nu_5). \end{aligned} \tag{3.23}$$

In general, 3-loop vacuum integrals related via tetrahedral symmetries are only equal if all propagators have the same mass. Otherwise, the tetrahedral symmetries may only be used to bring the integrals into some standard form. In this work, 3-loop integrals arise that contain two different masses (as well as massless propagators), but only have a maximum of four massive propagators. Therefore, symmetry relations were used to bring all integrals into the following basis:

$$\begin{aligned} &\mathcal{I}^{(3)}_{\text{vac}}(m^2, 0, 0, 0, 0, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6), \\ &\mathcal{I}^{(3)}_{\text{vac}}(0, 0, 0, 0, 0, m^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6), \\ &\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, 0, 0, 0, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6), \\ &\mathcal{I}^{(3)}_{\text{vac}}(m^2, 0, 0, 0, 0, m^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6), \\ &\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, m^2, 0, 0, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6), \\ &\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, 0, 0, m^2, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6), \\ &\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, 0, 0, m^2, m^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6), \\ &\mathcal{I}^{(3)}_{\text{vac}}(m^2, M^2, 0, 0, 0, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6), \\ &\mathcal{I}^{(3)}_{\text{vac}}(m^2, 0, 0, 0, 0, M^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6), \\ &\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, M^2, 0, 0, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6), \\ &\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, 0, 0, M^2, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6), \\ &\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, 0, 0, m^2, M^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6). \end{aligned} \tag{3.24}$$

This is a sufficiently large basis for the integrals generated in this work. It should be noted that M is not necessarily larger than m.

3.3 Dimensional Regularisation

The aim of renormalisation is to remove UV divergences from amplitudes. To do this, a necessary step is to extract the divergences from Feynman integrals in a systematic manner, in a process called regularisation. In this work dimensional regularisation is used exclusively [40, 41].[1] Dimensional regularisation relies on the principle that Feynman integrals would converge if the number of spacetime dimensions was less than four. For example, the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 - m^2)^2} \tag{3.25}$$

is logarithmically UV divergent by naive power counting. If the power of momentum in the numerator was less than four, then the integral would be finite. Therefore dimensional regularisation evaluates integrals in d complex dimensions, where $d = 4-2\epsilon$ with $\epsilon > 0$. The $d = 4$ case is then restored in the limit $\epsilon \to 0$. For example, the logarithmically divergent integral of Equation (3.25) can dimensionally regularised via [42]

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 - m^2)^2} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}^d q_E}{(2\pi)^d} \frac{i}{(q_E^2 + m^2)^2} \\
&= \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d} \frac{i}{2} \int_0^{\infty} \mathrm{d}q_E^2 \frac{(q_E^2)^{\frac{d}{2}-1}}{(q_E^2 + m^2)^2} \\
&= \frac{2}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} \frac{i}{2} \left(\frac{1}{m^2}\right)^{2-\frac{d}{2}} \int_0^1 \mathrm{d}x \; x^{1-\frac{d}{2}} (1-x)^{\frac{d}{2}-1} \\
&= \frac{i}{(4\pi)^d \Gamma\left(\frac{d}{2}\right)} \left(\frac{1}{m^2}\right)^{2-\frac{d}{2}} \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(2)} \\
&= \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)} \left(\frac{1}{m^2}\right)^{2-\frac{d}{2}} .
\end{aligned} \tag{3.26}$$

In the above, $\mathrm{d}\Omega_d$ is the surface area element of a d-sphere, and $\Gamma(z)$ is the Gamma function, which has poles at $z = \cup\{0, \mathbb{Z}^-\}$. Therefore, Equation (3.26) is divergent for $d = 4$, as expected. Similar results for different integrals can be found in [42]. Setting $d = 4 - 2\epsilon$ gives

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 - m^2)^2} = \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{\Gamma(2)} \left(\frac{1}{m^2}\right)^{\epsilon} . \tag{3.27}$$

[1] Other common regularisation schemes include cutoff regularisation and the discretisation of spacetime. Cutoff regularisation simply excludes high-momentum modes in Feynman integrals, and while it is mathematically simple, it breaks many symmetries of the theory, including Lorentz invariance. Spacetime discretisation is the regulator used in lattice QCD, where spacetime is divided into finite elements, with the physical limit being recovered as these elements shrink to zero size.

Taking ϵ to be small and positive, $\Gamma(\epsilon)$ may be expanded in powers of ϵ using the Weierstrass definition of the Gamma function [43],

$$\Gamma(z) = \frac{e^{-\gamma_E z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}}, \tag{3.28}$$

where γ_E is the Euler-Mascheroni constant. Then,

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon), \tag{3.29}$$

showing that in the limit $d \to 4$ ($\epsilon \to 0$), the UV divergence has been isolated as a simple pole in ϵ. Finally, using

$$x^{\epsilon} = e^{\epsilon \ln x} = 1 + \epsilon \ln x + \mathcal{O}(\epsilon^2), \tag{3.30}$$

the entire integral can be written as

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 - m^2)^2} = \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} - \ln{(m^2)} - \gamma_E + \ln{(4\pi)}\right) + \mathcal{O}(\epsilon), \tag{3.31}$$

where terms of order ϵ and higher are neglected, since we are interested in the $\epsilon \to 0$ limit. Therefore dimensional regularisation extracts UV divergences from Feynman integrals, and encodes them as poles in ϵ, with the regularisation being lifted in the limit $\epsilon \to 0$.

In general, 1-loop, 2-loop, and 3-loop UV divergent integrals can be dimensionally regularised to give expressions of the form [44]

$$\text{1-loop result} = \frac{a_1}{\epsilon} + b_1, \tag{3.32}$$

$$\text{2-loop result} = \frac{a_2}{\epsilon^2} + \frac{b_2}{\epsilon} + c_2, \tag{3.33}$$

$$\text{3-loop result} = \frac{a_3}{\epsilon^3} + \frac{b_3}{\epsilon^2} + \frac{c_3}{\epsilon} + d_3, \tag{3.34}$$

where a_i, b_i, c_i and d_i are all finite.

An important result of dimensionally regularised integrals is that integrals that do not contain some physical scale (masses or external momenta) are identically zero [45]:

$$\int_{-\infty}^{\infty} \mathrm{d}^d q (q^2)^{-\alpha} \equiv 0 \quad \forall \alpha. \tag{3.35}$$

Although such scaleless integrals are identically zero, they may contain UV and IR poles, which cancel. It is possible to isolate desired UV poles using IR rearrangement (see Section 3.7).

A key feature of dimensional regularisation is that it affects the mass dimensions of the fields in a Lagrangian. The action S of a theory must have zero mass dimension, and consequently in a renormalisable theory the Lagrangian has a mass dimension of four. From this observation, and the fact that for all d, masses and spacetime derivatives have mass dimension equal to one, the mass dimension of parameters and fields in a Lagrangian can be calculated. For example, for QCD

$$\begin{aligned} 4 &= \left[m\overline{\Psi}\Psi\right] = 1 + \left[\overline{\Psi}\Psi\right] = 1 + 2\left[\Psi\right] && \implies [\Psi] = \frac{3}{2}, \\ 4 &= \left[G_{\mu\nu}G^{\mu\nu}\right] = 2\left[G_{\mu\nu}\right] = 2\left[\partial_\mu G_\nu\right] = 2(1 + [G_\mu]) && \implies [G_\mu] = 1, \\ 4 &= \left[g_s\overline{\Psi}\gamma^\mu\Psi G_\mu\right] = 4 + [g_s] && \implies [g_s] = 0. \end{aligned} \tag{3.36}$$

Dimensional regularisation makes a Lagrangian d-dimensional, which modifies the mass dimensions of fields and parameters. Repeating the above calculation gives

$$\begin{aligned} d &= \left[m\overline{\Psi}\Psi\right] = 1 + \left[\overline{\Psi}\Psi\right] = 1 + 2\left[\Psi\right] && \implies [\Psi] = \frac{d-1}{2}, \\ d &= \left[G_{\mu\nu}G^{\mu\nu}\right] = 2\left[G_{\mu\nu}\right] = 2\left[\partial_\mu G_\nu\right] = 2(1 + [G_\mu]) && \implies [G_\mu] = \frac{d-2}{2}, \\ d &= \left[g_s\overline{\Psi}\gamma^\mu\Psi G_\mu\right] = 2\left(\frac{d-1}{2}\right) + \left(\frac{d-2}{2}\right) + [g_s] && \implies [g_s] = \frac{4-d}{2}. \end{aligned} \tag{3.37}$$

Note that these equations reproduce the previous set in the limit $d \to 4$. Of particular importance is the fact that in d-dimensions the gauge coupling becomes dimensionful. It is preferable to work with a dimensionless coupling, and so an arbitrary parameter μ with mass dimension equal to one is extracted from the coupling:

$$g_s \to g_s(\mu)\mu^{\frac{4-d}{2}} = g_s(\mu)\mu^{\epsilon}. \tag{3.38}$$

Note that since the original dimensionful g_s is μ-independent, the new dimensionless $g_s(\mu)$ must be μ-dependent. The parameter μ is arbitrary, and therefore cannot affect physical quantities. This μ-independence of Green's functions gives rise to the renormalisation group equations, as discussed in Section 3.8.

3.4 Integral Reduction and Integration By Parts

Finding analytic solutions for Feynman integrals is highly non-trivial, and becomes very difficult when working with multiple physical scales and higher loop orders. However, it can be shown that complicated Feynman integrals can be decomposed into a linear combination of a basis of simpler Feynman integrals, called *master integrals*. Since all integrals arising in a calculation may be decomposed into a sum of simpler master integrals, it only then remains to calculate the master integrals (which in a typical calculation form a much smaller set than all integrals generated in a calculation), as well as the coefficients of the master integrals in such a decomposition.

The process of linearly expanding a Feynman integral in terms of master integrals is called reduction, and relies on integration by parts (IBP) identities [46]:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\partial}{\partial q^\mu} \left(q^\mu f(q^2, p^2, m^2, \dots) \right) = 0, \tag{3.39}$$

where $f(q^2, p^2, m^2, \dots)$ denotes the integrand of some Feynman integral with loop momentum q, external momentum p and mass m. This identity holds by performing the integral by parts, and then discarding a surface term which must vanish at infinity. Explicitly,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\mathrm{d}^d q}{(2\pi)^d} 1 \cdot \frac{\partial}{\partial q^\mu} \left(q^\mu f(q^2, p^2, m^2, \dots) \right) \\ &\qquad = \left[1 \cdot \left(q^\mu f(q^2, p^2, m^2, \dots) \right) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial}{\partial q^\mu}(1) \cdot \left(q^\mu f(q^2, p^2, m^2, \dots) \right) \\ &\qquad = 0. \end{aligned} \tag{3.40}$$

This relation can be used to build recursion relations between integrals with different propagator powers. Consider the 1-loop vacuum integral example, containing a single mass and no external momenta [46],

$$F(a) = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2 - m^2)^a} \,. \tag{3.41}$$

Applying the IBP identity

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\partial}{\partial q^\mu} \frac{q^\mu}{(q^2 - m^2)^a} = 0, \tag{3.42}$$

it can be shown that

$$(d - 2a)F(a) - 2am^2 F(a + 1) = 0. \tag{3.43}$$

Rearranging this expression, and shifting $a \to a - 1$, one obtains the recurrence relation

$$F(a) = \frac{d - 2a + 2}{2(a - 1)m^2} F(a - 1). \tag{3.44}$$

This recurrence relation may be repeatedly used to relate any integral $F(a)$ to a single master integral $F(1)$. Therefore, it is only necessary to solve a single integral, and to calculate the prefactor that relates the master integral to the required integral.

The method of integral reduction has been implemented in many computer packages, for example [47–51]. We have exclusively used `FIRE5` [47] to perform integral reduction. This was used successfully for 1-loop, 2-loop, and 3-loop integrals containing up to two different masses, with up to four massive propagators.

3.5 Solutions to Vacuum Integrals

So far in this chapter, integrals have been reduced to scalar vacuum integrals, dimensionally regularised, and then reduced to master integrals. At this point it is necessary to solve the resulting master integrals. There are a number of different methods for solving Feynman integrals, including the use of Feynman parameters, Schwinger parameters, and evaluation by Mellin-Barnes representation. A simple 1-loop vacuum bubble was solved in Section 3.3. There are many cases of Feynman integrals that currently have no analytic solution, and such integrals can be estimated numerically, or by using appropriate expansions of the integrands to simplify integrals [52]. It has not been the aim of this work to solve loop integrals, rather applying solutions found in the literature to physical processes. Therefore, we list here those integrals that are relevant to our purposes.

Vacuum integrals at 1-loop can only be functions of a single mass. The general 1-loop vacuum integral has the solution [53]

$$\mathcal{I}^{(1)}_{\text{vac}}(m^2;\nu) = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(q^2-m^2)^\nu} = i^{1-d} \frac{\left(-m^2\right)^{\frac{d}{2}-\nu}}{(4\pi)^{d/2}} \frac{\Gamma\left(\nu-\frac{d}{2}\right)}{\Gamma(\nu)}. \tag{3.45}$$

This can be found by performing a Wick rotation to Euclidean spacetime, and then moving to spherical coordinates, as was done in Section 3.3.

The general 2-loop vacuum integral is

$$\mathcal{I}^{(2)}_{\text{vac}}(m_1^2,m_2^2,m_3^2;\nu_1,\nu_2,\nu_3) = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^d q_1}{(2\pi)^d} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}^d q_2}{(2\pi)^d} \frac{1}{(q_1^2-m_1^2)^{\nu_1}(q_2^2-m_2^2)^{\nu_2}((q_1-q_2)^2-m_3^2)^{\nu_3}}. \tag{3.46}$$

If one of the indices is equal to zero, this is simply the product of two 1-loop integrals, and so

$$\begin{aligned} \mathcal{I}^{(2)}_{\text{vac}}&(m_1^2,m_2^2,m_3^2;\nu_1,\nu_2,0) \\ &= \mathcal{I}^{(1)}_{\text{vac}}(m_1^2;\nu_1)\,\mathcal{I}^{(1)}_{\text{vac}}(m_2^2;\nu_2) &(3.47)\\ &= i^{2-2d}\frac{(-m_1^2)^{d/2-\nu_1}(-m_2^2)^{d/2-\nu_2}}{(4\pi)^d}\frac{\Gamma\left(\nu_1-\frac{d}{2}\right)\Gamma\left(\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)}. &(3.48)\end{aligned}$$

We are also interested in cases where there are two different mass scales, and as such need the solution of integrals of the form $\mathcal{I}^{(2)}_{\text{vac}}(m^2,m^2,M^2;\nu_1,\nu_2,\nu_3)$. It is sufficient to consider this integral without loss of generality due to the total symmetry of 2-loop vacuum integrals. This integral can be solved by first using IBP relations to reduce a general integral $\mathcal{I}^{(2)}_{\text{vac}}(m^2,m^2,M^2;\nu_1,\nu_2,\nu_3)$ in terms of the master integral

$\mathcal{I}^{(2)}_{\text{vac}}(m^2, m^2, M^2; 1, 1, 1)$. This master integral can then be solved using the Mellin-Barnes representation of a massive propagator, as done in [38]. The result is (expanding in $\epsilon = (4-d)/2$)

$$\mathcal{I}^{(2)}_{\text{vac}}(m^2, m^2, M^2; 1, 1, 1) = \frac{\pi^{2\epsilon-4}}{16^{2-\epsilon}} \left(m^2\right)^{1-2\epsilon} A(\epsilon) \times \Big\{ -\frac{1}{\epsilon^2}(1+2z) + \frac{1}{\epsilon}(4z \ln(z)) - 2z \ln^2(4z) + 2(1-z)\Phi(z) \Big\}, \tag{3.49}$$

where this result has been divided by $(2\pi)^{2(4-2\epsilon)}$ compared to [38] to agree with our conventions. Here,

$$z \equiv \frac{M^2}{4m^2},$$

$$A(\epsilon) \equiv \frac{\Gamma^2(1+\epsilon)}{(1-\epsilon)(1-2\epsilon)} = 1 + \epsilon(3 - 2\gamma_E) + \epsilon^2 \left(7 - 6\gamma_E + 2\gamma_E^2 + \frac{\pi^2}{6}\right) + \mathcal{O}(\epsilon^3)\,, \tag{3.50}$$

and

$$\Phi(z) = 4z \left[(2 - \ln(4z)) \;{}_2F_1\left(\left.\begin{matrix}1,1\\3/2\end{matrix}\right| z\right) - \partial_a \;{}_2F_1\left(\left.\begin{matrix}1,1\\3/2\end{matrix}\right| z\right) - \partial_c \;{}_2F_1\left(\left.\begin{matrix}1,1\\3/2\end{matrix}\right| z\right) \right]. \tag{3.51}$$

The definition of $\Phi(z)$ is in terms of hypergeometric functions and their derivatives,

$${}_PF_Q\left(\left.\begin{matrix}a_1, \ldots, a_P\\c1, \ldots, c_Q\end{matrix}\right| z\right) \equiv \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{(a_1)_j \ldots (a_P)_j}{(c_1)_j \ldots (c_Q)_j}, \tag{3.52}$$

$$(a)_j \equiv \frac{\Gamma(a+j)}{\Gamma(a)}, \tag{3.53}$$

$$\partial_a \;{}_2F_1\left(\left.\begin{matrix}a,b\\c\end{matrix}\right| z\right) \equiv \frac{\partial}{\partial a} \;{}_2F_1\left(\left.\begin{matrix}a,b\\c\end{matrix}\right| z\right) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{(a)_j (b)_j}{(c)_j} (\psi(a+j) - \psi(a)), \tag{3.54}$$

and

$$\partial_c \;{}_2F_1\left(\left.\begin{matrix}a,b\\c\end{matrix}\right| z\right) \equiv \frac{\partial}{\partial c} \;{}_2F_1\left(\left.\begin{matrix}a,b\\c\end{matrix}\right| z\right) = -\sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{(a)_j (b)_j}{(c)_j} (\psi(c+j) - \psi(a)), \tag{3.55}$$

where $\psi(a) \equiv (\mathrm{d}/\mathrm{d}a) \ln(\Gamma(a))$. Further details can be found in [38].

Finally, it is necessary to consider 3-loop vacuum integrals with two different masses. When initially considering this case, solutions for this class of integrals were not known, whereas solutions were known for 3-loop vacuum integrals with only a single mass scale [39]. Therefore, an initial strategy was to employ expansions to reduce all integrands to a series of integrals, each containing only a single mass scale, which could therefore be solved. Naively these expansions would take the form of Taylor expansions, with the expansion parameter being the ratio of the two masses present in the integrand. However, since masses serve the role of IR regulators for vacuum integrals, performing Taylor expansions in masses removes these regulators and hence introduces artificial IR divergences. Smirnov has outlined a method by which Taylor expansions can still be used to perform expansions without introducing artificial IR divergences, by the method of expansion by regions [52]. The essence of this program is to divide the integration regions of all integrals into "hard" and "soft" regions, where the loop momentum is respectively large and small, and then sum over all these regions. For each of the integrals thereby produced, there is a set of naturally small parameters that the integrand may be expanded in. Breaking up the integrals in this way and then expanding avoids the introduction of IR divergences. The method becomes more powerful since the integrals over a restricted domain of loop momenta may then be extended to integrals over all momenta without generating any new contributions, allowing the use of usual integral identities. However, it would be necessary to perform a large number of expansions for each integral to minimise the error associated with an expansion, and since for a 3-loop process there are many integrals to be expanded, such a strategy would be very computationally demanding. A significant amount of work was put into implementing expansion by regions as a `Mathematica` [54] routine, and it was found that the method created high-rank tensor integrals as an intermediate step, which needed to be decomposed into scalar integrals. While this was acceptable for low-rank tensor integrals, it very quickly became a serious computational problem which needed to be addressed. A discussion of the resolution to this problem is presented in Section 3.9.

Progress was made on computing 3-loop integrals with arbitrary masses without resorting to expansions [13, 55, 56]. In [55], a basis of three master integrals is selected that cannot be decomposed into a product of lower-loop integrals. These master integrals are then decomposed into sums of integrals, which are either divergent or finite. The divergent pieces can be computed analytically, whilst the finite pieces can be written as dispersion integrals and computed numerically. The routine for performing these calculations is presented in [56].

For the evaluation of 3-loop integrals, we used the results of Martin and Robertson [13]. They discuss vacuum diagrams for 1-, 2-, and 3-loops, and we shall outline their conventions and results here. We use these results extensively in Chapter 6. After Wick rotation, working in $d = 4 - 2\epsilon$ Euclidean dimensions, and defining integrals as

$$\int_p \equiv \mu^{4-d} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \tag{3.56}$$

such that integrals have a mass dimension of four, the 1-loop master integral may be written as

$$\mathbf{A}(x) = 16\pi^2 \int_p \frac{1}{p^2 + x} = \Gamma(-1+\epsilon) \left(\frac{4\pi\mu^2}{x}\right)^{\epsilon} x. \tag{3.57}$$

Note that μ here is the MS parameter (see Section 3.6 for a discussion of renormalisation schemes). Similarly, a general 2-loop integral is written in the form

$$\mathbf{I}(x,y,z) = (16\pi^2)^2 \int_p \int_q \frac{1}{[p^2+x]\,[q^2+y]\,[(p-q)^2+z]}. \tag{3.58}$$

Following the approach of [57], and in a similar manner to [55], $\mathbf{A}$ and $\mathbf{I}$ can be eliminated in favour of their respective renormalised integrals (i.e. finite integrals). However, expressions for integrals are also given in terms of coefficients of UV poles in [13], which we use. We list here the results for 1- and 2-loop integrals in the conventions of Martin and Robertson, before discussing 3-loop integrals. The 1-loop integral may be written as

$$\mathbf{A}(x) = -\frac{x}{\epsilon} + A(x) + \epsilon A_\epsilon(x) + \epsilon^2 A_{\epsilon^2}(x) + \dots\,, \tag{3.59}$$

where

$$A(x) = x[\overline{\ln}(x) - 1]\,, \tag{3.60}$$

$$A_\epsilon(x) = x\left[-\frac{1}{2}\overline{\ln}^2(x) + \overline{\ln}(x) - 1 - \frac{\pi^2}{12}\right], \tag{3.61}$$

$$A_{\epsilon^2}(x) = x\left[\frac{1}{6}\overline{\ln}^3(x) - \frac{1}{2}\overline{\ln}^2(x) + \left(1 + \frac{\pi^2}{12}\right)\overline{\ln}(x) - 1 - \frac{\pi^2}{12} + \frac{\zeta_3}{3}\right]. \tag{3.62}$$

The function $\overline{\ln}(x)$ is defined as

$$\overline{\ln}(x) \equiv \ln\left(\frac{x}{\mu^2_{\overline{\mathrm{MS}}}}\right), \tag{3.63}$$

where $\mu_{\overline{\mathrm{MS}}}$ is the renormalisation scale in the $\overline{\mathrm{MS}}$ scheme (which is related to μ via Equation (3.100)).

The 2-loop basis integral can be expanded as

$$\mathbf{I}(x,y,z) = \frac{I_2(x,y,z)}{\epsilon^2} + \frac{I_1(x,y,z)}{\epsilon} + I_0(x,y,z) + \epsilon I_\epsilon(x,y,z) + \dots\,, \tag{3.64}$$

where the pole pieces are

$$I_2(x,y,z) = -(x+y+z)/2\,, \tag{3.65}$$

$$I_1(x,y,z) = A(x) + A(y) + A(z) - (x+y+z)/2\,. \tag{3.66}$$

The finite piece is given by (for $z \geq x, y$)

$$\begin{aligned}
I_0(x,y,z) =& s\left[\mathrm{Li}_2(k_1) + \mathrm{Li}_2(k_2) - \ln(k_1)\ln(k_2) + \frac{1}{2}\ln(x/z)\ln(y/z) - \pi^2/6\right] \\
&+ \frac{1}{2}\left[(z-x-y)\overline{\ln}(x)\overline{\ln}(y) + (y-x-z)\overline{\ln}(x)\overline{\ln}(z) + (x-y-z)\overline{\ln}(y)\overline{\ln}(z)\right] \\
&+ 2x\overline{\ln}(x) + 2y\overline{\ln}(y) + 2z\overline{\ln}(z) - \frac{5}{2}(x+y+z) \\
&+ A_\epsilon(x) + A_\epsilon(y) + A_\epsilon(z)\,,
\end{aligned} \tag{3.67}$$

where

$$s = \sqrt{x^2 + y^2 + z^2 - 2xy - 2xz - 2yz}\,, \tag{3.68}$$

$$k_1 = \frac{x+z-y-s}{2z}\,, \tag{3.69}$$

$$k_2 = \frac{y+z-x-s}{2z}\,. \tag{3.70}$$

Due to the total symmetry of the 2-loop vacuum integral, the cases where $y \geq x, z$ and $x \geq y, z$ can be obtained by appropriate permutations of the argument of $I_0(x,y,z)$. The function $\mathrm{Li}_2(x)$ is the dilogarithm, defined as

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-u)}{u}\mathrm{d}u\,, \tag{3.71}$$

where x is a complex variable. The dilogarithm obeys a number of identities, including

$$\mathrm{Li}_2\left(\frac{x-1}{x}\right) = -\mathrm{Li}_2\left(\frac{1}{x}\right) - \ln\left(\frac{1}{x}\right)\ln\left(\frac{x-1}{x}\right) + \frac{\pi^2}{6}\,, \tag{3.72}$$

$$\mathrm{Li}_2\left(\frac{1}{x}\right) = -\mathrm{Li}_2(x) - \frac{1}{2}\ln^2(-x) - \frac{\pi^2}{6}\,, \tag{3.73}$$

which are useful in simplifying expressions.

The general 3-loop scalar integral with the Benz topology (see Figure 3.2) is denoted by

$$\mathbf{T}^{(n_1,n_2,n_3,n_4,n_5,n_6)}(x_1,x_2,x_3,x_4,x_5,x_6) = (16\pi^2)^3 \int_{p_1}\int_{p_2}\int_{p_3} \frac{1}{[p_1^2+x_1]^{n_1}[p_2^2+x_2]^{n_2}[p_3^2+x_3]^{n_3}[(p_1-p_2)^2+x_4]^{n_4}[(p_2-p_3)^2+x_5]^{n_5}[(p_3-p_1)^2+x_6]^{n_6}}\,,$$

where it should be noted that the authors of [13] have a different propagator ordering compared to our conventions.

By the application of IBP relations, any general integral $\mathbf{T}$ may be reduced to a sum of a basis of master integrals, given as

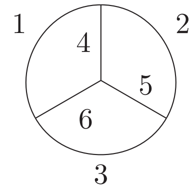
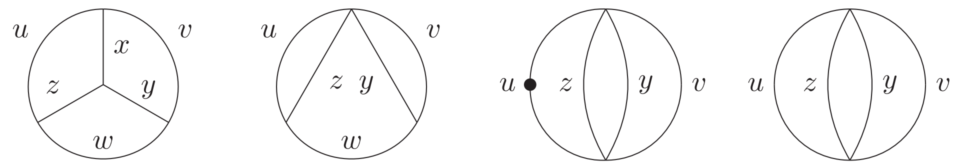


Fig. 3.2: General 3-loop topology in the conventions of [13].



$\mathbf{H}(u,v,w,x,y,z)$ $\mathbf{G}(w,u,z,v,y)$ $\mathbf{F}(u,v,y,z)$ $\mathbf{E}(u,v,y,z)$

Fig. 3.3: Topologies of the 3-loop master integrals. The dotted propagator in $\mathbf{F}(u,v,y,z)$ indicates that the propagator is raised by a power.

$$\begin{aligned}
\mathbf{H}(u,v,w,x,y,z) &= \mathbf{T}^{(1,1,1,1,1,1)}(u,v,w,x,y,z),\\
\mathbf{G}(w,u,z,v,y) &= \mathbf{T}^{(1,1,1,0,1,1)}(u,v,w,x,y,z),\\
\mathbf{F}(u,v,y,z) &= \mathbf{T}^{(2,1,0,0,1,1)}(u,v,w,x,y,z),\\
\mathbf{A}(u)\mathbf{I}(v,w,y) &= \mathbf{T}^{(1,1,1,0,1,0)}(u,v,w,x,y,z),\\
\mathbf{A}(u)\mathbf{A}(v)\mathbf{A}(w) &= \mathbf{T}^{(1,1,1,0,0,0)}(u,v,w,x,y,z).
\end{aligned} \tag{3.74}$$

An additional integral that is useful (but not a master integral) is given by

$$\mathbf{E}(u,v,y,z) = \mathbf{T}^{(1,1,0,0,1,1)}(u,v,w,x,y,z), \tag{3.75}$$

which can be expressed as a linear combination of $\mathbf{F}$-type integrals. $\mathbf{F}$-integrals may also simply be expressed through derivatives with respect to masses of $\mathbf{E}$-integrals:

$$\mathbf{F}(u,v,y,z) = -\frac{\partial}{\partial u}\mathbf{E}(u,v,y,z). \tag{3.76}$$

The topologies of $\mathbf{H},\mathbf{G},\mathbf{F}$, and $\mathbf{E}$ are shown in Figure 3.3.

The 3-loop integrals may be expanded in ϵ as

$$\begin{aligned}
\mathbf{E}(u,v,y,z) = {}& \frac{1}{\epsilon^3}E_3(u,v,y,z) + \frac{1}{\epsilon^2}E_2(u,v,y,z) + \frac{1}{\epsilon}E_1(u,v,y,z)\\
&+ E_0(u,v,y,z) + \dots\,,
\end{aligned} \tag{3.77}$$

$$\begin{aligned}
\mathbf{F}(u,v,y,z) = {}& \frac{1}{\epsilon^3}F_3(u,v,y,z) + \frac{1}{\epsilon^2}F_2(u,v,y,z) + \frac{1}{\epsilon}F_1(u,v,y,z)\\
&+ F_0(u,v,y,z) + \dots\,,
\end{aligned} \tag{3.78}$$

$$\mathbf{G}(w,u,z,v,y) = \frac{1}{\epsilon^3}G_3(w,u,z,v,y) + \frac{1}{\epsilon^2}G_2(w,u,z,v,y) + \frac{1}{\epsilon}G_1(w,u,z,v,y) + G_0(w,u,z,v,y) + \dots\,, \tag{3.79}$$

$$\mathbf{H}(u,v,w,x,y,z) = \frac{1}{\epsilon}H_1(u,v,w,x,y,z) + H_0(u,v,w,x,y,z) + \dots\,. \tag{3.80}$$

The ϵ-coefficient functions are:

$$E_3(u,v,y,z) = (uv + uy + uz + vy + vz + yz)/3\,, \tag{3.81}$$

$$\begin{aligned} E_2(u,v,y,z) = &-[(v+y+z)A(u) + (u+y+z)A(v) \\ &+ (u+v+z)A(y) + (u+v+y)A(z)]/2 \\ &+ (uv+uy+uz+vy+vz+yz)/3 - (u^2+v^2+y^2+z^2)/12\,, \end{aligned} \tag{3.82}$$

$$\begin{aligned} E_1(u,v,y,z) = &A(u)A(v) + A(u)A(y) + A(u)A(z) + A(v)A(y) + A(v)A(z) + A(y)A(z) \\ &- (v+y+z)[A_\epsilon(u) + A(u)]/2 - (u+y+z)[A_\epsilon(v) + A(v)]/2 \\ &- (v+v+z)[A_\epsilon(y) + A(y)]/2 - (u+v+y)[A_\epsilon(z) + A(z)]/2 \\ &+ [uA(u) + vA(v) + yA(y) + zA(z)]/4 \\ &+ (uv+uy+uz+vy+vz+yz)/3 - 3(u^2+v^2+y^2+z^2)/8\,, \end{aligned} \tag{3.83}$$

$$\begin{aligned} E_0(u,v,y,z) = &E(u,v,y,z) \\ &+ A(u)[A_\epsilon(v) + A_\epsilon(y) + A_\epsilon(z)] + A(v)[A_\epsilon(u) + A_\epsilon(y) + A_\epsilon(z)] \\ &+ A(y)[A_\epsilon(u) + A_\epsilon(v) + A_\epsilon(z)] + A(z)[A_\epsilon(u) + A_\epsilon(v) + A_\epsilon(y)] \\ &- (v+y+z)[A_\epsilon(u) + A_{\epsilon^2}(u)]/2 - (u+y+z)[A_\epsilon(v) + A_{\epsilon^2}(v)]/2 \\ &- (u+v+z)[A_\epsilon(y) + A_{\epsilon^2}(y)]/2 - (u+v+y)[A_\epsilon(z) + A_{\epsilon^2}(z)]/2 \\ &+ [uA_\epsilon(u) + vA_\epsilon(v) + yA_\epsilon(y) + zA_\epsilon(z)]/4\,, \end{aligned} \tag{3.84}$$

$$F_3(u,v,y,z) = -(v+y+z)/3\,, \tag{3.85}$$

$$\begin{aligned} F_2(u,v,y,z) = &(v+y+z)A(u)/2u + [A(v) + A(y) + A(z)]/2 \\ &+ (u+v+y+z)/6\,, \end{aligned} \tag{3.86}$$

$$\begin{aligned} F_1(u,v,y,z) = &-[A(v) + A(y) + A(z)]A(u)/u + (v+y+z)A_\epsilon(u)/2u \\ &+ [A_\epsilon(v) + A_\epsilon(y) + A_\epsilon(z) - A(u) - A(v) - A(y) - A(z)]/2 \\ &+ u/2 + (v+y+z)/6\,, \end{aligned} \tag{3.87}$$

$$\begin{aligned} F_0(u,v,y,z) = &F(u,v,y,z) + (v+y+z)A_{\epsilon^2}(u)/2u - [A(v) + A(y) + A(z)]A_\epsilon(u)/u \\ &+ [A(v) + A(y) + A(z) - A_\epsilon(v) - A_\epsilon(y) - A_\epsilon(z) \\ &+ u/4 - v/2 - y/2 - z/2]A(u)/u \\ &+ [A_{\epsilon^2}(v) + A_{\epsilon^2}(y) + A_{\epsilon^2}(z) - A_\epsilon(u) - A_\epsilon(v) - A_\epsilon(y) - A_\epsilon(z)]/2\,, \end{aligned} \tag{3.88}$$

$$G_3(w,u,z,v,y) = -(2w+u+v+y+z)/6\,, \tag{3.89}$$

$$\begin{aligned} G_2(w,u,z,v,y) = {} & [A(u)+A(v)+A(y)+A(z) \\ & - u - v - y - z]/2 + A(w) - 2w/3\,, \end{aligned} \tag{3.90}$$

$$\begin{aligned} G_1(w,u,v,y,z) = {} & I(u,w,z) + I(v,w,y) + A_\epsilon(w) + \big[A_\epsilon(u) + A_\epsilon(v) + A_\epsilon(y) + A_\epsilon(z) \\ & + A(u) + A(v) + A(y) + A(z)\big]/2 \\ & + (w - 2u - 2v - 2y - 2z)/3\,, \end{aligned} \tag{3.91}$$

$$\begin{aligned} G_0(w,u,z,v,y) = {} & G(w,u,z,v,y) + I_\epsilon(u,w,z) + I_\epsilon(v,w,y) - A_{\epsilon^2}(w) \\ & + \big[A_\epsilon(u) + A_\epsilon(v) + A_\epsilon(y) + A_\epsilon(z) \\ & - A_{\epsilon^2}(u) - A_{\epsilon^2}(v) - A_{\epsilon^2}(y) - A_{\epsilon^2}(z)\big]/2\,, \end{aligned} \tag{3.92}$$

$$H_1(u,v,w,x,y,z) = 2\zeta(3)\,, \tag{3.93}$$

$$H_0(u,v,w,x,y,z) = H(u,v,w,x,y,z)\,. \tag{3.94}$$

The finite integral $I(x,y,z)$ is known analytically, but the finite integrals $E(u,v,y,z)$, $F(u,v,y,z)$, $G(w,u,z,v,y)$ and $H(u,v,w,x,y,z)$ are not in general. These integrals must be calculated numerically.[2] This numerical evaluation is implemented in the program `3VIL`, developed by the authors of [13]. In addition, in the limit $u \to 0$, the finite integral $F(u,v,y,z)$ develops a logarithmic IR divergence. For these particular integrals, we use the results of [58].

To summarise, massive vacuum integrals can be solved analytically at 1- and 2-loop. At 3-loop, vacuum integrals may be written in terms of divergent pieces whose analytical form is known, and remainder finite pieces, which in general must be evaluated numerically.

3.6 Renormalisation and Schemes

Renormalisation is implemented by modifying a Lagrangian that produces divergent quantities by adding counterterms, whose inclusion subtracts the divergences from a Lagrangian, leading to finite predictions. Consider the case of QCD in Feynman gauge, with ghosts fields omitted:

$$\begin{aligned} \mathcal{L}_{QCD} = {} & \overline{\psi}^i_0 \left(i\partial\!\!\!/ - m_0\right) \psi^i_0 - g_{0,s} \overline{\psi}^i_0 T^A_{ij} \gamma^\mu \psi^j_0 G^A_{0,\mu} \\ & - \frac{1}{4}\left(\partial_\mu G^A_{0,\nu} - \partial_\nu G^A_{0,\mu}\right)\left(\partial^\mu G^{A\nu}_0 - \partial^\nu G^{A\mu}_0\right) - \frac{1}{2}\left(\partial^\mu G^A_{0,\mu}\right)^2 \end{aligned}$$

[2]Some special cases are known analytically, and are listed in [13].

$$+\frac{g_{0,s}}{2}f^{ABC}\left(\partial_\mu G^A_{0,\nu}-\partial_\nu G^A_{0,\mu}\right)G_0^{B\mu}G_0^{C\nu}-\frac{g^2_{0,s}}{4}f^{ABE}f^{CDE}G^A_{0,\mu}G^B_{0,\nu}G_0^{C\mu}G_0^{D\nu}.$$

The "0" subscripts denote that the quantities in this Lagrangian are "bare", and do not take into account quantum corrections. This bare Lagrangian gives rise to loop diagrams that are UV divergent, and so needs to be modified to subtract these divergences. This is done by rescaling the bare parameters by UV divergent renormalisation constants,

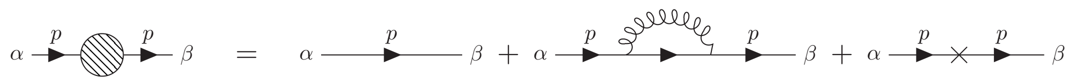
$$\begin{aligned}
\psi_0 &= Z_\psi^{1/2}\psi & G^A_{0,\mu} &= Z_G^{1/2}G^A_\mu \\
m_0 &= Z_m m & g_{0,s} &= Z_g g_s = Z_g g_s(\mu)\mu^\epsilon,
\end{aligned} \tag{3.95}$$

where the parameters on the right hand side of the equalities are called renormalised. Note that the renormalised coupling g_s is related to a dimensionless coupling $g_s(\mu)$ as in Equation (3.38).

Writing the QCD Lagrangian in terms of renormalised parameters (and using $Z = Z+1-1$) gives

$$\begin{aligned}
\mathcal{L}_{QCD} = \Big[&\overline{\psi}^i\left(i\partial\!\!\!/-m\right)\psi^i - g_s\overline{\psi}^i T^A_{ij}\gamma^\mu\psi^j G^A_\mu \\
&-\frac{1}{4}\left(\partial_\mu G^A_\nu-\partial_\nu G^A_\mu\right)\left(\partial^\mu G^{A\nu}-\partial^\nu G^{A\mu}\right)-\frac{1}{2}\left(\partial^\mu G^A_\mu\right)^2 \\
&+\frac{g_s}{2}f^{ABC}\left(\partial_\mu G^A_\nu-\partial_\nu G^A_\mu\right)G^{B\mu}G^{C\nu}-\frac{g_s^2}{4}f^{ABE}f^{CDE}G^A_\mu G^B_\nu G^{C\mu}G^{D\nu}\Big] \\
&+\left(Z_\psi-1\right)\overline{\psi}^i i\partial\!\!\!/\psi^i-\left(Z_\psi Z_m-1\right)m\overline{\psi}^i\psi^i-\left(\mu^\epsilon Z_g Z_\psi Z_G^{1/2}-1\right)g_s\overline{\psi}^i T^A_{ij}\gamma^\mu\psi^j G^A_\mu \\
&-\frac{Z_G-1}{4}\left(\partial_\mu G^A_\nu-\partial_\nu G^A_\mu\right)\left(\partial^\mu G^{A\nu}-\partial^\nu G^{A\mu}\right)-\frac{Z_G-1}{2}\left(\partial^\mu G^A_\mu\right)^2 \\
&+\left(\mu^\epsilon Z_g Z_G^{3/2}-1\right)\frac{g_s}{2}f^{ABC}\left(\partial_\mu G^A_\nu-\partial_\nu G^A_\mu\right)G^{B\mu}G^{C\nu} \\
&-\left(\mu^{2\epsilon}Z_g^2 Z_G-1\right)\frac{g_s^2}{4}f^{ABE}f^{CDE}G^A_\mu G^B_\nu G^{C\mu}G^{D\nu}.
\end{aligned} \tag{3.96}$$

The terms between the square brackets form the original QCD Lagrangian, but now in terms of renormalised quantities. The additional terms that appear are counterterms, which can be treated as interactions and included in the calculation of amplitudes. The undetermined renormalisation constants can be determined by demanding that amplitudes are finite order-by-order in perturbation theory. For example, at 1-loop,



$$\begin{aligned}
&= \frac{i\delta_{\alpha\beta}(p\!\!\!/+m)}{p^2-m^2+i\epsilon}+\left(iC_F\delta_{\alpha\beta}\frac{\alpha_s}{4\pi}\left(p\!\!\!/-4m\right)\frac{1}{\epsilon}+\text{ finite}\right) \\
&\quad+\left(i\left(Z_\psi-1\right)p\!\!\!/\delta_{\alpha\beta}-i\left(Z_\psi Z_m-1\right)m\delta_{\alpha\beta}\right),
\end{aligned} \tag{3.97}$$

where $\alpha_s = g_s^2/(4\pi)$.

Insistence that the dressed ψ propagator (the LHS of Equation (3.97)) is finite leads to two equations to solve for Z_ψ and Z_m (respectively by considering coefficients of $\not{p}/\epsilon$ and m/ϵ). These can be solved to find

$$\begin{aligned} Z_\psi &= 1 - \frac{\alpha_s}{4\pi} C_F \frac{1}{\epsilon} + \mathcal{O}(\alpha_s^2), \\ Z_m &= 1 - \frac{\alpha_s}{4\pi} 3C_F \frac{1}{\epsilon} + \mathcal{O}(\alpha_s^2). \end{aligned} \tag{3.98}$$

In general, renormalisation constants calculated perturbatively may be expanded in powers of coupling constants and powers of ϵ:

$$\begin{aligned} Z &= 1 + \sum_{i=1}^{\infty} \left(\frac{g}{4\pi}\right)^{2i} Z^{(i)}, \\ Z^{(i)} &= \sum_{j=1}^{i} \frac{1}{\epsilon^j} Z^{(i,j)}. \end{aligned} \tag{3.99}$$

The only requirement of renormalisation constants is that they subtract divergences from unrenormalised Lagrangians, leaving finite results. However, there is freedom to define renormalisation constants such that they additionally remove extra finite pieces, with each choice being a renormalisation scheme. Removal of only the pole terms in dimensional regularisation is called the Minimal Subtraction (MS) scheme [40, 59]. Closely related is the Modified Minimal Subtraction ($\overline{\text{MS}}$) scheme [60], in which renormalisation constants are chosen that also subtract finite pieces $\ln(4\pi)$ and $-\gamma_E$, which appear in loop calculations (see Equation (3.31)) and are relics of dimensional regularisation. Note that these terms also appear in the finite part of Equation (3.97). The $\overline{\text{MS}}$ scheme may be implemented practically by dropping $\ln(4\pi) - \gamma_E$ from loop calculations. Alternatively, it may be implemented by dimensionally regularising with the massive parameter $\mu_{\overline{\text{MS}}}$, which is related to μ via [60]

$$\mu = \mu_{\overline{\text{MS}}} \frac{e^{\gamma_E/2}}{\sqrt{4\pi}}. \tag{3.100}$$

Throughout this work we will use the $\overline{\text{MS}}$ scheme unless otherwise stated, and so we simply denote $\mu_{\overline{\text{MS}}}$ by μ.

3.7 Infrared Rearrangement

In mass-independent renormalisation schemes such as the $\overline{\text{MS}}$ scheme, the calculation of renormalisation constants only requires the UV poles of a divergent integral. Additionally, in the $\overline{\text{MS}}$ scheme all UV counterterms are polynomial in masses and momenta [61], which implies that it is possible to extract the UV poles from integrals in the $\overline{\text{MS}}$ scheme by first expanding the integrand in terms of masses and momenta, and then performing the simplified integrals. In general, expanding an integrand before integration can lead to spurious IR divergences, since the expansion removes IR regulators from the denominators of propagators. However, these spurious divergences can be avoided by using the method of IR rearrangement [62].

The method of IR rearrangement rests upon the exact decomposition of a scalar propagator as

$$\frac{1}{(q+p)^2 - m^2} = \frac{1}{q^2 - m_A^2} + \frac{m^2 - p^2 - 2q \cdot p - m_A^2}{q^2 - m_A^2} \frac{1}{(q+p)^2 - m^2}, \tag{3.101}$$

where q is a linear combination of loop momenta, p is a linear combination of external momenta, m is the propagator mass, and m_A is an auxiliary mass. Denoting the degree of divergence of a Feynman integral by D, the physical propagator on the left hand side of Equation (3.101) has a contribution to the degree of divergence of $\Delta D = -2$. The first term of the right hand side of Equation (3.101) also has $\Delta D = -2$, while the second term has $\Delta D = -3$ (due to the numerator term linear in q). Importantly, the second term contains the original propagator, which means this decomposition can be repeated arbitrarily many times. Repeatedly applying the decomposition leads to a series of terms, of which all except one have simple denominators involving only the loop momentum q and the auxiliary mass m_A. The final term has a more complicated denominator but a degree of divergence that is arbitrarily negative. For example,

$$\begin{aligned}
\underbrace{\frac{1}{(q+p)^2 - m^2}}_{\Delta D = -2} &\underset{1}{=} \underbrace{\frac{1}{q^2 - m_A^2}}_{\Delta D = -2} + \underbrace{\frac{m^2 - p^2 - 2q \cdot p - m_A^2}{q^2 - m_A^2} \frac{1}{(q+p)^2 - m^2}}_{\Delta D = -3} \\
&\underset{2}{=} \underbrace{\frac{1}{q^2 - m_A^2}}_{\Delta D = -2} + \underbrace{\frac{m^2 - p^2 - 2q \cdot p - m_A^2}{\left(q^2 - m_A^2\right)^2}}_{\Delta D = -3} \\
&\qquad + \underbrace{\left(\frac{m^2 - p^2 - 2q \cdot p - m_A^2}{q^2 - m_A^2}\right)^2 \frac{1}{(q+p)^2 - m^2}}_{\Delta D = -4} \\
&\underset{3}{=} \underbrace{\frac{1}{q^2 - m_A^2}}_{\Delta D = -2} + \underbrace{\frac{m^2 - p^2 - 2q \cdot p - m_A^2}{\left(q^2 - m_A^2\right)^2}}_{\Delta D = -3} + \underbrace{\frac{\left(m^2 - p^2 - 2q \cdot p - m_A^2\right)^2}{\left(q^2 - m_A^2\right)^3}}_{\Delta D = -4} \\
&\qquad \underbrace{\left(\frac{m^2 - p^2 - 2q \cdot p - m_A^2}{q^2 - m_A^2}\right)^3 \frac{1}{(q+p)^2 - m^2}}_{\Delta D = -5}
\end{aligned}$$

$$
\underset{n}{=} \sum_{i=1}^{n} \frac{\left(m^2 - p^2 - 2q \cdot p - m_A^2\right)^{i-1}}{\left(q^2 - m_A^2\right)^i}
$$

$$
+ \underbrace{\left(\frac{m^2 - p^2 - 2q \cdot p - m_A^2}{q^2 - m_A^2}\right)^n \frac{1}{(q+p)^2 - m^2}}_{\Delta D = -2-n} \,. \qquad (3.102)
$$

This explicitly shows that the expansion results in a sum of terms with denominators typical of massive vacuum diagrams, with a final term with an arbitrarily large negative degree of divergence. Note that the numbers under the equality denote the number of times the propagator decomposition has been performed.

IR rearrangement works by expanding every propagator of a Feynman integral using the exact propagator decomposition until the final (non-vacuum) integral is UV convergent. Since the final integral contains no UV pole, it can be ignored when extracting UV divergences. The expansion cannot introduce spurious IR divergences since the expansion is exact, and IR safety is guaranteed by the presence of the auxiliary mass m_A in denominators. Once the final term is dropped, the remaining integrals can be evaluated using standard methods to find the UV pole structure of the original integral, from which counterterms can be calculated.

Note that terms arise in the decomposition whose numerators are proportional to m_A^2, which result in UV divergences multiplied by m_A^2. These terms are local after subtraction of subdivergences, and must cancel other UV-divergent terms that are also proportional to m_A^2. This is evident since m_A is a non-physical mass that arises from an exact decomposition, and so there can be no dependence on m_A at the end of the calculation. This leads to the calculational trick whereby such integrals with m_A^2 in the numerator are not evaluated, but replaced by local counterterms proportional to m_A^2 that cancel any UV divergences proportional to m_A^2 that arise from integrals not containing m_A^2 in the numerator. The number of such counterterms is typically small, since they must have mass dimension two less than the dimension of the effective Lagrangian (since it is multiplied by m_A^2). In QCD, the only counterterm has the form of a gluon mass term [63],

$$
\frac{1}{2} m_A^2 Z_x G^{a\mu} G^a_\mu \,, \qquad (3.103)
$$

where at 1-loop in the Feynman-'t Hooft gauge [62]

$$
Z_x = -\frac{g_s^2}{16\pi^2}(n_c + 2n_f) \,. \qquad (3.104)
$$

3.8 Renormalisation Group Equations

In dimensional regularisation, renormalised couplings and masses become functions of the arbitrary dimensionful parameter μ. The dependence of couplings and masses with respect to μ is given by the renormalisation group equations (RGEs) [64–66]. RGEs for each coupling and mass may be found by using the fact that there is no μ dependence in bare parameters, and therefore the derivative of any bare parameter with respect to μ must be zero. Considering the bare QCD coupling, it follows that (using Equation (3.95))

$$0 = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_{0,s} = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \left(\mu^\epsilon Z_g(g_s) g_s(\mu) \right) \tag{3.105}$$

$$= \epsilon \mu^\epsilon Z_g(g_s) g_s(\mu) + \mu^\epsilon \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z_g(g_s) \right) g_s(\mu) + \mu^\epsilon Z_g(g_s) \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_s(\mu) \right), \tag{3.106}$$

where it is noted that the renormalisation constant $Z_g(g_s)$ may be expanded in terms of $g_s(\mu)$, and is therefore implicitly a function of μ. Hence,

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_s(\mu) \equiv \beta(g_s, \epsilon) = -\epsilon g_s(\mu) - g_s(\mu) \frac{1}{Z_{g_s}} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z_g(g_s). \tag{3.107}$$

The beta function describes the running of the coupling $g_s(\mu)$ with respect to μ. Completely analogously, considering the μ-independence of the bare mass m it can be shown that

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} m(\mu) \equiv -\gamma_m \left(g_s(\mu) \right) m(\mu) = -\frac{1}{Z_m(g_s)} \frac{\mathrm{d} Z_m(g_s)}{\mathrm{d}\mu} m(\mu), \tag{3.108}$$

where $\gamma_m(g_s)$ is the anomalous mass dimension. The renormalisation group functions $\beta(g_s, \epsilon)$ and $\gamma_m(g_s)$ only depend on the coupling g_s when calculating in the $\overline{\mathrm{MS}}$ scheme, and are mass-independent. They must also be finite everywhere since they are physical quantities. This observation allows the renormalisation group functions to be written in terms of the $1/\epsilon$ coefficient of their respective renormalisation constants [67]. For example, if $\beta(g_s, \epsilon)$ is finite, the quantity

$$\frac{1}{Z_g(g_s)} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z_g(g_s) \equiv f(g_s)$$

must also be finite. This can be rewritten as

$$f(g_s) Z_g(g_s) = \mu \frac{\mathrm{d} g_s}{\mathrm{d}\mu} \frac{\mathrm{d} Z_g(g_s)}{\mathrm{d} g_s} = \beta(g_s, \epsilon) \frac{\mathrm{d} Z_g(g_s)}{\mathrm{d} g_s}. \tag{3.109}$$

Expanding the renormalisation constants in powers of ϵ as

$$Z_g(g_s) = 1 + \sum_{i=1}^{\infty} \frac{Z_{g,i}(g_s)}{\epsilon^i}, \tag{3.110}$$

and using this expansion in Equation (3.109) gives

$$f(g_s) \left(1 + \frac{Z_{g,1}}{\epsilon} + \frac{Z_{g,2}}{\epsilon^2} + \dots \right) = \beta(g_s, \epsilon) \left(\frac{1}{\epsilon} \frac{\mathrm{d} Z_{g,1}}{\mathrm{d} g_s} + \frac{1}{\epsilon^2} \frac{\mathrm{d} Z_{g,2}}{\mathrm{d} g_s} + \dots \right). \tag{3.111}$$

Since both $\beta(g_s,\epsilon)$ and $f(g_s)$ are finite, the above equality must hold separately for every power of ϵ. Using $\beta(g_s,\epsilon) = -\epsilon g_s + \mathcal{O}(\epsilon^0)$, it follows that

$$f(g_s) = -g_s(\mu)\frac{\mathrm{d}Z_{g,1}}{\mathrm{d}g_s}, \tag{3.112}$$

and therefore that

$$\beta(g_s,\epsilon) = -\epsilon g_s(\mu) + g_s^2(\mu)\frac{\mathrm{d}Z_{g,1}}{\mathrm{d}g_s}. \tag{3.113}$$

Similarly it may be shown that

$$\gamma_m(g_s) = \frac{\beta(g_s,\epsilon)}{Z_m}\frac{\mathrm{d}Z_m(g_s)}{\mathrm{d}g_s} = -g_s(\mu)\frac{\mathrm{d}Z_{m,1}}{\mathrm{d}g_s}. \tag{3.114}$$

Therefore, renormalisation group functions may be found by computing renormalisation constants to whatever order is required, extracting the coefficient of the $1/\epsilon$ pole, and differentiating it with respect to $g_s(\mu)$.

Once RGEs are obtained to some order in a perturbative coupling, they may be solved. This is particularly simple at 1-loop, where the strong coupling is [42]

$$\frac{\alpha_s(\mu)}{4\pi} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2_{\mathrm{QCD}})}, \tag{3.115}$$

where

$$\beta_0 = \frac{11n_c - 2n_f}{3},$$

n_c is the number of colours, n_f is the number of active flavours, and Λ_{QCD} is the QCD scale, where perturbation theory fails. The running mass is given by [53]

$$m(\mu) = m(\mu_0)\exp\left[-\int_{g(\mu_0)}^{g(\mu)} \mathrm{d}g'\, \frac{\gamma_m(g')}{\beta(g')}\right], \tag{3.116}$$

which to leading order is given by [44, 53]

$$m(\mu) = m(\mu_0)\left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right]^{\frac{\gamma_m^{(0)}}{2\beta_0}}. \tag{3.117}$$

When we discuss the running of Wilson coefficients, we will see that they have analogous expressions.

3.9 Tensor Integral Decomposition

As discussed in Sections 3.1 and 3.5, during loop calculations, tensor integrals arise that can be decomposed into scalar integrals. In using the method of expansion by regions, integrals would be generated that need to be decomposed. For example, expanding the integral $\mathcal{I}^{(3)}_{\text{vac}}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2; 1, 1, 1, 1, 1, 1)$ in the hard-soft-soft (hss) region (where $q_1 \gg q_2, q_3$) would generate integrals of the form

$$\mathcal{I}(i,j,k,l,m) = \\ \iiint \frac{\mathrm{d}^d q_1\, \mathrm{d}^d q_2\, \mathrm{d}^d q_3}{((2\pi)^d)^3} \frac{(q_1 \cdot q_2)^i (q_1 \cdot q_3)^j (q_1^2)^k (q_2^2)^l (q_3^2)^m}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)((q_2 - q_3)^2 - m_4^2)}. \tag{3.118}$$

To bring this into a standard form it is necessary to remove all numerator factors, which involves the removal of 'crossed' scalar products of the form $q_a \cdot q_b$. This is done by decomposing the crossed products using $q_a \cdot q_b = g_{\mu\nu} q_a^\mu q_b^\nu$, such that

$$\mathcal{I}(i,j,k,l,m) = f(g) \int \frac{\mathrm{d}^d q_1}{(2\pi)^d} \frac{(q_1^2)^k \left(\prod_{a=1}^{i+j} q_1^{\mu_a}\right)}{(q_1^2 - m_1^2)} \\ \times \iint \frac{\mathrm{d}^d q_2 \mathrm{d}^d q_3}{((2\pi)^d)^2} \frac{(q_2^\nu)^i (q_3^\rho)^j (q_2^2)^l (q_3^2)^m}{(q_2^2 - m_2^2)(q_3^2 - m_3^2)((q_2 - q_3)^2 - m_4^2)}, \tag{3.119}$$

where $f(g)$ is a product of $(i+j)$ metrics. Note that this decomposition splits the 3-loop scalar integral into the product of a 1-loop tensor integral and a 2-loop tensor integral, which can be individually decomposed. If there are an odd number of q_1^μ momenta in the q_1 integral, then the integral will be odd and hence equal to zero. Therefore a non-zero integral requires

$$i + j = 2n, \qquad n \in \mathbb{Z}. \tag{3.120}$$

Consider the simple example $\mathcal{I}(2,2,0,0,0)$. For notational simplicity, we suppress integral measures and factors of 2π using the notation

$$\int_p \equiv \int \frac{\mathrm{d}^d p}{(2\pi)^d}.$$

Then,

$$\begin{aligned} \mathcal{I}(2,2,0,0,0) &= \iiint_{q_1,q_2,q_3} \frac{(q_1 \cdot q_2)^2 (q_1 \cdot q_3)^2}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)((q_2 - q_3)^2 - m_4^2)} \\ &= g_{\alpha\beta} g_{\lambda\mu} g_{\nu\rho} g_{\sigma\tau} \int_{q_1} \frac{q_1^\alpha q_1^\lambda q_1^\nu q_1^\sigma}{(q_1^2 - m_1^2)} \iint_{q_2,q_3} \frac{q_2^\beta q_2^\mu q_3^\rho q_3^\tau}{(q_2^2 - m_2^2)(q_3^2 - m_3^2)((q_2 - q_3)^2 - m_4^2)}. \end{aligned}$$

The 1-loop integral can be decomposed using the relation [42]

$$\int_p \frac{p^\mu p^\nu p^\rho p^\sigma}{(p^2 - m^2)^n} = \frac{g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}}{d(d+2)} \int_p \frac{(p^2)^2}{(p^2 - m^2)^n}, \tag{3.121}$$

which leads to

$$\mathcal{I}(2,2,0,0,0) = \frac{1}{d(d+2)} \int_{q_1} \frac{(q_1^2)^2}{(q_1^2 - m_1^2)} \iint_{q_2,q_3} \frac{q_2^2\, q_3^2 + 2(q_2 \cdot q_3)^2}{(q_2^2 - m_2^2)(q_3^2 - m_3^2)((q_2 - q_3)^2 - m_4^2)}.$$

The combination of metrics that arises from the tensor decomposition is the unique totally symmetric Lorentz structure, given the Lorentz indices of the tensor integral. The double integral may be brought into the required form using

$$q_2 \cdot q_3 = \frac{1}{2}\left(q_2^2 + q_3^2 - (q_2 - q_3)^2\right),$$

and then using partial fraction decomposition. This yields

$$\begin{aligned}
\mathcal{I}(2,2,0,0,0) = \frac{1}{d(d+2)} \Bigg\{ & \left(2m_1^4 m_2^2 + \frac{m_1^4 m_3^2}{2} - m_1^4 m_4^2\right) \\
& \times \int_{q_1} \frac{1}{q_1^2 - m_1^2} \iint\limits_{q_2,q_3} \frac{1}{(q_2^2 - m_2^2)((q_2 - q_3)^2 - m_4^2)} \\
& + \left(-m_1^4 m_2^2 - m_1^4 m_3^2 + \frac{m_1^4 m_4^2}{2}\right) \int_{q_1} \frac{1}{q_1^2 - m_1^2} \iint\limits_{q_2,q_3} \frac{1}{(q_2 - m_2^2)(q_3^2 - m_3^2)} \\
& + \frac{m_1^4}{2} \int_{q_1} \frac{1}{q_1^2 - m_1^2} \iint\limits_{q_2,q_3} \frac{(q_2 - q_3)^2}{(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \\
& + \left(\frac{m_1^4 m_2^2}{2} + 2m_1^4 m_3^2\right) \int_{q_1} \frac{1}{q_1^2 - m_1^2} \iint\limits_{q_2,q_3} \frac{1}{(q_3^2 - m_3^2)((q_2 - q_3)^2 - m_4^2)} \\
& + \left(\frac{m_1^4 m_2^4 + m_1^4 m_3^4 + m_1^4 m_4^4}{2} + 2m_1^4 m_2^2 m_3^2 - m_1^4 m_2^2 m_4^2 - m_1^4 m_3^2 m_4^2\right) \\
& \times \int_{q_1} \frac{1}{q_1^2 - m_1^2} \iint\limits_{q_2,q_3} \frac{1}{(q_2^2 - m_2^2)(q_3^2 - m_3^2)((q_2 - q_3)^2 - m_4^2)} \\
& + \frac{m_1^4}{2} \int_{q_1} \frac{1}{q_1^2 - m_1^2} \iint\limits_{q_2,q_3} \frac{q_3^2}{(q_2^2 - m_2^2)((q_2 - q_3)^2 - m_4^2)} \\
& + \frac{m_1^4}{2} \int_{q_1} \frac{1}{q_1^2 - m_1^2} \iint\limits_{q_2,q_3} \frac{q_2^2}{(q_3^2 - m_3^2)((q_2 - q_3)^2 - m_4^2)},
\end{aligned} \tag{3.122}$$

in which all integrals are now in a standard form (some with negative powers). Note that scaleless integrals are discarded since they are vanishing in dimensional regularisation. This process has taken a scalar integral that was not in standard form, and performed a tensor decomposition on a single integral in order to rewrite the original integral as a series of scalar integrals in standard form.

The crux of this procedure is the tensor decomposition of the 1-loop integral in terms of metrics, which determine how the remaining q_2 and q_3 are contracted. This tensor integral decomposition is simple for low-rank tensor integrals, but as the rank increases, the number of metric combinations increases, as shown in Table 3.2. Tensor integral decomposition was performed automatically in a brute force approach, by forming all possible metric combinations from the available Lorentz indices, and then contracting all remaining indices. Since it was intended to perform expansions up to the twentieth

Rank of 1-loop tensor integral	No. of metric combinations
2	1
4	3
6	15
8	105
10	945
12	10,395
14	135,135
16	2,027,025
18	34,459,425
20	654,729,075

Table 3.2: Table enumerating the number of unique ways of building a rank-$2n$ tensor from metrics.

power, the highest-rank integrals encountered required the generation of $\mathcal{O}(10^8)$ terms, and their contraction. This is highly computationally intensive, and consequently it is not possible to perform expansions to the desired order in this way.

To find the numbers listed in Table 3.2, consider the symmetries of tensor integrals. A rank-$2n$ tensor integral can be decomposed into sums of the products of n metrics. A group of $2n$ Lorentz indices may be ordered in $(2n)!$ ways. However, since each metric is symmetric, 2^n of the $(2n)!$ orderings are redundant. Furthermore, since metrics commute, there is an additional redundancy of $n!$, reflecting all the ways the metrics may be ordered. Therefore, there are

$$\frac{(2n)!}{2^n n!} = (2n-1)!! \tag{3.123}$$

unique ways of constructing a rank-$2n$ tensor from metrics. Equation (3.123) may be proved inductively, and introduces the double factorial operator, defined as

$$k!! = \begin{cases} \prod_{j=1}^{k/2} (2j) & \text{even } k, \\ \\ \prod_{j=1}^{\frac{k+1}{2}} (2j-1) & \text{odd } k. \end{cases} \tag{3.124}$$

For example, a rank-8 tensor integral has $n = 4$, and so there are $7 \cdot 5 \cdot 3 \cdot 1 = 105$ ways of forming a rank-8 tensor purely from metrics.

As well as forming $(2n-1)!!$ products of metrics, it is also necessary to know the constant of proportionality generated in tensor integral decomposition. This can be calculated for an individual case by contracting the metrics generated by tensor decomposition. For example, for a rank-2 tensor integral in d dimensions, Lorentz invariance requires

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{q^2 - m^2} = \kappa_2 g^{\mu\nu} \int_{-\infty}^{\infty} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q^2}{q^2 - m^2}, \tag{3.125}$$

and by contracting both sides by $g_{\mu\nu}$ it follows that $\kappa_2 = 1/d$. This strategy can be performed for arbitrary-rank tensors, and the pattern emerges that for a rank-$2n$ tensor integral,

$$\kappa_{2n} = \prod_{i=0}^{n-1} \frac{1}{d+2i}. \tag{3.126}$$

Using the results so far, a scalar integral involving scalar products of different loop momenta may be written as

$$\begin{aligned}
&\mathcal{I}(i,j,k,l,m) \\
&\quad = \iiint_{q_1,q_2,q_3} \frac{(q_1\cdot q_2)^i (q_1\cdot q_3)^j (q_1^2)^k (q_2^2)^l (q_3^2)^m}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)((q_2-q_3)^2-m_4^2)} \\
&\quad = f(g)\int_{q_1} \frac{(q_1^2)^k \left(\prod_{a=1}^{i+j} q_1^{\mu_a}\right)}{(q_1^2-m_1^2)} \iint_{q_2,q_3} \frac{(q_2^\nu)^i (q_3^\rho)^j (q_2^2)^l (q_3^2)^m}{(q_2^2-m_2^2)(q_3^2-m_3^2)((q_2-q_3)^2-m_4^2)} \\
&\quad = f(g)\Sigma(g)\left[\prod_{a=0}^{\frac{i+j}{2}-1} \frac{1}{d+2a}\right] \int_{q_1} \frac{(q_1^2)^{k+\frac{i+j}{2}}}{(q_1^2-m_1^2)} \iint_{q_2,q_3} \frac{(q_2^\nu)^i (q_3^\rho)^j (q_2^2)^l (q_3^2)^m}{(q_2^2-m_2^2)(q_3^2-m_3^2)((q_2-q_3)^2-m_4^2)},
\end{aligned} \tag{3.127}$$

where $\Sigma(g)$ represents the totally symmetric rank-$(i+j)$ Lorentz-invariant tensor. Note that the first equality follows from simply splitting up scalar products of loop momenta and removing the corresponding metrics into the product $f(g)$, and the second follows from performing a tensor decomposition of the 1-loop q_1 integral. The tensor $\Sigma(g)$ is totally symmetric on the $(i+j)$ Lorentz indices of the q_1 integral, and the tensor $f(g)$ is a product of $(i+j)$ metrics, each metric containing one index from the 1-loop integral and one index from the 2-loop integral. Since $g_{\mu\nu}g^{\nu\rho} = \delta_\mu^\rho$, the product $f(g)\Sigma(g)$ contracts the indices of the 1-loop integral, and replaces them with indices of the two loop integral. Consequently

$$h(g) \equiv f(g)\Sigma(g) \tag{3.128}$$

is a totally symmetric rank-$(i+j)$ tensor whose indices are those of the 2-loop tensor integral. It then remains to find how Lorentz indices contract in the product

$$h(g)\iint_{q_2,q_3} \frac{(q_2^\nu)^i (q_3^\rho)^j (q_2^2)^l (q_3^2)^m}{(q_2^2-m_2^2)(q_3^2-m_3^2)((q_2-q_3)^2-m_4^2)}.$$

Consider

$$\mathcal{S} = h(g)\iint_{q_2,q_3} (q_2^\nu)^i (q_3^\rho)^j, \tag{3.129}$$

which contains all relevant tensor quantities. Since $h(g)$ is totally symmetric over all of the Lorentz indices of the integral, all possible scalar products between q_2 and q_3 will be generated, and so

$$
\begin{aligned}
\mathcal{S} = \tilde{c}_0 \iint\limits_{q_2,q_3} (q_2^2)^{i/2}(q_3^2)^{j/2} &+ \tilde{c}_1 \iint\limits_{q_2,q_3} (q_2\cdot q_3)(q_2^2)^{\frac{i-1}{2}}(q_3^2)^{\frac{j-2}{2}} \\
&+ \tilde{c}_2 \iint\limits_{q_2,q_3} (q_2\cdot q_3)^2(q_2^2)^{\frac{i-2}{2}}(q_3^2)^{\frac{j-2}{2}} + \dots \\
= \sum_{x=0}^{\min(i,j)} \tilde{c}_x \iint\limits_{q_2,q_3} (q_2\cdot q_3)^x (q_2^2)^{\frac{i-x}{2}}(q_3^2)^{\frac{j-x}{2}}, \qquad (3.130)
\end{aligned}
$$

where the $\tilde{c}_x$ are integer coefficients to be determined. As stated previously, it is necessary for $i+j$ to be even, as otherwise the q_1 integral will be odd, and therefore zero.

The coefficients $\tilde{c}_x$ enumerate how many ways there are to form x "crossed pairs" of the form $(q_2 \cdot q_3)$. Denoting by X the number of ways of pairing up i lots of q_2 momenta and j lots of q_3 momenta into x pairs (where $x \leq \frac{i+j}{2}$), then

$$
X = \frac{[ij][(i-1)(j-1)][(i-2)(j-2)]\dots[(i-x+1)(j-x+1)]}{x!} = \frac{i!j!}{x!(i-x)!(j-x)!}, \qquad (3.131)
$$

where the factor of $x!$ arises since it does not matter in which order the momenta are paired up. Note that X is symmetric under $i \leftrightarrow j$ as expected. It is also necessary to enumerate the number of ways of pairing up the remaining momenta amongst themselves (to form products q_2^2 and q_3^2) after forming x "crossed" pairs. Consider the q_2 momenta, of which there remain $(i-x)$ momenta to be paired amongst themselves. Denote the number of ways of forming the product q_2^2 by Y_i. There are $(i-x)!$ ways of pairing the momenta, but since it doesn't matter in which order the pairs are formed, there is a suppression of $\frac{i-x}{2}!$. Additionally, the ordering of the momenta within each pair does not matter, leading to a further suppression of $2^{\frac{i-x}{2}}$. Therefore,

$$
Y_i = \frac{(i-x)!}{2^{\frac{i-x}{2}}\left(\frac{i-x}{2}\right)!} = Y_i = (i-x-1)!!, \qquad (3.132)
$$

where the second equality comes from Equation (3.123). This expression trivially generalises for self-pairing $(j-x)$ momenta of type q_3 to give $Y_j = (j-x-1)!!$.

Combining the expressions for X, Y_i, and Y_j, an expression is obtained for the total number of ways of pairing up i momenta of type q_2 and j momenta of type q_3 with x crossed pairs:

$$
\tilde{c}_x = \frac{i!j!(i-x-1)!!(j-x-1)!!}{x!(i-x)!(j-x)!} = \frac{i!j!}{x!(i-x)!!(j-x)!!}. \qquad (3.133)
$$

Therefore the integral

$$\mathcal{I}(i,j,k,l,m) = \iiint \frac{\mathrm{d}^d q_1\,\mathrm{d}^d q_2\,\mathrm{d}^d q_3}{((2\pi)^d)^3} \frac{(q_1\cdot q_2)^i (q_1\cdot q_3)^j (q_1^2)^k (q_2^2)^l (q_3^2)^m}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)((q_2-q_3)^2-m_4^2)},$$

may be rewritten in a standard form using tensor decomposition as

- i,j even

$$\mathcal{I}(i,j,k,l,m) = \left[\prod_{a=0}^{\frac{i+j}{2}-1} \frac{1}{d+2a}\right] \left(\sum_{\substack{x=0 \\ x \text{ even}}}^{\min(i,j)} \frac{i!j!}{x!(i-x)!!(j-x)!!} \right.$$
$$\left. \times \int_{q_1} \frac{(q_1^2)^{k+\frac{i+j}{2}}}{(q_1^2-m_1^2)} \iint_{q_2,q_3} \frac{(q_2^2)^{l+\frac{i-x}{2}} (q_3^2)^{m+\frac{j-x}{2}} (q_2\cdot q_3)^x}{(q_2^2-m_2^2)(q_3^2-m_3^2)((q_2-q_3)^2-m_4^2)} \right) \tag{3.134}$$

- i,j odd

$$\mathcal{I}(i,j,k,l,m) = \left[\prod_{a=0}^{\frac{i+j}{2}-1} \frac{1}{d+2a}\right] \left(\sum_{\substack{x=1 \\ x \text{ odd}}}^{\min(i,j)} \frac{i!j!}{x!(i-x)!!(j-x)!!} \right.$$
$$\left. \times \int_{q_1} \frac{(q_1^2)^{k+\frac{i+j}{2}}}{(q_1^2-m_1^2)} \iint_{q_2,q_3} \frac{(q_2^2)^{l+\frac{i-x}{2}} (q_3^2)^{m+\frac{j-x}{2}} (q_2\cdot q_3)^x}{(q_2^2-m_2^2)(q_3^2-m_3^2)((q_2-q_3)^2-m_4^2)} \right). \tag{3.135}$$

There is a consistency check for the expression for $\tilde{c}_x$. Since there are $(i+j-1)!!$ metric combinations in $h(g)$, there must be a total of $(i+j-1)!!$ possible contractions, which implies the sum of the $\tilde{c}_x$ coefficients must be $(i+j-1)!!$,

$$\sum_{x}^{\min(i,j)} \frac{i!j!}{x!(i-x)!!(j-x)!!} = (i+j-1)!!.$$

This was verified for all possible combinations of i and j for all values of x up to and including $x=20$, which was the desired maximum value of x.

This tensor decomposition result was simple to implement in `Mathematica`, allowing the decomposition of arbitrarily high-rank tensor integrals in millisecond time.

Chapter 4

Effective Field Theories

Effective Field Theories (EFTs) provide a framework in which calculations can be organised in a systematic manner to include relevant physics while ignoring higher-energy physics that is irrelevant at the scale of interest. This simplifies the calculation of low-energy physics, but exactly reproduces the results of the underlying full theory in the IR limit [68–70]. As well as simplifying calculations, EFTs can also be used as probes of new physics. In this approach, the SM is an EFT describing a more fundamental UV theory that is currently unknown, which can be used to calculate processes at low energies. Effects of heavy particles that exist beyond the SM are treated in Standard Model Effective Field Theory (SMEFT) [8, 9]. These calculations can then constrain what types of new physics may exist.

4.1 EFTs and Particle Decoupling

Effective Field Theories in particle physics are built on the principle that low-energy processes should be explicable in terms of physics of a similar scale - contributions from heavy particles should be negligible when there is not enough energy to produce such particles. At low energies, heavy particles cannot be produced on-shell, and therefore cannot exist as external states. Additionally, heavy particles arising as virtual particles have masses much larger than typical momentum transfer for low-energy processes. Consequently, propagators of heavy particles may be Taylor expanded in powers of momenta over mass, with fast convergence. Expanding propagators of heavy particles removes the heavy particles as degrees of freedom, and transforms renormalisable non-local interactions into non-renormalisable local interactions.

For example, the four-Fermi theory of weak interactions is an effective field theory. In the full theory (the Glashow-Salam-Weinberg (GSW) theory), there exists a massive W-boson that mediates flavour-changing processes. The W couples two fermion lines

Fig. 4.1: Integrating out the W-boson from GSW theory yields the four-Fermi theory, which is very successful at describing low-energy weak interactions.

together, with each coupling being dimension-four, and therefore renormalisable. Low-energy weak interactions are described by four-Fermi theory, in which there is no W-boson, and fermions interact at a point. The W-boson propagator (in Feynman gauge) is expanded as

$$\frac{-ig_{\mu\nu}}{p^2 - M_W^2} = \frac{-ig_{\mu\nu}}{-M_W^2}\left(\frac{1}{1-\frac{p^2}{M_W^2}}\right) = \frac{ig_{\mu\nu}}{M_W^2}\left(1 + \frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + \dots\right), \tag{4.1}$$

which leads to the creation of local non-renormalisable interactions, shown in Figure 4.1.

The local interactions of four-Fermi theory have mass dimension-six (in $d = 4$ dimensions), and are non-renormalisable. The coupling constant of four-Fermi theory, the Fermi constant G_F, contains factors of the full theory coupling, g_2, and a suppression by the W-mass, $G_F \propto g_2^2/M_W^2$. This expression for the Fermi constant is obtained via the process of matching, to be discussed below.

In general, an EFT is constructed in the following way [63, 68, 71]. Starting at a high scale with a renormalisable Lagrangian of light and heavy fields, $\mathcal{L}(\{\ell\}, H)$, the scale μ is reduced until it is less than the heavy mass m_H. Terms containing only the light fields $\{\ell\}$ are isolated, while the remaining part (involving both $\{\ell\}$ and H) is treated as a perturbation to the 'light' Lagrangian, with the heavy fields being integrated out:

$$\mathcal{L}(\{\ell\}, H) \xrightarrow{\mu < m_H} \mathcal{L}(\ell_i) + \delta\mathcal{L}(\ell_i). \tag{4.2}$$

Integrating out the heavy fields H creates a series of non-renormalisable operators, allowing $\delta\mathcal{L}(\{\ell\})$ to be written as an operator product expansion (in d dimensions),

$$\delta\mathcal{L}(\{\ell\}) = \sum_{i=1}^{\infty} \frac{1}{\Lambda^i} \sum_j C_j^{(d+i)} Q_j^{(d+i)}. \tag{4.3}$$

The parameter Λ is some massive parameter (often m_H), which ensures the Lagrangian remains d-dimensional. It acts as a cut-off for the validity of the expansion, with the effective theory expected to break down at scales of order Λ. The coefficients C_i are Wilson coefficients, and are effectively coupling constants for higher mass-dimension operators Q_i. The operators Q_i are exclusively built from the light fields ℓ_i of the effective theory. All possible Q_i that observe the symmetries of the theory (which typically include Lorentz and gauge symmetries) are included in the expansion of Equation (4.3). The expansion is organised by the mass-dimension of the operators, with a number of different operators possible at each order.

Equation (4.3) is written in terms of bare parameters, which need to be renormalised. In order to fully specify the theory, it is necessary to choose a renormalisation scheme. In a mass-dependent renormalisation scheme, the decoupling of heavy particles arises naturally, as at scales $\mu < m_H$ the contribution of large-mass particles to quantities such as the β-function is suppressed by powers of m_H. The effects of heavy particles to low energy processes then results in a rescaling of renormalisation and coupling constants, and the heavy particles do not arise as dynamical degrees of freedom that need to be included in calculations. This is the content of the Appelquist-Carazonne theorem [72]. Mass-independent schemes like $\overline{\text{MS}}$ do not include suppressions of heavy particles at low energies, and so do not automatically yield particle decoupling. In addition, the lack of decoupling leads to an 'incorrect' β-function, with an associated problem of large logarithms for low-momentum-transfers [68–70].

Both of these problems of the $\overline{\text{MS}}$ scheme are overcome by manually integrating out particles. Calculating in $\overline{\text{MS}}$, at the threshold $\mu = m_H$, the heavy particle H is removed as a dynamical degree of freedom, leading to a new low-energy theory with fewer degrees of freedom. The new theory is formed in accordance with Equations (4.2) and (4.3), with the requirement that amplitudes calculated in both theories at the scale $\mu = m_H$ must agree. These are called the matching conditions, and incorporate the residual effect of high-mass particles on low-energy parameters. Whenever a particle is integrated out, the new effective theory is strictly a new theory, and fields and parameters of the high- and low-energy theories may not be naively interchanged. For example, the QCD β-function in $\overline{\text{MS}}$ contains a term that is proportional to the number of quark flavours. If a quark is integrated out in passing between the full and effective theories, then there are fewer flavours in the effective theory than the full theory, leading to a different β-function [69, 70].

The advantage of using $\overline{\text{MS}}$ over mass-dependent schemes is two-fold. Firstly, calculations are simplified. More importantly, when evaluating loop diagrams involving higher-dimension operators in a mass-dependent scheme, the high-momentum region of the loop integrals are proportional to m_H^2 (at 1-loop), the mass of the heavy particle that is integrated out. This loop contribution cancels the suppression factor $1/m_H^2$ of the effective operator, and the effective operator is no longer suppressed. Therefore it is not possible in a mass-dependent scheme to truncate the operator product expansion, and predictivity is only achieved by considering the full infinite tower of effective operators. However, in a mass-independent scheme, all scale dependence resides in logarithms such as $\ln(\mu)$, which do not remove the suppression provided by m_H. This allows the operator product expansion of Equation (4.3) to be truncated to some appropriate power, allowing calculations to be predictive [68, 69].

4.2 Types of Effective Operators

The general prescription for building an effective theory is to include in the Lagrangian every possible operator allowed by the symmetries of the theory up to some chosen mass dimension. However, by following the approach of writing down all possible allowed operators, this basis will typically be linearly dependent, since operators (or combinations of operators) that superficially look different may be shown to be identical through equations of motion. Therefore, in order to build a linearly independent basis, it is important to study the equations of motion. For example, the equation of motion for the left-handed lepton doublet in the SM is

$$\frac{\partial \mathcal{L}_{\rm SM}}{\partial \bar{\ell}} = \partial_\mu \left(\frac{\partial \mathcal{L}_{\rm SM}}{\partial(\partial_\mu \bar{\ell})} \right) \implies i \not{D} \ell = \Gamma_e e \varphi \,, \tag{4.4}$$

while the EoM for the Higgs doublet is [8]

$$\frac{\partial \mathcal{L}_{\rm SM}}{\partial \varphi^\dagger} = \partial_\mu \left(\frac{\partial \mathcal{L}_{\rm SM}}{\partial(\partial_\mu \varphi)^\dagger} \right) \implies (D^\mu D_\mu \varphi)^j = m^2 \varphi^j - \lambda(\varphi^\dagger \varphi)\varphi^j - \bar{e}\Gamma_e^\dagger \ell^j + \varepsilon_{jk} \bar{q}^k \Gamma_u u - \bar{d}\Gamma_d^\dagger q^j \,.$$

Note that in the above, only the dimension-four Lagrangian has been used to derive the EoM. In general, it is necessary to derive EoMs using the entire Lagrangian of the effective theory, and in the case of SMEFT this would mean for example the EoM for the lepton field is

$$i \not{D} \ell_p = [\Gamma_e]_{pr} e_r \varphi + \frac{C_{e\varphi}^{pr}}{\Lambda^2} (\varphi^\dagger \varphi) e_r \varphi + \dots \,, \tag{4.5}$$

where p and r are lepton generation indices, and the ellipsis denotes further contributions from dimension-six operators. To eliminate a 'redundant' dimension-six operator involving the derivative of a lepton doublet, we may find, for example

$$\frac{1}{\Lambda^2} (\varphi^\dagger \varphi)(\overline{\ell_s} i \not{D} \ell_p) = \frac{1}{\Lambda^2} (\varphi^\dagger \varphi)(\overline{\ell_s} [\Gamma_e]_{pr} e_r \varphi) + \frac{C_{e\varphi}^{pr}}{\Lambda^4} (\varphi^\dagger \varphi)^2 (\overline{\ell_s} e_r \varphi) + \dots \,, \tag{4.6}$$

where it can be seen that the higher-dimension contributions to the leptonic EoM are higher order, and can be neglected when only working up to dimension-six operators.

While a physical basis should not contain operators that are related through equations of motion, it is possible to build non-physical operators that vanish on-shell through the equations of motion. While these operators are classically zero, they need to be included in effective calculations performed off-shell. This is done for the case of the

SMEFT in Chapter 5, where the equations of motion are used to construct EoM-vanishing operators that contribute to the renormalisation of diagrams with double-insertions of the Weinberg operator. For example, using the EoM for left-handed leptons (Equation (4.4)), the lepton-flavour violating off-shell operator

$$Q_{v(1)}^{pr} = (\varphi^\dagger\varphi)(\overline{\ell_p} i \overleftrightarrow{\not{D}} \ell_r) - (\varphi^\dagger\varphi)(\overline{\ell_p}\,[\Gamma_e]_{rs}\, e_s\varphi + \overline{e_s}\,[\Gamma_e^\dagger]_{sp}\ell_r\varphi^\dagger) \tag{4.7}$$

may be constructed, with a similar triplet operator also arising. These operators have derivatives acting on leptons, and consequently give Feynman rules dependent on lepton momenta. The inclusion of EoM-vanishing operators is also important in general when performing matching calculations off-shell, although they do not arise in the calculation of Chapter 6, since the gluons do not couple to leptons, and therefore the gluonic EoM cannot generate dimension-six operators that mediate the process in question.

An additional type of operator are evanescent operators, which are proportional to the ϵ of $d = 4 - 2\epsilon$, and are therefore vanishing in four dimensions. Further, since they are non-physical operators, their matrix elements give no contribution to physical amplitudes. However, these operators must be renormalised, and thus mix into physical operators through renormalisation matrices. Since evanescent operators are proportional to ϵ, and the momentum integral corresponding to a loop induces a $1/\epsilon$ pole, then at 1-loop evanescent operators are renormalised by finite counterterms. Consequently, at 1-loop evanescent operators do not contribute to the anomalous dimensions of physical operators. However, at 2-loop and above, evanescent operators affect the anomalous dimensions of physical operators, and so their inclusion is important [73]. It is possible to perform calculations without including evanescent operators, but to do so correctly it is not possible to use massless quarks within dimensional regularisation. The use of massless quarks introduces spurious infrared divergences that can only be properly handled with the addition of evanescent operators [12].

4.3 Renormalisation of EFTs

In usual QFT parlance, a Lagrangian in $d = 4$ dimensions is said to be renormalisable if it only contains operators with mass-dimension less than or equal to four. Such a Lagrangian may be renormalised to give finite Green's functions with the addition of a finite number of counterterms. Effective Lagrangians contain multiple higher-dimension operators, and so would normally be considered to be non-renormalisable. However, provided that calculations in EFTs are truncated at some order in Λ, only a finite number of counterterms are required to renormalise the effective theory, and so the theory is renormalisable in practice.

Effective operators are built out of light fields, and so effective operators will undergo wavefunction renormalisation. For example, a four-fermion operator of a generic fermion ψ will undergo wavefunction renormalisation as

$$\left(\overline{\psi_0}\Gamma\psi_0\right)\left(\overline{\psi_0}\Gamma\psi_0\right) = Z_\psi^2\left(\overline{\psi}\Gamma\psi\right)\left(\overline{\psi}\Gamma\psi\right). \tag{4.8}$$

However, this wavefunction renormalisation is insufficient to remove all divergences in effective theories, and a further renormalisation must be performed, called operator renormalisation [69, 74]. This is simply because the full and effective theories are different theories, with different UV structures. Therefore, it cannot be expected that the same procedure will renormalise the two different theories. The additional operator renormalisation leads to mixing between operators with the same quantum numbers and displaying the same symmetries. Instead of a single multiplicative renormalisation constant for each higher-dimension operator, a renormalisation matrix is required, with contributions from multiple operator renormalisation constants needed to renormalise a single operator. In this framework, and using dimensional regularisation, renormalisation of a higher-dimension operator is performed via

$$\frac{C_0^i}{\Lambda^{\dim[Q]-4}}Q_0^i = \mu^{a\epsilon}\frac{C^i(\mu)}{\Lambda^{\dim[Q]-4}}Z_{ij}(\mu)\left(Z_\chi(\mu)Q\right)^j(\mu), \tag{4.9}$$

where there is summation over the repeated indices i, j, the renormalisation constant Z_χ represents the relevant wavefunction renormalisations for the operator Q^j, and the subscript '0' denotes a bare parameter. Since dimensional regularisation is used, the scale $\mu \equiv \mu_{\overline{\text{MS}}}$ appears, and the factor $\mu^{a\epsilon}$ is introduced to ensure dimensionless Wilson coefficients. For example, considering the mass-dimension of a four-fermion operator with $a = 2$,

$$d = 4-2\epsilon = \left[\mu^{2\epsilon}\frac{C_{\psi 4}}{\Lambda^2}\left(\overline{\psi}\Gamma\psi\right)\left(\overline{\psi}\Gamma\psi\right)\right] = 4\frac{d-1}{2} - 2 + 2\epsilon + \left[C_{\psi^4}\right] = 4 - 2\epsilon + \left[C_{\psi^4}\right] \tag{4.10}$$

which implies that $\left[C_{\psi^4}\right] = 0$, and the Wilson coefficient is dimensionless as desired. Note that in the literature, relations of the form of Equation (4.9) are often written in a suppressed notation, typically as

$$C_0^i Q_0^i = C^i(\mu)Z_{ij}(\mu)Q^j(\mu). \tag{4.11}$$

Note that since Z_{ij} renormalises both the Wilson coefficient and the operator, it is standard to associate the renormalisation matrix with either the coefficient or the operator. We renormalise the Wilson coefficient as

$$C_0^i = C^i(\mu)Z_{ij}(\mu), \tag{4.12}$$

such that renormalisation of the Wilson coefficients additionally removes all divergences of its associated bare operator.

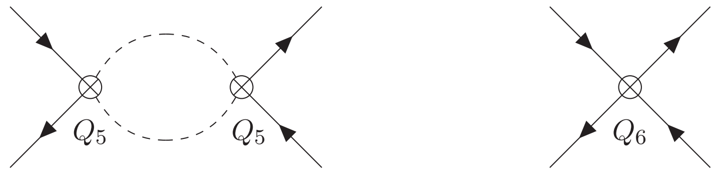


Fig. 4.2: Double-insertions of a dimension-five operator leads to mixing with dimension-six operators with the same external states.

Renormalisation matrices may be expanded in powers of both coupling constants and ϵ, and then evaluated in perturbation theory. For example, in calculating QCD corrections to weak processes, renormalisation constants may be expanded as

$$Z_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^k Z_{ij}^{(k)}, \qquad Z_{ij}^{(k)} = \sum_{m=0}^{k} \frac{1}{\epsilon^m} Z_{ij}^{(k,m)}. \tag{4.13}$$

4.3.1 Operator Mixing through Double-Insertions

The mixing of operators of the same dimension is described with renormalisation matrices Z_{ij}, but it is also possible for operators of different dimensionality to mix under renormalisation. For example, consider the simple Lagrangian

$$\mathcal{L} = \mathcal{L}^{(4)} + \frac{1}{\Lambda} C_5 Q_5 + \frac{1}{\Lambda^2} C_6 Q_6, \tag{4.14}$$

where there is a single operator of dimension-five and a single operator of dimension-six. If calculating a process to some order $1/\Lambda^2$, there can, in general, be contributions from diagrams with a single insertion of a dimension-six operator and two insertions of the dimension-five operator, since such amplitudes are both proportional to $1/\Lambda^2$. This is illustrated in Figure 4.2.

The loop that appears in Figure 4.2 generates a divergence that must be removed to obtain finite quantities. Since in general an effective theory contains all operators allowed by symmetries, no additional operators can be introduced to the Lagrangian to remove such divergences, and so divergences appearing from double insertions must be renormalised by operators already in the Lagrangian, in this case Q_6. The renormalisation is encoded in a renormalisation tensor $Z_{55,6}$, which specifies the UV structure required of Q_6 to remove the divergence generated by a double-insertion of Q_5.

In general, when discussing the mixing of operators of dimension-m into operators of dimension-n through double-insertions, there may be multiple operators at dimension-m and -n. For example, in the case of SMEFT, there is a single dimension-five operator that mixes into four LFV operators at dimension-six. However, if an additional Higgs doublet is added to the theory, there are a total of four dimension-five operators that mix into the four LFV dimension-six operators. Both of these cases are studied in Chapter 5 of this work. The general structure of renormalisation for such situations is discussed in [75]. Denoting quantities of dimension-n with a tilde and those of dimension-m without a tilde, the renormalisation equation of a dimension-n quantity may then be written as

$$\widetilde{C}_0^i \widetilde{Q}_0^i = \widetilde{C}^i \widetilde{Z}_{ij} \widetilde{Q}^j + C^k C^l Z_{kl,j} \widetilde{Q}^j, \tag{4.15}$$

where there is summation over i, j, k and l. Note that there are (here suppressed) factors of μ^ϵ on the right-hand-side of the equality to ensure all Wilson coefficients are dimensionless. Note also that there is a key difference in cases of intra- and inter-dimensional operator mixing. In the former case, where operators of the same dimension mix, mixing occurs when there is a vertex renormalisation induced by two external legs being connected by a gauge boson. This means that the renormalisation matrix Z_{ij} may be expanded in a gauge coupling g, with the leading order contribution $\propto g^2$. However, when renormalising diagrams containing a double-insertion of effective operators, at leading order there are no gauge couplings, and so the leading order contribution to the renormalisation tensor $Z_{mn,j}$ is g-independent (and therefore μ-independent). To address this situation, it is common to make a redefinition of terms in the effective Lagrangian by multiplying Wilson coefficients by g^2 and dividing effective operators by g^2 [75–77]. Such an action leaves the Lagrangian invariant, while introducing a g-dependence into the Wilson coefficients and effective operators that allows expansion in gauge couplings at leading order. However, this approach assumes that calculations are being performed in a theory where we calculate perturbatively in a specific small coupling. Since we consider only 1-loop processes with no gauge coupling necessarily present, this approach is not naturally suited to our case, and so we work with renormalisation tensors that are μ-independent at leading order. This is a novel approach, and is not present in the previous literature.

4.3.2 Renormalisation Group Equations for Wilson Coefficients

Wilson coefficients are couplings for effective operators, and are μ-dependent, and are therefore analogous to coupling constants of dimension-four operators. The running of Wilson coefficients is similarly given by anomalous dimensions, and due to operator mixing, the anomalous dimensions are encoded in anomalous dimension matrices (ADMs) and anomalous dimension tensors (ADTs).

In the following, we will consider the renormalisation of dimension-six Wilson coefficients, both among themselves, and from double-insertions of dimension-five operators. Dimension-six quantities are denoted by a tilde, while dimension-five quantities do not have a tilde. Consider a bare dimension-six Wilson coefficient $\widetilde{C}_0^i$. Considering only the mixing of dimension-six operators among themselves, and ignoring wavefunction renormalisation, the renormalisation equation for $\widetilde{C}_0^i$ is given by Equation (4.12). Since bare quantities are μ-independent, this implies

$$0 = \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\left(\widetilde{C}_0^i\right) = \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\left(\mu^{2\epsilon}\widetilde{C}^j \widetilde{Z}_{ji}\right), \tag{4.16}$$

where the factor of $\mu^{2\epsilon}$ from dimensional regularisation is included to make the Wilson coefficients $\widetilde{C}^i$ dimensionless. Using the product rule, it is then simple to find

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu}\widetilde{C}^i(\mu) = -2\epsilon \widetilde{C}^i(\mu) - \widetilde{C}^k(\mu)\left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu}\widetilde{Z}_{kj}\right)\widetilde{Z}_{ji}^{-1}. \tag{4.17}$$

This may be written as

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu}\widetilde{C}^i = \widetilde{C}^j \widetilde{\gamma}_{ji}, \tag{4.18}$$

where the dimension-six ADM is given by

$$\widetilde{\gamma}_{ji} = -2\epsilon \delta_{ji} - \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu}\widetilde{Z}_{jk}\right)\widetilde{Z}_{ki}^{-1}. \tag{4.19}$$

Similarly, an ADM for mixing among dimension-five operators is given by

$$\gamma_{ji} = -2\epsilon \delta_{ji} - \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu}Z_{jk}\right)Z_{ki}^{-1}. \tag{4.20}$$

Extending the treatment to also include double-insertions of dimension-five operators, dimension-six Wilson coefficients may be renormalised as

$$\widetilde{C}_0^i = \mu^{2\epsilon}\widetilde{C}^j \widetilde{Z}_{ji} + \mu^{2\epsilon} C^k Z_{kl,i} C^l. \tag{4.21}$$

Taking the derivative with respect to μ leads to the renormalisation group equation

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu}\widetilde{C}^i = \widetilde{C}^j \widetilde{\gamma}_{ji} + C^k \gamma_{kl,i} C^l, \tag{4.22}$$

where $\widetilde{\gamma}_{ji}$ is the dimension-six ADM of Equation (4.19). The quantity $\gamma_{kl,i}$ is the ADT for mixing of dimension-five operators into dimension-six via double-insertions, and is given by

$$\gamma_{kl,i} = -\left[2\epsilon Z_{kl,j}\widetilde{Z}_{ji}^{-1} + \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu}Z_{kl,j}\right)\widetilde{Z}_{ji}^{-1} + \left(\gamma_{kk'}\delta_{ll'} + \gamma_{ll'}\delta_{kk'}\right)Z_{k'l',j}\widetilde{Z}_{ji}^{-1}\right], \tag{4.23}$$

where $\gamma_{kk'}$ is the dimension-five ADM. This can be compared to the result in [75], where there is not a term proportional to ϵ. This is because the additional factor of $\mu^{2\epsilon}$ that arises in the Lagrangian from dimensional regularisation has not been considered.

It is useful to consider the perturbative expansion of Equation (4.22) to understand where the leading contributions come from. The dimension-six ADM has a leading order contribution of $-2\epsilon\delta_{ji}$ (see Equation (4.19)), since $\frac{\mathrm{d}\widetilde{Z}_{jk}}{\mathrm{d}\mu}$ starts at $\mathcal{O}(g^2)$. Then, considering the ADT of Equation (4.23), the first term is leading order since both $Z_{kl,j}$ and $\widetilde{Z}_{ji}$ start at $\mathcal{O}(g^0)$. The second term is not leading order, since $\frac{\mathrm{d}Z_{kl,j}}{\mathrm{d}\mu}$ starts at $\mathcal{O}(g^2)$. There is a contribution from the third term, since $Z_{kl,j}$ and $\widetilde{Z}_{ji}$ start at $\mathcal{O}(g^0)$, and the dimension-five ADMs (given in Equation (4.20)) have a part that is leading order. Bringing these contributions together, the leading order running of the dimension-six Wilson coefficient $\widetilde{C}^i$ is then given by

$$\begin{aligned}\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\widetilde{C}^i\right]^{(0)} &= -2\epsilon\widetilde{C}^i - 2\epsilon C^k\left[Z_{kl,j}\right]^{(0)}\delta_{ji}C^l \\ &\quad - C^k\left(\left(-2\epsilon\delta_{kk'}\delta_{ll'} - 2\epsilon\delta_{ll'}\delta_{kk'}\right)\left[Z_{k'l',j}\right]^{(0)}\delta_{ji}\right)C^l \\ &= -2\epsilon\widetilde{C}^i + 2\epsilon C^k\left[Z_{kl,i}\right]^{(0)}C^l. \end{aligned} \tag{4.24}$$

The expression $\left[Z_{k'l',j}\right]^{(0)}$ (where the (0) superscript denotes that this is the contribution at $\mathcal{O}(g^0)$) will contain a $1/\epsilon$ pole which will cancel the ϵ coefficient. Then returning to $d = 4$ dimensions, the term from the dimension-six ADM will no longer contribute, and the final expression for the leading order running is

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\widetilde{C}^i\right]^{(0)} = 2\epsilon C^k\left[Z_{kl,i}\right]^{(0)}C^l\,. \tag{4.25}$$

Therefore, the leading contribution to the running of the dimension-six Wilson coefficient $\widetilde{C}^i$ is given by the leading order term of the ADT $Z_{kl,i}$. This is a new result, and is given for the first time in [78]. We have also checked that we would obtain the same final results for the running of dimension-six Wilson coefficients if we used the method of [75], although the intermediate steps are different.

4.4 Solutions of RGEs and RG-Improved Perturbation Theory

Given the renormalisation group equation for a column of Wilson coefficients,

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}C^i = C^j\gamma_{ji}\,, \tag{4.26}$$

the Wilson coefficient at a scale μ may be solved as [44]

$$C^i(\mu) = C^j(M_W)\exp\left[\int_{g(M_W)}^{g(\mu)}\mathrm{d}g'\,\frac{\gamma_{ji}(g')}{\beta(g')}\right]\,, \tag{4.27}$$

where $\beta(g)$ is the QCD beta function. At leading order this gives

$$C^i(\mu) = C^j(M_W)\left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{\frac{\gamma_{ji}^{(0)}}{2\beta_0}}\,. \tag{4.28}$$

These solutions are crucial in determining the behaviour of Wilson coefficients. In a typical matching calculation at the weak scale we may obtain the result

$$C(\mu) \approx 1 + \mathcal{F}\frac{\alpha_s}{4\pi}\ln\left(\frac{M_W}{\mu}\right) , \tag{4.29}$$

where $\mathcal{F}$ is some numerical factor, typically of $\mathcal{O}(1)$. This is is well behaved for $\mu \sim M_W$, but breaks down for $\mu \sim 1$ GeV, where the leading correction becomes comparable to unity, and cannot be considered small in any way. Consequently, for this expression for C to be valid around $\mu = 1$ GeV, the matching would need to be computed to all orders in perturbation theory. An easy way to circumvent this problem is to calculate the matching to some finite order in perturbation theory, and then to use the renormalisation group equations to compute how the Wilson coefficient runs to different values of μ. This works because the RGEs automatically resum the logarithms from all orders, such that convergence is not a problem. This calculational approach is called renormalisation group-improved perturbation theory [44, 79].

4.5 Matching

As already discussed, calculations in effective field theory are usually performed in the mass-independent $\overline{\text{MS}}$ scheme, in which heavy particles do not manifestly decouple. This problem is dealt with by decoupling heavy particles by hand, in the process of matching. When integrating out a heavy particle H, a transition is made between a high-energy theory (valid at scales $\mu \geq m_H$) containing H and a low-energy theory (valid at scales $\mu \leq m_H$) not containing H. It is physically required that matrix elements calculated for light particles in the low-energy effective theory must be equal to the corresponding matrix elements for the light particles in the high-energy theory at the scale $\mu = m_H$, which gives a set of matching conditions. In this way, each theory can be used for calculations in its own domain of validity, with each theory smoothly transitioning with the preceding theory. A simple example is the matching the GSW theory to four-Fermi theory for the process $ud \to su$. At small momentum transfer ($\ll M_W$), and in the Feynman gauge at tree level,

$$\mathcal{A}^{\text{SM}}_{ud\to su} = -\frac{g_2^2}{8M_W^2} V_{ud}V_{us}^*(\overline{u_s}\gamma^\mu(1-\gamma_5)u_u)(\overline{u_u}\gamma_\mu(1-\gamma_5)u_d) , \tag{4.30}$$

where the overall sign is due to our choice of covariant derivative. In the four-Fermi theory with $\mathcal{L}_{\text{fermi}} \supset -\frac{G_F}{\sqrt{2}}V_{ud}V_{us}^*(\overline{u}\gamma^\mu(1-\gamma_5)s)(\overline{u}\gamma_\mu(1-\gamma_5)d)$, the tree-level amplitude is

$$\mathcal{A}^{\text{Fermi}}_{ud\to su} = -\frac{G_F}{\sqrt{2}} V_{ud}V_{us}^*(\overline{u_s}\gamma^\mu(1-\gamma_5)u_u)(\overline{u_u}\gamma_\mu(1-\gamma_5)u_d) . \tag{4.31}$$

Requirement that the full SM and effective amplitudes agree leads to the matching condition

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8M_W^2} \, . \tag{4.32}$$

This matching between Fermi theory and GSW theory at tree-level serves to define the Fermi constant in terms of GSW parameters. In a more general effective treatment of weak interactions (see Section 4.7), the four-fermi operator would be multiplied by a Wilson coefficient $C_{V,LL}^{suud}$ in the Lagrangian. The tree-level matching calculation would be the same as detailed above, except that Equation (4.32) would be used as an identity, and the matching condition would yield

$$C_{V,LL}^{suud} = 1 \tag{4.33}$$

at tree-level. Loop corrections would modify the Wilson coefficient, while leaving the Fermi constant unaffected.

Matching calculations can be made more precise by including QCD corrections in the calculation. This involves performing loop calculations in, and renormalising, the "full" theory, and in principle calculating loops in and renormalising the effective theory. In matching calculations, light quarks are usually treated as massless (since the heaviest "light" quark remaining in the theory has mass $m_b = 4.18$ GeV, which is much less than $M_W = 80.4$ GeV [32]), and so loops with massless quarks and gluons are identically zero in dimensional regularisation when external momenta are set to zero [12]. The matching calculation then proceeds by calculating the amplitude for some process in both the effective theory and full theory to some order in a perturbative coupling (for this work, the strong coupling, α_s). Each theory must be renormalised, which in the effective theory requires the calculation of the renormalisation matrix that mixes operators. This is done by calculating loop diagrams in the effective theory with some finite external momentum (such that the integrals are not scaleless), removing some subclass of divergences through a wavefunction renormalisation for the fields in the effective operators, and removing the remaining divergences with effective operator counterterms. Infrared divergences that arise from massless quarks also arise in the full theory calculation, and so their effect will cancel overall. The effective and full theories can then be matched order-by-order in the perturbative coupling, allowing the Wilson coefficients to be extracted.

A subtlety in the matching is that both the full theory and the effective theory must be expanded in the same coupling. In a mass-independent scheme such as $\overline{\text{MS}}$, couplings are dependent on the number of active flavours through their running, and so theories containing a different number of active quark flavours are naturally described by different couplings. In the case of matching the SM to a five-flavour theory below the weak scale, the SM has a QCD coupling $\alpha_s^{(6)}$ while the effective theory has the coupling $\alpha_s^{(5)}$.

This situation is resolved by using $\alpha_s^{(5)}$ in the effective and full theories, and applying threshold corrections in the full theory to compensate for this choice of coupling. These threshold corrections encode contributions of high-energy particles, and relate $\alpha_s^{(5)}$ and $\alpha_s^{(6)}$ (as well as other quantities such as quark masses and fields that are dependent on the number of flavours), allowing matching calculations to proceed. Threshold corrections may be calculated by appealing to the Appelquist-Carazzone theorem. Calculating the same Green's function using a mass-dependent and a mass-independent scheme, and requiring equality between them, allows one to identify the finite renormalisations required to make the Green's functions agree. These finite renormalisations are called threshold corrections, and are discussed in detail in [80, 81].

We quote here relevant threshold corrections in QCD from [81]. Denoting quantities in the effective theory by primes, and quantities in the full theory without primes, then

$$
\begin{aligned}
G'^a_\mu &= \sqrt{\zeta_G} G^a_\mu && \text{gluon field}\,, \\
\xi'_G &= \zeta_{\xi_G} \xi_G && \text{gauge parameter}\,, \\
\alpha'_s &= \zeta_{g^2} \alpha_s && \text{QCD coupling}\,, \\
\psi'_q &= \sqrt{\zeta_{\psi_q}} \psi_q && \text{quark field}\,.
\end{aligned} \tag{4.34}
$$

The threshold corrections are given by

$$
\begin{aligned}
\zeta_G &= 1 + \frac{\alpha_s(\mu)}{4\pi} \frac{2}{3} \ln\left(\frac{\mu^2}{m_H^2}\right)\,, \\
\zeta_{\xi_G} &= 1 + \frac{\alpha_s(\mu)}{4\pi} \frac{2}{3} \ln\left(\frac{\mu^2}{m_H^2}\right)\,, \\
\zeta_{g^2} &= 1 + \frac{\alpha_s(\mu)}{4\pi} \left[-\frac{2}{3} \ln\left(\frac{\mu^2}{m_H^2}\right) + \frac{\epsilon}{3} \left(\zeta_2 + \ln^2\left(\frac{\mu^2}{m_H^2}\right) \right) \right]\,, \\
\zeta_{\psi_q} &= 1 + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 C_F \left[\frac{5}{12} - \ln\left(\frac{\mu^2}{m_H^2}\right) \right]\,,
\end{aligned} \tag{4.35}
$$

where m_H is the mass of the heavy particle that is integrated out in going from the full to the effective theory, and $C_F = \frac{4}{3}$. It can be seen that $\zeta_G = \zeta_{\xi_G}$, which is required from the gauge-fixing term in general R_ξ gauge, $\frac{1}{2\xi}(\partial^\mu G^a_\mu)^2$.

4.6 SMEFT

The standard model is currently the best theory of particle physics that we have. However, it is well known that it is not a fundamental theory, and that it must be the low-energy theory of some more general UV-complete theory. In this sense, the SM can be seen as the IR limit of a higher-energy theory, whose heavy excitations have been integrated out (since they are too massive to be observed at the LHC). This allows one to build an effective theory where higher-dimension operators are built out of the fields of the SM, and where such operators respect the $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ symmetry of the unbroken SM. The unbroken SM is used since new physics must exist at energies above the weak scale. The operators of SMEFT at dimensions-five and -six were compiled by Buchmuller and Wyler [14], but as has been pointed out in the literature [82–84] this list contains redundant operators that are related by equations of motion. Consequently, a new basis was formed, known as the 'Warsaw' basis [8], which carefully considered equations of motion, and is used in this work.

The Lagrangian of SMEFT is given by

$$\mathcal{L}_{\mathrm{SM}} = \mathcal{L}^{(4)}_{\mathrm{SM}} + \frac{1}{\Lambda}\sum_k C^{(5)}_k Q^{(5)}_k + \frac{1}{\Lambda^2}\sum_k C^{(6)}_k Q^{(6)}_k + \mathcal{O}\left(\frac{1}{\Lambda^3}\right) , \tag{4.36}$$

where $\mathcal{L}_{\mathrm{SM}}$ is the Lagrangian of Equation (2.38), and Λ is the scale of new physics. In SMEFT there is only a single dimension-five operator: the lepton number violating (LNV) Weinberg operator,

$$Q_5^{pr} = \varepsilon_{jk}\varepsilon_{mn}\varphi^j\varphi^m(\ell^k_p)^T C\ell^n_r = \left(\overline{\ell^c_p}\varepsilon\varphi\right)\left(\ell_r\varepsilon\varphi\right) . \tag{4.37}$$

In the latter notation (which is used in Chapter 5), SU(2) indices j,k,m,n are contracted within brackets. The equality holds since (in the Dirac basis) $C^{-1} = -C$. The indices p and r are generation indices, and as such the Weinberg operator can mediate lepton flavour violating (LFV) processes, as well as LNV processes. An important property of the Weinberg operator is that it is symmetric in generation space:

$$\begin{aligned}
\left(\overline{\ell^c_p}\varepsilon\varphi\right)\left(\ell_r\varepsilon\varphi\right) &= -\left({\ell^j_p}^T C^{-1}\ell^m_r\right)\varepsilon_{jk}\varphi^k\varepsilon_{mn}\varphi^n \\
&= -\left({\ell^j_p}^T C^{-1}\ell^m_r\right)^T\varepsilon_{jk}\varphi^k\varepsilon_{mn}\varphi^n \\
&= +\left({\ell^m_r}^T {C^{-1}}^T\ell^j_p\right)\varepsilon_{jk}\varphi^k\varepsilon_{mn}\varphi^n \\
&= -\left({\ell^m_r}^T C^{-1}\ell^j_p\right)\varepsilon_{jk}\varphi^k\varepsilon_{mn}\varphi^n \\
&= \left(\overline{\ell^c_r}\varepsilon\varphi\right)\left(\ell_p\varepsilon\varphi\right) .
\end{aligned} \tag{4.38}$$

In the third line, a sign arises from fermion interchange, while an additional sign comes in the fourth line since $C^T = -C$. Therefore, Q_5^{pr} is a symmetric 3×3 matrix, with six independent degrees of freedom.

The Weinberg operator is physically very important. This is because at energies below the electroweak symmetry-breaking scale it gives rise to a Majorana mass term for the left-handed neutrinos of the SM. This can be seen by replacing the Higgs doublets in the Weinberg operator by their VEVs,

$$\varphi \to \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} ,$$

which leads to

$$\begin{aligned}
\mathcal{L}^{(5)} &= C_5^{pr}\left(\overline{\ell_p^c}\epsilon\varphi\right)\left(\ell_r\epsilon\varphi\right) \\
&\to \frac{C_5^{pr}}{2}\left(\,\overline{\nu_{Lp}^c} \quad \overline{e_{Lp}^c}\,\right)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}\left(\nu_{Lr} \quad e_{Lr}\right)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix} \\
&= \frac{C_5 v^2}{2}\overline{\nu_{Lp}^c}\nu_{Lr}\,.
\end{aligned} \tag{4.39}$$

Comparing this with the form of a Majorana mass term (Equation (2.89)),

$$\mathcal{L}_{\text{Majorana}} = -M\overline{\Psi_L^c}\Psi_L,$$

shows that the Weinberg operator generates in the broken phase a Majorana mass for the left-handed neutrinos, with mass

$$m_\nu = -\frac{C_5^{pr} v^2}{2}\,. \tag{4.40}$$

At dimension-six, there are many more possible operators that respect the gauge symmetries of the SM. In the Warsaw basis they are divided into different categories, depending on the field content of the operators. The categories are X^3, (φ^6 and $\varphi^4 D^2$), $\psi^2\varphi^3$, $X^2\varphi^2$, $\psi^2 X\varphi$, and $\psi^2\varphi^2 D$, where X, φ, D and ψ are generic labels for field strength tensors, scalars, derivatives, and fermions respectively. Note that the above categories exclude four-fermion operators. These additional operators are classified by their chirality structure, with the categories

$$(\overline{L}L)(\overline{L}L), \quad (\overline{R}R)(\overline{R}R), \quad (\overline{L}L)(\overline{R}R), \quad (\overline{L}R)(\overline{R}L) \text{ and } (\overline{L}R)(\overline{L}R)\,.$$

In total, there are 59 operators at dimension-six (ignoring Hermitian conjugates and flavour structures), but if one includes operators that allow B-number violation, an additional five operators appear. A list of the dimension-six operators of SMEFT is given in Appendix A.

The relevant dimension-six operators for this work are

$$\begin{aligned}
Q_{e\varphi}^{pr} &= \left(\varphi^\dagger\varphi\right)\left(\overline{\ell_p}e_r\varphi\right) && \in \psi^2\varphi^3\,, \\
Q_{\varphi\ell(1)}^{pr} &= \left(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi\right)\left(\overline{\ell_p}\gamma^\mu\ell_r\right) && \in \psi^2\varphi^2 D\,, \\
Q_{\varphi\ell(3)}^{pr} &= \left(\varphi^\dagger i\overleftrightarrow{D}^I{}_\mu\varphi\right)\left(\overline{\ell_p}\tau^I\gamma^\mu\ell_r\right) && \in \psi^2\varphi^2 D\,, \\
Q_{\ell\ell}^{prst} &= (\overline{\ell_p}\gamma_\mu\ell_r)(\overline{\ell_s}\gamma^\mu\ell_t) && \in (\overline{L}L)(\overline{L}L)\,, \\
Q_{\varphi\Box} &= \left(\varphi^\dagger\varphi\right)\Box\left(\varphi^\dagger\varphi\right) && \in \varphi^4 D^2\,, \\
Q_{\varphi D} &= \left(\varphi^\dagger D^\mu\varphi\right)^*\left(\varphi^\dagger D_\mu\varphi\right) && \in \varphi^4 D^2\,, \\
Q_{u\varphi}^{pr} &= \left(\varphi^\dagger\varphi\right)\left(\overline{q_p}u_r\widetilde{\varphi}\right) && \in \psi^2\varphi^3\,,
\end{aligned}$$

$$Q_{d\varphi}^{pr} = \left(\varphi^\dagger \varphi\right) (\overline{q_p} d_r \widetilde{\varphi}) \qquad\qquad \in \psi^2 \varphi^3 \,.$$

All of these operators can be generated by double-insertions of the Weinberg operator, and the dependence of the running of their respective Wilson coefficients due to such double-insertions is calculated in Chapter 5.[1]

4.6.1 Matching to Low-Energy Wilson Coefficients

While SMEFT is the correct framework above the electroweak symmetry breaking scale, it is useful to be able to relate this to effective theories below this scale. This allows constraints obtained from low-energy experiments (below the weak scale) to be converted into constraints on SMEFT Wilson coefficients. This is done through matching the two theories at the scale M_W. We will consider here the matching of one low-energy Wilson coefficient in terms of SMEFT Wilson coefficients to illustrate the procedure. A complete list of these matchings is given in Chapter 5, where we correct some results found in [85].

In EFTs below the weak scale, operators observe the gauge symmetry $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{U}(1)_\mathrm{em}$, and a basis of such operators is given in [86]. Consider in particular the Lagrangian term

$$\mathcal{L}_{\mu\to 3e}^{\mathrm{QED}\times\mathrm{QCD}} \supset -\frac{G_F}{\sqrt{2}} \mathcal{C}_{V,LL}^{e\mu ee} \mathcal{Q}_{V,LL}^{e\mu ee} \,, \qquad \mathcal{Q}_{V,LL}^{e\mu ee} = (\bar{e}\gamma^\rho P_L \mu)\,(\bar{e}\gamma_\rho P_L e) \,, \tag{4.41}$$

where $\mathcal{Q}_{V,LL}^{e\mu ee}$ is a vector-vector operator of left-handed fields. Note that low-energy operators are denoted by $\mathcal{Q}$, while SMEFT operators are denoted by Q. Similarly, low-energy Wilson coefficients and SMEFT coefficients are denoted by $\mathcal{C}$ and C respectively. There are two types of operator in SMEFT that match onto this operator. First, there is the four-lepton operator (plus its Hermitian conjugate)

$$\begin{aligned}
\sum_{p,r,s,t} \left(C_{\ell\ell}^{prst} Q_{\ell\ell}^{prst} + \mathrm{H.c.}\right) &\supset C_{\ell\ell}^{e\mu ee} Q_{\ell\ell}^{e\mu ee} + C_{\ell\ell}^{eee\mu} Q_{\ell\ell}^{eee\mu} + C_{\ell\ell}^{e\mu ee*} Q_{\ell\ell}^{ee\mu e} + C_{\ell\ell}^{eee\mu*} Q_{\ell\ell}^{\mu eee} \\
&= \left(C_{\ell\ell}^{e\mu ee} + C_{\ell\ell}^{eee\mu} + C_{\ell\ell}^{ee\mu e*} + C_{\ell\ell}^{\mu eee*}\right) Q_{\ell\ell}^{e\mu ee} \\
&= 4 C_{\ell\ell}^{e\mu ee} Q_{\ell\ell}^{e\mu ee} \\
&= 2 C_{\ell\ell}^{e\mu ee} (\overline{e_L}\gamma^\rho \mu_L)(\overline{e_L}\gamma_\rho e_L) \,,
\end{aligned} \tag{4.42}$$

where the equality holds since $C_{\ell\ell}^{prst} = C_{\ell\ell}^{rpts}$, $Q_{\ell\ell}^{prst} = Q_{\ell\ell}^{rpts}$, $C_{\ell\ell}^{prst\dagger} = C_{\ell\ell}^{prst*}$ and $Q_{\ell\ell}^{prst\dagger} = Q_{\ell\ell}^{tsrp}$. Note that the factor of 1/2 is introduced as a normalisation for the operator $Q_{\ell\ell}$, to be consistent with Chapter 5. This normalisation is used since we define all operators to enter a Lagrangian with their Hermitian conjugate (even if the operator is self-adjoint when ignoring flavour indices). Since $Q_{\ell\ell}^{prst\dagger} = Q_{\ell\ell}^{tsrp}$, this implies

[1] Note that although the final two operators of this list cannot be directly generated by double-insertions, they are related to operators that can through equations of motion.

that $C_{\ell\ell}^{prst} = C_{\ell\ell}^{rpts*}$, and therefore the four-lepton operator in SMEFT contributing to $\mathcal{Q}_{V,LL}^{e\mu ee}$ is

$$C_{\ell\ell}^{e\mu ee} Q_{\ell\ell}^{e\mu ee} .$$

The second contribution to $\mathcal{Q}_{V,LL}^{e\mu ee}$ from SMEFT operators is via Z-penguins that couple to a lepton current. These Z-penguins arise from the operators $Q_{\varphi\ell(1)}$ and $Q_{\varphi\ell(3)}$ when the Higgs receives a VEV. For example,

$$\begin{aligned} Q_{\varphi\ell(1)}^{pr} &= \left(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi\right) \left(\overline{\ell_p}\gamma^\mu \ell_r\right) \\ &\supset -\left(\varphi^\dagger \left[g_2 W_\mu^3 \tau^3 + g_1 B_\mu\right] \varphi\right) \left(\overline{\ell_p}\gamma^\mu \ell_r\right) \\ &\xrightarrow{\text{EWSB}} \frac{v^2}{2} \left(g_2 W_\mu^3 - g_1 B_\mu\right) \left(\overline{\ell_p}\gamma^\mu \ell_r\right) . \end{aligned} \tag{4.43}$$

Using the relations

$$g_1 = \frac{e}{\cos\Theta_W}, \quad g_2 = \frac{e}{\sin\Theta_W}, \quad Z_\mu = \cos\Theta_W W_\mu^3 - \sin\Theta_W B_\mu , \text{ and } M_Z^2 = \frac{v^2(g_1^2+g_2^2)}{4},$$

the above relation can be written as

$$Q_{\varphi\ell(1)}^{pr} \overset{\text{EWSB}}{\supset} v M_Z Z_\mu (\overline{e_{Lp}}\gamma^\mu e_{Lr}) , \tag{4.44}$$

where it is understood that this Z-penguin operator is only one of two operators generated by $Q_{\varphi\ell(1)}$ under EWSB (the other being a flavour-changing W-penguin), and there is additionally a neutrino part that we neglect here. Similarly, the triplet operator $Q_{\varphi\ell(3)}$ also gives

$$Q_{\varphi\ell(3)}^{pr} \overset{\text{EWSB}}{\supset} v M_Z Z_\mu (\overline{e_{Lp}}\gamma^\mu e_{Lr}) . \tag{4.45}$$

These two Z-penguin operators may couple to a lepton line as in Figure 4.3, and at low energies (where the Z is integrated out) contributes to the low-energy four-fermi operator. The matching is determined by evaluating the Feynman diagram of Figure 4.3.

The diagram $\mathcal{D}_Z$ may be evaluated using standard Feynman rules, as well as the Feynman rule for the Z-penguin read off from Equation (4.45), to find

$$\begin{aligned} \mathcal{M}|_{\mathcal{D}_Z} &= (-i)\left(\overline{u_e} i v M_Z \gamma^\rho C_{\varphi\ell(1)}^{e\mu} P_L u_\mu\right)\left(\frac{-ig_{\rho\sigma}}{-M_Z^2}\right)\left(\overline{u_e}\left(\frac{-ig_2}{\cos\Theta_W}\right)\left[-\frac{1}{2}+\sin^2\Theta_W\right]\gamma^\sigma P_L u_e\right) \\ &= \frac{v g_2 g_L^e}{2\cos\Theta_W M_Z} C_{\varphi\ell(1)}^{e\mu} \left(\overline{u_e}\gamma^\rho P_L u_\mu\right)\left(\overline{u_e}\gamma_\rho P_L u_e\right) \\ &= g_L^e C_{\varphi\ell(1)}^{e\mu} \left(\overline{u_e}\gamma^\rho P_L u_\mu\right)\left(\overline{u_e}\gamma_\rho P_L u_e\right) , \end{aligned} \tag{4.46}$$

where the relations

$$g_L^e = -1 + 2\sin^2\Theta_W , \qquad M_Z = \frac{v g_2}{2\cos\Theta_W} , \tag{4.47}$$

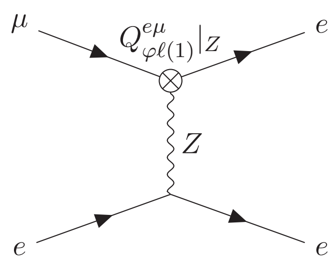
$\mathcal{D}_Z =$ 

Fig. 4.3: Below the weak scale the operator $Q_{\varphi\ell(1)}$ (and $Q_{\varphi\ell(3)}$) generates a Z-penguin operator that can couple to a leptonic current. Matching this to the low-energy theory, the Z is integrated out, resulting in the Z-penguin operators contributing to the four-lepton operators of the $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{U}(1)_\mathrm{em}$-invariant basis.

have been used. Note that the sign for the Feynman rule of the $\overline{e}eZ$ coupling is consistent with our sign convention for the covariant derivative. In the low-energy theory, the matrix element for the process $\mu e \to ee$ is trivially

$$\mathcal{M}|_{\mathrm{low-energy}} = \mathcal{C}^{e\mu ee}_{V,LL} \left(\overline{u_e}\gamma^\rho P_L u_\mu\right)\left(\overline{u_e}\gamma_\rho P_L u_e\right) , \tag{4.48}$$

and so the Wilson coefficients of the low-energy theory may be matched to the Wilson coefficients of SMEFT at M_W as

$$\mathcal{C}^{e\mu ee}_{V,LL}(M_W) = 2C^{e\mu ee}_{\ell\ell}(M_W) + g^e_L \left(C^{e\mu}_{\varphi\ell(1)}(M_W) + C^{e\mu}_{\varphi\ell(3)}(M_W) \right) . \tag{4.49}$$

All other Wilson coefficients $\mathcal{C}$ of the low-energy theory may be matched onto SMEFT coefficients at M_W in this manner. The matching for Wilson coefficients relevant to lepton flavour violation is given in Chapter 5.

4.7 The Weak Hamiltonian

The second effective theory considered here is the effective weak theory, used to describe weak interactions at scales below M_W. The starting point of such calculations is the effective weak Hamiltonian

$$\mathcal{H}_{\mathrm{eff}} = \frac{G_F}{\sqrt{2}} \sum_i V^i_{\mathrm{CKM}} C_i Q_i, \tag{4.50}$$

where G_F is the Fermi constant, and V^i_{CKM} are the relevant CKM matrix elements for the quarks contained in the effective operator Q_i. Amplitudes for transitions from a state $|I\rangle$ to a state $|F\rangle$ governed by effective operators are calculated as

$$\mathcal{A}(I \to F) = \langle F|\, \mathcal{H}_{\mathrm{eff}}\, |I\rangle = \frac{G_F}{\sqrt{2}} \sum_i V^i_{\mathrm{CKM}} C_i(\mu) \langle F|\, Q_i\, |I\rangle\, (\mu) , \tag{4.51}$$

where dimensional regularisation induces a μ-dependence in the Wilson coefficients and matrix elements. Since amplitudes are μ-independent, the μ-dependence of the Wilson coefficients $C_i(\mu)$ must cancel the μ-dependence of the matrix elements $\langle F| Q_i |I\rangle (\mu)$ when the sum over operators in the effective Hamiltonian is taken to extend to infinite mass dimension. Since the μ-cancellation typically involves several terms in the expansion, truncation may lead to some residual non-physical μ-dependence in amplitudes [44]. The Wilson coefficients are calculated by matching the low- and high-energy theories, and so encode the effects of heavy particles into the EFT, while the matrix elements contain the dynamics of the low-energy content of the effective theory.

Since calculations in the effective theory are performed at the arbitrary scale μ, the value of μ may be chosen for calculational convenience. It is often set at the scale of the decaying hadron, which simplifies the use of matching calculations. Usually these energy scales are high enough that the strong coupling constant $\alpha_s(\mu)$ is small, and perturbative techniques may be used to perform the matching calculations to find values for the coefficients $C_i(\mu)$. However, when considering the decays of K mesons, which are comprised of light quarks, it is common to take $\mu \sim 1-2$ GeV, which is greater than the mass m_K, since the strong coupling becomes non-perturbative below this region, making calculations considerably more difficult [44]. While the calculation of Wilson coefficients may be successfully performed within the framework of perturbation theory, their corresponding matrix elements typically cannot, and other methods such as lattice QCD and chiral perturbation theory must be used to evaluate them (see [87] for a review of lattice QCD, and [88] for an introduction to chiral perturbation theory).

The weak effective theory is useful for studying flavour-changing neutral current (FCNC) processes. These processes are forbidden at tree-level in the SM, and therefore experience loop suppression. Even at loop level, the unitarity of the CKM matrix can cause a further suppression through the Glashow-Iliopoulos-Maiani (GIM) mechanism [89]. Consequently, FCNC processes in the SM are highly suppressed, and so they are useful processes to search for signals of new physics.

In this work we use the weak effective Hamiltonian to discuss the decays $K^+ \to \pi^+\nu\overline{\nu}$ and $K_L \to \pi^0\nu\overline{\nu}$, which are mediated by the operator $Q_\nu = \sum_{l=e,\mu,\tau} (\overline{s}_L\gamma^\mu d_L)(\overline{\nu_l}_L\gamma_\mu\nu_{lL})$. We calculate the contribution to the corresponding Wilson coefficient at $\mathcal{O}(\alpha_s^2)$ arising from box diagrams, requiring a matching calulcation at three loops in the full theory (the SM). This calculation is presented in Chapter 6.

Chapter 5

Majorana Neutrino Masses in Renormalisation Group Equations for Lepton Flavour Violation

In the minimal SM, where neutrinos are massless and there are no sterile right-handed neutrinos, the number of e, μ, and τ-type leptons is individually conserved in all interactions. Any process in which these quantum numbers are not individually conserved indicates LFV and Beyond the Standard Model (BSM) physics. The observation of neutrino oscillations implies LFV must occur, where we define LFV as flavour-changing contact interactions of charged leptons (for a review, see *e.g.* [86]). Since observation of LFV is a signal of BSM physics, it is the subject of many experimental searches [32, 90–99]. While it is known that neutrinos are not massless, it is not known whether their masses are Dirac or Majorana (or some mixture of the two). A Majorana mass term for neutrinos is Lepton Number Violating (LNV), and if neutrino masses are indeed described by a Majorana mass, then they could mediate neutrinoless double beta decay [100]. Below the weak scale, such masses appear as renormalisable terms in the Lagrangian, but in the full SU(2) gauge-invariant Standard Model, they arise as a non-renormalisable, dimension-five operator.

We assume that neutrino masses are Majorana, and that the scale Λ of New Physics in the lepton sector is large. We focus on the theory at scales above M_W but below Λ, where it can be described in the framework of SMEFT (see Section 4.6). The neutrino masses can be parameterised by operators of dimension five, and LFV is parameterised by operators of dimension six. Our aim is to obtain the log-enhanced loop contributions of two LNV operators to LFV processes. These can be calculated via renormalisation group equations (RGEs), and in particular, we aim to calculate the anomalous dimensions that mix two dimension-five operators into a dimension-six operator. The renormalisation group running of the dimension-five operators has been extensively studied in the literature [101–103], and the mixing of the dimension-six operators among themselves have been evaluated at one-loop [104] in the Warsaw basis of SMEFT operators [8]. The mixing of two dimension-five operators into dimension-six operators was calculated in [11], using the Buchmuller-Wyler basis [14] at dimension-six. We perform this calculation in the Warsaw basis, and correct previous results in the literature [11] by considering in detail the relation between the Warsaw and Buchmuller-Wyler bases.

The mixing of neutrino masses into LFV amplitudes is $\mathcal{O}(m_\nu/M_W)^2 \ln(\Lambda/M_W)$, so negligibly small, but completes the anomalous dimensions required to perform a one-loop renormalisation-group analysis of the SMEFT at dimension-six. In addition, we explore an extension of SMEFT with two Higgs doublets [105], where the second Higgs doublet lives at a scale m_{22} between M_W and significantly below the lepton number/flavour-changing scale Λ, and impose that LFV at the weak scale is still described by the dimension-six operators of SMEFT. In this scenario, there are four LNV dimension-five operators above m_{22}, but only one combination of coefficients contributes to neutrino masses. We calculate the mixing of these LNV operators into the LFV operators of the SMEFT, and estimate the sensitivity of current LFV experiments to their coefficients.

5.1 Notation and Review

The SM Lagrangian for leptons can be written as

$$\mathcal{L}_{lep} = i\overline{\ell_\alpha}\,\gamma^\mu D_\mu\,\ell_\alpha + i\overline{e_\alpha}\gamma^\mu D_\mu\,e_\alpha - \left(\overline{\ell_\alpha}[\Gamma_e]_{\alpha\beta}e_\beta\varphi + \text{H.c.}\right) \tag{5.1}$$

where Greek letters attached to leptons represent generation indices in the charged-lepton mass eigenstate basis, $[\Gamma_e]$ is the diagonal charged-lepton Yukawa matrix, ℓ is a doublet of left-handed leptons, and e is a right-handed charged-lepton singlet. The explicit form of the lepton and Higgs doublets is

$$\ell = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi^+ \\ \varphi_0 \end{pmatrix}, \tag{5.2}$$

which have hypercharge $Y_\ell = -1/2$ and $Y_\varphi = 1/2$ respectively. The covariant derivative for a lepton doublet is

$$(D_\mu \ell)^i_\alpha = \left(\delta_{ij}\partial_\mu + i\frac{g_2}{2}\tau^a_{ij}W^a_\mu + i\delta_{ij}g_1 Y_\ell B_\mu\right)\ell^j_\alpha, \tag{5.3}$$

where τ^a are the Pauli matrices. This sign convention for the covariant derivative agrees with [104].

Heavy new physics can be parameterised by adding non-renormalisable operators to the SM Lagrangian that respect the SM gauge symmetries [14]. There is only a single operator at dimension-five in the SM, which is the Lepton Number Violating "Weinberg" operator [10], responsible for Majorana masses of left-handed neutrinos. The resulting effective Lagrangian at dimension-five is

$$\delta\mathcal{L}_5 = \frac{C_5^{\alpha\beta}}{2\Lambda}(\overline{\ell_\alpha}\varepsilon\varphi^*)(\ell_\beta^c\varepsilon\varphi^*) + \frac{C_5^{\alpha\beta *}}{2\Lambda}(\overline{\ell_\beta^c}\varepsilon\varphi)(\ell_\alpha\varepsilon\varphi)\,, \tag{5.4}$$

where ε is the totally antisymmetric rank-2 Levi-Civita symbol with $\varepsilon_{12} = +1$, all implicit SU(2) indices inside brackets are contracted, and the charge conjugation acts on the SU(2) component ℓ^i of the lepton doublet as $\left(\ell^i\right)^c = C\overline{\ell^i}^T$ (see Chapter 2 for a discussion of charge conjugation of fermions). The charge conjugation matrix C satisfies the properties of the charge-conjugation matrix used in [33].[1] The coefficient $C_5^{\alpha\beta}$ is symmetric under the interchange of the generation indices α, β, the New Physics scale Λ is assumed $\gg M_W$, and the second term is the Hermitian conjugate of the first.

In the broken theory, with $\varphi_0 = \frac{1}{\sqrt{2}}(v+h)$, $v \approx 246$ GeV, the Weinberg operator gives a Majorana neutrino mass matrix

$$\delta\mathcal{L} = -\frac{1}{2}[m_\nu]_{\alpha\beta}\overline{\nu_\alpha}\nu_\beta^c + \text{H.c.}\,, \qquad\qquad [m_\nu]_{\alpha\beta} = -\frac{v^2}{2\Lambda}C_5^{\alpha\beta}\,. \tag{5.5}$$

In the charged lepton mass eigenstate basis, this mass matrix is diagonalised by the PMNS matrix $[m_\nu]_{\alpha\beta} = U_{\alpha i}m_{\nu i}U_{\beta i}$.

At dimension-six, we are interested in SM-gauge invariant operators that violate lepton flavour, and a complete list is given in Section 5.1.1. Following the conventions of [8, 104], they are added to the Lagrangian as

$$\delta\mathcal{L}_6 = \sum_{X,\zeta}\frac{C_X^\zeta}{\Lambda^2}\mathcal{O}_X^\zeta + \text{H.c.}\,, \tag{5.6}$$

where X is an operator label and ζ represents all required generation indices which are summed over all generations. Of particular interest are the operators that can be generated at one-loop with two insertions of dimension-five operators, as illustrated in Figure 5.1. With SM particle content, these operators involve two Higgs doublets and two lepton doublets, four lepton doublets, or three Higgs doublets and leptons of both chiralities. In the Warsaw basis, the possibilities at dimension-six are

$$\begin{aligned}
\mathcal{O}_{\varphi\ell(1)}^{\alpha\beta} &= \frac{i}{2}(\varphi^\dagger \overleftrightarrow{D}_\mu\varphi)(\overline{\ell_\alpha}\gamma^\mu\ell_\beta) & \qquad \mathcal{O}_{\varphi\ell(3)}^{\alpha\beta} &= \frac{i}{2}(\varphi^\dagger \overleftrightarrow{D}_\mu^a\varphi)(\overline{\ell_\alpha}\gamma^\mu\tau^a\ell_\beta) \\
\mathcal{O}_{e\varphi}^{\alpha\beta} &= (\varphi^\dagger\varphi)\bar{\ell}_\alpha\varphi e_\beta & \qquad \mathcal{O}_{\ell\ell}^{\alpha\beta\gamma\delta} &= \frac{1}{2}(\bar{\ell}_\alpha\gamma_\mu\ell_\beta)(\bar{\ell}_\gamma\gamma^\mu\ell_\delta)\,,
\end{aligned} \tag{5.7}$$

where we normalise the "Hermitian" operators with a factor of 1/2 (see Section 5.1.1 for a discussion) in order to agree with [8, 104], and

[1] Note that this definition of the dimension-five operator is the Hermitian conjugate of the one used in [8], where $C = i\gamma^2\gamma^0$ in the Dirac representation. In the Dirac representation, $C^{-1} = -C$.

$$
\begin{aligned}
i(\varphi^\dagger \overleftrightarrow{D}_\mu \varphi) &\equiv i(\varphi^\dagger D_\mu \varphi) - i(D_\mu\varphi)^\dagger \varphi \\
&= \varphi^\dagger (i\partial_\mu \varphi) - i(\partial_\mu\varphi)^\dagger\varphi - g_2 \varphi^\dagger \tau^a W^a_\mu \varphi - 2Y_\varphi g_1 \varphi^\dagger B_\mu \varphi\,, \\
i(\varphi^\dagger \overleftrightarrow{D}^a_\mu \varphi) &\equiv i(\varphi^\dagger \tau^a D_\mu\varphi) - i(D_\mu\varphi)^\dagger \tau^a \varphi\,.
\end{aligned} \tag{5.8}
$$

The choice of operator basis implies a choice of operators that vanish by the Equations of Motion (EoMs). For example, $i\not{D}\ell_\alpha - [\Gamma_e]^{\alpha\sigma}\varphi e_\sigma = 0$ implies that the following operators

$$
\begin{aligned}
\mathcal{O}^{\alpha\beta}_{v(1)} &= (\varphi^\dagger\varphi)(\bar{\ell}_\alpha i\overleftrightarrow{\not{D}}\ell_\beta) - (\varphi^\dagger\varphi)(\bar{\ell}_\alpha \varphi e_\sigma [\Gamma^T_e]_{\sigma\beta} + [\Gamma^*_e]_{\alpha\sigma}\bar{e}_\sigma\varphi^\dagger\ell_\beta)\,, \\
\mathcal{O}^{\alpha\beta}_{v(3)} &= (\varphi^\dagger\tau^a\varphi)(\bar{\ell}_\alpha i\overleftrightarrow{\not{D}}^a\ell_\beta) - (\varphi^\dagger\varphi)(\bar{\ell}_\alpha \varphi e_\sigma [\Gamma^T_e]_{\sigma\beta} + [\Gamma^*_e]_{\alpha\sigma}\bar{e}_\sigma\varphi^\dagger\ell_\beta)\,,
\end{aligned} \tag{5.9}
$$

are EoM-vanishing operators. The role of these operators becomes clear by noting that in intermediate steps of our off-shell calculations, additional structures appear that can conveniently be matched onto combinations of EoM-vanishing operators and operators of the Warsaw basis. For example the structures involving two Higgs fields and a covariant derivative of a lepton doublet are expressed in terms of the above operators as

$$
\begin{aligned}
\mathcal{S}^{\alpha\beta}_{\varphi D\ell(1)} &= (\varphi^\dagger\varphi)(\bar{\ell}_\alpha i\overleftrightarrow{\not{D}}\ell_\beta) = \mathcal{O}^{\alpha\beta}_{v(1)} + \mathcal{O}^{\alpha\sigma}_{e\varphi}[\Gamma^T_e]_{\sigma\beta} + [\Gamma^*_e]_{\alpha\sigma}\mathcal{O}^{\dagger\sigma\beta}_{e\varphi}\,, \\
\mathcal{S}^{\alpha\beta}_{\varphi D\ell(3)} &= (\varphi^\dagger\tau^a\varphi)(\bar{\ell}_\alpha i\overleftrightarrow{\not{D}}^a\ell_\beta) = \mathcal{O}^{\alpha\beta}_{v(3)} + \mathcal{O}^{\alpha\sigma}_{e\varphi}[\Gamma^T_e]_{\sigma\beta} + [\Gamma^*_e]_{\alpha\sigma}\mathcal{O}^{\dagger\sigma\beta}_{e\varphi}\,.
\end{aligned} \tag{5.10}
$$

In practice, if the coefficients $C^{\beta\alpha}_{\varphi D\ell(1)}$ and $C^{\beta\alpha}_{\varphi D\ell(3)}$ of these structures are present, they are equivalent to $C^{\beta\sigma}_{e\varphi} = C^{\beta\alpha}_{\varphi D\ell(1)}[\Gamma_e]^{\alpha\sigma} + C^{\beta\alpha}_{\varphi D\ell(3)}[\Gamma_e]^{\alpha\sigma}$ (and the Hermitian conjugate relation).

5.1.1 LFV Operators of SMEFT

At dimension-six, we are interested in SM-gauge invariant operators that violate lepton flavour. The four-fermion operators involving $\beta \leftrightarrow \alpha$ lepton-flavour change and two quarks are

$$\mathcal{O}^{(1)\alpha\beta nm}_{\ell q} = \frac{1}{2}(\bar{\ell}_\alpha\gamma^\mu\ell_\beta)(\bar{q}_n\gamma_\mu q_m)\,, \tag{5.11}$$

$$\mathcal{O}^{(3)\alpha\beta nm}_{\ell q} = \frac{1}{2}(\bar{\ell}_\alpha\gamma^\mu\tau^a\ell_\beta)(\bar{q}_n\gamma_\mu\tau^a q_m)\,, \tag{5.12}$$

$$\mathcal{O}^{\alpha\beta nm}_{qe} = \frac{1}{2}(\bar{e}_\alpha\gamma^\mu e_\beta)(\bar{q}_n\gamma_\mu q_m)\,, \tag{5.13}$$

$$\mathcal{O}^{\alpha\beta nm}_{\ell u} = \frac{1}{2}(\bar{\ell}_\alpha\gamma^\mu\ell_\beta)(\bar{u}_n\gamma_\mu u_m)\,, \tag{5.14}$$

$$\mathcal{O}^{\alpha\beta nm}_{\ell d} = \frac{1}{2}(\bar{\ell}_\alpha\gamma^\mu\ell_\beta)(\bar{d}_n\gamma_\mu d_m)\,, \tag{5.15}$$

$$\mathcal{O}_{eu}^{\alpha\beta nm} = \frac{1}{2}(\overline{e}_\alpha\gamma^\mu e_\beta)(\overline{u}_n\gamma_\mu u_m)\,, \tag{5.16}$$

$$\mathcal{O}_{ed}^{\alpha\beta nm} = \frac{1}{2}(\overline{e}_\alpha\gamma^\mu e_\beta)(\overline{d}_n\gamma_\mu d_m)\,, \tag{5.17}$$

$$\mathcal{O}_{\ell equ}^{\alpha\beta nm} = (\overline{\ell}_\alpha^A e_\beta)\varepsilon_{AB}(\overline{q}_n^B u_m)\,, \tag{5.18}$$

$$\mathcal{O}_{\ell edq}^{\alpha\beta nm} = (\overline{\ell}_\alpha^A e_\beta)(\overline{d}_n q_m^A)\,, \tag{5.19}$$

$$\mathcal{O}_{T,\ell equ}^{\alpha\beta nm} = (\overline{\ell}_\alpha^A \sigma^{\mu\nu} e_\beta)\varepsilon_{AB}(\overline{q}_n^B \sigma_{\mu\nu} u_m)\,, \tag{5.20}$$

where ℓ and q are left-handed doublets, e and u are right-handed singlets, n and m are quark family indices, and A and B are SU(2) indices. Note that some of these operators differ from [8] by factors of 1/2, due to Hermiticity reasons discussed below.

In the case of four-lepton operators, the flavour change can be by one or two units:

$$\mathcal{O}_{\ell\ell}^{\alpha\beta\rho\sigma} = \frac{1}{2}(\overline{\ell}_\alpha\gamma^\mu \ell_\beta)(\overline{\ell}_\rho\gamma_\mu \ell_\sigma)\,, \tag{5.21}$$

$$\mathcal{O}_{\ell e}^{\alpha\beta\rho\sigma} = \frac{1}{2}(\overline{\ell}_\alpha\gamma^\mu \ell_\beta)(\overline{e}_\rho\gamma_\mu e_\sigma)\,, \tag{5.22}$$

$$\mathcal{O}_{ee}^{\alpha\beta\rho\sigma} = \frac{1}{2}(\overline{e}_\alpha\gamma^\mu e_\beta)(\overline{e}_\rho\gamma_\mu e_\sigma)\,. \tag{5.23}$$

Notice that in the case of $\mathcal{O}_{ee}$ and $\mathcal{O}_{\ell\ell}$, which are symmetric under interchange of the two bilinears (for example, $(\overline{e}\gamma^\mu\mu)(\overline{\tau}\gamma_\mu\tau) = (\overline{\tau}\gamma^\mu\tau)(\overline{e}\gamma_\mu\mu)$), there will be two equal coefficients that contribute to the Feynman rule.

There are also the operators allowing interactions with gauge bosons and Higgses. This includes dipole operators, which are normalised with the muon Yukawa coupling so as to match onto the normalisation of Kuno-Okada [86]:

$$\mathcal{O}_{e\varphi}^{\alpha\beta} = (\varphi^\dagger\varphi)(\overline{\ell}_\alpha\varphi e_\beta)\,, \tag{5.24}$$

$$\mathcal{O}_{eW} = \Gamma_\beta(\overline{\ell}_\beta\tau^a\varphi\sigma^{\mu\nu}e_\beta)W_{\mu\nu}^a\,, \tag{5.25}$$

$$\mathcal{O}_{eB} = \Gamma_\beta(\overline{\ell}_\beta\varphi\sigma^{\mu\nu}e_\beta)B_{\mu\nu}\,, \tag{5.26}$$

$$\mathcal{O}_{\varphi\ell(1)}^{\alpha\beta} = \frac{i}{2}(\varphi^\dagger \overleftrightarrow{D}_\mu \varphi)(\overline{\ell}_\alpha\gamma^\mu\ell_\beta)\,, \tag{5.27}$$

$$\mathcal{O}_{\varphi\ell(3)}^{\alpha\beta} = \frac{i}{2}(\varphi^\dagger \overleftrightarrow{D}_\mu^a \varphi)(\overline{\ell}_\alpha\gamma^\mu\tau^a\ell_\beta)\,, \tag{5.28}$$

$$\mathcal{O}_{\varphi e}^{\alpha\beta} = \frac{i}{2}(\varphi^\dagger \overleftrightarrow{D}_\mu \varphi)(\overline{e}_\alpha\gamma^\mu e_\beta)\,, \tag{5.29}$$

where Γ_β denotes the Yukawa coupling of a charged lepton e_β in the mass basis, the double derivatives are defined in Equation (5.8), and we include factors of 1/2 for Hermitian operators as discussed now.

We use the convention that all physical operators (i.e. included in the Warsaw basis) and their Hermitian conjugates are explicitly added to the Lagrangian

$$\delta\mathcal{L}_6 = \sum_{X,\zeta}\frac{C_X^\zeta}{\Lambda^2}\mathcal{O}_X^\zeta + \text{H.c.}\,, \tag{5.30}$$

where the flavour indices are represented by ζ, and are all summed over all generations. In the conventions of [8] and [104], the Hermitian conjugate is not added for "self-adjoint" operators, for which $\sum_\zeta C_X^\zeta \mathcal{O}_X^\zeta = [\sum_\zeta C_X^\zeta \mathcal{O}_X^\zeta]^\dagger$. (For instance, $\mathcal{O}_{\ell\ell}^{\alpha\beta\rho\sigma}$ of Equation (5.21) is "Hermitian", although $[(\overline{e}\gamma^\mu \mu)(\overline{\tau}\gamma_\mu \tau)]^\dagger = (\overline{\mu}\gamma^\mu e)(\overline{\tau}\gamma_\mu \tau)$). We therefore define such operators with a factor 1/2 to avoid this double-counting. However, it should be noted that the unphysical EoM-vanishing operators of Equation (5.9) do not enter the Lagrangian at tree-level, and therefore are not subject to this normalisation condition.

5.1.2 The Two Higgs Doublet Model

In this section, the addition of a second Higgs doublet φ_2 to the SM (of the same hypercharge as the SM Higgs, which is relabelled φ_1) is considered. The LFV induced by double-insertions of dimension-five operators could be more significant in this model, because there are several dimension-five operators, and so neutrino masses cannot constrain them all. However, a complete analysis of LFV in the Two Higgs Doublet Model (2HDM) would require extending the operator basis at dimension-six and calculating the additional terms in the RGEs, which is beyond the scope of this work. For simplicity, three restrictions are made:

1. Only the dimension-six LFV operators of SMEFT are considered. This is the appropriate set of dimension-six operators just above M_W, provided that φ_2 has a vanishing VEV, and that the mass m_{22} of the additional Higgses is sufficiently high: $M_W^2 \ll m_{22}^2 \ll \Lambda^2$. In our phenomenological analysis we extend this range to the scenario $M_W^2 \lesssim m_{22}^2 \ll \Lambda^2$, by considering a Higgs potential where the additional Higgses are not directly observable at the LHC, and where the Yukawa couplings of φ_2 are vanishing. Such a scenario would for example be realised in the inert two Higgs doublet model [106–109] and setting the scale m_{22} close to the electroweak scale does not require the consideration of additional renormalisation group effects in SMEFT.

2. It is supposed that at the high scale Λ, no dimension-six LFV operators are generated. This is unrealistic, but allows us to focus on the LFV generated by double-insertions of the dimension-five operators.

3. It is supposed there is no LFV in the renormalisable couplings of the 2HDM (in particular, in the lepton Yukawas), so that when matching the 2HDM + dimension-five operators onto SMEFT at the intermediate scale m_{22}, no additional LFV operators are generated.

Consider first the renormalisable Lagrangian. The Yukawa couplings can be written as [110]

$$\delta\mathcal{L}_{2\mathrm{HDM}} = -(\overline{\nu}, \overline{e_L})[\Gamma_e]\left(\begin{array}{c}\varphi_1^+\\ \varphi_1^0\end{array}\right)e - \overline{e}[\Gamma_e]^\dagger\varphi_1^\dagger\ell - (\overline{\nu}, \overline{e_L})[\Gamma_e^{(2)}]\left(\begin{array}{c}\varphi_2^+\\ \varphi_2^0\end{array}\right)e - \overline{e}[\Gamma_e^{(2)}]^\dagger\varphi_2^\dagger\ell\,,$$

where the flavour indices are implicit, and the basis in (φ_1, φ_2) space is taken to be the "Higgs basis" where $\langle\varphi_2\rangle = 0$. We suppose that $[\Gamma_e]$ and $[\Gamma_e^{(2)}]$ are simultaneously diagonalisable on their lepton flavour indices.

The second Yukawa coupling changes the equations of motion for the leptons, so the 2HDM version of the equation-of-motion vanishing operators (given in Equation (5.9) for the single Higgs model) should be modified. As a result, the operators $\mathcal{S}_{\varphi D\ell(1)}$ and $\mathcal{S}_{\varphi D\ell(3)}$ should not be replaced only by the SMEFT operator $\mathcal{O}_{e\varphi}$, as given in Equation (5.10), but also by an operator with an external φ_2 leg. However, since we neglect dimension-six operators with external φ_2, we use the relations (5.9) and (5.10) also in the 2HDM case.

In this "Higgs" basis, the most general Higgs potential is

$$\begin{aligned} V &= m_{11}^2\varphi_1^\dagger\varphi_1 + m_{22}^2\varphi_2^\dagger\varphi_2 - [m_{12}^2\varphi_1^\dagger\varphi_2 + \mathrm{H.c.}] \\ &\quad + \frac{1}{2}\lambda_1(\varphi_1^\dagger\varphi_1)^2 + \frac{1}{2}\lambda_2(\varphi_2^\dagger\varphi_2)^2 + \lambda_3(\varphi_1^\dagger\varphi_1)(\varphi_2^\dagger\varphi_2) + \lambda_4(\varphi_1^\dagger\varphi_2)(\varphi_2^\dagger\varphi_1) \\ &\quad + \left\{\frac{1}{2}\lambda_5(\varphi_1^\dagger\varphi_2)^2 + \left[\lambda_6\,(\varphi_1^\dagger\varphi_1) + \lambda_7(\varphi_2^\dagger\varphi_2)\right]\varphi_1^\dagger\varphi_2 + \mathrm{H.c.}\right\}\,. \end{aligned} \tag{5.31}$$

In order to decouple the additional Higgses, we can set $m_{12}^2 = 0$ and assume $m_{22}^2 \gg M_W^2$, or leave m_{22}^2 free, and impose $m_{12}^2 = \lambda_6 = \lambda_7 = [\Gamma_e^{(2)}] = 0$.

At dimension-five in the 2HDM, there are four operators [101]:

$$\begin{aligned} \delta\mathcal{L} &= +\frac{C_5^{\alpha\beta}}{2\Lambda}(\overline{\ell_\alpha}\varepsilon\varphi_1^*)(\ell_\beta^c\varepsilon\varphi_1^*) + \frac{C_5^{\alpha\beta*}}{2\Lambda}(\overline{\ell_\beta^c}\varepsilon\varphi_1)(\ell_\alpha\varepsilon\varphi_1) \\ &\quad + \frac{C_{21}^{\alpha\beta}}{2\Lambda}\Big((\overline{\ell_\alpha}\varepsilon\varphi_2^*)(\ell_\beta^c\varepsilon\varphi_1^*) + (\overline{\ell_\beta}\varepsilon\varphi_1^*)(\ell_\alpha^c\varepsilon\varphi_2^*)\Big) \\ &\qquad + \frac{C_{21}^{\alpha\beta*}}{2\Lambda}\Big((\overline{\ell_\beta^c}\varepsilon\varphi_2)(\ell_\alpha\varepsilon\varphi_1) + (\overline{\ell_\alpha^c}\varepsilon\varphi_1)(\ell_\beta\varepsilon\varphi_2)\Big) \\ &\quad + \frac{C_{22}^{\alpha\beta}}{2\Lambda}(\overline{\ell_\alpha}\varepsilon\varphi_2^*)(\ell_\beta^c\varepsilon\varphi_2^*) + \frac{C_{22}^{\alpha\beta*}}{2\Lambda}(\overline{\ell_\beta^c}\varepsilon\varphi_2)(\ell_\alpha\varepsilon\varphi_2) \\ &\quad - \frac{C_A^{\alpha\beta}}{2\Lambda}(\overline{\ell_\alpha}\varepsilon\ell_\beta^c)(\varphi_1^\dagger\varepsilon\varphi_2^*) - \frac{C_A^{\alpha\beta*}}{2\Lambda}(\overline{\ell_\beta^c}\varepsilon\ell_\alpha)(\varphi_2\varepsilon\varphi_1)\,, \end{aligned} \tag{5.32}$$

where $\{C_5, C_{22}, C_{21}\}$ are symmetric in their flavour indices (and so can contribute to neutrino masses, since Majorana mass matrices are also symmetric in flavour space). In the $\mathcal{O}_{21}$ operator, $(\overline{\ell_\alpha}\varepsilon\varphi_2^*)(\ell_\beta^c\varepsilon\varphi_1^*) = (\overline{\ell_\beta}\varepsilon\varphi_1^*)(\ell_\alpha^c\varepsilon\varphi_2^*)$, but both terms are retained here since they are convenient in our Feynman rule conventions.[2]

[2] The operator $\mathcal{O}_{21}$ can also be written as $2(\overline{\ell_\beta}\varepsilon\varphi_1^*)(\ell_\alpha^c\varepsilon\varphi_2^*)$ $+(\overline{\ell_\beta}\varepsilon\ell_\alpha^c)(\varphi_1^*\varepsilon\varphi_2^*)$ using the identity (5.52), as done in the first reference of [101].

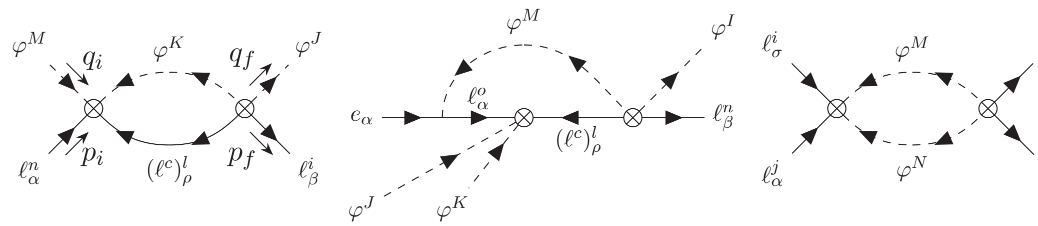


Fig. 5.1: Diagrams involving two insertions of dimension-five operators, which can contribute to dimension-six lepton-flavour-violating operators. SU(2) indices run from $I, ..., O$ and $i, ..., o$, lepton flavour indices are $\alpha, \beta, \rho, \sigma$.

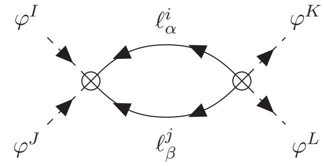


Fig. 5.2: Two insertions of dimension-five operators can also contribute to dimension-six operators involving four Higgses via this diagram.

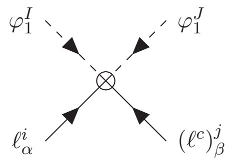
Tree-level LFV is often avoided in the 2HDM by imposing a Z_2 symmetry on the renormalisable Lagrangian: if under the Z_2 transformation, $\varphi_1 \to \varphi_1$ and $\varphi_2 \to -\varphi_2$, then $[\Gamma_e^{(2)}]$, λ_6 and λ_7 are forbidden. This case is discussed later, but we do not impose the Z_2 symmetry initially, as it also forbids the C_{21} and C_A coefficients at dimension-five.

5.2 The EFT Calculation

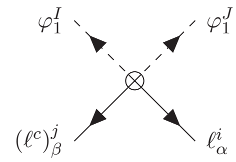
5.2.1 Diagrams, Divergences and the RGEs

Diagrams with two insertions of dimension-five operators are illustrated in Figures 5.1 and 5.2. We focus on the lepton flavour violating diagrams of Figure 5.1, but the four-Higgs diagram of Figure 5.2 is also briefly discussed in Section 5.6. The four-Higgs case is less interesting to us since it is lepton flavour conserving, and such interactions arise in the renormalisable SM.

The Feynman rules arising from the (tree-level) Lagrangian of Equation (5.4) are given in Figure 5.3. We use them to evaluate, using dimensional regularisation in $4 - 2\epsilon$ dimensions in $\overline{\text{MS}}$, the coefficient of the $1/\epsilon$ divergence of each diagram of Figure 5.1. These coefficients can be expressed as a sum of numerical factors multiplying the Feynman rules for the dimension-six operators of Equations (5.7) and (5.10) (the Feynman rules for dimension-six physical operators and EoM-vanishing operators are given in Figures 5.4 and 5.5 respectively). Then the EoMs are used to transform the operators of Equation (5.10) to $\mathcal{O}_{e\varphi}$ and $\mathcal{O}^\dagger_{e\varphi}$. The required counterterm ΔC_O for each of the



$$i\frac{C_5^{*\beta\alpha}}{\Lambda}(\varepsilon_{iI}\varepsilon_{jJ}+\varepsilon_{jI}\varepsilon_{iJ})P_L$$



$$i\frac{C_5^{\alpha\beta}}{\Lambda}(\varepsilon_{iI}\varepsilon_{jJ}+\varepsilon_{jI}\varepsilon_{iJ})P_R$$

Fig. 5.3: Feynman rules for the Weinberg operator, the only dimension-five operator of SMEFT, where φ_1 denotes the SM Higgs.

dimension-six operators given in Equation (5.7) can be identified as $(-1)\times$ the numerical factor that multiplies its Feynman rule. This counterterm is added in the Lagrangian to the operator coefficient $C_{\mathcal{O}}$, resulting in a "bare" coefficient $C_{\mathcal{O},\text{bare}} = \mu^{2\epsilon}(C_{\mathcal{O}}+\Delta C_{\mathcal{O}})$ that should be independent of the $\overline{\text{MS}}$ renormalisation scale μ. The factor $\mu^{2\epsilon}$ is chosen such that the bare Lagrangian remains d-dimensional.

We now discuss the renormalisation and RGEs of dimension-five and -six operators. This discussion is based on Section 4.3.2, but is extended to also consider flavour structures and conjugate Wilson coefficients, which is necessary for the consideration of LFV operators. The bare Wilson coefficients of dimension-five operators can be written as (noting that in preparation for a discussion of the 2HDM, multiple operators are allowed for at dimension-five)

$$\vec{C}^{\eta}_{X,\text{bare}} = \mu^{2\epsilon}\vec{C}^{\theta}_{Y}(\mu)Z^{\theta\eta}_{YX}(\mu), \tag{5.33}$$

where X, Y and η, θ are operator and flavour labels respectively, $\vec{C}^{\theta}_{Y}(\mu)$ is the renormalised Wilson coefficient, $Z^{\theta\eta}_{YX}(\mu)$ is the dimension-five renormalisation matrix, and μ is the renormalisation scale. The $\mu^{2\epsilon}$ introduces an additional term proportional to ϵ into the d-dimensional renormalisation group equation,

$$\mu\frac{d}{d\mu}\vec{C}^{\eta}_{X} = -\vec{C}^{\theta}_{Y}\left(\mu\frac{d}{d\mu}Z^{\theta\zeta}_{YZ}\right)\left[Z^{-1}\right]^{\zeta\eta}_{ZX} - 2\epsilon\vec{C}^{\eta}_{X}. \tag{5.34}$$

This reduces to the renormalisation group equation in $d = 4$ dimensions

$$(16\pi^2)\mu\frac{d}{d\mu}\vec{C}^{\eta}_{X} \overset{d=4}{=} \vec{C}^{\theta}_{Y}\gamma^{\theta\eta}_{YX}, \tag{5.35}$$

where the 4-dimensional anomalous dimension matrix

$$\gamma^{\theta\eta}_{YX} = -(16\pi^2)\left(\mu\frac{d}{d\mu}Z^{\theta\zeta}_{YZ}\right)\left[Z^{-1}\right]^{\zeta\eta}_{ZX} \tag{5.36}$$

is independent of the choice of the overall factor $\mu^{2\epsilon}$. Therefore, the $\mu^{2\epsilon}$ term can be neglected when only considering mixing amongst operators of equal dimensions. In the case of mixing between operators of different dimensions, a more careful treatment

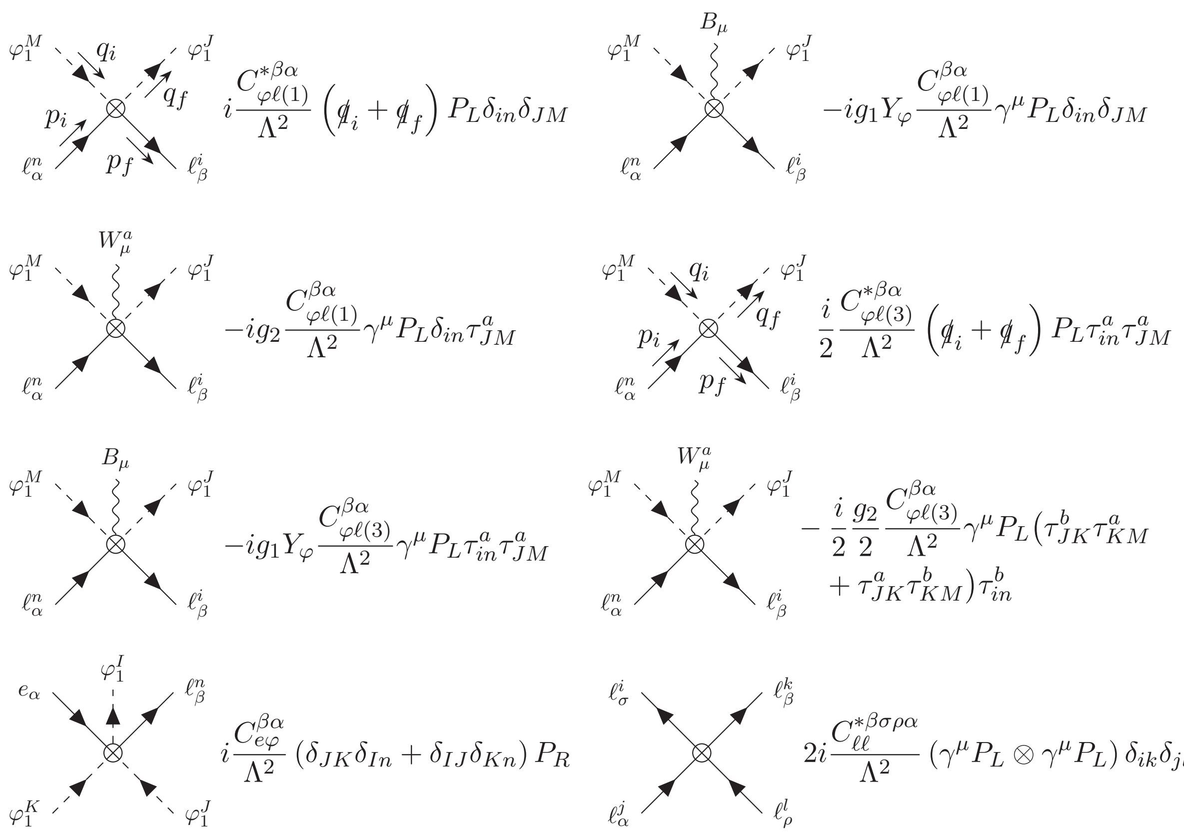


Fig. 5.4: Feynman rules for dimension-six operators of the SMEFT using the Warsaw basis. φ_1 is the SM Higgs. Note that the Feynman rules for the operators $\mathcal{O}_{\varphi\ell(1)}$ and $\mathcal{O}_{\varphi\ell(3)}$ come with a factor of 1/2, due to their normalisation condition arising from including in the Lagrangian every operator plus its Hermitian conjugate.

is required. Note that we (unconventionally) factor the $16\pi^2$ out of the anomalous dimension matrices.[3]

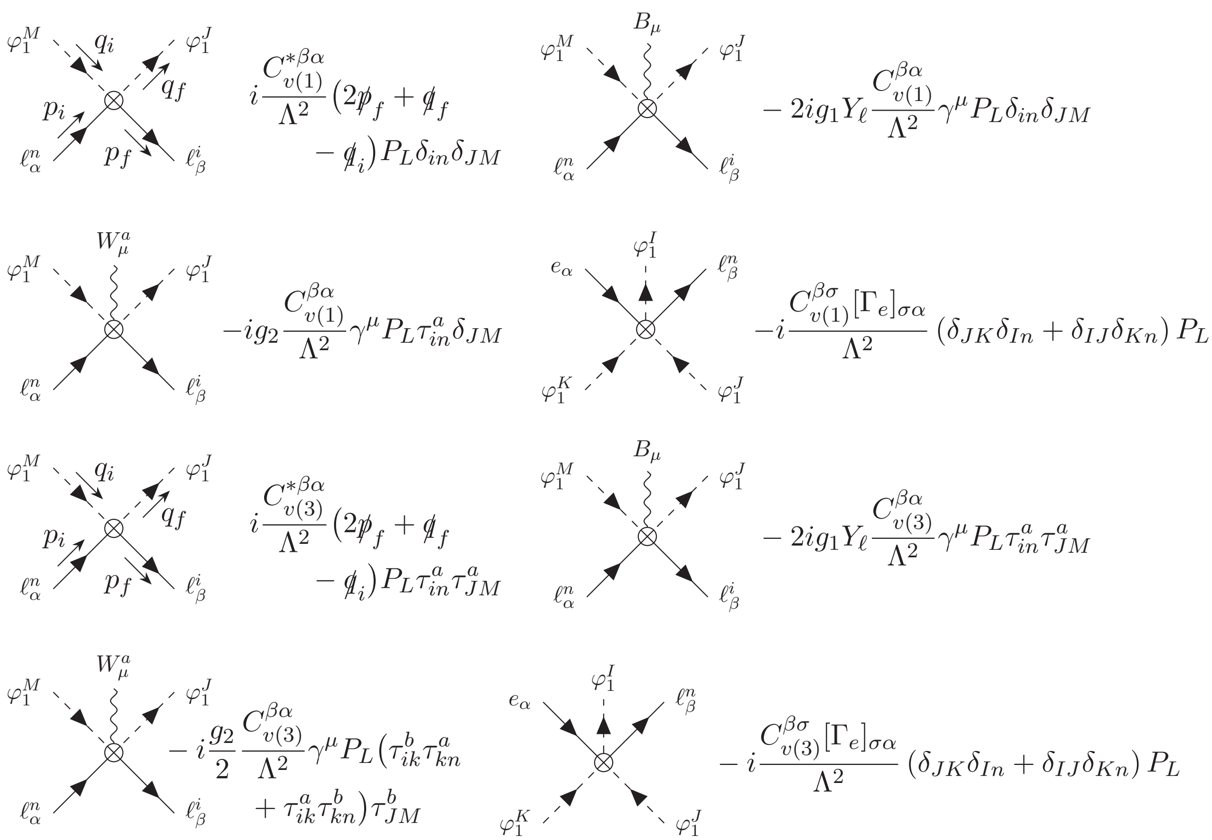
At loop level, operators of different dimensions can mix via multiple operator insertions [75]. Consider the specific case of loop diagrams involving two dimension-five operators mixing into diagrams with a single dimension-six operator insertion. We denote dimension-six Wilson coefficients by $\tilde{C}$, dimension-five Wilson coefficients by C, the dimension-six ADM by $\hat{Z}$, and the ADT for mixing dimension-five into dimension-six by $\tilde{Z}$. The bare dimension-six Wilson coefficient is

$$\tilde{C}^\eta_{X,\text{bare}} = \mu^{2\epsilon}\tilde{C}^\theta_Y(\mu)\hat{Z}^{\theta\eta}_{YX}(\mu) + \mu^{2\epsilon}C^\zeta_A(\mu)\tilde{Z}^{\zeta\theta,\eta}_{AB,X}(\mu)\left[C^\theta_B\right]^\dagger(\mu), \tag{5.37}$$

where $\tilde{C}_\text{bare}$ is μ-independent. Therefore, the renormalisation group equation is

$$(16\pi^2)\mu\frac{d}{d\mu}\tilde{C}^\eta_X = \tilde{C}^\theta_Y\hat{\gamma}^{\theta\eta}_{YX} + C^\zeta_A\tilde{\gamma}^{\zeta\theta,\eta}_{AB,X}\left[C^\theta_B\right]^\dagger, \tag{5.38}$$

[3] The usual definition is $\mu\frac{d}{d\mu}C = C\gamma$ [44], then γ is expanded in loops: $\gamma = \frac{\alpha_s}{4\pi}\gamma_0 + \ldots$. However, here we only work at one loop, and the one loop mixing of dimension-five-squared into dimension-six is not induced by a renormalisable coupling, so we factor out the $16\pi^2$.



$$i\frac{C^{*\beta\alpha}_{v(1)}}{\Lambda^2}\left(2\not p_f + \not q_f - \not q_i\right) P_L \delta_{in}\delta_{JM}$$

$$-\,2ig_1 Y_\ell \frac{C^{\beta\alpha}_{v(1)}}{\Lambda^2}\gamma^\mu P_L \delta_{in}\delta_{JM}$$

$$-ig_2 \frac{C^{\beta\alpha}_{v(1)}}{\Lambda^2}\gamma^\mu P_L \tau^a_{in}\delta_{JM}$$

$$-i\frac{C^{\beta\sigma}_{v(1)}[\Gamma_e]_{\sigma\alpha}}{\Lambda^2}\left(\delta_{JK}\delta_{In}+\delta_{IJ}\delta_{Kn}\right)P_L$$

$$i\frac{C^{*\beta\alpha}_{v(3)}}{\Lambda^2}\left(2\not p_f + \not q_f - \not q_i\right) P_L \tau^a_{in}\tau^a_{JM}$$

$$-\,2ig_1 Y_\ell \frac{C^{\beta\alpha}_{v(3)}}{\Lambda^2}\gamma^\mu P_L \tau^a_{in}\tau^a_{JM}$$

$$-\,i\frac{g_2}{2}\frac{C^{\beta\alpha}_{v(3)}}{\Lambda^2}\gamma^\mu P_L\left(\tau^b_{ik}\tau^a_{kn} + \tau^a_{ik}\tau^b_{kn}\right)\tau^b_{JM}$$

$$-\,i\frac{C^{\beta\sigma}_{v(3)}[\Gamma_e]_{\sigma\alpha}}{\Lambda^2}\left(\delta_{JK}\delta_{In}+\delta_{IJ}\delta_{Kn}\right)P_L$$

Fig. 5.5: Feynman rules for dimension-six operators that are vanishing by the equations of motion in the single Higgs Model (SMEFT). φ_1 is the SM Higgs.

where $\hat{\gamma}^{\theta\eta}_{YX}$ is defined analogously to Equation (5.34), and

$$\begin{aligned}\tilde{\gamma}^{\zeta\theta,\eta}_{AB,X} =&(16\pi^2)\left(2\epsilon\tilde{Z}^{\zeta\theta,\upsilon}_{AB,Y} - \mu\frac{d}{d\mu}\tilde{Z}^{\zeta\theta,\upsilon}_{AB,Y}\right)\left[\hat{Z}^{-1}\right]^{\upsilon\eta}_{YX}\\ &-(16\pi^2)\left(\left[\gamma^{\theta\omega}_{BD}\right]^\dagger\delta^{\zeta\chi}_{AC} + \gamma^{\zeta\chi}_{AC}\delta^{\theta\omega}_{BD}\right)\tilde{Z}^{\chi\omega,\upsilon}_{CD,Y}\left[\hat{Z}^{-1}\right]^{\upsilon\eta}_{YX}\,,\end{aligned} \tag{5.39}$$

where the explicit form in terms of generation indices is $[\gamma^{\alpha\beta\,\gamma\delta}_{AB}]^\dagger = [\gamma^{\beta\alpha\,\delta\gamma}_{AB}]^*$ and $\delta^{\alpha\beta\,\gamma\delta}_{AB} \equiv \delta_{AB}\delta_{\alpha\gamma}\delta_{\beta\delta}$. The terms in the second line of the above equation only contribute beyond 1-loop. Furthermore, the contribution to the renormalisation tensor $Z^{\zeta\theta,\upsilon}_{AB,Y}$ is μ-independent at one-loop and only the term proportional to 2ϵ contributes in our calculation. The factor in $\mu^{2\epsilon}$ in Equation (5.37) generates a term proportional to -2ϵ, while the derivative of the dimension-five Wilson coefficients generates a contribution proportional to $2\times 2\epsilon$ from Equation (5.34). Hence, the one-loop anomalous dimension matrix reads

$$\tilde{\gamma}^{\zeta\eta,\theta}_{AB,C} = 2\delta\tilde{Z}^{\zeta\eta,\theta}_{AB,C} \tag{5.40}$$

in terms of the 1-loop renormalisation constants defined in Equation (5.42). Correspondingly, we find $[\tilde{\gamma}] = 2(16\pi^2)\epsilon[\tilde{Z}]$. Consequently, to find the leading order contribution to the ADT $\tilde{\gamma}$, it is sufficient to calculate the renormalisation tensor $[\tilde{Z}]$, which may be done through renormalising the diagrams in Figure 5.1 using dimension-six operators.

5.3 Conventions of the Loop Calculations

5.3.1 Flavour Dependence

A key feature of the Warsaw basis is that the flavour dependence of operators is encoded in flavour indices, which need to be treated carefully. We therefore present the general structure of how the loop calculations are performed in order to deal with the flavour structures in a systematic way. We allow for multiple operators at both dimension-five and -six, and denote a particular Wilson coefficient by C_X^ζ, where X and ζ are the operator and flavour labels respectively. Then the bare Wilson coefficients of the dimension-six SMEFT Lagrangian can be written as

$$\sum_{\zeta,X} \widetilde{C}^\zeta_{X,\text{bare}} Q^\zeta_{X,\text{bare}} = \mu^{2\epsilon} \sum_{\theta,Y} \left(\sum_{\zeta,X} \widetilde{C}^\zeta_X \hat{Z}^{\zeta\theta}_{XY} + \sum_{\zeta,\eta} C^\zeta_5 \left[C^\eta_5\right]^\dagger \widetilde{Z}^{\zeta\eta\theta}_{5\bar{5},Y} \right) Q^\theta_{Y,\text{bare}}\,, \tag{5.41}$$

where ζ, η and θ represent generation indices of an operator, and the renormalisation constants $\hat{Z}^{\zeta\theta}_{XY}$ encode the mixing of dimension-six Wilson coefficients amongst themselves, which can be extracted from the anomalous dimensions of [104]. In the SM, the mixing of two dimension-five Wilson coefficients into a dimension-six coefficient is given by $\widetilde{Z}^{\zeta\eta\theta}_{5\bar{5},Y}$. They are induced by the double-insertion of dimension-five operators, as shown in Figure 5.1. In the case of a 2HDM effective field theory, we extend the summation of the dimension-five flavour indices to a sum over all dimension-five operators and their respective flavour components.

The renormalisation constants can be expanded in the number of loops and powers of ϵ. At 1-loop in the $\overline{\text{MS}}$ scheme, the counterterms of the physical and EOM-vanishing operators are pure $1/\epsilon$ poles, and the renormalisation of evanescent operators does not play a role. Hence, we can expand

$$\widetilde{Z}^{\zeta\eta\theta}_{5\bar{5},j} = \frac{1}{16\pi^2}\frac{1}{\epsilon}\delta\widetilde{Z}^{\zeta\eta\theta}_{5\bar{5},j} \tag{5.42}$$

and write the generation summation in the case of an operator involving four fermions explicitly as

$$C^\zeta_5 C^{\eta\dagger}_5 \delta\widetilde{Z}^{\zeta\eta\theta}_{5\bar{5},X} Q^\theta_X = C^{\alpha\beta}_5 C^{\delta\gamma*}_5 \delta\widetilde{Z}^{\alpha\beta\,\gamma\delta,\rho\sigma\tau\upsilon}_{5\bar{5},X} Q^{\rho\sigma\tau\upsilon}_X. \tag{5.43}$$

The sum over generation indices reduces trivially for operators that involve fewer fermions. The corresponding renormalisation equation ensures that the pole of the 1-loop off-shell matrix element of an insertion of two dimension-five operators is cancelled by its counterterm. Factoring out the common overall factor $C_5^{\alpha\beta}C_5^{\delta\gamma *}$, we write

$$\langle f|Q_5^{\alpha\beta}(Q_5^{\gamma\delta})^\dagger|i\rangle|^{(1)}_{1/\epsilon} + \langle f|\left(\delta\widetilde{Z}^{\alpha\beta\,\gamma\delta,\rho\sigma\tau\upsilon}_{5\bar{5},X} Q_X^{\rho\sigma\tau\upsilon} + \text{H.c.}\right)|i\rangle = 0\,, \tag{5.44}$$

where $|^{(1)}_{1/\epsilon}$ denotes the $1/\epsilon$ pole of a 1-loop diagram and $\langle f|$ and $|i\rangle$ are arbitrary off-shell final and initial states.

In calculations of the loop diagrams, the following generation structures arise:

$$\begin{aligned}
\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha} &= \frac{1}{4}\left(\delta_{\gamma\beta}\delta_{\delta\eta}\delta_{\kappa\alpha} + \delta_{\gamma\beta}\delta_{\delta\kappa}\delta_{\eta\alpha} + \delta_{\delta\beta}\delta_{\gamma\eta}\delta_{\kappa\alpha} + \delta_{\delta\beta}\delta_{\gamma\kappa}\delta_{\eta\alpha}\right)\,,\\
\Delta_{1,A\overline{A}}^{\gamma\delta\eta\kappa,\beta\alpha} &= \frac{1}{4}(\delta_{\gamma\beta}\delta_{\delta\kappa}\delta_{\eta\alpha} - \delta_{\gamma\kappa}\delta_{\delta\beta}\delta_{\eta\alpha} + \delta_{\gamma\eta}\delta_{\delta\beta}\delta_{\kappa\alpha} - \delta_{\gamma\beta}\delta_{\delta\eta}\delta_{\kappa\alpha})\,,\\
\Delta_{1,A\overline{S}}^{\gamma\delta\eta\kappa,\beta\alpha} &= \frac{1}{4}(\delta_{\gamma\beta}\delta_{\delta\kappa}\delta_{\eta\alpha} - \delta_{\gamma\kappa}\delta_{\delta\beta}\delta_{\eta\alpha} - \delta_{\gamma\eta}\delta_{\delta\beta}\delta_{\kappa\alpha} + \delta_{\gamma\beta}\delta_{\delta\eta}\delta_{\kappa\alpha})\,,\\
\Delta_{1,S\overline{A}}^{\gamma\delta\eta\kappa,\beta\alpha} &= \frac{1}{4}(\delta_{\gamma\beta}\delta_{\delta\kappa}\delta_{\eta\alpha} + \delta_{\gamma\kappa}\delta_{\delta\beta}\delta_{\eta\alpha} - \delta_{\gamma\eta}\delta_{\delta\beta}\delta_{\kappa\alpha} - \delta_{\gamma\beta}\delta_{\delta\eta}\delta_{\kappa\alpha})\,,\\
\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha} &= \frac{1}{4}\left(\delta_{\gamma\beta}\delta_{\delta\eta}[\Gamma_e]_{\kappa\alpha} + \delta_{\gamma\beta}\delta_{\delta\kappa}[\Gamma_e]_{\eta\alpha} + \delta_{\delta\beta}\delta_{\gamma\eta}[\Gamma_e]_{\kappa\alpha} + \delta_{\delta\beta}\delta_{\gamma\kappa}[\Gamma_e]_{\eta\alpha}\right)\,,\\
\Delta_{2,S}^{\gamma\delta\eta\kappa,\beta\alpha} &= \frac{1}{4}(\delta_{\gamma\beta}\delta_{\delta\kappa}[\Gamma_e^{(2)}]_{\eta\alpha} + \delta_{\gamma\beta}\delta_{\delta\eta}[\Gamma_e^{(2)}]_{\kappa\alpha} + \delta_{\gamma\kappa}\delta_{\delta\beta}[\Gamma_e^{(2)}]_{\eta\alpha} + \delta_{\gamma\eta}\delta_{\delta\beta}[\Gamma_e^{(2)}]_{\kappa\alpha})\,,\\
\Delta_{2,A}^{\gamma\delta\eta\kappa,\beta\alpha} &= \frac{1}{4}(\delta_{\gamma\beta}\delta_{\delta\kappa}[\Gamma_e^{(2)}]_{\eta\alpha} + \delta_{\gamma\beta}\delta_{\delta\eta}[\Gamma_e^{(2)}]_{\kappa\alpha} - \delta_{\gamma\kappa}\delta_{\delta\beta}[\Gamma_e^{(2)}]_{\eta\alpha} - \delta_{\gamma\eta}\delta_{\delta\beta}[\Gamma_e^{(2)}]_{\kappa\alpha})\,,\\
\Delta_3^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma} &= \frac{1}{4}(\delta_{\sigma\kappa}\delta_{\alpha\eta} + \delta_{\alpha\kappa}\delta_{\sigma\eta})(\delta_{\rho\gamma}\delta_{\beta\delta} + \delta_{\beta\gamma}\delta_{\rho\delta})\,,\\
\Delta_{3,A}^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma} &= \frac{1}{4}(\delta_{\gamma\alpha}\delta_{\delta\sigma} - \delta_{\gamma\sigma}\delta_{\delta\alpha})(\delta_{\kappa\beta}\delta_{\eta\rho} - \delta_{\kappa\rho}\delta_{\eta\beta})\,,\\
\Delta_4^{\gamma\delta\eta\kappa} &= \frac{1}{2}(\delta_{\gamma\eta}\delta_{\delta\kappa} + \delta_{\gamma\kappa}\delta_{\delta\eta}).
\end{aligned} \tag{5.45}$$

These are matched onto the generation structures of the dimension-six operators (the matching is more subtle for the four-lepton operator $\mathcal{O}_{\ell\ell}^{\alpha\beta\gamma\delta}$, where the matching is done via a Fierz-evanescent dimension-six operator $\mathcal{O}_{\text{eva}}^{\alpha\beta\gamma\delta}$), and the generation structure is therefore extracted from the renormalisation constants, which can then be written as a generation structure multiplied by a numerical factor.

5.3.2 SU(2) Identities and Dimension-Four Feynman Rules

In computing the loop diagrams to be discussed below, it was necessary to know the Feynman rules for dimension-four couplings that appear. The appearance of charge-conjugate fermions due to dimension-five operators means that their Feynman rules must be treated with some care, and so we use the Feynman rules of [33]. The Feynman rule for the Weinberg operator of Equation (5.4) can be obtained reliably by using Lehmann-Symanzik-Zimmermann (LSZ) reduction [111] or Wick's theorem, which gives the signs for fermion interchange. The fermion fields are expanded as [42]

$$\psi(x) = \sum_s \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E}}(a_k^s u_s(k)e^{-ik\cdot x} + b_k^{s\,\dagger} v_s(k)e^{+ik\cdot x})\,,$$

so the amplitude $\mathcal{M}_{fi}$ is

$$\begin{aligned}
\langle \ell^j_\alpha \varphi^I | i\frac{C_5^{\sigma\rho}}{2\Lambda}(\overline{\ell^n_\sigma}\varepsilon_{nN}\varphi^{N*})(\ell^{c,m}_\rho \varepsilon_{mM}\varphi^{M*})|\ell^{c,i}_\beta \varphi^{J*}\rangle \\
&= (-i)i\frac{C_5^{\alpha\beta}}{2\Lambda}\left(\overline{u}^j_\alpha P_R u_{\beta i} + \overline{u}^i_\beta P_R u^j_\alpha\right)(\varepsilon_{iI}\varepsilon_{jJ} + \varepsilon_{iJ}\varepsilon_{jI}) \\
&= (-i)i\frac{C_5^{\alpha\beta} + C_5^{\beta\alpha}}{2\Lambda}\overline{u}^j_\alpha P_R u^i_\beta(\varepsilon_{iI}\varepsilon_{jJ} + \varepsilon_{iJ}\varepsilon_{jI}) \\
&= (-i)i\frac{C_5^{\alpha\beta}}{\Lambda}\overline{u}^j_\alpha P_R u^i_\beta(\varepsilon_{iI}\varepsilon_{jJ} + \varepsilon_{iJ}\varepsilon_{jI}), \qquad (5.46)
\end{aligned}$$

where the SU(2) lepton indices are lower case, Higgs indices are upper case, and ℓ^j_α and $\ell^{c,j}_\alpha$ represent a final state lepton and an initial state anti-lepton respectively. The factor i is the usual factor for Feynman rules and the factor $(-i)$ is due to the calculation of $\mathcal{M}_{fi}$. This expression agrees with the Feynman rule of [11].

A Feynman rule to attach a W-boson to the ℓ^c line also will be needed. With the following identities [33]

$$\begin{gathered}
\ell^c = C\overline{\ell}^T \quad , \quad C = i\gamma_0\gamma_2 \quad , \; C^{-1} = C^\dagger \quad , \quad C^\dagger\gamma^{\mu T}C = -\gamma^\mu \\
\overline{\ell^c} = [C\gamma_0^T\ell^*]^\dagger\gamma_0 = \ell^T\gamma_0 C^\dagger\gamma_0 = \ell^T C^\dagger C\gamma_0 C^\dagger\gamma_0 = -\ell^T C^\dagger\gamma_0\gamma_0 = -\ell^T C^{-1} \qquad (5.47)
\end{gathered}$$

one obtains (where the (-1) is for interchanging fermions)

$$\begin{aligned}
\left[\overline{\ell_i}\tau_{ij}\not{W}P_L\ell_j\right]^T &= (-1)\Big[-\overline{\ell^c_j}C\tau^{a*}_{ji}P_L^T W^{\mu a}\gamma_\mu^T C^{-1}\ell^c\Big] \\
&= \overline{\ell^c}\tau^{a*}W^{\mu a}C\gamma^T_\mu C^{-1}P_R\ell^c \\
&= -\overline{\ell^c}\tau^{a*}\not{W}^a P_R\ell^c\,. \qquad (5.48)
\end{aligned}$$

Recall that $\tau = \tau^\dagger$, so $\tau^* = \tau^T$.

The relevant Feynman rules for dimension-four interactions are given in Figure 5.6. Note that the Feynman rules used in this calculation eliminate any dependence on the momentum of the incoming lepton, since not all momenta are independent.

Due to the presence of ε in the Feynman rules for dimension-five operators, the following SU(2) identities were useful:

$$\varepsilon_{ij}\varepsilon_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\,, \qquad (5.49)$$

$$2\varepsilon_{iI}\varepsilon_{jJ} = \delta_{ij}\delta_{IJ} - \tau^a_{ij}\tau_{a,IJ}\,, \qquad (5.50)$$

$$\varepsilon_{iJ}\varepsilon_{kJ} = \delta_{ik}\,, \qquad (5.51)$$

$$\varepsilon_{ab}\varepsilon_{cd} + \varepsilon_{bc}\varepsilon_{ad} + \varepsilon_{ac}\varepsilon_{bd} = 0\,, \qquad (5.52)$$

$$\varepsilon_{ij}\tau^a_{jk}\varepsilon_{kl} = \tau^a_{li}\,, \qquad (5.53)$$

$$\tau^a_{ij}\tau^a_{kl} = 2\delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl}\,, \qquad (5.54)$$

$$\delta_{ij}\tau^a_{kl} - \delta_{jl}\tau^a_{ki} + \delta_{kl}\tau^a_{ji} - \delta_{ik}\tau^a_{jl} = 0\,, \qquad (5.55)$$

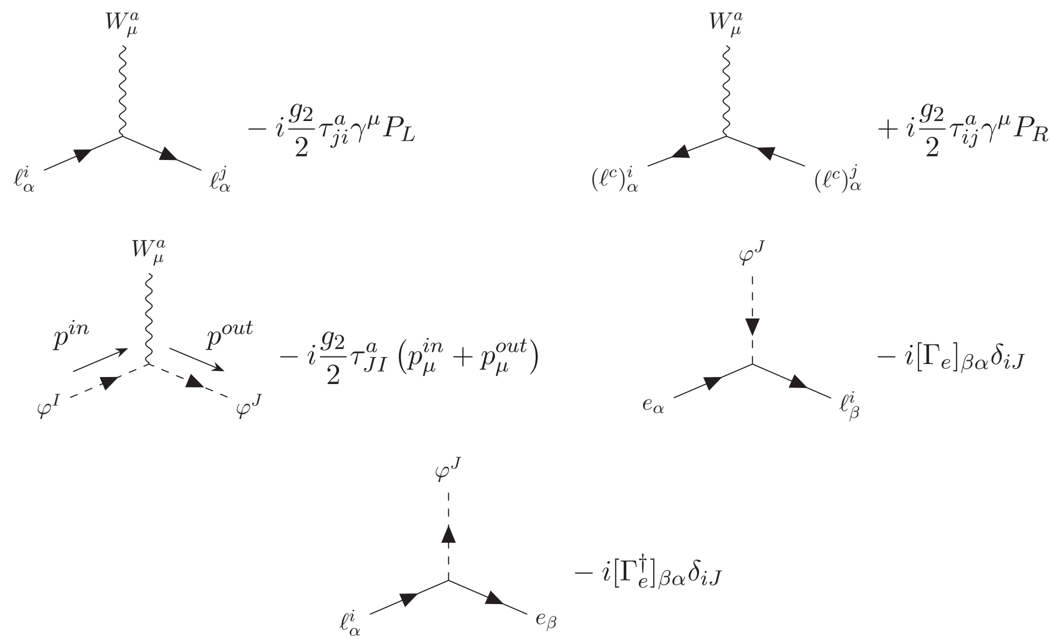


Fig. 5.6: Feynman rules for dimension-four interactions.

where

$$\varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad , \quad \vec{\tau} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

and the SU(2) generators used in the Warsaw basis are $S^a = \tau^a/2$. Note that while some of these SU(2) relations are commonly known, some are non-trivial and were found and manually checked for every combination of SU(2) indices.

5.4 Details of Loop Calculations in SMEFT

First, we discuss the calculation of the ADT within SMEFT, where there is only the Weinberg operator at dimension five, and later discuss the extension of this calculation to the 2HDM. All contributions to the ADT can be determined from the diagrams listed in Figures 5.1 and 5.2. The first operator in Figure 5.1 is renormalised by the dimension-six operators and structures $\mathcal{O}_{\varphi\ell(1)}, \mathcal{O}_{\varphi\ell(3)}, \mathcal{S}_{\varphi D\ell(1)}, \mathcal{S}_{\varphi D\ell(3)}$, which all involve a derivative, and hence have momentum dependence. Since these operators (and structures) involve covariant derivatives, there are also additional diagrams that can be used for the computation of those elements of the ADT, which involve the emission of an external B_μ or W_μ^a boson. These additional diagrams were also computed as a check of the renormalisation constants computed for the case of no boson emission. Calculation of the diagram in Figure 5.1 is performed first, and then followed by the additional consistency calculations.

Loop calculations were performed by hand, but also checked by automating the calculation using *FeynArts* [112] within `Mathematica`. This was done by writing a model file containing the interactions of SMEFT at dimensions four and five. This model file was then used to generate the diagrams and amplitudes of Figure 5.1, which were then simplified using in-house code.

5.4.1 $\varphi\ell \to \varphi\ell$

Consider the process $\varphi^M \ell^N_\alpha \to \varphi^J \ell^i_\beta$, which can be mediated through a double-insertion of the Weinberg operator (the first diagram of Figure 5.1), and by a single insertion of dimension-six operators. The renormalisation equation in $\overline{\text{MS}}$ is given by

$$
\begin{aligned}
0 = &\frac{C_5^{\gamma\delta}}{\Lambda}\frac{C_5^{\kappa\eta *}}{\Lambda}\langle \ell^i_\beta \varphi^J | \, \mathcal{O}_5^{\gamma\delta}(\mathcal{O}_5^{\eta\kappa})^\dagger \, | \ell^n_\alpha \varphi^M \rangle \, \Big|^{(1)}_{\frac{1}{\epsilon}} + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{5\bar{5},\varphi\ell(1)} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell^i_\beta \varphi^J | \, \mathcal{O}^{\rho\sigma}_{\varphi\ell(1)} \, | \ell^n_\alpha \varphi^M \rangle^{(0)} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{5\bar{5},\varphi\ell(3)} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell^i_\beta \varphi^J | \, \mathcal{O}^{\rho\sigma}_{\varphi\ell(3)} \, | \ell^n_\alpha \varphi^M \rangle^{(0)} + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma *}_{5\bar{5},\varphi\ell(1)} \frac{C_5^{\gamma\delta *} C_5^{\kappa\eta}}{\Lambda^2} \langle \ell^i_\beta \varphi^J | \, \mathcal{O}^{\sigma\rho}_{\varphi\ell(1)} \, | \ell^n_\alpha \varphi^M \rangle^{(0)} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma *}_{5\bar{5},\varphi\ell(3)} \frac{C_5^{\gamma\delta *} C_5^{\kappa\eta}}{\Lambda^2} \langle \ell^i_\beta \varphi^J | \, \mathcal{O}^{\sigma\rho}_{\varphi\ell(3)} \, | \ell^n_\alpha \varphi^M \rangle^{(0)} + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{5\bar{5},v(1)} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell^i_\beta \varphi^J | \, \mathcal{O}^{\rho\sigma}_{v(1)} \, | \ell^n_\alpha \varphi^M \rangle^{(0)} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{5\bar{5},v(3)} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell^i_\beta \varphi^J | \, \mathcal{O}^{\rho\sigma}_{v(3)} \, | \ell^n_\alpha \varphi^M \rangle^{(0)} , \qquad (5.56)
\end{aligned}
$$

where $|^{(1)}_{\frac{1}{\epsilon}}$ denotes the simple pole of a 1-loop expression, and the superscript (0) denotes the amplitude evaluated at tree level. Note that the Warsaw operators $\mathcal{O}_{\varphi\ell(1)}$ and $\mathcal{O}_{\varphi\ell(3)}$ appear with their Hermitian conjugates and consequent normalisation factors of 1/2, while the EoM-vanishing operators only appear once, in accordance with Section 5.1.1. Also note that we have used $\mathcal{O}^{\rho\sigma\dagger}_{\varphi\ell(1)} = \mathcal{O}^{\sigma\rho}_{\varphi\ell(1)}$, and a similar relation for $\mathcal{O}_{\varphi\ell(3)}$.

This expression is simplified by using the Hermiticity conditions of the renormalisation constants. If operators were not included with their Hermitian conjugate in the Lagrangian (as done in [11]), then the counterterms would arise as

$$
\widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{5\bar{5},\varphi\ell(1)} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell^i_\beta \varphi^J | \, \hat{\mathcal{O}}^{\rho\sigma}_{\varphi\ell(1)} \, | \ell^n_\alpha \varphi^M \rangle^{(0)} , \qquad (5.57)
$$

where $\mathcal{O}^{\rho\sigma}_{\varphi\ell(1)} = \frac{1}{2}\hat{\mathcal{O}}^{\rho\sigma}_{\varphi\ell(1)}$. To achieve consistency between this convention and the convention we use (illustrated in Equation (5.56)), we have the Hermiticity requirement that

$$
\widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{5\bar{5},\varphi\ell(1)} C_5^{\gamma\delta} C_5^{\kappa\eta *} \mathcal{O}^{\rho\sigma}_{\varphi\ell(1)} = \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma *}_{5\bar{5},\varphi\ell(1)} C_5^{\gamma\delta *} C_5^{\kappa\eta} \mathcal{O}^{\sigma\rho}_{\varphi\ell(1)} . \qquad (5.58)
$$

Relabelling the indices in the second term to extract a common factor of $C_5 C_5^* Q_{\varphi\ell(1)}$, the Hermiticity condition becomes

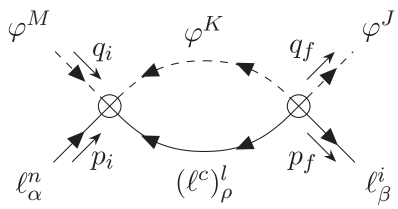
$$
\widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{5\bar{5},\varphi\ell(1)} = \widetilde{Z}^{\kappa\eta\delta\gamma,\sigma\rho *}_{5\bar{5},\varphi\ell(1)} . \qquad (5.59)
$$

Using this Hermiticity condition (plus the corresponding relation for $\mathcal{O}_{\varphi\ell(3)}$) in Equation (5.56), inserting the tree level Feynman rules for the dimension-six operators (see Figures 5.4 and 5.5), and dropping the common factor of $C_5 C_5^*/\Lambda^2$, this becomes

$$\begin{aligned}
0 = &\langle \ell^i_\beta \varphi^J |\, \mathcal{O}_5^{\gamma\delta} (\mathcal{O}_5^{\eta\kappa})^\dagger \,| \ell^n_\alpha \varphi^M \rangle \Big|^{(1)}_{\frac{1}{\epsilon}} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(1)} \overline{u_\ell}(\beta,i)(\not{q}_i + \not{q}_f) P_L u_\ell(\alpha,n) \delta_{in}\delta_{JM} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} \overline{u_\ell}(\beta,i)(\not{q}_i + \not{q}_f) P_L u_\ell(\alpha,n) \tau^a_{in}\tau^a_{JM} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(1)} \overline{u_\ell}(\beta,i)(2\not{p}_f + \not{q}_f - \not{q}_i) P_L u_\ell(\alpha,n) \delta_{in}\delta_{JM} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} \overline{u_\ell}(\beta,i)(2\not{p}_f + \not{q}_f - \not{q}_i) P_L u_\ell(\alpha,n) \tau^a_{in}\tau^a_{JM} \,. \qquad (5.60)
\end{aligned}$$

Only a single diagram needs to be evaluated to find the 1-loop contribution to the process $\varphi\ell \to \varphi\ell$ from a double-insertion of the Weinberg operator. It is found from

$$\langle \ell^i_\beta \varphi^J |\, \mathcal{O}_5^{\gamma\delta} (\mathcal{O}_5^{\eta\kappa})^\dagger \,| \ell^n_\alpha \varphi^M \rangle^{(1)} =$$



, (5.61)

of which only the $1/\epsilon$ pole is needed to renormalise the diagram. Using the Feynman rules of Figure 5.3 and simplifying factors of i and (-1), the amplitude is

$$\begin{aligned}
\langle \ell^i_\beta \varphi^J |\, \mathcal{O}_5^{\gamma\delta} (\mathcal{O}_5^{\eta\kappa})^\dagger \,| \ell^n_\alpha \varphi^M \rangle = \frac{-i}{16\pi^2} &\int \frac{\mathrm{d}^d q}{(2\pi)^4} \frac{\overline{u_\ell}(\beta,i)\not{q} P_L u_\ell(\alpha,n)}{(q^2)((q-q_f-p_f)^2)} \\
&\times (\varepsilon_{Ml}\varepsilon_{nK} + \varepsilon_{Mn}\varepsilon_{lK})(\varepsilon_{Jl}\varepsilon_{iK} + \varepsilon_{Ji}\varepsilon_{lK}) \qquad (5.62) \\
&\times \frac{1}{4}\left(\delta_{\gamma\beta}\delta_{\delta\eta}\delta_{\kappa\alpha} + \delta_{\gamma\beta}\delta_{\delta\kappa}\delta_{\eta\alpha} + \delta_{\delta\beta}\delta_{\gamma\eta}\delta_{\kappa\alpha} + \delta_{\delta\beta}\delta_{\gamma\kappa}\delta_{\eta\alpha}\right) .
\end{aligned}$$

Note that the flavour structure is fully symmetric under separate interchange of $\gamma \leftrightarrow \delta$ and $\eta \leftrightarrow \kappa$. Infrared rearrangement (see Section 3.7) is used to extract the UV divergence from the integral, and since this diagram is matched onto operators involving a derivative, the expansion is performed to first order in external momenta. Doing this, simplifying the SU(2) algebra using the relations in Section 5.3.2, and extracting the $1/\epsilon$ term gives

$$\begin{aligned}
\langle \ell^i_\beta \varphi^J |\, \mathcal{O}_5^{\gamma\delta} (\mathcal{O}_5^{\eta\kappa})^\dagger \,| \ell^n_\alpha \varphi^M \rangle \Big|^{(1)}_{\frac{1}{\epsilon}} = &\frac{(5\delta_{JM}\delta_{in} - 4\delta_{iM}\delta_{Jn})}{32\pi^2\epsilon} [\overline{u_\ell}(\beta,i)(\not{q}_f + \not{p}_f) P_L u_\ell(\alpha,n)] \\
&\times \frac{1}{4}\left(\delta_{\gamma\beta}\delta_{\delta\eta}\delta_{\kappa\alpha} + \delta_{\gamma\beta}\delta_{\delta\kappa}\delta_{\eta\alpha} + \delta_{\delta\beta}\delta_{\gamma\eta}\delta_{\kappa\alpha} + \delta_{\delta\beta}\delta_{\gamma\kappa}\delta_{\eta\alpha}\right) .
\end{aligned} \qquad (5.63)$$

Inserting this expression into Equation (5.60), and replacing the Pauli matrices from the tree-level amplitudes using

$$\tau^a_{ij}\tau^a_{kl} = 2\delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl} \,,$$

the coefficients of the momenta and SU(2) structures $\delta_{in}\delta_{JM}$ and $\delta_{iM}\delta_{Jn}$ can be used to construct a set of simultaneous equations. These are

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(1)} - \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} - \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(1)} + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} = 0 \qquad \text{from} \quad \not{q}_i\delta_{in}\delta_{JM}\,, \tag{5.64}$$

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} - \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} = 0 \qquad \text{from} \quad \not{q}_i\delta_{iM}\delta_{Jn}\,, \tag{5.65}$$

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(1)} - \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(1)} - \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} = -\frac{5\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha}}{32\pi^2\epsilon} \qquad \text{from} \quad \not{q}_f\delta_{in}\delta_{JM}\,, \tag{5.66}$$

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} = \frac{\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha}}{16\pi^2\epsilon} \qquad \text{from} \quad \not{q}_f\delta_{iM}\delta_{Jn}\,, \tag{5.67}$$

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(1)} - \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} = -\frac{5\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha}}{64\pi^2\epsilon} \qquad \text{from} \quad \not{p}_f\delta_{in}\delta_{JM}\,, \tag{5.68}$$

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} = \frac{\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha}}{32\pi^2\epsilon} \qquad \text{from} \quad \not{p}_f\delta_{iM}\delta_{Jn}\,, \tag{5.69}$$

where we use the shorthand

$$\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha} = \frac{1}{4}\left(\delta_{\gamma\beta}\delta_{\delta\eta}\delta_{\kappa\alpha} + \delta_{\gamma\beta}\delta_{\delta\kappa}\delta_{\eta\alpha} + \delta_{\delta\beta}\delta_{\gamma\eta}\delta_{\kappa\alpha} + \delta_{\delta\beta}\delta_{\gamma\kappa}\delta_{\eta\alpha}\right) .$$

These simultaneous equations can be uniquely solved to yield the solutions

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(1)} = -\frac{3}{4}\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha} \tag{5.70}$$

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} = +\frac{1}{2}\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha} \tag{5.71}$$

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(1)} = -\frac{3}{4}\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha} \tag{5.72}$$

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} = +\frac{1}{2}\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha}\,, \tag{5.73}$$

which, using the conventions set out in Equation (5.42), implies

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(1)} = -\frac{3}{4}\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha} \tag{5.74}$$

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} = +\frac{1}{2}\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha} \tag{5.75}$$

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(1)} = -\frac{3}{4}\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha} \tag{5.76}$$

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} = +\frac{1}{2}\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha}\,. \tag{5.77}$$

These renormalisation constants may be verified by considering similar processes with the associated emission of a B_μ or W^a_μ boson. We now consider both of these processes to show that they yield results that are consistent with those just presented.

5.4.2 $\varphi\ell \to \varphi\ell B$

We first consider the case of B_μ emission, since its trivial group structure makes it considerably simpler than the case of W^a_μ emission. Analogously to the $\varphi\ell \to \varphi\ell$ case, we have the renormalisation equation

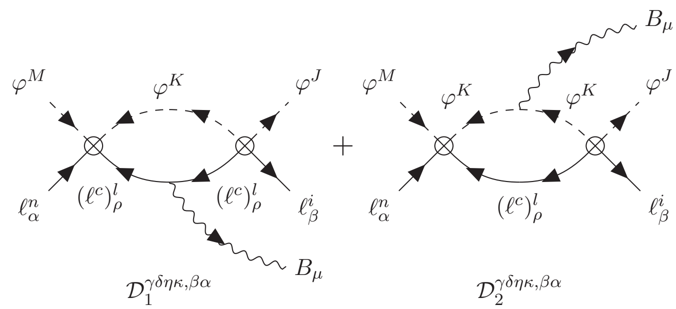
$$
\begin{aligned}
0 = {} & \frac{C_5^{\gamma\delta}}{\Lambda}\frac{C_5^{\kappa\eta *}}{\Lambda} \langle \ell_\beta^i \varphi^J B_\mu | \, \mathcal{O}_5^{\gamma\delta} (\mathcal{O}_5^{\eta\kappa})^\dagger \, | \ell_\alpha^n \varphi^M \rangle \, \big|_{\frac{1}{\epsilon}}^{(1)} \\
& + \widetilde{Z}_{5\bar{5},\varphi\ell(1)}^{\gamma\delta\eta\kappa,\rho\sigma} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell_\beta^i \varphi^J B_\mu | \, \mathcal{O}_{\varphi\ell(1)}^{\rho\sigma} \, | \ell_\alpha^n \varphi^M \rangle^{(0)} \\
& + \widetilde{Z}_{5\bar{5},\varphi\ell(3)}^{\gamma\delta\eta\kappa,\rho\sigma} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell_\beta^i \varphi^J B_\mu | \, \mathcal{O}_{\varphi\ell(3)}^{\rho\sigma} \, | \ell_\alpha^n \varphi^M \rangle^{(0)} \\
& + \widetilde{Z}_{5\bar{5},\varphi\ell(1)}^{\gamma\delta\eta\kappa,\rho\sigma *} \frac{C_5^{\gamma\delta *} C_5^{\kappa\eta}}{\Lambda^2} \langle \ell_\beta^i \varphi^J B_\mu | \, \mathcal{O}_{\varphi\ell(1)}^{\sigma\rho} \, | \ell_\alpha^n \varphi^M \rangle^{(0)} \\
& + \widetilde{Z}_{5\bar{5},\varphi\ell(3)}^{\gamma\delta\eta\kappa,\rho\sigma *} \frac{C_5^{\gamma\delta *} C_5^{\kappa\eta}}{\Lambda^2} \langle \ell_\beta^i \varphi^J B_\mu | \, \mathcal{O}_{\varphi\ell(3)}^{\sigma\rho} \, | \ell_\alpha^n \varphi^M \rangle^{(0)} \\
& + \widetilde{Z}_{5\bar{5},v(1)}^{\gamma\delta\eta\kappa,\rho\sigma} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell_\beta^i \varphi^J B_\mu | \, \mathcal{O}_{v(1)}^{\rho\sigma} \, | \ell_\alpha^n \varphi^M \rangle^{(0)} \\
& + \widetilde{Z}_{5\bar{5},v(3)}^{\gamma\delta\eta\kappa,\rho\sigma} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell_\beta^i \varphi^J B_\mu | \, \mathcal{O}_{v(3)}^{\rho\sigma} \, | \ell_\alpha^n \varphi^M \rangle^{(0)} \, . \qquad (5.78)
\end{aligned}
$$

Upon using the Hermiticity conditions of the renormalisation constants and using the Feynman rules to find the tree-level dimension-six amplitudes, this implies

$$
\begin{aligned}
0 = {} & \langle \ell_\beta^i \varphi^J B_\mu | \, \mathcal{O}_5^{\gamma\delta} (\mathcal{O}_5^{\eta\kappa})^\dagger \, | \ell_\alpha^n \varphi^M \rangle \, \big|_{\frac{1}{\epsilon}}^{(1)} \\
& + \widetilde{Z}_{5\bar{5},\varphi\ell(1)}^{\gamma\delta\eta\kappa,\beta\alpha} \overline{u_\ell}(\beta,i) \not{\epsilon}^* P_L u_\ell(\alpha,n) \delta_{in}\delta_{JM} \\
& + \widetilde{Z}_{5\bar{5},\varphi\ell(3)}^{\gamma\delta\eta\kappa,\beta\alpha} \overline{u_\ell}(\beta,i) \not{\epsilon}^* P_L u_\ell(\alpha,n) \tau_{in}^a \tau_{JM}^a \\
& + \widetilde{Z}_{5\bar{5},v(1)}^{\gamma\delta\eta\kappa,\beta\alpha} \overline{u_\ell}(\beta,i) \not{\epsilon}^* P_L u_\ell(\alpha,n) \delta_{in}\delta_{JM} \\
& + \widetilde{Z}_{5\bar{5},v(3)}^{\gamma\delta\eta\kappa,\beta\alpha} \overline{u_\ell}(\beta,i) \not{\epsilon}^* P_L u_\ell(\alpha,n) \tau_{in}^a \tau_{JM}^a \, . \qquad (5.79)
\end{aligned}
$$

In this equation, ϵ_μ^* is the polarisation vector for the external final state B_μ boson. To evaluate the amplitude from a double-insertion of the Weinberg operator, there are two diagrams that need to be considered:

$$\langle \ell_\beta^i \varphi^J B_\mu | \, \mathcal{O}_5^{\gamma\delta} (\mathcal{O}_5^{\eta\kappa})^\dagger \, | \ell_\alpha^n \varphi^M \rangle^{(1)} =$$



.

The divergences of these diagrams are extracted using infrared rearrangement, but since the external fields are of dimension-six, the expansion is performed to zeroth order in external momenta. Consequently, all external momenta can be set to zero from the outset. Extracting the divergences from each of these diagrams then yields

$$\mathcal{D}_1^{\gamma\delta\eta\kappa,\beta\alpha}\big|_{\frac{1}{\epsilon}} = -\frac{g_1}{64\pi^2\epsilon}(5\delta_{JM}\delta_{in} - 4\delta_{iM}\delta_{Jn})[\overline{u_\ell}(\beta,i)\not{\epsilon}^* P_L u_\ell(\alpha,n)]\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha} \, , \qquad (5.80)$$

$$\mathcal{D}_2^{\gamma\delta\eta\kappa,\beta\alpha}\big|_{\frac{1}{\epsilon}} = +\frac{g_1}{64\pi^2\epsilon}(5\delta_{JM}\delta_{in} - 4\delta_{iM}\delta_{Jn})[\overline{u_\ell}(\beta,i)\not{\epsilon}^* P_L u_\ell(\alpha,n)]\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha}\,, \tag{5.81}$$

where the relative minus sign arises from the opposite hypercharge of the lepton doublet and the Higgs doublet. These two diagrams trivially sum to give

$$\langle \ell^i_\beta \varphi^J B_\mu |\, \mathcal{O}_5^{\gamma\delta} (\mathcal{O}_5^{\eta\kappa})^\dagger \,| \ell^n_\alpha \varphi^M \rangle\, \big|^{(1)}_{\frac{1}{\epsilon}} = 0\,. \tag{5.82}$$

Inserting this result into Equation (5.79) and reducing all SU(2) structures to products of Kronecker deltas, extraction of the coefficients of $\delta_{in}\delta_{JM}$ and $\delta_{iM}\delta_{Jn}$ yields the two simultaneous equations

$$-\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(1)} + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(1)} - \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} = 0 \qquad \text{from} \quad \delta_{in}\delta_{JM}\,, \tag{5.83}$$

$$-\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} = 0 \qquad \text{from} \quad \delta_{iM}\delta_{Jn}\,. \tag{5.84}$$

These are only two equations to solve for four unknowns, and so by themselves are insufficient to yield a unique set of solutions. It is possible to substitute in the values of two renormalisation constants from the previous section to reduce the number of unknowns, and then solve for the remaining two "unknown" renormalisation constants. This has been done and verifies the solutions found above. However, it is sufficient to note that Equation (5.83) and Equation (5.84) are identical to Equation (5.64) and Equation (5.65) respectively. Consequently, the process with associated B_μ emission is automatically renormalised by the renormalisation constants in Equations (5.74 - 5.77).

5.4.3 $\varphi\ell \to \varphi\ell W$

A further check can be performed in which there is the emission of a W^a_μ boson. The renormalisation equation is

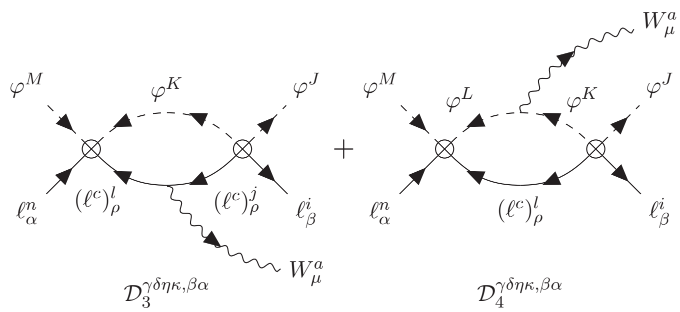
$$\begin{aligned}
0 = {} & \frac{C_5^{\gamma\delta}}{\Lambda}\frac{C_5^{\kappa\eta *}}{\Lambda} \langle \ell^i_\beta \varphi^J W^a_\mu |\, \mathcal{O}_5^{\gamma\delta} (\mathcal{O}_5^{\eta\kappa})^\dagger \,| \ell^n_\alpha \varphi^M \rangle\, \big|^{(1)}_{\frac{1}{\epsilon}} \\
& + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{5\bar{5},\varphi\ell(1)} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell^i_\beta \varphi^J W^a_\mu |\, \mathcal{O}^{\rho\sigma}_{\varphi\ell(1)} \,| \ell^n_\alpha \varphi^M \rangle^{(0)} \\
& + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{5\bar{5},\varphi\ell(3)} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell^i_\beta \varphi^J W^a_\mu |\, \mathcal{O}^{\rho\sigma}_{\varphi\ell(3)} \,| \ell^n_\alpha \varphi^M \rangle^{(0)} \\
& + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma *}_{5\bar{5},\varphi\ell(1)} \frac{C_5^{\gamma\delta *} C_5^{\kappa\eta}}{\Lambda^2} \langle \ell^i_\beta \varphi^J W^a_\mu |\, \mathcal{O}^{\sigma\rho}_{\varphi\ell(1)} \,| \ell^n_\alpha \varphi^M \rangle^{(0)} \\
& + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma *}_{5\bar{5},\varphi\ell(3)} \frac{C_5^{\gamma\delta *} C_5^{\kappa\eta}}{\Lambda^2} \langle \ell^i_\beta \varphi^J W^a_\mu |\, \mathcal{O}^{\sigma\rho}_{\varphi\ell(3)} \,| \ell^n_\alpha \varphi^M \rangle^{(0)} \\
& + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{5\bar{5},v(1)} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell^i_\beta \varphi^J W^a_\mu |\, \mathcal{O}^{\rho\sigma}_{v(1)} \,| \ell^n_\alpha \varphi^M \rangle^{(0)} \\
& + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{5\bar{5},v(3)} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell^i_\beta \varphi^J W^a_\mu |\, \mathcal{O}^{\rho\sigma}_{v(3)} \,| \ell^n_\alpha \varphi^M \rangle^{(0)}\,,
\end{aligned} \tag{5.85}$$

which upon using the Hermiticity conditions of the renormalisation constants and using the Feynman rules to find the tree-level dimension-six amplitudes implies

$$\begin{aligned}
0 = {} & \langle \ell^i_\beta \varphi^J W^a_\mu | \, \mathcal{O}^{\gamma\delta}_5 (\mathcal{O}^{\eta\kappa}_5)^\dagger \, | \ell^n_\alpha \varphi^M \rangle \, \big|^{(1)}_{\frac{1}{\epsilon}} \\
& + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\overline{5},\varphi\ell(1)} \, (-g_2) \, [\overline{u_\ell}(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n)] \, \tau^a_{JM} \delta_{in} \\
& + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\overline{5},H\ell(3)} \left(-\frac{g_2}{2}\right) [\overline{u_\ell}(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n)] \left(\tau^b_{JL}\tau^a_{LM} + \tau^a_{JL}\tau^b_{LM}\right) \tau^b_{in} \\
& + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\overline{5},v(1)} \, (-g_2) \, [\overline{u_\ell}(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n)] \, \delta_{JM} \tau^a_{in} \\
& + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\overline{5},v(3)} \left(-\frac{g_2}{2}\right) [\overline{u_\ell}(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n)] \left(\tau^b_{ij}\tau^a_{jn} + \tau^a_{ij}\tau^b_{jn}\right) \tau^b_{JM}. \qquad (5.86)
\end{aligned}$$

Again, there are two diagrams involving a double-insertion, and so

$$\langle \ell^i_\beta \varphi^J W^a_\mu | \, \mathcal{O}^{\gamma\delta}_5 (\mathcal{O}^{\eta\kappa}_5)^\dagger \, | \ell^n_\alpha \varphi^M \rangle^{(1)} =$$



Setting external momenta to zero and extracting the UV divergences using infrared rearrangement, we find

$$\mathcal{D}^{\gamma\delta\eta\kappa,\beta\alpha}_3 \big|_{\frac{1}{\epsilon}} = \frac{1}{\epsilon} \frac{g_2}{32\pi^2} \, [\overline{u_\ell}(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n)] \, \Delta^{\gamma\delta\eta\kappa,\beta\alpha}_1 S_\ell, \qquad (5.87)$$

$$\mathcal{D}^{\gamma\delta\eta\kappa,\beta\alpha}_4 \big|_{\frac{1}{\epsilon}} = \frac{1}{\epsilon} \frac{g_2}{32\pi^2} \, [\overline{u_\ell}(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n)] \, \Delta^{\gamma\delta\eta\kappa,\beta\alpha}_1 S_\varphi, \qquad (5.88)$$

where

$$S_\ell = (\varepsilon_{nK}\varepsilon_{Ml} + \varepsilon_{Mn}\varepsilon_{lK}) \, (\varepsilon_{iK}\varepsilon_{Jj} + \varepsilon_{Ji}\varepsilon_{jK}) \, \frac{\tau^a_{lj}}{2}, \qquad (5.89)$$

$$S_\varphi = (\varepsilon_{nL}\varepsilon_{Ml} + \varepsilon_{nM}\varepsilon_{L}) \, (\varepsilon_{iK}\varepsilon_{Jl} + \varepsilon_{iJ}\varepsilon_{Kl}) \, \frac{\tau^a_{LK}}{2}. \qquad (5.90)$$

Applying the SU(2) relations given in Section 5.3.2 to S_ℓ and S_φ yields

$$\begin{aligned}
S_\ell &= \frac{1}{2} \left(\delta_{iJ}\tau^a_{nM} + \delta_{MJ}\tau^a_{ni} - 2\delta_{Mi}\tau^a_{nJ} - 2\delta_{iJ}\tau^a_{Mn} - 2\delta_{nJ}\tau^a_{Mi} + 4\delta_{ni}\tau^a_{MJ} - 2\delta_{nM}\tau^a_{iJ} + \delta_{nM}\tau^a_{Ji}\right) \\
&= \frac{1}{2} \left(2\delta_{JM}\tau^a_{in} - \delta_{Jn}\tau^a_{iM} - \delta_{iM}\tau^a_{Jn} - \delta_{in}\tau^a_{JM}\right), \qquad (5.91)
\end{aligned}$$

$$\begin{aligned}
S_\varphi &= \frac{1}{2} \left(\delta_{iJ}\tau^a_{nM} + \delta_{MJ}\tau^a_{ni} - 2\delta_{Mi}\tau^a_{nJ} - 2\delta_{iJ}\tau^a_{Mn} - 2\delta_{nJ}\tau^a_{Mi} + 4\delta_{ni}\tau^a_{MJ} - 2\delta_{nM}\tau^a_{iJ} + \delta_{nM}\tau^a_{Ji}\right) \\
&= \frac{1}{2} \left(\delta_{iM}\tau^a_{Jn} + \delta_{Jn}\tau^a_{iM} - 3\delta_{JM}\tau^a_{in}\right), \qquad (5.92)
\end{aligned}$$

where the second equality in each of the above relations has been checked explicitly. The total amplitude of a double-insertion of dimension-five operators is then

$$\langle \ell^i_\beta \varphi^J W^a_\mu | \, \mathcal{O}^{\gamma\delta}_5 (\mathcal{O}^{\eta\kappa}_5)^\dagger \, | \ell^n_\alpha \varphi^M \rangle \Big|^{(1)}_{\frac{1}{\epsilon}} = \frac{1}{\epsilon} \frac{g_2}{64\pi^2} \left[\overline{u_\ell}(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n) \right] \times \Delta^{\gamma\delta\eta\kappa,\beta\alpha}_1 \left(-\delta_{JM} \tau^a_{in} - \delta_{in} \tau^a_{JM} \right). \tag{5.93}$$

Using the given SU(2) relations allows the tree-level amplitudes to be written as

$$\begin{aligned}
&\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(1)} (-g_2) \left[\overline{u_\ell}(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n) \right] \tau^a_{JM} \delta_{in} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} \left(-\frac{g_2}{2} \right) \left[\overline{u_\ell}(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n) \right] \left(2 \left[\delta_{Jn} \tau^a_{iM} - \delta_{in} \tau^a_{JM} + \delta_{iM} \tau^a_{Jn} \right] \right) \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(1)} (-g_2) \left[\overline{u_\ell}(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n) \right] \delta_{JM} \tau^a_{in} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} \left(-\frac{g_2}{2} \right) \left[\overline{u_\ell}(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n) \right] \left(2 \left[\delta_{Jn} \tau^a_{iM} - \delta_{JM} \tau^a_{in} + \delta_{iM} \tau^a_{Jn} \right] \right).
\end{aligned} \tag{5.94}$$

Comparing the SU(2) structures arising in the loop with the tree amplitudes, the tree amplitudes seemingly contain extra SU(2) structures. However, these can be eliminated using the relation

$$\delta_{ij} \tau^a_{km} - \delta_{jm} \tau^a_{ki} + \delta_{km} \tau^a_{ji} - \delta_{ik} \tau^a_{jm} = 0, \tag{5.95}$$

and the tree-level amplitudes become

$$\begin{aligned}
&\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(1)} (-g_2) \left[\bar{u}_\ell(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n) \right] \tau^a_{JM} \delta_{in} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} (-g_2) \left[\bar{u}_\ell(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n) \right] \delta_{JM} \tau^a_{in} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(1)} (-g_2) \left[\bar{u}_\ell(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n) \right] \delta_{JM} \tau^a_{in} \\
&+ \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} (-g_2) \left[\bar{u}_\ell(\beta, i) \not{\epsilon}^* P_L u_\ell(\alpha, n) \right] \tau^a_{JM} \delta_{in}.
\end{aligned} \tag{5.96}$$

In this form, it is simple to set up simultaneous equations for the renormalisation condition by comparing the loop and tree amplitudes,

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(1)} + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)} = -\frac{1}{\epsilon} \frac{1}{64\pi^2} \Delta^{\gamma\delta\eta\kappa,\beta\alpha}_1 \qquad \text{from} \quad \delta_{in} \tau^a_{JM}\,, \tag{5.97}$$

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(1)} + \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} = -\frac{1}{\epsilon} \frac{1}{64\pi^2} \Delta^{\gamma\delta\eta\kappa,\beta\alpha}_1 \qquad \text{from} \quad \delta_{JM} \tau^a_{in}. \tag{5.98}$$

This set of equations may be constrained by substituting in solutions for $\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(1)}$ and $\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},v(3)}$ from the momentum-dependent calculation, to verify the solutions

$$\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(1)} = -\frac{3}{4} \frac{1}{16\pi^2} \frac{1}{\epsilon} \Delta^{\gamma\delta\eta\kappa,\beta\alpha}_1\,, \qquad \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},\varphi\ell(3)} = +\frac{1}{2} \frac{1}{16\pi^2} \frac{1}{\epsilon} \Delta^{\gamma\delta\eta\kappa,\beta\alpha}_1\,. \tag{5.99}$$

Therefore, both the processes involving the emission of a B_μ boson or a W^a_μ boson verify the solutions found when considering the momentum-dependent diagram.

5.4.4 $e\varphi\varphi \to \ell\varphi$

Next, we consider the renormalisation constant $\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},e\varphi}$. This receives contributions from $\mathcal{O}_{e\varphi}$ and the EoM-vanishing operators $\mathcal{O}_{v(1)}$ and $\mathcal{O}_{v(3)}$, since they contain Yukawa interactions.

The renormalisation equation is

$$
\begin{aligned}
0 = {} & \frac{C_5^{\gamma\delta}}{\Lambda}\frac{C_5^{\kappa\eta *}}{\Lambda}\,\langle \ell_\beta^n \varphi^I |\, \mathcal{O}_5^{\gamma\delta}(\mathcal{O}_5^{\eta\kappa})^\dagger(\overline{\ell_\chi}[\Gamma_e]_{\chi\zeta} e_\zeta \varphi)\,| e_\alpha \varphi^J \varphi^K \rangle \,\Big|_{\frac{1}{\epsilon}}^{(1)} \\
& + \widetilde{Z}_{5\bar{5},e\varphi}^{\gamma\delta\eta\kappa,\rho\sigma}\frac{C_5^{\gamma\delta}C_5^{\kappa\eta *}}{\Lambda^2}\,\langle \ell_\beta^n \varphi^I |\, \mathcal{O}_{e\varphi}^{\rho\sigma}\,| e_\alpha \varphi^J \varphi^K \rangle^{(0)} \\
& + \widetilde{Z}_{5\bar{5},v(1)}^{\gamma\delta\eta\kappa,\rho\sigma}\frac{C_5^{\gamma\delta}C_5^{\kappa\eta *}}{\Lambda^2}\,\langle \ell_\beta^n \varphi^I |\, \mathcal{O}_{v(1)}^{\rho\sigma}\,| e_\alpha \varphi^J \varphi^K \rangle^{(0)} \\
& + \widetilde{Z}_{5\bar{5},v(3)}^{\gamma\delta\eta\kappa,\rho\sigma}\frac{C_5^{\gamma\delta}C_5^{\kappa\eta *}}{\Lambda^2}\,\langle \ell_\beta^n \varphi^I |\, \mathcal{O}_{v(3)}^{\rho\sigma}\,| e_\alpha \varphi^J \varphi^K \rangle^{(0)}\,,
\end{aligned}
\tag{5.100}
$$

which implies (after inserting tree-level amplitudes of dimension-six operators)

$$
\begin{aligned}
0 = {} & \langle \ell_\beta^n \varphi^I |\, \mathcal{O}_5^{\gamma\delta}(\mathcal{O}_5^{\eta\kappa})^\dagger(\overline{\ell_\chi}[\Gamma_e]_{\chi\zeta} e_\zeta \varphi)\,| e_\alpha \varphi^J \varphi^K \rangle \,\Big|_{\frac{1}{\epsilon}}^{(1)} \\
& + \widetilde{Z}_{5\bar{5},e\varphi}^{\gamma\delta\eta\kappa,\beta\alpha}\overline{u_\ell}(\beta,n) P_R u_e(\alpha)(\delta_{IJ}\delta_{Kn} + \delta_{IK}\delta_{Jn}) \\
& - \widetilde{Z}_{5\bar{5},v(1)}^{\gamma\delta\eta\kappa,\beta\sigma}[\Gamma_e]_{\sigma\alpha}\overline{u_\ell}(\beta,n) P_R u_e(\alpha)(\delta_{IJ}\delta_{Kn} + \delta_{IK}\delta_{Jn}) \\
& - \widetilde{Z}_{5\bar{5},v(3)}^{\gamma\delta\eta\kappa,\beta\sigma}[\Gamma_e]_{\sigma\alpha}\overline{u_\ell}(\beta,n) P_R u_e(\alpha)(\delta_{IJ}\delta_{Kn} + \delta_{IK}\delta_{Jn})\,.
\end{aligned}
\tag{5.101}
$$

Since $\widetilde{Z}_{5\bar{5},v(1)}^{\gamma\delta\eta\kappa,\beta\sigma}$ and $\widetilde{Z}_{5\bar{5},v(3)}^{\gamma\delta\eta\kappa,\beta\sigma}$ have been calculated previously, we find

$$
\begin{aligned}
\widetilde{Z}_{5\bar{5},v(1)}^{\gamma\delta\eta\kappa,\beta\sigma}[\Gamma_e]_{\sigma\alpha} &= -\frac{3}{4}\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_1^{\gamma\delta\eta\kappa,\beta\sigma}[\Gamma_e]_{\sigma\alpha} \\
&= -\frac{3}{4}\frac{1}{16\pi^2}\frac{1}{\epsilon}\frac{1}{4}\left(\delta_{\gamma\beta}\delta_{\delta\eta}\delta_{\kappa\sigma} + \delta_{\gamma\beta}\delta_{\delta\kappa}\delta_{\eta\sigma} + \delta_{\delta\beta}\delta_{\gamma\eta}\delta_{\kappa\sigma} + \delta_{\delta\beta}\delta_{\gamma\kappa}\delta_{\eta\sigma}\right)[\Gamma_e]_{\sigma\alpha} \\
&= -\frac{3}{4}\frac{1}{16\pi^2}\frac{1}{\epsilon}\frac{1}{4}\big(\delta_{\gamma\beta}\delta_{\delta\eta}[\Gamma_e]_{\kappa\alpha} + \delta_{\gamma\beta}\delta_{\delta\kappa}[\Gamma_e]_{\eta\alpha} \\
&\qquad\qquad + \delta_{\delta\beta}\delta_{\gamma\eta}[\Gamma_e]_{\kappa\alpha} + \delta_{\delta\beta}\delta_{\gamma\kappa}[\Gamma_e]_{\eta\alpha}\big)\,,
\end{aligned}
\tag{5.102}
$$

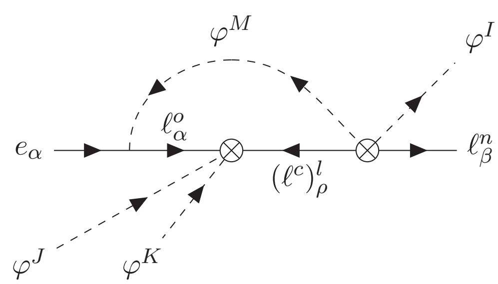
and similarly,

$$
\begin{aligned}
\widetilde{Z}_{5\bar{5},v(3)}^{\gamma\delta\eta\kappa,\beta\sigma}[\Gamma_e]_{\sigma\alpha} &= \frac{1}{2}\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_1^{\gamma\delta\eta\kappa,\beta\sigma}[\Gamma_e]_{\sigma\alpha} \\
&= \frac{1}{2}\frac{1}{16\pi^2}\frac{1}{\epsilon}\frac{1}{4}\big(\delta_{\gamma\beta}\delta_{\delta\eta}[\Gamma_e]_{\kappa\alpha} + \delta_{\gamma\beta}\delta_{\delta\kappa}[\Gamma_e]_{\eta\alpha} \\
&\qquad\qquad + \delta_{\delta\beta}\delta_{\gamma\eta}[\Gamma_e]_{\kappa\alpha} + \delta_{\delta\beta}\delta_{\gamma\kappa}[\Gamma_e]_{\eta\alpha}\big)\,.
\end{aligned}
\tag{5.103}
$$

Note that, as expected, these flavour structures are symmetric under $\gamma \leftrightarrow \delta$ and $\eta \leftrightarrow \kappa$ individually.

Now it is necessary to calculate the loop diagram, where

$$\langle \ell_\beta^n \varphi^I |\, \mathcal{O}_5^{\gamma\delta}(\mathcal{O}_5^{\eta\kappa})^\dagger(\overline{\ell_\chi}[\Gamma_e]_{\chi\zeta} e_\zeta \varphi)\,| e_\alpha \varphi^J \varphi^K \rangle^{(1)} =$$



.

Using the relevant Feynman rules, the pole part of the amplitude is given by

$$
\begin{aligned}
\langle \ell^n_\beta \varphi^I |\, \mathcal{O}^{\gamma\delta}_5 (\mathcal{O}^{\eta\kappa}_5)^\dagger (\overline{\ell_\chi} [\Gamma_e]_{\chi\zeta} e_\zeta \varphi)\, | e_\alpha \varphi^J \varphi^K \rangle \, \Big|^{(1)}_{\frac{1}{\epsilon}} = & \\
& - \frac{1}{16\pi^2} \frac{1}{\epsilon} \overline{u_\ell}(\beta, n) P_R u_e(\alpha) (\delta_{IJ}\delta_{Kn} + \delta_{Jn}\delta_{IK}) \\
& \times \frac{1}{4} \left(\delta_{\gamma\beta}\delta_{\delta\eta}[\Gamma_e]_{\kappa\alpha} + \delta_{\gamma\beta}\delta_{\delta\kappa}[\Gamma_e]_{\eta\alpha} + \delta_{\delta\beta}\delta_{\gamma\eta}[\Gamma_e]_{\kappa\alpha} + \delta_{\delta\beta}\delta_{\gamma\kappa}[\Gamma_e]_{\eta\alpha} \right) .
\end{aligned}
\tag{5.104}
$$

This is the same flavour structure that appears in the tree-level diagrams involving $\mathcal{O}_{v(1)}$ and $\mathcal{O}_{v(3)}$, and we therefore define

$$
\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha} = \frac{1}{4} \left(\delta_{\gamma\beta}\delta_{\delta\eta}[\Gamma_e]_{\kappa\alpha} + \delta_{\gamma\beta}\delta_{\delta\kappa}[\Gamma_e]_{\eta\alpha} + \delta_{\delta\beta}\delta_{\gamma\eta}[\Gamma_e]_{\kappa\alpha} + \delta_{\delta\beta}\delta_{\gamma\kappa}[\Gamma_e]_{\eta\alpha} \right) . \tag{5.105}
$$

With this definition, and inserting the result of Equation (5.104) along with the above expressions for $Z^{\gamma\delta\eta\kappa,\beta\sigma}_{5\bar{5},v(1)}[\Gamma_e]_{\sigma\alpha}$ and $Z^{\gamma\delta\eta\kappa,\beta\sigma}_{5\bar{5},v(3)}[\Gamma_e]_{\sigma\alpha}$ into Equation (5.101), we obtain

$$
-\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha} + Z^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},e\varphi} + \frac{3}{4}\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha} - \frac{1}{2}\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha} = 0 \,, \tag{5.106}
$$

which is solved to give

$$
Z^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},e\varphi} = +\frac{3}{4}\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha} \,, \tag{5.107}
$$

or equivalently,

$$
\delta Z^{\gamma\delta\eta\kappa,\beta\alpha}_{5\bar{5},e\varphi} = +\frac{3}{4}\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha} \,. \tag{5.108}
$$

5.4.5 $\ell\ell \to \ell\ell$

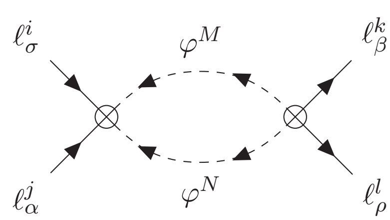
The final lepton-flavour violating process to consider is $\ell_\alpha \ell_\sigma \to \ell_\beta \ell_\rho$. The renormalisation equation for this process is

$$
\begin{aligned}
0 = & \frac{C_5^{\gamma\delta}}{\Lambda}\frac{C_5^{\kappa\eta *}}{\Lambda} \langle \ell^k_\beta \ell^l_\rho |\, \mathcal{O}^{\gamma\delta}_5 (\mathcal{O}^{\eta\kappa}_5)^\dagger \, | \ell^j_\alpha \ell^i_\sigma \rangle \, \Big|^{(1)}_{\frac{1}{\epsilon}} \\
& + 2 Z^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi}_{5\bar{5},\ell\ell} \frac{C_5^{\gamma\delta} C_5^{\kappa\eta *}}{\Lambda^2} \langle \ell^k_\beta \ell^l_\rho |\, \mathcal{O}^{\tau\upsilon\varphi\chi}_{\ell\ell} \, | \ell^j_\alpha \ell^i_\sigma \rangle^{(0)} \,,
\end{aligned}
\tag{5.109}
$$

and hence there is only a single dimension-six operator that is necessary to renormalise the loop diagram. Note that φ here is used as a flavour index, as opposed to referring to a Higgs doublet, and the factor of 2 in front of $Z_{5\bar{5},\ell\ell}$ is due to including the Hermitian conjugate of $\mathcal{O}_{\ell\ell}$. For this process, it is helpful to evaluate the loop diagram before considering the tree-level dimension-six amplitudes, as will be explained.

The loop amplitude is given by considering the diagram

$$
\langle \ell^k_\beta \ell^l_\rho |\, \mathcal{O}^{\gamma\delta}_5 (\mathcal{O}^{\eta\kappa}_5)^\dagger \, | \ell^j_\alpha \ell^i_\sigma \rangle =
$$



. (5.110)

We find that this diagram is equal to

$$\langle \ell^k_\beta \ell^l_\rho | \, \mathcal{O}^{\gamma\delta}_5 (\mathcal{O}^{\eta\kappa}_5)^\dagger \, | \ell^j_\alpha \ell^i_\sigma \rangle \, \Big|^{(1)}_{\frac{1}{\epsilon}} = \frac{(\overline{u_\ell}(\sigma,i) P_L v_\ell(\alpha,j))(\overline{v_\ell}(\beta,k) P_R u_\ell(\rho,l))}{64\pi^2\epsilon} \times (\delta_{\sigma\kappa}\delta_{\alpha\eta} + \delta_{\alpha\kappa}\delta_{\sigma\eta}) (\delta_{\rho\gamma}\delta_{\beta\delta} + \delta_{\beta\gamma}\delta_{\rho\delta}) (\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) \, , \tag{5.111}$$

which is completely symmetric under $\gamma \leftrightarrow \delta$ and $\eta \leftrightarrow \kappa$ (due to the symmetry of the Weinberg operator), and also under $\alpha \leftrightarrow \sigma$ and $\beta \leftrightarrow \rho$ (from the symmetry of the ingoing and outgoing states, respectively). We define this flavour structure as

$$\Delta_3^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma} = \frac{1}{4} (\delta_{\sigma\kappa}\delta_{\alpha\eta} + \delta_{\alpha\kappa}\delta_{\sigma\eta})(\delta_{\rho\gamma}\delta_{\beta\delta} + \delta_{\beta\gamma}\delta_{\rho\delta}) \, , \tag{5.112}$$

and therefore,

$$\langle \ell^k_\beta \ell^l_\rho | \, \mathcal{O}^{\gamma\delta}_5 (\mathcal{O}^{\eta\kappa}_5)^\dagger \, | \ell^j_\alpha \ell^i_\sigma \rangle \, \Big|^{(1)}_{\frac{1}{\epsilon}} = \frac{[\overline{u_\ell}(\sigma,i) P_L v_\ell(\alpha,j)][\overline{v_\ell}(\beta,k) P_R u_\ell(\rho,l)]}{16\pi^2\epsilon} \Delta_3^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma} \times (\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) \, . \tag{5.113}$$

It can be seen from Equation (5.113) that the loop diagram generates the spinors $\overline{u_\ell}, v_\ell, \overline{v_\ell}$ and u_ℓ, and has a left-right chirality structure. However, the amplitude $\langle \ell^k_\beta \ell^l_\rho | \, \mathcal{O}^{\tau\upsilon\varphi\chi}_{\ell\ell} \, | \ell^j_\alpha \ell^i_\sigma \rangle^{(0)}$ only generates the spinors $\overline{u_\ell}$ and u_ℓ, as can be seen from the Feynman rules listed in [33], and only generates a left-left chirality structure. Consequently, it is easier to find the renormalisation constant $\widetilde{Z}^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi}_{55,\ell\ell}$ by considering a different, but related, operator.

We introduce the operator

$$\mathcal{O}^{\tau\upsilon\varphi\chi}_{\text{eva}} = \delta_{ij}\delta_{kl} (\overline{\ell^i_\tau} \ell^{c,k}_\varphi)(\overline{\ell^{c,l}_\chi} \ell^j_\upsilon) - \mathcal{O}^{\tau\upsilon\varphi\chi}_{\ell\ell} \, , \tag{5.114}$$

where the first (scalar) term has a left-right chirality structure and i, j, k, l are SU(2) indices. This is an evanescent operator, since it vanishes in the limit $d \to 4$ due to Fierz relations and the properties of charge conjugation. Explicitly, using the symmetry $\mathcal{O}^{\tau\upsilon\varphi\chi}_{\ell\ell} = \mathcal{O}^{\varphi\chi\tau\upsilon}_{\ell\ell}$ and properties of charge conjugation,

$$\begin{aligned} \mathcal{O}^{\tau\upsilon\varphi\chi}_{\ell\ell} = \mathcal{O}^{\varphi\chi\tau\upsilon}_{\ell\ell} &\equiv +\frac{1}{2} (\bar{\ell}_\varphi \gamma^\mu \ell_\chi)(\overline{\ell_\tau} \gamma_\mu \ell_\upsilon) \\ &= -\frac{1}{2} ([\ell^c_\varphi]^T C^{-1} \gamma^\mu C \overline{\ell^c_\chi}^T)(\overline{\ell_\tau} \gamma_\mu \ell_\upsilon) \\ &= +\frac{1}{2} ([\ell^c_\varphi]^T (\gamma^\mu)^T \overline{\ell^c_\chi}^T)(\overline{\ell_\tau} \gamma_\mu \ell_\upsilon) \\ &= -\frac{1}{2} (\overline{\ell^c_\chi} \gamma^\mu \ell^c_\varphi)(\overline{\ell_\tau} \gamma_\mu \ell_\upsilon) \, , \end{aligned} \tag{5.115}$$

where the final minus sign comes from the anti-commutation of fermions when performing the transpose. We now use the Fierz relation in $d = 4$ dimensions (where the minus sign is for anticommuting fermions, see [113]),

$$(\overline{A} \gamma^\mu P_L B)(\overline{C} \gamma_\mu P_R D) = -2 (\overline{A} P_R D)(\overline{C} P_L B) \, , \tag{5.116}$$

to write

$$-\frac{1}{2}(\overline{\ell^{c,i}_\chi}\gamma^\mu\ell^{c,i}_\varphi)(\overline{\ell^k_\tau}\gamma_\mu\ell^k_\upsilon) = (\overline{\ell^i_\tau}\ell^{c,k}_\varphi)(\overline{\ell^{c,k}_\chi}\ell^i_\upsilon), \tag{5.117}$$

where spin indices are contracted within brackets, and SU(2) indices have been explicitly stated. Putting everything together, we have

$$\mathcal{O}^{\tau\upsilon\varphi\chi}_{\ell\ell} = (\overline{\ell^i_\tau}\ell^{c,k}_\varphi)(\overline{\ell^{c,k}_\chi}\ell^i_\upsilon)\,. \tag{5.118}$$

Therefore, the operator

$$\mathcal{O}^{\tau\upsilon\varphi\chi}_{\text{eva}} = \delta_{ij}\delta_{kl}(\overline{\ell^i_\tau}\ell^{c,k}_\varphi)(\overline{\ell^{c,l}_\chi}\ell^j_\upsilon) - \mathcal{O}^{\tau\upsilon\varphi\chi}_{\ell\ell} \tag{5.119}$$

is vanishing in four dimensions, and is thus an evanescent operator. This operator is useful since when it mediates the process $\ell\ell \to \ell\ell$, it generates a tree-level amplitude that has a left-right chirality structure, and so can be directly mapped onto the loop amplitude generated by the double-insertion of the Weinberg operator. This means the scalar part of $\mathcal{O}_{\text{eva}}$ can be used to renormalise the loop diagram, from which the renormalisation constant $Z_{5\bar{5},\ell\ell}$ can be extracted.

Following this strategy, we calculate

$$\begin{aligned}\langle \ell^k_\beta \ell^l_\rho|\, \mathcal{O}^{\tau\upsilon\varphi\chi}_{\text{eva,scalar}}\,|\ell^j_\alpha\ell^i_\sigma\rangle^{(0)} &= (\overline{u_\ell}(\sigma,i)P_L v_\ell(\alpha,j))(\overline{v_\ell}(\beta,k)P_R u_\ell(\rho,l)) \\ &\quad\times [\delta_{ik}\delta_{jl}(\delta_{\alpha\upsilon}\delta_{\sigma\chi}\delta_{\rho\tau}\delta_{\beta\varphi} + \delta_{\sigma\upsilon}\delta_{\alpha\chi}\delta_{\beta\tau}\delta_{\rho\varphi}) \\ &\quad\;\; + \delta_{il}\delta_{jk}(\delta_{\sigma\upsilon}\delta_{\alpha\chi}\delta_{\rho\tau}\delta_{\beta\varphi} + \delta_{\alpha\upsilon}\delta_{\sigma\chi}\delta_{\beta\tau}\delta_{\rho\varphi})]\,.\end{aligned} \tag{5.120}$$

Note that since $\mathcal{O}_{\text{eva}}$ is not a physical operator of the Warsaw basis, it is not required to also consider its Hermitian conjugate, and so there is no factor of 1/2 in the normalisation. Consider the modified renormalisation equation:

$$\begin{aligned}0 &= \frac{C_5^{\gamma\delta}}{\Lambda}\frac{C_5^{\kappa\eta*}}{\Lambda}\,\langle \ell^k_\beta\ell^l_\rho|\,\mathcal{O}_5^{\gamma\delta}(\mathcal{O}_5^{\eta\kappa})^\dagger\,|\ell^j_\alpha\ell^i_\sigma\rangle\,\Big|^{(1)}_{\frac{1}{\epsilon}} \\ &\quad+ \widetilde{Z}^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi}_{5\bar{5},\text{eva}}\frac{C_5^{\gamma\delta}C_5^{\kappa\eta*}}{\Lambda^2}\,\langle \ell^k_\beta\ell^l_\rho|\,\mathcal{O}^{\tau\upsilon\varphi\chi}_{\text{eva,scalar}}\,|\ell^j_\alpha\ell^i_\sigma\rangle^{(0)} \\ &\quad+ \widetilde{Z}^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi}_{5\bar{5},\text{eva}}\frac{C_5^{\gamma\delta}C_5^{\kappa\eta*}}{\Lambda^2}\,\langle \ell^k_\beta\ell^l_\rho|\,\mathcal{O}^{\tau\upsilon\varphi\chi}_{\text{eva},(V-A)\otimes(V-A)}\,|\ell^j_\alpha\ell^i_\sigma\rangle^{(0)} \\ &\quad+ 2\widetilde{Z}^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi}_{5\bar{5},\ell\ell}\frac{C_5^{\gamma\delta}C_5^{\kappa\eta*}}{\Lambda^2}\,\langle \ell^k_\beta\ell^l_\rho|\,\mathcal{O}^{\tau\upsilon\varphi\chi}_{\ell\ell}\,|\ell^j_\alpha\ell^i_\sigma\rangle^{(0)}\;.\end{aligned} \tag{5.121}$$

Considering only the first two terms of this expression (the terms which generate a left-right chiral structure), and inserting the results from above yields

$$
\begin{aligned}
0 = & \frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_3^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}\left(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}\right) \\
& + \widetilde{Z}_{5\bar{5},\text{eva}}^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi}[\delta_{ik}\delta_{jl}(\delta_{\alpha\upsilon}\delta_{\sigma\chi}\delta_{\rho\tau}\delta_{\beta\varphi}+\delta_{\sigma\upsilon}\delta_{\alpha\chi}\delta_{\beta\tau}\delta_{\rho\varphi})+ \\
& \qquad \delta_{il}\delta_{jk}(\delta_{\sigma\upsilon}\delta_{\alpha\chi}\delta_{\rho\tau}\delta_{\beta\varphi}+\delta_{\alpha\upsilon}\delta_{\sigma\chi}\delta_{\beta\tau}\delta_{\rho\varphi})] \\
= & \frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_3^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}\left(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}\right) \\
& + 2\widetilde{Z}_{5\bar{5},\text{eva}}^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}\delta_{ik}\delta_{jl}+2\widetilde{Z}_{5\bar{5},\text{eva}}^{\gamma\delta\eta\kappa,\rho\sigma\beta\alpha}\delta_{il}\delta_{jk}\,. \qquad (5.122)
\end{aligned}
$$

Here, we have used that $\widetilde{Z}_{5\bar{5},\text{eva}}^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma} = \widetilde{Z}_{5\bar{5},\text{eva}}^{\gamma\delta\eta\kappa,\beta\sigma\rho\alpha}$. This is manifestly true for the vector-vector part, and since the operator is evanescent, must also be true for the scalar-scalar part. Considering the coefficient of $\delta_{ik}\delta_{jl}$, we immediately obtain

$$
\widetilde{Z}_{5\bar{5},\text{eva}}^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma} = -\frac{1}{2}\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_3^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}\,. \qquad (5.123)
$$

Since the loop-amplitude $\langle \ell_\beta^k \ell_\rho^l|\,\mathcal{O}_5^{\gamma\delta}(\mathcal{O}_5^{\eta\kappa})^\dagger\,|\ell_\alpha^j\ell_\sigma^i\rangle$ has only a left-right structure, and no $(V-A)\otimes(V-A)$ structure, then the $(V-A)\otimes(V-A)$ contributions in Equation (5.121) must cancel. Moreover, since

$$
\mathcal{O}_{\text{eva},(V-A)\otimes(V-A)}^{\tau\upsilon\varphi\chi} = -\mathcal{O}_{\ell\ell}^{\tau\upsilon\varphi\chi}\,,
$$

then it follows that

$$
\widetilde{Z}_{5\bar{5},\text{eva}}^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi} = 2\widetilde{Z}_{5\bar{5},\ell\ell}^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi}. \qquad (5.124)
$$

Therefore, we immediately obtain

$$
\widetilde{Z}_{5\bar{5},\ell\ell}^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi} = -\frac{1}{4}\frac{1}{16\pi^2}\frac{1}{\epsilon}\Delta_3^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi}\,, \qquad (5.125)
$$

or equivalently,

$$
\delta\widetilde{Z}_{5\bar{5},\ell\ell}^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi} = -\frac{1}{4}\Delta_3^{\gamma\delta\eta\kappa,\tau\upsilon\varphi\chi}\,. \qquad (5.126)
$$

5.5 Loop Calculations in the 2HDM

We here briefly discuss the results obtained when considering the mixing between dimension-five operators of the 2HDM into the dimension-six operators of SMEFT, as discussed in Section 5.1.2. We do not present the calculations in detail as was done for SMEFT, since the procedure is the same, and there are a larger number of calculations to perform. Instead, we present the results and highlight any special cases that arise. Again, we consider each process in turn. Recall that the SM Higgs doublet is denoted by φ_1, and the additional (non-SM) Higgs doublet is denoted by φ_2. Since we only consider dimension-six SMEFT operators, we do not consider processes where φ_2 is present as an asymptotic state. The Feynman rules of the additional dimension-5 operators that arise in the 2HDM are given in Figure 5.7.

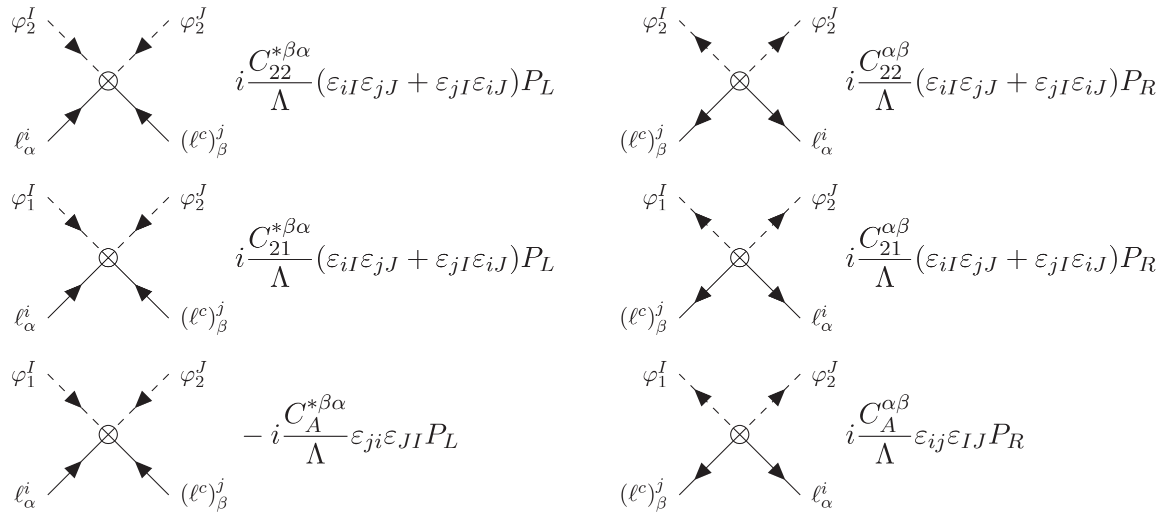


Fig. 5.7: Feynman rules we derived for additional dimension-5 operators arising in the 2HDM. φ_1 denotes the SM Higgs doublet, and φ_2 denotes the additional Higgs doublet, which we assume does not appear as an asymptotic state at the energy scales we are interested in.

5.5.1 $\varphi_1\ell \to \varphi_1\ell$

For the process $\varphi_1\ell \to \varphi_1\ell$, there are an additional four loop processes that arise in the 2HDM, from the amplitudes $\langle \ell^i_\beta\varphi^J_1|\,\mathcal{O}^{\gamma\delta}_{21}(\mathcal{O}^{\eta\kappa}_{21})^\dagger\,|\ell^n_\alpha\varphi^M_1\rangle$, $\langle \ell^i_\beta\varphi^J_1|\,\mathcal{O}^{\gamma\delta}_{A}(\mathcal{O}^{\eta\kappa}_{A})^\dagger\,|\ell^n_\alpha\varphi^M_1\rangle$, $\langle \ell^i_\beta\varphi^J_1|\,\mathcal{O}^{\gamma\delta}_{A}(\mathcal{O}^{\eta\kappa}_{21})^\dagger\,|\ell^n_\alpha\varphi^M_1\rangle$ and $\langle \ell^i_\beta\varphi^J_1|\,\mathcal{O}^{\gamma\delta}_{21}(\mathcal{O}^{\eta\kappa}_{A})^\dagger\,|\ell^n_\alpha\varphi^M_1\rangle$, shown in Figure 5.8. As before, these loop diagrams will mix into the physical operators $\mathcal{O}_{\varphi\ell(1)}$ and $\mathcal{O}_{\varphi\ell(3)}$, but since the calculation is performed off-shell, the EoM-vanishing operators $\mathcal{O}_{v(1)}$ and $\mathcal{O}_{v(3)}$ must also be included.

Calculating the loops and extracting the divergences gives

$$\langle \ell^i_\beta\varphi^J_1|\,\mathcal{O}^{\gamma\delta}_{21}(\mathcal{O}^{\eta\kappa}_{21})^\dagger\,|\ell^n_\alpha\varphi^M_1\rangle\,\Big|^{(1)}_{\frac{1}{\epsilon}} = \frac{(5\delta_{JM}\delta_{in}-4\delta_{iM}\delta_{Jn})}{32\pi^2\epsilon}[\overline{u_\ell}(\beta,i)(\not{q}_f+\not{p}_f)P_L u_\ell(\alpha,n)] \times \Delta_1^{\gamma\delta\eta\kappa,\beta\alpha}, \tag{5.127}$$

$$\langle \ell^i_\beta\varphi^J_1|\,\mathcal{O}^{\gamma\delta}_{A}(\mathcal{O}^{\eta\kappa}_{A})^\dagger\,|\ell^n_\alpha\varphi^M_1\rangle\,\Big|^{(1)}_{\frac{1}{\epsilon}} = -\frac{\delta_{in}\delta_{JM}}{32\pi^2\epsilon}[\overline{u_\ell}(\beta,i)(\not{q}_f+\not{p}_f)P_L u_\ell(\alpha,n)] \times \Delta_{1,A\overline{A}}^{\gamma\delta\eta\kappa,\beta\alpha}, \tag{5.128}$$

$$\langle \ell^i_\beta\varphi^J_1|\,\mathcal{O}^{\gamma\delta}_{A}(\mathcal{O}^{\eta\kappa}_{21})^\dagger\,|\ell^n_\alpha\varphi^M_1\rangle\,\Big|^{(1)}_{\frac{1}{\epsilon}} = -\frac{2\delta_{Jn}\delta_{iM}-\delta_{in}\delta_{JM}}{32\pi^2\epsilon}[\overline{u_\ell}(\beta,i)(\not{q}_f+\not{p}_f)P_L u_\ell(\alpha,n)] \times \Delta_{1,A\overline{S}}^{\gamma\delta\eta\kappa,\beta\alpha}, \tag{5.129}$$

$$\langle \ell^i_\beta\varphi^J_1|\,\mathcal{O}^{\gamma\delta}_{21}(\mathcal{O}^{\eta\kappa}_{A})^\dagger\,|\ell^n_\alpha\varphi^M_1\rangle\,\Big|^{(1)}_{\frac{1}{\epsilon}} = +\frac{2\delta_{Jn}\delta_{iM}-\delta_{in}\delta_{JM}}{32\pi^2\epsilon}[\overline{u_\ell}(\beta,i)(\not{q}_f+\not{p}_f)P_L u_\ell(\alpha,n)] \times \Delta_{1,S\overline{A}}^{\gamma\delta\eta\kappa,\beta\alpha}, \tag{5.130}$$

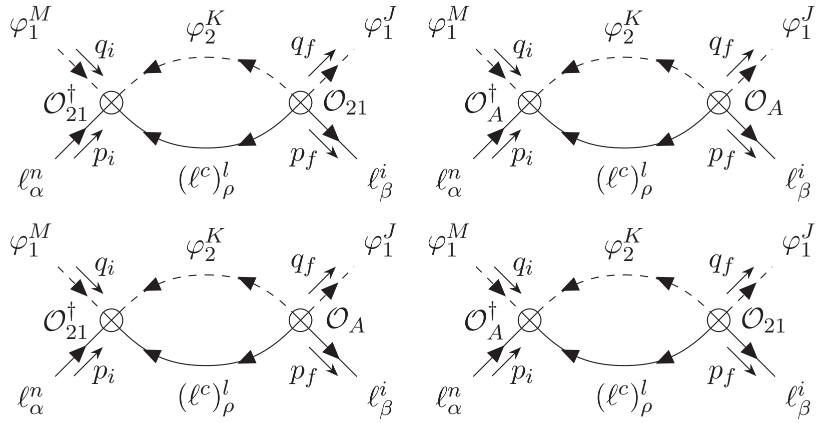


Fig. 5.8: Additional loop diagrams arising in the 2HDM that mix into the SMEFT operators $\mathcal{O}_{\varphi\ell(1)}$ and $\mathcal{O}_{\varphi\ell(3)}$.

	$\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha}$	$\Delta_{1,A\overline{A}}^{\gamma\delta\eta\kappa,\beta\alpha}$	$\Delta_{1,A\overline{S}}^{\gamma\delta\eta\kappa,\beta\alpha}$	$\Delta_{1,S\overline{A}}^{\gamma\delta\eta\kappa,\beta\alpha}$
$\gamma \leftrightarrow \delta$	$+$	$-$	$-$	$+$
$\eta \leftrightarrow \kappa$	$+$	$-$	$+$	$-$

Table 5.1: The symmetries of the flavour structures that arise in the process $\varphi_1\ell \to \varphi_1\ell$.

where we introduce the new flavour structures

$$\begin{aligned}
\Delta_{1,A\overline{A}}^{\gamma\delta\eta\kappa,\beta\alpha} &= \frac{1}{4}(\delta_{\gamma\beta}\delta_{\delta\kappa}\delta_{\eta\alpha} - \delta_{\gamma\kappa}\delta_{\delta\beta}\delta_{\eta\alpha} + \delta_{\gamma\eta}\delta_{\delta\beta}\delta_{\kappa\alpha} - \delta_{\gamma\beta}\delta_{\delta\eta}\delta_{\kappa\alpha})\,, \\
\Delta_{1,A\overline{S}}^{\gamma\delta\eta\kappa,\beta\alpha} &= \frac{1}{4}(\delta_{\gamma\beta}\delta_{\delta\kappa}\delta_{\eta\alpha} - \delta_{\gamma\kappa}\delta_{\delta\beta}\delta_{\eta\alpha} - \delta_{\gamma\eta}\delta_{\delta\beta}\delta_{\kappa\alpha} + \delta_{\gamma\beta}\delta_{\delta\eta}\delta_{\kappa\alpha})\,, \\
\Delta_{1,S\overline{A}}^{\gamma\delta\eta\kappa,\beta\alpha} &= \frac{1}{4}(\delta_{\gamma\beta}\delta_{\delta\kappa}\delta_{\eta\alpha} + \delta_{\gamma\kappa}\delta_{\delta\beta}\delta_{\eta\alpha} - \delta_{\gamma\eta}\delta_{\delta\beta}\delta_{\kappa\alpha} - \delta_{\gamma\beta}\delta_{\delta\eta}\delta_{\kappa\alpha})\,.
\end{aligned}$$

Each of the new flavour structures has specific symmetry properties, shown in Table 5.1 (where we include Δ_1 for illustrative purposes).

Since we consider only the dimension-six operators of SMEFT, the tree-level amplitudes are the same as given in the previous section. Considering the coefficients of specific SU(2) structures and momenta, systems of simultaneous equations can be formed, and solved to give

$$\delta\widetilde{Z}_{21\overline{21},\varphi\ell(1)}^{\gamma\delta\eta\kappa,\beta\alpha} = -\frac{3}{4}\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha}\,, \tag{5.131}$$

$$\delta\widetilde{Z}_{A\overline{A},\varphi\ell(1)}^{\gamma\delta\eta\kappa,\beta\alpha} = -\frac{1}{4}\Delta_{1,AA}^{\gamma\delta\eta\kappa,\beta\alpha}\,, \tag{5.132}$$

$$\delta\widetilde{Z}_{21\overline{21},\varphi\ell(3)}^{\gamma\delta\eta\kappa,\beta\alpha} = +\frac{1}{2}\Delta_1^{\gamma\delta\eta\kappa,\beta\alpha}\,, \tag{5.133}$$

$$\delta\widetilde{Z}_{A\overline{21},\varphi\ell(3)}^{\gamma\delta\eta\kappa,\beta\alpha} = +\frac{1}{4}\Delta_{1,AS}^{\gamma\delta\eta\kappa,\beta\alpha}\,, \tag{5.134}$$

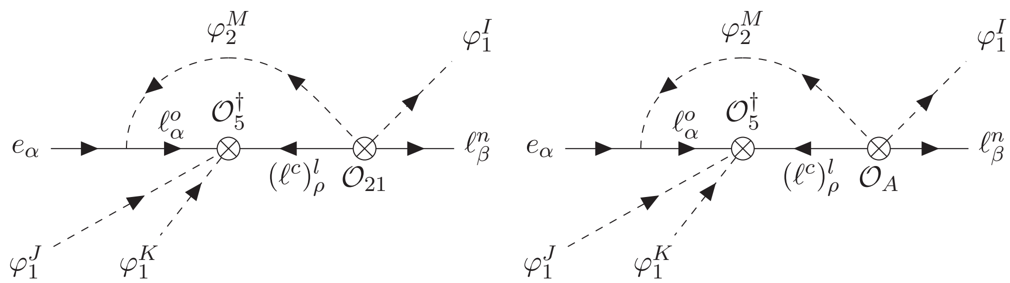


Fig. 5.9: Additional loop diagrams arising in the 2HDM that mix into the SMEFT operator $\mathcal{O}_{e\varphi}$.

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{21\overline{A},\varphi\ell(3)} = +\frac{1}{4}\Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{1,SA}\,, \tag{5.135}$$

with all other possible counterterms being vanishing. Note that as for the SMEFT calculation, these counterterms were checked by also considering the processes $\varphi_1\ell \to \varphi_1\ell B$ and $\varphi_1\ell \to \varphi_1\ell W$.

We also list for completeness the renormalisation constants for EoM-vanishing operators, as they are needed when considering the mixing into $\mathcal{O}_{e\varphi}$:

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{21\overline{21},v(1)} = -\frac{3}{4}\Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{1}\,, \tag{5.136}$$

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{A\overline{A},v(1)} = -\frac{1}{4}\Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{1,AA}\,, \tag{5.137}$$

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{21\overline{21},v(3)} = +\frac{1}{2}\Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{1}\,, \tag{5.138}$$

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{A\overline{21},v(3)} = +\frac{1}{4}\Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{1,AS}\,, \tag{5.139}$$

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{21\overline{A},v(3)} = +\frac{1}{4}\Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{1,SA}\,. \tag{5.140}$$

5.5.2 $e\varphi_1\varphi_1 \to \ell\varphi_1$

For the process $e\varphi_1\varphi_1 \to \ell\varphi_1$, two additional loop diagrams arise from the presence of internal φ_2 Higgses. Since the external Higgses must all be φ_1, there must always be an insertion of the Weinberg operator $\mathcal{O}_5^\dagger$. Consequently, new diagrams in which the internal Higgs is a φ_2 doublet can only have a single insertion of either $\mathcal{O}_{21}$ or $\mathcal{O}_A$, hence only two new diagrams arise. These are shown in Figure 5.9.

Calculating the diagrams of Figure 5.9 and extracting their divergences gives

$$\begin{aligned}\langle \ell^n_\beta\varphi^I|\,\mathcal{O}^{\gamma\delta}_{21}(\mathcal{O}^{\eta\kappa}_5)^\dagger(\overline{\ell_\chi}[\Gamma^{(2)}_e]_{\chi\zeta}e_\zeta\varphi)\,|e_\alpha\varphi^J\varphi^K\rangle\,\big|^{(1)}_{\frac{1}{\epsilon}} &= -\frac{1}{16\pi^2}\frac{1}{\epsilon}(\overline{u_\ell}(\beta,n)P_R u_e(\alpha)) \\ &\times(\delta_{IJ}\delta_{Kn}+\delta_{IK}\delta_{Jn})\Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{2,S}\,,\end{aligned} \tag{5.141}$$

$$\langle \ell^n_\beta \varphi^I |\, \mathcal{O}^{\gamma\delta}_A (\mathcal{O}^{\eta\kappa}_5)^\dagger (\overline{\ell_\chi} [\Gamma^{(2)}_e]_{\chi\zeta} e_\zeta \varphi)\, | e_\alpha \varphi^J \varphi^K \rangle \,\Big|^{(1)}_{\frac{1}{\epsilon}} = + \frac{1}{16\pi^2} \frac{1}{\epsilon} (\overline{u_\ell}(\beta, n) P_R u_e(\alpha)) \\ \times (\delta_{IJ}\delta_{Kn} + \delta_{IK}\delta_{Jn}) \Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{2,A}\,, \quad (5.142)$$

where $\Gamma^{(2)}_e$ denotes the leptonic Yukawa matrix for φ_2. The new flavour structures introduced are

$$\begin{aligned} \Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{2,S} &= \frac{1}{4} (\delta_{\gamma\beta}\delta_{\delta\kappa} [\Gamma^{(2)}_e]_{\eta\alpha} + \delta_{\gamma\beta}\delta_{\delta\eta} [\Gamma^{(2)}_e]_{\kappa\alpha} \\ &\quad + \delta_{\gamma\kappa}\delta_{\delta\beta} [\Gamma^{(2)}_e]_{\eta\alpha} + \delta_{\gamma\eta}\delta_{\delta\beta} [\Gamma^{(2)}_e]_{\kappa\alpha})\,, \quad (5.143) \\ \Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{2,A} &= \frac{1}{4} (\delta_{\gamma\beta}\delta_{\delta\kappa} [\Gamma^{(2)}_e]_{\eta\alpha} + \delta_{\gamma\beta}\delta_{\delta\eta} [\Gamma^{(2)}_e]_{\kappa\alpha} \\ &\quad - \delta_{\gamma\kappa}\delta_{\delta\beta} [\Gamma^{(2)}_e]_{\eta\alpha} - \delta_{\gamma\eta}\delta_{\delta\beta} [\Gamma^{(2)}_e]_{\kappa\alpha})\,. \quad (5.144) \end{aligned}$$

To renormalise these diagrams, we use dimension-six operators. In the equivalent SMEFT calculation of Section 5.4.4, the operators required were $\mathcal{O}_{e\varphi}$, and the EoM-vanishing operators $\mathcal{O}_{v(1)}$ and $\mathcal{O}_{v(3)}$. However, the corresponding renormalisation constants $\widetilde{Z}_{21\bar{5},v(1)}$, $\widetilde{Z}_{21\bar{5},v(3)}$, $\widetilde{Z}_{A\bar{5},v(1)}$ and $\widetilde{Z}_{A\bar{5},v(3)}$ are all zero. This is because loop diagrams involving a single $\mathcal{O}_5$ operator and a single $\mathcal{O}_{21}$ or $\mathcal{O}_A$ operator cannot give rise to a dimension-6 operator with only external φ_1 Higgses, as is the case for $\mathcal{O}_{v(1)}$ and $\mathcal{O}_{v(3)}$. Therefore, only the physical operator $\mathcal{O}_{e\varphi}$ is required to renormalise these additional diagrams. Doing this gives the results

$$\begin{aligned} \delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{21\bar{5},e\varphi} &= +\Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{2,S}\,, \quad (5.145) \\ \delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{A\bar{5},e\varphi} &= -\Delta^{\gamma\delta\eta\kappa,\beta\alpha}_{2,A}. \quad (5.146) \end{aligned}$$

There are additional contributions to $\mathcal{O}_{e\varphi}$ in the 2HDM, arising from EoM-vanishing operators. For example, consider the process $e\varphi_1 \to \ell\varphi_1\varphi_1$ mediated by the double-insertion of the operators $\mathcal{O}_{21}$. This has the renormalisation equation

$$\begin{aligned} 0 &= \frac{C^{\gamma\delta}_{21}}{\Lambda} \frac{C^{\kappa\eta*}_{21}}{\Lambda} \langle \ell^n_\beta \varphi^I |\, \mathcal{O}^{\gamma\delta}_{21} (\mathcal{O}^{\eta\kappa}_{21})^\dagger (\overline{\ell_\chi} [\Gamma_e]_{\chi\zeta} e_\zeta \varphi)\, | e_\alpha \varphi^J \varphi^K \rangle \,\Big|^{(1)}_{\frac{1}{\epsilon}} \\ &\quad + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{21\overline{21},e\varphi} \frac{C^{\gamma\delta}_{21} C^{\kappa\eta*}_{21}}{\Lambda^2} \langle \ell^n_\beta \varphi^I |\, \mathcal{O}^{\rho\sigma}_{e\varphi}\, | e_\alpha \varphi^J \varphi^K \rangle^{(0)} \\ &\quad + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{21\overline{21},v(1)} \frac{C^{\gamma\delta}_{21} C^{\kappa\eta*}_{21}}{\Lambda^2} \langle \ell^n_\beta \varphi^I |\, \mathcal{O}^{\rho\sigma}_{v(1)}\, | e_\alpha \varphi^J \varphi^K \rangle^{(0)} \\ &\quad + \widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{21\overline{21},v(3)} \frac{C^{\gamma\delta}_{21} C^{\kappa\eta*}_{21}}{\Lambda^2} \langle \ell^n_\beta \varphi^I |\, \mathcal{O}^{\rho\sigma}_{v(3)}\, | e_\alpha \varphi^J \varphi^K \rangle^{(0)}\,, \quad (5.147) \end{aligned}$$

where $\widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{21\overline{21},v(1)}$ and $\widetilde{Z}^{\gamma\delta\eta\kappa,\rho\sigma}_{21\overline{21},v(3)}$ are known from Section 5.5.1. The loop amplitude is manifestly zero, since a double-insertion of $\mathcal{O}_{21}$ cannot result in exclusively external φ_1 Higgses, and so we obtain the equation (after using the Feynman rules of the dimension-6 operators),

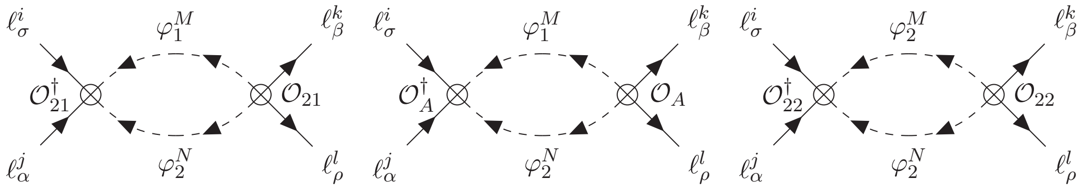


Fig. 5.10: Additional loop diagrams arising in the 2HDM that mix into the SMEFT operator $\mathcal{O}_{\ell\ell}$.

$$\begin{aligned} 0 &= \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{21\overline{21},e\varphi} - \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\sigma}_{21\overline{21},v(1)}[\Gamma_e]_{\sigma\alpha} - \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\sigma}_{21\overline{21},v(3)}[\Gamma_e]_{\sigma\alpha} \\ &= \widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{21\overline{21},e\varphi} + \frac{3}{4}\Delta_1^{\gamma\delta\eta\kappa,\beta\sigma}[\Gamma_e]_{\sigma\alpha} - \frac{1}{2}\Delta_1^{\gamma\delta\eta\kappa,\beta\sigma}[\Gamma_e]_{\sigma\alpha}\,. \end{aligned} \tag{5.148}$$

This is solved to give

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{21\overline{21},e\varphi} = -\frac{1}{4}\Delta_1^{\gamma\delta\eta\kappa,\beta\sigma}[\Gamma_e]_{\sigma\alpha} = -\frac{1}{4}\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha}\,. \tag{5.149}$$

A similar procedure can be used to find $\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{A\overline{A},e\varphi}$, $\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{A\overline{21},e\varphi}$, and $\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{21\overline{A},e\varphi}$, yielding

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{21\overline{21},e\varphi} = -\frac{1}{4}\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha}\,, \tag{5.150}$$

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{A\overline{A},e\varphi} = +\frac{1}{4}\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha}\,, \tag{5.151}$$

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{21\overline{A},e\varphi} = -\frac{1}{4}\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha}\,, \tag{5.152}$$

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\beta\alpha}_{A\overline{21},e\varphi} = +\frac{1}{4}\Delta_2^{\gamma\delta\eta\kappa,\beta\alpha}\,. \tag{5.153}$$

(5.154)

5.5.3 $\ell\ell \to \ell\ell$

For the process $\ell\ell \to \ell\ell$, there are three new diagrams that need to be considered for the 2HDM. These are listed in Figure 5.10.

In principle there could be an additional two diagrams included in Figure 5.10, involving the operator insertions $\mathcal{O}_{21}\mathcal{O}_A^\dagger$ and $\mathcal{O}_A\mathcal{O}_{21}^\dagger$. However, both of these diagrams are vanishing from their SU(2) structures, and so are not considered in any further detail here. Calculating the diagrams of Figure 5.10 and extracting their divergences one obtains

$$\begin{aligned} \langle \ell^k_\beta \ell^l_\rho|\,\mathcal{O}_{21}^{\gamma\delta}(\mathcal{O}_{21}^{\eta\kappa})^\dagger\,|\ell^j_\alpha\ell^i_\sigma\rangle\,\Big|^{(1)}_{\frac{1}{\epsilon}} &= +\frac{2}{16\pi^2}\frac{1}{\epsilon}(\overline{u_\ell}(\sigma,i)P_L v_\ell(\alpha,j))(\overline{v_\ell}(\beta,k)P_R u_\ell(\rho,l)) \\ &\quad (\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl})\,\Delta_3^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}\,, \end{aligned} \tag{5.155}$$

$$\langle \ell^k_\beta \ell^l_\rho |\, \mathcal{O}^{\gamma\delta}_{A} (\mathcal{O}^{\eta\kappa}_{A})^\dagger \,| \ell^j_\alpha \ell^i_\sigma \rangle \,\Big|^{(1)}_{\frac{1}{\epsilon}} = +\frac{2}{16\pi^2}\frac{1}{\epsilon}(\overline{u_\ell}(\sigma,i)P_L v_\ell(\alpha,j))(\overline{v_\ell}(\beta,k)P_R u_\ell(\rho,l))$$
$$(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})\, \Delta^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}_{3,A} \,, \tag{5.156}$$
$$\langle \ell^k_\beta \ell^l_\rho |\, \mathcal{O}^{\gamma\delta}_{22} (\mathcal{O}^{\eta\kappa}_{22})^\dagger \,| \ell^j_\alpha \ell^i_\sigma \rangle \,\Big|^{(1)}_{\frac{1}{\epsilon}} = -\frac{1}{16\pi^2}\frac{1}{\epsilon}(\overline{u_\ell}(\sigma,i)P_L v_\ell(\alpha,j))(\overline{v_\ell}(\beta,k)P_R u_\ell(\rho,l))$$
$$(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})\, \Delta^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}_{3} \,, \tag{5.157}$$

where the new flavour structure is

$$\Delta^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}_{3,A} = \frac{1}{4}(\delta_{\gamma\alpha}\delta_{\delta\sigma} - \delta_{\gamma\sigma}\delta_{\delta\alpha})(\delta_{\kappa\beta}\delta_{\eta\rho} - \delta_{\kappa\rho}\delta_{\eta\beta})\,. \tag{5.158}$$

These diagrams may all be renormalised using the Fierz-evanescent operator of Equation (5.114), as in the SMEFT case. Note that although the amplitude involving $\mathcal{O}^{\gamma\delta}_{A}(\mathcal{O}^{\eta\kappa}_{A})^\dagger$ contains a relative minus sign in the SU(2) structure compared to the other four-lepton amplitudes, the same evanescent operator is still able to renormalise the diagram, since the antisymmetric flavour structure provides an additional sign that cancels this. Therefore, these diagrams may be renormalised by the counterterms

$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}_{21\overline{21},\ell\ell} = -\frac{1}{2}\Delta^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}_{3} \,, \tag{5.159}$$
$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}_{A\overline{A},\ell\ell} = +\frac{1}{2}\Delta^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}_{3,A} \,, \tag{5.160}$$
$$\delta\widetilde{Z}^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}_{22\overline{22},\ell\ell} = -\frac{1}{4}\Delta^{\gamma\delta\eta\kappa,\rho\alpha\beta\sigma}_{3} \,. \tag{5.161}$$

5.5.4 Summary of Counterterms

In the preceding sections, the renormalisation constants of the renormalisation tensor for mixing between dimensions five and six have been calculated in terms of the full flavour structures. We present here the renormalisation constants in a slightly different form, where they are contracted with the relevant Wilson coefficients to form counterterms. In this form, we have:

$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\beta\alpha}_{\varphi\ell(1)} = -\frac{3}{4}\frac{1}{16\pi^2\epsilon}[C_5 C_5^*]^{\beta\alpha} \,, \tag{5.162}$$
$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\beta\alpha}_{\varphi\ell(1)} = -\frac{3}{4}\frac{1}{16\pi^2\epsilon}[C_{21} C_{21}^*]^{\beta\alpha} \,, \tag{5.163}$$
$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\beta\alpha}_{\varphi\ell(1)} = +\frac{1}{4}\frac{1}{16\pi^2\epsilon}[C_A C_A^*]^{\beta\alpha} \,, \tag{5.164}$$

$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\beta\alpha}_{\varphi\ell(3)} = +\frac{1}{2}\frac{1}{16\pi^2\epsilon}[C_5 C_5^*]^{\beta\alpha} \,, \tag{5.165}$$
$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\beta\alpha}_{\varphi\ell(3)} = +\frac{1}{2}\frac{1}{16\pi^2\epsilon}[C_{21} C_{21}^*]^{\beta\alpha} \,, \tag{5.166}$$
$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\beta\alpha}_{\varphi\ell(3)} = +\frac{1}{4}\frac{1}{16\pi^2\epsilon}[C_A C_{21}^* - C_{21} C_A^*]^{\beta\alpha} \,, \tag{5.167}$$

$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\beta\alpha}_{eH} = +\frac{3}{4}\frac{1}{16\pi^2\epsilon}[C_5 C_5^* \Gamma_e]^{\beta\alpha}\,, \tag{5.168}$$

$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\beta\alpha}_{eH} = +\frac{1}{4}\frac{1}{16\pi^2\epsilon}\big(4[(C_{21} - C_A)C_5^* \Gamma_e^{(2)}]^{\beta\alpha}\,, + [(C_A C_A^* + C_A C_{21}^* - C_{21} C_A^* - C_{21} C_{21}^*)\Gamma_e]^{\beta\alpha}\big)\,, \tag{5.169}$$

$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\rho\alpha\beta\sigma}_{\ell\ell} = -\frac{1}{4}\frac{1}{16\pi^2\epsilon}C_5^{\rho\beta}C_5^{*\sigma\alpha}\,, \tag{5.170}$$

$$\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\rho\sigma\beta\alpha}_{\ell\ell} = -\frac{1}{4}\frac{1}{16\pi^2\epsilon}C_{22}^{\rho\beta}C_{22}^{*\alpha\sigma} - \frac{1}{2}\frac{1}{16\pi^2\epsilon}C_{21}^{\rho\beta}C_{21}^{*\alpha\sigma} + \frac{1}{2}\frac{1}{16\pi^2\epsilon}C_A^{\rho\beta}C_A^{*\alpha\sigma}, \tag{5.171}$$

where each $\Delta(\vec{C}[\tilde{Z}]\vec{C}^\dagger)$ denotes a contribution to a counterterm. Note that in this notation, $\vec{C}$ denotes a vector of dimension-five Wilson coefficients (of which there is one in SMEFT and four in the 2HDM), and $[\tilde{Z}]$ denotes the renormalisation tensor (which is a vector in dimension-six space and a matrix in dimension-five space). This is the complete list of counterterms for lepton-flavour violating dimension-six operators of SMEFT.

5.6 Lepton Flavour Conserving Processes

For completeness, we also briefly consider processes which mix dimension-five and six operators without violating lepton flavour. These involve external states solely comprised of the SM Higgs doublet φ_1, and a lepton loop. Since all Higgses are external in these processes, the additional operators of the 2HDM do not play a role. There may be mixing into the dimension-six operators $\mathcal{O}_{\varphi\Box}$, $\mathcal{O}_{\varphi D}$, and $\mathcal{O}_\varphi$, where

$$\mathcal{O}_\varphi = (\varphi^\dagger\varphi)^3\,, \tag{5.172}$$

$$\mathcal{O}_{\varphi\Box} = (\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi)\,, \tag{5.173}$$

$$\mathcal{O}_{\varphi D} = (\varphi^\dagger D^\mu\varphi)^*(\varphi^\dagger D_\mu\varphi)\,. \tag{5.174}$$

There are nine 1-loop diagrams that can be drawn with six external Higgses, but when they are calculated and summed over, they cancel, and so do not require renormalisation.

There is a single diagram that needs to be renormalised, which involves four external SM Higgses, as shown in Figure 5.11.

Calculating the diagram of Figure 5.11, extracting the UV divergence using infrared rearrangement, and expanding in external momenta to quadratic order (since the dimension-six operators are double-derivative operators), we obtain the result

$$\langle\varphi^K\varphi^L|\,\mathcal{O}_5^{\gamma\delta}(\mathcal{O}_5^{\eta\kappa})^\dagger\,|\varphi^I\varphi^J\rangle\,\big|^{(1)}_{\frac{1}{\epsilon}} = \frac{1}{16\pi^2}\frac{1}{\epsilon}(p_3+p_4)^2(\delta_{IL}\delta_{JK}+\delta_{IK}\delta_{JL})\Delta_4^{\gamma\delta\eta\kappa}\,, \tag{5.175}$$

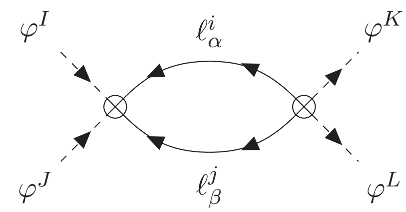


Fig. 5.11: Lepton-flavour-conserving mixing between dimensions-five and -six is via a diagram involving external Higgses.

where

$$\Delta_4^{\gamma\delta\eta\kappa} = \frac{1}{2}(\delta_{\gamma\eta}\delta_{\delta\kappa} + \delta_{\gamma\kappa}\delta_{\delta\eta}). \tag{5.176}$$

Note that when this structure is contracted with the Wilson coefficients $C_5^{\gamma\delta} C_5^{\kappa\eta *}$, it results in the trace of the Wilson coefficients, $\mathrm{Tr}[C_5 C_5^*]$. The operators that may mediate this process at tree level are $\mathcal{O}_{\varphi\Box}$ and $\mathcal{O}_{\varphi D}$. In addition, from the EoM of the Higgs field, there arises the EoM-vanishing operator

$$\begin{aligned}\mathcal{O}_{v\varphi} = (\varphi^\dagger\varphi)(\varphi^\dagger D^2\varphi) - m^2(\varphi^\dagger\varphi)^2 + \lambda(\varphi^\dagger\varphi)^3 \\ + (\varphi^\dagger\varphi)(\varphi^\dagger\overline{e}\Gamma_e^\dagger\ell) - (\varphi^\dagger\varphi)(\varphi^\dagger\varepsilon\overline{q}\Gamma_u u) + (\varphi^\dagger\varphi)(\varphi^\dagger\overline{d}\Gamma_d^\dagger q)\,,\end{aligned} \tag{5.177}$$

which can also contribute. Note that this operator contains $\mathcal{O}_{e\varphi}$, $\mathcal{O}_{u\varphi}$, $\mathcal{O}_{d\varphi}$ and $\mathcal{O}_{\varphi}$.

Performing the tree level calculations to renormalise the loop diagram results in the counterterms

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\beta\alpha}_{e\varphi} = -\frac{1}{16\pi^2\epsilon}\mathrm{Tr}[C_5C_5^*][\Gamma_e]_{\beta\alpha}\,, \tag{5.178}$$

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\beta\alpha}_{u\varphi} = -\frac{1}{16\pi^2\epsilon}\mathrm{Tr}[C_5C_5^*][\Gamma_u]_{\beta\alpha}\,, \tag{5.179}$$

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)^{\beta\alpha}_{d\varphi} = -\frac{1}{16\pi^2\epsilon}\mathrm{Tr}[C_5C_5^*][\Gamma_d]_{\beta\alpha}\,, \tag{5.180}$$

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{\varphi} = -2\frac{1}{16\pi^2\epsilon}\mathrm{Tr}[C_5C_5^*]\lambda\,, \tag{5.181}$$

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{\varphi D} = -2\frac{1}{16\pi^2\epsilon}\mathrm{Tr}[C_5C_5^*]\,, \tag{5.182}$$

$$(\vec{C}[\tilde{Z}]\vec{C}^\dagger)_{\varphi\Box} = -\frac{1}{16\pi^2\epsilon}\mathrm{Tr}[C_5C_5^*]\,. \tag{5.183}$$

5.7 Comparison with the Literature

The standard model calculation has been performed in [11] in a different operator basis. We disagree with their final results even after transforming our results to their basis. We specify our basis

$$\tilde{\mathcal{O}} = \left(\mathcal{O}_{\varphi\ell(1)}, \mathcal{O}_{\varphi\ell(3)}, \mathcal{O}_{e\varphi}, \mathcal{O}^\dagger_{e\varphi}, \mathcal{O}_{v(1)}, \mathcal{O}^\dagger_{v(1)}, \mathcal{O}_{v(3)}, \mathcal{O}^\dagger_{v(3)}\right)^T\,, \tag{5.184}$$

and the one used in [11]

$$\tilde{Q} = \left(Q_{\varphi\ell}^{(-)}, Q_{\varphi\ell}^{(+)}, Q_{e\varphi}, Q_{e\varphi}^\dagger, \mathcal{O}_{v(1)}, \mathcal{O}_{v(1)}^\dagger, \mathcal{O}_{v(3)}, \mathcal{O}_{v(3)}^\dagger\right)^T , \tag{5.185}$$

where the additional operators are defined as

$$\begin{aligned} Q_{\varphi\ell}^{(-)} &= \frac{1}{2}\left[(\varphi^\dagger D_\mu \varphi)(\bar{\ell}\gamma^\mu \ell) - (\varphi^\dagger D_\mu^a \varphi)(\bar{\ell}\tau^a\gamma^\mu \ell)\right] , & (5.186)\\ Q_{\varphi\ell}^{(+)} &= \frac{1}{2}\left[(\varphi^\dagger D_\mu \varphi)(\bar{\ell}\gamma^\mu \ell) + (\varphi^\dagger D_\mu^a \varphi)(\bar{\ell}\tau^a\gamma^\mu \ell)\right] , & (5.187)\\ Q_{e\varphi} &= \mathcal{O}_{e\varphi} \,. & (5.188)\end{aligned}$$

Here we drop the generation indices and note that the operators $Q_{\varphi\ell}^{(-)}$ and $Q_{\varphi\ell}^{(+)}$ are not Hermitian. For this reason, we treat the operator $\mathcal{O}_{e\varphi}$ and the EoM-vanishing operators as independent from their Hermitian conjugate in our basis transformation. Writing the resulting linear transformation as

$$\tilde{\mathcal{O}} = \hat{R}\tilde{Q} \,, \tag{5.189}$$

only the first two rows of $\hat{R}$ have entries that are not proportional to an identity transformation. These two rows are determined by the following linear transformation:[4]

$$\begin{pmatrix} \mathcal{O}_{\varphi\ell(1)} \\ \mathcal{O}_{\varphi\ell(3)} \end{pmatrix} = \begin{pmatrix} 2 & 2 & \Gamma_e & -\Gamma_e^\dagger & 1 & -1 & 0 & 0 \\ -2 & 2 & \Gamma_e & -\Gamma_e^\dagger & 0 & 0 & 1 & -1 \end{pmatrix} \tilde{Q} \,. \tag{5.190}$$

The Wilson coefficients and renormalisation constants will consequently fulfil our Hermiticity conditions in our basis, but not necessarily in the basis of [11]. The counterterms of the Wilson coefficients transform in the same way as the respective Wilson coefficients under our change of basis, i.e. as

$$\delta\tilde{c} = \hat{R}^T \delta\tilde{C} \,, \tag{5.191}$$

where $\delta\tilde{C} = (16\pi^2)\epsilon\vec{C}\tilde{Z}\vec{C}^\dagger$ represent the counterterms multiplied with $(16\pi^2)\epsilon$, while $\delta\tilde{c}$ correspond to the analogous expression in the $\tilde{Q}$ basis.

Using the counterterms presented in Equations (5.162), (5.165) and (5.168), we obtain

$$\delta\tilde{c} = \left(-\frac{5}{2}\left[C_5 C_5^*\right], -\frac{1}{2}\left[C_5 C_5^*\right], \frac{1}{2}\left[C_5 C_5^* \Gamma_e\right], \left[\Gamma_e^\dagger C_5^* C_5\right]\right)^T , \tag{5.192}$$

which fulfil the Hermiticity condition of the overall Lagrangian, even though this is not immediately apparent due to the choice of basis. These results are in disagreement with the final results quoted in [11]. Yet using the results quoted in the individual diagrams in Appendix B of [11] we find agreement with the expression of Equation (5.192), which

[4] To perform the change of basis we have to move covariant derivatives from one term to another. This can be done by noting that the total derivatives $D_\mu\left[\left(\varphi^\dagger\varphi\right)\left(\bar{\ell}\gamma^\mu\ell\right)\right]$ and $D_\mu\left[\left(\varphi^\dagger\tau^a\varphi\right)\left(\bar{\ell}\tau^a\gamma^\mu\ell\right)\right]$ are vanishing.

suggests that a different projection was performed. Following the explanations of the calculation, it appears that part (the $\delta\delta$ part) of the diagram evaluated in Appendix B.1 of [11] is projected onto an operator basis where the operators $Q_{\varphi\ell}^{(\pm)}$ are replaced by $Q_{\varphi\ell}^{(\pm)\prime} = Q_{\varphi\ell}^{(\pm)} + (Q_{\varphi\ell}^{(\pm)})^{\dagger}$, while another part (the $\varepsilon\varepsilon$ part) is projected onto the basis presented in Equation (5.185).

Transforming now to the primed basis, where the Hermitian conjugate is added to the first two operators of Equation (5.185) we find that the non-trivial transformation matrix involves only the first two elements of our basis and the primed basis. Explicitly writing

$$\begin{pmatrix} \mathcal{O}_{\varphi\ell(1)} \\ \mathcal{O}_{\varphi\ell(3)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} Q_{\varphi\ell}^{(-)\prime} \\ Q_{\varphi\ell}^{(+)\prime} \end{pmatrix}, \tag{5.193}$$

we find

$$\delta\tilde{c}' = \left(-\frac{5}{4}\big[C_5 C_5^*\big], -\frac{1}{4}\big[C_5 C_5^*\big], \frac{3}{4}\big[C_5 C_5^* \Gamma_e\big], \frac{3}{4}\big[\Gamma_e^\dagger C_5^* C_5\big]\right)^T . \tag{5.194}$$

Again, this result does not agree with [11]. Finally, note that projecting the results quoted for the individual diagrams in Appendix B of [11], except the $\varepsilon\varepsilon$ part, would give

$$\delta\tilde{c}'_{\text{not }\varepsilon\varepsilon} = \left(-\frac{1}{4}\big[C_5 C_5^*\big], -\frac{1}{4}\big[C_5 C_5^*\big], \frac{3}{4}\big[C_5 C_5^* \Gamma_e\big], \frac{3}{4}\big[\Gamma_e^\dagger C_5^* C_5\big]\right)^T , \tag{5.195}$$

while projecting only the $\varepsilon\varepsilon$ part on the non-Hermitian basis yields

$$\delta\tilde{c}_{\varepsilon\varepsilon} = \left(-2\big[C_5 C_5^*\big], 0, 0, 0\right)^T . \tag{5.196}$$

Summing these two terms would reproduce the results of [11].

5.8 The Renormalisation Group Equations

As discussed in Section 5.2.1, the leading order renormalisation constants and anomalous dimension tensor are related in a simple way:

$$\tilde{\gamma}_{AB,C}^{\zeta\eta,\theta} = 2\delta\tilde{Z}_{AB,C}^{\zeta\eta,\theta} , \tag{5.197}$$

where A, B, C are operator labels and ζ, η, θ represent collections of flavour indices. Recall that the normalisation of $\tilde{\gamma}$ is such that a factor of $(16\pi^2)$ is extracted. This means that we can immediately write down the anomalous dimensions from our previous calculations as

$$(\vec{C}[\tilde{\gamma}]\vec{C}^{\dagger})^{\beta\alpha}_{\varphi\ell(1)} = -C_5^{\beta\rho}\frac{3\delta_{\rho\sigma}}{2}C_5^{*\sigma\alpha}$$
$$- C_{21}^{\beta\rho}\frac{3\delta_{\rho\sigma}}{2}C_{21}^{*\sigma\alpha} + C_A^{\beta\rho}\frac{\delta_{\rho\sigma}}{2}C_A^{*\sigma\alpha}\,, \quad (5.198)$$
$$(\vec{C}[\tilde{\gamma}]\vec{C}^{\dagger})^{\beta\alpha}_{\varphi\ell(3)} = C_5^{\beta\rho}\delta_{\rho\sigma}C_5^{*\sigma\alpha}$$
$$+ C_{21}^{\beta\rho}\delta_{\rho\sigma}C_{21}^{*\sigma\alpha} + C_A^{\beta\rho}\frac{\delta_{\rho\sigma}}{2}C_{21}^{*\sigma\alpha} - C_{21}^{\beta\rho}\frac{\delta_{\rho\sigma}}{2}C_A^{*\sigma\alpha}\,, \quad (5.199)$$
$$(\vec{C}[\tilde{\gamma}]\vec{C}^{\dagger})^{\beta\alpha}_{e\varphi} = C_5^{\beta\rho}\frac{3[\Gamma_e]_{\eta\alpha}\delta_{\rho\sigma}}{2}C_5^{*\sigma\eta}$$
$$+ 2[(C_{21} - C_A)C_5^*\Gamma_e^{(2)}]^{\beta\alpha}$$
$$+ \frac{1}{2}[(C_AC_A^* + C_AC_{21}^* - C_{21}C_A^* - C_{21}C_{21}^*)\Gamma_e]^{\beta\alpha}\,, \quad (5.200)$$
$$(\vec{C}[\tilde{\gamma}]\vec{C}^{\dagger})^{\rho\sigma\beta\alpha}_{\ell\ell} = -C_5^{\beta\rho}\frac{1}{2}C_5^{*\sigma\alpha}$$
$$- C_{22}^{\beta\rho}\frac{1}{2}C_{22}^{*\sigma\alpha} - C_{21}^{\beta\rho}C_{21}^{*\sigma\alpha} + C_A^{\beta\rho}C_A^{*\sigma\alpha}\,, \quad (5.201)$$

where the operator label and flavour indices on the left-hand-side refer to the dimension-six operator (the dimension-five indices are summed over). The single Higgs model can be easily retrieved by setting $C_{21} = C_A = C_{22} = 0$ in the equations above.

In the next section, we will need the RGEs for dimension-five operators. Recall that in the single Higgs model, $[\gamma]$ is in principle a 9×9 matrix (or 6×6, if one uses the symmetry of $C_5^{\alpha\beta}$), mixing the elements of C_5 amongst themselves. However, in the basis where the charged leptons are diagonal, $[\gamma]$ is diagonal, and the anomalous dimension for the coefficient $C_5^{\alpha\beta}$ of the Weinberg operator is [101]

$$\gamma = -\frac{3}{2}([\Gamma_e]^2_{\alpha\alpha} + [\Gamma_e]^2_{\beta\beta}) + (\lambda - 3g_2 + 2\mathrm{Tr}(3[\Gamma_u]^\dagger[\Gamma_u] + 3[\Gamma_d]^\dagger[\Gamma_d] + [\Gamma_e]^\dagger[\Gamma_e]))\,, \quad (5.202)$$

where the Higgs self-interaction in the SM Lagrangian is $\frac{\lambda}{4}(\varphi^\dagger\varphi)^2$, and $[\Gamma_f]$ are the fermion Yukawa matrices.

5.9 Phenomenology

To solve the RGEs, it is convenient to define $t = \frac{1}{16\pi^2}\ln\frac{\mu}{M_W}$, in which case the 1-loop RGEs for dimension-five and -six operator coefficients can be written as

$$\begin{aligned}\frac{d}{dt}\tilde{C} &= \tilde{C}\cdot\hat{\gamma} + \vec{C}\cdot[\tilde{\gamma}]\cdot\vec{C}^{\dagger}\,,\\ \frac{d}{dt}\vec{C} &= \vec{C}\cdot[\gamma]\,.\end{aligned} \quad (5.203)$$

These differential equations have solutions of the form

$$\vec{C}(t_f) = \vec{C}(0)\exp\{\gamma t_f\} \simeq \vec{C}(0)\Big[1 + \gamma\frac{1}{16\pi^2}\ln\left(\frac{\Lambda}{M_W}\right) + ...\Big]\,, \quad (5.204)$$

$$\tilde{C}(t_f) \;=\; \Big[\int_0^{t_f} d\tau \vec{C}(0) e^{\gamma\tau} [\tilde{\gamma}] [e^{\gamma\tau}]^T \vec{C}^\dagger(0) e^{-\hat{\gamma}\tau} + \tilde{C}(0)\Big] e^{\hat{\gamma} t_f}\,, \tag{5.205}$$

where $16\pi^2 t_f = \ln\left(\frac{\Lambda}{M_W}\right)$. In these solutions, the anomalous dimension matrices are approximated as constant. This is not a good approximation, because the anomalous dimensions depend on running coupling constants. In particular, the Yukawa couplings can evolve significantly above M_W.

A simple solution to Equation (5.205) can be obtained by expanding the exponentials under the integral, as in Equation (5.204):

$$\tilde{C}(M_W) = \tilde{C}(\Lambda) - \tilde{C}(\Lambda)\hat{\gamma}\frac{1}{16\pi^2}\ln\frac{\Lambda}{M_W} - \vec{C}(\Lambda)[\tilde{\gamma}]\vec{C}^\dagger(\Lambda)\frac{1}{16\pi^2}\ln\frac{\Lambda}{M_W} + \ldots \tag{5.206}$$

5.9.1 The Single Higgs Model

In the SM case, where there is only one Higgs doublet, there is only the Weinberg operator at dimension-five: a symmetric 3×3 matrix, whose entries are determined by neutrino masses and mixing angles (in the mass basis of charged leptons). We now want to estimate the contribution of double-insertions of this dimension-five operator to lepton-flavour violating processes.

We neglect the "Majorana phases" (which cannot be experimentally determined from neutrino oscillations), suppose that the lightest neutrino mass is negligible, and neglect the lepton Yukawas in the RGEs. Then, the RG running of $C_5^{\alpha\beta}$ between M_W and Λ can be approximated as a rescaling, with $\gamma \approx \lambda - 3g_2 + 6y_t^2 \approx 3.5$:

$$C_5^{\alpha\beta}(\Lambda) = C_5^{\alpha\beta}(M_W)\Big[1 + 3.5\frac{1}{16\pi^2}\ln\frac{\Lambda}{M_W} + ...\Big]\,, \tag{5.207}$$

where y_t is the Yukawa eigenvalue of the top quark. For $\Lambda \leq 10^{16}$ GeV, $\ln\frac{\Lambda}{M_W} \leq 33$.

We can now estimate the contribution of the neutrino mass operator to lepton flavour violating processes from Equation (5.206). We neglect $\tilde{C}(\Lambda)$ and find that the contribution is $\frac{1}{16\pi^2}\ln\frac{\Lambda}{M_W} \times (\vec{C}[\tilde{\gamma}]\vec{C}^\dagger)$, where the coefficients are given in Equations (5.198) to (5.201). Therefore, the contribution is of order

$$\tilde{C}(M_W) \sim \frac{C_5^2}{16\pi^2}\ln\frac{\Lambda}{M_W}\,. \tag{5.208}$$

As expected, this is negligibly small, since $C_5^2/\Lambda^2 \sim m_\nu^2/v^4$.

5.9.2 The Two Higgs Doublet Model

Experimental neutrino data constrain the dimension-five operator in the one Higgs doublet model, so the lepton flavour violating effects estimated in Equation (5.208) are suppressed by the smallness of the neutrino masses. The situation changes in an extended Higgs sector, where more than one dimension-five operator is present. The operator $\mathcal{O}_A$ cannot contribute to neutrino masses as it is anti-symmetric in flavour space and is hence unconstrained. In addition, the neutrino mass contribution of operators $\mathcal{O}_{21}$ and $\mathcal{O}_{22}$ is suppressed if the VEV of the second Higgs doublet is small. Renormalisation group effects [101–103] will in general mix all operators, which could lift these suppression mechanisms at loop level. However, the mixing factorises in the limit where λ_6, λ_7 and $\Gamma_e^{(2)}$ (as defined in Equation (5.31)) tend to zero: then the operators $\mathcal{O}_{21}$ and $\mathcal{O}_A$ will not mix into $\mathcal{O}_5$ and $\mathcal{O}_{22}$ and are hence not constrained by the observed neutrino masses. Furthermore, the mixing of $\mathcal{O}_{22}$ into $\mathcal{O}_5$ vanishes in the limit where λ_5 also tends to zero (see [114] for a symmetry argument).

In the following we will study the sensitivity of lepton flavour violating decays to these additional operators. We assume that the Wilson coefficients of the dimension-five operators are generated at $\Lambda = 10$ TeV, while all other dimension-six Wilson coefficients are zero at this scale. To avoid constraints from the observed neutrino masses, we consider the scenario where the second Higgs doublet has a negligible VEV and a mass at the weak scale. The Higgs sector could be assumed to be close to that of an inert two-Higgs doublet model [106–109] and the dangerous couplings λ_6, λ_7 and $\Gamma_e^{(2)}$ are not generated radiatively. Renormalisation group running will then generate non-zero Wilson coefficients of several dimension-six operators at $\mu \sim v$. Only those dimension-six operators that involve standard model particles are of interest to us, since the vanishing VEV of the second Higgs doublet will suppress the contribution of the other operators after spontaneous symmetry breaking. Applying the constraints of Table 5.3, and neglecting the small log $\ln(m_{22}/M_W)$, we find that the $\mu \to 3e$ decays provide the greatest sensitivity to the additional dimension-five Wilson coefficients. In particular, the left-handed contribution implies

$$\left| C_{21}^{ee} C_{21}^{e\mu *} + 0.5 C_{22}^{ee} C_{22}^{e\mu *} + 0.1 \sum_\sigma \left(C_A^{e\sigma} - C_{21}^{e\sigma}\right)\left(C_A^{\sigma\mu *} + C_{21}^{\sigma\mu *}\right) \right| < \frac{1}{5.2 \ln\left(\Lambda/m_{22}\right)} \left(\frac{\Lambda}{10\,\mathrm{TeV}}\right)^2 , \tag{5.209}$$

where we neglected the mixing of the dimension-five operators amongst themselves, as this would contribute at 2-loop order to the lepton flavour violating processes. For the right-handed contribution, we find

$$\left| \sum_\sigma \left(C_A^{e\sigma} - C_{21}^{e\sigma}\right)\left(C_A^{\sigma\mu *} + C_{21}^{\sigma\mu *}\right) \right| < \frac{1.6}{\ln\left(\Lambda/m_{22}\right)} \left(\frac{\Lambda}{10\,\mathrm{TeV}}\right)^2 , \tag{5.210}$$

which exhibits a weaker sensitivity. The contribution of the $\mu \leftrightarrow e$ flavour changing Z vertex to $\mu \to e\gamma$ is relatively suppressed by a loop factor, so is beyond current experimental sensitivity. However, this Z vertex contributes at tree-level to $\mu \to e$ conversion, in interference with vector and scalar 2-quark 2-lepton operators. Indeed, the current sensitivity of $\mu \to e$ conversion in gold is $\left|C^{e\mu}_{\varphi\ell(1)} + C^{e\mu}_{\varphi\ell(3)}\right| \approx 1.4 \times 10^{-7}(\Lambda/m_t)^2$. The resulting constraints on the Wilson coefficients reads:

$$\left|\sum_\sigma (C^{e\sigma}_A - C^{e\sigma}_{21})(C^{\sigma\mu *}_A + C^{\sigma\mu *}_{21})\right| < \frac{1}{6.5\ln(\Lambda/m_{22})}\left(\frac{\Lambda}{10\,\text{TeV}}\right)^2 . \tag{5.211}$$

Taking the two most stringent bounds from above, and using the values $\Lambda = 10$ TeV, $m_{22} = M_W = 80.4$ GeV, we obtain the constraints

$$\begin{aligned} \left|C^{ee}_{21}C^{e\mu *}_{21} + 0.5C^{ee}_{22}C^{e\mu *}_{22} + 0.1\sum_\sigma \left(C^{e\sigma}_A - C^{e\sigma}_{21}\right)\left(C^{\sigma\mu *}_A + C^{\sigma\mu *}_{21}\right)\right| &< \frac{1}{25}, \\ \left|\sum_\sigma (C^{e\sigma}_A - C^{e\sigma}_{21})(C^{\sigma\mu *}_A + C^{\sigma\mu *}_{21})\right| &< \frac{1}{30}. \end{aligned} \tag{5.212}$$

5.10 Experimental Bounds on Coefficients

The aim of this section is to obtain experimental constraints on the coefficients of the LFV operators of Equation (5.7), evaluated at the weak scale M_W. We are interested in this subset of operators because they are generated at 1-loop by double-insertions of dimension-five LNV operators. Such constraints will allow an estimation of the sensitivity of LFV processes to the coefficients of LNV operators. We neglect the constraints on 2-lepton-2-quark operators, which are beyond the scope of this work, and focus on $\tau \leftrightarrow e$ and $\tau \leftrightarrow \mu$ flavour changes, since $\mu \leftrightarrow e$ is discussed in [85, 115]. Nonetheless, some $\mu \leftrightarrow e$ bounds are listed for completeness.

Three ways to relate low-energy experimental bounds to the coefficients of operators at a higher scale are:

1. To calculate the sensitivity of an experimental process to a particular operator coefficient.
2. To express an experimental rate as a function of high-scale coefficients. Each coefficient that contributes at the experimental scale will become a linear combination of high scale coefficients due the renormalisation group mixing.
3. To obtain constraints on coefficients at the high scale. A sufficient number of experimental constraints must be combined in order to obtain a finite allowed region in coefficient space (no "flat directions"). Then the allowed region must be projected onto the various axes, in order to obtain constraints.

The third option is the most useful, but beyond the scope of this work. Instead, we partially follow the second option, as a contribution to the third. We consider experimental bounds on the dimension-six operators which are generated in RGE evolution by double-insertions of dimension-five operators that change lepton number. We aim

Process	Br<	Process	Br<
$Z \to e^{\pm}\mu^{\mp}$	7.5×10^{-7} [91]	$Z \to \tau^{\pm}\mu^{\mp}$	1.2×10^{-5} [92]
$Z \to e^{\pm}\tau^{\mp}$	9.8×10^{-6} [93]	$h \to e^{\pm}\mu^{\mp}$	3.5×10^{-4} [94]
$h \to \tau^{\pm}\mu^{\mp}$	1.5×10^{-2} [95]	$h \to e^{\pm}\tau^{\mp}$	6.9×10^{-3} [94]
$\tau \to ee\overline{e}$	2.7×10^{-8} [96]	$\tau \to e\mu\overline{\mu}$	2.7×10^{-8} [96]
$\tau \to \mu e\overline{e}$	1.8×10^{-8} [96]	$\tau \to \mu\mu\overline{\mu}$	2.1×10^{-8} [96]
$\tau \to ee\overline{\mu}$	1.5×10^{-8} [96]	$\tau \to \mu\mu\overline{e}$	1.7×10^{-8} [96]
$\mu \to 3e$	1×10^{-12} [97]	$\tau \to e\gamma$	3.3×10^{-8} [98]
$\tau \to \mu\gamma$	4.4×10^{-8} [98, 99]	$\mu \to e\gamma$	4.2×10^{-13} [90]

Table 5.2: Experimentally measured bounds on branching ratios of lepton flavour violating processes. These may be used to determine bounds on Wilson coefficients of lepton flavour violating dimension-six operators (see Table 5.3).

to quote these bounds at M_W. The processes in question are LFV Higgs and Z decays (which occur at the weak scale), and flavour-changing lepton decays at low energy (these bounds must be translated to the weak scale via the RGEs of QED and QCD). Therefore we will not succeed in our aim of setting constraints on coefficients at M_W, because the low-energy experimental bounds depend on many coefficients at the weak scale, and we do not include enough experimental bounds.

5.10.1 Rates and Calculations

In Table 5.2 we present bounds on branching ratios of LFV processes that have been experimentally studied. We then use these to determine the implied bounds on the magnitudes of Wilson coefficients of LFV SMEFT operators, shown in Table 5.3. After presenting these results, we discuss how they are obtained, including a discussion of matching between SMEFT operators and low-energy effective operators.

$Z \to l_\alpha \bar{l}_\beta$ Decay

When the Higgs obtains a VEV, the "penguin" operators $\mathcal{O}_{\varphi\ell(1)}$ and $\mathcal{O}_{\varphi\ell(3)}$ generate a vertex involving the Z and two charged leptons. If the flavour-changing Z-fermion vertex is written in a SM-like form, $-\overline{l_\alpha} Z^\mu \frac{g_2}{2\cos\Theta_W}\gamma_\mu(g_V - g_A\gamma_5)l_\beta$, then

$$g_V = g_A = -(C_{\varphi\ell(1)} + C_{\varphi\ell(3)})\frac{v^2}{2\Lambda^2}. \tag{5.213}$$

The branching ratio can be written as

$$Br(Z \to l_\alpha \overline{l_\beta}) = \frac{M_Z}{2.5\,\mathrm{GeV}} \frac{g_2^2}{48\pi\cos^2\Theta_W}(|g_V|^2 + |g_A|^2) \tag{5.214}$$

Process	$\frac{v^2}{2\Lambda^2}\lvert\sum C\rvert <$
$Z\to e^\pm\mu^\mp$	$\lvert C^{e\mu}_{\varphi\ell(1)}+C^{e\mu}_{\varphi\ell(3)}\rvert < 1.2\times10^{-3}$
$Z\to \tau^\pm\mu^\mp$	$\lvert C^{\mu\tau}_{\varphi\ell(1)}+C^{\mu\tau}_{\varphi\ell(3)}\rvert < 4.6\times10^{-3}$
$Z\to e^\pm\tau^\mp$	$\lvert C^{e\tau}_{\varphi\ell(1)}+C^{e\tau}_{\varphi\ell(3)}\rvert < 4.1\times10^{-3}$
$h\to e^\pm\mu^\mp$	$\lvert C^{\mu e}_{e\varphi}\rvert, \lvert C^{e\mu}_{e\varphi}\rvert < 2.5\times10^{-4}$
$h\to \tau^\pm\mu^\mp$	$\lvert C^{\mu\tau}_{e\varphi}\rvert, \lvert C^{\tau\mu}_{e\varphi}\rvert < 1.6\times10^{-3}$
$h\to e^\pm\tau^\mp$	$\lvert C^{e\tau}_{e\varphi}\rvert, \lvert C^{\tau e}_{e\varphi}\rvert < 1.1\times10^{-3}$
$\tau\to ee\bar{e}$	$\lvert 2C^{e\tau ee}_{\ell\ell}+g^e_L[C^{e\tau}_{\varphi\ell(1)}+C^{e\tau}_{\varphi\ell(3)}]-\delta C^{e\tau}_{penguin}\rvert < 2.8\times10^{-4}$
	$\lvert C^{e\tau ee}_{\ell e}+g^e_R[C^{e\tau}_{\varphi\ell(1)}+C^{e\tau}_{\varphi\ell(3)}]-\delta C^{e\tau}_{penguin}\rvert < 4.0\times10^{-4}$
$\tau\to e\mu\bar{\mu}$	$\lvert 2C^{e\tau\mu\mu}_{\ell\ell}+2C^{e\mu\mu\tau}_{\ell\ell}+g^e_L[C^{e\tau}_{\varphi\ell(1)}+C^{e\tau}_{\varphi\ell(3)}]-\delta C^{e\tau}_{penguin}\rvert < 4.0\times10^{-4}$
	$\lvert C^{e\tau\mu\mu}_{\ell e}+g^e_R[C^{\mu\tau}_{\varphi\ell(1)}+C^{e\tau}_{\varphi\ell(3)}]-\delta C^{e\tau}_{penguin}\rvert < 4.0\times10^{-4}$
$\tau\to \mu e\bar{e}$	$\lvert 2C^{\mu\tau ee}_{\ell\ell}+2C^{\mu ee\tau}_{\ell\ell}+g^e_L[C^{\mu\tau}_{\varphi\ell(1)}+C^{\mu\tau}_{\varphi\ell(3)}]-\delta C^{\mu\tau}_{penguin}\rvert < 3.2\times10^{-4}$
	$\lvert C^{\mu\tau ee}_{\ell e}+g^e_R[C^{\mu\tau}_{\varphi\ell(1)}+C^{\mu\tau}_{\varphi\ell(3)}]-\delta C^{\mu\tau}_{penguin}\rvert < 3.2\times10^{-4}$
$\tau\to \mu\mu\bar{\mu}$	$\lvert 2C^{\mu\tau\mu\mu}_{\ell\ell}+g^e_L[C^{\mu\tau}_{\varphi\ell(1)}+C^{\mu\tau}_{\varphi\ell(3)}]-\delta C^{\mu\tau}_{penguin}\rvert < 2.5\times10^{-4}$
	$\lvert C^{\mu\tau\mu\mu}_{\ell e}+g^e_R[C^{\mu\tau}_{\varphi\ell(1)}+C^{\mu\tau}_{\varphi\ell(3)}]-\delta C^{\mu\tau}_{penguin}\rvert < 3.5\times10^{-4}$
$\tau\to ee\bar{\mu}$	$\lvert 2C^{e\tau e\mu}_{\ell\ell}\rvert < 3.2\times10^{-4}$
$\tau\to \mu\mu\bar{e}$	$\lvert 2C^{\mu\tau\mu e}_{\ell\ell}\rvert < 3.2\times10^{-4}$
$\mu\to 3e$	$\lvert 2C^{e\mu ee}_{\ell\ell}+g^e_L[C^{e\mu}_{\varphi\ell(1)}+C^{e\mu}_{\varphi\ell(3)}]-\delta C^{e\mu}_{penguin}\rvert < 7.1\times10^{-7}$
	$\lvert C^{e\mu ee}_{\ell e}+g^e_R[C^{e\mu}_{\varphi\ell(1)}+C^{e\mu}_{\varphi\ell(3)}]-\delta C^{e\mu}_{penguin}\rvert < 1.0\times10^{-6}$
$\tau\to e\gamma$	$\lvert C^{\tau e*}_{e\gamma}+\frac{e\alpha_e y_t}{8\pi^3 y_\mu}C^{\tau e*}_{e\varphi}+\frac{eg^e_L}{16\pi^2}C^{e\tau}_{\varphi e}\rvert < 7.3\times10^{-6}$
	$\lvert C^{e\tau}_{e\gamma}+\frac{e\alpha_e y_t}{8\pi^3 y_\mu}C^{e\tau}_{e\varphi}+\frac{eg^e_R}{16\pi^2}[C^{e\tau}_{\varphi\ell(1)}+C^{e\tau}_{\varphi\ell(3)}]\rvert < 7.3\times10^{-6}$
$\tau\to \mu\gamma$	$\lvert C^{\tau\mu*}_{e\gamma}+\frac{e\alpha_e y_t}{8\pi^3 y_\mu}C^{\tau\mu*}_{e\varphi}+\frac{eg^e_L}{16\pi^2}C^{\mu\tau}_{\varphi e}\rvert < 8.1\times10^{-6}$
	$\lvert C^{\mu\tau}_{e\gamma}+\frac{e\alpha_e y_t}{8\pi^3 y_\mu}C^{\mu\tau}_{e\varphi}+\frac{eg^e_R}{16\pi^2}[C^{\mu\tau}_{\varphi\ell(1)}+C^{\mu\tau}_{\varphi\ell(3)}]\rvert < 8.1\times10^{-6}$
$\mu\to e\gamma$	$\lvert C^{\mu e*}_{e\gamma}+\frac{e\alpha_e y_t}{8\pi^3 y_\mu}C^{\mu e*}_{e\varphi}+\frac{eg^e_L}{16\pi^2}C^{e\mu}_{\varphi e}\rvert < 1.05\times10^{-8}$
	$\lvert C^{e\mu}_{e\gamma}+\frac{e\alpha_e y_t}{8\pi^3 y_\mu}C^{e\mu}_{e\varphi}+\frac{eg^e_R}{16\pi^2}[C^{e\mu}_{\varphi\ell(1)}+C^{e\mu}_{\varphi\ell(3)}]\rvert < 1.05\times10^{-8}$

Table 5.3: Bounds on operator coefficients of the SMEFT, evaluated at M_W, from the bounds listed in Table 5.2 on the processes listed in the first column. The bounds on coefficients of Hermitian operators ($\mathcal{O}_{\varphi\ell(1)}$, $\mathcal{O}_{\varphi\ell(3)}$, $\mathcal{O}_{\ell\ell}, \mathcal{O}_{\ell e}$) also apply to the conjugate coefficient. All the bounds apply to running coefficients evaluated at M_W, and are for $\Lambda \simeq m_t \simeq v/\sqrt{2}$. The combination of coefficients $C_{penguin}$ is defined in Equation (5.225) and before Equation (5.236), δ is defined after Equation (5.236), and $g^e_R = 2s^2_W$, $g^e_L = -1+2\sin^2\Theta_W$.

where 2.5 GeV is the Z width in the SM. Since $\mathcal{O}_{\varphi\ell(1)}$ and $\mathcal{O}_{\varphi\ell(3)}$ are Hermitian, the conjugate process $Z \to l_\beta \overline{l_\alpha}$ necessarily occurs at the same rate, so the branching ratio to the experimental final state is

$$\begin{aligned} Br(Z \to l_\alpha^\pm l_\beta^\mp) &= Br(Z \to l_\alpha \overline{l_\beta}) + Br(Z \to l_\beta \overline{l_\alpha}) \\ &= \frac{M_Z}{2.5\,\text{GeV}} \frac{g_2^2}{12\pi \cos^2 \Theta_W} |C_{\varphi\ell(1)}^{\alpha\beta} + C_{\varphi\ell(3)}^{\alpha\beta}|^2 \frac{v^4}{4\Lambda^4} \end{aligned} \tag{5.215}$$

and the bounds we obtain on the operator coefficients, evaluated at $\sim M_W$, are given in Table 5.3.

$h \to \ell_\alpha^+ e_\beta^-, e_\alpha^+ \ell_\beta^-$ Decays

The flavour-changing Higgs decays occur via the non-Hermitian operator $\mathcal{O}_{e\varphi}$. When the Higgs has a VEV, it induces the Feynman rules for a flavour-changing Higgs vertex with two fermions:

$$\frac{1}{\Lambda^2} C_{e\varphi}^{\alpha\beta} \mathcal{O}_{e\varphi}^{\alpha\beta} \longrightarrow i \frac{3C_{e\varphi}^{\alpha\beta} v^2}{2\sqrt{2}\Lambda^2} P_R \quad , \quad \frac{1}{\Lambda^2} C_{e\varphi}^{\beta\alpha *} \mathcal{O}_{e\varphi}^{\beta\alpha *} \longrightarrow i \frac{3C_{e\varphi}^{\beta\alpha *} v^2}{2\sqrt{2}\Lambda^2} P_L \,. \tag{5.216}$$

We calculate the flavour-changing branching ratio by comparing to $Br(h \to b\bar{b}) = 0.575 \pm 0.32$ (from the Appendix of the Higgs Working Group Report [116], for $m_h = 125.1$ GeV), assuming the Feynman rule for $hb\bar{b}$ is $-\frac{i}{\sqrt{2}} y_b(m_h) P_{L,R}$. We use a one-loop approximation [44] for the running b mass

$$y_b(m_h) \frac{v}{\sqrt{2}} = m_b(m_b) \left[\frac{\alpha(m_h)}{\alpha(m_b)} \right]^{\gamma_m^{(0)}/2\beta^{(0)}} \simeq 3.0 \text{ GeV} \tag{5.217}$$

where $\alpha(m_h) \simeq 0.12, \alpha(m_b) \simeq 0.23$, $\gamma_m^{(0)} = 8$, $\beta^{(0)} = 23/3$ and $m_b(m_b) = 4.2$ GeV.

The operator $\mathcal{O}_{e\varphi}$ is not Hermitian, but is always included in the Lagrangian +H.c.. So $C_{e\varphi}^{e\mu} \mathcal{O}_{e\varphi}^{e\mu} + \text{H.c.}$ will induce both $h \to e_L \overline{\mu_R}$ and $h \to \mu_R \overline{e_L}$ at the same rate:

$$\frac{Br(h \to \overline{e_L} \mu_R)}{Br(h \to b\bar{b})} = \frac{9|C_{e\varphi}^{e\mu}|^2 v^4}{24 y_b^2 \Lambda^4} \,, \tag{5.218}$$

where in the denominator there is a factor of 3 for quark colour sums, and a factor of 2 from the chiral projectors in the lepton decay. The experimental search sums the $e_L \overline{\mu_R}$ and $\mu_R \overline{e_L}$ final states, so we obtain

$$\frac{3v^4}{4} \frac{|C_{e\varphi}^{\alpha\beta}|^2}{\Lambda^4} \,,\ \frac{3v^4}{4} \frac{|C_{e\varphi}^{\beta\alpha}|^2}{\Lambda^4} \leq y_b^2(m_h) \frac{Br(h \to l_\alpha^\pm l_\beta^\mp)}{Br(h \to b\bar{b})} \tag{5.219}$$

and the resulting constraints are given in Table 5.3.

Low Energy Decays

The flavour-changing τ and μ decays listed in Table 5.3 occur at energies $\sim m_\mu, m_\tau$, so the decay rates are usually written in terms of the coefficients of dimension-six operators from the QCD×QED invariant basis appropriate at low energies. These "low energy" coefficients, which we denote by $\mathcal{C}$, can be expressed in terms of SMEFT coefficients at M_W by running them up to M_W, then matching the QCD×QED-invariant operator basis onto the SMEFT. This was performed in [85] for $\mu \to e\gamma$, so we use the results of [85] for the radiative decays studied below. Reference [115] studied the renormalisation group evolution, below the weak scale, of the coefficients which mediate $\mu \to e\bar{e}e$ (as well those for as $\mu \to e\gamma$ and $\mu \to e$ conversion). We use these results, combined with the weak-scale matching conditions of [85], for the discussion of three body leptonic decays of the τ and μ. The minor differences between μ and τ decays are also discussed below.

In the EFT below M_W, we use the basis of lepton flavour changing four-fermion operators introduced in [85, 86] for $\mu \leftrightarrow e$ flavour change.[5] The operators and coefficients have as subscripts their Lorentz structure (V, S, T) and the chiral projection operators of the two fermion bilinears, and the flavour indices of the four fermions as superscripts. We restrict to the dipole and vector operators and neglect the scalars and tensors, which will turn out to be irrelevant for our study of LFV operators generated by double-insertions of LNV operators. The four-fermion operator basis below M_W is

$$\begin{aligned}\delta\mathcal{L}_{4f} \;=\;& \sum_{\alpha\beta}\sum_{f}\left[\mathcal{C}^{\alpha\beta ff}_{V,LL}(\overline{e_\alpha}\gamma^\omega P_L e_\beta)(\overline{f}\gamma_\omega P_L f) + \mathcal{C}^{\alpha\beta ff}_{V,LR}(\overline{e_\alpha}\gamma^\omega P_L e_\beta)(\overline{f}\gamma_\omega P_R f)\right] + \text{H.c.}\\ &+\sum_{\alpha\beta\sigma\rho}\left[\mathcal{C}^{\alpha\beta\sigma\rho}_{V,LL}(\overline{e_\alpha}\gamma^\omega P_L e_\beta)(\overline{e_\sigma}\gamma_\omega P_L e_\rho)\right] + \text{H.c.}\,, \end{aligned} \tag{5.220}$$

where $\alpha\beta \in \{e\mu, \mu\tau, e\tau\}$, $f \in \{e, \mu, \tau, u, d, s, c, b\}$, and $\alpha\beta\sigma\rho \in \{e\tau e\mu, \mu\tau\mu e\}$. In addition, below M_W, we consider the photon dipole operators

$$\delta\mathcal{L}_{dipole} = \frac{m_\beta}{\Lambda^2}\left(\mathcal{C}^{\alpha\beta}_{D,L}\overline{e^\alpha_R}\sigma^{\rho\sigma}e^\beta_L F_{\rho\sigma} + \mathcal{C}^{\alpha\beta}_{D,R}\overline{e^\alpha_L}\sigma^{\rho\sigma}e^\beta_R F_{\rho\sigma}\right) + \text{H.c.}\,, \tag{5.221}$$

because the SMEFT operators $\mathcal{O}_{\varphi\ell(1)}$, $\mathcal{O}_{\varphi\ell(3)}$ and $\mathcal{O}_{e\varphi}$ match onto the dipole at M_W. The current bounds on $\mu \to e\gamma$, $\tau \to e\gamma$ and $\tau \to \mu\gamma$ will give the best sensitivity to the coefficients $C_{\varphi\ell(1)}$, $C_{\varphi\ell(3)}$ and $C_{e\varphi}$.

$\tau \to 3l$ and $\mu \to 3e$

The first step is to translate the experimental bounds into constraints on operator coefficients at the experimental scale. For the three-body leptonic decays of the τ, it is

[5] In this basis, the flavour indices are written explicitly, so the normalisation of 1/2 is absent.

convenient to define

$$\widetilde{Br}(\tau \to 3l) \equiv \frac{Br(\tau \to 3l)}{Br(\tau \to \mu\bar{\nu}\nu)} \tag{5.222}$$

(where $Br(\tau \to \mu\bar{\nu}\nu) = 0.174$ [32]). Then, $\widetilde{Br}(\tau \to 3l)$ can be directly compared to the branching ratio for $\mu \to 3e$ [86]:

$$\begin{aligned}
Br(\mu \to e\bar{e}e)\frac{4\Lambda^4}{v^4} = &\frac{|\mathcal{C}^{e\mu ee}_{S,LL}|^2 + |\mathcal{C}^{e\mu ee}_{S,RR}|^2}{8} + 2|\mathcal{C}^{e\mu ee}_{V,RR} + 4e\mathcal{C}^{e\mu}_{D,L}|^2 + 2|\mathcal{C}^{e\mu ee}_{V,LL} + 4e\mathcal{C}^{e\mu}_{D,R}|^2 \\
&+ (64\ln\frac{m_\mu}{m_e} - 136)(|e\mathcal{C}^{e\mu}_{D,R}|^2 + |e\mathcal{C}^{e\mu}_{D,L}|^2) + |\mathcal{C}^{e\mu ee}_{V,RL} + 4e\mathcal{C}^{e\mu}_{D,L}|^2 \\
&+ |\mathcal{C}^{e\mu ee}_{V,LR} + 4e\mathcal{C}^{e\mu}_{D,R}|^2 ,
\end{aligned} \tag{5.223}$$

where $\sqrt{2}G_F = 1/v^2$, and the generalisation to τ decays is straightforward after accounting for factors of two.

We use the same branching ratio as in Equation (5.223) for the decays $\tau \to ee\bar{e}, \tau \to \mu\mu\overline{\mu}, \tau \to e\mu\overline{\mu}$, and $\tau \to \mu e\overline{e}$, but make allowances for factors of two due to whether final states contain identical particles. Note that we calculate the decay rates in the approximation that all final state fermions are massless. If there are two identically-flavoured fermions in the final state (as in Equation (5.223)), and those fermions have the same chirality, there is an enhancement in the branching ratio by a factor of two compared to if they have opposite chiralities [78]. This can be seen in Equation (5.223), where the coefficient $\mathcal{C}_{V,LL}$ comes with a factor of 2 compared to $\mathcal{C}_{V,LR}$.

We set the dipole coefficients to zero, because they are better constrained by the radiative decays discussed in the next subsection (see Table 5.3). Consequently, each upper bound on a three-body leptonic decay of the τ or μ implies six independent constraints on operator coefficients (evaluated at the experimental scale). This can be seen from Equation (5.223), by setting $\mathcal{C}^{e\mu}_{D,L} = \mathcal{C}^{e\mu}_{D,R} = 0$, and then considering each of the remaining coefficients in turn. Those of interest to us are given in Table 5.4.

The operator coefficients $\mathcal{C}_X(m_\tau)$ given in Table 5.4 can be expressed in terms of coefficients at M_W using the one-loop RGEs [85, 115]:

$$\mu\frac{d}{d\mu}\mathcal{C}_I = \frac{\alpha}{4\pi}\mathcal{C}_J[\gamma_e]_{JI} \Rightarrow \mathcal{C}_I(m_\tau) = \mathcal{C}_J(M_W)\left[\delta_{JI} - \frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}[\gamma_e]_{JI} + ...\right] , \tag{5.224}$$

where $[\gamma_e]$ is the one-loop anomalous dimension matrix of QED, $\ln\frac{M_W}{m_\tau} = 3.85$, $\ln\frac{M_W}{m_\mu} = 6.64$ and the approximate solution neglects the running of α. The one-loop QED corrections involve photon exchange between two legs of the operator, which does not change the flavour or chiral indices, and also "penguin" diagrams, where two legs of the operator are closed in a loop, and a photon is attached, which turns into two external leg fermions. The "penguins" can change the chirality and flavour, and allow 2-lepton-2-quark operators to mix with the four-lepton operators. We therefore need a prescription for dealing with the quark-sector thresholds m_b, m_c and $\Lambda_{\rm QCD}$. We make

Process	$\widetilde{\rm Br} <$	$\frac{v^2}{2\Lambda^2}\lvert\mathcal{C}\rvert <$
$\tau \to ee\bar{e}$	1.6×10^{-7}	$\widetilde{\mathcal{C}}^{e\tau ee}_{V,LL} < 2.8 \times 10^{-4}, \widetilde{\mathcal{C}}^{e\tau ee}_{V,LR} < 4 \times 10^{-4}$
$\tau \to e\mu\bar{\mu}$	1.6×10^{-7}	$\widetilde{\mathcal{C}}^{e\tau\mu\mu}_{V,LR}, \widetilde{\mathcal{C}}^{e\tau\mu\mu}_{V,LL} < 4 \times 10^{-4}$
$\tau \to \mu e\bar{e}$	1.0×10^{-7}	$\widetilde{\mathcal{C}}^{\mu\tau ee}_{V,LR}, \widetilde{\mathcal{C}}^{\mu\tau ee}_{V,LL} < 3.2 \times 10^{-4}$
$\tau \to \mu\mu\bar{\mu}$	1.2×10^{-7}	$\widetilde{\mathcal{C}}^{e\tau\mu\mu}_{V,LL} < 2.5 \times 10^{-4}, \widetilde{\mathcal{C}}^{e\tau\mu\mu}_{V,LR} < 3.5 \times 10^{-4}$
$\tau \to ee\bar{\mu}$	8.6×10^{-8}	$\widetilde{\mathcal{C}}^{e\tau e\mu}_{V,LL} < 3.2 \times 10^{-4},$
$\tau \to \mu\mu\bar{e}$	1.0×10^{-7}	$\widetilde{\mathcal{C}}^{\mu\tau\mu e}_{V,LL} < 3.2 \times 10^{-4}$
$\mu \to ee\bar{e}$	1.0×10^{-12}	$\widetilde{\mathcal{C}}^{e\mu ee}_{V,LL} < 7.1 \times 10^{-7}, \widetilde{\mathcal{C}}^{e\mu ee}_{V,LR} < 10^{-6}$

Table 5.4: Bounds on some operator coefficients from three-body lepton decays, evaluated at the experimental scale.

the simplest approximation, which is to have a single low-energy threshold at m_τ, and run from $M_W \to m_\tau$ with five quark flavours, and use this low-energy scale also for the decays of the μ. In this approximation, it is convenient to define the combination of operator coefficients

$$\mathcal{C}^{\alpha\beta}_{penguin} = -\frac{4N_c}{3}\sum_q Q_q(\mathcal{C}^{\alpha\beta qq}_{V,LL} + \mathcal{C}^{\alpha\beta qq}_{V,LR}) + \frac{4}{3}\sum_l([1 + \delta_{\alpha l} + \delta_{\beta l}]\mathcal{C}^{\alpha\beta ll}_{V,LL} + \mathcal{C}^{\alpha\beta ll}_{V,LR})\,, \quad (5.225)$$

where $l \in \{e, \mu, \tau\}$, $q \in \{u, d, s, c, b\}$, and Q_q is the electric charge of the quark. Then the coefficients constrained in table 5.4 can be written

$$\mathcal{C}^{e\mu ee}_{V,LR}(m_\tau) = \left[1 + 12\frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\right]\mathcal{C}^{e\mu ee}_{V,LR}(M_W) - \frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\mathcal{C}^{e\mu}_{penguin}(M_W)\,, \quad (5.226)$$

$$\mathcal{C}^{e\mu ee}_{V,LL}(m_\tau) = \left[1 - 12\frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\right]\mathcal{C}^{e\mu ee}_{V,LL}(M_W) - \frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\mathcal{C}^{e\mu}_{penguin}(M_W)\,, \quad (5.227)$$

$$\mathcal{C}^{e\tau ll}_{V,LR}(m_\tau) = \left[1 + 12\frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\right]\mathcal{C}^{e\tau ll}_{V,LR}(M_W) - \frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\mathcal{C}^{e\tau}_{penguin}(M_W)\,, \quad (5.228)$$

$$\mathcal{C}^{e\tau ll}_{V,LL}(m_\tau) = \left[1 - 12\frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\right]\mathcal{C}^{e\tau ll}_{V,LL}(M_W) - \frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\mathcal{C}^{e\tau}_{penguin}(M_W)\,, \quad (5.229)$$

$$\mathcal{C}^{\mu\tau ll}_{V,LR}(m_\tau) = \left[1 + 12\frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\right]\mathcal{C}^{\mu\tau ll}_{V,LR}(M_W) - \frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\mathcal{C}^{\mu\tau}_{penguin}(M_W)\,, \quad (5.230)$$

$$\mathcal{C}^{\mu\tau ll}_{V,LL}(m_\tau) = \left[1 - 12\frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\right]\mathcal{C}^{\mu\tau ll}_{V,LL}(M_W) - \frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\mathcal{C}^{\mu\tau}_{penguin}(M_W)\,, \quad (5.231)$$

$$\mathcal{C}^{\mu\tau\mu e}_{V,LL}(m_\tau) = \left[1 - 12\frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\right]\mathcal{C}^{\mu\tau\mu e}_{V,LL}(M_W)\,, \quad (5.232)$$

$$\mathcal{C}^{e\tau e\mu}_{V,LL}(m_\tau) = \left[1 - 12\frac{\alpha}{4\pi}\ln\frac{M_W}{m_\tau}\right]\mathcal{C}^{e\tau e\mu}_{V,LL}(M_W)\,. \quad (5.233)$$

Finally, the combinations of coefficients that are constrained by data can be matched at M_W onto coefficients of SMEFT operators [85]:[6]

$$\begin{aligned}
\mathcal{C}^{e\tau e\mu}_{V,LL}(M_W) &= 2C^{e\tau e\mu}_{\ell\ell}(M_W)\,,\\
\mathcal{C}^{\mu\tau\mu e}_{V,LL}(M_W) &= 2C^{\mu\tau\mu e}_{\ell\ell}(M_W)\,,\\
\mathcal{C}^{\mu\tau ee}_{V,LL}(M_W) &= 2C^{\mu\tau ee}_{\ell\ell}(M_W) + 2C^{\mu ee\tau}_{\ell\ell}(M_W) + g^e_L[C^{\mu\tau}_{\varphi\ell(1)}(M_W) + C^{\mu\tau}_{\varphi\ell(3)}(M_W)]\,,\\
\mathcal{C}^{\mu\tau\mu\mu}_{V,LL}(M_W) &= 2C^{\mu\tau\mu\mu}_{\ell\ell}(M_W) + g^e_L[C^{\mu\tau}_{\varphi\ell(1)}(M_W) + C^{\mu\tau}_{\varphi\ell(3)}(M_W)]\,,\\
\mathcal{C}^{\mu\tau ll}_{V,LR}(M_W) &= C^{\mu\tau ll}_{\ell e}(M_W) + g^e_R[C^{\mu\tau}_{\varphi\ell(1)}(M_W) + C^{\mu\tau}_{\varphi\ell(3)}(M_W)]\,,\\
\mathcal{C}^{e\tau\mu\mu}_{V,LL}(M_W) &= 2C^{e\tau\mu\mu}_{\ell\ell}(M_W) + 2C^{e\mu\mu\tau}_{\ell\ell}(M_W) + g^e_L[C^{e\tau}_{\varphi\ell(1)}(M_W) + C^{e\tau}_{\varphi\ell(3)}(M_W)]\,,\\
\mathcal{C}^{e\tau ee}_{V,LL}(M_W) &= 2C^{e\tau ee}_{\ell\ell}(M_W) + g^e_L[C^{e\tau}_{\varphi\ell(1)}(M_W) + C^{e\tau}_{\varphi\ell(3)}(M_W)]\,,\\
\mathcal{C}^{e\tau ll}_{V,LR}(M_W) &= C^{e\tau ll}_{\ell e}(M_W) + g^e_R[C^{e\tau}_{\varphi\ell(1)}(M_W) + C^{e\tau}_{\varphi\ell(3)}(M_W)]\,,\\
\mathcal{C}^{e\mu ee}_{V,LL}(M_W) &= 2C^{e\mu ee}_{\ell\ell}(M_W) + g^e_L[C^{e\mu}_{\varphi\ell(1)}(M_W) + C^{e\mu}_{\varphi\ell(3)}(M_W)]\,,\\
\mathcal{C}^{e\mu ee}_{V,LR}(M_W) &= C^{e\mu ee}_{\ell e}(M_W) + g^e_R[C^{e\mu}_{\varphi\ell(1)}(M_W) + C^{e\mu}_{\varphi\ell(3)}(M_W)]\,,
\end{aligned} \tag{5.234}$$

where $l \in \{e, \mu\}$ in the above equations, and $g^e_R = 2\sin^2\Theta_W$, $g^e_L = -1 + 2\sin^2\Theta_W$. In order to match the "penguin" coefficient of Equation (5.225) onto coefficients of the SMEFT, matching conditions for operators with a quark bilinear are also required:

$$\begin{aligned}
\mathcal{C}^{\alpha\beta uu}_{V,LL}(M_W) &= C^{\alpha\beta uu}_{\ell q(1)}(M_W) - C^{\alpha\beta uu}_{\ell q(3)}(M_W) + g^u_L[C^{\alpha\beta}_{\varphi\ell(1)}(M_W) + C^{\alpha\beta}_{\varphi\ell(3)}(M_W)]\,,\\
\mathcal{C}^{\alpha\beta dd}_{V,LL}(M_W) &= C^{\alpha\beta dd}_{\ell q(1)}(M_W) + C^{\alpha\beta dd}_{\ell q(3)}(M_W) + g^d_L[C^{\alpha\beta}_{\varphi\ell(1)}(M_W) + C^{\alpha\beta}_{\varphi\ell(3)}(M_W)]\,,\\
\mathcal{C}^{\alpha\beta uu}_{V,LR}(M_W) &= C^{\alpha\beta uu}_{\ell u}(M_W) + g^u_R[C^{\alpha\beta}_{\varphi\ell(1)}(M_W) + C^{\alpha\beta}_{\varphi\ell(3)}(M_W)]\,,\\
\mathcal{C}^{\alpha\beta dd}_{V,LR}(M_W) &= C^{\alpha\beta dd}_{\ell d}(M_W) + g^d_R[C^{\alpha\beta}_{\varphi\ell(1)}(M_W) + C^{\alpha\beta}_{\varphi\ell(3)}(M_W)]\,,
\end{aligned} \tag{5.235}$$

where $\alpha\beta \in \{\mu\tau, e\tau, e\mu\}$, $g^u_L = 1 - \frac{4}{3}\sin^2\Theta_W$, $g^u_R = -\frac{4}{3}\sin^2\Theta_W$, $g^d_L = -1 + \frac{2}{3}\sin^2\Theta_W$ and, $g^d_R = \frac{2}{3}\sin^2\Theta_W$. Combining the definition (5.225) with the matching conditions of Equation (5.235) allows the definition of a combination of SMEFT coefficients $C^{\alpha\beta}_{penguin}(M_W)$. Then, the experimental constraint on, for instance $\mathcal{C}^{e\tau\mu\mu}_{V,LL}(m_\tau)$, gives

$$\left|\Big[1 - 12\delta\Big]\Big[2C^{e\tau\mu\mu}_{\ell\ell} + 2C^{e\mu\mu\tau}_{\ell\ell} + g^e_L[C^{e\tau}_{\varphi\ell(1)} + C^{e\tau}_{\varphi\ell(3)}]\Big] - \delta C^{e\tau}_{penguin}\right| < 4\times 10^{-4}\,, \tag{5.236}$$

where all the coefficients are evaluated at M_W, and $\delta = \frac{\alpha}{4\pi}\log\frac{M_W}{m_\tau} \sim 1/400$. This, and other constraints from 3-body τ decays, are given in Table 5.3, where for compactness, $[1 \pm 12\delta]$ is approximated as 1.

[6] These equations differ from [85] due to different conventions for operator normalisation and signs, and also due to some errors in [85]. The SMEFT basis used here is normalised according to [8], where there are "redundant" flavour changing four-fermion operators, which are absent from the basis used below M_W in [85]. Then, the sign convention used here for the $g^f_{L,R}$ and the Z-vertex Feynman rule agrees with the PDG [32] but is opposite to that of [85]. Finally, in [85], there is an incorrect factor of 2 multiplying the penguin coefficients.

$l_\beta \to l_\alpha\gamma$

The radiative decays $l_\beta \to l_\alpha\gamma$ provide some of the most restrictive bounds on lepton flavour violation. The branching ratio at m_β can be written

$$\begin{aligned}\widetilde{Br}(l_\beta \to l_\alpha\gamma) &\equiv \frac{Br(l_\beta \to l_\alpha\gamma)}{Br(l_\beta \to l_\alpha\bar{\nu}\nu)} \\ &= 384\pi^2\frac{v^4}{4\Lambda^4}(|\mathcal{C}_{D,L}^{\alpha\beta}|^2 + |\mathcal{C}_{D,R}^{\alpha\beta}|^2) \leq \begin{cases} 4.2\times10^{-13} & \mu\to e\gamma \\ 2.0\times10^{-7} & \tau\to e\gamma \\ 2.5\times10^{-7} & \tau\to\mu\gamma \end{cases} , \qquad (5.237)\end{aligned}$$

where the low energy dipole operators are added to the Lagrangian as in Equation (5.221).

The dipole coefficients evaluated at the experimental scale can be expressed in terms of SMEFT coefficients at the weak scale as [85]

$$\mathcal{C}_{D,L}^{\alpha\beta}(m_\tau) = C_{e\gamma}^{\beta\alpha *}(M_W) + \frac{e\alpha y_t}{8\pi^3 y_\mu}C_{e\varphi}^{\beta\alpha *}(M_W) + \frac{eg_L^e}{16\pi^2}C_{\varphi e}^{\alpha\beta}(M_W) + \ldots, \qquad (5.238)$$

$$\begin{aligned}\mathcal{C}_{D,R}^{\alpha\beta}(m_\tau) = C_{e\gamma}^{\alpha\beta}(M_W) + \frac{e\alpha y_t}{8\pi^3 y_\mu}C_{e\varphi}^{\alpha\beta}(M_W) \\ + \frac{eg_R^e}{16\pi^2}[C_{\varphi\ell(1)}^{\alpha\beta}(M_W) + C_{\varphi\ell(3)}^{\alpha\beta}(M_W)] + \ldots, \qquad (5.239)\end{aligned}$$

where the contributions of scalar and tensor four-fermion operators were neglected, g_R^e and g_L^e are defined after Equation (5.234), and

$$C_{e\gamma}^{\alpha\beta} = \cos\Theta_W C_{eB}^{\alpha\beta} - \sin\Theta_W C_{eW}^{\alpha\beta} \,. \qquad (5.240)$$

Chapter 6

Three-Loop QCD Corrections to $K \to \pi\nu\overline{\nu}$

6.1 Introduction

Flavour-changing neutral currents (FCNC) are a sensitive probe of new physics. Such processes are forbidden in the SM at tree-level, and therefore receive loop suppression. In addition, when considering $\Delta S = 1$ processes with an internal top quark, there is a further suppression arising from CKM contributions: since the CKM matrix is nearly diagonal, FCNC processes of hadrons receive a large suppression [89]. Consequently, any SM contribution to such processes is very small, making it a promising channel for the detection of new physics, since even a small new physics contribution stands a chance of being distinguished from the SM background [117].

Two such processes are the semi-leptonic decay $K^+ \to \pi^+\nu\overline{\nu}$ and the CP-violating decay $K_L \to \pi^0\nu\overline{\nu}$. At the quark level, the processes are described by $s \to d\nu\overline{\nu}$, with the additional quarks being spectators to the weak decay. These are shown at leading order (which is an electroweak loop process) in Figure 6.1. These channels have been measured experimentally to a high precision [118, 119], whilst the theoretical calculation is also very clean, due to the minimal contribution from long-range physics [44]. This suppression of long-range contributions stems from the dominant contribution of internal top-quarks, which also introduce the CKM factors $V_{ts}^* V_{td}$, where [32]

$$|V_{td}| = (8.2 \pm 0.6) \times 10^{-3}, \qquad |V_{ts}| = (40.0 \pm 2.7) \times 10^{-3}, \tag{6.1}$$

providing an overall suppression of $\mathcal{O}(10^{-4})$.

Measured branching ratios for rare Kaon decays are collected by the Particle Data Group [87], with current experimental branching ratios being

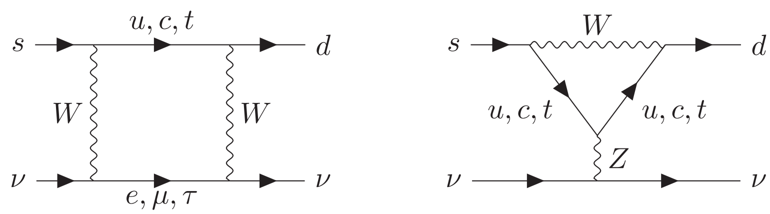


Fig. 6.1: Example box and penguin diagrams that contribute to the decays $K^+ \to \pi^+\nu\overline{\nu}$ and $K_L \to \pi^0\nu\overline{\nu}$ at leading order. Only box diagrams and Z-penguins contribute to the decays, and we only consider the box diagrams in this work.

$$Br(K^+ \to \pi^+\nu\overline{\nu}) = (1.7 \pm 1.1) \times 10^{-10} \qquad [118]\,,$$
$$Br(K_L \to \pi^0\nu\overline{\nu}) < 2.6 \times 10^{-8} \qquad [119]\,.$$

The KOTO experiment at JPARC is now online, and will be able to provide an update to $Br(K_L \to \pi^0\nu\overline{\nu})$. It has already seen a signal above the expected background, which gives a branching ratio of $Br(K_L \to \pi^0\nu\overline{\nu}) < 5.1 \times 10^{-8}$ [120], which is less stringent than the bound measured by the E391a collaboration and used by the PDG. We therefore use the limit from [87], and note that KOTO should be able to provide updated experimental values in the future.

In the SM the amplitude for $s \to d\nu\bar{\nu}$ may be split into three parts depending on the internal quark, u, c, t. At 1-loop level, this yields the structure [121]

$$\mathcal{A}_{\text{full}}(s \to d\nu\bar{\nu}) = \sum_{q=u,c,t} V_{qs}^* V_{qd} \mathcal{A}_q \sim \begin{cases} \mathcal{O}\big(\lambda^5 \frac{m_t^2}{M_W^2}\big) + i\mathcal{O}\big(\lambda^5 \frac{m_t^2}{M_W^2}\big) & (q = t) \\ \mathcal{O}\big(\lambda \frac{m_c^2}{M_W^2} \ln \frac{M_W}{m_c}\big) + i\mathcal{O}\big(\lambda^5 \frac{m_c^2}{M_W^2} \ln \frac{M_W}{m_c}\big) & (q = c) \\ \mathcal{O}\big(\lambda \frac{\Lambda_{\text{QCD}}^2}{M_W^2}\big) & (q = u) \end{cases}$$

where λ is the Cabbibo angle ($\lambda = 0.22$). Despite the CKM suppression, it can be seen that the top contributions are dominant due to the large top mass. For the $K^+ \to \pi^+\nu\bar{\nu}$ decay, whose branching ratio contains both $\text{Re}(\lambda_c)$ and λ_t, the charm sector provides a subleading contribution which cannot be neglected. However, the CP violating decay $K_L \to \pi^0\nu\bar{\nu}$ is sensitive to the imaginary parts only, and the imaginary part of the charm sector receives the same CKM suppression as the top sector. Therefore, for $K_L \to \pi^0\nu\bar{\nu}$, the top sector is completely dominant and up- and charm-contributions may be safely neglected.

K decays are described in EFT via the formalism of the weak Hamiltonian. Specifically, the 3-flavour weak Hamiltonian for these decays is given by [122, 123]

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_W} \sum_{l=e,\mu,\tau} \Big(\lambda_c X^l + \lambda_t X_t\Big) \left(\overline{s}_L \gamma^\mu d_L\right) \left(\overline{\nu}_{lL} \gamma_\mu \nu_{lL}\right) + \text{H.c.}, \qquad (6.2)$$

where $\lambda_i = V_{is}^* V_{id}$, α is the QED fine-structure constant, and Θ_W is the Weinberg angle. The functions X^l and X_t are the contributions from the charm- and top-sectors respectively, where $X^l \ll X_t$. It is interesting to note that the top contribution does not run after the top-quark is integrated out, whereas X^l runs above the charm scale, as semileptonic operators involving the charm-quark mix into X^l. Note that the effective Hamiltonian has no contribution proportional to λ_u, since it is eliminated using the GIM mechanism, in which the up-quark contribution is encoded in X^l and X_t. The disparity between X^l and X_t (which are separated by three orders of magnitude [44]) suggests that the charm contribution can be safely neglected in favour of the top sector. As discussed above, this is true for the decay $K_L \to \pi^0 \nu\overline{\nu}$, but for the case $K^+ \to \pi^+ \nu\overline{\nu}$ the top contribution receives a CKM suppression relative to the charm sector, which makes the inclusion of the charm sector necessary. Consequently, the effective Hamiltonian of Equation (6.2) is used for K^+ decay, while for K_L decay we use the Hamiltonian

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_W} \sum_{l=e,\mu,\tau} \lambda_t X_t \left(\overline{s}_L \gamma^\mu d_L\right) \left(\overline{\nu_l}_L \gamma_\mu \nu_{lL}\right) + \text{H.c.} \tag{6.3}$$

From the Hamiltonians above, it is possible to derive branching ratios for K decays in the effective theory. Branching ratios require the evaluation of hadronic matrix elements, typically performed on the lattice. However, it is possible to circumvent this difficulty in rare K decays through the use of isospin relations, which allow $Br(K^+ \to \pi^+ \nu\bar{\nu})$ and $Br(K_L \to \pi^0 \nu\bar{\nu})$ to be related to $Br(K^+ \to \pi^0 e^+ \nu)$, which is experimentally well known [87].

First, consider the K^+ decay. The effective Hamiltonian for $K^+ \to \pi^0 e^+ \nu$ is

$$\mathcal{H}_{\text{eff}}(K^+ \to \pi^0 e^+ \nu) = \frac{4G_F}{\sqrt{2}} V_{us}^* (\overline{s}_L \gamma^\mu u_L)(\overline{\nu_e}_L \gamma_\mu e_L) \tag{6.4}$$

and from isospin symmetry (assuming $m_u = m_d = 0$), there is the relation

$$\langle \pi^+ | \left(\overline{s}_L \gamma^\mu d_L\right) | K^+ \rangle = \sqrt{2} \langle \pi^0 | \left(\overline{s}_L \gamma^\mu u_L\right) | K^+ \rangle . \tag{6.5}$$

This leads to the ratio

$$\frac{Br(K^+ \to \pi^+ \nu\bar{\nu})}{Br(K^+ \to \pi^0 e^+ \nu)} = \frac{\alpha^2}{|V_{us}|^2 2\pi^2 \sin^4 \Theta_W} \sum_{l=e,\mu,\tau} \left| V_{cs}^* V_{cd} X^l + V_{ts}^* V_{td} X_t \right|^2 , \tag{6.6}$$

from which $Br(K^+ \to \pi^+ \nu\bar{\nu})$ can be obtained. Including isospin breaking effects, it is given by [122–125]

$$\begin{aligned} &Br(K^+ \to \pi^+ \nu\bar{\nu}(\gamma)) \\ &\qquad = \kappa_+ (1 + \Delta_{\text{EM}}) \left[\left(\frac{\text{Im}\lambda_t}{\lambda^5} X_t \right)^2 + \left(\frac{\text{Re}\lambda_c}{\lambda} (P_c + \delta P_{c,u}) + \frac{\text{Re}\lambda_t}{\lambda^5} X_t \right)^2 \right] , \end{aligned} \tag{6.7}$$

where κ_+ encodes the branching ratio of $K^+ \to \pi^0 e^+ \nu$ including isospin breaking effects, calculated in [126]. The parameters $P_c(X)$ and $\delta P_{c,u}$ (which are functions of the charm function X^l) describe the charm contributions, where $P(X)$ includes the short distance contributions, and $\delta P_{c,u}$ encodes long range contributions as well as contributions from dimension-eight operators. These are given by

$$P_c(X) = \frac{1}{\lambda^4}\left(\frac{2}{3}X^e + \frac{1}{3}X^\tau\right) , \tag{6.8}$$

which is given in [127], and

$$\delta P_{c,u} = 0.04 \pm 0.02 \, , \tag{6.9}$$

given in [124, 125, 128].

Similarly, the process $K_L \to \pi^0 \nu\bar{\nu}$ has the branching ratio [44, 124]

$$Br(K_L \to \pi^0 \nu\bar{\nu}) = \kappa_L \left(\frac{\text{Im}\lambda_t}{\lambda^5} X_t\right)^2 , \tag{6.10}$$

where the simple form is because only the top sector contributes. The parameter κ_L contains the hadronic matrix element (as κ_+ did for the K^+ decay), and is given in [129]. To include non-negligible contributions from channels that violate CP indirectly, we multiply the branching ratio by the factor [130]

$$1 - \sqrt{2}|\epsilon_K| \frac{1 + P_c(X)/(A^2 X_t) - \rho}{\eta} , \tag{6.11}$$

where ϵ_K describes indirect CP violation in the neutral Kaon system. A, λ, ρ and η are CKM parameters from the Wolfenstein parameterisation of the CKM matrix, defined as (see Section 2.6)

$$\lambda = |V_{us}| = 0.22 \, , \qquad A = V_{cb}/\lambda^2 \, , \qquad \rho = \frac{s_{13}}{s_{12}s_{23}} \cos\delta \, , \qquad \eta = \frac{s_{13}}{s_{12}s_{23}} \sin\delta \, . \tag{6.12}$$

The theoretical branching ratios given above both depend on the function X_t.

The loop functions X^l and X_t have been calculated previously, considering both QCD and electroweak corrections. The function X^l has been calculated to NNLO in QCD [131–133] and NLO in the electroweak theory [127], the result of which is a 2.5% theoretical uncertainty. Since the charm contribution to the $K^+ \to \pi^+ \nu\bar{\nu}$ branching ratio is only $\approx 30\%$, this represents only a small (sub-percent) uncertainty. The dominant function X_t has been calculated to NLO in both QCD [123,134,135] and the electroweak theory [124]. With these results the numerical value (to NLO in QCD and electroweak) is [124]

$$X_t = 1.469 \pm 0.017 \pm 0.002 \, , \tag{6.13}$$

where the first error is from QCD uncertainties and the second error is from electroweak uncertainties. Note that X_t is scale-invariant, since it is the Wilson coefficient of an operator involving a conserved quark current. Therefore, it has a vanishing anomalous dimension, and is μ-independent. As can be seen, whereas the electroweak uncertainty is at the per mille level, the QCD uncertainty still stands at the percent level. We here calculate the contribution of box diagrams to NNLO QCD corrections to X_t. The penguin contribution is also currently being calculated and the results will be presented in a future paper. Once these are brought together, an update to the value of X_t in Equation (6.13) will be possible, reducing the QCD uncertainties.

6.2 The Matching Calculation

The NNLO QCD correction of X_t is performed via a matching equation between the full SM and the 5-flavour effective theory to $\mathcal{O}(\alpha_s^2)$. The tree-level Hamiltonian for this theory is conventionally defined as

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \frac{\alpha}{2\pi\ \sin^2\Theta_W} V_{ts}^* V_{td} C_\nu Q_\nu + \text{H.c.}\,, \tag{6.14}$$

where C_ν can be identified with X_t, and Q_ν is

$$Q_\nu = \sum_{l=e,\mu,\tau} (\overline{s}_L \gamma^\mu d_L)(\overline{\nu_l}_L \gamma_\mu \nu_{lL})\,. \tag{6.15}$$

For the purposes of our calculation, we use the equivalent Hamiltonian

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \frac{\alpha}{2\pi\ \sin^2\Theta_W} \sum_{i=u,c,t} V_{is}^* V_{id} C_\nu^i Q_\nu + \text{H.c.}\,, \tag{6.16}$$

which reproduces Equation (6.14) upon utilising the unitarity of the CKM matrix. Explicitly,

$$\begin{aligned} &\lambda_t C_\nu = \lambda_u C_\nu^u + \lambda_c C_\nu^c + \lambda_t C_\nu^t = ((\lambda_u + \lambda_c) C_\nu^u + \lambda_t C_\nu^t) = \lambda_t (C_\nu^t - C_\nu^u) \\ \implies &C_\nu = C_\nu^t - C_\nu^u . \end{aligned} \tag{6.17}$$

The Wilson coefficients may be expanded in the strong coupling as

$$C_\nu^i = C_\nu^{i,(0)} + \frac{\alpha_s}{4\pi} C_\nu^{i,(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 C_\nu^{i,(2)} + \mathcal{O}(\alpha_s^3)\,, \qquad i \in \{u, t\}\,, \tag{6.18}$$

where the NLO QCD coefficients $C_\nu^{i,(1)}$ have been calculated in [123, 134, 135]. Note that the strong coupling $\alpha_s \equiv \alpha_s^{(5)}(\mu)$, which we also use in the full theory calculation. Unless stated otherwise, α_s is always evaluated at the scale μ. Since we use $\alpha_s^{(5)}$ in the full theory, we have to include threshold corrections that relate $\alpha_s^{(5)}$ and $\alpha_s^{(6)}$, as outlined in [80]. This is also true for $\overline{\text{MS}}$ renormalisation constants, and is a result of manually integrating out the top quark. Since the effective theory matches onto W-boxes and Z-penguins from the full theory (as in Figure 6.1), the Wilson coefficients may be written in terms of contributions from each of these sources,

$$C_\nu^{(n)} = C_\nu^{W,(n)} + C_\nu^{Z,(n)}\,. \tag{6.19}$$

We calculate here $C_\nu^{W,(n)}$ by matching box diagrams of the full theory onto the effective theory up to $n = 2$. However, since we do not consider penguin diagrams, we use the notation $C_\nu^{(n)} \equiv C_\nu^{W,(n)}$.

The box diagrams that arise in the full theory have either an up, charm, or top quark running inside the loop. Each of these sectors separately matches onto the Wilson coefficients C_ν^u, C_ν^c, and C_ν^t respectively. However, as already discussed, it is only necessary to calculate the up and top sectors, due to the unitarity of the CKM matrix.

The matching requirement is that both theories give the same amplitude for the process $K^+ \to \pi^+ \nu\overline{\nu}$ at the matching scale μ (where we recall that $\mu \equiv \mu_{\overline{\rm MS}}$),

$$\mathcal{A}_{\rm full}(s \to d\nu\overline{\nu}) = \mathcal{A}_{\rm eff}(s \to d\nu\overline{\nu}). \tag{6.20}$$

Therefore, there are two calculations that must be performed, in the full theory and the effective theory.

6.2.1 The Effective Theory Calculation

In addition to the physical operator Q_ν, we also need to consider the evanescent operator Q_E,

$$Q_E = \sum_{l=e,\mu,\tau} (\overline{s}_L \gamma^\mu \gamma^\nu \gamma^\lambda d_L)(\overline{\nu_l}_L \gamma_\mu \gamma_\nu \gamma_\lambda \nu_{lL}) - (16 + a\epsilon) Q_\nu \,. \tag{6.21}$$

The evanescent operator Q_E is a relic of dimensional regularisation, and while it identically vanishes in $d = 4$ dimensions, its inclusion is necessary when matching at loop-level in d-dimensions. This is because using massless quarks generates spurious IR divergences in both the effective and full theories, which require the presence of evanescent operators in intermediate steps [12]. The constant a is arbitrary, and amounts to a scheme definition [75]. We include it in the calculation, keeping it fully general, and see that it cancels in the final result. To renormalise the effective theory, it is necessary to replace the combination $\sum_{i=u,t} \lambda_i (C_\nu^i Q_\nu + C_E^i Q_E)$ by the expression [12, 135]

$$\sum_{i=u,t} \lambda_i (C_\nu^i Q_\nu + C_E^i Q_E) \to Z_\psi \sum_{i=u,t} \lambda_i \big(C_\nu^i Z_{\nu\nu} Q_\nu + C_\nu^i Z_{\nu E} Q_E + C_E^i Z_{E\nu} Q_\nu + C_E^i Z_{EE} Q_E \big) \,, \tag{6.22}$$

which is the full operator renormalisation mixing in a two-operator system, where Z_ψ is the $\overline{\rm MS}$ quark wavefunction renormalisation. The renormalisation matrix elements Z_{ij} may be expanded in the strong coupling,

$$Z_{ij} = \delta_{ij} + \left(\frac{\alpha_s}{4\pi}\right) Z_{ij}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z_{ij}^{(2)} + \mathcal{O}(\alpha_s^3), \tag{6.23}$$

where we can write the matrix as

$$Z = \begin{pmatrix} Z_{\nu\nu} & Z_{\nu E} \\ Z_{E\nu} & Z_{EE} \end{pmatrix} . \tag{6.24}$$

As a result of Equations (6.23) and (6.24), we can write the renormalisation constants as

$$\begin{aligned}
Z_{\nu\nu} &= 1 + \left(\frac{\alpha_s}{4\pi}\right) Z^{(1)}_{\nu\nu} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(2)}_{\nu\nu} + \mathcal{O}(\alpha_s^3)\,, \\
Z_{\nu E} &= \left(\frac{\alpha_s}{4\pi}\right) Z^{(1)}_{\nu E} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(2)}_{\nu E} + \mathcal{O}(\alpha_s^3)\,, \\
Z_{E\nu} &= \left(\frac{\alpha_s}{4\pi}\right) Z^{(1)}_{E\nu} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(2)}_{E\nu} + \mathcal{O}(\alpha_s^3)\,, \\
Z_{EE} &= 1 + \left(\frac{\alpha_s}{4\pi}\right) Z^{(1)}_{EE} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(2)}_{EE} + \mathcal{O}(\alpha_s^3)\,,
\end{aligned} \tag{6.25}$$

where it remains to calculate the loop contributions. However, there are immediately simplifications that can be made. Since the physical operator Q_ν contains a quark current that is conserved in the massless limit, the corresponding conserved charge must be renormalisation independent. Therefore, this operator cannot receive renormalisation corrections, since if these are non-zero, the value of the conserved charge would depend on the calculation order, which cannot be the case. Therefore, $Z_{\nu\nu}$ and $Z_{\nu E}$, which both renormalise the Q_ν operator, must have the values 1 and 0 respectively at all loop orders. We checked this explicitly up to $\mathcal{O}(\alpha_s^2)$, and write

$$\begin{aligned}
Z_{\nu\nu} &= 1\,, \\
Z_{\nu E} &= 0\,, \\
Z_{E\nu} &= \left(\frac{\alpha_s}{4\pi}\right) Z^{(1)}_{E\nu} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(2)}_{E\nu} + \mathcal{O}(\alpha_s^3)\,, \\
Z_{EE} &= 1 + \left(\frac{\alpha_s}{4\pi}\right) Z^{(1)}_{EE} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(2)}_{EE} + \mathcal{O}(\alpha_s^3)\,.
\end{aligned} \tag{6.26}$$

Other quantities in Equation (6.22) may also be expanded in powers of α_s:

$$Z_\psi = 1 + \left(\frac{\alpha_s}{4\pi}\right) Z^{(1)}_{\psi} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(2)}_{\psi} + \mathcal{O}(\alpha_s^3)\,, \tag{6.27}$$

$$C^i_E = C^{i,(0)}_E + \left(\frac{\alpha_s}{4\pi}\right) C^{i,(1)}_E + \left(\frac{\alpha_s}{4\pi}\right)^2 C^{i,(2)}_E + \mathcal{O}(\alpha_s^3)\,, \qquad i \in \{u, t\}\,, \tag{6.28}$$

and recall that C^i_ν may be expanded as in Equation (6.18). When performing the matching we consider the matrix elements $\langle Q_\nu \rangle$ and $\langle Q_E \rangle$, which can also be expanded in loops. However, since the effective theory contains only massless particles, and we work with vanishing external momenta, all such loop diagrams are vanishing in dimensional regularisation. Consequently, only the tree-level amplitudes $\langle Q_\nu \rangle^{(0)}$ and $\langle Q_E \rangle^{(0)}$ appear in the matching, and we do not need to consider higher-order matrix elements such as $\langle Q_\nu \rangle^{(1)}$ explicitly. Therefore, the quantities needed for the matching that arise in the effective theory are

$$Z^{(1)}_\psi, Z^{(2)}_\psi, Z^{(1)}_{E\nu}, Z^{(2)}_{E\nu}, Z^{(1)}_{EE}, Z^{(2)}_{EE}, C^{i,(0)}_\nu, C^{i,(1)}_\nu, C^{i,(2)}_\nu, C^{i,(0)}_E, C^{i,(1)}_E, C^{i,(2)}_E .$$

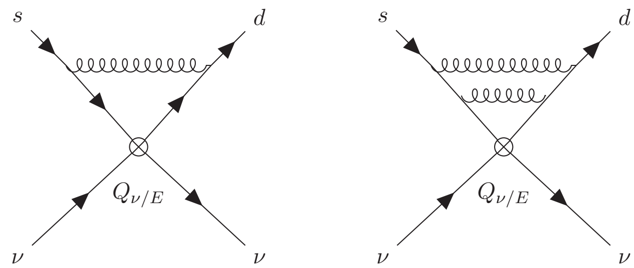


Fig. 6.2: Loop diagrams that must be evaluated to renormalise the operators Q_ν and Q_E.

The renormalisation matrix elements are obtained by renormalising diagrams involving QCD loops and the operators Q_ν and Q_E (see Figure 6.2), where external momenta must be included as a scale to ensure non-vanishing integrals. The Wilson coefficients are obtained order-by-order in the matching calculation that we perform here, with the ultimate aim being the calculation of $C_\nu^{(2)}$, which has not been calculated previously.

We briefly discuss the calculation of the renormalisation matrix of effective operators. We generate loop diagrams and amplitudes using *FeynArts*, which does not naturally handle four-fermion operators. Therefore, we use a trick in which we do not actually calculate the diagrams in Figure 6.2, but instead the corresponding diagrams in Figure 6.3. This is because the amplitudes are closely related, and the loop amplitudes of the effective operator can be reconstructed from the amplitudes found from these (SM) processes. This can be seen from the following. In the effective theory, the leptonic line is a spectator in QCD, and so gluons do not couple to it. Therefore, there is a two-to-one mapping between QCD loop diagrams in the effective theory and QCD diagrams involved in $s \to uW$. (The mapping is two-to-one since there is one physical and one evanescent insertion for each SM diagram drawn). Additionally, both approaches contain only massless quarks, and the W coupling forces the amplitudes of $s \to uW$ to have the same chirality structure as Q_ν and Q_E. The diagrams generated from $s\nu \to d\nu$ have the same Dirac structure as the quark current of Q_ν, and so to reconstruct the evanescent amplitude we insert additional Dirac matrices as required. The amplitudes from $s \to uW$ will have a different normalisation compared to the effective amplitudes. However, since these will be common to all amplitudes in each case, they do not affect the renormalisation constants computed.

It should be noted that this technique brings one more subtlety. The amplitudes obtained correspond to an insertion Q_ν and the "operator" $\sum_l (\overline{s}_L \gamma^\mu \gamma^\nu \gamma^\rho d_L)(\overline{\nu_l}_L \gamma_\mu \gamma_\nu \gamma_\rho \nu_{lL})$, which is part of the evanescent operator, but not equal to it. In fact, it is equal to $Q_E + (16 + a\epsilon)Q_\nu$ (see Equation (6.21)), and as such this needs to be accounted for when projecting the loop amplitudes onto the operators Q_ν and Q_E.

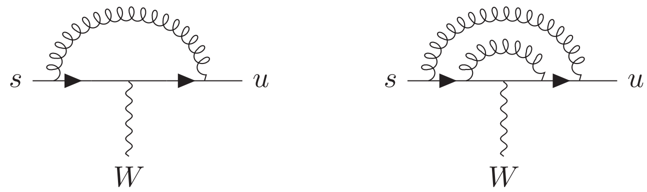


Fig. 6.3: Loop diagrams that were actually evaluated to renormalise the effective theory. The amplitudes from such processes can be simply related to those of Figure 6.2.

In performing this calculation, it was also necessary to include the renormalisation of SM parameters, which are presented below to $\mathcal{O}(\alpha_s^2)$ in the Feynman gauge [63]:

$$
\begin{aligned}
Z_g^{(1,1)} &= -\frac{11}{6}n_c + \frac{1}{3}n_f\,, \\
Z_G^{(1,1)} &= \frac{5}{3}n_c - \frac{2}{3}n_f\,, \\
Z_\psi^{(1,1)} &= -C_F\,, \\
Z_\psi^{(2,1)} &= \frac{3}{4}C_F^2 - \frac{17}{4}C_F n_c + \frac{1}{2}C_F n_f\,, \\
Z_\psi^{(2,2)} &= \frac{1}{2}C_F^2 + C_F n_c\,,
\end{aligned} \tag{6.29}
$$

where $n_c = 3$ denotes the number of colours, $n_f = 5$ is the number of active quark flavours, and the Casimir $C_F = \frac{4}{3}$.

Renormalisation of the effective theory up to $\mathcal{O}(\alpha_s^2)$ gives the renormalisation constants

$$
\begin{aligned}
Z_{\nu\nu} &= 1 + \mathcal{O}(\alpha_s^3)\,, \\
Z_{\nu E} &= 0 + \mathcal{O}(\alpha_s^3)\,, \\
Z_{E\nu} &= \frac{\alpha_s}{4\pi}\left(-\frac{12(n_c^2-1)}{n_c}\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{\epsilon}\frac{n_c^2-1}{n_c^2}(22n_c^2 - 4n_c n_f)\right. \\
&\qquad\left. + \frac{n_c^2-1}{3n_c^2}\left(-27 + (11a-14)n_c^2 - 2(5+a)n_c n_f\right)\right] + \mathcal{O}(\alpha_s^3)\,, \\
Z_{EE} &= 1 - \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{\epsilon}\frac{(n_c^2-1)(11n_c - 2n_f)}{3n_c}\right] + \mathcal{O}(\alpha_s^3)\,,
\end{aligned} \tag{6.30}
$$

which completes the renormalisation of the effective theory up to the required order.

6.2.2 The SM Calculation

Having completed the effective theory calculation, it remains to perform the full theory calculation up to $\mathcal{O}(\alpha_s^2)$, which corresponds to three loops in total. Example 3-loop diagrams that are evaluated are shown in Figure 6.4. We work in Feynman gauge, in which diagrams with Goldstone exchanges should, in principle, be included. However,

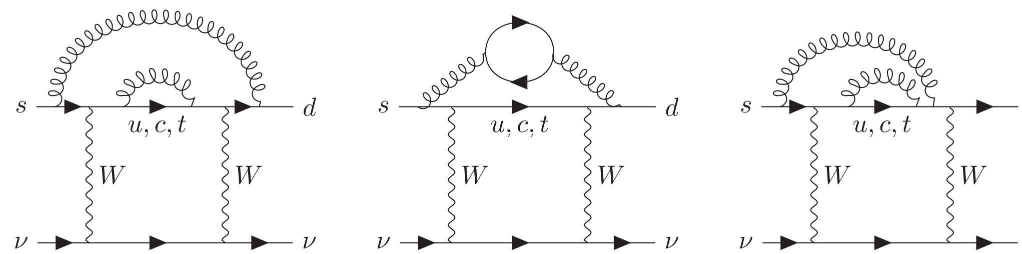


Fig. 6.4: Example 3-loop diagrams that arise from the $\mathcal{O}(\alpha_s^2)$ calculation in the SM. In total there are 248 diagrams that contribute to the process: 2 at $\mathcal{O}(\alpha_s^0)$, 14 at $\mathcal{O}(\alpha_s^1)$, and 232 at $\mathcal{O}(\alpha_s^2)$. The relatively small number of diagrams is partly due the presence of only a single quark line for gluons to couple to, restricting the number of topologies available.

since the coupling of Goldstones to fermions is proportional to the mass of the fermion, then Goldstones do not couple to the massless neutrinos, and so do not arise in the calculation.

As discussed previously, when working with massless up and charm quarks, diagrams that only differ by having an internal up or charm quark are equal up to CKM factors. Therefore, it is only necessary to compute with either the up or charm quark, in addition to the top quark. We choose to calculate the up and top sectors. Each of these sectors were computed and renormalised separately. We generated diagrams and amplitudes in *FeynArts*, using in-house code to perform manipulations to these inputs. We took all particles to be massless except the top quark and the W boson, and so generate 3-loop integrals with two different mass scales. By expanding in external momenta, these integrals were written in terms of vacuum integrals, with at most four massive propagators. See Chapter 3 for details of Feynman integral manipulation.

It was necessary to perform a reduction of the integrals to a basis set of master integrals. The first step in this was to identify all the different families of integrals that arose. Families were identified according to the mass content of their propagators, with, for example, an integral with two propagators with mass m and two propagators with mass M being distinct from an integral with four propagators of mass m. Note that m and M do not directly correspond to m_t and M_W, since in a single scale integral m can be used to indicate either m_t or M_W. In addition, integrals with the same mass content may belong to different families depending on which propagators the masses entered. Two different families with the same mass content can exist, where vacuum integral symmetries cannot be used to move between one family and the other. See Section 3.2 for a relevant discussion. We reproduce here the integral families that arose:

$$\mathcal{I}^{(3)}_{\text{vac}}(m^2, 0, 0, 0, 0, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),$$
$$\mathcal{I}^{(3)}_{\text{vac}}(0, 0, 0, 0, 0, m^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),$$

$$
\begin{aligned}
&\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, 0, 0, 0, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),\\
&\mathcal{I}^{(3)}_{\text{vac}}(m^2, 0, 0, 0, 0, m^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),\\
&\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, m^2, 0, 0, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),\\
&\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, 0, 0, m^2, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),\\
&\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, 0, 0, m^2, m^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),\\
&\mathcal{I}^{(3)}_{\text{vac}}(m^2, M^2, 0, 0, 0, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),\\
&\mathcal{I}^{(3)}_{\text{vac}}(m^2, 0, 0, 0, 0, M^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),\\
&\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, M^2, 0, 0, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),\\
&\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, 0, 0, M^2, 0; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),\\
&\mathcal{I}^{(3)}_{\text{vac}}(m^2, m^2, 0, 0, m^2, M^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6),
\end{aligned}
\tag{6.31}
$$

where

$$
\begin{aligned}
&\mathcal{I}^{(3)}_{\text{vac}}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6) = \\
&\qquad \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_1}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_2}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 q_3}{(2\pi)^4} \frac{1}{(q_1^2 - m_1^2)^{\nu_1} (q_2^2 - m_2^2)^{\nu_2} (q_3^2 - m_3^2)^{\nu_3}} \\
&\qquad \times \frac{1}{((q_1 - q_2)^2 - m_4^2)^{\nu_4} ((q_1 - q_3)^2 - m_5^2)^{\nu_5} ((q_2 - q_3)^2 - m_6^2)^{\nu_6}} .
\end{aligned}
\tag{6.32}
$$

Once all 3-loop integrals had been sorted into families, and vacuum integral symmetries had been used to bring them into the chosen form (one of the integrals of Equation (6.31)), it remained to reduce the integrals to a set of master integrals. We did this using the program `FIRE5` [47] in `Mathematica`, since it provided a simple pre-built module that performed the reduction in a good time, even without using the available C++ back-end machinery. However, the master integrals of `FIRE5` do not correspond to the master integrals of Martin and Robertson [13], whose results we wished to use on the final master integrals obtained. Therefore, we obtained linear relations connecting the master integrals of `FIRE5` and of [13], and used them to write our master integrals in the form of [13].

To find the above linear relations, we took basis master integrals used in [13], and reduced them in `FIRE5` in terms of `FIRE5` master integrals. From these reductions a matrix could be formed to relate masters belonging to the two different bases, which could be inverted to rewrite `FIRE5` master integrals in terms of master integrals used in [13]. For example, consider the case where we have two integrals (denoted by F_i) in the basis of `FIRE5` that do not coincide with the basis (denoted by G_i) of [13]. Then, reducing the integrals G_i in `FIRE5` gives

$$
\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} .
\tag{6.33}
$$

This approach can be easily generalised to n integrals. Given n master integrals of the Martin and Robertson basis, arranged in a vector $\vec{G}$, then upon reduction in `FIRE5`, these will be rewritten as a linear combination of master integral of the `FIRE5` basis, which can be arranged in a vector $\vec{F}$. Algebraically,

$$\vec{G} = \mathbf{M}\vec{F}\,, \tag{6.34}$$

where $\mathbf{M}$ specifies the linear relation. Since we wish to obtain $\vec{F}$, we can simply invert the above relation to find

$$\vec{F} = \mathbf{M}^{-1}\vec{G}\,. \tag{6.35}$$

This was done for each integral family, so that all integral results were in the correct basis to use the results of [13].

These integrals were evaluated using the results of [13], discussed in Section 3.5. These had analytic solutions for a large number of cases, but also provided a facility for numerical evaluation of integrals for which there are no known analytic solutions. Recall that the reduction of integrals writes an initial integral in terms of a linear combination of master integrals, with coefficients that are functions of the spacetime dimension d. Since we worked in $d = 4 - 2\epsilon$ dimensions, the prefactors were expanded in the small parameter ϵ up to $\mathcal{O}(\epsilon^3)$. This ensured that when the product of the prefactors and the 3-loop vacuum integrals was performed, only residual terms proportional to $\epsilon, \epsilon^2, \ldots$ were vanishing when the physical limit $d \to 4$ was taken, with no divergences remaining.

As mentioned before, the full theory calculation was performed using the strong coupling $\alpha_s^{(5)}$, despite there being six active flavours in the full theory. To compensate for this choice, it is necessary to include threshold corrections as outlined in Section 4.5. Following [12], we incorporate these threshold corrections in a simple way, by modifying renormalisation constants. We introduce the quantity

$$N_\epsilon = \left(\frac{\mu^2}{m_t^2}\right)^\epsilon e^{\gamma_E}\Gamma(1+\epsilon)\,, \tag{6.36}$$

which makes the renormalised α_s in the full SM equal to the $\overline{\text{MS}}$-renormalised α_s in the five-flavour effective theory, to all orders in ϵ. To implement the threshold corrections, in the full theory we use the renormalisation constants

$$Z_g = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left(-\frac{11}{6}n_c + \frac{1}{3}(n_f + N_\epsilon)\right) + \mathcal{O}(\alpha_s^2)\,, \tag{6.37}$$

$$\begin{aligned} Z_\psi = 1 &+ \frac{\alpha_s}{4\pi}(-C_F) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(\frac{1}{2}C_F^2 + C_F n_c\right)\right. \\ &\left. + \frac{1}{\epsilon}\left(\frac{3}{4}C_F^2 - \frac{17}{4}C_F n_c + \frac{1}{2}C_F n_f + \left(\frac{n_c^2-1}{4n_c}\right)N_\epsilon^2\right) - \frac{5}{6}\left(\frac{n_c^2-1}{4n_c}\right)N_\epsilon^2\right] \\ &+ \mathcal{O}(\alpha_s^3, \epsilon)\,, \end{aligned} \tag{6.38}$$

$$
\begin{aligned}
Z_{m_t} &= 1 + \frac{\alpha_s}{4\pi}\left(\frac{1}{\epsilon}\frac{-3(n_c^2-1)}{2n_c}\right) \\
&+ \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{1}{\epsilon^2}\left[\frac{70}{3} + \frac{n_c^2-1}{2n_c} - \frac{n_c^2-1}{n_c}N_\epsilon\right]\right. \\
&\qquad\left. + \frac{1}{\epsilon}\left[-\frac{253}{9} + \frac{5(n_c^2-1)}{12n_c}\right]\right) + \mathcal{O}(\alpha_s^3)\,. \qquad (6.39)
\end{aligned}
$$

These definitions of the renormalisation constants in the full theory are sufficient to consistently use $\alpha_s^{(5)}$ in the full theory.

6.2.3 The Matching Calculation

The matching is performed order-by-order in the strong coupling α_s, with a separate matching performed in the up and top sectors. We first consider the up sector. The full theory calculation produces very lengthy expressions, and so we schematically represent the matching equations here. Loop calculations in the full theory produce amplitudes containing the structures

$$
\mathcal{S}^\gamma = (\overline{u_d}\gamma^\mu P_L u_s)\otimes(\overline{u_\nu}\gamma_\mu P_L u_\nu)\,, \qquad \mathcal{S}^{3\gamma} = (\overline{u_d}\gamma^\mu\gamma^\nu\gamma^\rho P_L u_s)\otimes(\overline{u_\nu}\gamma_\mu\gamma_\nu\gamma_\rho P_L u_\nu)\,, \quad (6.40)
$$

with coefficients denoted by $\mathcal{A}^\gamma_{\text{full}}$ and $\mathcal{A}^{3\gamma}_{\text{full}}$ respectively. These coefficients may be expanded as

$$
\begin{aligned}
\mathcal{A}^{\gamma,u}_{\text{full}} &= \mathcal{A}^{\gamma,u(0)}_{\text{full}} + \frac{\alpha_s}{4\pi}\mathcal{A}^{\gamma,u(1)}_{\text{full}} + \left(\frac{\alpha_s}{4\pi}\right)^2\mathcal{A}^{\gamma,u(2)}_{\text{full}} + \mathcal{O}(\alpha_s^3)\,, \\
\mathcal{A}^{3\gamma,u}_{\text{full}} &= \mathcal{A}^{3\gamma,u(0)}_{\text{full}} + \frac{\alpha_s}{4\pi}\mathcal{A}^{3\gamma,u(1)}_{\text{full}} + \left(\frac{\alpha_s}{4\pi}\right)^2\mathcal{A}^{3\gamma,u(2)}_{\text{full}} + \mathcal{O}(\alpha_s^3)\,. \qquad (6.41)
\end{aligned}
$$

The leading order matching equation is

$$
\mathcal{A}^{\gamma,u(0)}_{\text{full}}\mathcal{S}^\gamma + \mathcal{A}^{3\gamma,u(0)}_{\text{full}}\mathcal{S}^{3\gamma} = C_\nu^{u,(0)}\langle Q_\nu\rangle^{(0)} + C_E^{u,(0)}\langle Q_E\rangle^{(0)}\,, \qquad (6.42)
$$

where it is necessary to project $\mathcal{A}^{3\gamma,u(0)}_{\text{full}}$ onto the physical and evanescent parts as in Equation (6.21). The leading order matching gives

$$
\begin{aligned}
C_\nu^{u,(0)} &= -1 - \epsilon\left(\frac{24+a}{16} + \ln\left(\frac{\mu^2}{M_W^2}\right)\right) \\
&\quad - \frac{\epsilon^2}{96}\left(168 + 8\pi^2 + 9a + (144+6a)\ln\left(\frac{\mu^2}{M_W^2}\right) + 48\ln^2\left(\frac{\mu^2}{M_W^2}\right)\right) + \mathcal{O}(\epsilon^3)\,, \\
C_E^{u,(0)} &= -\frac{1}{16} - \epsilon\left(\frac{3}{32} + \frac{1}{16}\ln\left(\frac{\mu^2}{M_W^2}\right)\right) \\
&\quad - \frac{\epsilon^2}{192}\left(21 + \pi^2 + 18\ln\left(\frac{\mu^2}{M_W^2}\right) - 6\ln^2\left(\frac{\mu^2}{M_W^2}\right)\right) + \mathcal{O}(\epsilon^3)\,. \qquad (6.43)
\end{aligned}
$$

We here keep terms up to $\mathcal{O}(\epsilon^2)$, since they are needed for the matching equation at $\mathcal{O}(\alpha_s^2)$.

The $\mathcal{O}(\alpha_s)$ matching equation is

$$\left(\mathcal{A}_{\text{full}}^{\gamma,u(1)}\mathcal{S}^{\gamma}+\mathcal{A}_{\text{full}}^{3\gamma,u(1)}\mathcal{S}^{3\gamma}\right)=\big(C_{\nu}^{u,(1)}\langle Q_{\nu}\rangle^{(0)}+Z_{\psi}^{(1)}C_{\nu}^{u,(0)}\langle Q_{\nu}\rangle^{(0)}$$
$$+C_{E}^{u,(1)}\langle Q_{E}\rangle^{(0)}+C_{E}^{u,(0)}Z_{\psi}^{(1)}\langle Q_{E}\rangle^{(0)}+C_{E}^{u,(0)}Z_{EE}^{(1)}\langle Q_{E}\rangle^{(0)}$$
$$+C_{E}^{u,(0)}Z_{E\nu}^{(1)}\langle Q_{E}\rangle^{(0)}\big)\,. \tag{6.44}$$

In principle, there could be more terms on the effective side, but due to the conservation of the quark current in the massless limit, the additional renormalisation constants are zero. All terms in this equation have been calculated in the preceding sections, apart from $C_{\nu}^{u,(1)}$ and $C_{E}^{u,(1)}$. These are determined to be

$$C_{\nu}^{u,(1)}=\frac{3(n_c^2-1)}{4n_c}+\epsilon\frac{(n_c^2-1)\left(280+14a+144\ln\left(\frac{\mu^2}{M_W^2}\right)\right)}{64n_c}+\mathcal{O}(\epsilon^2)\,, \tag{6.45}$$

$$C_{E}^{u,(1)}=\frac{7(n_c^2-1)}{32n_c}+\epsilon\frac{(n_c^2-1)\left(39+28\ln\left(\frac{\mu^2}{M_W^2}\right)\right)}{64n_c}+\mathcal{O}(\epsilon^2)\,. \tag{6.46}$$

Here only terms up to $\mathcal{O}(\epsilon)$ are necessary to be kept for the $\mathcal{O}(\alpha_s^2)$ matching equation.

Finally, there is the $\mathcal{O}(\alpha_s^2)$ matching equation,

$$\mathcal{A}_{\text{full}}^{\gamma,u(2)}\mathcal{S}^{\gamma}+\mathcal{A}_{\text{full}}^{3\gamma,u(2)}\mathcal{S}^{3\gamma}=C_{\nu}^{u,(2)}\langle Q_{\nu}\rangle^{(0)}+C_{\nu}^{u,(1)}Z_{\psi}^{(1)}\langle Q_{\nu}\rangle^{(0)}+C_{\nu}^{u,(0)}Z_{\psi}^{(2)}\langle Q_{\nu}\rangle^{(0)}$$
$$+C_{E}^{u,(2)}\langle Q_{E}\rangle^{(0)}+C_{E}^{u,(1)}Z_{\psi}^{(1)}\langle Q_{E}\rangle^{(0)}+C_{E}^{u,(1)}Z_{E\nu}^{(1)}\langle Q_{\nu}\rangle^{(0)}$$
$$+C_{E}^{u,(0)}Z_{\psi}^{(2)}\langle Q_{E}\rangle^{(0)}+C_{E}^{u,(0)}Z_{EE}^{(2)}\langle Q_{E}\rangle^{(0)}+C_{E}^{u,(0)}Z_{E\nu}^{(2)}\langle Q_{\nu}\rangle^{(0)}$$
$$+C_{E}^{u,(0)}Z_{\psi}^{(1)}Z_{E\nu}^{(1)}\langle Q_{\nu}\rangle^{(0)}\,. \tag{6.47}$$

The only unknown quantities in the above are $C_{\nu}^{u,(2)}$ and $C_{E}^{u,(2)}$, and so we can solve for these quantities. In the 3-loop case, things are considerably more complicated than lower order matching calculations. Firstly, integrals arise for which there are not analytic expressions for the finite pieces, and so these integrals can only be evaluated numerically. Secondly, those integrals that can be represented analytically are functions of dilogarithms. Additionally, the Wilson coefficients at $\mathcal{O}(\alpha_s^2)$ become functions of the top mass m_t, even in the up sector. This is because at 3-loop level, diagrams with a top loop can arise (see the middle diagram of Figure 6.4), which generates a mass dependence on the mass of the quark in the loop. Since $m_u=m_c=0$, only a dependence on the top mass is generated in this way, which we parameterise through the variable x as

$$x\equiv\frac{m_t^2}{M_W^2}\,. \tag{6.48}$$

We present here the NNLO QCD Wilson coefficient $C_{\nu}^{u,(2)}$ for the up sector. We have also calculated $C_{E}^{u,(2)}$, but since it is extremely lengthy, and only contributes to physical quantities at $\mathcal{O}(\alpha_s^3)$, we do not present it here. $C_\nu^{u,(2)}$ involves integrals for which analytic solutions are not available, and we leave them explicitly unevaluated:

$$
\begin{aligned}
C_\nu^{u,(3)} = & \frac{\left(n_c^2-1\right)\left(12x^2+10x+1\right)E\left(0,m_t^2,m_t^2,M_W^2\right)}{15M_W^4 n_c x} \\
& - \frac{\left(n_c^2-1\right)\left(6x^2-7x+1\right)G\left(0,0,m_t^2,m_t^2,M_W^2\right)}{15M_W^2 n_c} \\
& - \frac{\left(n_c^2-1\right)(24x+1)G\left(0,m_t^2,m_t^2,0,M_W^2\right)}{15M_W^2 n_c} \\
& + n_f\left(-\frac{\left(n_c^2-1\right)\ln\left(\frac{\mu^2}{M_W^2}\right)}{2n_c} - \frac{13\left(n_c^2-1\right)}{12n_c}\right) \\
& + \mathrm{Li}_2\left(1-\frac{1}{x}\right)\left(-\frac{\left(n_c^2-1\right)(6x-1)(x-1)^2\ln\left(\frac{\mu^2}{M_W^2}\right)}{15n_c}\right. \\
& \qquad \left. - \frac{\left(n_c^2-1\right)(6x-1)(x-1)^2}{15n_c} + \frac{\left(n_c^2-1\right)(6x-1)(x-1)^2\ln(x)}{15n_c}\right) \\
& - \frac{\left(n_c^2-1\right)\left(12x^3+36x^2+21x+2\right)\ln^3\left(\frac{\mu^2}{M_W^2}\right)}{30n_c} \\
& + \ln^2(x)\left(-\frac{\left(n_c^2-1\right)\left(36x^3+48x^2+13x-1\right)\ln\left(\frac{\mu^2}{M_W^2}\right)}{30n_c}\right. \\
& \qquad \left. - \frac{\left(n_c^2-1\right)\left(72x^3+114x^2-30x-1\right)}{30n_c}\right) \\
& + \ln(x)\left(\frac{\left(n_c^2-1\right)\left(18x^3+39x^2+13x+1\right)\ln^2\left(\frac{\mu^2}{M_W^2}\right)}{15n_c}\right. \\
& \quad + \frac{\left(n_c^2-1\right)\left(72x^3+180x^2+55x+4\right)\ln\left(\frac{\mu^2}{M_W^2}\right)}{15n_c} \\
& \quad \left. + \frac{\left(n_c^2-1\right)\left(2\pi^2\left(6x^3-x^2+24x+1\right)+3\left(300x^3+970x^2+707x-10\right)\right)}{180n_c}\right) \\
& + \frac{\left(n_c^2-1\right)\left(12x^3+6x^2+8x-1\right)\ln^3(x)}{30n_c} \\
& - \frac{\left(n_c^2-1\right)\left(144x^4+492x^3+280x^2+18x-1\right)\ln^2\left(\frac{\mu^2}{M_W^2}\right)}{60n_c x} \\
& + \frac{\left(n_c^2-1\right)\left(990n_c x-24\left(75+\pi^2\right)x^4-\left(7764-28\pi^2\right)x^3\right)}{360n_c x} \\
& + \frac{\left(n_c^2-1\right)\left(-2\left(2121+50\pi^2\right)x^2-\left(66+4\pi^2\right)x+39\right)\ln\left(\frac{\mu^2}{M_W^2}\right)}{360n_c x} \\
& - \frac{\left(n_c^2-1\right)\left(2184x^4+4x^3(960\zeta(3)+3683)-4838x^2\right)}{720n_c x}
\end{aligned}
$$

$$
+\frac{\left(n_c^2-1\right)\left(\pi^2\left(48x^4-56x^3+8x^2-4\right)+394x-133\right)}{720n_cx}
$$

$$
+\frac{1}{24}\left(113n_c^2-\frac{36}{n_c^2}-77\right)+\mathcal{O}(\epsilon)\,. \tag{6.49}
$$

The above result is presented at the generic scale μ. It is useful to check the μ-independence of the Wilson coefficients, which is guaranteed by the conserved quark current in the effective operator. For this purpose, it is useful to present $C_\nu^{u,(3)}$ in an alternative form, where the integrals are evaluated at the scale $\mu = M_W$ (which is the natural scale for the up sector matching), thereby moving the μ-dependence of the integrals into logarithms. This is done by using the differential equations [13]

$$
\begin{aligned}
\mu^2\frac{\partial}{\partial\mu^2}E(u,v,y,z) &= A(u)A(v)+A(u)A(y)+A(u)A(z)\\
&\quad +A(v)A(y)+A(v)A(z)+A(y)A(z)\\
&\quad +(u/2-v-y-z)A(u)+(v/2-u-y-z)A(v)\\
&\quad +(y/2-u-v-z)A(y)+(z/2-u-v-y)A(z)\\
&\quad +uv+uy+uz+vy+vz+yz\\
&\quad -9(u^2+v^2+y^2+z^2)/8\,,\\
\mu^2\frac{\partial}{\partial\mu^2}G(w,u,z,v,y) &= I(w,u,z)+I(w,v,y)+A(u)+A(v)+A(y)+A(z)\\
&\quad -2u-2v-2y-2z+w\,,
\end{aligned} \tag{6.50}
$$

where

$$
\begin{aligned}
A(x) &= x\left[\overline{\ln}(x)-1\right]\,,\\
\overline{\ln}(x) &= \ln\left(\frac{x}{\mu^2_{\overline{\rm MS}}}\right)\,,\\
I(x) &= I_0(x,y,z)-A_\epsilon(x)-A_\epsilon(u)-A_\epsilon(z)\,,
\end{aligned} \tag{6.51}
$$

and $I_0(x,y,z)$ is given in Equation (3.67). Since the integrals $A(x)$ and $I(x,y,z)$ have μ-depedence through their logarithms, the above differentials may simply be integrated to relate the integrals evaluated at different values of the scale μ. Therefore, we can write

$$
\begin{aligned}
C_\nu^{u,(3)}\big|_{\mu=M_W} &= -\frac{\left(n_c^2-1\right)48\left(6x^2-7x+1\right)G\left(0,0,m_t^2,m_t^2,M_W^2\right)\big|_{\mu=M_W}}{720M_W^2n_c}\\
&\quad +\frac{\left(n_c^2-1\right)48(24x+1)G\left(0,m_t^2,m_t^2,0,M_W^2\right)\big|_{\mu=M_W}}{720M_W^2n_c}\\
&\quad +\frac{\left(n_c^2-1\right)\left(2184x^3+4x^2(960\zeta(3)+3683)-4838x\right)}{720n_c}\\
&\quad +\frac{\left(n_c^2-1\right)\left(\pi^2\left(48x^4-56x^3+8x^2-4\right)+394x-133\right)}{720n_cx}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\left(n_c^2-1\right)\left(48\left(12x^2+10x+1\right)E\left(0,m_t^2,m_t^2,M_W^2\right)\big|_{\mu=M_W}\right)}{720M_W^4 n_c x}\\
&+n_f\left(-\frac{\left(n_c^2-1\right)\ln\left(\frac{\mu^2}{M_W^2}\right)}{2n_c}-\frac{13\left(n_c^2-1\right)}{12n_c}\right)\\
&+\mathrm{Li}_2\left(1-\frac{1}{x}\right)\left(-\frac{\left(n_c^2-1\right)(6x-1)(x-1)^2\ln\left(\frac{\mu^2}{M_W^2}\right)}{15n_c}\right.\\
&\left.-\frac{\left(n_c^2-1\right)(6x-1)(x-1)^2}{15n_c}+\frac{\left(n_c^2-1\right)(6x-1)(x-1)^2\ln(x)}{15n_c}\right)\\
&-\frac{\left(n_c^2-1\right)(x-1)^2(6x-1)\mathrm{Li}_2(1-x)\ln\left(\frac{\mu^2}{M_W^2}\right)}{15n_c}\\
&+\ln^2(x)\left(-\frac{\left(n_c^2-1\right)\left(72x^3+114x^2-30x-1\right)}{30n_c}\right.\\
&\left.-\frac{\left(n_c^2-1\right)(6x-1)(x-1)^2\ln\left(\frac{\mu^2}{M_W^2}\right)}{30n_c}\right)\\
&+\frac{\left(n_c^2-1\right)\left(12x^3+6x^2+8x-1\right)\ln^3(x)}{30n_c}\\
&+\frac{\left(n_c^2-1\right)\left(2\pi^2\left(6x^3-x^2+24x+1\right)\right)\ln(x)}{180n_c}\\
&+\frac{\left(n_c^2-1\right)\left(3\left(300x^3+970x^2+707x-10\right)\right)\ln(x)}{180n_c}\\
&+\frac{11}{4}\left(n_c^2-1\right)\ln\left(\frac{\mu^2}{M_W^2}\right)+\frac{1}{24}\left(113n_c^2-\frac{36}{n_c^2}-77\right)+\mathcal{O}(\epsilon)\,. \qquad (6.52)
\end{aligned}
$$

We now consider the μ-dependence of C_ν^u at leading order in ϵ, up to $\mathcal{O}(\alpha_s^2)$. Consider the derivative

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}C_\nu^u=\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left(C_\nu^{u,(0)}+\frac{\alpha_s}{4\pi}C_\nu^{u,(1)}+\left(\frac{\alpha_s}{4\pi}\right)^2C_\nu^{u,(2)}(\mu,x(\mu))\right), \qquad (6.53)$$

where we note from our above results that $C_\nu^{u,(0)}$ and $C_\nu^{u,(1)}$ are μ-independent. Then, we have

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}C_\nu^u=-2\beta_0\frac{\alpha_s^2}{4\pi}C_\nu^{u,(1)}+\left(\frac{\alpha_s}{4\pi}\right)^2\mu\frac{\partial}{\partial\mu}C_\nu^{u,(2)}(\mu,x(\mu))\,, \qquad (6.54)$$

where β_0 is the leading order QCD beta function,

$$\beta_0=\frac{11n_c-2n_f}{3}=\frac{23}{3}\,, \qquad \text{for } n_c=3,\ n_f=5\,. \qquad (6.55)$$

Note that since the mass anomalous dimension γ_{m_t} has a leading order at $\mathcal{O}(\alpha_s)$, such a term does not contribute at $\mathcal{O}(\alpha_s^2)$, since it is also multiplied by an α_s^2 from the expansion of C_ν^u. Then, it is just necessary to calculate $\mu\frac{\partial}{\partial\mu}C_\nu^{u,(2)}=\frac{\partial}{\partial\ln\mu}C_\nu^{u,(2)}$. Doing this gives

$$
\begin{aligned}
\frac{\partial}{\partial \ln \mu} C_\nu^{u,(2)} &= \frac{\left(n_c^2-1\right)\left(3960 n_c^2 x - 720 n_c n_f x - 96 n_c (x-1)^2 (6x-1) x \mathrm{Li}_2(1-x)\right)}{720 n_c^2 x} \\
&\quad + \frac{\left(n_c^2-1\right)\left(-96 n_c (x-1)^2 (6x-1) x \mathrm{Li}_2\left(\frac{x-1}{x}\right) - 288 n_c x^4 \ln^2(x)\right)}{720 n_c^2 x} \\
&\quad + \frac{\left(n_c^2-1\right)\left(624 n_c x^3 \ln^2(x) - 384 n_c x^2 \ln^2(x) + 48 n_c x \ln^2(x)\right)}{720 n_c^2 x} \\
&= \frac{\left(n_c^2-1\right)\left(11 n_c - 2 n_f\right)}{2 n_c} \,. \qquad (6.56)
\end{aligned}
$$

Combining this result with Equations (6.54), (6.45) and (6.55), we obtain

$$
\begin{aligned}
\mu \frac{\mathrm{d}}{\mathrm{d}\mu} C_\nu^u = \left(\frac{\alpha_s}{4\pi}\right)^2 \Bigg[&-2\left(\frac{11 n_c - 2 n_f}{3}\right)\left(\frac{3(n_c^2-1)}{4 n_c}\right) \\
&+ \frac{\left(n_c^2-1\right)\left(11 n_c - 2 n_f\right)}{2 n_c}\Bigg] = 0 + \mathcal{O}(\alpha_s^3, \epsilon) \,, \qquad (6.57)
\end{aligned}
$$

showing that the Wilson coefficient C_ν^u is μ-independent, as required.

So far we have only discussed the matching in the up sector. There is a similar matching equation in the top sector, where the effective theory calculation is exactly the same (but with λ_t instead of λ_u), and the full theory calculation has the same structure, but is more involved due to the presence of the heavy top mass at all orders. The matching equation in the top and the up sectors are exactly the same, with the replacement

$$
\mathcal{A}_{\text{full}}^{(3)\gamma,u} \to \mathcal{A}_{\text{full}}^{(3)\gamma,t}, \qquad C_\nu^u \to C_\nu^t, \qquad C_E^u \to C_E^t, \qquad (6.58)
$$

and so we do not reproduce the matching equations for the top sector explicitly. It should be noted that the renormalisation of the top sector requires the renormalisation constant Z_{m_t}, which receives threshold corrections, as given in Equation (6.39). The leading order matching calculation gives

$$
\begin{aligned}
C_\nu^{t,(0)} &= -\left[\frac{1}{1-x} + \frac{x \ln x}{(1-x)^2}\right] + \mathcal{O}(\epsilon) \,, \\
C_E^{t,(0)} &= -\frac{1}{16}\left[\frac{1}{1-x} + \frac{x \ln x}{(1-x)^2}\right] + \mathcal{O}(\epsilon) \,, \qquad (6.59)
\end{aligned}
$$

and the next-to-leading order gives

$$
\begin{aligned}
C_\nu^{t,(1)} &= \frac{\left(n_c^2-1\right)\left(-12(x-1)x\mathrm{Li}_2\left(\frac{1}{x}\right)+12x^2\ln(x)\ln\left(\frac{\mu^2}{m_t^2}\right)-24x^2\ln\left(\frac{\mu^2}{m_t^2}\right)\right)}{4n_c(x-1)^3} \\
&+\frac{\left(n_c^2-1\right)\left(2\pi^2x^2-35x^2+7x^2\ln(x)+12x\ln(x)\ln\left(\frac{\mu^2}{m_t^2}\right)+24x\ln\left(\frac{\mu^2}{m_t^2}\right)\right)}{4n_c(x-1)^3} \\
&+\frac{\left(n_c^2-1\right)\left(-2\pi^2x+38x-12(x-1)x\ln\left(\frac{1}{x}\right)\ln\left(\frac{x-1}{x}\right)+25x\ln(x)-3\right)}{4n_c(x-1)^3}+\mathcal{O}(\epsilon)\,, \\
C_E^{t,(1)} &= \frac{\left(n_c^2-1\right)\left(-6(x-1)x\mathrm{Li}_2\left(\frac{1}{x}\right)+6x^2\ln(x)\ln\left(\frac{\mu^2}{m_t^2}\right)-12x^2\ln\left(\frac{\mu^2}{m_t^2}\right)+\pi^2x^2\right)}{32n_c(x-1)^3} \\
&+\frac{\left(n_c^2-1\right)\left(-23x^2+9x^2\ln(x)+6x\ln(x)\ln\left(\frac{\mu^2}{m_t^2}\right)+12x\ln\left(\frac{\mu^2}{m_t^2}\right)-\pi^2x+30x\right)}{32n_c(x-1)^3} \\
&+\frac{\left(n_c^2-1\right)\left(-6(x-1)x\ln\left(\frac{1}{x}\right)\ln\left(\frac{x-1}{x}\right)+7x\ln(x)-7\right)}{32n_c(x-1)^3}+\mathcal{O}(\epsilon)\,. \qquad (6.60)
\end{aligned}
$$

We only give the parts of the Wilson coefficients proportional to ϵ^0, since the higher order terms in ϵ are too lengthy to present here. The Wilson coefficient $C_\nu^{t,(2)}$, evaluated at the matching scale m_t, is given by

$$
\begin{aligned}
C_\nu^{t,(2)}\big|_{\mu=m_t} &= \frac{1}{1620}\Bigg[-\frac{368640\mathrm{Li}_4\left(\frac{1}{2}\right)x^2}{(x-1)^3}+\frac{4320(x-1)(11x+61)}{(x-1)^4} \\
&+\frac{2160((x-96)x-49)\ln(x)\ln^2\left(\frac{\mu^2}{m_t^2}\right)x}{(x-1)^4}+19440\mathrm{Ls}_2^2x+20088\sqrt{3}\mathrm{Ls}_2x \\
&+\frac{720(x(x(81x-461)+837)-361)\mathrm{Li}_3(1-x)x}{(x-1)^3} \\
&+\frac{720(x+1)(x(27x-122)+143)\mathrm{Li}_3\left(\frac{1}{x}\right)x}{(x-1)^3} \\
&-\frac{720(x(x(x(16x-175)+603)-515)-25)\mathrm{Li}_3\left(\frac{x-1}{x}\right)x}{(x-1)^3} \\
&+\frac{5760(x(27x-122)+143)\mathrm{Li}_3\left(\frac{1}{\sqrt{x}}\right)x}{(x-1)^2} \\
&+\frac{5760(x(27x-122)+143)\mathrm{Li}_3\left(-\sqrt{x}\right)x}{(x-1)^2} \\
&+\frac{1440(x+1)((x-4)x-1)xH\left(m_t^2,m_t^2,0,0,M_W^2,m_t^2\right)\big|_{\mu=m_t}}{(x-1)^3} \\
&+\frac{1440(9(x-2)x+16)xH\left(m_t^2,m_t^2,M_W^2,0,0,0\right)\big|_{\mu=m_t}}{(x-1)^2} \\
&+\frac{240\left(x\left(x\left(48x^2-885x+2629\right)-3467\right)+1003\right)\zeta(3)x}{(x-1)^3} \\
&+\frac{360\ln\left(\frac{\mu^2}{m_t^2}\right)\left((1-x)\left(x\left(193x+2\pi^2(x+47)-2566\right)+69\right)\right)}{(x-1)^4}
\end{aligned}
$$

$$+\frac{360\ln\left(\frac{\mu^2}{m_t^2}\right)\left(x\left(x(191x-834)-24(x-1)(x+47)\coth^{-1}(1-2x)-1661\right)\ln(x)\right)}{(x-1)^4}$$
$$+\frac{360\ln\left(\frac{\mu^2}{m_t^2}\right)\left(12(x-1)x(x+47)\mathrm{Li}_2\left(\frac{1}{x}\right)\right)}{(x-1)^4}$$
$$+\frac{144(x(x(x(3x(6x(15x-52)+895)-1004)-4)+12)+1)F\left(m_t^2,m_t^2,m_t^2,M_W^2\right)\big|_{\mu=m_t}}{M_W^2(x-1)^5(5x-1)}$$
$$-\frac{144(x(x(x(x(5x-238)-191)+327)-26)-5)G\left(0,m_t^2,m_t^2,m_t^2,M_W^2\right)\big|_{\mu=m_t}}{M_W^2(x-1)^4(5x-1)}$$
$$+\frac{4\pi^4x^2(5x-1)(x(195(x-3)x+133)-447)(x-1)^2}{(x-1)^5(5x-1)x}$$
$$-\frac{24\pi^2(5x-1)\left(x\left(x\left(4x\left(10x(2x-19)+160\ln^2(2)-271\right)+3677\right)+8\right)-1\right)(x-1)^2}{(x-1)^5(5x-1)x}$$
$$+\frac{-167265x^8+1480842x^7+3x^6\left(25600\ln^4(2)-1533505\right)+3x^5\left(3519146-56320\ln^4(2)\right)}{(x-1)^5(5x-1)x}$$
$$+\frac{3x^4\left(35840\ln^4(2)-5169499\right)+3x^3\left(2987114-5120\ln^4(2)\right)-1410153x^2+28602x+798}{(x-1)^5(5x-1)x}$$
$$+\frac{x^8\left(-81000\ln^3(x)+360(405\ln(x-1)-608)\ln^2(x)+360\left(608\ln(x-1)+315\pi^2\right)\ln(x)\right)}{(x-1)^5(5x-1)x}$$
$$+\frac{x^7\left(642000\ln^3(x)-4896(235\ln(x-1)-156)\ln^2(x)\right)}{(x-1)^5(5x-1)x}$$
$$-\frac{x^7\left(144\left(5304\ln(x-1)\right)+5995\pi^2-1580\right)\ln(x)}{(x-1)^5(5x-1)x}$$
$$+\frac{x^6\left(-1979160\ln^3(x)+360(9823\ln(x-1)-7048)\ln^2(x)\right)}{(x-1)^5(5x-1)x}$$
$$+\frac{x^6\left(6\left(422880\ln(x-1)+426920\pi^2-35883\right)\ln(x)\right)}{(x-1)^5(5x-1)x}$$
$$+\frac{x^5\left(2872800\ln^3(x)+(6904656-5155200\ln(x-1))\ln^2(x)\right)}{(x-1)^5(5x-1)x}$$
$$+\frac{x^5\left(-12\left(575388\ln(x-1)+307640\pi^2-164655\right)\ln(x)\right)}{(x-1)^5(5x-1)x}$$
$$+\frac{x^4\left(-2040600\ln^3(x)+360(10291\ln(x-1)-24270)\ln^2(x)\right)}{(x-1)^5(5x-1)x}$$
$$+\frac{x^4\left(+72\left(121350\ln(x-1)+37275\pi^2-99499\right)\ln(x)\right)}{(x-1)^5(5x-1)x}$$
$$+\frac{x^3\left(655440\ln^3(x)-1440(841\ln(x-1)-3057)\ln^2(x)\right)}{(x-1)^5(5x-1)x}$$
$$+\frac{x^3\left(-48\left(91710\ln(x-1)+18805\pi^2-136287\right)\ln(x)\right)}{(x-1)^5(5x-1)x}$$
$$+\frac{x^2\left(-69480\ln^3(x)+72(1805\ln(x-1)-7988)\ln^2(x)\right)}{(x-1)^5(5x-1)x}$$

$$
\begin{aligned}
&+\frac{x^2\left(6\left(95856\ln(x-1)+16480\pi^2-189207\right)\ln(x)\right)}{(x-1)^5(5x-1)x}\\
&+\frac{x\left(36(60\ln(x-1)+491)\ln(x)-2160\ln^2(x)\right)+144\ln^2(x)+6(78-24\ln(x-1))\ln(x)}{(x-1)^5(5x-1)x}\\
&+\frac{144\left(5(x-1)(34x-109)\ln(x)x^2+\left(x\left(4061-4x\left(76x^2-98x+559\right)\right)+8\right)x-1\right)\mathrm{Li}_2\left(\frac{1}{x}\right)}{(x-1)^3x}\\
&+\left.\frac{144(x(x(x(x(x(35x-334)+64)-252)-29)+2)+2)E\left(m_t^2,m_t^2,m_t^2,M_W^2\right)\big|_{\mu=m_t}}{M_W^4(x-1)^5(5x-1)x}\right]\,.
\end{aligned}
\tag{6.61}
$$

This expression has also been checked to be μ-independent. Note that since the top mass appears at leading order in C_ν^t, the anomalous mass dimension γ_{m_t} also enters the Callan-Symanzik equation for the top sector. In the expression for $C_\nu^{t,(2)}$, the following functions arise [13]:

$$
\begin{aligned}
\mathrm{Li}_n(z) &= \sum_{k=1}^{\infty}\frac{z^k}{k^n}\,,\\
\mathrm{Li}_4(1/2) &\approx 0.517479\,,\\
\mathrm{Ls}_2 \equiv \mathrm{Ls}_2(2\pi/3) &= -\int_0^{2\pi/3} \mathrm{d}x\ \ln^2[2\sin(x/2)] \approx -2.144767\,.
\end{aligned}
\tag{6.62}
$$

The Wilson coefficient of Equation (6.14) can be found from our results by the relation $C_\nu = C_\nu^t - C_\nu^u$, which yields C_ν to $\mathcal{O}(\alpha_s^2)$.

6.3 Conclusions and Future Work

We have calculated for the first time the matching of box diagrams in the SM to the effective theory below the weak scale for the process $K \to \pi\nu\bar{\nu}$, up to $\mathcal{O}(\alpha_s^2)$. In calculating theoretical predictions of branching ratios, we are interested in the function X_t, which receives contributions from both the box and penguin diagrams. At leading order [44],

$$
X_0^t(x) = C_0(x) - 4B_0(x)\,, \tag{6.63}
$$

where C_0 and B_0 are Inami-Lim functions [136], which are respectively found from matching penguin and box diagrams. Explicitly,

$$
B_0(x) = \frac{1}{4}\left[\frac{x}{1-x}+\frac{x\ln x}{(x-1)^2}\right]\,, \tag{6.64}
$$

$$
C_0(x) = \frac{x}{8}\left[\frac{x-6}{x-1}+\frac{3x+2}{(x-1)^2}\ln x\right]\,, \tag{6.65}
$$

where

$$
C_\nu^{(0)} = -4B_0(x). \tag{6.66}
$$

Similarly, at 1-loop

$$X_t(x) = C(x) - 4B(x)\,, \tag{6.67}$$

where $C(x)$ and $B(x)$ are given to $\mathcal{O}(\alpha_s)$ in [135]. To construct $X_t(x)$ to $\mathcal{O}(\alpha_s^2)$, it is therefore necessary to obtain the function $C(x)$ to $\mathcal{O}(\alpha_s^2)$, which we have yet to do. A similar calculation of the decay $B_s \to \mu^+\mu^-$ [12] has had to consider a similar set of penguin diagrams, and in principle one can extract $C(x)$ to $\mathcal{O}(\alpha_s^2)$ from their result. This will be a useful check for our calculation. Once this has been done, an updated value of X_t can be given, as well as updated theoretical predictions for the branching ratios of $K^+ \to \pi^+\nu\overline{\nu}$ and $K_L \to \pi^0\nu\overline{\nu}$.

Chapter 7

Conclusions

In this work we have considered the dimension-five Weinberg operator of SMEFT, calculating for the first time its mixing into the dimension-six lepton flavour violating operators of the Warsaw basis. This process is responsible for the leading order contributions to lepton flavour violating decays such as $\mu \to ee\bar{e}$ in SMEFT. This calculation involved a new derivation of the renormalisation group equations for dimension-six operators, including double-insertions of dimension-five operators, which does not involve the redefinition of operators through the QCD coupling. We then compared this result to a previous calculation [11], where the mixing of the Weinberg operator into the dimension-six operators of the Buchmuller-Wyler basis was calculated. We found that translating our results into this basis did not replicate the findings of [11], and discovered that the authors of that paper made a mistake in the projection of their results. We then extended the calculation in a novel way, by including a second Higgs doublet, which admitted additional dimension-five operators that mixed into the dimension-six SMEFT operators. Following this, we determined bounds on the Wilson coefficients of these new operators by translating experimental bounds on low-energy Wilson coefficients into bounds on SMEFT coefficients. This matching had been done previously in [85], but we have corrected some mistakes from that work. We found that current experimental data is already sufficient to place significant bounds on the additional dimension-five Wilson coefficients arising from a second Higgs doublet.

We then moved on to discuss the $\mathcal{O}(\alpha_s^2)$ matching calculation for the rare decays $K \to \pi\nu\bar{\nu}$. We calculated the corresponding box diagrams in the SM for the first time, and also calculated the full renormalisation matrix of the effective theory up to $\mathcal{O}(\alpha_s^2)$ for the first time, completing the results of [12]. Finally, we checked the μ-independence of our results, which is required by the conservation of the massless quark current. This result, when combined with the matching of $\mathcal{O}(\alpha_s^2)$ penguin diagrams of the full theory onto the effective theory, may be used to provide theoretical updates on the branching ratios for the decays $K^+ \to \pi^+\nu\bar{\nu}$ and $K_L \to \pi^0\nu\bar{\nu}$.

We also outlined work done on the decomposition of tensor integrals arising when using the strategy of expansion by regions on 3-loop vacuum integrals. The decomposition was reduced to a combinatorics problem, and an analytic expression for the decomposition of the relevant class of tensor integrals was given. This new relation allowed tensor integrals to be decomposed extremely quickly, which were too complex to be decomposed in a brute force manner.

We intend to use the framework used in the matching calculation for $K \to \pi\nu\overline{\nu}$ to perform a matching calculation for the neutral meson mixing $K^0 - \overline{K^0}$ and $B^0 - \overline{B^0}$, to $\mathcal{O}(\alpha_s^2)$. This is a more involved calculation, since it involves two quark lines that may interact with gluons. This results in more diagrams to be evaluated, as well as a greater number of operators in the effective theory. However, only a further two integral families arise in the reduction, and so the additional complexity involved in this calculation is tractable.

Appendix A

Dimension-6 Operators of the Warsaw Basis

We present here a list of the dimension-6 operators of SMEFT, in the Warsaw basis [8]. These may be categorised according to the field and derivative content of the operators, into the categories

$$\begin{array}{ccc} X^3 & \varphi^6 \text{ and } \varphi^4 D^2 & \psi^2\varphi^3 \\ X^2\varphi^2 & \psi^2 X\varphi & \psi^2\varphi^2 D \end{array} ,$$

where X, φ, D and ψ are generic labels for field strength tensors, scalars, derivatives, and fermions respectively. These operators (which have at most two fermions) are given in Table A.1. There are additionally the four-fermion operators, which are categorised according to their chirality structure. These are given in Table A.2. We only include the operators that conserve baryon number, of which there are 59 in total (neglecting flavour structures).

The matter content of SMEFT is given in Section 2.5, while the left-right derivatives in the operator class $\psi^2\varphi^2 D$ are given in Section 4.6. In addition, the dual tensors are defined by

$$\widetilde{X}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}X^{\rho\sigma}, \qquad X \in \{G^A, W^I, B\}\,. \tag{A.1}$$

The operators of the Warsaw basis are a minimal set, with redundant operators from equations of motion eliminated. Here, flavour indices on the operators are suppressed, and in this form (with no normalisation constants), any "non-Hermitian" operator must be added to the Lagrangian + H.c., whereas Hermitian operators are added on their own. Here, Hermitian is defined to mean the operator is unchanged under Hermitian conjugation when neglecting flavour indices.

X^3		φ^6 and $\varphi^4 D^2$		$\psi^2\varphi^3$	
Q_G	$f^{ABC} G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	Q_φ	$(\varphi^\dagger\varphi)^3$	$Q_{e\varphi}$	$(\varphi^\dagger\varphi)(\overline{\ell}_p e_r \varphi)$
$Q_{\widetilde{G}}$	$f^{ABC} \widetilde{G}_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	$Q_{\varphi\Box}$	$(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi)$	$Q_{u\varphi}$	$(\varphi^\dagger\varphi)(\overline{q}_p u_r \widetilde{\varphi})$
Q_W	$\varepsilon^{IJK} W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$	$Q_{\varphi D}$	$\left(\varphi^\dagger D^\mu\varphi\right)^* \left(\varphi^\dagger D_\mu\varphi\right)$	$Q_{d\varphi}$	$(\varphi^\dagger\varphi)(\overline{q}_p d_r \varphi)$
$Q_{\widetilde{W}}$	$\varepsilon^{IJK} \widetilde{W}_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$				
$X^2\varphi^2$		$\psi^2 X\varphi$		$\psi^2\varphi^2 D$	
$Q_{\varphi G}$	$\varphi^\dagger\varphi\, G_{\mu\nu}^A G^{A\mu\nu}$	Q_{eW}	$(\overline{\ell}_p \sigma^{\mu\nu} e_r)\tau^I \varphi W_{\mu\nu}^I$	$Q_{\varphi\ell}^{(1)}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi)(\overline{\ell}_p \gamma^\mu \ell_r)$
$Q_{\varphi\widetilde{G}}$	$\varphi^\dagger\varphi\, \widetilde{G}_{\mu\nu}^A G^{A\mu\nu}$	Q_{eB}	$(\overline{\ell}_p \sigma^{\mu\nu} e_r)\varphi B_{\mu\nu}$	$Q_{\varphi\ell}^{(3)}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu^I \varphi)(\overline{\ell}_p \tau^I \gamma^\mu \ell_r)$
$Q_{\varphi W}$	$\varphi^\dagger\varphi\, W_{\mu\nu}^I W^{I\mu\nu}$	Q_{uG}	$(\overline{q}_p \sigma^{\mu\nu} T^A u_r)\widetilde{\varphi}\, G_{\mu\nu}^A$	$Q_{\varphi e}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi)(\overline{e}_p \gamma^\mu e_r)$
$Q_{\varphi\widetilde{W}}$	$\varphi^\dagger\varphi\, \widetilde{W}_{\mu\nu}^I W^{I\mu\nu}$	Q_{uW}	$(\overline{q}_p \sigma^{\mu\nu} u_r)\widetilde{\varphi}\, W_{\mu\nu}^I$	$Q_{\varphi q}^{(1)}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi)(\overline{q}_p \gamma^\mu q_r)$
$Q_{\varphi B}$	$\varphi^\dagger\varphi\, B_{\mu\nu} B^{\mu\nu}$	Q_{uB}	$(\overline{q}_p \sigma^{\mu\nu} u_r)\widetilde{\varphi}\, B_{\mu\nu}$	$Q_{\varphi q}^{(3)}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu^I \varphi)(\overline{q}_p \tau^I \gamma^\mu q_r)$
$Q_{\varphi\widetilde{B}}$	$\varphi^\dagger\varphi\, \widetilde{B}_{\mu\nu} B^{\mu\nu}$	Q_{dG}	$(\overline{q}_p \sigma^{\mu\nu} T^A d_r)\widetilde{\varphi}\, G_{\mu\nu}^A$	$Q_{\varphi u}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi)(\overline{u}_p \gamma^\mu u_r)$
$Q_{\varphi WB}$	$\varphi^\dagger \tau^I \varphi\, W_{\mu\nu}^I B^{\mu\nu}$	Q_{dW}	$(\overline{q}_p \sigma^{\mu\nu} d_r)\tau^I \varphi W_{\mu\nu}^I$	$Q_{\varphi d}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi)(\overline{d}_p \gamma^\mu d_r)$
$Q_{\varphi\widetilde{W}B}$	$\varphi^\dagger \tau^I \varphi\, \widetilde{W}_{\mu\nu}^I B^{\mu\nu}$	Q_{dB}	$(\overline{q}_p \sigma^{\mu\nu} d_r)\varphi B_{\mu\nu}$	$Q_{\varphi ud}$	$i(\widetilde{\varphi}^\dagger D_\mu \varphi)(\overline{u}_p \gamma^\mu d_r)$

Table A.1: Dimension-6 operators of SMEFT which involve bosons, scalars and derivatives.

$(\overline{L}L)(\overline{L}L)$		$(\overline{R}R)(\overline{R}R)$	
$Q_{\ell\ell}$	$(\overline{\ell_p}\gamma_\mu \ell_r)(\overline{\ell_s}\gamma^\mu \ell_t)$	Q_{ee}	$(\overline{e_p}\gamma_\mu e_r)(\overline{e_s}\gamma^\mu e_t)$
$Q_{qq}^{(1)}$	$(\overline{q_p}\gamma_\mu q_r)(\overline{q_s}\gamma^\mu q_t)$	Q_{uu}	$(\overline{u_p}\gamma_\mu u_r)(\overline{u_s}\gamma^\mu u_t)$
$Q_{qq}^{(3)}$	$(\overline{q_p}\gamma_\mu \tau^I q_r)(\overline{q_s}\gamma^\mu \tau^I q_t)$	Q_{dd}	$(\overline{d_p}\gamma_\mu d_r)(\overline{d_s}\gamma^\mu d_t)$
$Q_{\ell q}^{(1)}$	$(\overline{\ell_p}\gamma_\mu \ell_r)(\overline{q_s}\gamma^\mu q_t)$	Q_{eu}	$(\overline{e_p}\gamma_\mu e_r)(\overline{u_s}\gamma^\mu u_t)$
$Q_{\ell q}^{(3)}$	$(\overline{\ell_p}\gamma_\mu \tau^I \ell_r)(\overline{q_s}\gamma^\mu \tau^I q_t)$	Q_{ed}	$(\overline{e_p}\gamma_\mu e_r)(\overline{d_s}\gamma^\mu d_t)$
		$Q_{ud}^{(1)}$	$(\overline{u_p}\gamma_\mu u_r)(\overline{d_s}\gamma^\mu d_t)$
		$Q_{ud}^{(8)}$	$(\overline{u_p}\gamma_\mu T^A u_r)(\overline{d_s}\gamma^\mu T^A d_t)$

$(\overline{L}L)(\overline{R}R)$		$(\overline{L}R)(\overline{R}L)$ and $(\overline{L}R)(\overline{L}R)$	
$Q_{\ell e}$	$(\overline{\ell_p}\gamma_\mu \ell_r)(\overline{e_s}\gamma^\mu e_t)$	$Q_{\ell edq}$	$(\overline{\ell_p^j} e_r)(\overline{d_s} q_t^j)$
$Q_{\ell u}$	$(\overline{\ell_p}\gamma_\mu \ell_r)(\overline{u_s}\gamma^\mu u_t)$	$Q_{quqd}^{(1)}$	$(\overline{q_p^j} u_r)\varepsilon_{jk}(\overline{q_s^k} d_t)$
$Q_{\ell d}$	$(\overline{\ell_p}\gamma_\mu \ell_r)(\overline{d_s}\gamma^\mu d_t)$	$Q_{quqd}^{(8)}$	$(\overline{q_p^j} T^A u_r)\varepsilon_{jk}(\overline{q_s^k} T^A d_t)$
Q_{qe}	$(\overline{q_p}\gamma_\mu q_r)(\overline{e_s}\gamma^\mu e_t)$	$Q_{\ell equ}^{(1)}$	$(\overline{\ell_p^j} e_r)\varepsilon_{jk}(\overline{q_s^k} u_t)$
$Q_{qu}^{(1)}$	$(\overline{q_p}\gamma_\mu q_r)(\overline{u_s}\gamma^\mu u_t)$	$Q_{\ell equ}^{(3)}$	$(\overline{\ell_p^j}\sigma_{\mu\nu} e_r)\varepsilon_{jk}(\overline{q_s^k}\sigma^{\mu\nu} u_t)$
$Q_{qu}^{(8)}$	$(\overline{q_p}\gamma_\mu T^A q_r)(\overline{u_s}\gamma^\mu T^A u_t)$		
$Q_{qd}^{(1)}$	$(\overline{q_p}\gamma_\mu q_r)(\overline{d_s}\gamma^\mu d_t)$		
$Q_{qd}^{(8)}$	$(\overline{q_p}\gamma_\mu T^A q_r)(\overline{d_s}\gamma^\mu T^A d_t)$		

Table A.2: Four-fermion dimension-6 operators of SMEFT.

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