Master Dissertation

Exact Global Symmetry Generators for Restricted Schur Polynomials

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Declaration

I declare that the work on which this dissertation is based is my own, unaided work (except where acknowledgements indicate otherwise). It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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Abstract

The six scalar fields in $\mathcal{N} = 4$ super Yang-Mills theory enjoy a global $SO(6)$ symmetry, and large N but non-planar limits of this theory are well-described by adopting a group representation approach. Studies have shown that the one-loop dilatation operator is highly determined by the action of the $su(2)/su(3)$ subalgebras on restricted Schur polynomials. These actions involve the traces of products of projection operators. In this dissertation, exact analytical formulae for these traces are found which in turn are used to find the exact action of these algebras on restricted Schur polynomials. The potential of the $su(2)$ algebra to determine the one-loop dilatation operator is also explored. This is done by exploiting necessary symmetry conditions and moving to a continuum limit in order to derive a number of partial differential equations which determine the dilatation operator. The ultimate goal of this work is to provide tools to find the exact one-loop dilatation operator in the non-planar limit.

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Contents

1
Background and Introduction

Modern physics has been successful in creating quantum theories which explain three of the four fundamental forces of nature. However, it has been much more of a challenge to reconcile general relativity, the non-quantum theory successfully explaining the fourth, with quantum field theories mainly because general relativity is a classical theory which becomes ill-defined when quantised. Despite this, the ultimate goal of unifying the four forces of nature into one, coherent theory still remains a large goal of modern theoretical physics.

A great step forward in this direction was made when Maldacena proposed the AdS/CFT correspondence ([1]). This idea states that quantum gravity, defined on anti-de Sitter space, is dual to a conformal field theory. While many types of anti-de Sitter spaces and conformal field theories exist, the most common example of this correspondence is that string theory on an $\mathrm{AdS}_5\times\mathrm{S}^5$ background is equivalent to $\mathcal{N}=4$ super Yang-Mills theory. Using this correspondence, calculations made in the one theory can be translated into calculations in the other.

When working in $\mathcal{N}=4$ super Yang-Mills theory, the fields are $N\times N$ matrices. The generating functional (the quantum field theory analogue of the statistical mechanics partition function) based on these matrices can then be used to compute various correlation functions. By generating a complete set of correlation functions, one can compute any observable one is interested in. In a non-interacting, zero-dimensional matrix model, doing all this is relatively easy: one sums a finite number of ribbon diagrams (diagrams akin to Feynman diagrams) in order to compute the correlation functions. However, in an interacting model, the usual way to compute the resulting more complicated generating functional is to employ a perturbative expansion by expanding the functional as a power series in the coupling constant. It consequently becomes impossible to calculate any correlation function exactly since one encounters an ever-increasing number of ribbon diagrams to sum over as the order of the expansion increases.

To tackle this situation, a different expansion, called the 't Hooft expansion, can be used. In this method, the coupling constant g and the matrix size N are scaled in specific ways. N is made to tend to infinity, and the perturbative corrections to the non-interacting model are computed. This expansion produces an appealing interpretation in terms of string theory: each ribbon diagram resulting from the expansion triangulates a surface representing a worldsheet traced out by a string as it moves in spacetime. The topology of these worldsheets are used to determine the specific N dependence of the ribbon diagram in question, and summing over all the ribbon diagrams gives the correlator we seek ([2]). A nice result of the large N planar limit (i.e. the limit in which the studied observables are constructed from either a fixed number of fields or in which the number of fields grows like $\sqrt{N}$ as $N \to \infty$) is that correlation functions factorise, which eases calculations ([3]). This happens to be the case in both free theories and interacting ones. Because of the large size of N, in the more realistic interacting model, one typically only sums planar ribbon diagrams instead of all diagrams. Furthermore, for a single matrix model, a procedure based on the eigenvalues of the matrix field allows for the construction of the system's dynamics, which can be used to recover the large N limit of the model (which is equivalent to a classical theory of strings) and its correlators ([4]). Since $\mathcal{N} = 4$ super Yang-Mills theory is a conformal field theory, when computing the said correlators, the answer factorises into a combinatoric factor and a spacetime dependent factor. For the questions we will consider, the spacetime factor is easy to compute. Hence, most of the work goes into computing the combinatoric factor, a process which involves summing over all the Wick contractions present in the correlator. In the planar limit of a single matrix model, we know how to find an orthogonal basis for the local operators. This in turn makes it possible to find the scaling dimensions of the operators. These scaling dimensions are the basic observables of any conformal field theory and have the potential to connect the theoretical work to real-world measurements.

This approach still has various problems however. Firstly, it concerns a single matrix model; more interesting real-world models would be multi-matrix models. Secondly, the niceties of the planar limit break down as we allow the number of fields in each trace structure (i.e. observable) to grow like N (rather than $\sqrt{N}$) as $N \to \infty$; this specific *non-planar* limit is the one we will concern ourselves with. In such a limit, non-planar diagrams become non-negligible and must also be included when calculating correlators. And thirdly, our original diagonal basis for the operators becomes inappropriate and loses a lot of its simplifying properties since different multi-trace structures mix freely in the non-planar limit. In light of these problems, group representation theory has provided many insights and techniques, allowing physicists to sum over all of the ribbon diagrams and take all possible trace structures into account. Using group representation theory in the context of the discussed single matrix model, the carrier space of the matrix operator in the model, V_N, is extended to a much large space, $V_N^{\otimes n}$ ($n \in \mathbb{N}$), and any multi-trace structure can be written as a single trace in the larger space by making use of permutations belonging to the symmetric group, S_n. Free-field Wick contractions of n fields are found to be equal to a simple summation of elements of the action of S_n on the larger space $V_N^{\otimes n}$. The original problem of finding a convenient orthogonal basis becomes equivalent to finding a complete set of projection operators projecting onto different representations of S_n. Schur polynomials, defined in terms of these projectors, play a fundamental role here ([5]). It is possible to write any multi-trace operator in terms of sums of these Schur polynomials (and vice versa) by means of equations analogous to Fourier transforms. Our problems in quantum field theory therefore become replaced with equivalent problems in group theory. The Schur polynomials are in turn used to find the two-point correlation functions. Within this framework, these two-point correlation functions sum over all the ribbon graphs in question (not just the planar ones), avoiding the problems typically associated with perturbative methods.

The above discussion, as mentioned, relates to a single matrix model. In the context of the AdS/CFT correspondance, the single matrix model provides a description of the $\frac{1}{2}$-BPS sector of the theory ([6]). However, also as mentioned, multi-matrix models are more interesting. Indeed, they allow us to go beyond the $\frac{1}{2}$ BPS sector of the theory. Free-field answers are exact in models comprised of a single matrix due to the enormous amount of symmetry inherent in them. Introducing more matrices breaks enough symmetries that curious results arise in the interacting theory. Luckily, a number of useful results from the single matrix case carry over into the multi-matrix case: a multi-matrix model is dual to a string theory, the large N planar limit of the model is the classical limit of a string theory, and different trace structures do not mix in the large N limit. However, as soon as the number of matrices we include in our model is of the order N, we need to sum over all ribbon diagrams, and different trace structures do mix together ([7]). We are interested in studying this large N but non-planar limit.

For the case of more than one matrix field, as before, different multi-trace structures are treated on the same footing by employing group representation theory. The different multi-trace structures (which represent observables) are labeled by permutations of the specific symmetry group in question. It turns out that permutations in the same conjugacy class result in the same observable when bosonic fields are considered. This is a result of bosonic statistics, in which swapping two bosonic fields leaves the trace structure invariant. However, in order to not mix different matrix fields with one another when calculating multi-trace structures, one needs to introduce the notion of a ‘restricted character’ ([8]). For example, in the case of two different fields (let’s say n Z fields and m Y fields), we restrict the symmetric group S_{n+m} (the permutations being elements of this group) to the subgroup $S_n \times S_m$. The restricted trace in this case is found by only summing over indices of the possible states which are part of a specific irreducible representation of the subgroup $S_n \times S_m$, rather than the whole group (as was the case above). This type of restriction, and therefore the restricted characters, can be generalised to the case with more than two matrix fields in the model.

Restricted Schur polynomials and restricted projectors are defined using these restricted characters (by replacing the usual characters in their definitions with the restricted characters). Two point functions can also be constructed based on these restricted Schur polynomials. These can in turn be used to determine observables which have the potential to link the theory with real world applications.

Until now, the ideas discussed were essentially a wide but very shallow background leading up to the purpose of this dissertation; it is from this point onwards that our new work will begin. Among other aims, the main goals of this dissertation are to:

- Construct the exact $SU(2)$ generators acting on restricted Schur polynomials built from two complex bosonic fields
- Explain how to derive a number of partial differential equations (PDE’s) for this two complex bosonic field case by exploiting certain symmetry conditions
- Construct the exact $SU(3)$ generators acting on restricted Schur polynomials built from three complex bosonic fields
- Numerically verify these results

In a broader view, the ultimate goal of using a group theory approach is to calculate correlators in an interacting super Yang-Mills theory. In $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, the Lagrangian is ([9]):

$$\begin{aligned} L =& \mathrm{Tr}\left(-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\overline{F}^{\mu\nu} - i\lambda^a\sigma^\mu D_\mu\lambda_a - D_\mu\phi^i D^\mu\phi^i \right. \\ &\left. + gC_i^{ab}\lambda_a[\phi^i,\lambda_b] + C_{iab}\lambda^a[\phi^i,\lambda^b] + \frac{g^2}{2}[\phi^i,\phi^j]^2\right) \end{aligned} \tag{1.1}$$

where $F^k_{\mu\nu} = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu + gf^k_{lm}A^l_\mu A^m_\nu$; i, j run from 1 to 6; a, b run from 1 to 4; μ, ν run from 0 to 3; and all the f’s are the structure constants of the chosen gauge group. In this expression, the A’s are components of the gauge field; g, θ_I are coupling constants; ϕ^i are scalar fields of spin 0; and the λ’s are spinor fields of spin $\frac{1}{2}$. Also, D_μ is the usual covariant derivative, σ^μ the Pauli matrices, and C_i^{ab} are $SU(4)$ Clebsch-Gordon coefficients, which couple the fermions, λ^a, to the bosons, ϕ^i, in an $SU(4)$ invariant way. Using this Lagrangian L, we will be calculating correlation functions of the ϕ^i fields.

Algebras also play an extremely important role in the development of this area of physics. The superconformal algebra is a graded Lie algebra which combines supersymmetry and the conformal algebra. We know that many systems have certain types of symmetries. In particular, some are invariant under translations (with translations being generated by the momentum operators), some are invariant under Lorentz boosts or spacetime rotations (which are generated by the angular momentum operators), some are invariant under scaling (which is generated by the dilatation operator) and some are invariant under 'inverted translations' (generated by the special conformal operators). All of these operators are combined in the superconformal algebra in 3 + 1D. In particular, we will be focusing on the dilatation operator, D. In this superalgebra, if $M_{\mu\nu}$ represents the Lorentz operator, P_μ the momentum operator, and K_μ the special conformal operator, then the commutator of these operators with D are:

$$\begin{aligned}
&[M_{\mu\nu}, D] = 0 \\
&[D, P_\mu] = -P_\mu \\
&[D, K_\mu] = K_\mu
\end{aligned} \tag{1.2}$$

These operators, along with quite a number of others, comprise the $\mathcal{N} = 1$ superconformal algebra in $3 + 1$D. $SU(2)$ and $SU(3)$ generators form a part of this much larger algebra. We will not be considering this much larger algebra (or symmetry) but rather a smaller symmetry which keeps the calculations inside a smaller subspace which will be easier to work with. For example, we do not wish to consider the gauge fields A_μ. If we worked in the full algebra, we would be forced to, since some symmetry transformations would change some of the fermion fields into gauge fields. However, we prefer to build up the theory slowly, moving from $su(2)$ into $su(3)$, and so forth. Not having to consider the full set of symmetry transformations at this stage is convenient.

In this study, we will also construct global symmetry generators which generate different special unitary groups' algebras. These algebras are important as the $\mathcal{N} = 4$ super Yang-Mills theory has certain symmetries under the action of the corresponding groups. Symmetries have become one of the most important tools in physics since it seems that many of the laws of physics originate from, or are tightly constrained by, symmetries. So, by studying the symmetries of systems further, very interesting questions and results are almost sure to follow.

The main aims and accomplishments of this study were listed in point form above. However, it is worth noting that the actual accomplishment of constructing the exact generators (the task which the bulk of this study is dedicated to) is admittedly a technical precursor of the more interesting and larger goal which will be briefly touched on here and hopefully examined further in later study: that of constructing the one- and higher-loop dilatation operators in the $su(2)$ and $su(3)$ sectors by requiring that the dilatation operator (acting on different restricted Schur polynomials, depending on the case) closes the relevant algebras. The dilatation operator is important for a number of reasons, one being that it can be directly interpreted as the Hamiltonian of a quantum mechanical system ([10]). It can be found by summing Feynman diagrams. However, at higher and higher loop orders, one must sum over an increasing number of diagrams, making the exact calculation of the dilatation operator difficult. The one loop dilatation operator has been constructed previously by brute force calculations ([11, 12]). However, the goal here is to show that the imposed symmetry condition is enough to determine the dilatation operator (up to a constant), at both one and higher loops. This idea is touched on at the end of chapter 3. While this ultimate goal of determining the dilatation operator this way would be a Herculean task, finding it would allow us to calculate the exact conformal dimensions of certain operators and would be a significant step towards understanding the non-planar limit of $\mathcal{N} = 4$ SYM. And finally, as a bonus, thanks to the AdS/CFT correspondence, calculations in $\mathcal{N} = 4$ SYM theory can be translated into calculations in the dual quantum gravity setting. Therefore, such results will have bearing in string theory.

The equations in chapter 3.3 are some of the important results of this dissertation. The equations in this chapter are used to find some of the actual novel results, given in chapter 3.5. This chapter contains the exact action of the $su(2)$ generators on restricted Schur polynomials. Similarly, other important novel results in chapter 4.2 give rise to the results in chapter 4.3. In this chapter of the dissertation, the exact action of some $su(3)$ generators on restricted Schur polynomials are given and the techniques to calculate the others are laid out. This is important since large N but non-planar limits of $\mathcal{N} = 4$ super Yang-Mills theory, a most auspicious theory which promises to be a breeding ground for approaches for solving problems in more realistic theories, are well described by restricted Schur polynomials and our group theory approach. The explanation towards the end of chapter 3 also details how these results can be used to find the dilatation operator. If one were to use exact results, one could exactly close the appropriate algebras. Studying the dilatation operator is important, amongst other reasons, for its strong connection with the Hamiltonian in 2-dimensional conformal field theories.

The results in this dissertation are explained in great detail. The actual, more concise, paper reporting these novel results can be found at: arXiv:1602.05675 ([13]).

2
Review

This chapter of the dissertation will give further basic background details of specific topics that will be needed, but in a little bit more depth than was discussed above. Ideas such as symmetric groups, projectors, correlation functions, (restricted) characters, and Schur polynomials will be described.

The symmetric group of the set of n integers, denoted S_n, will play an important role. If a permutation of the integers $(1, 2, 3, ..., n)$ is denoted by σ, then $\sigma \in S_n$, with S_n being the complete set of all possible permutations. S_n clearly contains $n!$ elements. In the most conventional approach, a representation of S_n (or, in fact, any group) is a way of assigning a matrix to each element of this abstract group in a way that preserves the group operation. The group operation then becomes matrix multiplication. More technically, a representation of a group is a homeomorphism from the group to the automorphism of some vector space. There are a finite number of inequivalent, irreducible representations of S_n, and each representation is normally denoted by a Young diagram with n blocks. We will use letters of the Latin alphabet to denote Young diagrams. In a specific representation, R, of S_n, the matrix representing the permutation σ will be denoted by $\Gamma_R(\sigma)$ and we will denote the trace of such a matrix with the Greek letter χ, i.e. $\chi_R(\sigma) = \text{Tr}(\Gamma_R(\sigma))$.

As mentioned in the previous chapter, we extend the vector space V_N on which the operators in $\mathcal{N} = 4$ super Yang-Mills theory act, to $V_N^{\otimes n}$, where n is the number of matrices (which we typically denote by Z) comprising the multi-trace structure. This is for the case of a single matrix model; the multi-matrix model will be discussed shortly. A very important fact is that the actions of symmetric group (S_n) elements and unitary group ($U(N)$) elements on $V_N^{\otimes n}$ commute ([14]). Because of this, $V_N^{\otimes n}$ reduces into subspaces labeled by Young diagrams, with each subspace simultaneously carrying the irreducible representation of the symmetric group and the irreducible representation of the unitary group of the Young diagram labeling the specific subspace in question. In addition, whenever one restricts S_n under subgroups (for example, restricting S_n under S_{n-1}, or under $S_{n_1} \times S_{n_2}$ where $n_1 + n_2 = n$), an originally irreducible representation of S_n often becomes reducible, with S_n becoming the direct sum of certain irreducible subspaces. In light of this, it is reasonable to think that it's possible to project from $V_N^{\otimes n}$ onto a subspace, and indeed, it is. The projection operator responsible for projecting onto the R subspace is:

$$\hat{P}_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma \quad \text{or} \quad P_R = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma \tag{2.1}$$

where σ is an action on $V_N^{\otimes n}$ and d_R is the dimension of R. Depending on the type of carrier space we are acting on, we either have σ on the right-hand side of the above summation, or the matrix representation for σ, i.e. $\Gamma_R(\sigma)$. $\chi_R(\sigma)$ is known as the *character* of the group element σ in the R representation, and since the groups we will consider are symmetric groups, it is relatively easy to compute a desired projection operator within this context since tables of these characters exists for symmetric groups (for example, see [15]). The situation will become involved when we consider more fields instead of only a single field Z. Note also that we use either of the equations in equation (2.1) when working with Schur polynomials. In linear algebra, projectors $\hat{P}_R, \hat{P}_S$ would normally obey $\hat{P}_R \hat{P}_S = \delta_{RS} \hat{P}_R$. This is the case if we choose to use the first of the two equations. However, another convention is the second: projectors which do not obey the usual normalisation requirement, but rather obey the following:

$$P_R P_S = \frac{\delta_{RS}}{d_R} P_R \tag{2.2}$$

where d_R is the dimension of the symmetric group representation R. It doesn't matter much which we choose to use. Apart from the above being true, the operators also satisfy $P_R \sigma = \sigma P_R \ \forall \sigma \in S_n$, and we also require $P_R^\dagger = P_R$, for convenience.

Now, one of the benefits of the group representation theory approach is that it allows us to write any multi-matrix operator, constructed from one matrix Z which acts on V_N, as a single matrix operator on the larger space $V_N^{\otimes n}$. By slowly but easily developing the index notation used below, it can be shown that:

$$\text{Tr}(\sigma Z^{\otimes n}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \dots Z^{i_n}_{i_{\sigma(n)}} \tag{2.3}$$

where $\sigma(i)$ is the integer that i is permuted to under the action of the permutation σ. Different trace structures appear when calculating correlation functions, and by making a prudent choice for σ, it is possible to express any multi-trace structure/operator in the form of the left-hand side of equation (2.3). For example, if we take $n = 4$ and $\sigma = (12)(3)(4)$, then we would have:

$$\mathrm{Tr}((12)(3)(4)Z^{\otimes n}) = Z^{i_1}_{i_2} Z^{i_2}_{i_1} Z^{i_3}_{i_3} Z^{i_4}_{i_4} = \mathrm{Tr}(Z^2)\mathrm{Tr}(Z)^2$$

In the above equation, the left-hand side is a single trace over $V_N^{\otimes n}$, whereas the right-hand side is a multi-trace structure with each trace being over V_N. Another important concept which we will use repeatedly and develop is the idea of a *Schur polynomial*. There are a number of ways of defining Schur polynomials; more mathematical studies unrelated to representation theory often define them in terms of determinants of matrices (see [16] for example). However, we will define our Schur polynomials, denoted by $\chi_R(Z)$, using equation (2.3) as follows:

$$\begin{aligned} \chi_R(Z) = \mathrm{Tr}(P_R Z^{\otimes n}) &= \mathrm{Tr}\left(\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\sigma Z^{\otimes n}\right) \\ &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\mathrm{Tr}(\sigma Z^{\otimes n}) \end{aligned} \tag{2.4}$$

where R is a representation of S_n and Z is a matrix field. So, our Schur polynomial is a linear combination of different multi-trace structures. A few different orthogonality relations exist in group representation theory. These link different representations of a symmetric group or different equivalence classes of symmetric group elements ([15]). Using them, it is possible to swap equation (2.4) around and express a multi-trace structure as a linear combination of Schur polynomials as follows:

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = \sum_{R\vdash n}\chi_R(\sigma)\chi_R(Z) \tag{2.5}$$

The notation $R \vdash n$ means that the Young diagram R is a partition of the integer n, i.e. R is a Young diagram comprised of n boxes. This type of inversion shown in equation (2.5) is akin to the more familiar Fourier and inverse Fourier transform concepts. Finally, to link to quantum field theory, it can easily be shown that the two-point correlation function based on these Schur polynomials is ([5]):

$$\langle\chi_R(Z)\chi_S(Z)^\dagger\rangle = \delta_{RS} f_R \tag{2.6}$$

where f_R is the product of the factors of the boxes comprising the Young diagram labeled R. The factor of a box in a Young diagram is defined to be N plus the content of the box, where the content of a box is the column number (counting from the left) of the box minus the row number (counting from the top) of the box. For example, we'd have:

$$\langle \chi_{\yng(4,2)}(Z)\chi_{\yng(4,2)}(Z)^\dagger \rangle = N^2(N+1)(N+2)(N+3)(N-1)$$

The astonishing thing about the result in equation (2.6) is that this answer is exact: we have summed over all diagrams, not only the planar ones.

As mentioned, the above discussion relates to a zero-dimensional single matrix model. These results can easily be extended to a higher dimensional single matrix model by multiplying the answer given in equation (2.6) with a spacetime dependent factor, which is easy to find (in our case). For example, the $3+1$ dimensional 2-point function would become:

$$\langle \chi_R(Z)(x_1)\chi_S(Z^\dagger)(x_2)\rangle = \frac{\delta_{RS} f_R}{|x_1 - x_2|^{2J}}$$

where J is the scale dimension of the restricted Schur polynomial, and $R \vdash J$.

We now move on to a more complicated case. Consider a model constructed from two bosonic fields Z and Y, instead of just Z as we had done previously. The single-trace structure analogous to equation (2.3) in the case of operators constructed from m Y fields and n Z fields (in $V_N^{\otimes n+m}$) is $\mathrm{Tr}(\sigma Y^{\otimes m} \otimes Z^{\otimes n})$, where $\sigma \in S_{n+m}$. In this discussion we will keep n on the order of N and m on the order of $\sqrt{N}$. We have:

$$\mathrm{Tr}(\sigma Y^{\otimes m} \otimes Z^{\otimes n}) = Y^{i_1}_{i_{\sigma(1)}} Y^{i_2}_{i_{\sigma(2)}} \dots Y^{i_m}_{i_{\sigma(m)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}} Z^{i_{m+2}}_{i_{\sigma(m+2)}} \dots Z^{i_{m+n}}_{i_{\sigma(m+n)}} \tag{2.7}$$

Now, the observable $\mathrm{Tr}(\sigma Y^{\otimes m} \otimes Z^{\otimes n})$ is invariant when we swap the Z fields amongst themselves and the Y fields amongst themselves. Indeed, swapping a letter Z on the right-hand side of the above equation with another Z would not change the multi-trace structure. This is a result of the fact that the Z and Y fields are *bosonic*. This is not the case with fermionic fields. In the case of equation (2.3), it can easily be shown that permutations in the same conjugacy classes give rise to the same observables. However, in this two-matrix case, we need to consider permutations σ and τ that are related to one another not through the usual conjugacy relation $\tau = \rho^{-1}\sigma\rho$ with $\rho \in S_{n+m}$, but rather through the *restricted* conjugacy class relation $\tau = \rho^{-1}\sigma\rho$ with $\rho \in S_n \times S_m$. ρ in this case is a general permutation of the Z's amongst each other and the Y's amongst each other. If we have $\tau = \rho^{-1}\sigma\rho$ with $\rho \in S_n \times S_m$, then it can be shown that $\mathrm{Tr}(\tau Y^{\otimes m} \otimes Z^{\otimes n}) = \mathrm{Tr}(\sigma Y^{\otimes m} \otimes Z^{\otimes n})$: the two permutations, when members of the same restricted conjugacy class (restricted in the sense that we restrict ρ to the subgroup $S_n \times S_m$, so that in general $\rho \notin S_{n+m}$) give rise to the same observable.

By restricting to this subgroup, a given irreducible representation $R \vdash n+m$ of S_{n+m} will decompose into a number of irreducible representations of the subgroup $S_n \times S_m$. To label a representation of the subgroup $S_n \times S_m$, we need a Young diagram $r \vdash n$ to label the S_n part, a second Young diagram $s \vdash m$ to label the S_m part, and a multiplicity label α. Since a number of copies of (r,s) can appear when restricting to the subgroup $S_n \times S_m$, α labels a specific copy of (r,s). Finally, let I label a specific state in the carrier space of $(r,s)\alpha$ so that we can abstractly represent our states by $|R,(r,s)\alpha;I\rangle$. We take into account our restriction to $S_n \times S_m$ in the trace structures by now only summing over the state indices I which belong to the specific irreducible representation of the subgroup in question. Denoting this restricted trace by $\chi_{R,(r,s)\alpha\beta}(\sigma)$, we finally have the formula for the *restricted character* ([8]):

$$\begin{aligned}\chi_{R,(r,s)\alpha\beta}(\sigma) &= \mathrm{Tr}_{R,(r,s)\alpha\beta}(\sigma) = \mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma_R(\sigma)) \\ &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\sigma)|R,(r,s)\beta;I\rangle \end{aligned} \tag{2.8}$$

In the above formula, we use the α copy of (r,s) for the row index and the β copy of (r,s) for the column index.

It is equation (2.8), instead of the usual definition of characters of elements in the single matrix case, i.e. $\chi_R(\sigma) = \mathrm{Tr}(\Gamma_R(\sigma))$, upon which many of the following definitions, which are parallel to those already defined, are based. Unfortunately, this complicates matters since it is a much more tedious task to calculate restricted characters compared to usual characters. Luckily, the situations we will consider are such that calculating the restricted characters won't be excessively complicated. So, pressing forward and using this notion of restricted characters, we define the *restricted Schur polynomial*, $\chi_{R,(r,s)\overline{\mu}}(Z,Y)$ ([17]). This restricted Schur polynomial is built from two bosonic fields (Z and Y) and has the specific representation $R,(r,s)\overline{\mu}$, with $r \vdash n$, $s \vdash m$ and $\overline{\mu} = (\mu_1, \mu_2)$. It is given by:

$$\chi_{R,(r,s)\overline{\mu}}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)\overline{\mu}}(\sigma)\mathrm{Tr}(\sigma Y^{\otimes m} \otimes Z^{\otimes n}) \tag{2.9}$$

Equation (2.9) is analogous to equation (2.4). Another result we will need is one akin to equation (2.5), i.e. an equation which is the inverse Fourier transform of equation (2.9):

$$\mathrm{Tr}(\sigma Y^{\otimes m} \otimes Z^{\otimes n}) = \sum_{T,(t,u)\overline{\mu}} \frac{d_T n! m!}{d_t d_u (n+m)!} \chi_{T,(t,u)\overline{\mu}^*}(\sigma^{-1}) \chi_{T,(t,u)\overline{\mu}}(Z,Y) \tag{2.10}$$

where $\overline{\mu} = (\mu_1, \mu_2)$ and $\overline{\mu}^* = (\mu_2, \mu_1)$ are multiplicity labels. And, similarly, we can define projection operators analogous to equation (2.1):

$$P_{R,(r,s)\overline{\mu}} = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)\overline{\mu}}(\sigma)\sigma \tag{2.11}$$

Strictly speaking, equation (2.11) is not a projector: it intertwines two representations of (r, s). Concretely, it obeys the following:

$$P_{R,(r,s)\overline{\mu}}\ \Gamma_{(r,s)\mu_2}(\sigma) = \Gamma_{(r,s)\mu_1}(\sigma)\ P_{R,(r,s)\overline{\mu}} \tag{2.12}$$

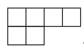
$\forall \sigma \in S_n \times S_m$. For this reason, we need two multiplicity labels to label $P_{R,(r,s)\overline{\mu}}$. Using equation (2.11), we can also define the restricted trace structures in another way:

$$\mathrm{Tr}_{R,(r,s)\overline{\mu}}(\cdots) = \mathrm{Tr}(P_{R,(r,s)\overline{\mu}} \cdots) \tag{2.13}$$

Projection operators will play a vital role in the work to follow since we will need them in order to find different generators which close the $su(2)$ and $su(3)$ algebras. Also, even though we won't show the details here (see [17] for these details), the two-point function based on these restricted Schur polynomials of equation (2.9), the result akin to equation (2.6), is:

$$\langle \chi_{R,(r,s)\alpha\beta}(Z,Y)\chi_{T,(t,u)\gamma\delta}(Z,Y)^\dagger \rangle = \frac{f_R \mathrm{hooks}_R}{\mathrm{hooks}_r \mathrm{hooks}_s} \delta_{RT}\delta_{rt}\delta_{su}\delta_{\beta\delta}\delta_{\alpha\gamma} \tag{2.14}$$

where hooks_R (respectively, r, s) is the products of all of the hook lengths of the boxes comprising the Young diagram R (respectively, r, s). The hook length of a box labeled x in a Young diagram is the number of boxes in the same row and to the right of x, plus the number of boxes in the same column and underneath box x, plus one. As an example, consider the following:

If $R =$ , then $\mathrm{hooks}_R = 5 \cdot 4 \cdot 2 \cdot 1 \cdot 2 \cdot 1$

Next, a quick review of the dilatation operator, D, is useful. The dilatation operator is one of the four operators which generates the superconformal algebra, and it can be viewed as the Hamiltonian operator in certain theories (for example, in radial quantisation of two dimensional CFT's). In the $su(2)$ sector of our theory, it is defined (up to a constant) as:

$$D_2 = \mathrm{Tr}\left([Y, Z] \left[\frac{d}{dY}, \frac{d}{dZ} \right] \right) \tag{2.15}$$

where $Z = \phi_1 + i\phi_2$ and $Y = \phi_3 + i\phi_4$ are complex combinations of four of the six scalar fields ϕ_i ($i = 1, \ldots, 6$) in $\mathcal{N} = 4$ SYM ($X = \phi_5 + i\phi_6$ appears in the $su(3)$ sector). These scalar fields are $N \times N$ hermitian matrices which transform in the adjoint representation of $U(N)$ ([13]). In the $su(3)$ sector (which actually forms part of the larger $su(2|3)$ sector), at one-loop, the form of the dilatation operator is closed:

$$D_3 = \sum_{I,J}^{3} \mathrm{Tr}\left([\phi_I, \phi_J] \left[\frac{d}{d\phi_I}, \frac{d}{d\phi_J} \right] \right) \tag{2.16}$$

We are interested in a subset of $\mathcal{N} = 4$ SYM dealing with *giant gravitons* ([18], [19], [20]), and giant gravitons correspond with the special class of operators called the (restricted) Schur polynomials. The conformal dimensions of these operators equal the energies of the giant graviton states, and since acting with the dilatation operator on an operator gives that operator's conformal dimension, finding the dilatation operator would allow us to calculate the energies of giant graviton states. The dilatation operator has been studied in much previous work (such as [21]), but mainly using approximations and/or in certain subsectors of $\mathcal{N} = 4$ SYM.

Finally, it is worth mentioning why our large N non-planar limit is important. If the number of fields comprising the observables grows like N as $N \to \infty$, we see the emergence of giant gravitons in our theory. If we had to let the number of fields go like N^2 as $N \to \infty$ for example, we would in fact see the emergence of new types of geometries, not gravitons. Since this study wishes to consider giant gravitons, we look at the former of these two types of large N non-planar limits in chapters 3 and 4. It is also important to note that we will exclusively consider Young diagrams with only two rows. This is equivalent to considering a system with two giant gravitons. Such a restriction greatly simplifies the calculations we will encounter because, in such cases, there is no multiplicity of the (r, s) subspaces. As such, we can drop the multiplicity labels in equations (2.8) to (2.14). For a discussion of gravitons and Young diagrams with more than two rows, see ([12]). Having more than two rows in a Young diagram complicates the system tremendously; a system with only one row (i.e. only one giant graviton) would be BPS, and the higher loop corrections to the dilatation operator turn out to be 0. An easy way to see this is to consider what the higher loop corrections to the dilatation operator do to a restricted Schur polynomial on which it is acting: it shifts one or more boxes between the rows in the Young diagram. If the Young diagrams in question have only one row, then the dilatation operator cannot shift boxes between rows when the diagrams are only allowed to have a single row of boxes.

With this requisite background material, we are now in a position to begin tackling the problem at hand. Our main premise is that it is possible to uniquely determine the one- and higher-loop dilatation operators by requiring that they close the specific algebra in question. The number of bosonic fields we will use to construct the restricted Schur polynomial depends on the algebra. We will consider two such algebras: the $su(2)$ and $su(3)$ algebras.

3
$su(2)$ Sector

The basic outline of this chapter is as follows: we first require the exact action of the three $su(2)$ generators, J_-, J_+ and J_3, on restricted Schur polynomials; this is the focus until chapter 3.5. Once we have these results, we can compute the dilatation operator, D_2, since it obeys the commutation relations $[D_2, J_+] = [D_2, J_-] = [D_2, J_3] = 0$ ([21]). Acting these commutators on restricted Schur polynomials (while knowing the action of the three generators) results in a number of equations for the matrix elements of D_2 (chapter 3.6). Solving these equations then finally determines D_2 (chapter 3.7).

Before beginning, it is worth making some remarks about the differences between the $su(2)$ and $su(3)$ sectors. The $su(2)$ algebra is generated by three independent generators, while the $su(3)$ algebra is generated by eight. In the present case, the $su(2)$ generators act on restricted Schur polynomials built from two complex bosonic fields. The projection operators in such a case are not too complex to compute because equation (2.11) assumes a simple form in this case. However, the $su(3)$ generators act on restricted Schur polynomials built from three complex bosonic fields. This extra field, although not complicating the computation of the formulae for the projection operators much, does make the computation of the products of the requisite traces quite complicated. We will also use the $su(2)$ calculations to work towards providing formulae in the $su(3)$ case.

In terms of the complex variables z and y, the $su(2)$ generators are:

$$J_- = z\frac{\partial}{\partial y}, \quad J_+ = y\frac{\partial}{\partial z}, \quad J_3 = y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} \tag{3.1}$$

These generators act on restricted Schur polynomials built from two complex bosonic fields, i.e. equation (2.9). As was the case in the explanation leading to up this equation, $R \vdash n+m, r \vdash n$ and $s \vdash m$. When acting on such objects, the three generators become:

$$J_- = \text{Tr}\left(Z\frac{d}{dY}\right),\ J_+ = \text{Tr}\left(Y\frac{d}{dZ}\right),\ J_3 = \text{Tr}\left(Y\frac{d}{dY} - Z\frac{d}{dZ}\right) \tag{3.2}$$

The action of J_- is:

$$\begin{aligned}
J_-\chi_{R,(r,s)}(Z,Y) &= \text{Tr}\left(Z\frac{d}{dY}\right)\chi_{R,(r,s)}(Z,Y) \\
&= \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)}(\sigma)\text{Tr}\left(Z\frac{d}{dY}\right)\text{Tr}(\sigma Y^{\otimes m}\otimes Z^{\otimes n}) \\
&= \frac{m}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)}(\sigma)\text{Tr}(\sigma Y^{\otimes m-1}\otimes Z^{\otimes n+1}) \\
&= \frac{m}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)}(\sigma)\sum_{T,(t^+,u^-)}\frac{d_T(n+1)!(m-1)!}{d_{t^+}d_{u^-}(n+m)!} \\
&\quad \times\chi_{T,(t^+,u^-)}(\sigma^{-1})\chi_{T,(t^+,u^-)}(Z,Y) \\
&= \frac{m}{n!m!}\sum_{T,(t^+,u^-)}\left[\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)}(\sigma)\chi_{T,(t^+,u^-)}(\sigma^{-1})\right] \\
&\quad \times\frac{d_T(n+1)!(m-1)!}{d_{t^+}d_{u^-}(n+m)!}\chi_{T,(t^+,u^-)}(Z,Y)
\end{aligned} \tag{3.3}$$

In the above calculation, $T \vdash n+m$, $t^+ \vdash n+1$ and $u^- \vdash m-1$ with t^+ subducing r and u^- being subduced by s (a diagram r is said to *subduce* another diagram t if removing one block from r produces t). We have made use of equation (2.10) to arrive at the above. Now, note the following:

$$\begin{aligned}
\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)}(\sigma)\chi_{T,(t^+,u^-)}(\sigma^{-1}) &= \sum_{\sigma\in S_{n+m}} \mathrm{Tr}(P_{R,(r,s)}\Gamma_R(\sigma))\mathrm{Tr}(P_{T,(t^+,u^-)}\Gamma_T(\sigma^{-1})) \\
&= \sum_{\sigma\in S_{n+m}} \left[P_{R,(r,s)}\right]_{a,b} \left[\Gamma_R(\sigma)\right]_{b,a} \left[P_{T,(t^+,u^-)}\right]_{c,d} \left[\Gamma_T(\sigma^{-1})\right]_{d,c} \\
&= \left[P_{R,(r,s)}\right]_{a,b} \left[P_{T,(t^+,u^-)}\right]_{c,d} \sum_{\sigma\in S_{n+m}} \left[\Gamma_R(\sigma)\right]_{b,a} \left[\Gamma_T(\sigma^{-1})\right]_{d,c} \\
&= \left[P_{R,(r,s)}\right]_{a,b} \left[P_{T,(t^+,u^-)}\right]_{c,d} \times \frac{(n+m)!}{d_R}\delta_{RT}\delta_{bc}\delta_{ad} \\
&= \frac{(n+m)!}{d_R}\delta_{RT}\mathrm{Tr}_{R\oplus T}(P_{R,(r,s)}P_{T,(t^+,u^-)}) \qquad (3.4)
\end{aligned}$$

The above is a result of a fundamental orthogonality relation from group theory. Using this to simplify our calculation, we continue:

$$\begin{aligned}
J_-\chi_{R,(r,s)}(Z,Y) &= \frac{m}{n!m!}\sum_{T,(t^+,u^-)}\left[\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)}(\sigma)\chi_{T,(t^+,u^-)}(\sigma^{-1})\right] \\
&\quad \times\frac{d_T(n+1)!(m-1)!}{d_{t^+}d_{u^-}(n+m)!}\chi_{T,(t^+,u^-)}(Z,Y) \\
&= \frac{m}{n!m!}\sum_{T,(t^+,u^-)}\frac{d_T(n+1)!(m-1)!}{d_{t^+}d_{u^-}(n+m)!}\frac{(n+m)!}{d_R}\delta_{RT} \\
&\quad \times\mathrm{Tr}_{R\oplus T}(P_{R,(r,s)}P_{T,(t^+,u^-)})\chi_{T,(t^+,u^-)}(Z,Y) \\
&= \sum_{T,(t^+,u^-)}\frac{d_T(n+1)}{d_{t^+}d_{u^-}}\frac{1}{d_R}\delta_{RT}\mathrm{Tr}_{R\oplus T}(P_{R,(r,s)}P_{T,(t^+,u^-)}) \\
&\quad \times\chi_{T,(t^+,u^-)}(Z,Y) \\
&= \sum_{(t^+,u^-)}\frac{(n+1)}{d_{t^+}d_{u^-}}\mathrm{Tr}(P_{R,(r,s)}P_{R,(t^+,u^-)})\chi_{R,(t^+,u^-)}(Z,Y) \qquad (3.5)
\end{aligned}$$

The action of J_- is to add a box to r and remove a box from s, so the summation runs over those diagrams where t^+ subduces r, s subduces u^- and the diagrams are restricted to having a maximum of two rows. Similar calculations can be carried out for J_+ and J_3 so that, in summary, we have the following ([21]):

$$J_-\chi_{R,(r,s)}(Z,Y) = \sum_{(t^+,u^-)}\frac{(n+1)}{d_{t^+}d_{u^-}}\mathrm{Tr}(P_{R,(r,s)}P_{R,(t^+,u^-)})\chi_{R,(t^+,u^-)}(Z,Y) \qquad (3.6)$$

$$J_+\chi_{R,(r,s)}(Z,Y) = \sum_{(t^-,u^+)} \frac{(m+1)}{d_{t^-}d_{u^+}} \mathrm{Tr}(P_{R,(r,s)}P_{R,(t^-,u^+)})\chi_{R,(t^-,u^+)}(Z,Y) \tag{3.7}$$

$$J_3\chi_{R,(r,s)}(Z,Y) = (m-n)\chi_{R,(r,s)}(Z,Y) \tag{3.8}$$

However, we wish to find the action of these generators on *normalized* versions of our restricted Schur polynomials. Using equation (2.14), we rescale our operators and define new ones, $O_{R,(r,s)}(Z,Y)$ as follows ([21]):

$$\chi_{R,(r,s)}(Z,Y) = \sqrt{\frac{f_R \mathrm{hooks}_R}{\mathrm{hooks}_r \mathrm{hooks}_s}} O_{R,(r,s)}(Z,Y) \tag{3.9}$$

With that, equations (3.6) to (3.8) become:

$$J_-O_{R,(r,s)}(Z,Y) = \sum_{(t^+,u^-)} \frac{(n+1)}{d_{t^+}d_{u^-}} \sqrt{\frac{\mathrm{hooks}_r \mathrm{hooks}_s}{\mathrm{hooks}_{t^+}\mathrm{hooks}_{u^-}}} \times \mathrm{Tr}(P_{R,(r,s)}P_{R,(t^+,u^-)})O_{R,(t^+,u^-)}(Z,Y) \tag{3.10}$$

$$J_+O_{R,(r,s)}(Z,Y) = \sum_{(t^-,u^+)} \frac{(m+1)}{d_{t^-}d_{u^+}} \sqrt{\frac{\mathrm{hooks}_r \mathrm{hooks}_s}{\mathrm{hooks}_{t^-}\mathrm{hooks}_{u^+}}} \times \mathrm{Tr}(P_{R,(r,s)}P_{R,(t^-,u^+)})O_{R,(t^-,u^+)}(Z,Y) \tag{3.11}$$

$$J_3O_{R,(r,s)}(Z,Y) = (m-n)O_{R,(r,s)}(Z,Y) \tag{3.12}$$

Equation (3.10) expresses the action of J_- on $O_{R,(r,s)}(Z,Y)$ as a linear combination of other (normalised) restricted Schur polynomials which themselves are obtained by adding a block to r to form t^+ and removing a block from s to form u^-. Equation (3.11) expresses the action of J_+ on $O_{R,(r,s)}(Z,Y)$ as a linear combination of other (normalised) restricted Schur polynomials which themselves are obtained by removing a block from r to form t^- and adding a block to s to form u^+ ([21]). Since there are a finite number of ways this can be done in both cases, there are a finite number of terms in each summation. So, in order to determine the actions of these two operators (the first two, since the action of J_3 on $O_{R,(r,s)}(Z,Y)$ is trivial), we need to determine the trace of the products of the two projection operators for each term in the summation. In fact, we will see that we only need to compute the matrix elements for either J_- or J_+, since it's possible to determine the matrix elements of the one, given the other. This is because J_+ and J_- are related via hermitian conjugation ([21]). In the next chapter we will derive analytical formulae for the traces of these products of projectors in the case of J_-. Numerical results for specific cases (when m, the number of boxes making up s, is $3, 4$ and 5) appear in the Appendix.

3.1 $su(2)$ Analytical Results

We will here derive analytical formula for the traces of products of projection operators appearing in equations (3.10) and (3.11). These results are novel, extending the formulae given in ([21]) which were derived using the displaced corners approximation. To simplify the formulae, however, we need to learn to express our old variables (which described the shape of the subspace $R, (r, s)$) in terms of the new variables r_1, j, j_3, n, m. To understand this translation of variables, let R_1, R_2; r_1, r_2 and s_1, s_2 be the number of boxes in the first (subscript 1) and second (subscript 2) rows of R, r and s, respectively (these are the old variables with which we worked). These variables represent the row lengths of the Young diagrams. We then make the change:

$$s_1 = \frac{m}{2} + j,\ s_2 = \frac{m}{2} - j,\ R_1 = r_1 + \frac{m}{2} + j_3,\ R_2 = r_2 + \frac{m}{2} - j_3 \tag{3.13}$$

Using the above (and the fact that $r_1 + r_2 = n$), given r_1, j, j_3, n, m, it is possible to return to the $R, (r, s)$ variables. Apart from the above formulae incorporating j_3, another quick way to calculate it is $j_3 = \frac{m_1 - m_2}{2}$, where m_i denotes the number of boxes which are removed from row i of R when forming r and s.

This notation also makes it easy to see exactly how boxes are added, removed or just shifted about when various operators act on the restricted Schur polynomials. For example, when J_+ acts on a restricted Schur polynomial, it adds a Y box and takes away a Z box when comparing the new restricted Schur polynomial to the one the operator is acting on. It does this without changing the sum $n + m$. However, the Y box added to s can be added to the first or second row of s (s_1 or s_2 respectively). If it is added to s_1, by noting (3.13) above, $s_1 \to s_1 + 1$, and $m \to m + 1$, so necessarily, we must have $j \to j + \frac{1}{2}$ if the equations are to balance. Since no box is added to s_2, it remains constant, and the second equation in (3.13) is still obeyed with the new m and j values. If the box is added to s_2 instead, $s_2 \to s_2 + 1$, $m \to m + 1$ and so $j \to j - \frac{1}{2}$. A box is also simultaneously taken away from either r_1 or r_2 when compared with the original restricted Schur polynomial. If it is taken away from r_1, then $n \to n - 1$, $r_1 \to r_1 - 1$, and so $j_3 \to j_3 + \frac{1}{2}$ (because $m \to m + 1$ and R_1 is a constant). Taken away from r_2, we see that $n \to n - 1$, $r_2 \to r_2 - 1$, and so $j_3 \to j_3 - \frac{1}{2}$ (again, because the box was added to s, $m \to m + 1$ and R_2 is a constant). Similar logic can be used when considering the action of J_- and the dilatation operator (although, as we shall see, the dilatation operator only moves boxes about within the diagrams themselves so that n

and m do not change). With this notation, we can compute the four formulae which we will need.

We introduce indices with the following ranges:

$$I, J = 1, 2, ..., m+n; \qquad \alpha, \beta = m+1, m+2, ..., m+n; \qquad a, b = 1, 2, ..., m$$

We will compute the restricted character

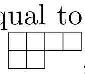
$$\chi_{R,(r,s)}(\,(a,\alpha)\,) = \mathrm{Tr}(P_{R,(r,s)}(a,\alpha)\,) \tag{3.14}$$

By $\sum_{IJ}(I,J)$ we will mean the sum of all distinct two cycles over the corresponding index range. For example, for S_4 with $m=2, n=2$, $\sum_{IJ}(I,J) = (1,2)+(1,3)+(1,4)+(2,3)+(2,4)+(3,4)$, while $\sum_{ab}(a,b) = (1,2)$, $\sum_{\alpha\beta}(\alpha,\beta) = (3,4)$ and $\sum_{a,\alpha}(a,\alpha) = (1,3)+(1,4)+(2,3)+(2,4)$. From this, it is easy to establish the identity:

$$\sum_{IJ}(I,J) = \sum_{ab}(a,b) + \sum_{\alpha\beta}(\alpha,\beta) + \sum_{a,\alpha}(a,\alpha) \tag{3.15}$$

We will use the following formula:

$$\sum_{a,\alpha}(a,\alpha) = \sum_{IJ}(I,J) - \sum_{ab}(a,b) - \sum_{\alpha\beta}(\alpha,\beta) \tag{3.16}$$

The sum over all two cycles is a Casimir with the eigenvalue equal to the number of row pairs minus the number of column pairs ([22]). Given the Young diagram , for example, the number of row pairs minus the number of column pairs is $(6+1)-2=5$. If the row lengths are R_1 and R_2 respectively (where in our example, $R_1 = 4$ and $R_2 = 2$), it can easily be shown that the number of row pairs minus the number of column pairs is $\frac{R_1(R_1-1)}{2} + \frac{R_2(R_2-1)}{2} - R_2$. Furthermore, taking the trace of the sum over all distinct two cycles then amounts to multiplying this eigenvalue by the dimension of the identity operator, which in all cases is $d_r d_s$ when we restrict the symmetry group S_{n+m} to $S_n \times S_m$. As a result it follows that:

$$\sum_{IJ} \mathrm{Tr}(P_{R,(r,s)}(I,J)) = \left(\frac{R_1(R_1-1)}{2} + \frac{R_2(R_2-1)}{2} - R_2\right) d_r d_s \tag{3.17}$$

$$\sum_{\alpha\beta} \mathrm{Tr}(P_{R,(r,s)}(\alpha,\beta)) = \left(\frac{r_1(r_1-1)}{2} + \frac{r_2(r_2-1)}{2} - r_2\right) d_r d_s \tag{3.18}$$

$$\sum_{ab} \mathrm{Tr}(P_{R,(r,s)}(a,b)) = \left(\frac{s_1(s_1-1)}{2} + \frac{s_2(s_2-1)}{2} - s_2\right) d_r d_s \tag{3.19}$$

Using our translation (3.13) and taking the trace of (3.16), it can be shown that:

$$\begin{aligned} \sum_{a,\alpha} \chi_{R,(r,s)}(\,(a,\alpha)\,) &= \frac{2j_3^2 - 2j(j+1) + 2j_3(r_1 - r_2 + 1) + m(r_1 + r_2)}{2} d_r d_s \\ &= nm\chi_{R,(r,s)}(\,(m, m+1)\,) \end{aligned} \tag{3.20}$$

The second equality comes from the fact that we are summing over characters which all belong to the same conjugacy class and are hence equal, and there are nm such characters.

We will be considering J_-, so by studying equation (3.10), we see that there are four traces possible: given $P_{R,(r,s)}$, we can add a block to either the first or second row of r to form t^+, and simultanesouly remove a block from either the first or second row of s to form u^- (resulting in $P_{R,(t^+,u^-)}$).

3.1.1 First and Second Basic Trace Formulae

As usual, we will remove boxes from R, leaving behind r. Assume that we remove m_1 boxes from the first row of R and m_2 boxes from the second row of R. After having done this (to form $R,(r,s)$), the first basic trace we will need assumes that t^+ is given by adding one box to the second row of r. The Young diagram labeling s will be a single row of m boxes. The Young diagram labeling u^- is thus a single row of $m-1$ boxes. We want to compute:

$$T = \mathrm{Tr}(P_{R,(r,s)} P_{R,(t^+,u^-)}) = \sum_I \langle R,(r,s);I|P_{R,(t^+,u^-)}|R,(r,s);I\rangle \tag{3.21}$$

The only states which participate on the right hand side of the above equation have the mth box (the rightmost box of s which, after shuffling, becomes the rightmost box in the second row of t^+) in the second row. Furthermore, only the subspace of s corresponding to u^- contributes. Thus, we need to do the sum over I making sure that we arrange two things: box m must sit in the second row of t^+ and we must project onto the u^- subspace.

Now, projecting to u^- is easy: we know how to make a projector. To fix the position of box m, note that the last box of s, when moved, can be moved to the first or second row of r to form t^+. By fixing the content of this box, we can ensure that it's in the second row. That is the function of the operator in the first set of square brackets in the equation below: when $\sum_{i=m+1}^{m+n}(m,i)$ acts on a state, it gives the content of the mth box. If the state has the mth box in the first row, it's content is r_1, while if it's in the second row, it's content is $r_2 - 1$. So, the operator in square brackets gives either 0 or 1, respectively. Clearly then:

$$\begin{aligned}
T &= \sum_I \langle R,(r,s);I| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \left[\frac{1}{(m-1)!}\sum_{\sigma\in S_{m-1}} \chi_{u^-}(\sigma)\Gamma_R(\sigma)\right] \\
&\quad \times |R,(r,s);I\rangle \\
&= \sum_I \langle R,(r,s);I| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \left[\frac{1}{(m-1)!}\sum_{\sigma\in S_{m-1}} \Gamma_R(\sigma)\right] \\
&\quad \times |R,(r,s);I\rangle \\
&= \sum_I \langle R,(r,s);I| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \left[\frac{1}{(m-1)!}\sum_{\sigma\in S_{m-1}} \Gamma_s(\sigma)\right] \\
&\quad \times |R,(r,s);I\rangle \\
&= \sum_I \langle R,(r,s);I| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] |R,(r,s);I\rangle \\
&= \frac{r_1 d_r d_s}{r_1 - r_2 + 1} - n\frac{\chi_{R,(r,s)}(\,(m,m+1)\,)}{r_1 - r_2 + 1} \\
&= \frac{m - 2j_3}{4m}\left[2 + \frac{m + 2j_3}{r_1 - r_2 + 1}\right] d_r d_s
\end{aligned} \tag{3.22}$$

To get the second equality, we used the fact that the characters in the u^- representation (which is just a row of $m-1$ boxes) are all 1. Also, in the second to last equality, we multiply $\chi_{R,(r,s)}((m,m+1))$ by n rather than nm because the sum $\sum_{i=m+1}^{m+n}(m,i)$ runs over n identical characters. To get the final line, we use equation (3.20) and our translation (equation (3.13)).

Above, we assumed that t^+ is given by adding a box to the second row of r (and a box is removed from the first, and only, row of s). If we instead add a box to the first row of r to form t^+, the above derivation remains the same, except that the first operator in square brackets above changes as follows:

$$\left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \rightarrow \left[\frac{\sum_{i=m+1}^{m+n}(m,i) - r_2 + 1}{r_1 - r_2 + 1}\right]$$

Following the same steps as above, we consequently get:

$$T = \frac{m + 2j_3}{4m}\left[2 + \frac{2j_3 - m}{r_1 - r_2 + 1}\right] d_r d_s \tag{3.23}$$

So, we now have formulae for $\mathrm{Tr}(P_{R,(r,s)}P_{R,(t^+,u^-)})$, where a box is removed from the first row of s to form u^-, and t^+ is formed by adding a box to the first or second row of r. Note that $d_s = 1$ in the above equations. For the case when a box is removed from the second row of s, we need formulae based on sums of traces.

3.1.2 First and Second Sum Formulae

We assume t^+ is given by adding a box to the second row of r (if we wish to add a box to the first row, the operator in square brackets below changes in the same way as before). Now, any given s subduces two possible u^-. We call them u_1^- and u_2^-, where u_1^- is obtained by removing a box from row 1 of s and u_2^- is obtained by removing a box from row 2 of s. Finally, letting $s = u_1^- \oplus u_2^-$, we compute:

$$\begin{aligned}
T &= \sum_{K=1}^{2} \mathrm{Tr}\left(P_{R,(r,s)}P_{R,(t^+,u_K^-)}\right) \\
&= \sum_I \langle R,(r,s);I| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \times \\
&\quad \times \sum_K \left[\frac{1}{(m-1)!}\sum_{\sigma\in S_{m-1}} \chi_{u_K^-}(\sigma)\Gamma_R(\sigma)\right] |R,(r,s);I\rangle \\
&= \sum_I \langle R,(r,s);I| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] \times \\
&\quad \times \left[\frac{1}{m!}\sum_{\sigma\in S_m} \chi_s(\sigma)\Gamma_R(\sigma)\right] |R,(r,s);I\rangle \\
&= \sum_I \langle R,(r,s);I| \left[\frac{r_1 - \sum_{i=m+1}^{m+n}(m,i)}{r_1 - r_2 + 1}\right] |R,(r,s);I\rangle \\
&= \left[\frac{m - 2j_3}{2m} + \frac{2j(j+1) - 2j_3^2 - m}{2m(r_1 - r_2 + 1)}\right] d_r d_s
\end{aligned} \tag{3.24}$$

Note that in the third equality we replaced our sum over projectors onto subspaces with a single projector onto s since we made the choice $s = u_1^- \oplus u_2^-$. If we repeat the above algorithm but instead add a box to the first row of r, we get:

$$T \;=\; \left[\frac{m+2j_3}{2m} - \frac{2j(j+1)-2j_3^2-m}{2m(r_1-r_2+1)}\right] d_r d_s \tag{3.25}$$

We need one more formula.

3.1.3 Third Sum Formula

Any given u^- can be suduced by two possible s, call them s_1 and s_2. If we remove a box from the first row of s_1 we get u^-, and if we remove a box from the second row of s_2 we get u^-. We want to compute:

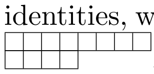
$$T = \sum_{J=1}^{2} \mathrm{Tr}(P_{R,(r,s_J)} P_{R,(t^+,u^-)}) = \mathrm{Tr}(\sum_{J=1}^{2} P_{R,(r,s_J)} P_{R,(t^+,u^-)}) \tag{3.26}$$

We can extend the sum above to a sum over all the possible s's: since the only irreducible representations that can subduce u^- are the two s_J's, the additional terms all vanish. Doing this, $\sum_s P_{R,(r,s)}$ projects us from R to r. Thus, the product $\sum_{J=1}^{2} P_{R,(r,s_J)} P_{R,(t^+,u^-)}$ projects us to $r \oplus \square \oplus u^-$, and hence:

$$T = d_r d_{u^-} \tag{3.27}$$

Using these formulae, we have all we need to calculate the traces appearing in equation (3.10). Before we state the final results, we give an example of how various combinations of these formulae need to be used in succession to get the traces we want.

3.2 Example of Algorithm

Using the two basic and the three sum identities, we can compute any trace we want. Let's illustrate this with an example. Suppose $R =$ , and imagine that we remove 3 boxes from the first row and 2 boxes from the second row of R. We use the removed boxes to form s, and what is left of R is r. The projectors we will need are:

$$P_1 = P_{\yng(8,4),\,\yng(5,2),\,\yng(5)}$$

$$P_2 = P_{\yng(8,4),\,\yng(5,2),\,\yng(4,1)}$$

$$P_3 = P_{\yng(8,4),\,\yng(5,2),\,\yng(3,2)}$$

and, considering the possible $R, (t^+, u^-)$ labels:

$$P_A = P_{\yng(8,4),\,\yng(5,3),\,\yng(4)}$$

$$P_B = P_{\yng(8,4),\,\yng(5,3),\,\yng(3,1)}$$

Note that P_2 will not appear in (3.10) because it does not respect the boundaries of the removed boxes, but we need it in an intermediate step. Firstly, we compute $\text{Tr}(P_1 P_A)$ using our first basic trace formula. Given this, we compute $\text{Tr}(P_2 P_A)$ using our third sum identity. Given this, we compute $\text{Tr}(P_2 P_B)$ using our first sum identity. Given this, we compute $\text{Tr}(P_3 P_B)$ using our third sum identity. This is the complete set of traces needed.

3.3 Final $su(2)$ Trace Results

Equations (3.22) and (3.23) can have their m dependence removed if we note that, in their cases, $0 = s_2 = \frac{m}{2} - j$. Using this to simplify them and moving from the $R, (r, s)$ notation to r_1, j, j_3, n, m notation, we have:

$$\text{Tr}\left(P(r_1, j, j_3, n, m) P(r_1, j - \frac{1}{2}, j_3 + \frac{1}{2}, n + 1, m - 1) \right)$$
$$= \left[\frac{j - j_3}{2j} + \frac{j^2 - j_3^2}{2j(r_1 - r_2 + 1)} \right] d_r d_{u^-} \tag{3.28}$$

$$\text{Tr}\left(P(r_1, j, j_3, n, m) P(r_1 + 1, j - \frac{1}{2}, j_3 - \frac{1}{2}, n + 1, m - 1) \right)$$
$$= \left[\frac{j + j_3}{2j} - \frac{j^2 - j_3^2}{2j(r_1 - r_2 + 1)} \right] d_r d_{u^-} \tag{3.29}$$

For the two remaining formulae, we make use of the sum identities and the two above formulae. To use a sum identity we need to have two of the three terms in the equations. Using them, we have:

$$\begin{aligned}\mathrm{Tr}\left(P(r_1,j,j_3,n,m)P(r_1,j+\frac{1}{2},j_3+\frac{1}{2},n+1,m-1)\right)\\ =\left[\frac{j+j_3+1}{2j+2}-\frac{(j+1)^2-j_3^2}{(2j+2)(r_1-r_2+1)}\right]d_r d_{u^-}\end{aligned} \tag{3.30}$$

$$\begin{aligned}\mathrm{Tr}\left(P(r_1,j,j_3,n,m)P(r_1+1,j+\frac{1}{2},j_3-\frac{1}{2},n+1,m-1)\right)\\ =\left[\frac{j-j_3+1}{2j+2}+\frac{(j+1)^2-j_3^2}{(2j+2)(r_1-r_2+1)}\right]d_r d_{u^-}\end{aligned} \tag{3.31}$$

In the next chapter of this dissertation we shall use these results to construct the exact matrix elements of J_-. But first, we shall evaluate these formulae for specific cases (where m = 3, 4 and 5) below. These results can then be compared with those obtained numerically in the Appendix, which also gives a more detailed explanation of the notation used.

3.3.1 $m = 3$ case

We consider the case where $m = 3$. As an example, take the projector:

$$P^{(2,1)}_{\ydiagram{3}} = P_{\ytableaushort{{}{}{}{}{}{}**,{}{}{}*},\ydiagram{3}} \tag{3.32}$$

The stars represent the boxes removed from R. Note also that we will always consider projectors where one block is removed from the second row of R, and the rest from the first. Since we end up only considering the traces of these projectors, it would not matter if we chose to take more blocks from the second row. This case is merely the simplest to consider. Using the translation (3.13) and the representation for s, i.e. $\ydiagram{3}$, we find $j=\frac{m}{2}-s_2=\frac{3}{2}-0=\frac{3}{2}, j_3=\frac{m_1-m_2}{2}=\frac{2-1}{2}=\frac{1}{2}$, and for the other possible projector:

$$P^{(2,1)}_{\ydiagram{2,1}} = P_{\ytableaushort{{}{}{}{}{}{}**,{}{}{}*},\ydiagram{2,1}} \tag{3.33}$$

we have $j=\frac{m}{2}-s_2=\frac{3}{2}-1=\frac{1}{2}, j_3=\frac{m_1-m_2}{2}=\frac{2-1}{2}=\frac{1}{2}$. By using the variables relevant to projector (3.32) and using equations (3.28) and (3.29), we respectively get:

$$\mathrm{Tr}(P^{(2,1)}_{\yng(3)} P^{(2)}_{\yng(2)}) = \frac{1}{3}\left(1 + \frac{2}{r_1 - r_2 + 1}\right) d_r$$

$$\mathrm{Tr}(P^{(2,1)}_{\yng(3)} P^{(1,1)}_{\yng(2)}) = \frac{2}{3}\left(1 - \frac{1}{r_1 - r_2 + 1}\right) d_r$$

Note that in these specific examples, we know what d_{u^-} is. Another important point to note is that only traces in which s subduces u^- can be non-zero (see the Appendix for a proof of this). Indeed, only representations in which s subduces u^- appear in equation (3.10). If s does not subduce u^-, then the trace has to be zero. Since $\yng(3)$ does not subduce $\yng(1,1)$ when a single box is removed from $\yng(3)$, the following trace is necessarily zero:

$$\mathrm{Tr}(P^{(2,1)}_{\yng(3)} P^{(1,1)}_{\yng(1,1)}) = 0$$

Next, by substituting the variables relevant to projector (3.33) into equations (3.30), (3.31) and (3.29) respectively, we get:

$$\mathrm{Tr}(P^{(2,1)}_{\yng(2,1)} P^{(2)}_{\yng(2)}) = \frac{2}{3}\left(1 - \frac{1}{r_1 - r_2 + 1}\right) d_r$$

$$\mathrm{Tr}(P^{(2,1)}_{\yng(2,1)} P^{(1,1)}_{\yng(2)}) = \frac{1}{3}\left(1 + \frac{2}{r_1 - r_2 + 1}\right) d_r$$

$$\mathrm{Tr}(P^{(2,1)}_{\yng(2,1)} P^{(1,1)}_{\yng(1,1)}) = d_r$$

Comparing these results with those numerically obtained in the Appendix, we see that they match perfectly (save for the d_r factor). We repeat the above process for the $m = 4$ and $m = 5$ cases.

3.3.2 $m = 4$ case

For the projector:

$$P^{(3,1)}_{\yng(4)} = P_{\young(\ \ \ \ \ ***,\ \ \ *),\yng(4)} \tag{3.34}$$

we have $j = 2, j_3 = 1$, and for the projector:

$$P^{(3,1)}_{\yng(3,1)} = P_{\young(\ \ \ \ \ ***,\ \ \ *),\yng(3,1)} \tag{3.35}$$

we have $j = 1, j_3 = 1$. By substituting the variables relevant to projector (3.34) into equations (3.28) and (3.29), we get:

$$\mathrm{Tr}(P^{(3,1)}_{\yng(4)} P^{(3)}_{\yng(3)}) = \frac{1}{4}\left(1 + \frac{3}{r_1 - r_2 + 1}\right) d_r$$

$$\mathrm{Tr}(P^{(3,1)}_{\yng(4)} P^{(2,1)}_{\yng(3)}) = \frac{3}{4}\left(1 - \frac{1}{r_1 - r_2 + 1}\right) d_r$$

Since $\yng(4)$ does not subduce $\yng(2,1)$, the following trace is zero:

$$\mathrm{Tr}(P^{(3,1)}_{\yng(4)} P^{(2,1)}_{\yng(2,1)}) = 0$$

Using projector (3.35)'s variables and equations (3.30), (3.31) and (3.29), we respectively get:

$$\mathrm{Tr}(P^{(3,1)}_{\yng(3,1)} P^{(3)}_{\yng(3)}) = \frac{3}{4}\left(1 - \frac{1}{r_1 - r_2 + 1}\right) d_r$$

$$\mathrm{Tr}(P^{(3,1)}_{\yng(3,1)} P^{(2,1)}_{\yng(3)}) = \frac{1}{4}\left(1 + \frac{3}{r_1 - r_2 + 1}\right) d_r$$

$$\mathrm{Tr}(P^{(3,1)}_{\yng(3,1)} P^{(2,1)}_{\yng(2,1)}) = 2d_r$$

As before, except for the d_r factor, these results match the numerical ones.

3.3.3 $m = 5$ case

For the projector:

$$P^{(4,1)}_{\yng(5)} = P_{\young(\ \ \ \ \ \ast\ast\ast\ast,\ \ \ \ast),\yng(5)} \tag{3.36}$$

we have $j = \frac{5}{2}, j_3 = \frac{3}{2}$, and for the projector:

$$P^{(4,1)}_{\yng(4,1)} = P_{\young(\ \ \ \ \ \ast\ast\ast\ast,\ \ \ \ast),\yng(4,1)} \tag{3.37}$$

we have $j = \frac{3}{2}, j_3 = \frac{3}{2}$.

Repeating the same process, using equations (3.28) and (3.29) we get:

$$\mathrm{Tr}(P^{(4,1)}_{\yng(5)} P^{(4)}_{\yng(4)}) = \frac{1}{5}\left(1 + \frac{4}{r_1 - r_2 + 1}\right) d_r$$

$$\mathrm{Tr}(P^{(4,1)}_{\yng(5)} P^{(3,1)}_{\yng(4)}) = \frac{4}{5}\left(1 - \frac{1}{r_1 - r_2 + 1}\right) d_r$$

As before, the following trace is zero:

$$\mathrm{Tr}(P^{(4,1)}_{\yng(5)} P^{(3,1)}_{\yng(3,1)}) = 0$$

For projector (3.37), equations (3.30), (3.31) and (3.29) give:

$$\mathrm{Tr}(P^{(4,1)}_{\yng(4,1)} P^{(4)}_{\yng(4)}) = \frac{4}{5}\left(1 - \frac{1}{r_1 - r_2 + 1}\right) d_r$$

$$\mathrm{Tr}(P^{(4,1)}_{\yng(4,1)} P^{(3,1)}_{\yng(4)}) = \frac{1}{5}\left(1 + \frac{4}{r_1 - r_2 + 1}\right) d_r$$

$$\mathrm{Tr}(P^{(4,1)}_{\yng(4,1)} P^{(3,1)}_{\yng(3,1)}) = 3d_r$$

We can substitute these traces into equation (3.10) to find the action of J_- on a restricted Schur polynomial for these specific cases.

3.4 Hermitian Conjugation

Given J_-, it is possible to determine J_+ (or vice versa). Using an abstract vector notation, equation (3.10) is:

$$J_-|O_{R,(r,s)}(Z,Y)\rangle = \sum_{(t^+,u^-)} F(R,r,s,t^+,u^-)|O_{R,(t^+,u^-)}(Z,Y)\rangle \tag{3.38}$$

where $F(R,r,s,t^+,u^-)$ is the matrix element of $O_{R,(t^+,u^-)}(Z,Y)$ in equation (3.10). Assuming an orthonormal basis, we see that:

$$F(R,r,s,t^+,u^-) = \langle O^\dagger_{R,(t^+,u^-)}(Z,Y)|\mathrm{Tr}\left(Z\frac{d}{dY}\right)|O_{R,(r,s)}(Z,Y)\rangle \tag{3.39}$$

using equations (3.2) to rewrite J_-. Now, consider two results from quantum field theory (specifically, when studying Wick contractions):

$$(Y^\dagger)^i_j Y^k_l = \delta^k_j \delta^i_l \tag{3.40}$$

$$\frac{d}{dY^j_i} Y^k_l = \delta^k_j \delta^i_l \tag{3.41}$$

Clearly, the two are closely related. Using this, we find the conjugate of equation (3.39):

$$
\begin{aligned}
&(\langle O^{\dagger}_{R,(t^+,u^-)}(Z,Y)|\mathrm{Tr}\left(Z\frac{d}{dY}\right)|O_{R,(r,s)}(Z,Y)\rangle)^* \\
&\qquad = (\langle O^{\dagger}_{R,(t^+,u^-)}(Z,Y)|\mathrm{Tr}\left(ZY^{\dagger}\right)|O_{R,(r,s)}(Z,Y)\rangle)^* \\
&\qquad = \langle O^{\dagger}_{R,(r,s)}(Z,Y)|\mathrm{Tr}\left(YZ^{\dagger}\right)|O_{R,(t^+,u^-)}(Z,Y)\rangle \\
&\qquad = \langle O^{\dagger}_{R,(r,s)}(Z,Y)|\mathrm{Tr}\left(Y\frac{d}{dZ}\right)|O_{R,(t^+,u^-)}(Z,Y)\rangle
\end{aligned} \tag{3.42}
$$

Notice that $\mathrm{Tr}(Y\frac{d}{dZ}) = J_+$, and hence it is easy to see that the above is the matrix element of the 'reverse' restricted Schur polynomial in the summation when J_+ acts on $O_{R,(r,s)}(Z,Y)$. For example, given a representation $R,(r,s)$, suppose that $R,(t^+,u^-)$ is the representation formed when a box is removed from the top row of s (forming u^-) and a box is added to the bottom row of r (forming t^+). This is one of the terms in the result of J_- acting on $O_{R,(r,s)}(Z,Y)$. Equation (3.39) is the coefficient of $O_{R,(t^+,u^-)}(Z,Y)$ in equation (3.10). However, equation (3.42) shows that the conjugate of this matrix element gives the coefficient of $O_{R,(t^-,u^+)}(Z,Y)$ in equation (3.11), where, given a representation $R,(r,s)$, $R,(t^-,u^+)$ is the representation formed (with J_+ acting on $O_{R,(r,s)}(Z,Y)$) when a box is removed from the bottom row of r (forming t^-) and a box is added to the top row of s (forming u^+). Using this simple rule, one can compute the action of J_+ immediately upon calculating J_-.

The trace structures found in the $m = 3, 4, 5$ cases above, when multiplied with the other relevant factors in equations (3.10) and (3.11), give matrix elements which are exact but were found by choosing specific representations from the outset. In the next chapter we will give general formulae for the matrix elements of the two generators expressed in terms of our r_1, j, j_3, n, m variables. These formulae combine the traces mentioned in the previous chapter with the extra factors present in formulae (3.10) and (3.11), resulting in the complete form of the matrix elements.

3.5 $su(2)$ Generators' Exact Matrix Elements

The previous chapter considered specific diagrams in an attempt to learn the applicable methods in such calculations. Before continuing, we will rewrite $O_{R,(r,s)}(Z,Y)$ in terms of our new variables as such:

$$
O_{R,(r,s)}(Z,Y) \equiv O^{(n,m)}(r_1,j,j_3)(Z,Y) \tag{3.43}
$$

We will drop the (Z,Y) dependence from now on; it is implied. Equations (3.10) and (3.11) then become:

$$
\begin{aligned}
J_- O^{(n,m)}(r_1, j, j_3) = & A_- O^{(n+1,m-1)}(r_1+1, j+\frac{1}{2}, j_3-\frac{1}{2}) \\
& + B_- O^{(n+1,m-1)}(r_1+1, j-\frac{1}{2}, j_3-\frac{1}{2}) \\
& + C_- O^{(n+1,m-1)}(r_1, j+\frac{1}{2}, j_3+\frac{1}{2}) \\
& + D_- O^{(n+1,m-1)}(r_1, j-\frac{1}{2}, j_3+\frac{1}{2})
\end{aligned}
\tag{3.44}
$$

and

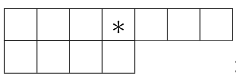
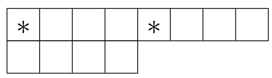
$$
\begin{aligned}
J_+ O^{(n,m)}(r_1, j, j_3) = & A_+ O^{(n-1,m+1)}(r_1-1, j+\frac{1}{2}, j_3+\frac{1}{2}) \\
& + B_+ O^{(n-1,m+1)}(r_1-1, j-\frac{1}{2}, j_3+\frac{1}{2}) \\
& + C_+ O^{(n-1,m+1)}(r_1, j+\frac{1}{2}, j_3-\frac{1}{2}) \\
& + D_+ O^{(n-1,m+1)}(r_1, j-\frac{1}{2}, j_3-\frac{1}{2})
\end{aligned}
\tag{3.45}
$$

In ([21]), the displaced corners approximation was used to find the four matrix elements of J_- and J_+. We will derive exact expressions here.

A_- corresponds to equation (3.31). If we absorb the $d_r d_{u^-}$ factors in equation (3.31) into the $\frac{(n+1)}{d_{t^+} d_{u^-}} \sqrt{\frac{\text{hooks}_r \text{hooks}_s}{\text{hooks}_{t^+} \text{hooks}_{u^-}}}$ factor in equation (3.10) (using the outline given in ([21])), A_- is:

$$
A_- = \sqrt{\frac{\text{hooks}_{t^+} \text{hooks}_s}{\text{hooks}_r \text{hooks}_{u^-}}} \left[\frac{j - j_3 + 1}{2j+2} + \frac{(j+1)^2 - j_3^2}{(2j+2)(r_1 - r_2 + 1)} \right]
$$

A_- corresponds with the diagrams found by removing a box from the second row of s to form u^- and adding a box to the first row of r to form t^+. So, $\sqrt{\frac{\text{hooks}_{t^+}}{\text{hooks}_r}}$ will be the square root of the hook lengths of boxes in t^+ with hook lengths not canceling with equal hook lengths of boxes in t^+, over the hook lengths of boxes in r which don't cancel with the hook lengths of boxes in t^+ (and similarly for $\sqrt{\frac{\text{hooks}_s}{\text{hooks}_{u^-}}}$). For example, consider r, with row lengths r_1 and r_2, and t^+ which, in the case of A_-, has row lengths of $r_1 + 1$ and r_2, respectively:

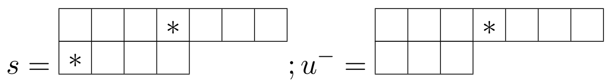
$r =$  ; $t^+ =$ 

When considering the ratio of the hooks, the hook length of the box in r marked with a $*$ (which, in general, is $r_1 - r_2 + 2$) and the hook lengths of the boxes in t^+ marked with $*$'s (which are $r_1 + 2$

and $r_1 - r_2 + 1$ in general) will not cancel. This can easily be checked by labeling the hook lengths of a Young diagram with arbitrary row lengths and noting which do not cancel with one another. In our graphs above which are drawn with $r_1 = 7$ and $r_2 = 4$, the starred box in r has a hook length of 5, while no box in t^+ has a hook length of 5. Similarly, the starred boxes in t^+ have hook lengths of 9 and 4, while no boxes in r have hook lengths of 9 or 4. So, we have:

$$\sqrt{\frac{\text{hooks}_{t^+}}{\text{hooks}_r}} = \sqrt{\frac{(r_1+2)(r_1-r_2+1)}{(r_1-r_2+2)}}$$

We follow an identical procedure for $\sqrt{\frac{\text{hooks}_s}{\text{hooks}_{u^-}}}$. Let s have row lengths of s_1 and s_2, and in the case of A_-, u^- will have row lengths of s_1 and $s_2 - 1$, respectively:



The hook lengths of the boxes marked with $*$'s above do not cancel. In s, they have hook lengths of s_2 and $s_1 - s_2 + 2$, and in u^-, $s_1 - s_2 + 1$. The hook lengths of all other boxes cancel. So, using equations (3.13), we have:

$$\sqrt{\frac{\text{hooks}_s}{\text{hooks}_{u^-}}} = \sqrt{\frac{s_2(s_1-s_2+2)}{s_1-s_2+1}} = \sqrt{\frac{m-2j}{2}\frac{2j+2}{2j+1}}$$

Combining all of this, we thus have:

$$\begin{aligned} A_- &= \sqrt{\frac{(r_1+2)(r_1-r_2+1)}{(r_1-r_2+2)}}\sqrt{\frac{m-2j}{2}\frac{2j+2}{2j+1}} \\ &\quad \times \left[\frac{j-j_3+1}{2j+2} + \frac{(j+1)^2-j_3^2}{(2j+2)(r_1-r_2+1)}\right] \end{aligned} \tag{3.46}$$

The computation of B_-, C_- and D_- follows exactly the same outline. We calculate $\sqrt{\frac{\text{hooks}_{t^+}}{\text{hooks}_r}}$ and $\sqrt{\frac{\text{hooks}_s}{\text{hooks}_{u^-}}}$ for the three remaining cases. The results depend on whether blocks are added to either the first or second row of r to form t^+, and are taken away from either the first or second row of s to form u^-. The product of these two square roots is then multiplied with the right-hand sides of equations (3.29), (3.30) or (3.28) (without the d_r factor), for B_-, C_- and D_- respectively.

Doing this, we finally get:

$$\begin{aligned} B_- &= \sqrt{\frac{(r_1+2)(r_1-r_2+1)}{(r_1-r_2+2)}}\sqrt{\frac{m+2j+2}{2}\frac{2j}{2j+1}} \\ &\quad\times\left[\frac{j+j_3}{2j}-\frac{j^2-j_3^2}{2j(r_1-r_2+1)}\right] \end{aligned} \tag{3.47}$$

$$\begin{aligned} C_- &= \sqrt{\frac{(r_2+1)(r_1-r_2+1)}{(r_1-r_2)}}\sqrt{\frac{m-2j}{2}\frac{2j+2}{2j+1}} \\ &\quad\times\left[\frac{j+j_3+1}{2j+2}-\frac{(j+1)^2-j_3^2}{(2j+2)(r_1-r_2+1)}\right] \end{aligned} \tag{3.48}$$

$$\begin{aligned} D_- &= \sqrt{\frac{(r_2+1)(r_1-r_2+1)}{(r_1-r_2)}}\sqrt{\frac{m+2j+2}{2}\frac{2j}{2j+1}} \\ &\quad\times\left[\frac{j-j_3}{2j}+\frac{j^2-j_3^2}{2j(r_1-r_2+1)}\right] \end{aligned} \tag{3.49}$$

So, we have found $J_-O^{(n,m)}(r_1,j,j_3)$ (equation (3.44)). To get the four matrix elements of J_+, we could repeat the above algorithm, except now removing a box from r to form t^- and adding a box to s to form u^+. However, a quicker method would be to simply find the hermitian conjugates of the J_- matrix elements.

To tackle the problem using this trick, we first need to identify which matrix elements of J_+ are the hermitian conjugates of the matrix elements of J_-. A matrix element of J_+ is the hermitian conjugate of the matrix element of J_- which 'undoes' it. For example, consider C_+. It is the coefficient of the normalised restricted Schur polynomial with diagrams obtained by removing a box from the second row of r and adding a box to the top row of s. The 'opposite' of this would be removing a box from the top row of s and adding a box to the second row of r, which corresponds with the $D_-O^{(n+1,m-1)}(r_1,j-\frac{1}{2},j_3+\frac{1}{2})$ term in equation (3.44). So, C_+ is the matrix element corresponding to D_-. We have a formula for D_- above. To find C_+ from this, note that C_+ corresponds to removing a box from the second row of r and adding a box to the top row of s. Using equations (3.13), this means $s_1\to s_1+1$, $m\to m+1$, which implies that $j\to j+\frac{1}{2}$. Furthermore, $r_2\to r_2-1$, and since R_2 is constant, $j_3\to j_3-\frac{1}{2}$. Inserting these new values for r_1,r_2,j,j_3 and m into the expression for D_- gives us an expression for C_+. The same procedure can be followed to find expressions for A_+, B_+ and D_+ so that, finally, we have:

$$\begin{aligned} A_+ &= \sqrt{\frac{(r_1+1)(r_1-r_2)}{(r_1-r_2+1)}}\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}} \\ &\quad \times \left[\frac{j+j_3+1}{2j+1} - \frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2)}\right] \end{aligned} \tag{3.50}$$

$$\begin{aligned} B_+ &= \sqrt{\frac{(r_1+1)(r_1-r_2)}{(r_1-r_2+1)}}\sqrt{\frac{m-2j+2}{2}\frac{2j+1}{2j}} \\ &\quad \times \left[\frac{j-j_3}{2j+1} + \frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2)}\right] \end{aligned} \tag{3.51}$$

$$\begin{aligned} C_+ &= \sqrt{\frac{r_2(r_1-r_2+2)}{(r_1-r_2+1)}}\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}} \\ &\quad \times \left[\frac{j-j_3+1}{2j+1} + \frac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2)}\right] \end{aligned} \tag{3.52}$$

$$\begin{aligned} D_+ &= \sqrt{\frac{r_2(r_1-r_2+2)}{(r_1-r_2+1)}}\sqrt{\frac{m-2j+2}{2}\frac{2j+1}{2j}} \\ &\quad \times \left[\frac{j+j_3}{2j+1} - \frac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2)}\right] \end{aligned} \tag{3.53}$$

This completes the derivation of the exact $su(2)$ elements, since equation (3.12) (the action of J_3 on $O^{(n,m)}(r_1, j, j_3)$) is of no real concern because of its extremely simple form.

In the next chapter we show that using the fact that the dilatation operator, D_2, closes the $su(2)$ algebra allows one to determine D_2 itself.

3.6 Symmetry Conditions in the $su(2)$ Sector

It is posited that the dilatation operator, D_2, can be found by exploiting the following ([21]):

$$[D_2, J_+] = [D_2, J_-] = [D_2, J_3] = 0 \tag{3.54}$$

This condition does not determine D_2 uniquely: if D_2 is a solution to the above, then so is

$\lambda D_2 + c\mathbb{I}$. However, we can fix c by requiring that the BPS state has a vanishing anomalous dimension. This implies that the smallest eigenvalue is 0. So, the above condition determines D_2 up to λ, which is arbitrary (our analysis cannot fix it). We have the matrix elements for the three generators. To continue, we first need the general structure of D_2 acting on $O^{(n,m)}(r_1, j, j_3)$.

The p-loop dilatation operator, when acting on a restricted Schur polynomial, shifts at most p boxes around in each of the Young diagrams labeling the restricted Schur polynomial ([21]). The boxes, if shifted, are shifted around within their original diagrams; boxes are not shifted between diagrams (as was the case when calculating the $su(2)$ generators). For the one-loop dilatation operator, when considering the first row of r, a box can be added to it, taken away from it, or it can be left as is. So, the possible values r_1 can assume under the action of the one-loop dilatation operator are r_1 or $r_1 \pm 1$ (note that, given the change in r_1, we can deduce the new r_2 value since $r_1 + r_2 = n$). Furthermore, studying the translations given in (3.13), we see that if the shape of s is left alone then j remains j, if a box is added to the first row of s then j becomes $j + 1$, and if a box is taken away from the first row of s then j becomes $j - 1$. Similar logic gives the possible new values of j_3 as j_3 or $j_3 \pm 1$ ([21]). Therefore, when the one-loop dilatation operator acts on $O^{(n,m)}(r_1, j, j_3)$, 27 outcomes are possible:

$$D_2 O^{(n,m)}(r_1, j, j_3) = \sum_{c=-1}^{1} \sum_{d=-1}^{1} \sum_{e=-1}^{1} \beta^{(n,m)}_{r_1,j,j_3}(c, d, e) O^{(n,m)}(r_1 + c, j + d, j_3 + e) \tag{3.55}$$

The $\beta^{(n,m)}_{r_1,j,j_3}(c, d, e)$ coefficients are the matrix elements of the one-loop dilatation operator. More generally, a similar argument gives the following action of the p-loop dilatation operator:

$$D_2 O^{(n,m)}(r_1, j, j_3) = \sum_{c=-p}^{p} \sum_{d=-p}^{p} \sum_{e=-p}^{p} \beta^{(n,m)}_{r_1,j,j_3}(c, d, e) O^{(n,m)}(r_1 + c, j + d, j_3 + e) \tag{3.56}$$

Equation (3.55) is not the technical definition of the one-loop dilatation operator: this equation merely shows that the result of it acting on the restricted Schur polynomial is to shift at most one box around in each Young diagram in the representation. The actual definition is ([21], [23], [24]):

$$D_2 = -g^2_{YM}[Y, Z][\partial_Y, \partial_Z] \tag{3.57}$$

We can see that the commutators in the above definition result in a varied combination of derivatives and Z, Y factors acting on the restricted Schur polynomial. The pair of derivatives themselves result in a Kronecker delta function. This Kronecker delta, when summing over S_{n+m} (this summation coming from the restricted Schur polynomial definition), results in a summation over the S_{n+m-1} subgroup instead. A consequence of this is that one of the representations of S_{n+m-1} subduced by T (where T is the representation of one of the restricted Schur polynomials resulting from the action of the dilatation operator on the restricted Schur polynomial with representation R) equals one of those subduced by R. Summing over S_{n+m-1} singles out one of the boxes in the diagrams, and D_2 shifts the position of a single box. This shifting of a single box in the one-loop dilatation operator is related to summing over one-loop Feynman diagrams because at one loop the diagrams have a single vertex, and Wick contracting with the vertex creates a Kronecker delta because of two indices being set equal ([23]).

There are 27 matrix elements. However, most of them turn out to be zero. In the large N limit we're interested in, the string coupling constant (a dynamical scalar field that determines the strength of forces in an interaction) goes to zero. Within the context of strings, this means that there is no splitting of closed strings or joining of open string ends into closed strings. The open string Chan-Paton factors (factors which make the strings transform under gauge groups) are therefore unaltered. And, within our two giant graviton system, j_3 tells us the number of open strings' end points attached to these gravitons. Since the number of open strings on each giant graviton does not change, j_3 is conserved and hence the matrix elements of the dilatation operator which change the j_3 value must be zero. This is the case when $e = \pm 1$, so $\beta^{(n,m)}_{r_1,j,j_3}(c,d,\pm 1) = 0$. This leaves us with the following form for the one-loop dilatation operator:

$$D_2 O^{(n,m)}(r_1, j, j_3) = \sum_{c=-1}^{1} \sum_{d=-1}^{1} \beta^{(n,m)}_{r_1,j,j_3}(c,d) O^{(n,m)}(r_1 + c, j + d, j_3) \tag{3.58}$$

where $\beta^{(n,m)}_{r_1,j,j_3}(c,d,0) = \beta^{(n,m)}_{r_1,j,j_3}(c,d)$. There are now 9 terms to consider. As it turns out, a further 5 of these are also zero. However, to simplify the calculations, this will be taken into account later.

Before we continue, note that equation (3.45) can be rewritten as:

$$J_+ O^{(n,m)}_{r_1,j,j_3} = \sum_{a=-1}^{0} \sum_{b=-\frac{1}{2}}^{\frac{1}{2}} \alpha^{(n,m)}_{r_1,j,j_3}(a,b) O^{(n-1,m+1)}_{r_1+a,j+b,j_3-\frac{1}{2}-a} \tag{3.59}$$

with $\alpha^{(n,m)}_{r_1,j,j_3}(-1,-\frac{1}{2}) = B_+$, $\alpha^{(n,m)}_{r_1,j,j_3}(-1,\frac{1}{2}) = A_+$, $\alpha^{(n,m)}_{r_1,j,j_3}(0,-\frac{1}{2}) = D_+$ and $\alpha^{(n,m)}_{r_1,j,j_3}(0,\frac{1}{2}) = C_+$. Using equation (3.45) and (3.58), we will impose the first symmetry condition in (3.54):

$$
\begin{aligned}
[D_2,J_+]O^{(n,m)}_{r_1,j,j_3} =& D_2J_+O^{(n,m)}_{r_1,j,j_3} - J_+D_2O^{(n,m)}_{r_1,j,j_3} \\
=& \sum_{a=-1}^{0}\sum_{b=-\frac{1}{2}}^{\frac{1}{2}} \alpha^{(n,m)}_{r_1,j,j_3}(a,b)D_2O^{(n-1,m+1)}_{r_1+a,j+b,j_3-\frac{1}{2}-a} \\
& - \sum_{c=-1}^{1}\sum_{d=-1}^{1} \beta^{(n,m)}_{r_1,j,j_3}(c,d)J_+O^{(n,m)}_{r_1+c,j+d,j_3} \\
=& \sum_{a=-1}^{0}\sum_{b=-\frac{1}{2}}^{\frac{1}{2}} \alpha^{(n,m)}_{r_1,j,j_3}(a,b)\sum_{c=-1}^{1}\sum_{d=-1}^{1}\Big[\beta^{(n-1,m+1)}_{r_1+a,j+b,j_3-\frac{1}{2}-a}(c,d) \\
& \times O^{(n-1,m+1)}_{r_1+a+c,j+b+d,j_3-\frac{1}{2}-a}\Big] - \sum_{c=-1}^{1}\sum_{d=-1}^{1} \beta^{(n,m)}_{r_1,j,j_3}(c,d) \\
& \times \sum_{a=-1}^{0}\sum_{b=-\frac{1}{2}}^{\frac{1}{2}} \alpha^{(n,m)}_{r_1+c,j+d,j_3}(a,b)O^{(n-1,m+1)}_{r_1+a+c,j+b+d,j_3-\frac{1}{2}-a} \\
=& \sum_{a=-1}^{0}\sum_{b=-\frac{1}{2}}^{\frac{1}{2}}\sum_{c=-1}^{1}\sum_{d=-1}^{1}\Big[\alpha^{(n,m)}_{r_1,j,j_3}(a,b)\beta^{(n-1,m+1)}_{r_1+a,j+b,j_3-\frac{1}{2}-a}(c,d) \\
& - \alpha^{(n,m)}_{r_1+c,j+d,j_3}(a,b)\beta^{(n,m)}_{r_1,j,j_3}(c,d)\ \Big]\ O^{(n-1,m+1)}_{r_1+a+c,j+b+d,j_3-\frac{1}{2}-a} \\
=& 0
\end{aligned}
\tag{3.60}
$$

Equation (3.44) can be written as an expression akin to equation (3.59):

$$
J_-O^{(n,m)}_{r_1,j,j_3} = \sum_{a=0}^{1}\sum_{b=-\frac{1}{2}}^{\frac{1}{2}} \gamma^{(n,m)}_{r_1,j,j_3}(a,b)O^{(n+1,m-1)}_{r_1+a,j+b,j_3+\frac{1}{2}-a} \tag{3.61}
$$

The above calculation could be carried out in the $[D_2, J_-]$ case too. However, we do not need to consider this case since no new information would result from it. The $[D_2, J_3]$ case is trivial, given the simplicity of equation (3.12).

Now, since $O^{(n,m)}_{r_1,j,j_3}$ are all linearly independent, their coefficients in the above summation are separately zero. However, it is important to keep in mind that different combinations of a, b, c and d can result in the same operator $O^{(n-1,m+1)}_{r_1+a+c,j+b+d,j_3-\frac{1}{2}-a}$ because of the combinations of variables that appear. Since only a appears in the expression $j_3 - \frac{1}{2} - a$, different values of a will result in independent restricted Schur polynomials. Therefore, different c values also result in independent restricted Schur polynomials (since the only appearance of c in $O^{(n-1,m+1)}_{r_1+a+c,j+b+d,j_3-\frac{1}{2}-a}$ is in the expression r_1+a+c). If $b=\frac{1}{2}$ and $d=1$, then the coefficient of $O^{(n-1,m+1)}_{r_1+a+c,j+\frac{3}{2},j_3-\frac{1}{2}-a}$ in equation (3.60) consists of only two terms. But, $b=-\frac{1}{2}, d=0$ or $b=\frac{1}{2}, d=-1$ will both give the same restricted Schur polynomial $O^{(n-1,m+1)}_{r_1+a+c,j-\frac{1}{2},j_3-\frac{1}{2}-a}$ (for any a,c values), so this restricted Schur polynomial will have four terms in its coefficient.

3.7 Continuum Limit, Leading Order in n and Partial Differential Equations

Consider the term where $a=0, b=\frac{1}{2}, c=1=d$. The coefficient of $O^{(n-1,m+1)}_{r_1+1,j+\frac{3}{2},j_3-\frac{1}{2}}$, which equals 0, is then:

$$\alpha^{(n,m)}_{r_1,j,j_3}(0,\frac{1}{2})\beta^{(n-1,m+1)}_{r_1,j+\frac{1}{2},j_3-\frac{1}{2}}(1,1)-\alpha^{(n,m)}_{r_1+1,j+1,j_3}(0,\frac{1}{2})\beta^{(n,m)}_{r_1,j,j_3}(1,1) \tag{3.62}$$

We are interested in the limit $N\to\infty$. n is kept on the order of N and m on the order of $\sqrt{N}$, so in this limit our variables r_1, r_2, j and j_3 can be taken to be continuous because of their large sizes. The following substitutions are made:

$$s_1-s_2=2j=2\sqrt{m}x_j,\ r_1-r_2=2k=2\sqrt{n}x,\ m_1-m_2=2j_3=2\sqrt{m}x_{j_3} \tag{3.63}$$

We make the change of notation $\alpha^{(n,m)}_{r_1,j,j_3}(a,b)\leftrightarrow\alpha_{a,b}(x,x_j,x_{j_3},n,m)$, $\beta^{(n,m)}_{r_1,j,j_3}(c,d)\leftrightarrow\beta_{c,d}(x,x_j,x_{j_3},n,m)$, so that:

$$\alpha^{(n,m)}_{r_1,j,j_3}(0,\frac{1}{2})\to\alpha_{0,\frac{1}{2}}(x,x_j,x_{j_3},n,m) \tag{3.64}$$

$$\beta^{(n,m)}_{r_1,j,j_3}(1,1)\to\beta_{1,1}(x,x_j,x_{j_3},n,m) \tag{3.65}$$

In the case of $\alpha^{(n,m)}_{r_1+1,j+1,j_3}(0,\frac{1}{2})$, since $x=\frac{r_1-r_2}{2\sqrt{n}}$, if r_1 becomes r_1+1, r_2 becomes r_2-1, so $x_{new}=\frac{(r_1+1)-(r_2-1)}{2\sqrt{n}}=x+\frac{1}{\sqrt{n}}$. Since $x_j=\frac{j}{\sqrt{m}}$, $x_{j,new}=\frac{j+1}{\sqrt{m}}=x_j+\frac{1}{\sqrt{m}}$. Consequently:

$$\begin{aligned}\alpha^{(n,m)}_{r_1+1,j+1,j_3}(0,\frac{1}{2}) &\to \alpha_{0,\frac{1}{2}}(x_{new},x_{j,new},x_{j_3},n,m)\\ &= \alpha_{0,\frac{1}{2}}(x+\frac{1}{\sqrt{n}},x_j+\frac{1}{\sqrt{m}},x_{j_3},n,m)\end{aligned} \tag{3.66}$$

In the case of $\beta^{(n-1,m+1)}_{r_1,j+\frac{1}{2},j_3-\frac{1}{2}}(1,1)$, since $x=\frac{r_1-r_2}{2\sqrt{n}}$ originally, we now have $x_{new}=\frac{r_1-(r_2-1)}{2\sqrt{n-1}}=$

$(\frac{r_1-r_2}{2\sqrt{n}}+\frac{1}{2\sqrt{n}})\sqrt{\frac{n}{n-1}} = (x+\frac{1}{2\sqrt{n}})\sqrt{\frac{n}{n-1}}$; $x_j=\frac{j}{\sqrt{m}}$ becomes $x_{j,new}=\frac{j+\frac{1}{2}}{\sqrt{m+1}} = (\frac{j}{\sqrt{m}}+\frac{1}{2\sqrt{m}})\sqrt{\frac{m}{m+1}} = (x_j+\frac{1}{2\sqrt{m}})\sqrt{\frac{m}{m+1}}$; $x_{j_3}=\frac{j_3}{\sqrt{m}}$ becomes $x_{j_3,new}=\frac{j_3-\frac{1}{2}}{\sqrt{m+1}} = (x_{j_3}-\frac{1}{2\sqrt{m}})\sqrt{\frac{m}{m+1}}$, so:

$$\begin{aligned}\beta^{(n-1,m+1)}_{r_1,j+\frac{1}{2},j_3-\frac{1}{2}}(1,1) &\to \beta_{1,1}(x_{new},x_{j,new},x_{j_3,new},n-1,m+1)\\ &= \beta_{1,1}((x+\frac{1}{2\sqrt{n}})\sqrt{\frac{n}{n-1}},(x_j+\frac{1}{2\sqrt{m}})\sqrt{\frac{m}{m+1}},\\ &\qquad (x_{j_3}-\frac{1}{2\sqrt{m}})\sqrt{\frac{m}{m+1}},n-1,m+1)\end{aligned} \tag{3.67}$$

Equation (3.62) hence becomes:

$$\begin{aligned}0 = {}&\beta_{1,1}(x,x_j,x_{j_3},n,m)\alpha_{0,\frac{1}{2}}(x+\frac{1}{\sqrt{n}},x_j+\frac{1}{\sqrt{m}},x_{j_3},n,m)-\\ &\Big[\alpha_{0,\frac{1}{2}}(x,x_j,x_{j_3},n,m)\beta_{1,1}((x+\frac{1}{2\sqrt{n}})\sqrt{\frac{n}{n-1}},(x_j+\frac{1}{2\sqrt{m}})\sqrt{\frac{m}{m+1}},\\ &(x_{j_3}-\frac{1}{2\sqrt{m}})\sqrt{\frac{m}{m+1}},n-1,m+1)\Big]\end{aligned} \tag{3.68}$$

This is only one of the possible equations that follows from equation (3.60). There are 24 equations in total. In order to determine all the β functions, all coefficients of the linearly independent restricted Schur polynomials are needed.

It is possible to solve equation (3.62) without taking the continuum limit. Substituting equation (3.52) into equation (3.62) would lead to recursion relations (see [21] for a similar calculation). Solving all such recursion relations coming from equation (3.60), one would arrive at formulae for the matrix elements of D_2 (save for a normalisation constant). However, at higher loops this procedure becomes quite cumbersome. So, it is prudent to instead take the continuum limit. In this limit, equation (3.68) gives a partial differential equation (PDE), and simultaneously solving all the PDE's coming from equation (3.60) will give the β's we seek. All the resulting PDE's contain the exact same information as that contained in the recursion relations. This can be checked by calculating the spectrum in the continuous case, and noting that it would exactly equal that of the discrete case ([11], [12]). As seen in ([21]), the eigenvectors of the discrete case would also agree with the eigenvectors eventually obtained from the continuous case.

We make the ansatz:

$$\beta_{1,1}(x,x_j,x_{j_3},n,m) = nmf_{1,1}^{(0)}(x_j,x_{j_3}) + n\sqrt{m}f_{1,1}^{(1)}(x_j,x_{j_3}) + nf_{1,1}^{(2)}(x_j,x_{j_3}) + \frac{nf_{1,1}^{(3)}(x_j,x_{j_3})}{\sqrt{m}} + O(\frac{n}{m}) \quad (3.69)$$

The reason for this particular choice of ansatz is simply because it gives nontrivial answers. If β were expanded in integer powers of m, for example, we would arrive at the trivial solution with all β's vanishing. The need for the expansion in terms of half integer powers of m is possibly linked to the appearance of the roots of m and n appearing in the arguments of α and β. Furthermore, substituting j, j_3 by their continuum counterparts $\sqrt{m}x_j, \sqrt{m}x_{j_3}$ respectively into the displaced corners approximation form of equation (3.52) gives:

$$\alpha_{0,\frac{1}{2}}(x,x_j,x_{j_3},n,m) = \sqrt{r_2}\sqrt{\frac{m}{2}}\sqrt{1+2\frac{x_j}{\sqrt{m}}+\frac{4}{m}}\frac{x_j - x_{j_3} + \frac{1}{\sqrt{m}}}{\sqrt{2x_j+\frac{2}{\sqrt{m}}}\sqrt{2x_j+\frac{1}{\sqrt{m}}}} \quad (3.70)$$

We now perform an expansion in $\frac{1}{\sqrt{m}}$ in $\alpha_{0,\frac{1}{2}}(x,x_j,x_{j_3},n,m)$ using the explicit form of equation (3.70). In such an expansion, $\frac{1}{m}$ and $\frac{1}{\sqrt{m}}$ are very small. The expansion is done up to a reasonable order in m. This result is then used to find an expanded form for $\alpha_{0,\frac{1}{2}}(x+\frac{1}{\sqrt{n}}, x_j+\frac{1}{\sqrt{m}}, x_{j_3}, n, m)$, and the result is expanded further using a Taylor series about the point (x,x_j,x_{j_3},n,m). Hence we arrive at expansions for the two α's. A further Taylor expansion is performed for $\beta_{1,1}((x+\frac{1}{2\sqrt{n}})\sqrt{\frac{n}{n-1}}, (x_j+\frac{1}{2\sqrt{m}})\sqrt{\frac{m}{m+1}}, (x_{j_3}-\frac{1}{2\sqrt{m}})\sqrt{\frac{m}{m+1}}, n-1, m+1)$ (about the point (x,x_j,x_{j_3},n,m)) using the ansatz (equation (3.69)). This Taylor expansion is done up to a reasonable order (typically first or second order) and results in derivatives of the f functions appearing. All of these expansions are then finally inserted into equation (3.68).

The resultant equation is very unwieldy as it involves dozens if not hundreds of terms, depending on the orders chosen; although the above algorithm is relatively simple, computing the expansions and multiplying them together by hand is very tedious and time-consuming. Computer software certainly streamlines the computations.

Next, the equation is grouped into decreasing powers of n with only the highest power of n (which turns out to be n itself) being kept and equated to 0. Only the highest power of n is kept because, again, keeping just it gives a nontrivial answer. It is possible that only this highest power is relevant due to the fact that, in the limit of large n, the factors multiplied with n would 'outweigh' those multiplied by lower powers of n. Next, the coefficient of n can further be grouped into different powers of m. The highest power of m is in fact m, the next highest $\sqrt{m}$, the next m^0, and so on. To recover the same information contained in the recursion relations (equation (3.60)), the coefficients of m, $\sqrt{m}$ and m^0 are all equated to 0. Negative half-integer powers of m

would vanish and have no bearing in the large N limit since we have taken m to be on the order of $\sqrt{N}$. Simplifying these three equations (by removing any common numerical factors, for example) finally results in three PDE's ([21]). For example, carrying out these steps and setting the order m term to zero results in the following:

$$2x_{j_3}f_{1,1}^{(0)}(x_j,x_{j_3}) + x_jx_{j_3}\frac{\partial f_{1,1}^{(0)}(x_j,x_{j_3})}{\partial x_j} + x_j^2\frac{\partial f_{1,1}^{(0)}(x_j,x_{j_3})}{\partial x_{j_3}}$$
$$-x_j^2\frac{\partial f_{1,1}^{(0)}(x_j,x_{j_3})}{\partial x_j} - x_jx_{j_3}\frac{\partial f_{1,1}^{(0)}(x_j,x_{j_3})}{\partial x_{j_3}} = 0$$

or, equivalently:

$$2x_{j_3}f_{1,1}^{(0)} + x_j(x_{j_3}-x_j)(\frac{\partial f_{1,1}^{(0)}}{\partial x_j} - \frac{\partial f_{1,1}^{(0)}}{\partial x_{j_3}}) = 0 \tag{3.71}$$

The order $\sqrt{m}$ and order m^0 terms will give two further, more lengthy, PDE's. Simultaneously solving them will give the different f functions appearing in the ansatz and, in turn, $\beta_{1,1}(x,x_j,x_{j_3},n,m)$. Repeating this procedure for all the linearly independent restricted Schur polynomials in equation (3.60), we will have determined the one-loop dilatation operator D_2 (note that we do not have to consider the $[D_2,J_-]=0$ imposed condition, since if $[D_2,J_+]=0$ holds, then $0=0^\dagger=[D_2,J_+]^\dagger=[(J_+)^\dagger,D_2^\dagger]=[J_-,D_2]$). For the p-loop dilatation operator, this procedure can be generalised by using equation (3.56).

This procedure gives the correct, known result ([21]). We have hence explicitly demonstrated how the symmetry generators can be used to determine the dilatation operator D_2 (up to some constant). Chapter 3 starts off by giving the general formula of the action of the generators on restricted Schur polynomials (which contain unknown traces), chapter 3.1 derives some of the explicit formulae for these traces, chapter 3.2 gives an example of how these trace formulae are used, and chapter 3.3 summarises all of the required trace formulae appearing in the generators' actions and compares the analytical formulae with the numerical calculations in the Appendix. Chapter 3.4 makes some useful side comments, and chapter 3.5 takes the trace formulae and uses them to find the exact matrix elements of the $su(2)$ generators. These exact $su(2)$ generators are the new, non-trivial result. Chapter 3.6 gives the dilatation operator's matrix elements' recursion relations and chapter 3.7 gives an example solving a simple case of these relations (using the original approximated matrix elements from [21]), which shows how the dilatation operator's matrix elements can be found. These last two chapters draw highly from ([21]) and mimic the steps followed therein in obtaining the dilatation operator. However, that paper uses approximations for the generators' matrix elements. We have now found their exact forms, and although repeating the method of

([21]) would be an arduous task indeed (involving a cumbersome simplification of the relations and long partial differential equations), using these exact forms should give the exact dilatation operator to one-loop. This would be a worthwhile goal for a future study.

This ends chapter 3, wherein we considered the $su(2)$ sector. We move on to the $su(3)$ sector in the next chapter.

4

$su(3)$ Sector

The $su(2)$ sector is contained within the larger $su(3)$ sector, and hence a similar procedure is followed here as in chapter 3. We first find the general form for the action of the eight $su(3)$ generators on restricted Schur polynomials (constructed now from three complex bosonic fields Z, Y and X). These actions will involve unknown traces of products of projection operators, like before. Using non-trivial arguments to find a number of trace identities, we next find exact formulae for all the requisite traces, and hence the exact matrix elements of the $su(3)$ generators themselves. Next, even though it is not done here, one could then use the fact that the eight generators commute with the dilatation operator (combined with the fact that we now know the exact $su(3)$ generators' matrix elements) to find recursion relations. These relations should be enough to determine the exact dilatation operator itself.

Before continuing, it should be noted that we still consider a system of only two giant gravitons. In this case, the Young diagrams labeling the restricted Schur polynomial only have two rows, and as such, we do not have to consider any multiplicity labels. This greatly simplifies the situation.

The three-field analogue of equations (2.4) and (2.9) is simple enough to understand: the restricted Schur polynomials in this case are:

$$\chi_{R,(r,s,w)}(Z,Y,X) = \frac{1}{n!m!p!} \sum_{\sigma \in S_{n+m+p}} \chi_{R,(r,s,w)}(\sigma) \mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m} \otimes X^{\otimes p}) \tag{4.1}$$

where $R \vdash n+m+p, r \vdash n, s \vdash m$ and $w \vdash p$. Note that in the $su(2)$ case we used (r,s) when referring to labels of one restricted Schur polynomial or projector, and (t,u) when referring to another one. In the $su(3)$ sector, we will use the labels (r,s,w) for one restricted Schur polynomial or projector, and (t,u,v) for another (with appropriate superscripts, like $+$ or $-$ to indicate the number of boxes comprising the diagrams).

The $su(3)$ algebra has eight generators. When acting on the restricted Schur polynomial above, they assume the following forms:

$$J_{Z,X} := \mathrm{Tr}\left(Z\frac{d}{dX}\right),\ J_{Z,Y} := \mathrm{Tr}\left(Z\frac{d}{dY}\right),\ J_{Y,X} := \mathrm{Tr}\left(Y\frac{d}{dX}\right)$$
$$J_{Y,Z} := \mathrm{Tr}\left(Y\frac{d}{dZ}\right),\ J_{X,Z} := \mathrm{Tr}\left(X\frac{d}{dZ}\right),\ J_{X,Y} := \mathrm{Tr}\left(X\frac{d}{dY}\right)$$

and

$$J_1 := \mathrm{Tr}\left(Z\frac{d}{dZ} - Y\frac{d}{dY}\right),\ J_2 := \mathrm{Tr}\left(Z\frac{d}{dZ} + Y\frac{d}{dY} - 2X\frac{d}{dX}\right)$$

In order to calculate the action of these operators on $\chi_{R,(r,s,w)}(Z,Y,X)$, we will make use of the following identity (the three-field analogue of equation (2.10)):

$$\mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m} \otimes X^{\otimes p}) = \sum_{T,(t,u,v)} \frac{d_T n! m! p!}{d_t d_u d_v (n+m+p)!} \times \chi_{T,(t,u,v)}(\sigma^{-1})\chi_{T,(t,u,v)}(Z,Y,X) \tag{4.2}$$

where $T \vdash (n+m+p), t \vdash n, u \vdash m$ and $v \vdash p$. Acting the operator $J_{Z,X}$ on equation (4.1) gives:

$$\begin{aligned}
J_{Z,X}\chi_{R,(r,s,w)}(Z,Y,X) &= \mathrm{Tr}\left(Z\frac{d}{dX}\right)\chi_{R,(r,s,w)}(Z,Y,X) \\
&= \frac{p}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\chi_{R,(r,s,w)}(\sigma) \\
&\quad\times \mathrm{Tr}(\sigma Z^{\otimes n+1}\otimes Y^{\otimes m}\otimes X^{\otimes p-1}) \\
&= \frac{p}{n!m!p!}\sum_{\sigma\in S_{n+m+p}}\chi_{R,(r,s,w)}(\sigma) \\
&\quad\times \sum_{T,(t^+,u,v^-)}\frac{d_T(n+1)!m!(p-1)!}{d_{t^+}d_u d_{v^-}(n+m+p)!} \\
&\quad\times \chi_{T,(t^+,u,v^-)}(\sigma^{-1})\chi_{T,(t^+,u,v^-)}(Z,Y,X) \\
&= \frac{p}{n!m!p!}\sum_{T,(t^+,u,v^-)}\left[\sum_{\sigma\in S_{n+m+p}}\chi_{R,(r,s,w)}(\sigma)\chi_{T,(t^+,u,v^-)}(\sigma^{-1})\right] \\
&\quad\times \frac{d_T(n+1)!m!(p-1)!}{d_{t^+}d_u d_{v^-}(n+m+p)!}\chi_{T,(t^+,u,v^-)}(Z,Y,X) \\
&= \frac{p}{n!m!p!}\sum_{T,(t^+,u,v^-)}\frac{d_T(n+1)!m!(p-1)!}{d_{t^+}d_u d_{v^-}(n+m+p)!}\frac{(n+m+p)!}{d_R}\delta_{RT} \\
&\quad\times \mathrm{Tr}_{R\oplus T}(P_{R,(r,s,w)}P_{T,(t^+,u,v^-)})\chi_{T,(t^+,u,v^-)}(Z,Y,X) \\
&= \sum_{T,(t^+,u,v^-)}\frac{(n+1)d_T}{d_{t^+}d_u d_{v^-}}\frac{1}{d_R}\delta_{RT}\mathrm{Tr}_{R\oplus T}(P_{R,(r,s,w)}P_{T,(t^+,u,v^-)}) \\
&\quad\times \chi_{T,(t^+,u,v^-)}(Z,Y,X) \\
&= \sum_{(t^+,u,v^-)}\frac{n+1}{d_{t^+}d_u d_{v^-}}\mathrm{Tr}(P_{R,(r,s,w)}P_{R,(t^+,u,v^-)}) \\
&\quad\times \chi_{R,(t^+,u,v^-)}(Z,Y,X)
\end{aligned} \tag{4.3}$$

where $T \vdash n+m+p, t^+ \vdash n+1, u \vdash m$ and $v^- \vdash p-1$. Akin to equation (3.4), we used the following result in the above calculation:

$$
\begin{aligned}
\sum_{\sigma\in S_{n+m+p}} & \chi_{R,(r,s,w)}(\sigma)\chi_{T,(t^+,u,v^-)}(\sigma^{-1}) \\
&= \sum_{\sigma\in S_{n+m+p}} \mathrm{Tr}(P_{R,(r,s,w)}\Gamma_R(\sigma))\mathrm{Tr}(P_{T,(t^+,u,v^-)}\Gamma_T(\sigma^{-1})) \\
&= \sum_{\sigma\in S_{n+m+p}} \left[P_{R,(r,s,w)}\right]_{a,b}\left[\Gamma_R(\sigma)\right]_{b,a}\left[P_{T,(t^+,u,v^-)}\right]_{c,d} \\
&\quad \times \left[\Gamma_T(\sigma^{-1})\right]_{d,c} \\
&= \left[P_{R,(r,s,w)}\right]_{a,b}\left[P_{T,(t^+,u,v^-)}\right]_{c,d}\sum_{\sigma\in S_{n+m+p}}\left[\Gamma_R(\sigma)\right]_{b,a} \\
&\quad \times \left[\Gamma_T(\sigma^{-1})\right]_{d,c} \\
&= \left[P_{R,(r,s,w)}\right]_{a,b}\left[P_{T,(t^+,u,v^-)}\right]_{c,d} \times \frac{(n+m+p)!}{d_R}\delta_{RT}\delta_{bc}\delta_{ad} \\
&= \frac{(n+m+p)!}{d_R}\delta_{RT}\mathrm{Tr}_{R\oplus T}(P_{R,(r,s,w)}P_{T,(t^+,u,v^-)})
\end{aligned} \tag{4.4}
$$

Following very similar logic, we can find the action of each of the eight generators on the restricted Schur polynomial $\chi_{R,(r,s,w)}(Z,Y,X)$. However, as before, we wish to instead consider these actions on normalised versions of the restricted Schur polynomials. The three-field analogue of equation (2.14) is:

$$
\langle \chi_{R,(r,s,w)\alpha\beta}(Z,Y,X)\chi_{T,(t,u,v)\gamma\delta}(Z,Y,X)^\dagger\rangle = \frac{f_R \mathrm{hooks}_R}{\mathrm{hooks}_r\mathrm{hooks}_s\mathrm{hooks}_w}\delta_{RT}\delta_{rt}\delta_{su}\delta_{wv}\delta_{\alpha\gamma}\delta_{\beta\delta} \tag{4.5}
$$

Therefore, we define our normalised restricted Schur polynomials, $O_{R,(r,s,w)}(Z,Y,X)$, as:

$$
\chi_{R,(r,s,w)}(Z,Y,X) = \sqrt{\frac{f_R\mathrm{hooks}_R}{\mathrm{hooks}_r\mathrm{hooks}_s\mathrm{hooks}_w}}O_{R,(r,s,w)}(Z,Y,X) \tag{4.6}
$$

With this, equation (4.3), along with the actions of the other generators, can be written in terms of $O_{R,(r,s,w)}(Z,Y,X)$:

$$
\begin{aligned}
J_{Z,X}O_{R,(r,s,w)}(Z,Y,X) = &\sum_{(t^+,u,v^-)}\frac{n+1}{d_{t^+}d_u d_{v^-}}\sqrt{\frac{\mathrm{hooks}_r\mathrm{hooks}_s\mathrm{hooks}_w}{\mathrm{hooks}_{t^+}\mathrm{hooks}_u\mathrm{hooks}_{v^-}}} \\
&\times \mathrm{Tr}(P_{R,(r,s,w)}P_{R,(t^+,u,v^-)})O_{R,(t^+,u,v^-)}(Z,Y,X)
\end{aligned} \tag{4.7}
$$

$$J_{Z,Y}O_{R,(r,s,w)}(Z,Y,X) = \sum_{(t^+,u^-,v)} \frac{n+1}{d_{t^+}d_{u^-}d_v}\sqrt{\frac{\text{hooks}_r\text{hooks}_s\text{hooks}_w}{\text{hooks}_{t^+}\text{hooks}_{u^-}\text{hooks}_v}} \times \text{Tr}(P_{R,(r,s,w)}P_{R,(t^+,u^-,v)})O_{R,(t^+,u^-,v)}(Z,Y,X) \tag{4.8}$$

$$J_{Y,X}O_{R,(r,s,w)}(Z,Y,X) = \sum_{(t,u^+,v^-)} \frac{m+1}{d_t d_{u^+}d_{v^-}}\sqrt{\frac{\text{hooks}_r\text{hooks}_s\text{hooks}_w}{\text{hooks}_t\text{hooks}_{u^+}\text{hooks}_{v^-}}} \times \text{Tr}(P_{R,(r,s,w)}P_{R,(t,u^+,v^-)})O_{R,(t,u^+,v^-)}(Z,Y,X) \tag{4.9}$$

$$J_{Y,Z}O_{R,(r,s,w)}(Z,Y,X) = \sum_{(t^-,u^+,v)} \frac{m+1}{d_{t^-}d_{u^+}d_v}\sqrt{\frac{\text{hooks}_r\text{hooks}_s\text{hooks}_w}{\text{hooks}_{t^-}\text{hooks}_{u^+}\text{hooks}_v}} \times \text{Tr}(P_{R,(r,s,w)}P_{R,(t^-,u^+,v)})O_{R,(t^-,u^+,v)}(Z,Y,X) \tag{4.10}$$

$$J_{X,Z}O_{R,(r,s,w)}(Z,Y,X) = \sum_{(t^-,u,v^+)} \frac{p+1}{d_{t^-}d_u d_{v^+}}\sqrt{\frac{\text{hooks}_r\text{hooks}_s\text{hooks}_w}{\text{hooks}_{t^-}\text{hooks}_u\text{hooks}_{v^+}}} \times \text{Tr}(P_{R,(r,s,w)}P_{R,(t^-,u,v^+)})O_{R,(t^-,u,v^+)}(Z,Y,X) \tag{4.11}$$

$$J_{X,Y}O_{R,(r,s,w)}(Z,Y,X) = \sum_{(t,u^-,v^+)} \frac{p+1}{d_t d_{u^-}d_{v^+}}\sqrt{\frac{\text{hooks}_r\text{hooks}_s\text{hooks}_w}{\text{hooks}_t\text{hooks}_{u^-}\text{hooks}_{v^+}}} \times \text{Tr}(P_{R,(r,s,w)}P_{R,(t,u^-,v^+)})O_{R,(t,u^-,v^+)}(Z,Y,X) \tag{4.12}$$

$$J_1 O_{R,(r,s,w)}(Z,Y,X) = (n-m)O_{R,(r,s,w)}(Z,Y,X) \tag{4.13}$$

$$J_2 O_{R,(r,s,w)}(Z,Y,X) = (n+m-2p)O_{R,(r,s,w)}(Z,Y,X) \tag{4.14}$$

The main factors we now need to compute in order to find the matrix elements of these operators are the various traces. In the previous $su(2)$ sector case, we labeled the m boxes we needed to remove from R, leaving behind r. The m boxes could then be arranged in various diagrams to form s. Now, in the three complex bosonic field case, we need to label the $m+p$ boxes we need to remove from R, leaving behind r. We first remove p boxes, which are used to form w, and then we remove a further m boxes, which are used to form s. However, unlike in the $su(2)$ sector case, we need to be mindful of multiplicity in the $su(3)$ case. We need an extra multiplicity label to indicate the intermediate diagram formed after removing p boxes from R. We need this because removing different p and m boxes from R can still result in the same set of $R,(r,s,w)$ labels. For example, suppose we start with $R = \yng(7,4)$ and we remove one box from the first row, one box from the second row, and use them to form the label $w = \yng(2)$. We then remove a box from the first row and a box from the second row and use them to form the label $s = \yng(2)$. So, if we do not include a label for the 'intermediate' diagram (which would be $\yng(6,3)$), we have the labels $R,(r,s,w) = \yng(7,4), (\yng(5,2), \yng(2), \yng(2))$. However, we could have arrived at the same final label by removing two boxes from the first row of R to form w, and then two boxes from the second row of the 'intermediate' diagram, $\yng(5,4)$, to form s. So, we clearly need an intermediate diagram label.

As before, we will see that we only need to compute the matrix elements for one generator. If we find them, the other generators' matrix elements can be gleaned from the $su(2)$ results, hermitian conjugation, or the $su(3)$ algebra itself.

4.1 $su(3)$ Analytical Results

Here we derive analytical formulae for the traces appearing in the generators' actions' equations. These results are exact and have been reported in ([13]). As before, we restrict all diagrams to having two rows. This is then dual to a system with two giant gravitons. We again use a change of variables (akin to the change given in equation (3.13)). Let R_1, R_2; r_1, r_2; s_1, s_2 and w_1, w_2 be the number of boxes in the first (subscript 1) and second (subscript 2) rows of R, r, s and w, respectively. The new translation is:

$$
\begin{aligned}
s_1 = \frac{m}{2} + j,\ s_2 = \frac{m}{2} - j, w_1 = \frac{p}{2} + k,\ w_2 = \frac{p}{2} - k \\
R_1 = r_1 + \frac{m+p}{2} + j_3 + k_3, R_2 = r_2 + \frac{m+p}{2} - j_3 - k_3
\end{aligned}
\tag{4.15}
$$

Using the above, given $r_1, j, j_3, k, k_3, n, m, p$, it is possible to return to the $R,(r,s,w)$ labels and vice versa. Also, the way the variables change under the action of the various generators is exactly the same as in the $su(2)$ case, except that we now have an extra diagram to consider. For example, suppose that we consider the operator $J_{X,Z}$ and the term in the action where a box is taken away from the second row of r and is added to the first row of w. In this case, $p \to p+1$, $w_1 \to w_1+1$ which implies that $k \to k + \frac{1}{2}$. Also, $r_2 \to r_2 - 1$, and since R_2, m and j_3 remain unchanged, $k_3 \to k_3 - \frac{1}{2}$. As before, if we first remove p boxes from R (and use them to form w) and then remove m boxes from the intermediate diagram (and use them to form s), leaving behind r, if p_i (respectively, m_i) is the number of boxes removed from the ith row of R (the intermediate diagram, R/w), then $k_3 = \frac{p_1 - p_2}{2}$ ($j_3 = \frac{m_1 - m_2}{2}$).

Note that, at first, it appears that we in fact have too many labels. Indeed, given $r_1, j, k, j_3 + k_3, n, m, p$, we could reconstruct the Young diagram labels $R, (r, s, w)$. It seems we only need $j_3 + k_3$, not the individual numbers. In the $su(2)$ case, we only needed 5 pieces of information (r_1, j, j_3, n, m) to calculate 6 row lengths $(R_1, R_2, r_1, r_2, s_1, s_2)$. As mentioned before, the point is that even when we only consider the two row case, when we restrict S_{p+m+n} to $S_p \times S_m \times S_n$, we do need a multiplicity label. If we specify both j_3 and k_3 separately, we resolve this multiplicity issue. To see this, note that we, in steps, restrict S_{p+m+n} to $S_p \times S_{m+n}$ (without multiplicity), which gives (k, k_3), and then we restrict S_{m+n} to $S_m \times S_n$ (again without multiplicity), which then introduces (j, j_3)

In the distant corners approximation, the computation of the traces needed to find the matrix elements of the generators reduces to the computation of $SU(2)$ Clebsch-Gordan coefficients ([13]). The results we compute below for one of the generators are exact however. Since we have decided to choose the convention where we first remove the p boxes which are used to form w (these boxes are associated with the X fields), the generators $J_{Z,Y} = \mathrm{Tr}\left(Z \frac{d}{dY}\right)$ and $J_{Y,Z} = \mathrm{Tr}\left(Y \frac{d}{dZ}\right)$ will be unchanged from the results we obtained in the $su(2)$ sector. The exact matrix elements we calculated in the case of those two generators carry over exactly and can be used to find the action of $J_{Y,Z}$ and $J_{Z,Y}$ on $O_{R,(r,s,w)}(Z, Y, X)$. If we can calculate either $J_{Y,X} = \mathrm{Tr}\left(Y \frac{d}{dX}\right)$ or $J_{X,Y} = \mathrm{Tr}\left(X \frac{d}{dY}\right)$, the other can be found by hermitian conjugation and any other remaining generators can be found using the $su(3)$ algebra (the J_1 and J_2 generators are trivial) or hermitian conjugation. So, as in the $su(2)$ case, we only need to calculate one generator. Therefore, we set about finding analytical formulae for the traces appearing in the action of $\mathrm{Tr}\left(X \frac{d}{dY}\right)$.

For the $su(2)$ case, we found identities for the traces over sums of projectors and we found specific traces which were easy to compute exactly. Using these, we were able to find all the traces appearing in the action of J_-, and hence the exact matrix elements in general. These results were then checked against some numerical calculations. It can also be seen that the matrix elements reduce to those given in ([21]) in the displaced corners approximation. The same type of approach will be used in the present case.

We freely interchange between the $R, R/w, r, s, w$ and $r_1, j, j_3, k, k_3, n, m, p$ notations.

4.1.1 Tensor Product Structure

If we restrict S_{p+m+n} to $S_p \times S_m \times S_n$, states of the irreducible representation (r, s, w) of $S_p \times S_m \times S_n$ are obtained from the carrier space of the irreducible representation R of S_{p+m+n} by associating boxes in R with r, s or w (Z, Y or X boxes, respectively). The X boxes are filled with labels $1, 2, \ldots, p$, the Y boxes are filled with labels $p+1, p+2, \ldots, p+m$, and the Z boxes are filled with labels $p+m+1, p+m+2, \ldots, p+m+n$. Since each of the type of boxes are filled independently from one another, it can be seen that vectors in this basis are tensor products of a Z vector, a Y vector and an X vector. Also, the Z boxes are organised by the r Young diagram, and since we are considering $\mathrm{Tr}\left(X \frac{d}{dY}\right)$, which leaves the r part of our label unchanged, we can write a general projector in this case as:

$$\begin{aligned} P_{R,(r,s,w)} &= \frac{1}{m!}\sum_{\sigma\in S_m}\chi_s(\sigma)\Gamma_R(\sigma)\frac{1}{p!}\sum_{\tau\in S_p}\chi_w(\tau)\Gamma_R(\tau) \\ &\equiv \mathbf{1}_r^R \otimes P_s^R \otimes P_w^R \end{aligned} \tag{4.16}$$

where above we have restricted the projector to the subspace spanned by states filled according to the convention described. Also, the S_m part acts on the $p+1, p+2, \ldots, p+m$ boxes, and the S_p part acts on the $1, 2, \ldots, p$ boxes. The tensor product structure of the problem is evident from the projector. We wish to compute:

$$\mathrm{Tr}(P^{n,m,p}_{R,(r,s,w)}\, P^{n,m-1,p+1}_{R,(r,u^-,v^+)}) = \mathrm{Tr}(\mathbf{1}_r^R \otimes P_s^R \otimes P_w^R \cdot \mathbf{1}_r^R \otimes P_{u^-}^R \otimes P_{v^+}^R) \tag{4.17}$$

where $u^- \vdash m-1$ and $v^+ \vdash p+1$. To compute this, we decompose P_s^R into a sum of 2 projectors onto S_{m-1} irreducible representations:

$$P_s^R = \alpha_1 P^R_{s_1'} + \alpha_2 P^R_{s_2'} \tag{4.18}$$

and $P^R_{v^+}$ into a sum of 2 projectors onto S_p irreducible representations:

$$P^R_{v^+} = \beta_1 P^R_{v_1^{+\prime}} + \beta_2 P^R_{v_2^{+\prime}} \tag{4.19}$$

where the notation T_i' denotes the Young diagram obtained by dropping a box from row i of T. Now assuming, for example, that $s_1' = u^-$ and $v_2^{+\prime} = w$, we find:

$$\mathrm{Tr}(P^{n,m,p}_{R,(r,s,w)}\, P^{n,m-1,p+1}_{R,(r,u^-,v^+)}) = d_r d_{u^-} d_w \alpha_1 \beta_2 \tag{4.20}$$

So, we can see that our traces will have been determined once we know how to decompose the projectors. The problems of finding equations (4.18) and (4.19) are clearly independent of one another. Now, the problem of finding the decomposition in equation (4.19) can be answered by studying the problem of restricting S_{n+m+p} to $S_{n+m} \times S_p$. Indeed, treating the problem this way, S_{n+m} can now be treated like S_n was treated in the $su(2)$ case, and S_p can be treated like S_m was treated. This is a result of our convention to first remove w boxes. This being the case, n is replaced by $n+m$, m by p, j by k, j_3 by k_3, r_1 by r_1+m_1, r_2 by r_2+m_2. This is more or less the case, but is not exactly so since there are more equations in translation (4.15) than in translation (3.13). However, one can see the connection between the $su(2)$ and $su(3)$ sector translations. Given R/w, we remove m_1 boxes from the first row and m_2 boxes from the second row to form r (and use the removed boxes to form s). As a result of this connection between the two cases, the results of the projector decomposition in equation (4.19) can be read directly from the $su(2)$ results (i.e. equations (3.28) to (3.31)) after replacing r_1-r_2 with $r_1-r_2+2j_3$, j with k, j_3 with k_3, d_r with $d_r d_{u^-}$, and d_{u^-} with d_w. So, we only need to study the decomposition of equation (4.18).

4.1.2 Trace of Sums Formulae

As mentioned, equations (4.18) and (4.19) are independent of one another. As a result of this, we can greatly simplify our calculations when studying equation (4.18) if we set $p=0$. This does not cause any loss of generality. On account of this, we only need study the problem of computing:

$$\mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\, P^{n,m-1,1}_{R,(r,u^-,\square)}) \tag{4.21}$$

All of the traces encompassed by the above equation can be found by deriving two trace identities based on sums. First, consider $\sum_s \mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\, P^{n,m-1,1}_{R,(r,u^-,\square)})$. If we sum over all s, only two terms above will be non-zero (those where s subduces u^-). Now, $\sum_s P^{n,m,0}_{R,(r,s)}$ projects us from R to r, and so $\sum_s P^{n,m,0}_{R,(r,s)}\, P^{n,m-1,1}_{R,(r,u^-,\square)}$ projects us to $r \oplus u^- \oplus \square$, and so:

$$\sum_s \mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\, P^{n,m-1,1}_{R,(r,u^-,\square)}) = d_r d_{u^-} \tag{4.22}$$

The next identity we shall consider is:

$$\sum_{u^-} \mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\, P^{n,m-1,1}_{R,(r,u^-,\square)}) = \mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\, \hat{C}) \tag{4.23}$$

where $\hat{C} = \sum_{u^-} P^{n,m-1,1}_{R,(r,u^-,\square)}$. Next, assume that $v^+ = \square$, which is the first block to be removed from R in our convention and which appears in $P^{n,m-1,1}_{R,(r,u^-,\square)}$, is initially in the first row of R (a calculation similar to the following one could be carried out for the row 2 case). On the left-hand side of equation (4.23), since we are summing over all u^-, $\sum_{u^-} P^{n,m-1,1}_{R,(r,u^-,\square)}$ will project all the Z

boxes to r and the first box (the only one forming v^+), $\Box$, to the rightmost box of the first row of R (which, under the action of $P^{n,m,0}_{R,(r,s)}$, will be removed with the further $m-1$ boxes to form s). Now, since $P^{n,m,0}_{R,(r,s)}$ already projects all the Z boxes to r, the net effect of $\hat{C} = \sum_{u^-} P^{n,m-1,1}_{R,(r,u^-,\Box)}$ is to project $\Box$ to the rightmost box position in the first row of R. We want to solve equation (4.23), i.e.:

$$T = \mathrm{Tr}(P^{n,m,0}_{R,(r,s)}\hat{C}) = \sum_I \langle R,(r,s);I|\hat{C}|R,(r,s);I\rangle \tag{4.24}$$

We can follow a similar argument as that leading up to equation (3.22), using a Jucys-Murphy element. Let m_i denote the number of boxes removed from row i of R which are used to form s (and leave behind r). The only states which appear on the right-hand side of equation (4.24) are those where $v^+ = \Box$ is in the first row of R. So, as we take the sum over I we need only fix the position of $\Box$. To fix its position, note that it could sit in the first or second row of R. As before, we fix its content to ensure that it falls in the first row. So, we can see that:

$$\hat{C} = \frac{\sum_{i=2}^{m+n}(i,1) - (r_2+m_2-2)}{r_1 - r_2 + m_1 - m_2 + 1} \tag{4.25}$$

since $\sum_{i=2}^{m+n}(i,1)$ gives the content of $\Box$ when it appears in R, the content of which would be $r_1 + m_1 - 1$ if it lay in the first row, and $r_2 + m_2 - 2$ if it lay in the second row. So, $\hat{C}$ gives 1 if $\Box$ is in the first row and 0 if it's in the second row of R. So:

$$\begin{aligned}
T &= \sum_I \langle R,(r,s);I| \left[\frac{\sum_{i=2}^{m+n}(i,1) - (r_2+m_2-2)}{r_1-r_2+m_1-m_2+1}\right] |R,(r,s);I\rangle \\
&= \frac{\sum_I \langle R,(r,s);I|\sum_{i=2}^{m+n}(i,1)|R,(r,s);I\rangle}{r_1-r_2+m_1-m_2+1} - \frac{(r_2+m_2-2)d_r d_s}{r_1-r_2+m_1-m_2+1} \\
&= \frac{\sum_I \langle R,(r,s);I| \left(\sum_{i=2}^{m}(i,1) + \sum_{i=m+1}^{m+n}(i,1)\right) |R,(r,s);I\rangle}{r_1-r_2+m_1-m_2+1} - \frac{(r_2+m_2-2)d_r d_s}{r_1-r_2+m_1-m_2+1} \\
&= \frac{d_r(m-1)\chi_s((1,m-1)) + n\chi_{R,(r,s)}((1,m+1)) - (r_2+m_2-2)d_r d_s}{r_1-r_2+m_1-m_2+1} \\
&= \frac{d_r(m-1)\chi_s((m,m-1)) + n\chi_{R,(r,s)}((m,m+1)) - (r_2+m_2-2)d_r d_s}{r_1-r_2+2j_3+1}
\end{aligned} \tag{4.26}$$

since $\chi_s((1,m-1)) = \chi_s((m,m-1))$, $\chi_{R,(r,s)}((1,m+1)) = \chi_{R,(r,s)}((m,m+1))$, and $m_1 - m_2 = 2j_3$. This is very similar to what we had in the $su(2)$ calculation; even the exact same restricted character, $\chi_{R,(r,s)}((m,m+1))$, appears. Hence we can substitute equation (3.20) into the above. However, we see the appearance of a new, S_m character, $\chi_s((m,m-1))$. It can be evaluated using the Murnaghan-Nakayama rule ([15]) (recall that $s = 2j$):

$$\chi_s((m, m-1)) = d_s \frac{2j(j+1) + \frac{m}{2}(m-4)}{m(m-1)} \tag{4.27}$$

We have everything we need to evaluate equation (4.26), so that we find:

$$\sum_{u^-} \text{Tr}(P^{n,m,0}_{R,(r,s)}\, P^{n,m-1,1}_{R,(r,u^-,\square)}) = \left[\frac{m+2j_3}{2m} + \frac{2j(j+1) - 2j_3^2 - m}{2m(r_1 - r_2 + 2j_3 + 1)}\right] d_r d_s \tag{4.28}$$

Compare our two derived identities with equations (3.27) and (3.25). Note the extreme similarity between the above result and equation (3.25), except with $r_1 - r_2$ replaced with $r_1 - r_2 + 2j_3$ and a sign difference.

So, we finally have our two sum identities: equations (4.22) and (4.28). There are only two non-zero terms on the left-hand sides of both of these two identities.

4.1.3 Simple Trace Formulae

In addition to the above sum formulae, there are a number of traces that are simple enough that they can be computed immediately. For example, the two traces:

$$\text{Tr}(P^{n,m,p}_{(r_1,j,j,k,k)} P^{n,m-1,p+1}_{(r_1,j-1,j-1,k+1,k+1)}) = d_r \tag{4.29}$$

$$\text{Tr}(P^{n,m,p}_{(r_1,j,-j,k,-k)} P^{n,m-1,p+1}_{(r_1,j-1,-j+1,k+1,-k-1)}) = d_r \tag{4.30}$$

follow because the subspaces we project to are the same for both projectors appearing in both of the traces above ($d_s = d_w = 1$ since s and w are described by Young diagrams with a single row). These results would be specific examples of terms under consideration when looking at the action of $\text{Tr}\left(X \frac{d}{dY}\right)$. Also, consider the following case in which we let $r_1 = r_2$. In this situation, the Y boxes would already be organised into an irreducible representation so we can use the $su(2)$ sector formulae (equations (3.28) to (3.31)) by replacing $r_1 - r_2$ with $r_1 - r_2 + 2j$ (and leaving j as is, not changing it to k), j with k, j_3 with k_3, d_r with $d_r d_s$, d_{u^-} with d_{v^-} (where $v^- \vdash p-1$ is the diagram subduced by w, labeling the appropriate projector), and, as mentioned, setting $r_1 = r_2$. The traces we obtain in this way, corresponding to equations (3.28) to (3.31) respectively, are:

$$\text{Tr}(P^{n,m,p}_{(r_1,j,j,k,k_3)} P^{n,m+1,p-1}_{(r_1,j-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2})}) = \left[\frac{k-k_3}{2k} + \frac{k^2-k_3^2}{2k(2j+1)}\right] d_r d_s d_{v^-} \tag{4.31}$$

$$\text{Tr}(P^{n,m,p}_{(r_1,j,j,k,k_3)} P^{n,m+1,p-1}_{(r_1,j+\frac{1}{2},j+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2})}) = \left[\frac{k+k_3}{2k} - \frac{k^2-k_3^2}{2k(2j+1)}\right] d_r d_s d_{v^-} \tag{4.32}$$

$$\mathrm{Tr}(P^{n,m,p}_{(r_1,j,j,k,k_3)}P^{n,m+1,p-1}_{(r_1,j-\frac{1}{2},j-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2})}) = \left[\frac{k+k_3+1}{2k+2} - \frac{(k+1)^2-k_3^2}{(2k+2)(2j+1)}\right] d_r d_s d_{v^-} \tag{4.33}$$

$$\mathrm{Tr}(P^{n,m,p}_{(r_1,j,j,k,k_3)}P^{n,m+1,p-1}_{(r_1,j+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2})}) = \left[\frac{k-k_3+1}{2k+2} + \frac{(k+1)^2-k_3^2}{(2k+2)(2j+1)}\right] d_r d_s d_{v^-} \tag{4.34}$$

Equation (3.28) corresponds to removing a box from the top row of s, forming u^-, and adding a box to the bottom row of r, forming t^+ (with the labels r, s, t, u read using the conventions set out in the $su(2)$ chapter). Equation (4.31), as expected, then corresponds to removing a box from the top row of w, forming v^-, and adding a box to the bottom row of s, forming u^+ (with these labels read within the $su(3)$ context). These four formulae are specific examples of formulae which might appear when considering the action of $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$. However, remember that we can always swap between $\mathrm{Tr}\left(X\frac{d}{dY}\right)$ and $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$ by taking the hermitian conjugate of matrix elements. Indeed, most of the derivations in chapter 4.1 of this dissertation, save for the four above results, have had the generator $\mathrm{Tr}\left(X\frac{d}{dY}\right)$ in mind. Using the following:

$$\begin{aligned}&\langle r_1, j', j_3', k', k_3', n, m-1, p+1|\mathrm{Tr}\left(X\frac{d}{dY}\right)|r_1, j, j_3, k, k_3, n, m, p\rangle \\ &= \langle r_1, j, j_3, k, k_3, n, m, p|\mathrm{Tr}\left(Y\frac{d}{dX}\right)|r_1, j', j_3', k', k_3', n, m-1, p+1\rangle\end{aligned} \tag{4.35}$$

and shifting variables, the change becomes quite trivial.

We now have enough to write the general results for the $su(3)$ sector.

4.2 Final $su(3)$ Trace Results

The following results were calculated using the same type of algorithm as in chapter 3.2 for the $su(2)$ sector. They relate to $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$, and any changes from the previous chapters' $\mathrm{Tr}\left(X\frac{d}{dY}\right)$ results were found using appropriate variable shifts in the above results. The $su(3)$ trace results are:

$$\mathrm{Tr}(P^{n,m,p}_{(r_1,j,j_3,k,k_3)}P^{n,m+1,p-1}_{(r_1,j+\frac{1}{2},j_3+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2})}) = \left[\frac{k+k_3}{2k} - \frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times \left[\frac{j+j_3+1}{2j+1} + \frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] d_r d_s d_{v^-} \quad (4.36)$$

$$\mathrm{Tr}(P^{n,m,p}_{(r_1,j,j_3,k,k_3)}P^{n,m+1,p-1}_{(r_1,j-\frac{1}{2},j_3+\frac{1}{2},k-\frac{1}{2},k_3-\frac{1}{2})}) = \left[\frac{k+k_3}{2k} - \frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times \left[\frac{j-j_3}{2j+1} - \frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] d_r d_s d_{v^-} \quad (4.37)$$

$$\mathrm{Tr}(P^{n,m,p}_{(r_1,j,j_3,k,k_3)}P^{n,m+1,p-1}_{(r_1,j+\frac{1}{2},j_3+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2})}) = \left[\frac{k-k_3+1}{2k+2} + \frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times \left[\frac{j+j_3+1}{2j+1} + \frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] d_r d_s d_{v^-} \quad (4.38)$$

$$\mathrm{Tr}(P^{n,m,p}_{(r_1,j,j_3,k,k_3)}P^{n,m+1,p-1}_{(r_1,j-\frac{1}{2},j_3+\frac{1}{2},k+\frac{1}{2},k_3-\frac{1}{2})}) = \left[\frac{k-k_3+1}{2k+2} + \frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times \left[\frac{j-j_3}{2j+1} - \frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] d_r d_s d_{v^-} \quad (4.39)$$

$$\mathrm{Tr}(P^{n,m,p}_{(r_1,j,j_3,k,k_3)}P^{n,m+1,p-1}_{(r_1,j+\frac{1}{2},j_3-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2})}) = \left[\frac{k-k_3}{2k} + \frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times \left[\frac{j-j_3+1}{2j+1} - \frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] d_r d_s d_{v^-} \quad (4.40)$$

$$\mathrm{Tr}(P^{n,m,p}_{(r_1,j,j_3,k,k_3)}P^{n,m+1,p-1}_{(r_1,j-\frac{1}{2},j_3-\frac{1}{2},k-\frac{1}{2},k_3+\frac{1}{2})}) = \left[\frac{k-k_3}{2k} + \frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times \left[\frac{j+j_3}{2j+1} + \frac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] d_r d_s d_{v^-} \quad (4.41)$$

$$\mathrm{Tr}(P^{n,m,p}_{(r_1,j,j_3,k,k_3)}P^{n,m+1,p-1}_{(r_1,j+\frac{1}{2},j_3-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2})}) = \left[\frac{k+k_3+1}{2k+2} - \frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times \left[\frac{j-j_3+1}{2j+1} - \frac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] d_r d_s d_{v^-} \quad (4.42)$$

and:

$$\mathrm{Tr}(P^{n,m,p}_{(r_1,j,j_3,k,k_3)}P^{n,m+1,p-1}_{(r_1,j-\frac{1}{2},j_3-\frac{1}{2},k+\frac{1}{2},k_3+\frac{1}{2})}) = \left[\frac{k+k_3+1}{2k+2} - \frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times \left[\frac{j+j_3}{2j+1} + \frac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] d_r d_s d_{v^-} \quad (4.43)$$

where $r \vdash n$ and $s \vdash m$ are the Z and Y boxes' diagrams given by the labels of the left-hand projectors on the left-hand sides in each of the above 8 equations, while $v^- \vdash p-1$ is the X boxes' diagram given by the labels of the right-hand projectors on the left-hand sides in each of the above 8 equations. Note the striking similarity between the expressions given in the square brackets and the $su(2)$ results (equations (3.28) to (3.31)). Indeed, we used the latter equations to construct the former.

These traces are those appearing in equation (4.9). Looking at this equation, we would naïvely expect there to be four terms in the summation. Indeed, under the action of $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$, a box is removed from the first or second row of w (forming v^-) and a box is added to the first or second row of s (forming u^+), giving four possible combinations. We also only had four terms in the $su(2)$ case. However, in this $su(3)$ case we have labels of the intermediate states to worry about. Consider the $su(2)$ translations, equations (3.13). There, if we consider the operator where a box is taken away from r and added to s, either r_1 or r_2 decreases by 1, and j either increases or decreases by $\frac{1}{2}$, giving four combinations. Given these two pieces of information, using the constraints of equations (3.13) we can work out if j_3 increases or decreases by $\frac{1}{2}$. In this $su(3)$ case, considering $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$, equations (4.15) show that j and k either increase or decrease by $\frac{1}{2}$, giving four possibilities. However, under $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$, R_i, r_i and $m+p$ are constant, so the combination j_3+k_3 must be constant too. However, these variables themselves are not constant and do change since they denote the boxes removed from either R or R/w. Indeed, j_3 can either increase or decrease by $\frac{1}{2}$ (in which case k_3 decreases or increases by $\frac{1}{2}$ accordingly). So, we have eight combinations in total. This is also the reason why we have eight and not sixteen terms: given j_3, we can deduce k_3.

In the next chapter of this dissertation we shall combine these traces with the other factors appearing in equation (4.9) to find the exact action of Tr $\left(Y\frac{d}{dX}\right)$ on $O^{n,m,p}_{(r_1,j,j_3,k,k_3)}$. However, these results will first be evaluated for specific cases (where $m=2, p=1$ and $m=2, p=3$) in the subchapters below. As before, this all can then be compared with the numerical results in the Appendix.

4.2.1 $m=2, p=1$ case

In this case we consider Tr $\left(X\frac{d}{dY}\right)$. See the Appendix for an explanation of the notation to follow. Take the projector:

$$P^{(1s,1w),(1s)}_{\Box\Box,\Box} = P_{\begin{array}{l}\Box\Box\Box\Box\Box\Box\boxed{s}\boxed{w}\\ \Box\Box\Box\boxed{s}\end{array},\Box\Box,\Box} \tag{4.44}$$

In this case, $j=1$, $j_3=0$, $k=\frac{1}{2}$ and $k_3=\frac{1}{2}$. Plugging these values into equations (4.40) and (4.42) (after having shifted variables accordingly using translation (4.15), since the trace results above, in the forms written, refer to the Tr $\left(Y\frac{d}{dX}\right)$ generator), we get:

$$\mathrm{Tr}(P^{(1s,1w),(1s)}_{\Box\Box,\Box}P^{(1s,1w),(1w)}_{\Box,\Box\Box}) = \frac{1}{4}\left[1+\frac{1}{r_1-r_2+2}\right]d_r$$

$$\mathrm{Tr}(P^{(1s,1w),(1s)}_{\Box\Box,\Box}P^{(1s,1w),(1w)}_{\Box,\substack{\Box\\\Box}}) = \frac{1}{4}\left[1-\frac{1}{r_1-r_2+2}\right]\left[1-\frac{1}{r_1-r_2+1}\right]d_r$$

Next, consider:

$$P^{(1s,1w),(1s)}_{\substack{\Box\\\Box},\Box} = P_{\begin{array}{l}\Box\Box\Box\Box\Box\Box\boxed{s}\boxed{w}\\ \Box\Box\Box\boxed{s}\end{array},\substack{\Box\\\Box},\Box} \tag{4.45}$$

where we have $j=0$, $j_3=0$, $k=\frac{1}{2}$ and $k_3=\frac{1}{2}$. Plugging these values into equations (4.41) and (4.43) (after having shifted variables) we get:

$$\mathrm{Tr}(P^{(1s,1w),(1s)}_{\substack{\Box\\\Box},\Box}P^{(1s,1w),(1w)}_{\Box,\Box\Box}) = \frac{1}{4}\left[1+\frac{1}{r_1-r_2+2}\right]\left[1+\frac{1}{r_1-r_2+1}\right]d_r$$

$$\mathrm{Tr}(P^{(1s,1w),(1s)}_{\substack{\Box\\\Box},\Box}P^{(1s,1w),(1w)}_{\Box,\substack{\Box\\\Box}}) = \frac{1}{4}\left[1-\frac{1}{r_1-r_2+2}\right]\left[1+\frac{1}{r_1-r_2+1}\right]d_r$$

These results can be compared with those in the Appendix. At first is appears that some do not agree. However, upon simplifcation of the above, it's clear that they do (save for the d_r factor, like before).

4.2.2 $m = 2, p = 3$ case

Here we consider $\text{Tr}\left(Y\frac{d}{dX}\right)$ (and we can hence use the trace results without shifting like before). Consider the projector:

$$P^{(2s,2w),(1w)}_{\yng(2),\yng(3)} = P_{\young(\ \ \ \ ssww,\ \ \ w),\yng(2),\yng(3)} \tag{4.46}$$

In this case, $j = 1$, $j_3 = 1$, $k = \frac{3}{2}$ and $k_3 = \frac{1}{2}$. Plugging these values into equations (4.40) and (4.41), we get:

$$\text{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(3)} P^{(2s,2w),(1s)}_{\yng(3),\yng(2)}) = \frac{1}{9}\left[1 + \frac{2}{r_1 - r_2 + 3}\right] d_r$$

$$\text{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(3)} P^{(2s,2w),(1s)}_{\yng(2,1),\yng(2)}) = \frac{2}{9}\left[1 + \frac{2}{r_1 - r_2 + 3}\right]\left[1 + \frac{1}{r_1 - r_2 + 2}\right] d_r$$

As before, we need diagrams to subduce one another when calculating these traces. In this case, the projector will be zero unless u^+ subduces s and w subduces v^-. Therefore, we theoretically expect the folllowing:

$$\text{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(3)} P^{(3s,1w),(1w)}_{\yng(3),\yng(1,1)}) = 0$$

$$\text{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(3)} P^{(2s,1w,1s),(1w)}_{\yng(2,1),\yng(1,1)}) = 0$$

For the next projector:

$$P^{(2s,2w),(1w)}_{\yng(2),\yng(2,1)} = P_{\young(\ \ \ \ ssww,\ \ \ w),\yng(2),\yng(2,1)} \tag{4.47}$$

we have $j = 1$, $j_3 = 1$, $k = \frac{1}{2}$ and $k_3 = \frac{1}{2}$. Plugging these values into equations (4.42), (4.43) and (4.36), we get:

$$\text{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(2,1)} P^{(2s,2w),(1s)}_{\yng(3),\yng(2)}) = \frac{2}{9}\left[1 - \frac{1}{r_1 - r_2 + 3}\right]\left[1 - \frac{2}{r_1 - r_2 + 2}\right] d_r$$

$$\text{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(2,1)} P^{(2s,2w),(1s)}_{\yng(2,1),\yng(2)}) = \frac{4}{9}\left[1 - \frac{1}{r_1 - r_2 + 3}\right]\left[1 + \frac{1}{r_1 - r_2 + 2}\right] d_r$$

$$\mathrm{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(2,1)}\, P^{(3s,1w),(1w)}_{\yng(3),\yng(1,1)}) = d_r$$

And finally, because the right-hand side projector below is invalid (blocks were removed incorrectly from R), we expect the following:

$$\mathrm{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(2,1)}\, P^{(2s,1w,1s),(1w)}_{\yng(2,1),\yng(1,1)}) = 0$$

These results agree with those in the Appendix.

4.3 $su(3)$ Generators' Exact Matrix Elements

With the above traces, we can finally compute the exact action of $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$ on $O^{n,m,p}_{(r_1,j,j_3,k,k_3)}$ (where we have suppressed the (Z,Y,X) dependence of the normalised restricted Schur polynomial). This action is exact because we have not made the displaced corners approximation. First, equations (4.36) to (4.43) (only those which are necessary in a particular case) are substituted into equation (4.9). Then, the dimensions of the representations and other factors are combined together in the exact same way as in chapter 3.5 of this dissertation. This, as before, leaves ratios of hook lengths, the factors of which mostly cancel, leaving behind relatively simple expressions in the numerator and denominator of these ratios. After substituting our final results into equation (4.9), we find the exact matrix elements of $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$:

$$\mathrm{Tr}\left(Y\frac{d}{dX}\right) O^{n,m,p}_{(r_1,j,j_3,k,k_3)} = \sum_{a,b,c=-\frac{1}{2}}^{\frac{1}{2}} C(a,b,c)\, O^{n,m+1,p-1}_{(r_1,j+a,j_3+b,k+c,k_3-b)} \tag{4.48}$$

where:

$$C(\frac{1}{2},\frac{1}{2},-\frac{1}{2}) = \sqrt{\frac{p+2k+2}{2}\frac{2k}{2k+1}}\left[\frac{k+k_3}{2k} - \frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\left[\frac{j+j_3+1}{2j+1} + \frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \tag{4.49}$$

$$C(-\frac{1}{2},\frac{1}{2},-\frac{1}{2}) = \sqrt{\frac{p+2k+2}{2}\frac{2k}{2k+1}}\left[\frac{k+k_3}{2k} - \frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times\sqrt{\frac{m-2j+2}{2}\frac{2j+1}{2j}}\left[\frac{j-j_3}{2j+1} - \frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \tag{4.50}$$

$$C(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = \sqrt{\frac{p-2k}{2}\frac{2k+2}{2k+1}}\left[\frac{k-k_3+1}{2k+2}+\frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\left[\frac{j+j_3+1}{2j+1}+\frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \tag{4.51}$$

$$C(-\frac{1}{2},\frac{1}{2},\frac{1}{2}) = \sqrt{\frac{p-2k}{2}\frac{2k+2}{2k+1}}\left[\frac{k-k_3+1}{2k+2}+\frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times\sqrt{\frac{m-2j+2}{2}\frac{2j+1}{2j}}\left[\frac{j-j_3}{2j+1}-\frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3+2)}\right] \tag{4.52}$$

$$C(\frac{1}{2},-\frac{1}{2},-\frac{1}{2}) = \sqrt{\frac{p+2k+2}{2}\frac{2k}{2k+1}}\left[\frac{k-k_3}{2k}+\frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\left[\frac{j-j_3+1}{2j+1}-\frac{(j+\frac{1}{2})^2-(j_3+\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] \tag{4.53}$$

$$C(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}) = \sqrt{\frac{p+2k+2}{2}\frac{2k}{2k+1}}\left[\frac{k-k_3}{2k}+\frac{k^2-k_3^2}{2k(r_1-r_2+2j_3+1)}\right] \times\sqrt{\frac{m-2j+2}{2}\frac{2j+1}{2j}}\left[\frac{j+j_3}{2j+1}+\frac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] \tag{4.54}$$

$$C(\frac{1}{2},-\frac{1}{2},\frac{1}{2}) = \sqrt{\frac{p-2k}{2}\frac{2k+2}{2k+1}}\left[\frac{k+k_3+1}{2k+2}-\frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times\sqrt{\frac{m+2j+4}{2}\frac{2j+1}{2j+2}}\left[\frac{j-j_3+1}{2j+1}-\frac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right] \tag{4.55}$$

and:

$$C(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}) = \sqrt{\frac{p-2k}{2}\frac{2k+2}{2k+1}}\left[\frac{k+k_3+1}{2k+2} - \frac{(k+1)^2-k_3^2}{(2k+2)(r_1-r_2+2j_3+1)}\right] \times\sqrt{\frac{m-2j+2}{2}\frac{2j+1}{2j}}\left[\frac{j+j_3}{2j+1} + \frac{(j+\frac{1}{2})^2-(j_3-\frac{1}{2})^2}{(2j+1)(r_1-r_2+2j_3)}\right]. \tag{4.56}$$

So, we finally have the exact action of $\mathrm{Tr}\left(Y\frac{d}{dX}\right)$ on $O^{n,m,p}_{(r_1,j,j_3,k,k_3)}$. With the above results, taking the hermitian conjugate (i.e. using equation (4.35)) would give the action of $\mathrm{Tr}\left(X\frac{d}{dY}\right)$, i.e. equation (4.12). The other operators, as mentioned, are then either given by using the $su(3)$ algebra itself or are already given by the exact $su(2)$ results.

It is straightforward to check the above results numerically (indeed, we have already checked a few examples of the traces in the Appendix), to check that the generators reduce to the displaced corners generators in the correct limit (see ([13]) for the displaced corners results), to explicitly check their actiona for small values of n, m, p and to verify that they close the correct algebra.

This concludes this chapter of the dissertation. Chapter 4 begins by giving general formulae for the action of the $su(3)$ generators on restricted Schur polynomials. Chapter 4.1 details trace identities, which are used in chapter 4.2 to find the requisite traces appearing in the action of one of the generators on the restricted Schur polynomials (as was argued, we only needed to consider one generator). Chapter 4.2 also checks these analytical results with the numerical ones in the Appendix. Finally, chapter 4.3 gives the exact matrix elements of one of the $su(3)$ generators, which could in turn be used to find them all. Chapter 4 as a whole is akin to the first five chapters of chapter 3. An interesting study would be to take these $su(3)$ results and use them to find the one-loop dilatation operator in a similar way to the process followed in chapters 3.6 and 3.7.

This achieves the aims of this dissertation.

5
Discussion and Conclusion

In this dissertation, an introduction to the group theory approach to $\mathcal{N} = 4$ super Yang-Mills theory was given. Next, the exact actions of the $su(2)$ and $su(3)$ algebras on restricted Schur polynomials were found by deriving analytical formulae for the generators' matrix elements. This process involved finding formulae for the traces over products of projectors, which in turn was done by making use of sly arguments to find certain identities and then combining them to find all the requisite traces. In the case of the $su(2)$ sector, an explanation of how these generators can be used to find the one-loop dilatation operator was also given. However, this explanation made use of the displaced corners approximation matrix elements of the generators and is hence only valid up to leading order in $\frac{m}{n}$.

A large goal of this work was to find the exact matrix elements of the $su(2)$ and $su(3)$ generators; many continuations are now possible. With the exact results for the $su(2)$ and $su(3)$ generators, it should be possible to extend the approach given in chapters 3.6 and 3.7 and use the exact results to explore the structure of the exact, one-loop dilatation operator of $\mathcal{N} = 4$ SYM. Understanding the dilatation operator is important because, amongst other reasons, having it would allow us to find the energy spectrum of the multi-graviton states corresponding to the restricted Schur polynomial operators. These exact results also allow us to go beyond small perturbations of the $\frac{1}{2}$-BPS operators. The calculations one would encounter if attempting this would be horrendous. Indeed, the calculations are already extremely cumbersome in the displaced corners approximation limit. However, if successful, the reward would be well worth the effort. Another extension would be to consider a larger subalgebra of the full algebra enjoyed by $\mathcal{N} = 4$ super Yang-Mills theory. This would result in fields rotating into other types of fields (for example, fermion fields into gauge fields).

6 Appendix

6.1 Numerical Results: $su(2)$

We first argue that equation (2.11) can be written in an alternative but equivalent way, given that Young diagrams representing the symmetric groups are allowed at most two rows (and hence the projection operators have no need for multiplicity labels):

$$P_{R,(r,s)} = \frac{d_r d_s}{n!m!} \sum_{\sigma_1 \circ \sigma_2 \in S_n \times S_m} \chi_r(\sigma_1)\chi_s(\sigma_2)\Gamma_R(\sigma_1 \circ \sigma_2) \tag{6.1}$$

Equation (6.1) is the formula for the projection operator $P_{R,(r,s)}$ in the $su(2)$ sector. In these numerical calculations, we use the version of $P_{R,(r,s)}$ which includes the $d_r d_s$ factors. We do this in order to arrive at correctly normalised results which match the normalised analytical formulae. In our case, without the multiplicity indices to worry about, there is no ambiguity in which representation of (r, s) we project onto. Also, equation (2.11) is the more general formula which acts on all possible states in R. When we restrict ourselves to the subspaces we will consider below, the action of equation (2.11) on certain states outside of these subspaces is zero. Only states in these subspaces, when acted upon by equation (2.11), don't give zero, and only the 'equation (6.1) part' of equation (2.11) has any effect on these states. We also know from group representation theory that the mapping from the group elements to the corresponding matrices is a homeomorphism, so

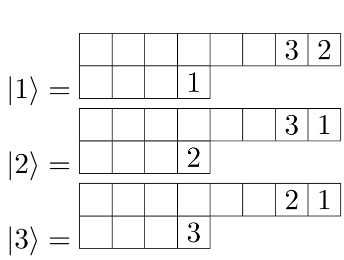
$\Gamma_R(\sigma_1 \circ \sigma_2) = \Gamma_R(\sigma_1)\Gamma_R(\sigma_2)$. This allows for a convenient factorisation of equation (6.1) into parts acting on the r blocks and the s blocks separately. Because the subspaces we are interested in are the ones where we only label the m boxes removed from R to form s, the 'r' part of the projector can be ignored. Therefore, the projector we will use below is the 's' part of equation (6.1), i.e.:

$$P_{R,(r,s)} = \frac{d_s}{m!} \sum_{\sigma \in S_m} \chi_s(\sigma)\Gamma_R(\sigma) \tag{6.2}$$

In the following calculations we label either the three, four or five boxes which we remove from R, leaving behind r and using the removed boxes to form s. These will be used as an example of the terms appearing in the action of J_-, i.e. equation (3.10). The action of J_+ case would be similar. We need to respect pre-existing boundaries which exist between boxes which are removed. For example, if we remove three boxes from R, two of which share a common boundary in the original Young diagram, then, when forming s from those boxes, two of the three boxes must share that common boundary. It can be shown that projectors which don't obey this rule end up giving results of zero and so have no bearing. Also, we will consider the subspaces of R where one box is removed from the second row of R, with the rest being removed from the top row of R (assuming, of course, that the top row contains enough boxes to remove the required number and still be left with a valid Young diagram for r). One could choose other subspaces of course; this choice just simplifies the situation. The end result (i.e. the traces) will be the same either way. In addition, we will denote the number of boxes in the first and second rows of r as r_1 and r_2 respectively. Even though the diagrams to be used below have definite numerical values for these two variables, we prefer to keep the variables as r_1, r_2 for generality's sake. All the numerically-calculated results below can be compared with the analytical examples of chapter 3 of this dissertation.

6.1.1 $m = 3$

Label the subspaces as:



(6.3)

These partially-labeled Young-Yamanouchi diagrams label subspaces (and not individual vectors in R) because they include any and all possible labeling of the blank boxes.

Define the following projectors:

$$P^{(2,1)}_{\ydiagram{3}} = P_{\ytableaushort{{}{}{}{}{}{}**,{}{}{}*},\ydiagram{3}} \tag{6.4}$$

$$P^{(2,1)}_{\ydiagram{2,1}} = P_{\ytableaushort{{}{}{}{}{}{}**,{}{}{}*},\ydiagram{2,1}} \tag{6.5}$$

$$P^{(2)}_{\ydiagram{2}} = P_{\ytableaushort{{}{}{}{}{}{}**,{}{}{}{}},\ydiagram{2}} \tag{6.6}$$

$$P^{(1,1)}_{\ydiagram{2}} = P_{\ytableaushort{{}{}{}{}{}{}{}*,{}{}{}*},\ydiagram{2}} \tag{6.7}$$

$$P^{(1,1)}_{\ydiagram{1,1}} = P_{\ytableaushort{{}{}{}{}{}{}{}*,{}{}{}*},\ydiagram{1,1}} \tag{6.8}$$

In the above notation, the asterisks in the larger Young diagrams (on the right-hand sides) indicate the blocks removed from R. The remaining blocks in these larger Young diagrams form r, and the second, smaller Young diagrams indicate the representation, s, formed from the removed 'asterisk' blocks. This notation will extend to the $m = 4$ and $m = 5$ cases.

Firstly, we use equation (6.2) to find the requisite projection operators. Suppose, for example, that we wish to compute the projector:

$$P^{(2,1)}_{\ydiagram{2,1}} = P_{\ydiagram{8,4},(\ydiagram{6,3},\ydiagram{2,1})}$$

Using equation (6.2) and a character table such as the one given in ([15]), we get:

$$
\begin{aligned}
P^{(2,1)}_{\tiny\yng(2,1)} &= \frac{2}{3!}\sum_{\sigma\in S_3}\chi_{\tiny\yng(2,1)}(\sigma)\Gamma_R(\sigma) \\
&= \frac{2}{3!}\Big(\chi_{\tiny\yng(2,1)}(e)\Gamma_R(e)+\chi_{\tiny\yng(2,1)}((12))\Gamma_R((12)) \\
&\quad +\chi_{\tiny\yng(2,1)}((13))\Gamma_R((13))+\chi_{\tiny\yng(2,1)}((23))\Gamma_R((23)) \\
&\quad +\chi_{\tiny\yng(2,1)}((123))\Gamma_R((123))+\chi_{\tiny\yng(2,1)}((132))\Gamma_R((132))\Big) \\
&= \frac{2}{3!}\left(2\Gamma_R(e)-\Gamma_R((123))-\Gamma_R((132))\right)
\end{aligned}
$$

where e here is the identity permutation. Similar formulae can be found for the other projection operators above. Hence, all that remains is to compute the matrix representations of the group elements, $\Gamma_R(\sigma)$. From group theory, we know that any permutation can be written as a product of transpositions, and any transposition can be written as a product of adjacent transpositions (see [25] for the details and proofs of these claims). So, we need only calculate the matrix representations $\Gamma_R((k,k+1))$, for $k = 1\dots m-1$. This can be done by using the formula:

$$\Gamma_R((k,k+1))|R\rangle = \frac{1}{c_k-c_{k+1}}|R\rangle + \sqrt{1-\frac{1}{(c_k-c_{k+1})^2}}|R_{(k,k+1)}\rangle \tag{6.9}$$

where $|R\rangle$ is one of the subspaces labeled in (6.3), $|R_{(k,k+1)}\rangle$ is the subspace obtained by swapping the labels of boxes k and $k+1$ in $|R\rangle$, and c_k is the content of the box labeled k ([26]). For example, for $\Gamma_R((12))$:

$$
\begin{aligned}
\Gamma_R((12))|1\rangle &= \frac{1}{2-7}|1\rangle + \sqrt{1-\frac{1}{(2-7)^2}}|2\rangle \\
&= \frac{-1}{r_1-r_2+2}|1\rangle + \sqrt{1-\frac{1}{(r_1-r_2+2)^2}}|2\rangle \\
\Gamma_R((12))|2\rangle &= \frac{1}{7-2}|2\rangle + \sqrt{1-\frac{1}{(7-2)^2}}|1\rangle \\
&= \frac{1}{r_1-r_2+2}|2\rangle + \sqrt{1-\frac{1}{(r_1-r_2+2)^2}}|1\rangle \\
\Gamma_R((12))|3\rangle &= |3\rangle
\end{aligned}
$$

Using the above results, we can construct the matrix representation of the permutation (12):

$$\Gamma_R((12)) = \begin{bmatrix} \frac{-1}{r_1-r_2+2} & \sqrt{1-\frac{1}{(r_1-r_2+2)^2}} & 0 \\ \sqrt{1-\frac{1}{(r_1-r_2+2)^2}} & \frac{1}{r_1-r_2+2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Repeating the above procedure gives $\Gamma_R((23))$:

$$\Gamma_R((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{r_1-r_2+1} & \sqrt{1-\frac{1}{(r_1-r_2+1)^2}} \\ 0 & \sqrt{1-\frac{1}{(r_1-r_2+1)^2}} & \frac{1}{r_1-r_2+1} \end{bmatrix}$$

Also, note that the identity permutation is always represented by the identity matrix, so $\Gamma_R(e)$ is the 3×3 identity matrix. Using these matrices, we can compute the matrix representations of all the other group elements. For example, since $(123) = (12)(23)$, we compute $\Gamma_R((123)) = \Gamma_R((12)(23)) = \Gamma_R((12))\Gamma_R((23))$. With all these results, we find explicit formulae for the projectors and multiply them together in the requisite combinations appearing in equation (3.10) for the $m = 3$ case. Finally, taking the traces, we arrive at the following:

$$\mathrm{Tr}(P^{(2,1)}_{\square\square\square} P^{(2)}_{\square\square}) = \frac{1}{3}\left(1 + \frac{2}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(2,1)}_{\square\square\square} P^{(1,1)}_{\square\square}) = \frac{2}{3}\left(1 - \frac{1}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(2,1)}_{\square\square\square} P^{(1,1)}_{\substack{\square\\\square}}) = 0$$

$$\mathrm{Tr}(P^{(2,1)}_{\substack{\square\square\\\square}} P^{(2)}_{\square\square}) = \frac{2}{3}\left(1 - \frac{1}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(2,1)}_{\substack{\square\square\\\square}} P^{(1,1)}_{\square\square}) = \frac{1}{3}\left(1 + \frac{2}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(2,1)}_{\substack{\square\square\\\square}} P^{(1,1)}_{\substack{\square\\\square}}) = 1$$

Note that the two possible $s \vdash 3$ have been considered in this $m = 3$ case. Not all of the above traces appear in equation (3.10), but only those corresponding to the chosen s diagram: given specific $R, (r, s)$ representations, only a maximum of four terms will appear in equation (3.10). The same holds in the $m = 4$ and $m = 5$ cases. Also note the absence of the d_r factor. This is a result of the fact that our numerical computation did not sum over all the ways of labeling the empty boxes.

6.1.2 $m = 4$

Label the subspaces as:

$$
\begin{aligned}
|1\rangle &= \begin{ytableau} \; & \; & \; & \; & \; & 4 & 3 & 2 \\ \; & \; & \; & 1 \end{ytableau} \\
|2\rangle &= \begin{ytableau} \; & \; & \; & \; & \; & 4 & 3 & 1 \\ \; & \; & \; & 2 \end{ytableau} \\
|3\rangle &= \begin{ytableau} \; & \; & \; & \; & \; & 4 & 2 & 1 \\ \; & \; & \; & 3 \end{ytableau} \\
|4\rangle &= \begin{ytableau} \; & \; & \; & \; & \; & 3 & 2 & 1 \\ \; & \; & \; & 4 \end{ytableau}
\end{aligned} \tag{6.10}
$$

Define the projectors:

$$P^{(3,1)}_{\begin{ytableau} \; & \; & \; & \; \end{ytableau}} = P_{\begin{ytableau} \; & \; & \; & \; & \; & * & * & * \\ \; & \; & \; & * \end{ytableau}, \begin{ytableau} \; & \; & \; & \; \end{ytableau}} \tag{6.11}$$

$$P^{(3,1)}_{\begin{ytableau} \; & \; & \; \\ \; \end{ytableau}} = P_{\begin{ytableau} \; & \; & \; & \; & \; & * & * & * \\ \; & \; & \; & * \end{ytableau}, \begin{ytableau} \; & \; & \; \\ \; \end{ytableau}} \tag{6.12}$$

$$P^{(3)}_{\begin{ytableau} \; & \; & \; \end{ytableau}} = P_{\begin{ytableau} \; & \; & \; & \; & \; & * & * & * \\ \; & \; & \; & \; \end{ytableau}, \begin{ytableau} \; & \; & \; \end{ytableau}} \tag{6.13}$$

$$P^{(2,1)}_{\begin{ytableau} \; & \; & \; \end{ytableau}} = P_{\begin{ytableau} \; & \; & \; & \; & \; & \; & * & * \\ \; & \; & \; & * \end{ytableau}, \begin{ytableau} \; & \; & \; \end{ytableau}} \tag{6.14}$$

$$P^{(2,1)}_{\begin{ytableau} \; & \; \\ \; \end{ytableau}} = P_{\begin{ytableau} \; & \; & \; & \; & \; & \; & * & * \\ \; & \; & \; & * \end{ytableau}, \begin{ytableau} \; & \; \\ \; \end{ytableau}} \tag{6.15}$$

Equation (6.2) is again used to compute the relevant projectors, using S_4 (instead of S_3 as we did in the $m = 3$ case). Also, now $s \vdash 4$, not 3, so the characters corresponding to the new representations for s need to be used. Furthermore, the matrix representations of the permutations themselves will be different from those in the $m = 3$ case, but equation (6.9) and the same procedure as before are still used to find them. Once all possible projectors have been computed, the combinations of the products of the projectors appearing in equation (3.10), given a specific representation for s, are computed. Finally, taking their traces, we arrive at the following results (note that, as for the $m = 3$ and $m = 5$ cases, more traces, corresponding to different s representations, are given here than would be needed for any one s representation):

$$\mathrm{Tr}(P^{(3,1)}_{\ydiagram{4}} P^{(3)}_{\ydiagram{3}}) = \frac{1}{4}\left(1 + \frac{3}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(3,1)}_{\ydiagram{4}} P^{(2,1)}_{\ydiagram{3}}) = \frac{3}{4}\left(1 - \frac{1}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(3,1)}_{\ydiagram{4}} P^{(2,1)}_{\ydiagram{2,1}}) = 0$$

$$\mathrm{Tr}(P^{(3,1)}_{\ydiagram{3,1}} P^{(3)}_{\ydiagram{3}}) = \frac{3}{4}\left(1 - \frac{1}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(3,1)}_{\ydiagram{3,1}} P^{(2,1)}_{\ydiagram{3}}) = \frac{1}{4}\left(1 + \frac{3}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(3,1)}_{\ydiagram{3,1}} P^{(2,1)}_{\ydiagram{2,1}}) = 2$$

6.1.3 $m = 5$

Label the subspaces as:

$$\begin{aligned}
|1\rangle &= \begin{ytableau} \ & \ & \ & \ & \ & 5 & 4 & 3 & 2 \\ \ & \ & \ & 1 \end{ytableau} \\
|2\rangle &= \begin{ytableau} \ & \ & \ & \ & \ & 5 & 4 & 3 & 1 \\ \ & \ & \ & 2 \end{ytableau} \\
|3\rangle &= \begin{ytableau} \ & \ & \ & \ & \ & 5 & 4 & 2 & 1 \\ \ & \ & \ & 3 \end{ytableau} \\
|4\rangle &= \begin{ytableau} \ & \ & \ & \ & \ & 5 & 3 & 2 & 1 \\ \ & \ & \ & 4 \end{ytableau} \\
|5\rangle &= \begin{ytableau} \ & \ & \ & \ & \ & 4 & 3 & 2 & 1 \\ \ & \ & \ & 5 \end{ytableau}
\end{aligned} \tag{6.16}$$

Define the following projectors:

$$P^{(4,1)}_{\ydiagram{5}} = P_{\ytableaushort{\ \ \ \ \ ****,\ \ \ *},\,\ydiagram{5}} \tag{6.17}$$

$$P^{(4,1)}_{\ydiagram{4,1}} = P_{\ytableaushort{\ \ \ \ \ ****,\ \ \ *},\,\ydiagram{4,1}} \tag{6.18}$$

$$P^{(4)}_{\ydiagram{4}} = P_{\ytableaushort{\ \ \ \ \ ****,\ \ \ \ },\,\ydiagram{4}} \tag{6.19}$$

$$P^{(3,1)}_{\ydiagram{4}} = P_{\ytableaushort{\ \ \ \ \ \ ***,\ \ \ *},\,\ydiagram{4}} \tag{6.20}$$

$$P^{(3,1)}_{\ydiagram{3,1}} = P_{\ytableaushort{\ \ \ \ \ \ ***,\ \ \ *},\,\ydiagram{3,1}} \tag{6.21}$$

Following the now familiar scheme, we get the following results:

$$\mathrm{Tr}(P^{(4,1)}_{\ydiagram{5}} P^{(4)}_{\ydiagram{4}}) = \frac{1}{5}\left(1 + \frac{4}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(4,1)}_{\ydiagram{5}} P^{(3,1)}_{\ydiagram{4}}) = \frac{4}{5}\left(1 - \frac{1}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(4,1)}_{\ydiagram{5}} P^{(3,1)}_{\ydiagram{3,1}}) = 0$$

$$\mathrm{Tr}(P^{(4,1)}_{\ydiagram{4,1}} P^{(4)}_{\ydiagram{4}}) = \frac{4}{5}\left(1 - \frac{1}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(4,1)}_{\ydiagram{4,1}} P^{(3,1)}_{\ydiagram{4}}) = \frac{1}{5}\left(1 + \frac{4}{r_1 - r_2 + 1}\right)$$

$$\mathrm{Tr}(P^{(4,1)}_{\ydiagram{4,1}} P^{(3,1)}_{\ydiagram{3,1}}) = 3$$

6.2 Numerical Results: $su(3)$

Before we can continue, we need a general formula for the projector. So, we extend equation (6.1) in the obvious way:

$$P_{R,(r,s,w)} = \frac{d_r d_s d_w}{n!m!p!} \sum_{\sigma_1\circ\sigma_2\circ\sigma_3 \in S_n\times S_m\times S_p} \chi_r(\sigma_1)\chi_s(\sigma_2)\chi_w(\sigma_3)\Gamma_R(\sigma_1\circ\sigma_2\circ\sigma_3) \tag{6.22}$$

However, this projector must act on the space of states found by labeling the $m+p$ boxes we must remove from R, leaving behind r. As before, this projector can be factorised into the r, s and w parts, and we can forgo the r part. The formula for the projector then becomes:

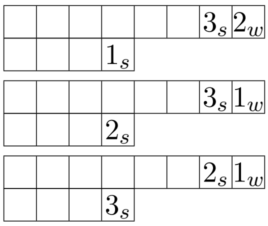
$$\begin{aligned} P_{R,(r,s,w)} &= \frac{d_s d_w}{m!p!} \sum_{\sigma_1\circ\sigma_2 \in S_m\times S_p} \chi_s(\sigma_1)\chi_w(\sigma_2)\Gamma_R(\sigma_1\circ\sigma_2) \\ &= \frac{d_s d_w}{m!p!} \sum_{\sigma_1 \in S_m} \chi_s(\sigma_1)\Gamma_R(\sigma_1) \sum_{\sigma_2\in S_p} \chi_w(\sigma_2)\Gamma_R(\sigma_2) \end{aligned} \tag{6.23}$$

In the following diagrams, it is important to establish a convention regarding which type of boxes (s or w) are removed first and which second. We will always first remove the w boxes, and then the s boxes from R, leaving behind r. It is also important to remember which integers are permuted under which group. S_p is the group of all permutations of the integers $(1,\ldots,p)$, and S_m is the group of all permutations of the integers $(p+1,\ldots,p+m)$.

6.2.1 $m = 2, p = 1$

We now need to not only assign a number to each box to be removed, but also a label indicating whether the box will be used to construct the s or w Young diagram label. We will also consider subspaces in which only one box is removed from the bottom row of R, with the rest being removed from the top row of R. So, label the subspaces:

$|1\rangle =$, $|2\rangle =$, $|3\rangle =$



(6.24)

In these subspaces, the boxes labeled s or w are used to form the s or w diagrams, respectively.

All w boxes are removed first, and then s boxes. In this example we will consider the action of $J_{X,Y} = \mathrm{Tr}\left(X\frac{d}{dY}\right)$, i.e. equation (4.12). So, given $R, (r, s, w)$, we need projectors where a block is removed from s to form u^-, and a block is added to w to form v^+. With this in mind, define the following projectors:

$$P^{(1s,1w),(1s)}_{\ydiagram{2},\ydiagram{1}} = P_{\begin{ytableau} \; & \; & \; & \; & \; & \; & s & w \\ \; & \; & \; & s \end{ytableau},\ydiagram{2},\ydiagram{1}} \tag{6.25}$$

$$P^{(1s,1w),(1s)}_{\ydiagram{1,1},\ydiagram{1}} = P_{\begin{ytableau} \; & \; & \; & \; & \; & \; & s & w \\ \; & \; & \; & s \end{ytableau},\ydiagram{1,1},\ydiagram{1}} \tag{6.26}$$

$$P^{(1s,1w),(1w)}_{\ydiagram{1},\ydiagram{2}} = P_{\begin{ytableau} \; & \; & \; & \; & \; & \; & s & w \\ \; & \; & \; & w \end{ytableau},\ydiagram{1},\ydiagram{2}} \tag{6.27}$$

$$P^{(1s,1w),(1w)}_{\ydiagram{1},\ydiagram{1,1}} = P_{\begin{ytableau} \; & \; & \; & \; & \; & \; & s & w \\ \; & \; & \; & w \end{ytableau},\ydiagram{1},\ydiagram{1,1}} \tag{6.28}$$

In this notation, on the left-hand side of the equations, the first bracket in the numerator of P indicates the number and type (either an s or w) of blocks removed from the top row of R, while the second one is indicative of the number and type removed from the second row of R. On the right-hand side of the equations, the first diagram is R, and the blocks labeled with either an s or w are blocks removed from R (leaving behind r) and are used to form the s and w diagrams. This is not the full set of possible projectors, given these diagrams. We could have labeled the boxes to be removed differently. Here just a sample have been chosen to illustrate the methods used when numerically computing the traces.

Now, using equation (6.23), we find the following:

$$\begin{aligned} P^{(1s,1w),(1s)}_{\ydiagram{2},\ydiagram{1}} &= \frac{1}{2!1!}\sum_{\sigma_1 \in S_2} \chi_{\ydiagram{2}}(\sigma_1)\Gamma_R(\sigma_1) \sum_{\sigma_2 \in S_1} \chi_{\ydiagram{1}}(\sigma_2)\Gamma_R(\sigma_2) \\ &= \frac{1}{2}\left(\chi_{\ydiagram{2}}(e)\Gamma_R(e) + \chi_{\ydiagram{2}}((23))\Gamma_R((23))\right)\left(\chi_{\ydiagram{1}}(e)\Gamma_R(e)\right) \\ &= \frac{1}{2}\left(\Gamma_R(e) + \Gamma_R((23))\right) \end{aligned}$$

since $\Gamma_R(e)$ is the identity matrix. Also:

$$\begin{aligned}
P^{(1s,1w),(1s)}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},\square} &= \frac{1}{2!1!}\sum_{\sigma_1\in S_2}\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(\sigma_1)\Gamma_R(\sigma_1)\sum_{\sigma_2\in S_1}\chi_{\square}(\sigma_2)\Gamma_R(\sigma_2)\\
&= \frac{1}{2}\left(\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(e)\Gamma_R(e)+\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}((23))\Gamma_R((23))\right)\left(\chi_{\square}(e)\Gamma_R(e)\right)\\
&= \frac{1}{2}\left(\Gamma_R(e)-\Gamma_R((23))\right)
\end{aligned}$$

$$\begin{aligned}
P^{(1s,1w),(1w)}_{\square,\square\square} &= \frac{1}{1!2!}\sum_{\sigma_1\in S_1}\chi_{\square}(\sigma_1)\Gamma_R(\sigma_1)\sum_{\sigma_2\in S_2}\chi_{\square\square}(\sigma_2)\Gamma_R(\sigma_2)\\
&= \frac{1}{2}\left(\chi_{\square}(e)\Gamma_R(e)\right)\left(\chi_{\square\square}(e)\Gamma_R(e)+\chi_{\square\square}((12))\Gamma_R((12))\right)\\
&= \frac{1}{2}\left(\Gamma_R(e)+\Gamma_R((12))\right)
\end{aligned}$$

$$\begin{aligned}
P^{(1s,1w),(1w)}_{\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}} &= \frac{1}{1!2!}\sum_{\sigma_1\in S_1}\chi_{\square}(\sigma_1)\Gamma_R(\sigma_1)\sum_{\sigma_2\in S_2}\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(\sigma_2)\Gamma_R(\sigma_2)\\
&= \frac{1}{2}\left(\chi_{\square}(e)\Gamma_R(e)\right)\left(\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(e)\Gamma_R(e)+\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}((12))\Gamma_R((12))\right)\\
&= \frac{1}{2}\left(\Gamma_R(e)-\Gamma_R((12))\right)
\end{aligned}$$

As before, we use equation (6.9) to find all the adjacent transposition matrix representations required. Now, since the mapping to the matrix representations of permutations is a homeomorphism:

$$\Gamma_R(e)\Gamma_R(e)=\Gamma_R(e)\Rightarrow\Gamma_R(e)(\Gamma_R(e)-\mathbb{I}_{3\times3})=0$$

which implies that $\Gamma_R(e)=\mathbb{I}_{3\times3}$. Equation (6.9) gives:

$$\begin{aligned}
\Gamma_R((23))|1\rangle &= |1\rangle \\
\Gamma_R((23))|2\rangle &= \frac{1}{2-6}|2\rangle + \sqrt{1-\frac{1}{(2-6)^2}}|3\rangle \\
&= \frac{-1}{r_1-r_2+1}|2\rangle + \sqrt{1-\frac{1}{(r_1-r_2+1)^2}}|3\rangle \\
\Gamma_R((23))|3\rangle &= \frac{1}{6-2}|3\rangle + \sqrt{1-\frac{1}{(6-2)^2}}|2\rangle \\
&= \frac{1}{r_1-r_2+1}|3\rangle + \sqrt{1-\frac{1}{(r_1-r_2+1)^2}}|2\rangle
\end{aligned}$$

So, we get:

$$\Gamma_R((23)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{r_1-r_2+1} & \sqrt{1-\frac{1}{(r_1-r_2+1)^2}} \\ 0 & \sqrt{1-\frac{1}{(r_1-r_2+1)^2}} & \frac{1}{r_1-r_2+1} \end{bmatrix}$$

Similarly:

$$\Gamma_R((12)) = \begin{bmatrix} \frac{-1}{r_1-r_2+2} & \sqrt{1-\frac{1}{(r_1-r_2+2)^2}} & 0 \\ \sqrt{1-\frac{1}{(r_1-r_2+2)^2}} & \frac{1}{r_1-r_2+2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting these matrices into the above formulae and multiplying the requisite combination of projection operators together, we get the following results:

$$\mathrm{Tr}(P^{(1s,1w),(1s)}_{\square\square,\square} P^{(1s,1w),(1w)}_{\square,\square\square}) = \frac{1}{4}\left(1+\frac{1}{r_1-r_2+2}\right)$$

$$\mathrm{Tr}(P^{(1s,1w),(1s)}_{\square\square,\square} P^{(1s,1w),(1w)}_{\square,\substack{\square\\\square}}) = \frac{1}{4}\left(1-\frac{2}{r_1-r_2+2}\right)$$

$$\mathrm{Tr}(P^{(1s,1w),(1s)}_{\substack{\square\\\square},\square} P^{(1s,1w),(1w)}_{\square,\square\square}) = \frac{1}{4}\left(1+\frac{2}{r_1-r_2+1}\right)$$

$$\mathrm{Tr}(P^{(1s,1w),(1s)}_{\substack{\square\\\square},\square} P^{(1s,1w),(1w)}_{\square,\substack{\square\\\square}}) = \frac{1}{4}$$

6.2.2 $m = 2, p = 3$

In this case, we will find the traces relevant to the $J_{Y,X} = \mathrm{Tr}\left(Y \frac{d}{dX}\right)$ operator. In other words, given the representation $R, (r, s, w)$, we consider projectors where a block is removed from w to form v^- and a block is added to s to form u^+. Label the subspaces:

$$
\begin{aligned}
|1\rangle &= \begin{ytableau} \, & \, & \, & \, & \, & 5_s & 4_s & 3_w & 2_w \\ \, & \, & \, & 1_w \end{ytableau} \\
|2\rangle &= \begin{ytableau} \, & \, & \, & \, & \, & 5_s & 4_s & 3_w & 1_w \\ \, & \, & \, & 2_w \end{ytableau} \\
|3\rangle &= \begin{ytableau} \, & \, & \, & \, & \, & 5_s & 4_s & 2_w & 1_w \\ \, & \, & \, & 3_w \end{ytableau} \\
|4\rangle &= \begin{ytableau} \, & \, & \, & \, & \, & 5_s & 3_s & 2_w & 1_w \\ \, & \, & \, & 4_w \end{ytableau} \\
|5\rangle &= \begin{ytableau} \, & \, & \, & \, & \, & 4_s & 3_s & 2_w & 1_w \\ \, & \, & \, & 5_w \end{ytableau}
\end{aligned}
\tag{6.29}
$$

Define the projectors:

$$P^{(2s,2w),(1w)}_{\yng(2),\yng(3)} = P_{\begin{ytableau} \, & \, & \, & \, & \, & s & s & w & w \\ \, & \, & \, & w \end{ytableau},\yng(2),\yng(3)} \tag{6.30}$$

$$P^{(2s,2w),(1w)}_{\yng(2),\yng(2,1)} = P_{\begin{ytableau} \, & \, & \, & \, & \, & s & s & w & w \\ \, & \, & \, & w \end{ytableau},\yng(2),\yng(2,1)} \tag{6.31}$$

$$P^{(2s,2w),(1s)}_{\yng(3),\yng(2)} = P_{\begin{ytableau} \, & \, & \, & \, & \, & s & s & w & w \\ \, & \, & \, & s \end{ytableau},\yng(3),\yng(2)} \tag{6.32}$$

$$P^{(2s,2w),(1s)}_{\yng(2,1),\yng(2)} = P_{\begin{ytableau} \, & \, & \, & \, & \, & s & s & w & w \\ \, & \, & \, & s \end{ytableau},\yng(2,1),\yng(2)} \tag{6.33}$$

$$P^{(3s,1w),(1w)}_{\yng(3),\yng(1,1)} = P_{\begin{ytableau} \, & \, & \, & \, & \, & s & s & s & w \\ \, & \, & \, & w \end{ytableau},\yng(3),\yng(1,1)} \tag{6.34}$$

$$P^{(2s,1w,1s),(1w)}_{\yng(2,1),\yng(1,1)} = P_{\young(\ \ \ \ \ ssws,\ \ \ w),\yng(2,1),\yng(1,1)} \tag{6.35}$$

Again, the above does not constitute all possible projectors. Others are possible if we remove the s and w boxes differently. Note also that this last projection operator is actually not valid, since all the w blocks, which we have decided to remove first, are not all on the right of the s blocks. We will see that, because of this, it does not contribute at all.

Using equation (6.23), we get:

$$\begin{aligned}
P^{(2s,2w),(1w)}_{\yng(2),\yng(3)} &= \frac{1}{2!3!}\sum_{\sigma_1\in S_2}\chi_{\yng(2)}(\sigma_1)\Gamma_R(\sigma_1)\sum_{\sigma_2\in S_3}\chi_{\yng(3)}(\sigma_2)\Gamma_R(\sigma_2) \\
&= \frac{1}{12}[\chi_{\yng(2)}(e)\Gamma_R(e) + \chi_{\yng(2)}((45))\Gamma_R((45))][\chi_{\yng(3)}(e)\Gamma_R(e) \\
&\quad +\chi_{\yng(3)}((12))\Gamma_R((12)) + \chi_{\yng(3)}((23))\Gamma_R((23)) \\
&\quad +\chi_{\yng(3)}((13))\Gamma_R((13)) + \chi_{\yng(3)}((123))\Gamma_R((123)) \\
&\quad +\chi_{\yng(3)}((132))\Gamma_R((132))] \\
&= \frac{1}{12}[\Gamma_R(e) + \Gamma_R((45))][\Gamma_R(e) + \Gamma_R((12)) + \Gamma_R((23)) \\
&\quad +\Gamma_R((13)) + \Gamma_R((123)) + \Gamma_R((132))]
\end{aligned}$$

$$\begin{aligned}
P^{(2s,2w),(1w)}_{\yng(2),\yng(2,1)} &= \frac{2}{2!3!}\sum_{\sigma_1\in S_2}\chi_{\yng(2)}(\sigma_1)\Gamma_R(\sigma_1)\sum_{\sigma_2\in S_3}\chi_{\yng(2,1)}(\sigma_2)\Gamma_R(\sigma_2) \\
&= \frac{1}{6}[\chi_{\yng(2)}(e)\Gamma_R(e) + \chi_{\yng(2)}((45))\Gamma_R((45))][\chi_{\yng(2,1)}(e)\Gamma_R(e) \\
&\quad +\chi_{\yng(2,1)}((12))\Gamma_R((12)) + \chi_{\yng(2,1)}((23))\Gamma_R((23)) \\
&\quad +\chi_{\yng(2,1)}((13))\Gamma_R((13)) + \chi_{\yng(2,1)}((123))\Gamma_R((123)) \\
&\quad +\chi_{\yng(2,1)}((132))\Gamma_R((132))] \\
&= \frac{1}{6}[\Gamma_R(e) + \Gamma_R((45))][2\Gamma_R(e) - \Gamma_R((123)) - \Gamma_R((132))]
\end{aligned}$$

$$\begin{aligned}
P^{(2s,2w),(1s)}_{\square\square\square,\square\square} &= \frac{1}{3!2!}\sum_{\sigma_1\in S_3}\chi_{\square\square\square}(\sigma_1)\Gamma_R(\sigma_1)\sum_{\sigma_2\in S_2}\chi_{\square\square}(\sigma_2)\Gamma_R(\sigma_2)\\
&= \frac{1}{12}[\chi_{\square\square\square}(e)\Gamma_R(e)+\chi_{\square\square\square}((34))\Gamma_R((34))+\chi_{\square\square\square}((35))\Gamma_R((35))\\
&\quad+\chi_{\square\square\square}((45))\Gamma_R((45))+\chi_{\square\square\square}((345))\Gamma_R((345))\\
&\quad+\chi_{\square\square\square}((354))\Gamma_R((354))][\chi_{\square\square}(e)\Gamma_R(e)+\chi_{\square\square}((12))\Gamma_R((12))]\\
&= \frac{1}{12}[\Gamma_R(e)+\Gamma_R((34))+\Gamma_R((35))+\Gamma_R((45))+\Gamma_R((345))\\
&\quad+\Gamma_R((354))]\times[\Gamma_R(e)+\Gamma_R((12))]
\end{aligned}$$

$$\begin{aligned}
P^{(2s,2w),(1s)}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\square\square} &= \frac{2}{3!2!}\sum_{\sigma_1\in S_3}\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}(\sigma_1)\Gamma_R(\sigma_1)\sum_{\sigma_2\in S_2}\chi_{\square\square}(\sigma_2)\Gamma_R(\sigma_2)\\
&= \frac{1}{6}[\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}(e)\Gamma_R(e)+\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}((34))\Gamma_R((34))+\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}((35))\Gamma_R((35))\\
&\quad+\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}((45))\Gamma_R((45))+\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}((345))\Gamma_R((345))\\
&\quad+\chi_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}((354))\Gamma_R((354))][\chi_{\square\square}(e)\Gamma_R(e)+\chi_{\square\square}((12))\Gamma_R((12))]\\
&= \frac{1}{6}[2\Gamma_R(e)-\Gamma_R((345))-\Gamma_R((354))][\Gamma_R(e)+\Gamma_R((12))]
\end{aligned}$$

$$\begin{aligned}
P^{(3s,1w),(1w)}_{\square\square\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}} &= \frac{1}{3!2!}\sum_{\sigma_1\in S_3}\chi_{\square\square\square}(\sigma_1)\Gamma_R(\sigma_1)\sum_{\sigma_2\in S_2}\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(\sigma_2)\Gamma_R(\sigma_2)\\
&= \frac{1}{12}[\chi_{\square\square\square}(e)\Gamma_R(e)+\chi_{\square\square\square}((34))\Gamma_R((34))+\chi_{\square\square\square}((35))\Gamma_R((35))\\
&\quad+\chi_{\square\square\square}((45))\Gamma_R((45))+\chi_{\square\square\square}((345))\Gamma_R((345))\\
&\quad+\chi_{\square\square\square}((354))\Gamma_R((354))][\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(e)\Gamma_R(e)+\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}((12))\Gamma_R((12))]\\
&= \frac{1}{12}[\Gamma_R(e)+\Gamma_R((34))+\Gamma_R((35))+\Gamma_R((45))+\Gamma_R((345))\\
&\quad+\Gamma_R((354))]\times[\Gamma_R(e)-\Gamma_R((12))]
\end{aligned}$$

$$\begin{aligned}
P^{(2s,1w,1s),(1w)}_{\yng(2,1),\yng(1,1)} &= \frac{2}{3!2!}\sum_{\sigma_1\in S_3}\chi_{\yng(2,1)}(\sigma_1)\Gamma_R(\sigma_1)\sum_{\sigma_2\in S_2}\chi_{\yng(1,1)}(\sigma_2)\Gamma_R(\sigma_2)\\
&= \frac{1}{6}[\chi_{\yng(2,1)}(e)\Gamma_R(e)+\chi_{\yng(2,1)}((34))\Gamma_R((34))+\chi_{\yng(2,1)}((35))\Gamma_R((35))\\
&\quad +\chi_{\yng(2,1)}((45))\Gamma_R((45))+\chi_{\yng(2,1)}((345))\Gamma_R((345))\\
&\quad +\chi_{\yng(2,1)}((354))\Gamma_R((354))][\chi_{\yng(1,1)}(e)\Gamma_R(e)+\chi_{\yng(1,1)}((12))\Gamma_R((12))]\\
&= \frac{1}{6}[2\Gamma_R(e)-\Gamma_R((345))-\Gamma_R((354))][\Gamma_R(e)-\Gamma_R((12))]
\end{aligned}$$

All that remains is to find the matrix representations of the permutations. In this case, $\Gamma_R(e)$ is the 5×5 identity matrix. Using equation (6.9), we can find all the necessary adjacent transpositions (using the subspaces in equation (6.29)). We get:

$$\begin{aligned}
\Gamma_R((12))|1\rangle &= \frac{1}{2-8}|1\rangle+\sqrt{1-\frac{1}{(2-8)^2}}|2\rangle\\
&= \frac{-1}{r_1-r_2+4}|1\rangle+\sqrt{1-\frac{1}{(r_1-r_2+4)^2}}|2\rangle\\
\Gamma_R((12))|2\rangle &= \frac{1}{8-2}|2\rangle+\sqrt{1-\frac{1}{(8-2)^2}}|1\rangle\\
&= \frac{1}{r_1-r_2+4}|2\rangle+\sqrt{1-\frac{1}{(r_1-r_2+4)^2}}|1\rangle\\
\Gamma_R((12))|3\rangle &= |3\rangle\\
\Gamma_R((12))|4\rangle &= |4\rangle\\
\Gamma_R((12))|5\rangle &= |5\rangle
\end{aligned}$$

So:

$$\Gamma_R((12)) = \begin{bmatrix}
\frac{-1}{r_1-r_2+4} & \sqrt{1-\frac{1}{(r_1-r_2+4)^2}} & 0 & 0 & 0\\
\sqrt{1-\frac{1}{(r_1-r_2+4)^2}} & \frac{1}{r_1-r_2+4} & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

The other adjacent transpositions can be found using the same algorithm, and all other matrix representations of the permutations (such as $\Gamma_R(13)$ or $\Gamma_R(354)$) can be constructed using matrix multiplication of the adjacent transpositions. Finally, once all the explicit formulae for the projectors have been found, we get the following values for the traces:

$$\mathrm{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(3)} P^{(2s,2w),(1s)}_{\yng(3),\yng(2)}) = \frac{1}{9}\left(1 + \frac{2}{r_1 - r_2 + 3}\right)$$

$$\mathrm{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(3)} P^{(2s,2w),(1s)}_{\yng(2,1),\yng(2)}) = \frac{2}{9}\left(1 + \frac{3}{r_1 - r_2 + 2}\right)$$

$$\mathrm{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(3)} P^{(3s,1w),(1w)}_{\yng(3),\yng(1,1)}) = 0$$

$$\mathrm{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(3)} P^{(2s,1w,1s),(1w)}_{\yng(2,1),\yng(1,1)}) = 0$$

$$\mathrm{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(2,1)} P^{(2s,2w),(1s)}_{\yng(3),\yng(2)}) = \frac{2}{9}\left(1 - \frac{3}{r_1 - r_2 + 3}\right)$$

$$\mathrm{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(2,1)} P^{(2s,2w),(1s)}_{\yng(2,1),\yng(2)}) = \frac{4}{9}$$

$$\mathrm{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(2,1)} P^{(3s,1w),(1w)}_{\yng(3),\yng(1,1)}) = 1$$

$$\mathrm{Tr}(P^{(2s,2w),(1w)}_{\yng(2),\yng(2,1)} P^{(2s,1w,1s),(1w)}_{\yng(2,1),\yng(1,1)}) = 0$$

These results agree perfectly with the analytical results. Note how some of the traces are zero. This is either because the diagrams do not subduce one another, or because the trace includes the invalid projector (as mentioned earlier). This was our analytical argument. The numerical computations confirm that these traces are zero.

6.3 Subducing and Non-subducing Diagrams

In this chapter, given $R(r,s)$ and $R(r^-,s^+)$, we show that $\mathrm{Tr}(P_{R,(r,s)}P_{R(r^-,s^+)}) = 0$ if s^+ does not subduce s.

We can write $s^+ = s_1 \oplus s_2$, where s_1 is obtained by removing a box from the first row of s^+, and s_2 is obtained by removing a box from the second row of s^+. We can then write:

$$\begin{aligned}\mathrm{Tr}(P_{R,(r,s)}P_{R(r^-,s^+)}) &= \mathrm{Tr}(P_{R,(r,s)}[P_{R(r^-,s_1)} + P_{R(r^-,s_2)}]) \\ &= \mathrm{Tr}(P_{R,(r,s)}P_{R(r^-,s_1)}) + \mathrm{Tr}(P_{R,(r,s)}P_{R(r^-,s_2)})\end{aligned} \tag{6.36}$$

Without loss of generality, consider the s_1 term. We will study:

$$\mathrm{Tr}(P_{R,(r,s)}\sum \Gamma_R((\cdot,\cdot))P_{R(r^-,s_1)}) \tag{6.37}$$

where $(\cdot,\cdot)$ is an arbitrary 2-cycle, and the sum runs over all 2-cycles in S_m. First, take the sum to be acting rightwards, on $P_{R(r^-,s_1)}$. This being the case, the only subspace of R for which $\sum \Gamma_R((\cdot,\cdot))$ won't produce a zero value, is s^+. So, we can replace R with s^+, and acting the summation rightwards, we get:

$$\begin{aligned}\mathrm{Tr}(P_{R,(r,s)}\sum \Gamma_R((\cdot,\cdot))P_{R(r^-,s_1)}) &= \mathrm{Tr}(P_{R,(r,s)}\sum \Gamma_{s^+}((\cdot,\cdot))P_{R(r^-,s_1)}) \\ &= (\text{no. of row pairs of}\, s_1 - \text{no. of column pairs of}\, s_1) \\ &\quad \times \mathrm{Tr}(P_{R,(r,s)}P_{R(r^-,s_1)})\end{aligned} \tag{6.38}$$

As mentioned in the beginning of this dissertation, $P_R\Gamma_R(\sigma) = \Gamma_R(\sigma)P_R$, $\forall\,\sigma$. Using this fact, we can act the summation leftwards (i.e. on $P_{R,(r,s)}$) and get:

$$\begin{aligned}\mathrm{Tr}(P_{R,(r,s)}\sum \Gamma_R((\cdot,\cdot))P_{R(r^-,s_1)}) &= \mathrm{Tr}(P_{R,(r,s)}\sum \Gamma_{s}((\cdot,\cdot))P_{R(r^-,s_1)}) \\ &= (\text{no. of row pairs of}\, s - \text{no. of column pairs of}\, s) \\ &\quad \times \mathrm{Tr}(P_{R,(r,s)}P_{R(r^-,s_1)})\end{aligned} \tag{6.39}$$

The number of row pairs minus the number of column pairs of two diagrams will only be equal if those diagrams themselves are identical. Since the two results above are equal, we can assert that if the trace (in the last line of equations (6.38) and (6.39)) is non-zero, then $s_1 = s$. Of course this trace could be zero, but then, assuming (6.36) is non-zero, the same argument applied to the second term in (6.36) would have to be non-zero. In this case s_2 would be identical to s. So, in conclusion, since we know that s^+ subduces both s_1 and s_2, if $\mathrm{Tr}(P_{R,(r,s)}P_{R(r^-,s^+)})$ is non-zero, then s is identical to either s_1 or s_2, and hence s^+ subduces s. If $\mathrm{Tr}(P_{R,(r,s)}P_{R(r^-,s^+)})$ is zero, then we know that s^+ does not subduce s. This ends the argument.

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