AXIALLY SYMMETRIC E-LENS BASED ON MCMILLAN MAP

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Axially symmetric McMillan map & Round beams

The most general map can be reduced to the form

$$r' = \sqrt{p_r^2 + \frac{p_\theta^2}{r^2}}, \qquad \theta' = \theta + \arctan\frac{p_\theta}{p_r\, r},$$
$$p_r' = -p_r \frac{r}{r'} + \frac{a\, r'}{1 + \mathrm{sgn}[\Gamma]\, r'^2}, \qquad p_\theta' = p_\theta,$$

with two independent invariants

$$\mathcal{K}_r[p_r, r] = \underbrace{p_r^2 - a\, p_r\, r + r^2}_{\propto \mathrm{Courant-Snyder}} + \underbrace{\mathrm{sgn}(\Gamma)\, p_r^2\, r^2}_{\text{nonlinearity}} + \underbrace{\frac{p_\theta^2}{r^2}}_{\text{rotation}}, \qquad \mathcal{K}_\theta[p_\theta, \theta] = p_\theta,$$

and one intrinsic parameter equal to unperturbed betatron tune

$$\nu_0 = \frac{\Phi}{2\pi} = \frac{1}{2\pi} \arccos\frac{a}{2}.$$

Considering general axially symmetric lattice with thin nonlinear kick $\delta\dot{R}$, we can rewrite the map as

$$r' = \sqrt{p_r^2 + \frac{p_\theta^2}{r^2}}, \qquad \theta' = \theta + \arctan\frac{p_\theta}{r\, p_r},$$
$$p_r' = -p_r \frac{r}{r'} + \delta\dot{r}(r'), \qquad p_\theta' = p_\theta,$$

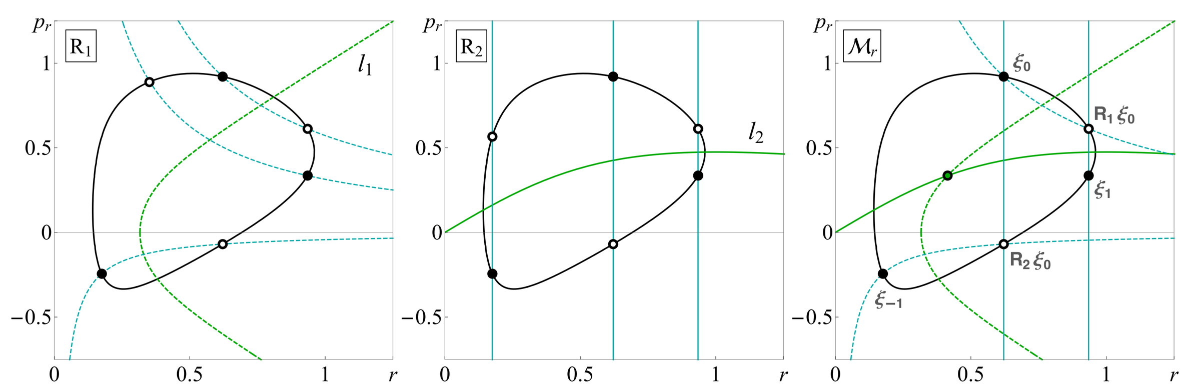
where $\delta\dot{r}(r) = 2\, r \cos\Phi + \beta\, \delta\dot{R}(r) \sin\Phi$. It has one exact, $\mathcal{K}_\theta = p_\theta$, and one approximated invariant in McMillan form

$$\mathcal{K}_r[p_r, r] \approx \mathrm{C.S.}_r - \frac{c}{3!} \frac{p_r^2\, r^2}{a} + \frac{p_\theta^2}{r^2},$$

where the parameters a and c are defined as

$$a = 2\cos\Phi + \beta \sin\Phi\, \partial_r \delta\dot{R}(0) \qquad \text{and} \qquad c = \beta \sin\Phi\, \partial_{rrr} \delta\dot{R}(0).$$

Symmetry lines



The map and its inverse can be broken down into a composition of two nonlinear reflections, denoted by $\mathrm{R}_{1,2}$

$$\mathcal{M}_r^{\pm} = \mathrm{R}_2 \circ \mathrm{R}_1, \qquad (\mathcal{M}_r^{\pm})^{-1} = \mathrm{R}_1 \circ \mathrm{R}_2,$$

where

$$\mathrm{R}_1 : r' = \sqrt{p_r^2 + \frac{p_\theta^2}{r^2}}, \quad p_r' = p_r \frac{r}{r'}, \qquad \text{and} \qquad \mathrm{R}_2 : r' = r, \quad p_r' = -p_r + f(r).$$

Both transformations are anti-area preserving involutions

$$\mathrm{R}_{1,2} = \mathrm{R}_{1,2}^{-1}, \qquad \mathrm{R}_{1,2}^2 = \mathrm{I}_2, \qquad \det \mathbf{J}_{\mathrm{R}_{1,2}} = -1,$$

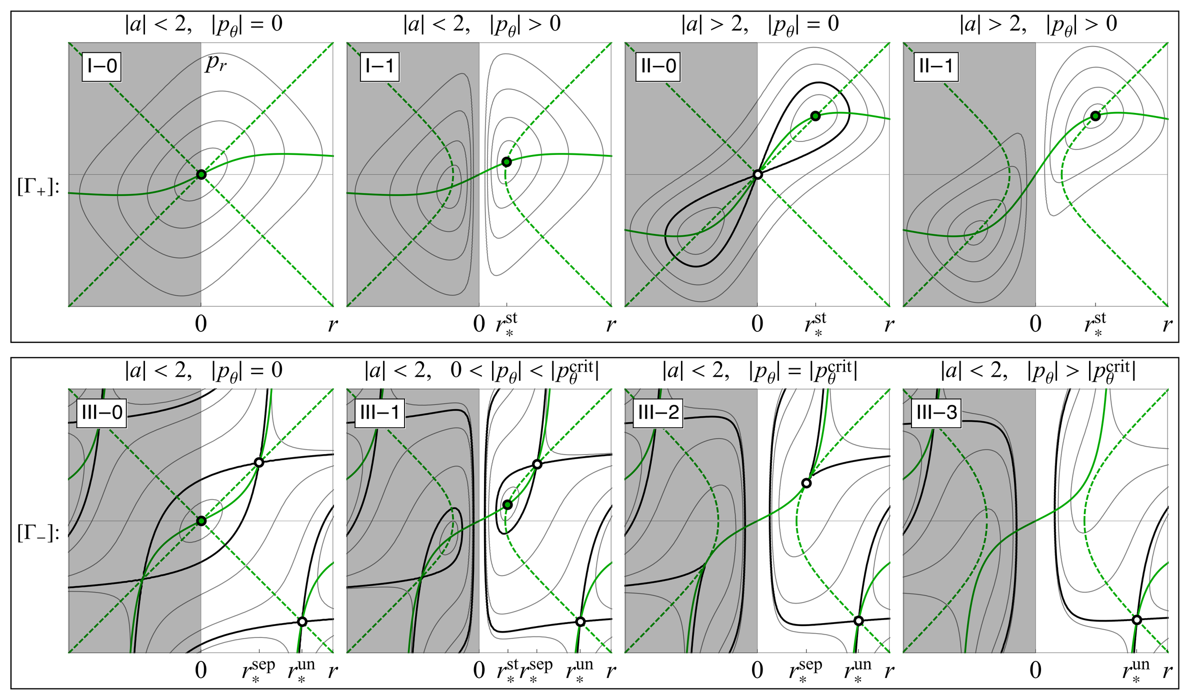
each (and in composition) preserving the radial invariant:

$$\mathcal{K}_r[p_r, r] - \mathcal{K}_r[\mathrm{R}_{1,2}(p_r, r)] = 0.$$

Fixed points of $\mathrm{R}_{1,2}$ form continuous lines of equilibrium solutions, the *first* and *second symmetry lines*, respectively:

$$l_1 : \; p_r^2 = r^2 - \frac{p_\theta^2}{r^2} \qquad l_2 : \; p_r = \frac{f(r)}{2}.$$

Fixed points & Regimes of motion



- In the case $[\Gamma_+]$, map has only one positive root and motion is stable for any value of a and almost all values of angular momentum (except $p_\theta = 0$). Based on the absolute value of a we distinguish two different regimes: $|a| < 2$ [I] and $|a| > 2$ [II].
- In the case $[\Gamma_-]$, the motion is stable only for $|a| < 2$ and $|p_\theta| < p_\theta^{\mathrm{crit}}$ [III]

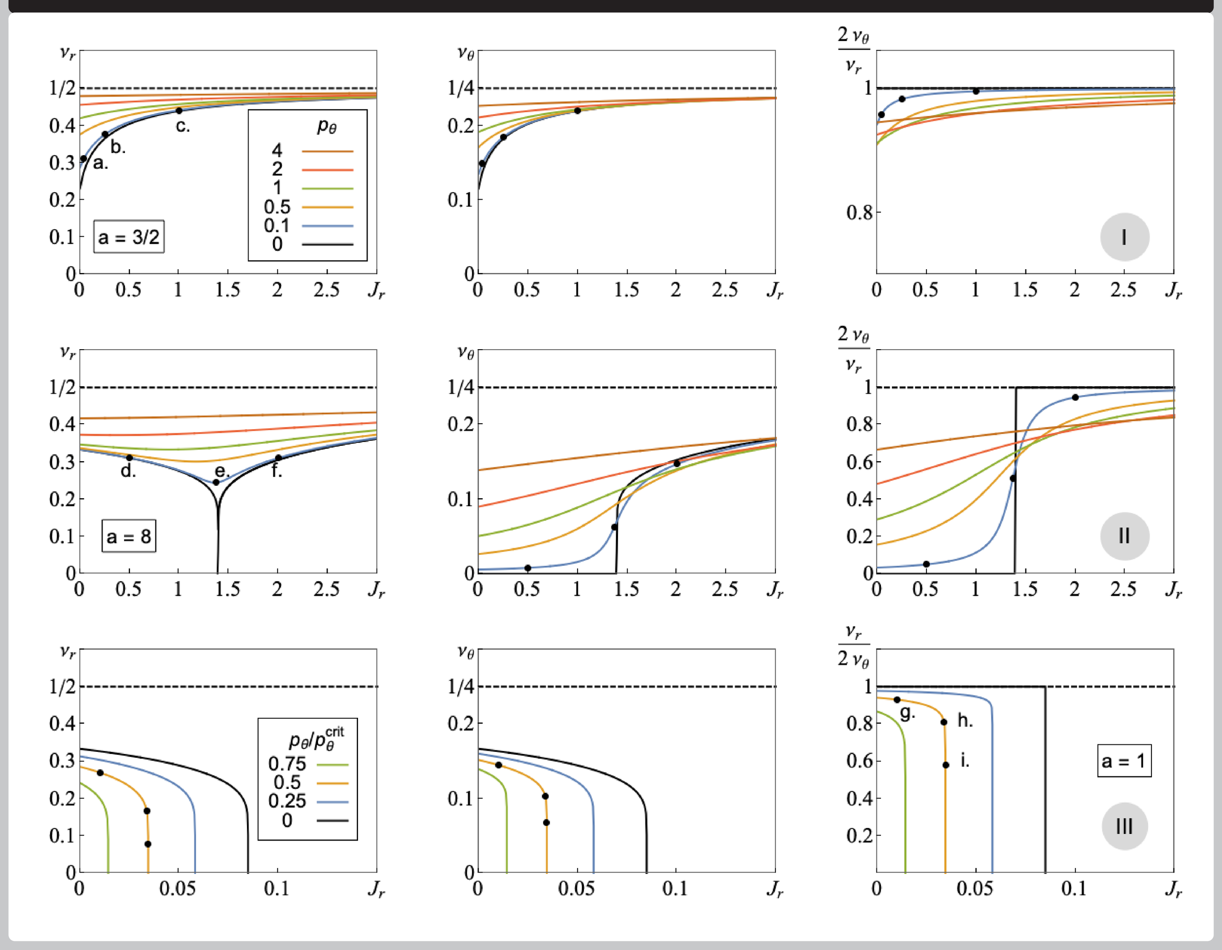
$$p_\theta^{\mathrm{crit}} = \left[1 - (|a|/2)^{2/3}\right]^{3/2} < 1.$$

In this scenario the map has three stationary solutions:

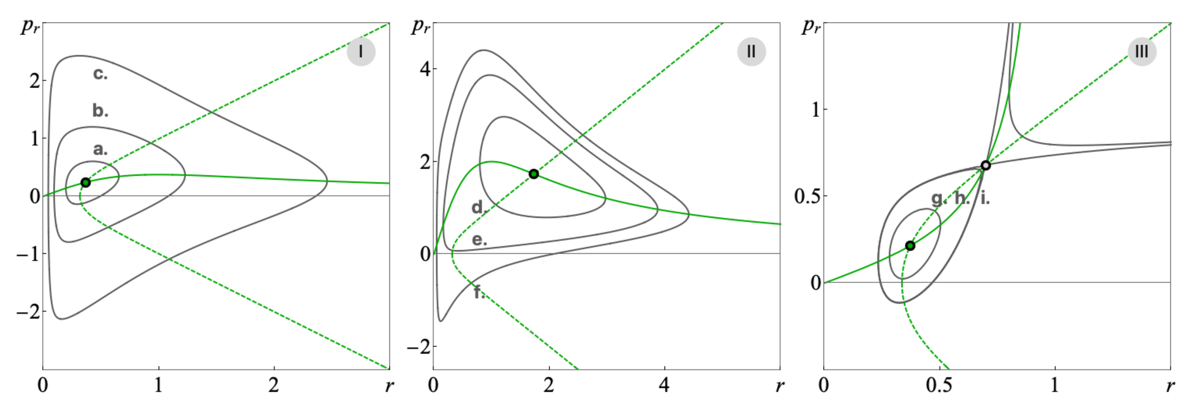
$$0 < r_*^{\mathrm{st}} < r_*^{\mathrm{sep}} < 1 < r_*^{\mathrm{un}},$$

where r_*^{sep} is unstable fixed point with a separatrix isolating stable trajectories and r_*^{un} is the second unstable fixed point.

Action-angle variables



Case studies



Motion invariants $\mathcal{K}_r$ and p_θ, action variables $J_{t,\theta}$ and rotation numbers $\nu_{r,\theta}$ for each case study:

	$[\Gamma_+]$						$[\Gamma_-]$		
	I			II			III		
	$a = 3/2,\ p_\theta = -0.1$			$a = 8,\ p_\theta = 0.1$			$a = 1,\ p_\theta = 0.5\, p_\theta^{\mathrm{crit}} = 0.1125$		
	(a.)	(b.)	(c.)	(d.)	(e.)	(f.)	(g.)	(h.)	(i.)
J_r	0.04	0.25	1	0.5	1.3695	2	0.01	0.033692	0.0344042
$\mathcal{K}_r$	0.28228	1.17187	5.54626	-5.54758	0	4.2368	0.217948	0.2746	0.275683
ν_r	0.310521	0.375294	0.440763	0.310913	0.244355	0.310346	0.269487	0.166497	0.0785715
ν_θ	0.148603	0.184318	0.219254	0.007675	0.062641	0.146941	0.144787	0.102572	0.0680104

Cartesian frequencies

x-, y-planes have identical spectra with 2 families of overtones:

$$(\nu_r - \nu_\theta) + n\,\nu_r : \quad \nu_r - \nu_\theta, \quad 2\,\nu_r - \nu_\theta, \quad 3\,\nu_r - \nu_\theta, \quad \ldots,$$
$$\nu_\theta + n\,\nu_r : \quad \nu_\theta, \quad \nu_\theta + \nu_r, \quad \nu_\theta + 2\,\nu_r, \quad \ldots,$$

Fundamental tunes $\nu_r - \nu_\theta$ and ν_θ play the role of the sum and difference of the "carrier" and "modulating" frequencies:

$$\nu_\Sigma = \nu_1 + \nu_2 \qquad \nu_\Delta = \nu_1 - \nu_2$$

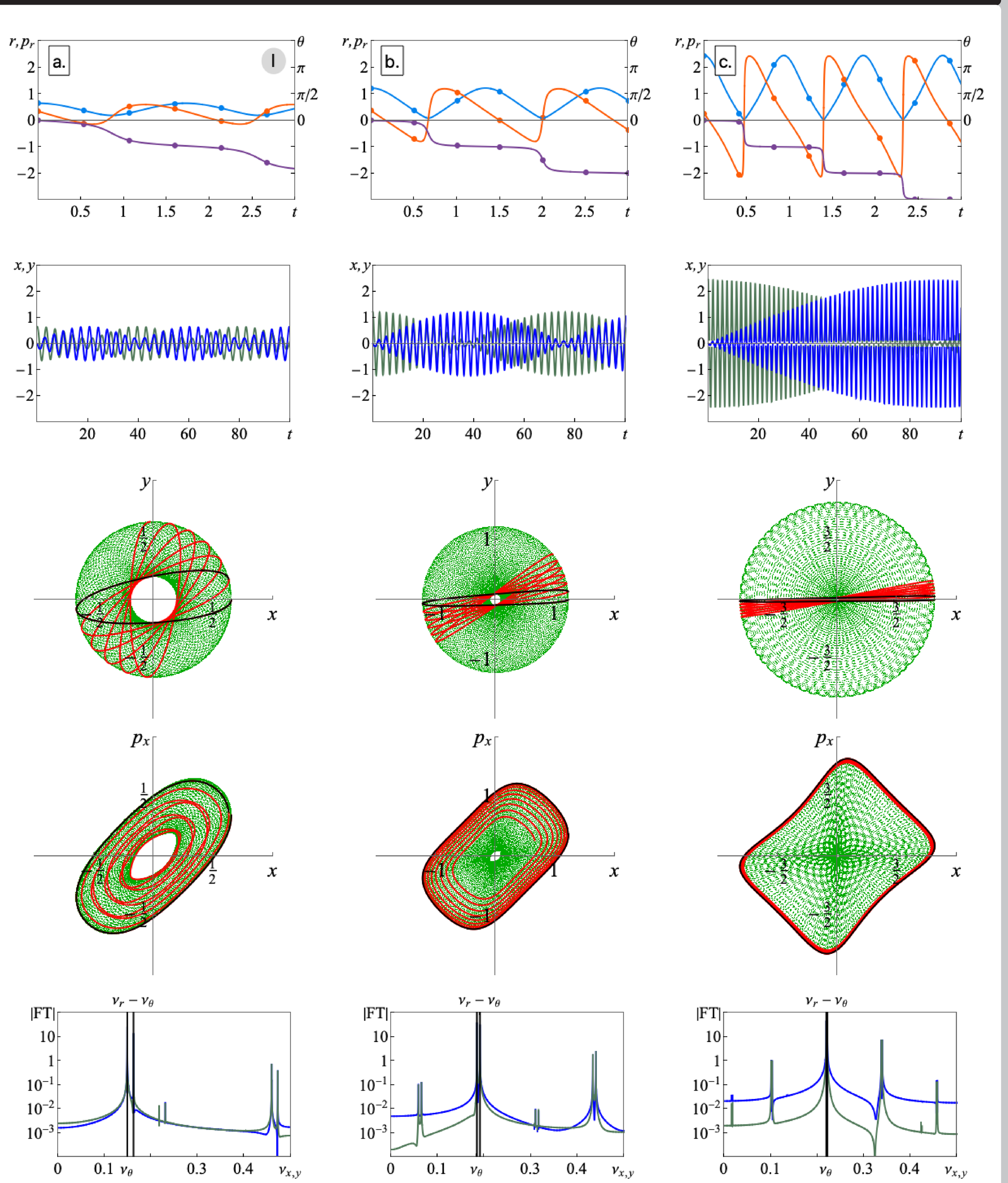
For configuration $[\Gamma_+]$, where $2\,\nu_\theta < \nu_r$, we define:

$$\nu_\Sigma = \nu_r - \nu_\theta, \qquad \nu_1 = \frac{\nu_r}{2},$$
$$\nu_\Delta = \nu_\theta, \qquad \nu_2 = \frac{\nu_r}{2} - \nu_\theta.$$

While for configuration $[\Gamma_-]$, where $2\,\nu_\theta > \nu_r$, we will use:

$$\nu_\Sigma = \nu_\theta, \qquad \nu_1 = \frac{\nu_r}{2},$$
$$\nu_\Delta = \nu_r - \nu_\theta, \qquad \nu_2 = \nu_\theta - \frac{\nu_r}{2}.$$

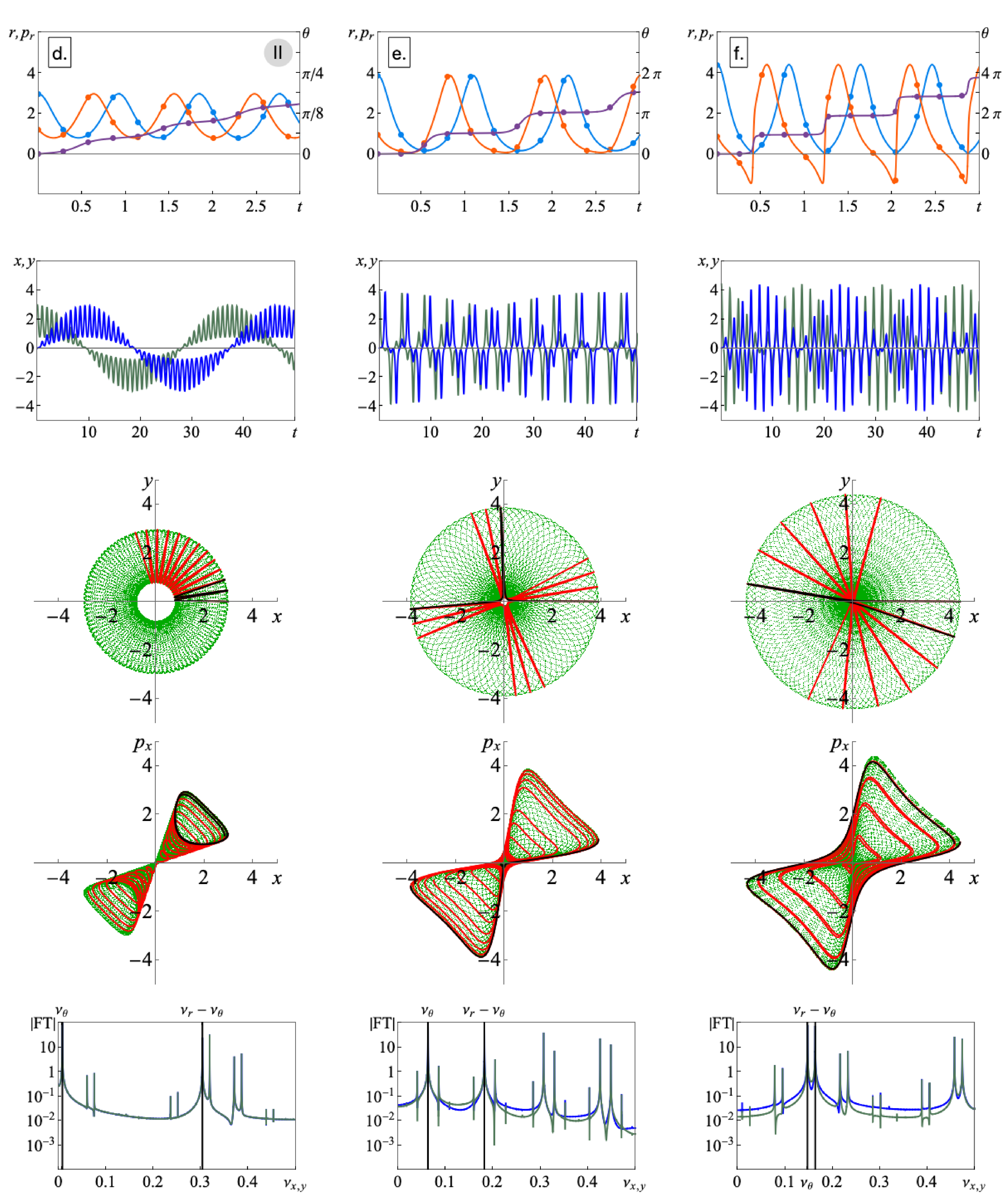
Case I



The presence of beats indicates that the sum and difference modes have nearly the same frequencies:

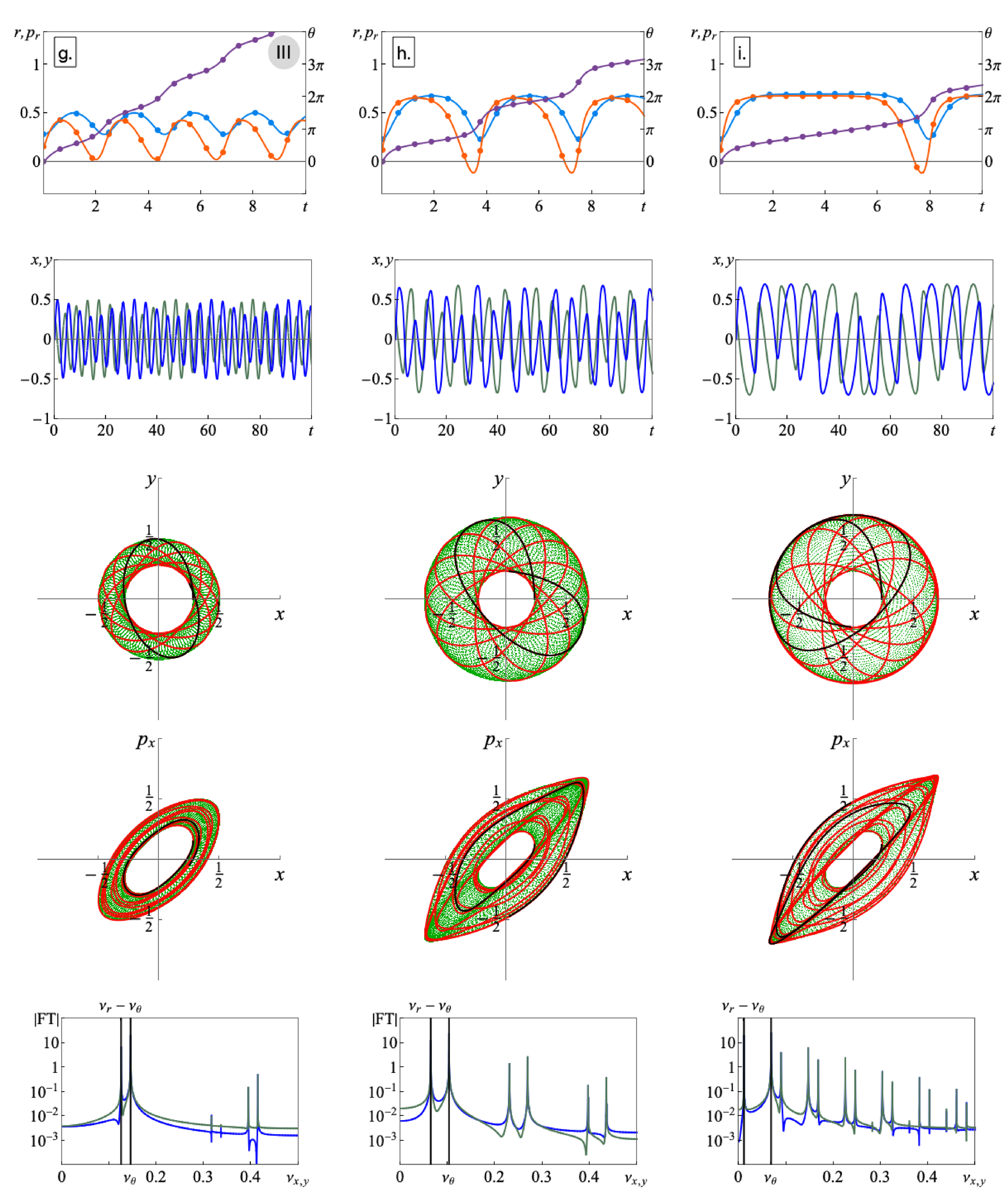
$$\nu_\Sigma \approx \nu_\Delta \qquad (\text{or } \nu_r \approx 2\,\nu_\theta)$$
$$\nu_1 \approx \nu_{\Sigma,\Delta} \qquad \text{and} \qquad \nu_2 \approx 0$$

Case II



$(d.)$	$\nu_\theta,\ \frac{\nu_\theta}{\nu_r} \approx 0$	$\nu_\Sigma \gg \nu_\Delta \approx 0$	$\nu_1 \approx \nu_2 \approx \nu_\Sigma/2$
$(e.)$	$0 < \frac{\nu_\theta}{\nu_r} < \frac{1}{2}$	$\nu_\Sigma > \nu_\Delta$	$\nu_1 > \nu_2 \approx 0$
$(f.)$	$\frac{\nu_\theta}{\nu_r} \approx \frac{1}{2}$	$\nu_\Sigma \approx \nu_\Delta$	$(\nu_1 \approx \nu_{\Sigma,\Delta}) \gg (\nu_2 \approx 0)$

Case III



$(g.)$	$\frac{\nu_\theta}{\nu_r} \approx \frac{1}{2}$	$\nu_\Sigma \approx \nu_\Delta$	$(\nu_1 \approx \nu_{\Sigma,\Delta}) \gg (\nu_2 \approx 0)$
$(h.)$	$\frac{1}{2} < \frac{\nu_\theta}{\nu_r} < 1$	$\nu_\Sigma > \nu_\Delta$	$\nu_1 > \nu_2 \approx 0$
$(i.)$	$\frac{\nu_\theta}{\nu_r} \approx 1$	$\nu_\Sigma \gg \nu_\Delta \approx 0$	$\nu_1 \approx \nu_2 \approx 0$

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