$\mathcal{O}(\alpha_s^4)$ loop-by-loop contributions to heavy quark pair production in hadronic collisions

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Abstract

The present state of the theoretical predictions for the hadronic heavy hadron production is not quite satisfactory. The full next-to-leading order (NLO) $\mathcal{O}(\alpha_s^3)$ corrections to the hadroproduction of heavy quarks have raised the leading order (LO) $\mathcal{O}(\alpha_s^2)$ estimates but the NLO predictions are still slightly below the experimental numbers. Moreover, the theoretical NLO predictions suffer from the usual large uncertainty resulting from the freedom in the choice of renormalization and factorization scales of perturbative QCD. In this light there are hopes that a next-to-next-to-leading order (NNLO) $\mathcal{O}(\alpha_s^4)$ calculation will bring theoretical predictions even closer to the experimental data. Also, the dependence on the factorization and renormalization scales of the physical process is expected to be greatly reduced at NNLO. This would reduce the theoretical uncertainty and therefore make the comparison between theory and experiment much more significant.

In this thesis I have concentrated on that part of NNLO corrections for hadronic heavy quark production where one-loop integrals contribute in the form of a loop-by-loop product. In the first part of the thesis I use dimensional regularization to calculate the $\mathcal{O}(\varepsilon^2)$ expansion of scalar one-loop one-, two-, three- and four-point integrals. The Laurent series of the scalar integrals is needed as an input for the calculation of the one-loop matrix elements for the loop-by-loop contributions. Since each factor of the loop-by-loop product has negative powers of the dimensional regularization parameter ε up to $\mathcal{O}(\varepsilon^{-2})$, the Laurent series of the scalar integrals has to be calculated up to $\mathcal{O}(\varepsilon^2)$. The negative powers of ε are a consequence of ultraviolet and infrared/collinear (or mass) divergences. Among the scalar integrals the four-point integrals are the most complicated. The $\mathcal{O}(\varepsilon^2)$ expansion of the three- and four-point integrals contains in general classical polylogarithms up to Li_4 and L-functions related to multiple polylogarithms of maximal weight and depth four. All results for the scalar integrals are also available in electronic form.

In the second part of the thesis I discuss the properties of the classical polylogarithms. I present the algorithms which allow one to reduce the number of the polylogarithms in an expression. I derive identities for the L-functions which have been intensively used in order to reduce the length of the final results for the scalar integrals. I also discuss the properties of multiple polylogarithms. I derive identities to express the L-functions in terms of multiple polylogarithms.

In the third part I investigate the numerical efficiency of the results for the scalar integrals. The dependence of the evaluation time on the relative error is discussed.

In the forth part of the thesis I present the larger part of the $\mathcal{O}(\varepsilon^2)$ results on one-loop matrix elements in heavy flavor hadroproduction containing the full spin information. The $\mathcal{O}(\varepsilon^2)$ terms arise as a combination of the $\mathcal{O}(\varepsilon^2)$ results for the scalar integrals, the spin algebra and the Passarino-Veltman decomposition. The one-loop matrix elements will be needed as input in the determination of the loop-by-loop part of NNLO for the hadronic heavy flavor production.

Brevity is the sister of talent.

Die Kürze ist die Schwester des Talents.

Anton P. Chekhov

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Chapter 1

Introduction

At the leading order (LO) Born term level, heavy quark production mechanisms have been studied some time ago [1, 2, 3, 4, 5, 6]. The next-to-leading order (NLO) corrections to unpolarized heavy quark hadroproduction were first presented in [7, 8, 9, 10], and in [11, 12] for photoproduction. Corresponding results with initial particles being longitudinally polarized were calculated in [13] and [14, 15, 16, 17]. The calculation of NLO corrections to top-quark hadroproduction with spin correlations of the final quarks was performed in [18]. Analytical results for the so called "virtual plus soft" terms were presented in [9, 10, 12, 15] for the photoproduction and unpolarized hadroproduction of heavy quarks. Complete analytic results for the polarized and unpolarized photoproduction of heavy quarks, including real bremsstrahlung, can be found in [17].

The full next-to-leading order (NLO) corrections to the hadroproduction of heavy quarks have raised the leading order (LO) estimates but several initial analysis' showed a serious disagreement with experimental results [19, 20, 21]. Recently the situation has considerably improved in that a more refined NLO analysis (due to considerably more precise experimental input for the b-quark fragmentation function as well as other QCD parameters) now shows signs of rapprochement between theory and the new experimental data (see [22] and references therein for the new CDF measurements as well as [23, 24, 25]). However, the NLO predictions are still slightly below the experimental numbers. Moreover, the theoretical NLO predictions suffer from the usual large uncertainty resulting from the freedom in the choice of renormalization and factorization scales of perturbative QCD. In this light there are hopes that a next-to-next-to-leading order (NNLO) calculation will bring theoretical predictions even closer to the experimental data. Also, the dependence on the factorization and renormalization scales of the physical process is expected to be greatly reduced at NNLO. This would reduce the theoretical uncertainty and therefore make the comparison between theory and experiment much more significant. In Fig. 1.1 I show one generic diagram each for the four classes of gluon-induced contributions that need to be calculated for the NNLO corrections to hadroproduction of heavy flavors. They involve the two-loop contribution (Fig. 1.1a), the loop-by-loop contribution (Fig. 1.1b), the one-loop gluon emission contribution (Fig. 1.1c) and, finally, the two gluon emission contribution (Fig. 1.1d). A similar classification holds for the quark-induced contributions.

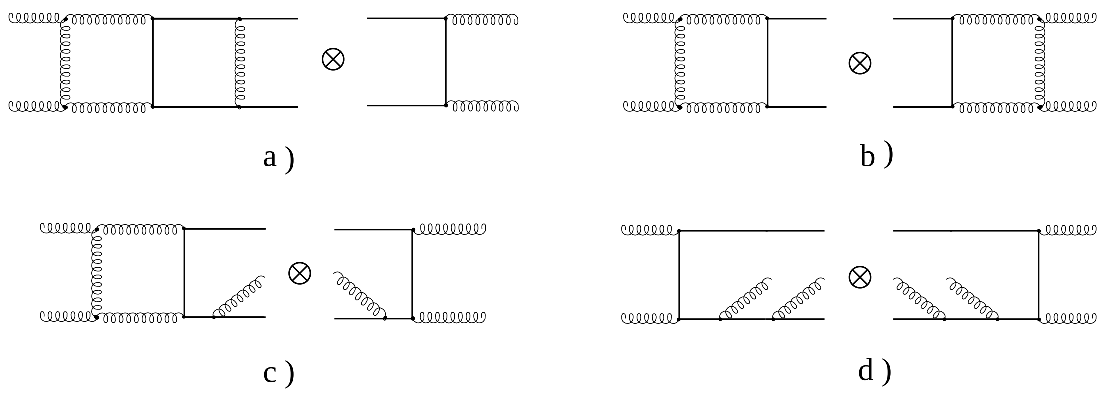

Figure 1.1: Exemplary gluon fusion diagrams for the NNLO calculation of heavy hadron production.

In this thesis I concentrate on the loop-by-loop contribution Fig. 1.1b. Specifically, working in the framework of the dimensional regularization scheme [26, 27, 28] with $n = 4 - 2\varepsilon$ dimensions, I will present $\mathcal{O}(\varepsilon^2)$ results on all scalar one-loop one-, two-, three- and four-point integrals that are needed in the calculation of hadronic heavy flavour production. Based on these results I present also the matrix elements for quark-induced and part of the contributions to gluon-induced heavy flavour production up to $\mathcal{O}(\varepsilon^2)$. The coefficients of the Laurent series expansion up to $\mathcal{O}(\varepsilon^0)$ were already presented in [29]. In this thesis I present analytical results for the ε- and ε^2-coefficients of the ε-expansion including also their imaginary parts. Since the one-loop integrals exhibit ultraviolet (UV) and infrared (IR)/collinear (or mass (M)) singularities up to $\mathcal{O}(\varepsilon^{-2})$ one needs to know the one-loop integrals up to $\mathcal{O}(\varepsilon^2)$ because the one-loop contributions appear in product form in the loop-by-loop contributions[1] (Fig. 1.1b). It is clear that not only the calculation of the scalar integrals has to be performed up to $\mathcal{O}(\varepsilon^2)$ but also the spin algebra and the Passarino-Veltman decomposition of tensor integrals (required for the calculation of matrix elements for heavy flavour production) in the one-loop contributions also have to be done up to $\mathcal{O}(\varepsilon^2)$.

Let me briefly describe the procedure for the calculation of scalar one-loop one-, two-, three- and four-point integrals. One introduces Feynman parameter representations for each of the two-, three- and four-point integrals. The one-point integral does not need Feynman parameterization. Then one performs a Wick rotation and integrates over the whole Euclidean momentum space. For the one-point integral the result is ready. For the two-, three- and four-point integrals one is left with one-, two- and three-dimensional integration over Feynman parameters, respectively. The general idea is to integrate an integrand over Feynman parameters keeping the full ε-dependence of the result as long as

[1] In a more general setting the Laurent-series expansion of the scalar integrals is needed if the integration-by-parts technique [30, 31] is employed. The reason is that the solution of the recursion relations induced by the integration-by-parts technique can bring in negative powers of ε.

possible. When one faces the impossibility for further analytical integration, one expands the respective integrand in ε up to $\mathcal{O}(\varepsilon^2)$ and then analytically integrates the expanded integrand term by term.

The general case of massive scalar one-loop integrals was studied some time ago [32, 33], where a general one-loop N-point integral was expressed in terms of hypergeometric functions of several variables. Recently, there have been a number of papers where the authors took a more general attitude to calculate the ε-expansion of massive one-loop integrals. They write down general representations of the ε-expansion of one-loop integrals for general kinematic configurations. In [34] an attempt was made to compare the results of [34] with the results of the more general approaches whenever possible. In papers [35, 36] the all-order ε-expansion of one-loop two-point and of certain three-point functions was done explicitly by expanding the relevant hypergeometric functions. One-fold integral representations for general three- and four-point functions, as well as ways to get expansion terms of order ε for 3-point functions, were worked out in a recent paper [37]. The publications [32, 33, 36, 37] also contain a comprehensive list of references on the subject.

However, in general, the required ε-expansion (including ε^2-terms) is not readily available for all the integrals needed in the hadronic heavy flavour production process. Also, the analytic continuation of the above mentioned hypergeometric functions in [32, 33, 37] to the appropriate kinematical regions of validity is not always possible. This mainly concerns the four-point functions. In addition, it is more convenient to present results for the ε-expansion in terms of simpler special functions, in the form convenient for numerical evaluation. And finally, collecting together all the necessary scalar integrals needed for the derivation of tensor integrals entering the loop-by-loop contribution constitutes a first step in the difficult task of obtaining the NNLO corrections to heavy flavor hadroproduction cross section.

The notation of this thesis will remain very close to the notation introduced and used in [9, 10, 29]. For the calculation of the NNLO virtual corrections to hadroproduction of heavy flavors one needs the same set of scalar master integrals as given in the Appendix A of [9, 10] (the relevant set of master integrals is listed in Table 1.1). However, as explained above, knowledge of their singular and finite terms is not sufficient for the calculation of NNLO loop-by-loop corrections. For that purpose one needs to know the one-loop integrals up to $\mathcal{O}(\varepsilon^2)$ including also their imaginary parts which equally well contribute to the modulus squared of the one-loop amplitudes. The imaginary parts of the one-loop integrals are really needed only up to $\mathcal{O}(\varepsilon)$ since the highest singularity of the imaginary parts is only $\mathcal{O}(\varepsilon^{-1})$ compared to $\mathcal{O}(\varepsilon^{-2})$ for the real parts. Nevertheless it was decided to include $\mathcal{O}(\varepsilon^2)$ results also for the imaginary parts which may be of interest in other applications. Consequently, in this thesis I present the relevant expressions for all scalar integrals needed in the calculation of the NNLO loop-by-loop corrections to hadroproduction of heavy flavors. For reasons of comprehensiveness I decided to include also the singular and finite (i.e. $\mathcal{O}(\varepsilon^0)$) parts of the scalar integrals in this presentation. They agree with the results of the real contributions presented in [9, 10].

A comment on the length of the formula expressions for the scalar one-loop integrals is appropriate. The untreated computer output of the integrations is generally quite lengthy.

Table 1.1: List of one-, two-, three- and four-point massive one-loop functions calculated up to $\mathcal{O}(\varepsilon^2)$.

	Nomenclature of [9, 10]	This nomenclature	Novelty	Comments
1-point	$A(m)$	A	–	Re
2-point	$B(p_4 - p_2, 0, m)$	B_1	–	Re
	$B(p_3 + p_4, m, m)$	B_2	–	Re, Im
	$B(p_4, 0, m)$	B_3	–	Re
	$B(p_2, m, m)$	B_4	–	Re
	$B(p_3 + p_4, 0, 0)$	B_5	–	Re, Im
3-point	$C(p_4, p_3, 0, m, 0)$	C_1	new	Re, Im
	$C(p_4, -p_2, 0, m, m)$	C_2	new	Re
	$C(-p_2, p_4, 0, 0, m)$	C_3	–	Re
	$C(-p_2, -p_1, 0, 0, 0)$	C_4	–	Re, Im
	$C(-p_2, -p_1, m, m, m)$	C_5	–	Re, Im
	$C(p_3, p_4, m, 0, m)$	C_6	–	Re, Im
4-point	$D(p_4, -p_2, -p_1, 0, m, m, m)$	D_1	new	Re, Im
	$D(-p_2, p_4, p_3, 0, 0, m, 0)$	D_2	new	Re, Im
	$D(-p_2, p_4, -p_1, 0, 0, m, m)$	D_3	new	Re

The hard work is to simplify these expressions. I have written semi–automatic computer codes that achieve the simplifications using known identities among polylogarithms and using a number of identities for the L–functions introduced in this thesis.

As was already mentioned above one obtains the $\mathcal{O}(\varepsilon^2)$ results of the corresponding matrix elements for the hadronic heavy flavour production based on the results for the massive scalar one-loop integrals. Let me emphasize the importance of knowing one-loop matrix elements which contain the full spin information of the relevant subprocess. When the one-loop contributions are folded with the Born term contributions and spin is summed, as in a NLO rate calculation, the information on the spin content of the one-loop contribution is lost and cannot be reconstructed from the rate expressions. On the other hand, having expressions for matrix elements allows one to easily derive the one-loop contributions to partonic cross section including any polarization of the incoming or outgoing particles. Also, it allows one to obtain any of the crossed processes, including the ones with a heavy incoming particle. This thesis presents almost complete results on a NNLO calculation of partonic matrix elements for the set of one-loop Feynman graphs present in hadroproduction of heavy flavors, separately for every Feynman diagram in order to facilitate the use of the results for the photon-induced processes $\gamma + g \to Q + \overline{Q}$ and $\gamma + \gamma \to Q + \overline{Q}$ that differ by color factors.

The hadroproduction of heavy flavors proceeds through the following two partonic channels:

$$g + g \to Q + \overline{Q}, \tag{1.1}$$

where g denotes a gluon and $Q(\overline{Q})$ denotes a heavy quark (antiquark), and

$$q + \bar{q} \to Q + \overline{Q}, \tag{1.2}$$

where $q(\bar{q})$ is a light massless quark (antiquark).

As mentioned above the Abelian part of the NLO result for (1.1) provides the NLO corrections to heavy flavor production by the collision of two on-shell photons

$$\gamma + \gamma \to Q + \overline{Q}, \tag{1.3}$$

with the appropriate color factor substitutions. The results for (1.1) can be also used to determine corresponding amplitudes for heavy flavor photoproduction

$$\gamma + g \to Q + \overline{Q}. \tag{1.4}$$

One should mention that the partonic processes (1.1) and (1.2) are needed for the calculation of the contributions of single- and double-resolved photons in the photonic processes (1.3) and (1.4).

NLO cross sections for the process (1.3) have already been determined in [38, 39, 40] for unpolarized and in [40, 41] for polarized initial photons. Note that the authors of [41] used a nondimensional regularization scheme to regularize the poles of divergent integrals. In the papers [38, 41] analytic results were presented for "virtual plus soft" contributions alone. It should be noted that complete analytical results including hard gluon contributions can be found only in [40]. The reaction (1.3) will be investigated at future e^+- e^-– linear colliders. NLO corrections for the heavy quark production cross section (1.3) are of interest in themselves as they represent an background to the intermediate Higgs boson searches for Higgs masses in the range of 90 to 160 GeV.

The thesis is organized as follows. In Chap. 2 I present the results for all scalar one-loop one-, two-, three- and four-point integrals that are needed in the calculation of hadronic heavy flavour production. Some calculational details can also be found there. In Chap. 3 I discuss identities involving classical polylogarithms. I develop algorithms based on these identities which allow one to reduce the number of the polylogarithms in the expressions containing polylogarithms. The scalar master integrals contain a large number of different polylogarithms after integration. Therefore the algorithms are very useful for the simplification of the results. The algorithms are written in the internal programming language of the computer algebra system **Mathematica** [42]. They are presented in Appendix B.1, B.2 and C. In Chap. 4 I discuss properties and identities involving the single- and triple-index $L-$functions introduced in Chap. 2. A judicious use of these identities has allowed me to considerably reduce the length of the final analytical results for the three– and four–point functions. In addition in Chap. 4 I discuss the multiple polylogarithms introduced by Goncharov [43] and discuss how they are related to the classical polylogarithms [44], Nielsen's generalized polylogarithms, the harmonic polylogarithms of Remiddi and Vermaseren [45] and the two-dimensional harmonic polylogarithms [46]. It is also explicitly shown how the $L-$functions can be expressed in terms of multiple polylogarithms. In Chap. 5 I discuss the numerical efficiency of the results for the massive one-loop scalar integrals obtained in Chap. 2. In Chap. 6 I present the one-loop matrix elements for quark-induced and larger part of the contributions to gluon-induced heavy flavour production up to $\mathcal{O}(\varepsilon^2)$ basing on the results of Chap. 2. Finally, in Chap. 7 I give a summary and an outlook.

Chapter 2

Calculation of the scalar master integrals

2.1 One- and two-point functions

I start with the one-loop one-point function which is defined by

$$A(m) = \mu^{2\varepsilon} \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2 - m^2}. \tag{2.1}$$

Before giving the results for the integral (2.1) we first consider a generalization of the integral (2.1), which is needed in the calculation of the other N-point functions (N=2,...,4):

$$\mu^{2\varepsilon} \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - K)^N}. \tag{2.2}$$

The parameter function K appears due to the Feynman parametrization of the two-, three- and four-point functions. It is a function of the external momenta and the internal masses of the corresponding Feynman graphs (see later Eqs. (2.19), (2.40) and (2.63)). Working in $n = 4 - 2\varepsilon$ dimensions, one performs a so-called Wick rotation of the energy component of the n-momentum q (see e.g. [47]). In the integral (2.2) the integration for all components $q_i (i = 0, ..., n-1)$ of the momentum runs from $-\infty$ to $+\infty$. Consider the integration over the q_0 component. In Fig. 2.1 one can see how the integration from $-\infty$ to $+\infty$ can be changed to the integration from $-i\infty$ to $+i\infty$. In the parameter K one always assumes an infinitely small imaginary shift $-i\delta$ (it corresponds to the "causal" infinitely small positive imaginary shift $+i\delta$ in the propagators). The points in Fig. 2.1 show the poles for the variable q_0 on the complex plane for the case of positive K $(K > 0)$. The pole in the second quadrant of the complex plane is located at

$$-\sqrt{q_1^2 + ... + q_{n-1}^2 + K} + \frac{i\delta}{2\sqrt{q_1^2 + ... + q_{n-1}^2 + K}} + \mathcal{O}(\delta^2) \tag{2.3}$$

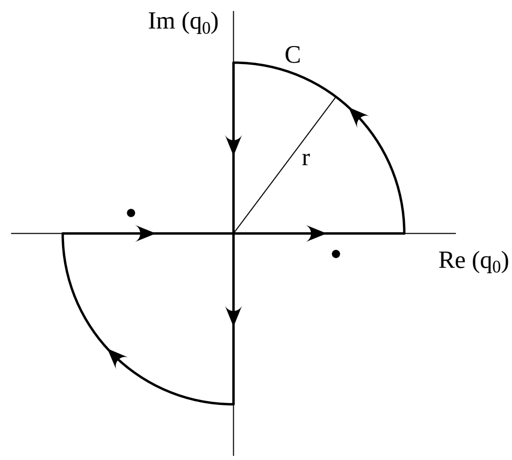

Figure 2.1: Singularities and integration contour for the integral (2.2) in the q_0 plane.

and the pole in the fourth quadrant is located at

$$+\sqrt{q_1^2+...+q_{n-1}^2+K}-\frac{i\delta}{2\sqrt{q_1^2+...+q_{n-1}^2+K}}+\mathcal{O}(\delta^2). \tag{2.4}$$

In the case of negative K there are no poles. Let us consider the contour **C** in Fig. 2.1. Inside of this contour there are no poles for any K. Therefore the integral over the closed contour **C** vanishes due to Cauchy's theorem. Then one takes the limiting case $r \to +\infty$. For $n < 2N$ the integrand for the variable q_0 vanishes fast enough for large $|q_0|$ so that the contributions of the integration on the arcs drops out for $|q_0| = r \to +\infty$. Therefore the integral over q_0 from $-\infty$ to $+\infty$ plus the integral over q_0 from $+i\infty$ to $-i\infty$ are equal to zero. Thus one can replace the integral over the real axis from $-\infty$ to $+\infty$ by one over the imaginary axes $-i\infty$ to $+i\infty$. After change of the integration variables

$$q_0 \to ix_1,\ q_{1,2,...,n-1} \to x_{2,3,...,n}\,. \tag{2.5}$$

the integral (2.2) transforms into

$$\mu^{2\varepsilon}\int\frac{id^nx}{(2\pi)^n}\frac{1}{\left((ix_1)^2-x_2^2-...-x_n^2-K\right)^N}=$$
$$\mu^{2\varepsilon}i(-1)^N\int\frac{d^nx}{(2\pi)^n}\frac{1}{\left(x_1^2+x_2^2+...+x_n^2+K\right)^N}\,, \tag{2.6}$$

where the integration runs over the whole n-dimensional Euclidean space: $-\infty < x_i < +\infty$. One introduces n-dimensional spherical coordinates

$$x_i \equiv r\prod_{k=i}^{n-1}\sin\theta_k\cos\theta_{k-1}\,,\ i=2,...,n-1$$
$$x_1 \equiv r\prod_{k=1}^{n-1}\sin\theta_k\,,\ x_n \equiv r\cos\theta_{n-1} \tag{2.7}$$

with the integration ranges

$$r \in [0, +\infty), \quad \theta_1 \in [0, 2\pi], \quad \theta_{2,\dots,n} \in [0, \pi]\,. \tag{2.8}$$

From the Jacobian one obtains the integration measure $dx_n = r^{n-1} dr d\Omega_{n-1}$ with

$$d\Omega_{n-1} \equiv \prod_{k=1}^{n-1} \sin^{k-1}\theta_k d\theta_k\,. \tag{2.9}$$

Using Eqs. (2.7) and (2.9) one writes the integral (2.6) in the form

$$\mu^{2\varepsilon} i(-1)^N \int\limits_0^{+\infty} \frac{r^{n-1}dr}{(r^2+K)^N} \int \frac{d\Omega_{n-1}}{(2\pi)^n}. \tag{2.10}$$

Using

$$\int\limits_0^{\pi} d\theta \sin^m(\theta) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \tag{2.11}$$

one integrates the angular part of (2.10):

$$\begin{gathered}\int \frac{d\Omega_{n-1}}{(2\pi)^n} = \frac{1}{(2\pi)^n} \int\limits_0^{2\pi} d\theta_1 \prod_{k=2}^{n-1} \int\limits_0^{\pi} \sin^{k-1}\theta_k d\theta_k = \\ \frac{2\pi}{(2\pi)^n} \prod_{k=2}^{n-1} \frac{\sqrt{\pi}\,\Gamma\left(\frac{k-1+1}{2}\right)}{\Gamma\left(\frac{k-1+2}{2}\right)} = \frac{(\sqrt{\pi})^{n-2}}{(2\pi)^{n-1}} \prod_{k=2}^{n-1} \frac{\sqrt{\pi}\,\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)} = \\ \pi^{-\frac{n}{2}} 2^{-n+1} \left\{ \frac{\Gamma\left(\frac{2}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \times \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{4}{2}\right)} \times \dots \times \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \right\} = \frac{\pi^{-\frac{n}{2}} 2^{-n+1}}{\Gamma\left(\frac{n}{2}\right)}\,.\end{gathered} \tag{2.12}$$

The integral for the variable r can be expressed through the beta function

$$\int\limits_0^{+\infty} \frac{r^{n-1}dr}{(r^2+K)^N} = \frac{K^{\frac{n}{2}-N} B\left(N-\frac{n}{2}, \frac{n}{2}\right)}{2} = \frac{K^{\frac{n}{2}-N}\Gamma\left(N-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}{2\Gamma(N)}\,. \tag{2.13}$$

Using Eqs. (2.12) and (2.13) one arrives at the following result for the integral (2.10) which is equal to the integral (2.2):

$$\begin{gathered}\mu^{2\varepsilon} \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2-K)^N} = \mu^{2\varepsilon} i(-1)^N \times \frac{K^{\frac{n}{2}-N}\Gamma\left(N-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}{2\Gamma(N)} \times \frac{\pi^{-\frac{n}{2}}2^{-n+1}}{\Gamma\left(\frac{n}{2}\right)} = \\ \frac{\mu^{2\varepsilon} i(-1)^N \pi^{-\frac{n}{2}} 2^{-n} K^{\frac{n}{2}-N}\Gamma\left(N-\frac{n}{2}\right)}{\Gamma(N)} = \frac{i\mu^{2\varepsilon}(-1)^N(4\pi)^{-2+\varepsilon}K^{2-N-\varepsilon}\Gamma(-2+N+\varepsilon)}{\Gamma(N)}\,,\end{gathered} \tag{2.14}$$

where in the last step one uses $n = 4 - 2\varepsilon$. Conventionally, one also extracts the coefficient $C_\varepsilon(m^2)$ defined as

$$C_\varepsilon(m^2) \equiv \frac{\Gamma(1+\varepsilon)}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^\varepsilon . \tag{2.15}$$

Combining Eqs. (2.14) and (2.15) one writes the final result for the integral (2.2) as

$$\begin{aligned}\mu^{2\varepsilon}\int\frac{d^nq}{(2\pi)^n}\frac{1}{(q^2-K)^N} = iC_\varepsilon(m^2)\,\frac{(-1)^N\Gamma(-2+N+\varepsilon)}{\Gamma(N)\Gamma(1+\varepsilon)}\,m^{2\varepsilon}K^{2-N-\varepsilon} = \\ iC_\varepsilon(m^2)\,\frac{(-1)^N\Gamma(-2+N+\varepsilon)}{\Gamma(N)\Gamma(1+\varepsilon)}\,(m^2)^{2-N}\tilde{K}^{2-N-\varepsilon}\,,\end{aligned} \tag{2.16}$$

where the dimensionless parameter $\tilde{K} \equiv K/m^2$ was introduced. Note that the result Eq. (2.16) *does not depend* on m^2. If one counts all powers of m^2 also from $C_\varepsilon(m^2)$ and $\tilde{K}$ one obtains that the resulting power is equal to zero. This statement is in the full agreement with the expression for the initial integral (2.2). Looking at Eq. (2.2) one realizes that this integral does not depend on m^2. The formula (2.16) will be used for the calculation of all the N-point functions needed in this thesis.

Let us now return to the calculation of the one-point function Eq. (2.1). In this case one can directly use Eq. (2.16) with $N = 1$ and $\tilde{K} = m^2/m^2 = 1$. One immediately arrives at the result for the one-point function

$$A(m) = iC_\varepsilon(m^2)\frac{m^2}{\varepsilon(1-\varepsilon)}\,, \tag{2.17}$$

where m is the internal loop mass. The ε-expansion for this one-point function Eq. (2.17) can be written in the general form:

$$A(m) = iC_\varepsilon(m^2)\frac{m^2}{\varepsilon(1-\varepsilon)} = iC_\varepsilon(m^2)m^2\left\{\frac{1}{\varepsilon}+1+\varepsilon+\varepsilon^2+\mathcal{O}(\varepsilon^3)\right\}. \tag{2.18}$$

The one-loop two-point functions are defined by [9, 10]

$$B(q_1, m_1, m_2) = \mu^{2\varepsilon}\int\frac{d^nq}{(2\pi)^n}\frac{1}{(q^2-m_1^2)[(q+q_1)^2-m_2^2]}, \tag{2.19}$$

where the m_i $(i = 1, 2)$ can be either m or 0. In the denominators of the relevant functions I always imply the "causal" $+i\delta$ prescription to deal with the singularities in Minkowski space. One applies *Feynman parametrization* (see e.g. [47]):

$$\begin{aligned}\frac{1}{D_1D_2\times\ldots\times D_N} = (N-1)!\int_0^1\frac{\{dx\}_N}{[x_1D_1+\ldots+x_ND_N]^N}, \\ \{dx\}_N \equiv dx_1dx_2\ldots dx_N\delta(1-x_1-x_2-\ldots-x_N).\end{aligned} \tag{2.20}$$

After that the two-point function will be written as

$$B(q_1, m_1, m_2) = \mu^{2\varepsilon} \int \frac{d^n q}{(2\pi)^n} \int_0^1 \int_0^1 \frac{dx_1 dx_2 \delta(1 - x_1 - x_2)}{\left(x_1(q^2 - m_1^2) + x_2[(q + q_1)^2 - m_2^2]\right)^2} =$$
$$\mu^{2\varepsilon} \int \frac{d^n q}{(2\pi)^n} \int_0^1 \frac{dx}{\left(a(q^2 - m_1^2) + b\,[(q + q_1)^2 - m_2^2]\right)^2}\,, \qquad (2.21)$$

where the set of parameters $\{a, b\}$ corresponds to an arbitrary choice from the permutations of the set $\{x, 1-x\}$. The fact that the choice is arbitrary is a consequence of the symmetry of the l.h.s. of Eq. (2.20) under the permutations of D_i. Shifting the integration variable $q \rightarrow q - bq_1$ one obtains

$$a(q^2 - m_1^2) + b\left[(q + q_1)^2 - m_2^2\right] \stackrel{q \rightarrow q - bq_1}{\longrightarrow} q^2 - \left(-abq_1^2 + am_1 + bm_2\right) \qquad (2.22)$$

for the denominator of Eq. (2.21), where one has used the fact that $a + b = 1$. Using Eq. (2.22) one obtains

$$B(q_1, m_1, m_2) = \mu^{2\varepsilon} \int \frac{d^n q}{(2\pi)^n} \int_0^1 \frac{dx}{\left(q^2 - K_B\right)^2} = \int_0^1 dx\, \mu^{2\varepsilon} \int \frac{d^n q}{(2\pi)^n} \frac{1}{\left(q^2 - K_B\right)^2} \qquad (2.23)$$

with

$$K_B \equiv -abq_1^2 + am_1 + bm_2 - i\delta\,. \qquad (2.24)$$

Finally applying formula (2.16), one is left with a one-fold parametric integral for the two-point functions:

$$B(q_1, m_1, m_2) = \frac{iC_\varepsilon(m^2)}{\varepsilon} \int_0^1 dx\ \tilde{K}_B^{-\varepsilon}, \qquad (2.25)$$

with the kernel $\tilde{K}_B$ given by

$$m^2 \tilde{K}_B \quad = \quad -abq_1^2 + am_1^2 + bm_2^2 - i\delta\,. \qquad (2.26)$$

For each particular two-point function one should make a judicious choice of the set $\{a, b\}$ from the permutations of the set $\{x, 1 - x\}$ in order to get the most convenient kernel for the integration.

In what follows, I will always present the results for the scalar functions separately for the real and imaginary contributions. I introduce the Mandelstam-type variables

$$s \equiv (p_1 + p_2)^2, \qquad t \equiv T - m^2 \equiv (p_1 - p_3)^2 - m^2, \qquad u \equiv U - m^2 \equiv (p_2 - p_3)^2 - m^2, \quad (2.27)$$

with the kinematical condition on external momenta being $p_1 + p_2 = p_3 + p_4$ (i.e. $s + t + u = 0$) and the on-shell conditions are $p_1^2 = p_2^2 = 0,\ p_3^2 = p_4^2 = m^2$ (see also Chap. 5 for the physical

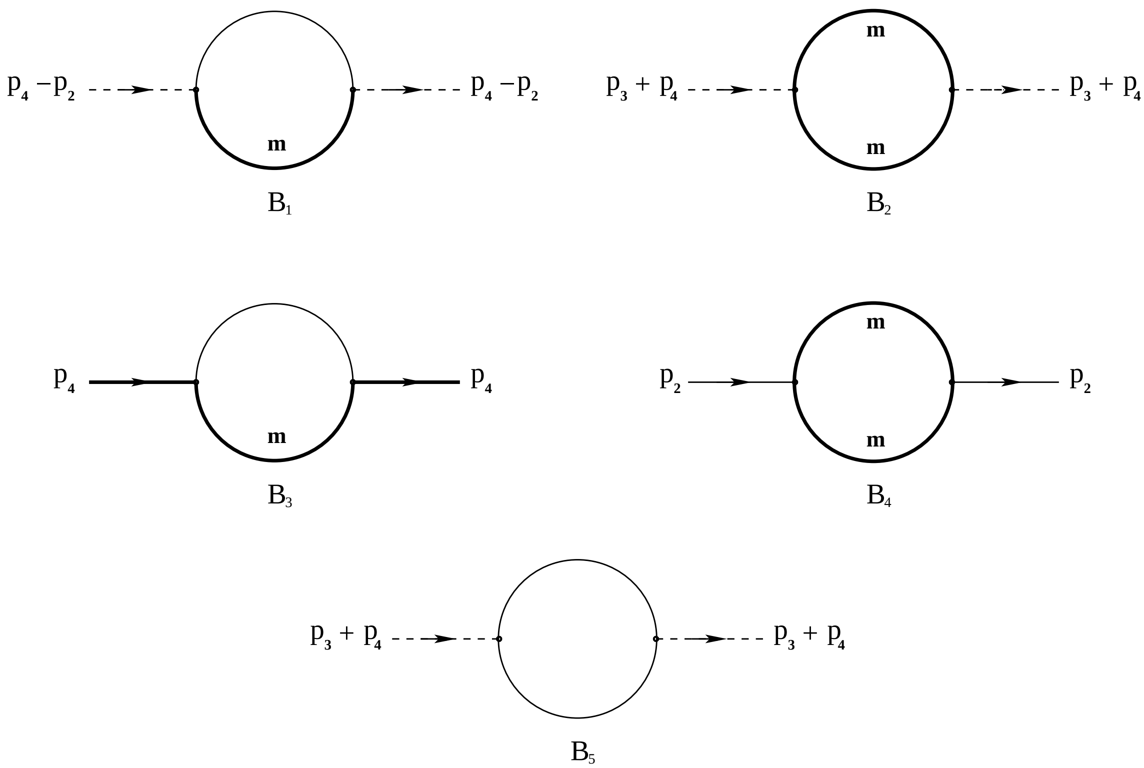

Figure 2.2: Two-point functions. Thick and thin internal lines correspond to massive and massless propagators, respectively. Thick legs represent massive momenta *on-shell*. Thin legs represent massless momenta *on-shell*. Dashed external lines correspond to *off-shell* momenta.

restrictions of the s and t variables). Momenta p_1 and p_2 correspond to the two incoming massless particles. Momenta p_3 and p_4 correspond to the two outgoing particles (see for example Fig. 2.4). Note that the variables t and u defined in (2.27) are *not* the usual Mandelstam variables.

There are altogether five different two-point scalar functions B_i ($i = 1, 2, ..., 5$) (see Fig. 2.2) needed for hadronic heavy flavor production [9, 10]. Again I choose to extract a common factor $iC_\varepsilon(m^2)$, where $C_\varepsilon(m^2)$ is defined in (2.15). The coefficients of the ε-expansion are denoted by $B_i^{(j)}$, i.e. one writes

$$B_i = i\,C_\varepsilon(m^2)\left\{\frac{1}{\varepsilon}B_i^{(-1)} + B_i^{(0)} + \varepsilon B_i^{(1)} + \varepsilon^2 B_i^{(2)} + \mathcal{O}(\varepsilon^3)\right\}. \tag{2.28}$$

The ε-expansion of the two-point functions starts at ε^{-1}. It turns out that $B_i^{(-1)} = 1$ for all i. The first two-point function

$$B_1 \equiv B(p_4 - p_2, 0, m) \tag{2.29}$$

is real for the kinematics of the reaction which can be seen by drawing the appropriate Feynman diagram for B_1 and applying the Landau-Cutkosky rules. The same statement holds true for the two-point functions B_3 and B_4 to be discussed later on. One has

$$\begin{aligned}
\mathrm{Re}\, B_1^{(-1)} &= 1, && (2.30)\\
\mathrm{Re}\, B_1^{(0)} &= 2 - \frac{t}{T}\ln\frac{-t}{m^2},\\
\mathrm{Re}\, B_1^{(1)} &= 2\mathrm{Re}\, B_1^{(0)} + \frac{t}{T}\ln^2\frac{-t}{m^2} + \frac{t}{T}\mathrm{Li}_2(\frac{T}{m^2}),\\
\mathrm{Re}\, B_1^{(2)} &= 2\mathrm{Re}\, B_1^{(1)} - \frac{t}{T}\left[\frac{1}{3}\ln^3\frac{-t}{m^2} + 2\mathrm{Li}_3(\frac{T}{t}) + \mathrm{Li}_3(\frac{T}{m^2})\right];\\
\mathrm{Im}\, B_1^{(j)} &= 0. && (2.31)
\end{aligned}$$

The second scalar two-point function

$$B_2 \equiv B(p_3 + p_4, m, m) \tag{2.32}$$

has both real and imaginary parts. Defining $\beta = (1 - 4m^2/s)^{1/2}$ and $x = (1-\beta)/(1+\beta)$ we obtain

$$\begin{aligned}
\mathrm{Re}\, B_2^{(-1)} &= 1, && (2.33)\\
\mathrm{Re}\, B_2^{(0)} &= 2 + \beta\ln x,\\
\mathrm{Re}\, B_2^{(1)} &= \beta\left[\frac{4}{\beta} - 2\ln(\frac{s\beta^2}{m^2}) + \frac{1}{2}\ln^2(\frac{s\beta^2}{m^2}) + 4\ln(1-x) - 2\ln^2(1-x) - 4\zeta(2) - 2\mathrm{Li}_2(x)\right],\\
\mathrm{Re}\, B_2^{(2)} &= 2\mathrm{Re}\, B_2^{(1)} + \beta\left[4\zeta(2)\ln(\frac{s\beta^2}{m^2}) - \frac{1}{6}\ln^3(\frac{s\beta^2}{m^2}) + 4\zeta(2)\ln(1-x) - 2\ln x\ln^2(1-x)\right.\\
&\qquad\left. + \frac{4}{3}\ln^3(1-x) + 2\zeta(3) - 4\mathrm{Li}_3(1-x) - 2\mathrm{Li}_3(x)\right];\\
\mathrm{Im}\, B_2^{(0)} &= \pi\beta, \qquad \mathrm{Im}\, B_2^{(1)} = \pi\beta\left[2 - \ln(\frac{s\beta^2}{m^2})\right], && (2.34)\\
\mathrm{Im}\, B_2^{(2)} &= 2\mathrm{Im}\, B_2^{(1)} + \pi\beta\left[\frac{1}{2}\ln^2(\frac{s\beta^2}{m^2}) - 2\zeta(2)\right].
\end{aligned}$$

The remaining three two-point functions have a simple structure:

$$\begin{aligned}
B_3 \equiv B(p_4, 0, m) &= iC_\varepsilon(m^2)\frac{1}{\varepsilon(1-2\varepsilon)}\\
&= iC_\varepsilon(m^2)\left\{\frac{1}{\varepsilon} + 2 + 4\varepsilon + 8\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right\}; && (2.35)\\
B_4 \equiv B(p_2, m, m) &= iC_\varepsilon(m^2)\frac{1}{\varepsilon}; && (2.36)\\
B_5 \equiv B(p_3 + p_4, 0, 0) &= iC_\varepsilon(m^2)\frac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\left(-\frac{s+i\delta}{m^2}\right)^{-\varepsilon}\frac{1}{\varepsilon}. && (2.37)
\end{aligned}$$

The results for B_3 and B_4 in (2.35), (2.36) are not separately listed in the standard format $B_i^{(j)}$ which can of course be read off from the relevant expressions (2.35), (2.36). The two-point function B_5 (2.37) has both real and imaginary parts:

$$\begin{aligned}
\mathrm{Re}\, B_5^{(-1)} &= 1, \\
\mathrm{Re}\, B_5^{(0)} &= 2 - \ln\frac{s}{m^2}, \\
\mathrm{Re}\, B_5^{(1)} &= 2\mathrm{Re}\, B_5^{(0)} - 4\zeta(2) + \frac{1}{2}\ln^2\frac{s}{m^2}, \\
\mathrm{Re}\, B_5^{(2)} &= 2\mathrm{Re}\, B_5^{(1)} + 4\zeta(2)\ln\frac{s}{m^2} - \frac{1}{6}\ln^3\frac{s}{m^2} - 2\zeta(3);
\end{aligned} \tag{2.38}$$

$$\begin{aligned}
\mathrm{Im}\, B_5^{(0)} &= \pi, \qquad \mathrm{Im}\, B_5^{(1)} = \pi\left[2 - \ln\frac{s}{m^2}\right], \\
\mathrm{Im}\, B_5^{(2)} &= 2\mathrm{Im}\, B_5^{(1)} + \pi\left[\frac{1}{2}\ln^2\frac{s}{m^2} - 2\zeta(2)\right].
\end{aligned} \tag{2.39}$$

I have done various checks on the above results. First of all, they were double-checked, i.e. the results were obtained by two independent calculations. Secondly, they were checked numerically by verifying that the original integrals (after Feynman parametrization and integrating out the loop momentum, and for B_1 and B_2 also expanding in ε) are equal numerically to the final integrals. I have also verified these results by extracting the relevant expressions from general formulae given in [32, 33, 35]. In particular, the coefficients (2.30) may be obtained from Eq. (10) of [32] and then using Eq. (2.14) of [35]. The results (2.33), (2.34) can be obtained from Eqs. (2.10) and (2.14) of [35]. Finally, the expressions (2.35) – (2.37) can also be obtained from Eqs. (10), (17) and (8), respectively, of [32].

There is one more special case of the two-point integral which is needed for the calculation of the self-energy insertion into external massive fermion lines. This integral is used for the definition of the fermion mass and wave function renormalization constants in the on-shell renormalization scheme. This specific two-point function is given in App. A of this thesis.

2.2 Three-point functions

The one-loop three-point functions are defined by [9, 10]

$$C(q_1, q_2, m_1, m_2, m_3) = \mu^{2\varepsilon}\int\frac{d^n q}{(2\pi)^n}\frac{1}{(q^2 - m_1^2)[(q+q_1)^2 - m_2^2][(q+q_1+q_2)^2 - m_3^2]}. \tag{2.40}$$

The three masses m_1, m_2 and m_3 come in various combinations of zero and nonzero masses where all nonzero masses are equal to m as before. After applying Feynman parametrization Eq. (2.20) one obtains

$$C(q_1, q_2, m_1, m_2, m_3) =$$
$$2\mu^{2\varepsilon}\int\frac{d^n q}{(2\pi)^n}\int_0^1\int_0^1\int_0^1\frac{dx_1 dx_2 dx_3 \delta(1 - x_1 - x_2 - x_3)}{(x_1 D_1 + x_2 D_2 + x_3 D_3)^3} =$$

$$2\mu^{2\varepsilon}\int\frac{d^nq}{(2\pi)^n}\int_0^1 dx_1\int_0^{1-x_1}dx_2\frac{1}{(x_1D_1+x_2D_2+(1-x_1-x_2)D_3)^3}\overset{x_2\to(1-x_1)x_2}{=}$$
$$2\mu^{2\varepsilon}\int\frac{d^nq}{(2\pi)^n}\int_0^1 dx_1\int_0^1 dx_2(1-x_1)\times$$
$$\times\frac{1}{(x_1D_1+x_2(1-x_1)D_2+(1-x_1-x_2(1-x_1))D_3)^3}\overset{x_1\to(1-x_1)}{=}$$
$$2\mu^{2\varepsilon}\int\frac{d^nq}{(2\pi)^n}\int_0^1\frac{dx_1dx_2x_1}{((1-x_1)D_1+x_2x_1D_2+x_1(1-x_2)D_3)^3}\,, \tag{2.41}$$

where D_i corresponds to the arbitrary choice between the propagators in the denominator of Eq. (2.40). Substituting D_i one obtains

$$C(q_1,q_2,m_1,m_2,m_3)=$$
$$2\mu^{2\varepsilon}\int\frac{d^nq}{(2\pi)^n}\int_0^1\frac{dx_1dx_2x_1}{(a(q^2-m_1^2)+b\,[(q+q_1)^2-m_2^2]+c[(q+q_1+q_2)^2-m_3^2])^3} \tag{2.42}$$

for the three-point function, where the set of parameters $\{a,b,c\}$ above corresponds to an arbitrary choice from the permutations of the set $\{x_1x_2,\ x_1(1-x_2),\ 1-x_1\}$. One shifts the integration variable $q\to q-bq_1-c(q_1+q_2)$. Then one makes use of the fact that $a+b+c=1$. Finally applying the formula (2.16) one is left with a two-fold parametric integral for the three-point functions:

$$C(q_1,q_2,m_1,m_2,m_3)=-iC_\varepsilon(m^2)(m^2)^{-1}\int_0^1 dx_1dx_2\ x_1\tilde{K}_C^{-1-\varepsilon}, \tag{2.43}$$

with the kernel $\tilde{K}_C$ given by

$$\begin{aligned} m^2\tilde{K}_C &= -abq_1^2-ac(q_1+q_2)^2-bcq_2^2 \\ &\quad +am_1^2+bm_2^2+cm_3^2-i\delta. \end{aligned} \tag{2.44}$$

For each particular three-point function one should make a judicious choice of the set $\{a,b,c\}$ from the set $\{x_1x_2,\ x_1(1-x_2),\ 1-x_1\}$ in order to get the most convenient kernel for the subsequent integrations.

There are six different types of three-point functions C_i ($i=1,2,...,6$) (see Fig. 2.3) needed for the calculation purposes [9, 10]. They have both real and imaginary parts except for C_2 and C_3 which are real. This can also be seen from the Feynman diagrams representing C_2 and C_3 and applying the Landau-Cutkosky rules. Their ε-expansion is again written in the following universal format

$$C_i=i\,C_\varepsilon(m^2)\left\{\frac{1}{\varepsilon^2}C_i^{(-2)}+\frac{1}{\varepsilon}C_i^{(-1)}+C_i^{(0)}+\varepsilon C_i^{(1)}+\varepsilon^2C_i^{(2)}+\mathcal{O}(\varepsilon^3)\right\}, \tag{2.45}$$

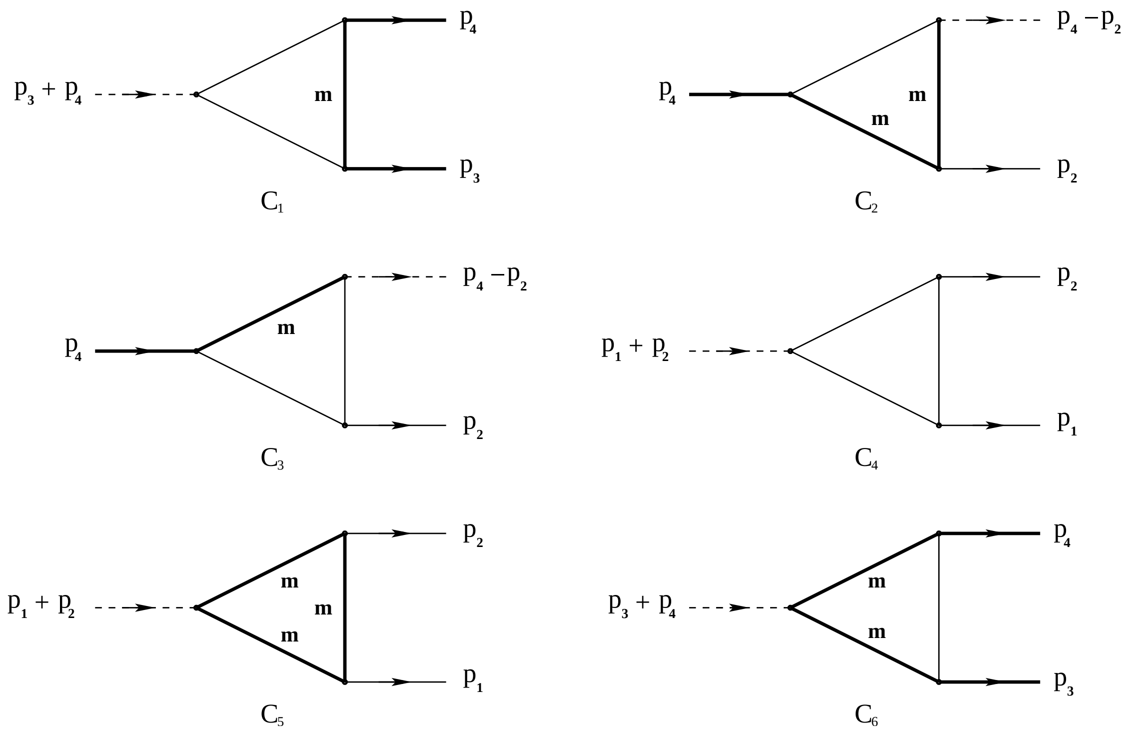

Figure 2.3: Three-point functions. Thick and thin internal lines correspond to massive and massless propagators, respectively. Thick legs represent massive momenta *on-shell*. Thin legs represent massless momenta *on-shell*. Dashed external lines correspond to *off-shell* momenta.

where the ε-expansion now starts at $\varepsilon^{(-2)}$. Note that the coefficients $C_i^{(-2)}$ are purely real.

It turns out that the $\mathcal{O}(\varepsilon^2)$ results for the three-point functions can no longer be presented in terms of classical polylogarithms but require a new class of functions given by the one-fold integral representations defined below. To write down the results in a short and convenient form, I introduce the following functions:

$$L_{\sigma_1\sigma_2\sigma_3}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1+\sigma_1 y)\ln(\alpha_2+\sigma_2 y)\ln(\alpha_3+\sigma_3 y)}{\alpha_4+y}, \tag{2.46}$$

and

$$L_{\sigma_1}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1+\sigma_1 y)\mathrm{Li}_2(\alpha_2+\alpha_3 y)}{\alpha_4+y}. \tag{2.47}$$

Here the σ_i ($i=1,2,3$) take values ± 1 and the α_j's are either integers $\{1,0,-1\}$ or else kinematical variables. The above L-functions arise naturally in the calculational framework[1]. They can all be expressed in terms of so-called multiple polylogarithms of maximum weight four [43, 48] (see Sec. 4.3 for details). However, I choose to write the results

[1]According to A. Davydychev the functions analogous to the triple-index functions $L_{\sigma_1\sigma_2\sigma_3}$ also arise in the approach of [35] when one analytically continues their Eq. (3.2) for the $\mathcal{O}(\varepsilon^2)$ terms.

in terms of the above single- and triple-index L-functions for several reasons. The results look simpler, e.g. they can be expressed as one-fold integrals of products of logarithms and dilogarithms, and are shorter. I have also found that the L-functions are much easier to evaluate numerically than the corresponding multiple polylogarithms (see[49] for relevant details).

There exist simple algebraic relations between these L-functions based on either symmetry relations regarding permutations of indices and change of integration variables or on relations based on integration-by-parts techniques. I describe them in Secs. 4.1 and 4.2. In particular this means that the results on the three- and four-point functions can all be written in terms of the L_{-++} and L_{+++} variants of the triple–index $L_{\sigma_1\sigma_2\sigma_3}$ functions in Eq. (2.46), and of the L_+ variant of the single–index L_{σ_1} function of Eq. (2.47).

I start with the three-point function C_1 defined by

$$C_1 \equiv C(p_4, p_3, 0, m, 0).$$

One obtains

$$\begin{aligned}
\mathrm{Re}\, C_1^{(-2)} &= \mathrm{Re}\, C_1^{(-1)} = 0, \qquad (2.48)\\
\mathrm{Re}\, C_1^{(0)} &= \frac{1}{2s\beta}[\ln^2 x + 4\mathrm{Li}_2(-x) + 2\zeta(2)],\\
\mathrm{Re}\, C_1^{(1)} &= -\frac{1}{s\beta}\Big[\frac{1}{6}\ln^3\frac{s}{m^2} + 2\ln\frac{s}{m^2}\ln(1-x)\ln x + \ln(1-x)\ln^2 x - 4\zeta(2)\ln\frac{s}{m^2}\\
&+ 2\ln\frac{s}{m^2}\mathrm{Li}_2(x) + 5\zeta(3) - 4\mathrm{Li}_3(1-x) + 2\mathrm{Li}_3(x) + 2\mathrm{Li}_3(1-x^2)\\
&- 8\mathrm{Li}_3(\frac{1}{1+x})\Big],\\
\mathrm{Re}\, C_1^{(2)} &= \frac{1}{s\beta}\Big[\frac{7}{48}\ln^4\frac{s}{m^2} - \frac{11}{24}\ln^3\frac{s}{m^2}\ \ln x - \frac{1}{4}\ln^2\frac{s}{m^2}\Big(\ln^2(1-x) + 6\ln(1-x)\ln x\\
&+ 5\ln^2 x\Big) + \ln\frac{s}{m^2}\Big(\frac{1}{3}\ln^3(1-x) - \frac{7}{2}\ln^2(1-x)\ln x + \frac{11}{24}\ln^3 x\Big)\\
&- \frac{1}{3}\ln^3(1-x)\ln x - \frac{1}{4}\ln^2(1-x)\ln^2 x + \frac{5}{2}\ln(1-x)\ln^3 x + \frac{55}{48}\ln^4 x\\
&- \frac{7}{8}\Big(3\ln^2\frac{s}{m^2} + 10\ln\frac{s}{m^2}\ln x + 16\ln(1-x)\ln x + 3\ln^2 x\Big)\zeta(2)\\
&+ \frac{1}{2}\Big(\ln\frac{s}{m^2} - 11\ln x\Big)\zeta(3) + \frac{5}{2}\zeta(4) - 2\mathrm{Li}_2^2(-x) + \mathrm{Li}_2(x)\Big(-\ln^2\frac{s}{m^2}\\
&+ 2\ln(1-x)\ln x + 7\ln^2 x - 2\ln\frac{s}{m^2}(\ln(1-x) - \ln x) - 10\zeta(2)\Big)\\
&+ \mathrm{Li}_2(-x)\Big(-\frac{13}{4}\ln^2\frac{s}{m^2} - 2\ln\frac{s}{m^2}(\ln(1-x) - \ln x) + 2\ln(1-x)\ln x\\
&+ \frac{31}{4}\ln^2 x - 6\zeta(2)\Big) + 4\mathrm{Li}_3(1-x)\Big(\ln\frac{s}{m^2} + 2\ln(1-x) - \ln x\Big)\\
&- \mathrm{Li}_3(-x)\Big(\frac{21}{2}\ln\frac{s}{m^2} + 20\ln(1-x) + \frac{29}{2}\ln x\Big) - 4\mathrm{Li}_3(x)\Big(\ln\frac{s}{m^2} + 2\ln(1-x)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{5}{2}\ln(x)\Big) - \mathrm{Li}_3\left(\frac{1}{1+x}\right)\left(3\ln\frac{s}{m^2} + 4\ln(1-x) - 2\ln x\right) - 2\mathrm{Li}_3\left(1-x^2\right) \times \\
&\quad\; \left(\ln\frac{s}{m^2} + 2\ln(1-x) - \ln x\right) - 4\,\mathrm{Li}_4(1-x) - 4\,\mathrm{Li}_4\left(\frac{-x}{1-x}\right) + 12\,\mathrm{Li}_4(-x) \\
&+ \quad 28\mathrm{Li}_4(x) + 17\mathrm{Li}_4\left(\frac{1}{1+x}\right) - 17\mathrm{Li}_4\left(\frac{x}{1+x}\right) + 4\mathrm{Li}_4\left(1-x^2\right) + 4\mathrm{Li}_4\left(\frac{-x^2}{1-x^2}\right) \\
&- \quad L_{-++}\left(1, x^{-1}, x^{-1}, 0\right) + L_{-++}\left(1, x^{-1}, x^{-1}, x\right) + 2\,L_{-++}\left(1, x, x^{-1}, 0\right) \\
&- \quad 2\,L_{-++}\left(1, x, x^{-1}, x^{-1}\right) - 2\,L_{-++}\left(1, x, x^{-1}, x\right) + L_{-++}(1, x, x, 0) \\
&- \quad L_{-++}\left(1, x, x, x^{-1}\right) + 2\,L_{-++}\left(1+x^{-1}, 0, 0, -1\right) - 2\,L_{-++}(1+x, 0, 0, -1) \\
&- \quad 2\,L_{-++}\left(1+x^{-1}, 0, 0, -1-x\right) + 2\,L_{-++}\left(1+x, 0, 0, -1-x^{-1}\right) \\
&+ \quad 2\,L_{+}\left(0, 0, -x^{-1}, x\right) - 2\,L_{+}\left(0, 0, -x, x^{-1}\right) - 4\,L_{+}(0, 0, -x, x) \\
&- \quad 4\,L_{+}\left(0, -x^{-1}, x^{-1}, -1\right) + 4\,L_{+}\left(0, -x^{-1}, x^{-1}, -1-x\right) + 4\,L_{+}(0, -x, x, -1) \\
&+ \quad 4\,L_{+}\left(0, -x^{-1}, x^{-1}, -1-x^{-1}\right) - 4\,L_{+}(0, -x, x, -1-x) \\
&- \quad 4\,L_{+}\left(0, -x, x, -1-x^{-1}\right) - 4\,L_{+}\left(0, \frac{-x}{1-x}, \frac{x}{1-x^2}, -1\right) \\
&+ \quad 4\,L_{+}\left(0, \frac{-x}{1-x}, \frac{x}{1-x^2}, -1-x\right) + 4\,L_{+}\left(0, \frac{-x}{1-x}, \frac{x}{1-x^2}, -1-x^{-1}\right) \\
&+ \quad 2\,L_{+}\left(x^{-1}, 0, -x^{-1}, x\right) + 2\,L_{+}\left(x^{-1}, 0, -x, x\right) - 3\,L_{+++}\left(0, 0, x^{-1}, x\right) \\
&- \quad L_{+++}\left(0, 0, x, x^{-1}\right) + \frac{3}{2}\,L_{+++}\left(0, x^{-1}, x^{-1}, x\right) + 3\,L_{+++}(0, x^{-1}, x, x) \\
&+ \quad L_{+++}\left(0, x, x^{-1}, x^{-1}\right) + \frac{1}{2}\,L_{+++}\left(0, x, x, x^{-1}\right) - 2\,L_{+++}\left(x^{-1}, x, x^{-1}, x\right)\Big];
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Im}\, C_1^{(-1)} &= 0, \qquad \mathrm{Im}\, C_1^{(0)} = \frac{\pi}{s\beta}\ln x, \qquad (2.49) \\
\mathrm{Im}\, C_1^{(1)} &= -\frac{\pi}{2s\beta}\left[2\ln\frac{s}{m^2}\ln x - 4\ln(1-x)\ln x + \ln^2 x + 4\zeta(2) - 4\mathrm{Li}_2(x)\right], \\
\mathrm{Im}\, C_1^{(2)} &= \frac{\pi}{6s\beta}\left[3\ln^2\frac{s}{m^2}\ln x - 12\ln\frac{s}{m^2}\ln(1-x)\ln x + 3\ln\frac{s}{m^2}\ln^2 x - 6\ln(1-x)\ln^2 x\right. \\
&+ \left.\ln^3 x - 12\ln\frac{s}{m^2}\mathrm{Li}_2(x) - 24\mathrm{Li}_3(1-x) - 12\mathrm{Li}_3(x) + 12\zeta(2)\ln\frac{s}{m^2} + 12\zeta(3)\right].
\end{aligned}
$$

I have not been able to derive the corresponding result from the known general hypergeometric function that represents the above integral in [32, 33]. On the other hand, for a general three-point function, an expression for the order ε-terms was obtained in [50] in terms of simple polylogarithms up to Li_3. However, I believe that the result Eq. (5.21) in [50] is not applicable to the case of the function C_1 as one faces singularities resulting from vanishing denominators in the arguments of the relevant logarithms and polylogarithms. I have checked the final result numerically against the original two-fold and one-fold Feynman parameter integrals (after ε-expanding the corresponding integrand). This was done term by term for coefficients at the corresponding orders in ε. Although the result for the ε^2-coefficient looks lengthy, the final analytic results (2.48), (2.49) for the three-point function C_1 integrate numerically very fast (in fraction of a second on a desktop computer for a chosen numerical point) and without any problems. In comparison, the numerical integration of the one-fold integral by the computer algebra system **Mathematica** [42] took eight times longer, and that of the two-fold integral even 200 times longer to evaluate. In addition, because of various branch cuts, the one- and two-fold integrals would only allow integrations in the complex plane of kinematical variables, while for the physical region they have severe problems.

The integral C_2 is real and finite:

$$C_2 \equiv C(p_4, -p_2, 0, m, m);$$

$$\begin{aligned}
\mathrm{Re}\, C_2^{(-2)} &= \mathrm{Re}\, C_2^{(-1)} = 0, \qquad (2.50)\\
\mathrm{Re}\, C_2^{(0)} &= \frac{1}{t}\left[\zeta(2) - \mathrm{Li}_2(\frac{T}{m^2})\right],\\
\mathrm{Re}\, C_2^{(1)} &= \frac{1}{3t}\left[\ln^3\frac{-t}{m^2} + 6\ln\frac{-t}{m^2}\mathrm{Li}_2(\frac{T}{m^2}) + 9\zeta(3) - 6\mathrm{Li}_3(\frac{T}{t}) - 9\mathrm{Li}_3(\frac{T}{m^2})\right],\\
\mathrm{Re}\, C_2^{(2)} &= \frac{1}{24t}\left[-5\ln^4\frac{-t}{m^2} + 12\zeta(2)\ln^2\frac{-t}{m^2} + 12\zeta(2)\ln^2\frac{-T}{m^2} - 12\ln^2\frac{-t}{m^2}\mathrm{Li}_2(\frac{T}{m^2})\right.\\
&\quad -24\ln\frac{-t}{m^2}\ln\frac{-T}{m^2}\mathrm{Li}_2(\frac{T}{m^2}) - 24\zeta(3)\ln\frac{-t}{m^2} + 24\zeta(3)\ln\frac{-T}{m^2}\\
&\quad +24\ln\frac{-t}{m^2}\mathrm{Li}_3(\frac{T}{t}) + 24\ln\frac{-T}{m^2}\mathrm{Li}_3(\frac{T}{t}) + 24\ln\frac{-t}{m^2}\mathrm{Li}_3(\frac{T}{m^2})\\
&\quad +48\ln\frac{-T}{m^2}\mathrm{Li}_3(\frac{T}{m^2}) + 192\zeta(4) - 24\mathrm{Li}_4(\frac{m^2}{-t})\\
&\quad \left. +12L_{-++}(1, -\frac{m^2}{T}, -\frac{m^2}{T}, 0) - 12L_{-++}(\frac{t}{T}, 0, 0, -1)\right];\\
\mathrm{Im}\, C_2^{(j)} &= 0. \qquad (2.51)
\end{aligned}$$

This result was checked numerically against the original double parametric representation (obtained after doing Feynman parametrization) of this integral expanded in powers of ε. I could not obtain similar expressions from known general results for this integral, as the ε-expansion of the relevant hypergeometric function is problematic. In addition, it turns out that the general result for the order ε-terms for the massive three-point function of [50] does not allow for a straightforward extraction of the corresponding expression for this particular case. More exactly, the equation $Q_3(y) = 0$ originating from the table in [50] (on page 608), does not have solutions for the relevant kinematics. In this sense, the expressions for the coefficients of the ε- and ε^2-terms for C_2 represent a new result.

The integration of the function C_3 defined by

$$C_3 \equiv C(-p_2, p_4, 0, 0, m)$$

requires the construction of a subtraction term since an ε-expansion of the relevant integrand does not straightforwardly lead to the desired ε-expansion of the integral. This is best illustrated in a simple example which nevertheless captures the essential idea of the subtraction method. Consider the integral

$$\int_0^1 dx\, x^{-1+\varepsilon} f(x,\varepsilon) \tag{2.52}$$

where $f(x,\varepsilon)$ is an integrable function in the interval $[0,1]$ and has derivatives in ε. For the sake of the argument take $f(x,\varepsilon)$ to have a Laurent series expansion starting at the zeroth order in ε, i.e. $f(x,\varepsilon) = f^{(0)}(x) + \varepsilon f^{(1)}(x) + \ldots$. It is clear that expanding the integrand

$$x^{-1+\varepsilon} f(x,\varepsilon) = \frac{1}{x} f^{(0)}(x) + \varepsilon \left(\frac{\ln x}{x} f^{(0)}(x) + \frac{1}{x} f^{(1)}(x) \right) + \ldots$$

does not in general render the integral integrable. However, if one writes

$$\int_0^1 dx x^{-1+\varepsilon} f(x,\varepsilon) = \int_0^1 dx \left(x^{-1+\varepsilon} \right)_+ f(x,\varepsilon) + f(0,\varepsilon) \int_0^1 dx x^{-1+\varepsilon} \tag{2.53}$$

the terms on the r.h.s. of (2.53) are now integrable. In (2.53) I have introduced a "plus" prescription

$$\int_0^1 dx \left(x^{-1+\varepsilon} \right)_+ f(x,\varepsilon) = \int_0^1 dx x^{-1+\varepsilon} (f(x,\varepsilon) - f(0,\varepsilon))$$

not unlike the "plus" prescription usually introduced when discussing parton splitting functions. The ε-expansion of the integral (2.52) can now be obtained since the first integrand on the r.h.s. of (2.53) can be expanded in ε and then be integrated term by term whereas the second integral can be computed in closed form. The task is then to find the appropriate subtraction terms for the integrals encountered in the calculation. This is required for the three-point function C_3 and the three four-point functions to be discussed in the next section.

As exemplified above I derive the subtraction terms by substituting the value of the integration variable (usually the lower or upper limit of integration) at which the given integrand diverges into the nonsingular part of the singular integrand. Adding and subtracting the subtraction term does all the job: e.g. the subtraction term contains all the poles in a given Feynman parameter but can be easily integrated due to its simpler analytic structure, while the rest of the integrand is now finite with respect to the same parameter and can therefore be integrated as well. When dealing with such a finite but complicated integration one often makes use of the integration-by-parts method to evaluate and simplify the expressions.

Applying the subtraction method to the evaluation of the three-point function C_3 one obtains:

$$\mathrm{Re}\, C_3^{(-2)} = \frac{1}{2t}, \quad \mathrm{Re}\, C_3^{(-1)} = -\frac{1}{t} \ln \frac{-t}{m^2}, \quad \mathrm{Re}\, C_3^{(0)} = \frac{1}{t} \left[\ln^2 \frac{-t}{m^2} + \mathrm{Li}_2(\frac{T}{m^2}) \right],$$

$$\begin{aligned}
\mathrm{Re}\, C_3^{(1)} &= -\frac{1}{t}\left[\frac{1}{3}\ln^3\frac{-t}{m^2} + 2\mathrm{Li}_3(\frac{T}{t}) + \mathrm{Li}_3(\frac{T}{m^2})\right], \\
\mathrm{Re}\, C_3^{(2)} &= \frac{1}{3t}\left[\ln^4\frac{-t}{m^2} - \ln^3\frac{-t}{m^2}\ln\frac{-T}{m^2} - 3\zeta(2)\ln^2\frac{-t}{m^2} + 6\zeta(3)\ln\frac{-t}{m^2} - 6\zeta(4)\right. \\
&\quad \left. -3\mathrm{Li}_4(\frac{T}{m^2}) + 6\mathrm{Li}_4(\frac{m^2}{-t}) - 6\mathrm{Li}_4(\frac{T}{t})]\right];
\end{aligned} \tag{2.54}$$

$$\mathrm{Im}\, C_3^{(j)} = 0. \tag{2.55}$$

Note that one can obtain corresponding expressions in terms of generalized Nielsen polylogarithms from Eq. (27) of [32, 33]. The corresponding hypergeometric function of three variables Φ_1 can be reduced to a hypergeometric function ${}_2F_1$ of one variable and one can then use Eq. (2.14) of [35] to get the relevant ε-expansion. I have verified agreement with [32, 33] analytically up to $\mathcal{O}(\varepsilon)$. The agreement for the ε^2-terms was verified numerically.

The three-point function C_4 has a closed form solution:

$$C_4 \equiv C(-p_2, -p_1, 0, 0, 0) = \frac{iC_\varepsilon(m^2)}{s}\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(-\frac{s+i\delta}{m^2}\right)^{-\varepsilon} \tag{2.56}$$

which is straightforward to obtain. For the ε-expansion of the real and imaginary parts of (2.56) one gets:

$$\begin{aligned}
\mathrm{Re}\, C_4^{(-2)} &= \frac{1}{s}, \quad \mathrm{Re}\, C_4^{(-1)} = -\frac{1}{s}\ln\frac{s}{m^2}, \quad \mathrm{Re}\, C_4^{(0)} = \frac{1}{2s}\left[\ln^2\frac{s}{m^2} - 8\zeta(2)\right], \\
\mathrm{Re}\, C_4^{(1)} &= \frac{1}{s}\left[4\zeta(2)\ln\frac{s}{m^2} - \frac{1}{6}\ln^3\frac{s}{m^2} - 2\zeta(3)\right], \\
\mathrm{Re}\, C_4^{(2)} &= \frac{1}{s}\left[\frac{1}{24}\ln^4\frac{s}{m^2} - 2\zeta(2)\ln^2\frac{s}{m^2} + 2\zeta(3)\ln\frac{s}{m^2} + 9\zeta(4)\right];
\end{aligned} \tag{2.57}$$

$$\begin{aligned}
\mathrm{Im}\, C_4^{(-1)} &= \frac{\pi}{s}, \quad \mathrm{Im}\, C_4^{(0)} = -\frac{\pi}{s}\ln\frac{s}{m^2}, \\
\mathrm{Im}\, C_4^{(1)} &= \frac{\pi}{2s}\left[\ln^2\frac{s}{m^2} - 4\zeta(2)\right], \\
\mathrm{Im}\, C_4^{(2)} &= \frac{\pi}{s}\left[2\zeta(2)\ln\frac{s}{m^2} - \frac{1}{6}\ln^3\frac{s}{m^2} - 2\zeta(3)\right].
\end{aligned} \tag{2.58}$$

For the fifth three-point integral C_5 defined by

$$C_5 \equiv C(-p_2, -p_1, m, m, m)$$

I first obtain a one-fold integral representation similar to Eq. (3.13) of [35]. As before, the main difficulty is the derivation of the coefficient for the ε^2-term. The corresponding coefficient has a complicated singularity structure as well as two branch points on its integration path. Therefore, in order to analytically separate the real and imaginary parts for the final result, I have divided the integration regions for the relevant terms into three parts. After analytical integration these terms are free of numerical instabilities and converge very fast. One obtains

$$
\begin{aligned}
\mathrm{Re}\, C_5^{(-2)} &= \mathrm{Re}\, C_5^{(-1)} = 0, \\
\mathrm{Re}\, C_5^{(0)} &= \frac{1}{2s}\left[\ln^2 x - 6\zeta(2)\right], \\
\mathrm{Re}\, C_5^{(1)} &= \frac{1}{2s}\left[\frac{1}{3}\ln^3 x - 8\zeta(2)\ln x + 4\ln x \mathrm{Li}_2(x) - 6\zeta(3) - 8\mathrm{Li}_3(x)\right], \\
\mathrm{Re}\, C_5^{(2)} &= \frac{1}{2s}\left[4\ln x \ln^3(1-x) - \frac{9}{2}\ln^2 x \ln^2(1-x) - \frac{1}{3}\ln^3 x \ln(1-x) + \frac{1}{12}\ln^4 x\right. \\
&\quad -\zeta(2)\left(3\ln^2 x - 10\ln x \ln(1-x) + 6\ln^2(1-x)\right) - \ln^2 x\, \mathrm{Li}_2(-x) \\
&\quad +2\left(\ln^2 x - 6\ln x\ln(1-x) + 3\ln^2(1-x) - 6\zeta(2)\right)\mathrm{Li}_2(x) + 10\zeta(3)\ln x \\
&\quad -6\zeta(3)\ln(1-x) + 2\left(\ln x + 3\ln(1-x)\right)\mathrm{Li}_3(x) + 2\ln x\, \mathrm{Li}_3(-x) \\
&\quad -12\left(\ln x - \ln(1-x)\right)\mathrm{Li}_3(1-x) + \frac{45}{2}\zeta(4) - L_{-++}\left(1,0,0,\frac{1}{-1+x}\right) \\
&\quad +L_{-++}\left(1,0,0,\frac{x}{1-x}\right) - L_{+++}\left(0,0,\frac{1-x}{x},-1\right) + L_{+++}\left(0,0,\frac{1-x}{x},\frac{1}{x}\right) \\
&\quad \left. -L_{+++}\left(0,\frac{1-x}{x},\frac{1-x}{x},-1\right) + L_{+++}\left(0,\frac{1-x}{x},\frac{1-x}{x},\frac{1}{x}\right)\right];
\end{aligned} \tag{2.59}
$$

$$
\begin{aligned}
\mathrm{Im}\, C_5^{(-1)} &= 0, \quad \mathrm{Im}\, C_5^{(0)} = \frac{\pi}{s}\ln x, \\
\mathrm{Im}\, C_5^{(1)} &= \frac{\pi}{2s}\left[\ln^2 x - 4\zeta(2) + 4\mathrm{Li}_2(x)\right], \\
\mathrm{Im}\, C_5^{(2)} &= \frac{\pi}{6s}\left[-12\ln x\ln^2(1-x) + \ln^3 x - 12(\ln x - 2\ln(1-x))(\zeta(2) - \mathrm{Li}_2(x))\right. \\
&\quad \left. +12\zeta(3) - 12\mathrm{Li}_3(x) - 24\mathrm{Li}_3(1-x)\right].
\end{aligned} \tag{2.60}
$$

Explicit result for this integral was given very recently in Eq. (4.4) of [36]. I have checked agreement with [36] analytically up to $\mathcal{O}(\varepsilon)$. The agreement for the ε^2-terms was verified numerically.

Finally, I write down real and imaginary parts for the last required three-point function C_6 defined by

$$C_6 \equiv C(p_3, p_4, m, 0, m).$$

One has

$$
\begin{aligned}
\mathrm{Re}\, C_6^{(-2)} &= 0, \quad \mathrm{Re}\, C_6^{(-1)} = \frac{1}{s\beta}\ln x, \\
\mathrm{Re}\, C_6^{(0)} &= \frac{1}{2s\beta}\left[-4\ln x\ln(1-x) + \ln^2 x - 8\zeta(2) - 4\mathrm{Li}_2(x)\right], \\
\mathrm{Re}\, C_6^{(1)} &= \frac{1}{6s\beta}\left[-6\ln^2 x\ln(1-x) + \ln^3 x - 24\zeta(2)\ln x + 72\zeta(2)\ln(1-x) + 12\zeta(3)\right. \\
&\quad \left. -12\mathrm{Li}_3(x) - 24\mathrm{Li}_3(1-x)\right],
\end{aligned} \tag{2.61}
$$

$$
\begin{aligned}
\mathrm{Re}\, C_6^{(2)} &= \frac{1}{24s\beta}\Big[-16\ln x\ln^3(1-x)+24\ln^2 x\ln^2(1-x)-8\ln^3 x\ln(1-x)+\ln^4 x \\
&\quad +4\ln^4(1-x)+192\zeta(2)\ln x\ln(1-x)-48\zeta(2)\ln^2 x-240\zeta(2)\ln^2(1-x) \\
&\quad -48\zeta(3)\ln x+120\zeta(4)+48\mathrm{Li}_4(x)+96\mathrm{Li}_4(\frac{-x}{1-x})+96\mathrm{Li}_4(1-x)\Big]\,; \\
\mathrm{Im}\, C_6^{(-1)} &= \frac{\pi}{s\beta}, \quad \mathrm{Im}\, C_6^{(0)} = \frac{\pi}{s\beta}\left[\ln x-2\ln(1-x)\right], \qquad (2.62) \\
\mathrm{Im}\, C_6^{(1)} &= \frac{\pi}{6s\beta}\Big[-12\ln x\ln(1-x)+3\ln^2 x+12\ln^2(1-x)-12\zeta(2)\Big], \\
\mathrm{Im}\, C_6^{(2)} &= \frac{\pi}{6s\beta}\Big[12\ln x\ln^2(1-x)-6\ln^2 x\ln(1-x)+\ln^3 x-8\ln^3(1-x)-12\zeta(2)\ln x \\
&\quad +24\zeta(2)\ln(1-x)-12\zeta(3)\big].
\end{aligned}
$$

Corresponding results for C_6 may be obtained from Eqs. (3.5), (3.7), (2.10) and (2.14) of [35]. I have done an order by order numerical comparisons for the coefficients of the ε- and ε^2-terms, while other terms can be easily compared analytically. I have obtained exact agreement.

I mention that I have checked all analytical results for the three–point functions against numerical results provided by M.M. Weber [51] (see also [52]). I found agreement.

2.3 Four-point functions

The scalar four-point one-loop integrals with one, two or three heavy quarks running in the loop are the most difficult to evaluate. The one-loop four-point functions are defined by [9, 10]

$$
\begin{aligned}
&D(q_1,q_2,q_3,m_1,m_2,m_3,m_4) = \qquad (2.63) \\
&\quad \mu^{2\varepsilon}\int\frac{d^n q}{(2\pi)^n}\frac{1}{(q^2-m_1^2)[(q+q_1)^2-m_2^2][(q+q_1+q_2)^2-m_3^2][(q+q_1+q_2+q_3)^2-m_4^2]}.
\end{aligned}
$$

As before, the $+i\delta$ terms in the denominators have not been written out. Again, there is only one internal mass scale m for the calculation purposes.

For heavy flavor production one needs three different types of four-point functions D_i $(i=1,2,3)$ which are expanded as

$$
D_i = i\, C_\varepsilon(m^2)\left\{\frac{1}{\varepsilon^2}D_i^{(-2)}+\frac{1}{\varepsilon}D_i^{(-1)}+D_i^{(0)}+\varepsilon D_i^{(1)}+\varepsilon^2 D_i^{(2)}+\mathcal{O}(\varepsilon^3)\right\}. \qquad (2.64)
$$

Again the coefficient of the most singular part of the four-point functions is purely real, i.e. $\mathrm{Im}\, D_i^{(-2)}=0$.

Before giving the results for the four-point functions it is necessary to discuss some general technical features. After applying Feynman parametrization Eq. (2.20) one obtains for the four-point function

$$
D(q_1,q_2,q_3,m_1,m_2,m_3,m_4) =
$$

$$6\mu^{2\varepsilon}\int\frac{d^nq}{(2\pi)^n}\times \tag{2.65}$$
$$\times\int_0^1\frac{dx_1dx_2dx_3x_1^2x_2}{(a(q^2-m_1^2)+b\left[(q+q_1)^2-m_2^2\right]+c[(q+q_1+q_2)^2-m_3^2]+d[(q+q_1+q_2+q_3)^2-m_4^2])^4},$$

where the set of parameters $\{a,b,c,d\}$ above corresponds to an arbitrary choice from the permutations of the set $\{x_1x_2x_3,\ x_1x_2(1-x_3),\ x_1(1-x_2),\ 1-x_1\}$. Shifting the integration variable $q\to q-bq_1-c(q_1+q_2)-d(q_1+q_2+q_3)$ and using the fact, that $a+b+c+d=1$, one obtains

$$D(q_1,q_2,q_3,m_1,m_2,m_3,m_4)=6\int_0^1 dx_1dx_2dx_3x_1^2x_2\int\frac{d^nq}{(2\pi)^n}\frac{1}{(q^2-K_D)^4}\,, \tag{2.66}$$

with

$$\begin{aligned}K_D &= abq_1^2-ac(q_1+q_2)^2-ad(q_1+q_2+q_3)^2-bcq_2^2\\ &\quad -bd(q_2+q_3)^2-cdq_3^2+am_1^2+bm_2^2+cm_3^2+dm_4^2-i\delta\,.\end{aligned} \tag{2.67}$$

Finally applying the formula (2.16) one is left with a three-fold parametric integral for the four-point functions:

$$D(q_1,q_2,q_3,m_1,m_2,m_3,m_4)=iC_\varepsilon(m^2)(1+\varepsilon)(m^2)^{-2}\int_0^1 dx_1dx_2dx_3\ x_1^2x_2\tilde{K}_D^{-2-\varepsilon}, \tag{2.68}$$

with the kernel $\tilde{K}_D$ given by

$$\begin{aligned}m^2\tilde{K}_D &= -abq_1^2-ac(q_1+q_2)^2-ad(q_1+q_2+q_3)^2-bcq_2^2\\ &\quad -bd(q_2+q_3)^2-cdq_3^2+am_1^2+bm_2^2+cm_3^2+dm_4^2-i\delta.\end{aligned} \tag{2.69}$$

For each particular four-point function one should make a judicious choice of the set $\{a,b,c,d\}$ from the set $\{x_1x_2x_3,\ x_1x_2(1-x_3),\ x_1(1-x_2),\ 1-x_1\}$ in order to get the most convenient kernel for the subsequent integrations. Below I present calculational details and results for all three four-point functions needed.

2.3.1 Four-point function with three massive propagators

First I consider the four-point function D_1 with three massive propagators shown in Fig. 2.4 which is defined by

$$D_1\equiv D(p_4,-p_2,-p_1,0,m,m,m).$$

Substitution of the corresponding values of momenta and masses for D_1 into the expression for the kernel (2.69) gives

$$\tilde{K}_D=ac\tilde{t}-bd\tilde{s}+(1-a)^2-i\delta, \tag{2.70}$$

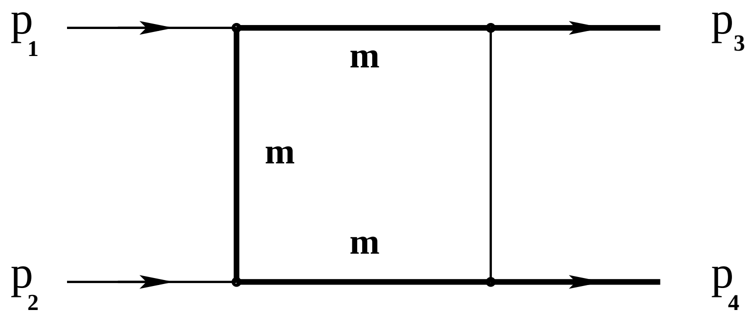

Figure 2.4: Massive box D_1 with three massive propagators. Thick and thin internal lines correspond to massive and massless propagators, respectively. Thick legs represent massive momenta *on-shell*. Thin legs represent massless momenta *on-shell*.

where I have introduced positive valued dimensionless variables

$$\tilde{s} \equiv \frac{s}{m^2}, \qquad \tilde{t} \equiv -\frac{t}{m^2}. \tag{2.71}$$

The kinematical conditions $\tilde{s} \geq 4$, $\tilde{t} \geq 1$, $\tilde{s} \geq \tilde{t}$ constrain the allowable region of phase space for the present physical $2 \to 2$ process (see also Sec. 5.1). The choice for the parameters $\{a, b, c, d\}$ is $\{1 - x_1,\, x_1 x_2 x_3,\, x_1 x_2(1 - x_3),\, x_1(1 - x_2)\}$. For D_1, the integration of the corresponding integrand over x_3 results in two terms:

$$I^{D_1}_{x_1 x_2} = -\frac{x_1^{-\varepsilon}\left[x_1 + \tilde{t}(1 - x_1)x_2)\right]^{-1-\varepsilon}}{(1 + \varepsilon)[\tilde{s}x_1(1 - x_2) + \tilde{t}(1 - x_1)]}, \tag{2.72}$$

$$II^{D_1}_{x_1 x_2} = \frac{x_1^{-1-2\varepsilon}\left[1 - \tilde{s}x_2(1 - x_2) - i\delta\right]^{-1-\varepsilon}}{(1 + \varepsilon)[\tilde{s}x_1(1 - x_2) + \tilde{t}(1 - x_1)]}, \tag{2.73}$$

which then have to be integrated over the remaining parameters x_1 and x_2. Eqs. (2.72) and (2.73) correspond to the indefinite integral (or primitive) evaluated at the upper and lower limit of x_3, respectively. The term $I^{D_1}_{x_1 x_2}$ in (2.72) does not change sign on the integration path, e.g. does not have a branch cut in the interval $[0, 1]$ for both variables x_1 and x_2. Consequently, it does not give an imaginary contribution and it is thus safe to drop the $i\delta$ shift in $I^{D_1}_{x_1 x_2}$. Furthermore, since $I^{D_1}_{x_1 x_2}$ does not have poles in ε, one expands it up to ε^2 and straightforwardly integrates over the second variable x_2 to obtain $I^{D_1}_{x_1}$. Concerning the second term $II^{D_1}_{x_1 x_2}$ in (2.73), one can see that there is a branch cut for the variable x_2 in its numerator as well as a divergence due to the factor $x_1^{-1-2\varepsilon}$ at the lower limit of the integration $x_1 = 0$ (I have dropped the $i\delta$ shift in the denominator as it does not affect the further calculation). At this point one introduces a subtraction term for $II^{D_1}_{x_1 x_2}$ in the simplest possible way: I set $x_1 = 0$ in $II^{D_1}_{x_1 x_2}$ everywhere except for the divergent term $x_1^{-1-2\varepsilon}$. This results in the following subtraction term:

$$II^{D_1,s}_{x_1 x_2} = \frac{x_1^{-1-2\varepsilon}\left[1 - \tilde{s}x_2(1 - x_2) - i\delta\right]^{-1-\varepsilon}}{(1 + \varepsilon)\tilde{t}}, \tag{2.74}$$

which, in the framework of the dimensional regularization scheme, integrates over x_1 to

$$II^{D_1,s}_{x_2} = -\frac{\left[1-\tilde{s}x_2(1-x_2)-i\delta\right]^{-1-\varepsilon}}{2\varepsilon(1+\varepsilon)\tilde{t}}. \tag{2.75}$$

Then I expand the above expression up to ε^2 and reexpress the argument of subsequent logarithms as

$$1-\tilde{s}x_2(1-x_2)-i\delta = \frac{(x_2-x_2^{(0)})(x_2-1+x_2^{(0)})}{x_2^{(0)}(1-x_2^{(0)})} \tag{2.76}$$

with

$$x_2^{(0)} = \frac{1+\sqrt{1-4/\tilde{s}}}{2} + i\delta = \frac{1+\beta}{2} + i\delta\,. \tag{2.77}$$

A final integration of the subsequent series can be done analytically in the complex plane and its result is expressed in terms of logarithms and classical polylogarithms up to Li_4. Analytic continuation of the result for $\delta \to 0$ is then straightforward.

Lastly, I calculate the finite difference

$$\Delta II^{D_1}_{x_1x_2} = II^{D_1}_{x_1x_2} - II^{D_1,s}_{x_1x_2} = \frac{x_1^{-1-2\varepsilon}\left(\tilde{t}-\tilde{s}(1-x_2)\right)\left[1-\tilde{s}x_2(1-x_2)-i\delta\right]^{-1-\varepsilon}}{(1+\varepsilon)\,\tilde{t}\,[\tilde{s}x_1(1-x_2)+\tilde{t}(1-x_1)]}$$

by again expanding the difference up to ε^2 and using (2.76) for the arguments of the logarithms. Then I first integrate over the variable x_2, leading to a reduction of the integrand to simple fractions w.r.t. x_2. In this way one avoids spurious poles in the remaining integral which would otherwise arise in case of integration over x_1 first.

To complete the derivation of the first four-point (box) integral, I combine the two terms

$$I^{D_1}_{x_1} + \Delta II^{D_1}_{x_1}$$

and perform the last integration over the variable x_1.

At this point I would like to comment on some technical details of the calculation which are used throughout this work. For instance, the integrand for the last integration contains expressions such as

$$f(x_1)\cdot \mathrm{Li}_{2,3}\left(\frac{a_1x_1^2+a_2x_1+a_3}{a_4x_1^2+a_5x_1+a_6}\right), \tag{2.78}$$

where $f(x_1)$ is a rational function or a product of a rational function and a logarithm. Using recursively the method of integration-by-parts as much as necessary I render their arguments to be linear functions of x_1. In addition, in the case of Li_3, I can reduce the weight of Li_3 by one. At the same time, the sources of imaginary contributions are transferred into logarithms (or remain in Li_2's and Li_3's with arguments that are independent of the integration variable). Finally, performing the last integration and adding up all the relevant contributions we arrive at the result for the box integral with three massive lines, containing polylogs up to Li_4 and the single- and triple-index L-functions introduced in Eqs. (2.46) and (2.47). As mentioned before Eq. (2.48) the results are written in terms of the L-functions L_{-++}, L_{+++} and L_+ only using the identities derived in Secs. 4.1 and 4.2.

In order to keep the results at reasonable length I introduce the abbreviations

$$\begin{aligned}
&z_3 \equiv (s+2t+s\beta)/2, \qquad z_4 \equiv (s+2t-s\beta)/2,\\
&z_5 \equiv (2m^2+t+t\beta)/2, \qquad z_6 \equiv (2m^2+t-t\beta)/2,\\
&l_s \equiv \ln\frac{s}{m^2}, \quad l_t \equiv \ln\frac{-t}{m^2}, \quad l_T \equiv \ln\frac{-T}{m^2}, \quad l_x \equiv \ln x,\\
&l_\beta \equiv \ln\beta, \quad l_{z3} \equiv \ln\frac{z_3}{m^2}, \quad l_{z4} \equiv \ln\frac{-z_4}{m^2}
\end{aligned} \tag{2.79}$$

and obtain:

$$\begin{aligned}
\mathrm{Re}\,D_1^{(-2)} &= 0, \qquad \mathrm{Re}\,D_1^{(-1)} = \ln x/(st\beta),\\
\mathrm{Re}\,D_1^{(0)} &= -\left[2\ln x\ln(-t\beta/m^2)+2\mathrm{Li}_2(x)-2\mathrm{Li}_2(-x)+3\zeta(2)\right]/(st\beta),\\
\mathrm{Re}\,D_1^{(1)} &= \frac{1}{st\beta}\Bigg[\frac{l_s^3}{12}-\frac{l_s^2\,l_x}{2}-\frac{5\,l_x^3}{6}-l_x^2\,l_{z3}+l_x^2\,l_{z4}-l_x\,l_{z4}^2-\frac{l_t^3}{3}+l_t^2\,(3\,l_x+l_{z4})-\\
&l_t\,\left(l_T^2+l_x^2-l_{z3}^2+2l_T\,(l_x+l_{z3}-l_{z4})+l_{z4}^2+l_x\,(2\,l_{z3}-4\,l_\beta)\right)+l_x\,l_\beta^2-l_\beta^3+\\
&l_s\,\left(l_t\,l_x+l_x^2/4+l_x\,(l_{z3}+l_{z4})-l_\beta^2\right)+(-l_s+10l_t-l_x+6l_\beta)\,\zeta(2)-\\
&5\zeta(3)+4\mathrm{Li}_2(x)\,l_t-2\mathrm{Li}_2\left(\frac{m^2\,x}{-T}\right)l_t+2\mathrm{Li}_2\left(\frac{T}{z_3}\right)l_t-2\mathrm{Li}_2\left(\frac{m^2}{z_5}\right)l_x-\\
&2\mathrm{Li}_2\left(\frac{-t\,(1-\beta)}{2\,m^2}\right)l_x-2\mathrm{Li}_3(-x)-2\mathrm{Li}_3\left(\frac{-x^2}{1-x^2}\right)-2\mathrm{Li}_3\left(\frac{z_3}{t}\right)+\\
&2\mathrm{Li}_3\left(\frac{z_4}{t}\right)+4\mathrm{Li}_3\left(\frac{1-\beta}{2}\right)+2\mathrm{Li}_3\left(\frac{m^2\,(1-\beta)}{2\,z_5}\right)-2\mathrm{Li}_3\left(\frac{-t\,(1-\beta)}{2\,z_5}\right)+\\
&8\mathrm{Li}_3\left(\frac{-1+\beta}{2\,\beta}\right)-2\mathrm{Li}_3\left(\frac{2\,z_6}{m^2(1+\beta)}\right)+2\mathrm{Li}_3\left(\frac{2\,z_6}{-t(1+\beta)}\right)\Bigg],
\end{aligned} \tag{2.80}$$

$$\begin{aligned}
\mathrm{Re}\,D_1^{(2)} &= \frac{1}{st\beta}\Bigg[-\frac{5\,l_s^4}{64}-\frac{43}{24}\,l_s^3\,l_t-\frac{7}{4}\,l_s^2\,l_t^2+\frac{2}{3}\,l_s\,l_t^3+\frac{5\,l_t^4}{12}-\frac{11}{24}\,l_s^3\,l_T-\frac{3}{2}\,l_s^2\,l_t\,l_T-\\
&\frac{5}{2}\,l_s\,l_t^2\,l_T-\frac{4}{3}\,l_t^3\,l_T-\frac{5}{16}\,l_s^2\,l_T^2+\frac{1}{4}\,l_s\,l_t\,l_T^2-\frac{1}{2}\,l_s\,l_T^3+l_t\,l_T^3+\frac{5}{8}\,l_s^3\,l_x-\frac{3}{8}\,l_s^2\,l_t\,l_x-\\
&4\,l_s\,l_t^2\,l_x-l_t^3\,l_x-\frac{7}{8}\,l_s^2\,l_T\,l_x+\frac{3}{2}\,l_s\,l_t\,l_T\,l_x-\frac{3}{2}\,l_t^2\,l_T\,l_x+\frac{3}{8}\,l_s\,l_T^2\,l_x+\frac{13}{4}\,l_t\,l_T^2\,l_x-\\
&\frac{1}{2}l_T^3\,l_x-\frac{13}{32}\,l_s^2\,l_x^2+\frac{1}{8}\,l_s\,l_t\,l_x^2-\frac{1}{4}\,l_t^2\,l_x^2+\frac{17}{8}\,l_s\,l_T\,l_x^2+\frac{1}{2}\,l_t\,l_T\,l_x^2+\frac{7}{16}\,l_T^2\,l_x^2+\frac{1}{2}l_s\,l_x^3+\\
&\frac{3}{8}\,l_t\,l_x^3+\frac{17}{24}\,l_T\,l_x^3+\frac{7\,l_x^4}{64}+\frac{29}{48}\,l_s^3\,l_{z3}+3l_s^2\,l_t\,l_{z3}+\frac{5}{2}\,l_s\,l_t^2\,l_{z3}-\frac{3}{2}\,l_t^3\,l_{z3}-\frac{3}{4}\,l_s^2\,l_T\,l_{z3}+\\
&\frac{17}{2}\,l_s\,l_t\,l_T\,l_{z3}+\frac{5}{2}\,l_t^2\,l_T\,l_{z3}+2\,l_t\,l_T^2\,l_{z3}-\frac{19}{16}\,l_s^2\,l_x\,l_{z3}+l_s\,l_t\,l_x\,l_{z3}+\frac{5}{2}\,l_t^2\,l_x\,l_{z3}+
\end{aligned}$$

$$\frac{9}{2}\,l_s\,l_T\,l_x\,l_{z3}-\frac{1}{2}\,l_t\,l_T\,l_x\,l_{z3}-\frac{1}{2}\,l_T^2\,l_x\,l_{z3}+\frac{13}{16}\,l_s\,l_x^2\,l_{z3}+l_t\,l_x^2\,l_{z3}-\frac{3}{4}\,l_T\,l_x^2\,l_{z3}-$$
$$\frac{11}{48}\,l_x^3\,l_{z3}-\frac{1}{8}\,l_s^2\,l_{z3}^2-4\,l_s\,l_t\,l_{z3}^2+\frac{1}{2}\,l_t^2\,l_{z3}^2+l_s\,l_T\,{l_{z3}}^2-5\,l_t\,l_T\,l_{z3}^2-\frac{3}{4}\,l_s\,l_x\,l_{z3}^2+$$
$$l_t\,l_x\,l_{z3}^2-2\,l_T\,l_x\,l_{z3}^2-\frac{1}{8}\,l_x^2\,l_{z3}^2+l_t\,l_{z3}^3+\frac{61}{48}\,l_s^3\,l_{z4}+\frac{27}{4}\,l_s^2\,l_t\,l_{z4}+\frac{1}{2}\,l_s\,l_t^2\,l_{z4}-$$
$$\frac{13}{6}\,l_t^3\,l_{z4}+\frac{9}{4}\,l_s^2\,l_T\,l_{z4}-\frac{1}{2}\,l_s\,l_t\,l_T\,l_{z4}+\frac{5}{2}\,l_t^2\,l_T\,l_{z4}+l_s\,l_T^2\,l_{z4}-\frac{5}{2}\,l_t\,l_T^2\,l_{z4}+$$
$$\frac{9}{16}\,l_s^2\,l_x\,l_{z4}+\frac{11}{2}\,l_s\,l_t\,l_x\,l_{z4}+\frac{3}{2}\,l_t^2\,l_x\,l_{z4}-\frac{3}{2}\,l_s\,l_T\,l_x\,l_{z4}-\frac{5}{2}\,l_t\,l_T\,l_x\,l_{z4}+\frac{1}{2}l_T^2\,l_x\,l_{z4}-$$
$$\frac{11}{16}\,l_s\,l_x^2\,l_{z4}-\frac{1}{4}\,l_t\,l_x^2\,l_{z4}-\frac{9}{4}\,l_T\,l_x^2\,l_{z4}-\frac{5}{16}\,l_x^3\,l_{z4}-2\,l_s^2\,l_{z3}\,l_{z4}-6\,l_s\,l_t\,l_{z3}\,l_{z4}-$$
$$l_t^2\,l_{z3}\,l_{z4}-3l_s\,l_T\,l_{z3}\,l_{z4}-6l_t\,l_T\,l_{z3}\,l_{z4}-4l_t\,l_x\,l_{z3}\,l_{z4}-3l_T\,l_x\,l_{z3}\,l_{z4}-\frac{1}{2}l_x^2\,l_{z3}\,l_{z4}+$$
$$l_s\,l_{z3}^2\,l_{z4}+\frac{5}{2}\,l_t\,l_{z3}^2\,l_{z4}+l_x\,l_{z3}^2\,l_{z4}-\frac{77}{16}\,l_s^2\,l_{z4}^2-\frac{13}{2}\,l_s\,l_t\,l_{z4}^2+3\,l_t^2\,l_{z4}^2-\frac{5}{4}\,l_s\,l_T\,l_{z4}^2+$$
$$2\,l_t\,l_T\,l_{z4}^2+\frac{1}{4}\,l_T^2\,l_{z4}^2-\frac{33}{8}\,l_s\,l_x\,l_{z4}^2-\frac{5}{2}\,l_t\,l_x\,l_{z4}^2+\frac{7}{4}\,l_T\,l_x\,l_{z4}^2+\frac{15}{16}\,l_x^2\,l_{z4}^2+3\,l_s\,l_{z3}\,l_{z4}^2+$$
$$\frac{7}{2}\,l_t\,l_{z3}\,l_{z4}^2+3\,l_T\,l_{z3}\,l_{z4}^2+2\,l_x\,l_{z3}\,l_{z4}^2-l_{z3}^2\,l_{z4}^2+\frac{71}{12}\,l_s\,l_{z4}^3+\frac{2}{3}\,l_t\,l_{z4}^3-\frac{5}{6}\,l_T\,l_{z4}^3+$$
$$\frac{31}{12}\,l_x\,l_{z4}^3-2l_{z3}\,l_{z4}^3-\frac{49}{24}l_{z4}^4-\frac{13}{24}\,l_s^3\,l_\beta-\frac{7}{2}\,l_s^2\,l_t\,l_\beta-2\,l_s\,l_t^2\,l_\beta-\frac{5}{3}\,l_t^3\,l_\beta-\frac{9}{4}\,l_s^2\,l_T\,l_\beta-$$
$$3\,l_s\,l_t\,l_T\,l_\beta-4\,l_t^2\,l_T\,l_\beta+\frac{13}{8}\,l_s^2\,l_x\,l_\beta-2\,l_s\,l_t\,l_x\,l_\beta-5\,l_t^2\,l_x\,l_\beta-\frac{1}{2}\,l_s\,l_T\,l_x\,l_\beta-$$
$$3\,l_t\,l_T\,l_x\,l_\beta+l_T^2\,l_x\,l_\beta-\frac{9}{8}\,l_s\,l_x^2\,l_\beta+\frac{1}{2}\,l_t\,l_x^2\,l_\beta+\frac{7}{4}\,l_T\,l_x^2\,l_\beta+\frac{17}{24}\,l_x^3\,l_\beta+\frac{9}{4}\,l_s^2\,l_{z3}\,l_\beta+$$
$$4\,l_s\,l_t\,l_{z3}\,l_\beta+4\,l_t^2\,l_{z3}\,l_\beta+2\,l_s\,l_T\,l_{z3}\,l_\beta+6\,l_t\,l_T\,l_{z3}\,l_\beta-\frac{5}{2}\,l_s\,l_x\,l_{z3}\,l_\beta+4\,l_t\,l_x\,l_{z3}\,l_\beta+$$
$$4l_T\,l_x\,l_{z3}\,l_\beta+\frac{5}{4}l_x^2\,l_{z3}\,l_\beta-l_s\,l_{z3}^2\,l_\beta-3l_t\,l_{z3}^2\,l_\beta-l_x\,l_{z3}^2\,l_\beta+\frac{11}{4}l_s^2\,l_{z4}\,l_\beta+7l_s\,l_t\,l_{z4}\,l_\beta+$$
$$l_t^2\,l_{z4}\,l_\beta+3\,l_s\,l_T\,l_{z4}\,l_\beta+2\,l_t\,l_T\,l_{z4}\,l_\beta+\frac{1}{2}\,l_s\,l_x\,l_{z4}\,l_\beta+3\,l_t\,l_x\,l_{z4}\,l_\beta-l_T\,l_x\,l_{z4}\,l_\beta-$$
$$\frac{1}{4}\,l_x^2\,l_{z4}\,l_\beta-3\,l_s\,l_{z3}\,l_{z4}\,l_\beta-6\,l_t\,l_{z3}\,l_{z4}\,l_\beta-4\,l_T\,l_{z3}\,l_{z4}\,l_\beta+l_x\,l_{z3}\,l_{z4}\,l_\beta+2\,l_{z3}^2\,l_{z4}\,l_\beta-$$
$$5\,l_s\,l_{z4}^2\,l_\beta-l_t\,l_{z4}^2\,l_\beta-2\,l_x\,l_{z4}^2\,l_\beta+l_{z3}\,l_{z4}^2\,l_\beta+\frac{7}{3}l_{z4}^3\,l_\beta-\frac{7}{8}l_s^2\,l_\beta^2-\frac{5}{2}l_s\,l_t\,l_\beta^2-2\,l_t^2\,l_\beta^2-$$
$$\frac{5}{2}\,l_s\,l_T\,l_\beta^2-3\,l_t\,l_T\,l_\beta^2+\frac{3}{4}\,l_s\,l_x\,l_\beta^2-\frac{5}{2}\,l_t\,l_x\,l_\beta^2-\frac{1}{2}\,l_T\,l_x\,l_\beta^2-\frac{3}{8}\,l_x^2\,l_\beta^2+\frac{5}{2}\,l_s\,l_{z3}\,l_\beta^2+$$
$$3\,l_t\,l_{z3}\,l_\beta^2+2\,l_T\,l_{z3}\,l_\beta^2-\frac{1}{2}\,l_x\,l_{z3}\,l_\beta^2-l_{z3}^2\,l_\beta^2+\frac{5}{2}\,l_s\,l_{z4}\,l_\beta^2+3\,l_t\,l_{z4}\,l_\beta^2+2\,l_T\,l_{z4}\,l_\beta^2+$$
$$\frac{1}{2}\,l_x\,l_{z4}\,l_\beta^2-2\,l_{z3}\,l_{z4}\,l_\beta^2-\frac{3}{2}\,l_{z4}^2\,l_\beta^2-\frac{1}{3}\,l_s\,l_\beta^3-\frac{2}{3}\,l_t\,l_\beta^3-l_T\,l_\beta^3-\frac{1}{3}\,l_x\,l_\beta^3+l_{z3}\,l_\beta^3+$$
$$l_{z4}l_\beta^3+\frac{l_\beta^4}{6}+\Bigg(\frac{19}{8}l_x^2-\frac{l_s^2}{8}-9l_t^2+5l_Tl_x+l_Tl_{z3}+\frac{11}{2}l_xl_{z3}-l_{z3}^2-6l_Tl_{z4}+2l_xl_{z4}-$$

$$3\,l_{z3}\,l_{z4} - 7\,l_{z4}^2 + 2\,l_t\,\left(3\,l_T + 7\,l_{z3} + 5\,l_{z4} - 12\,l_\beta\right) + 6\,l_T\,l_\beta - 10\,l_x\,l_\beta - 2\,l_{z3}\,l_\beta +$$
$$14\,l_{z4}\,l_\beta - 13\,l_\beta^2 - \frac{1}{4}\,l_s\,\left(64\,l_t - 8\,l_T + 35\,l_x + 18\,l_{z3} - 44\,l_{z4} + 32\,l_\beta\right)\Big)\,\zeta(2) +$$
$$(-2\,l_s - 4\,l_t - l_T + l_x + 7\,l_{z3} + l_{z4})\,\zeta(3) - \frac{35}{4}\zeta(4) - 2\,\mathrm{Li}_2^2\left(\frac{m^2}{z_5}\right) +$$
$$2\,\mathrm{Li}_2^2\left(\frac{-t(1-\beta)}{2\,m^2}\right) + \mathrm{Li}_2(x)\,\left(2\,\mathrm{Li}_2\left(\frac{m^2}{z_5}\right) + 2\,\mathrm{Li}_2\left(\frac{-t(1-\beta)}{2\,m^2}\right) + \frac{l_s^2}{4} -\right.$$
$$2\,l_t^2 + l_T^2 + l_T\,l_x + \frac{l_x^2}{4} + 2\,l_T\,l_{z3} + 3\,l_x\,l_{z3} + l_t\,\left(-4\,l_T - 3\,l_x + 2\,l_{z3}\right) +$$
$$l_s\,\left(l_t - l_T - \frac{9}{2}l_x - l_{z3} - l_{z4}\right) + 5\,l_x\,l_{z4} + l_{z4}^2 - 4\,l_x\,l_\beta - 6\,\zeta(2)\Big) + \mathrm{Li}_2\left(\frac{z_3}{z_4}\right) \times$$
$$\left(-2\mathrm{Li}_2\left(\frac{m^2}{z_5}\right) - \frac{l_s^2}{4} + 4l_t^2 + l_T\,l_x + \frac{3}{4}l_x^2 - \frac{l_s}{2}\left(6l_t + 2l_T + 3l_x - 4l_{z3} - 4l_{z4}\right) +\right.$$
$$2\,l_x\,l_{z4} - l_{z4}^2 - l_t\,\left(l_x - 2\,l_T + 2\,l_{z3} + 2\,l_{z4}\right) - 2\,l_x\,l_\beta - 6\,\zeta(2)\Big) +$$
$$\mathrm{Li}_2\left(\frac{-t(1-\beta)}{2\,m^2}\right)\,\left(-2\mathrm{Li}_2\left(\frac{z_3}{z_4}\right) + \frac{1}{4}\left(l_s^2 - 12l_t^2 - 2l_T^2 - 6l_T\,l_x + l_x^2 - 12l_T\,l_{z3} +\right.\right.$$
$$8\,l_x\,l_{z3} - 4\,l_x\,l_{z4} - 4\,l_{z4}^2 + l_s\,\left(-4\,l_t + 6\,l_T + 2\,l_x - 8\,l_{z3} + 4\,l_{z4} - 4\,l_\beta\right) +$$
$$4\,l_t\,\left(l_x + 4\,l_{z3} + 2\,l_{z4} - 2\,l_\beta\right) + 12\,l_x\,l_\beta + 8\,l_{z4}\,l_\beta - 4\,l_\beta^2\Big) + 12\,\zeta(2)\Big) +$$
$$\frac{1}{8}\,\mathrm{Li}_2\left(\frac{m^2 x}{-T}\right)\left(-l_s^2 + 16l_tl_T - l_x\left(4l_T + l_x - 4l_{z4}\right) - 2l_s\left(2l_T + l_x - 2l_{z4}\right)\right) + \frac{1}{4}\times$$
$$\mathrm{Li}_2(\frac{m^2}{z_5})\left(3l_s^2 + 4l_t^2 + 2l_T^2 + 2l_Tl_x + 7l_x^2 + 8l_Tl_{z3} - 2l_s(8l_t + 3l_T - l_x + 4l_{z3} - 6l_{z4})\right.$$
$$+4l_Tl_{z4} + 4l_xl_{z4} - 12l_{z4}^2 + 8l_t(l_{z3} + l_{z4} - l_\beta) + 8l_xl_\beta - 8l_{z3}l_\beta + 8l_{z4}l_\beta - 4l_\beta^2\Big) +$$
$$\frac{1}{8}\,\mathrm{Li}_2\left(\frac{T}{z_3}\right)\,\left(9\,l_s^2 - 16\,l_t^2 + l_x\,\left(-4\,l_T + l_x + 12\,l_{z3} + 8\,l_{z4} - 8\,l_\beta\right) -\right.$$
$$2l_s(4l_t - 2l_T + 5l_x + 6l_{z3} + 4l_{z4} - 4l_\beta) - 8l_t(2l_T + l_x - 2l_{z3} - 2l_{z4} + 2l_\beta)) +$$
$$\frac{1}{2}\mathrm{Li}_2(\frac{T}{m^2})\left(2l_s^2 - 2l_Tl_x - 2l_Tl_{z3} + 3l_xl_{z3} + 5l_{z3}^2 - 2l_t(l_x + 2l_{z3}) + 2l_Tl_{z4} + 3l_xl_{z4} +\right.$$
$$4l_{z3}l_{z4} - l_{z4}^2 - 2l_xl_\beta - 4l_{z3}l_\beta + l_s(2l_t - 3l_x - 7l_{z3} - l_{z4} + 2l_\beta)\Big) + \frac{1}{4}\mathrm{Li}_2(-x)\times$$
$$\left(5l_s^2 - 16l_t^2 + l_x^2 - 4l_Tl_{z3} + 6l_xl_{z3} - 6l_{z3}^2 - 4l_Tl_{z4} + 2l_xl_{z4} - 8l_{z3}l_{z4} + 2l_{z4}^2 - 4l_x\,l_\beta +\right.$$
$$8l_{z3}l_\beta + 2l_s(-6l_t + 2l_T - 2l_x + l_{z3} - 3l_{z4} + 2l_\beta) - 4l_t(l_x - 6l_{z3} - 4l_{z4} + 4l_\beta)) +$$
$$\mathrm{Li}_3\left(\frac{z_4}{T}\right)\,\left(-l_T + l_{z4}\right) + 2\left(\mathrm{Li}_3\left(\frac{z_3}{s\beta}\right) + \mathrm{Li}_3\left(\frac{z_5}{t\beta}\right)\right)\,\left(2\,l_t + 2\,l_T - l_{z3} - l_{z4}\right) +$$
$$2\mathrm{Li}_3\left(\frac{-1+\beta}{2\beta}\right)\,\left(2\,l_s - 2\,l_t - 2\,l_T - l_{z3} - l_{z4}\right) +$$

$$\mathrm{Li}_3\left(\frac{m^2}{z_5}\right)\left(-\frac{17}{2}l_s - 8\,l_t + 4\,l_T - \frac{l_x}{2} + 8\,l_{z3} + 9\,l_{z4} - 8\,l_\beta\right) +$$
$$\frac{1}{2}\,\mathrm{Li}_3\left(\frac{T}{z_6}\right)(-5\,l_s - 4\,l_t - 2\,l_T + l_x + 6\,l_{z3} + 4\,l_{z4} - 4\,l_\beta) +$$
$$2\,\mathrm{Li}_3\left(\frac{z_3}{z_4}\right)(-l_s + l_t + l_T + l_x - l_\beta) + \mathrm{Li}_3\left(\frac{z_3}{t}\right)(-l_s - 2\,l_T - 3\,l_x + 2\,l_{z3}) +$$
$$\frac{1}{2}\mathrm{Li}_3\left(\frac{z_6}{m^2}\right)(l_s + 8l_t - 8l_T - l_x - 6l_{z3} - 4l_{z4}) + \mathrm{Li}_3\left(\frac{z4}{t}\right)(l_s + 4l_t + 2l_T - 3l_x -$$
$$2l_{z3} - 4l_{z4}) + \frac{1}{2}\mathrm{Li}_3\left(\frac{z_5}{T}\right)(l_s + 2l_T + l_x - 2l_{z4}) + 2\left(\mathrm{Li}_3\left(\frac{-x^2}{1-x^2}\right) -\right.$$
$$\left.2\mathrm{Li}_3\left(\frac{1-\beta}{2}\right) - \mathrm{Li}_3\left(\frac{m^2(1-\beta)}{2\,z_5}\right) + \mathrm{Li}_3\left(\frac{-t(1-\beta)}{2\,z_5}\right) - 2\mathrm{Li}_3\left(\frac{T}{m^2}\right)\right) l_T +$$
$$2\left(2\mathrm{Li}_3\left(\frac{T}{m^2}\right) + \mathrm{Li}_3\left(\frac{2\,z_6}{m^2(1+\beta)}\right) - \mathrm{Li}_3\left(\frac{-2\,z_6}{t(1+\beta)}\right) + \mathrm{Li}_3\left(\frac{z_6}{z_5}\right)\right)(l_s + l_t +$$
$$l_T - l_{z3} - l_{z4} + l_\beta) - 2\mathrm{Li}_3\left(\frac{z_6}{z_5}\right) l_x - \mathrm{Li}_3\left(\frac{T}{z_3}\right) l_{z3} + \left(\mathrm{Li}_3\left(\frac{T}{z_3}\right) + 2\,\mathrm{Li}_3(-x)\right) \times$$
$$(2l_s + 2l_t + l_T - 2l_{z3} - 2l_{z4} + 2l_\beta) + 2\mathrm{Li}_3(x)\,(4l_s + 3l_t + l_T + l_x - 2l_{z3} - 6l_{z4} +$$
$$3l_\beta) + 2\mathrm{Li}_4(x) - \mathrm{Li}_4\left(\frac{T}{z_3}\right) - 4\mathrm{Li}_4\left(\frac{z_3}{t}\right) + 4\mathrm{Li}_4\left(\frac{z_4}{t}\right) - \mathrm{Li}_4\left(\frac{z_4}{T}\right) + \mathrm{Li}_4\left(\frac{z_5}{T}\right) +$$
$$2\mathrm{Li}_4\left(\frac{s(1-\beta)}{-2t}\right) + 3\mathrm{Li}_4\left(\frac{s(1-\beta)}{2\,z_4}\right) + 4\mathrm{Li}_4\left(\frac{-1+\beta}{2\,\beta}\right) + 2\mathrm{Li}_4\left(\frac{-2t}{s(1+\beta)}\right) +$$
$$4\mathrm{Li}_4\left(\frac{2\,\beta}{1+\beta}\right) + \mathrm{Li}_4\left(\frac{sT(1+\beta)}{2\,m^2\,z_3}\right) + 3\mathrm{Li}_4\left(\frac{2\,z_3}{s(1+\beta)}\right) +$$
$$\frac{1}{2}L_{-++}\left(1, \frac{m^2}{-T}, \frac{m^2}{-T}, \frac{s(1-\beta)}{-2z_4}\right) - \frac{1}{2}L_{-++}\left(1, \frac{m^2}{-T}, \frac{m^2}{-T}, \frac{s(1+\beta)}{-2z_3}\right) +$$
$$2L_{-++}\left(1, \frac{m^2}{-T}, \frac{s(1-\beta)}{-2z_4}, 0\right) + L_{-++}\left(1, -\frac{t}{z_3}, -\frac{t}{z_3}, -\frac{t}{T}\right) -$$
$$L_{-++}\left(1, \frac{s(1-\beta)}{-2z_4}, \frac{m^2}{-T}, \frac{s(1-\beta)}{-2z_4}\right) - L_{-++}\left(1, \frac{s(1-\beta)}{-2z_4}, \frac{m^2}{-T}, \frac{s(1+\beta)}{-2z_3}\right) +$$
$$L_{-++}\left(1, \frac{s(1-\beta)}{-2z_4}, \frac{s(1-\beta)}{-2z_4}, 0\right) - \frac{1}{2}L_{-++}\left(1, \frac{s(1-\beta)}{-2z_4}, \frac{s(1-\beta)}{-2z_4}, \frac{s(1+\beta)}{-2z_3}\right) +$$
$$\frac{1}{2}L_{-++}\left(\frac{t}{T}, 0, 0, -\frac{t}{z_3}\right) - \frac{1}{2}L_{-++}\left(\frac{t}{T}, 0, 0, -\frac{t}{z_4}\right) + 2L_{-++}\left(\frac{t}{T}, 0, -\frac{t}{z_3}, -1\right) -$$
$$L_{-++}\left(\frac{t}{T}, 0, -\frac{t}{z_3}, -\frac{t}{z_3}\right) - L_{-++}\left(\frac{t}{T}, 0, -\frac{t}{z_3}, -\frac{t}{z_4}\right) - L_{-++}\left(\frac{t}{T}, -\frac{t}{z_3}, -\frac{t}{z_3}, -\frac{t}{z_4}\right) +$$
$$3L_{-++}(\frac{t}{z_4}, 0, 0, -1) - \frac{3}{2}L_{-++}\left(\frac{t}{z_4}, 0, 0, -\frac{t}{z_3}\right) - 2L_{-++}\left(\frac{t}{z_4}, 0, -\frac{t}{z_3}, -1\right) +$$
$$L_{-++}\left(\frac{t}{z_4}, 0, -\frac{t}{z_3}, -\frac{t}{z_3}\right) + 3L_{-++}\left(\frac{t}{z_4}, 0, -\frac{t}{z_3}, -\frac{t}{z_4}\right) +$$

$$\frac{1}{2}L_{-++}\left(\frac{t}{z_4},-\frac{t}{z_3},-\frac{t}{z_3},-\frac{t}{T}\right)+2L_{-++}\left(\frac{s(1+\beta)}{2\,z_3},\frac{m^2}{-T},0,\frac{m^2}{-T}\right)-$$
$$2L_{-++}\left(\frac{s(1+\beta)}{2\,z_3},\frac{m^2}{-T},0,\frac{s(1-\beta)}{-2z_4}\right)-2L_{-++}\left(\frac{s(1+\beta)}{2\,z_3},\frac{m^2}{-T},0,\frac{s(1+\beta)}{-2z_3}\right)+$$
$$\frac{1}{2}L_{-++}\left(\frac{s(1+\beta)}{2\,z_3},\frac{m^2}{-T},\frac{m^2}{-T},\frac{s(1-\beta)}{-2z_4}\right)+2L_{-++}\left(\frac{s(1+\beta)}{2\,z_3},\frac{s(1-\beta)}{-2z_4},0,\frac{m^2}{-T}\right)-$$
$$\frac{1}{2}L_{-++}\left(\frac{s(1+\beta)}{2\,z_3},\frac{s(1-\beta)}{-2z_4},\frac{s(1-\beta)}{-2z_4},\frac{m^2}{-T}\right)-4L_{+}\left(0,\frac{z_3}{s\beta},-\frac{z_3z_4}{st\beta},-1\right)+$$
$$4L_{+}\left(0,\frac{z_3(1-\beta)}{-2t\beta},\frac{z_3z_4}{st\beta},\frac{m^2}{-T}\right)+2L_{+}\left(\frac{m^2}{-T},\frac{z_3(1-\beta)}{-2t\beta},\frac{z_3z_4}{st\beta},\frac{s(1-\beta)}{-2z_4}\right)+$$
$$2L_{+}\left(-\frac{t}{z_3},0,\frac{z_4}{t},-\frac{t}{z_4}\right)+2L_{+}\left(-\frac{t}{z_3},0,\frac{z_3}{t},-\frac{t}{z_4}\right)+2L_{+}\left(-\frac{t}{z_3},\frac{z_3}{s\beta},-\frac{z_3z_4}{st\beta},-\frac{t}{T}\right)+$$
$$\frac{3}{2}L_{+++}\left(0,0,-\frac{t}{z_3},-\frac{t}{z_4}\right)-2L_{+++}\left(0,\frac{m^2}{-T},\frac{s(1-\beta)}{-2z_4},\frac{m^2}{-T}\right)+$$
$$2L_{+++}\left(0,\frac{m^2}{-T},\frac{s(1-\beta)}{-2z_4},\frac{s(1-\beta)}{-2z_4}\right)+2L_{+++}\left(0,\frac{m^2}{-T},\frac{s(1-\beta)}{-2z_4},\frac{s(1+\beta)}{-2z_3}\right)+$$
$$L_{+++}\left(0,-\frac{t}{z_3},-\frac{t}{z_3},-1\right)-L_{+++}\left(\frac{m^2}{-T},\frac{m^2}{-T},0,\frac{s(1-\beta)}{-2z_4}\right)+$$
$$L_{+++}\left(\frac{m^2}{-T},\frac{m^2}{-T},0,\frac{s(1+\beta)}{-2z_3}\right)-\frac{1}{2}L_{+++}\left(\frac{m^2}{-T},\frac{m^2}{-T},\frac{s(1-\beta)}{-2z_4},\frac{s(1+\beta)}{-2z_3}\right)-$$
$$3L_{+++}\left(-\frac{t}{z_3},0,0,-1\right)+\frac{3}{2}L_{+++}\left(-\frac{t}{z_3},-\frac{t}{z_3},0,-\frac{t}{z_4}\right)+$$
$$L_{+++}\left(\frac{s(1-\beta)}{-2z_4},\frac{s(1-\beta)}{-2z_4},0,\frac{m^2}{-T}\right)\Bigg];$$

$$\mathrm{Im}\,D_1^{(-1)} = \pi/(st\beta), \qquad \mathrm{Im}\,D_1^{(0)} = -2\pi\ln(-t\beta/m^2)/(st\beta), \tag{2.81}$$
$$\mathrm{Im}\,D_1^{(1)} = \frac{\pi}{st\beta}\left[-\frac{3}{4}\,l_s^2+2\,l_t^2-\frac{3}{4}\,l_x^2-l_xl_{z3}+l_xl_{z4}-l_{z4}^2+l_s\,(l_t+l_x/2+l_{z3}+l_{z4})-\right.$$
$$\left.l_t\,(l_x+2\,l_{z3}-4\,l_\beta)+2\,l_\beta^2-2\zeta(2)-2\mathrm{Li}_2\left(\frac{m^2}{z_5}\right)-2\mathrm{Li}_2\left(\frac{t(-1+\beta)}{2m^2}\right)\right],$$
$$\mathrm{Im}\,D_1^{(2)} = \frac{\pi}{st\beta}\left[\frac{1}{12}\left(7\,l_s^3-l_x^3-3\,l_s^2\,(3\,l_x+4\,l_{z3}+4\,l_{z4}-6\,l_\beta)+6\,l_x^2\,(-2\,l_{z3}+3\,l_\beta)+\right.\right.$$
$$12\,l_x\left(l_{z4}^2-2\,(-l_{z3}+l_{z4})\,l_\beta+2\,l_t\,(-l_{z3}+l_\beta)\right)-$$
$$8\,l_\beta\left(6\,l_t^2-3\,l_{z4}^2+2\,l_\beta^2+6\,l_t\,(-l_{z3}+l_\beta)\right)-3\,l_s\left(8\,l_t^2-5\,l_x^2-4\,l_{z4}^2-\right.$$
$$\left.\left.8\,l_t\,(l_x+l_{z3}-l_\beta)+8\,l_{z3}\,l_\beta+8\,l_{z4}\,l_\beta+4\,l_x\,(-2\,l_{z3}+2\,l_{z4}+l_\beta)\right)\right)+$$
$$4\,(l_t+l_x+l_\beta)\,\zeta(2)+2\,\mathrm{Li}_2(x)\,l_x+2\,\mathrm{Li}_2\left(\frac{z3}{z4}\right)\,l_x+$$

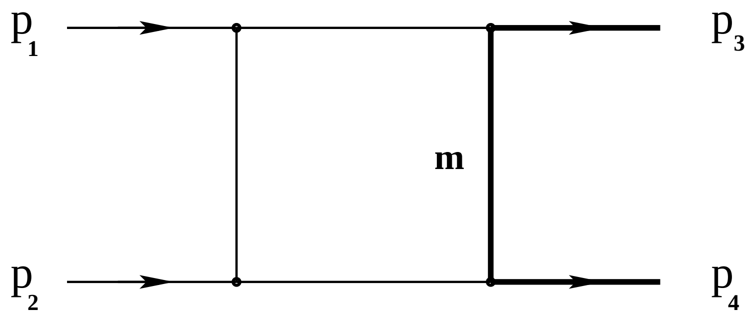

Figure 2.5: Massive box D_2 with one massive propagator. Thick and thin internal lines correspond to massive and massless propagators, respectively. Thick legs represent massive momenta *on-shell*. Thin legs represent massless momenta *on-shell*.

$$2\,\mathrm{Li}_2\left(\frac{t(-1+\beta)}{2\,m^2}\right)(l_s - l_x + 2\,l_\beta) + 2\,\mathrm{Li}_2\left(\frac{m^2}{z_5}\right)(l_s + l_x + 2\,l_\beta) -$$
$$2\,\mathrm{Li}_3(x) - 4\,\mathrm{Li}_3\left(\frac{z_3}{t}\right) + 2\,\mathrm{Li}_3\left(\frac{z_3}{z_4}\right) - 4\,\mathrm{Li}_3\left(\frac{z_4}{t}\right) - 2\,\mathrm{Li}_3\left(\frac{z_6}{z_5}\right)\Bigg].$$

Even though the result for the four-point function D_1 appears to be rather lengthy I *have to emphasize* that it has already been considerably simplified by a factor of ~ 30 with the help of the algorithms described in Chap. 3 and the identities discussed in Sec. 4.1 and Sec. 4.2 compared to the untreated output of the original integration. The same holds true for the four-point functions D_2 and D_3 discussed below as well as for the results of the three-point functions in Sec. 2.2.

2.3.2 Four-point function with one massive propagator

Next I turn to the second four-point function D_2 with one massive propagator shown in Fig. 2.5 which is defined by

$$D_2 \equiv D(-p_2, p_4, p_3, 0, 0, m, 0).$$

I substitute the appropriate values of momenta and masses for the D_2 integral into the general kernel expression (2.69) and obtain

$$\tilde{K}_D = ac\tilde{t} - bd\tilde{s} + c^2 - i\delta. \tag{2.82}$$

In order to simplify the first integration over the Feynman parameter x_3 I choose $\{a, b, c, d\}$ as $\{x_1x_2(1-x_3),\ x_1x_2x_3,\ x_1(1-x_2),\ 1-x_1\}$. After x_3-integration I write the result for the integrand as a sum of two parts $I^{D_2}_{x_1x_2} + II^{D_2}_{x_1x_2}$:

$$I^{D_2}_{x_1x_2} = -\frac{x_1\left[x_1^2(1-x_2)(1+(\tilde{t}-1)x_2) - i\delta\right]^{-1-\varepsilon}}{(1+\varepsilon)[\tilde{s}(1-x_1) + \tilde{t}x_1(1-x_2)]}, \tag{2.83}$$

$$II^{D_2}_{x_1x_2} = \frac{x_1\left[-\tilde{s}x_1x_2 + x_1^2(1+(\tilde{s}-2)x_2 + x_2^2) - i\delta\right]^{-1-\varepsilon}}{(1+\varepsilon)[\tilde{s}(1-x_1) + \tilde{t}x_1(1-x_2)]}, \tag{2.84}$$

where again the two terms derive from the indefinite integral (or primitive) evaluated at the upper and lower boundary of x_3, respectively. The indices x_1 and x_2 in $I^{D_2}_{x_1x_2}$ and $II^{D_2}_{x_1x_2}$ indicate that one remains with a two-dimensional integration over x_1 and x_2. First consider the integration of $I^{D_2}_{x_1x_2}$. One notes that its numerator is not negative on the integration path since $\tilde{t} > 1$, which implies there is no imaginary contribution coming from $I^{D_2}_{x_1x_2}$. Therefore, one omits the $i\delta$ term for the remaining integration. After integration over x_1 one arrives at

$$I^{D_2}_{x_2} = \frac{\Big[(1-x_2)\,\Big(1+(\tilde{t}-1)\,x_2\Big)\Big]^{-1-\varepsilon}\;{}_2F_1(1,-2\varepsilon,1-2\varepsilon,1-A)}{2\tilde{s}\,(1+\varepsilon)\,\varepsilon}, \tag{2.85}$$

where I have defined $A \equiv \tilde{t}\,(1-x_2)/\tilde{s}$ and ${}_2F_1$ is a hypergeometric function. The above expression is singular at the upper integration limit $x_2 = 1$. In order to regularize this singularity one has to find a suitable subtraction term.

First note that the ε-expansion of the hypergeometric function reads

$$\begin{aligned} F_1(1,-2\varepsilon,1-2\varepsilon,1-A) = 1 + 2\varepsilon\,\ln A - 4\varepsilon^2\,\mathrm{Li}_2(1-A) - 8\varepsilon^3\,\mathrm{Li}_3(1-A) \\ -16\varepsilon^4\,\mathrm{Li}_4(1-A) + \mathcal{O}(\varepsilon^5). \end{aligned} \tag{2.86}$$

To obtain a suitable subtraction term one substitutes (2.86) into (2.85) and replace x_2 by 1 everywhere in $I^{D_2}_{x_2}$ except for terms that diverge. Therefore, the subtraction term can be defined as

$$I^{D_2,s}_{x_2} = \frac{(1-x_2)^{-1-\varepsilon}\,\tilde{t}^{-1-\varepsilon}\,[1+2\varepsilon\,\ln A - 4\varepsilon^2\zeta(2) - 8\varepsilon^3\,\zeta(3) - 16\varepsilon^4\zeta(4)]}{2\tilde{s}\,(1+\varepsilon)\,\varepsilon}. \tag{2.87}$$

The subtraction term is simple enough to be integrated analytically over x_2 giving the result

$$I^{D_2,s} = \frac{\tilde{t}^{-1-\varepsilon}\,\Big(-3 - 2\varepsilon\,\ln\frac{\tilde{t}}{\tilde{s}} + 4\varepsilon^2\,\zeta(2) + 8\varepsilon^3\,\zeta(3) + 16\varepsilon^4\,\zeta(4)\Big)}{2\,(1+\varepsilon)\,\varepsilon^2\,\tilde{s}} \tag{2.88}$$

which can readily be expanded in ε.

Next I turn to the remaining finite integral $I^{D_2}_{x_2} - I^{D_2,s}_{x_2}$. I expand $I^{D_2}_{x_2} - I^{D_2,s}_{x_2}$ up to second order in ε and integrate over x_2 after the expansion.

Next consider the second integrand $II^{D_2}_{x_1x_2}$ (2.84). The term in the numerator in square brackets raised to the power $(-1-\varepsilon)$ changes sign on the integration path. It means that the corresponding integral has an imaginary contribution. One can rewrite $II^{D_2}_{x_1x_2}$ as follows:

$$II^{D_2}_{x_1x_2} = \frac{x_1^{-\varepsilon}(\tilde{s}x_2)^{-1-\varepsilon}\,\Big(-1 - i\delta + x_1(\frac{(1-x_2)^2}{\tilde{s}x_2} + 1)\Big)^{-1-\varepsilon}}{(1+\varepsilon)\tilde{s}\,\Big(1 - x_1(1 - \frac{\tilde{t}}{\tilde{s}}(1-x_2))\Big)}, \tag{2.89}$$

The integration of $II^{D_2}_{x_1x_2}$ over x_1 is more difficult because of the additional term $x_1^{-\varepsilon}$. I proceed by expanding $x_1^{-\varepsilon}$ as $(1 - \varepsilon\ln x_1 + \frac{\varepsilon^2}{2}\ln^2 x_1 + ...)$. One can see that only the first term of this expansion gives rise to a divergence in the subsequent integration. As one needs to find a subtraction term for this term I will treat it separately. For the remaining terms I do an overall ε-expansion of (2.89) and then perform the remaining integrations. I therefore substitute $x_1^{-\varepsilon}$ by 1. The integral looks simpler (I denote it by $II^0_{x_1x_2}$). Integration over x_1 yields

$$II^0_{x_2} = \frac{(\tilde{s}x_2)^{-1-\varepsilon}x_2(-1-i\delta)^{-\varepsilon}\,{}_2F_1(1,-\varepsilon,1-\varepsilon,-\frac{x_2(\tilde{s}+\tilde{t}(-1+x_2))}{(1-x_2)(1+(\tilde{t}-1)x_2)})}{\varepsilon(1+\varepsilon)(1-x_2)(1+(\tilde{t}-1)x_2)} - \frac{(\tilde{s}x_2)^{-1-\varepsilon}x_2\,(1-x_2)^{-2\varepsilon}\,{}_2F_1(1,-\varepsilon,1-\varepsilon,\frac{(1-x_2)(\tilde{s}+\tilde{t}(-1+x_2))}{\tilde{s}(1+(-1+\tilde{t})x_2)})}{\varepsilon(1+\varepsilon)(1-x_2)(1+(\tilde{t}-1)x_2)(\tilde{s}x_2)^{-\varepsilon}}. \tag{2.90}$$

Note that I have omitted the imaginary shifts $i\delta$ in the arguments of the hypergeometric functions ${}_2F_1$, as the branch cuts of ${}_2F_1$ are never crossed in the physical region. If one would directly integrate the above expression one would have a divergence at $x_2 \to 1$. One must therefore define a subtraction term. If one uses $II^0_{x_2}$ in the present form the definition is rather difficult: as $x_2 \to 1$ the argument of the first function ${}_2F_1$ goes to infinity. To circumvent this problem one can use one of the relations between hypergeometric functions to transform the argument of the function. As a result one pulls out the divergent term as an overall factor multiplying the hypergeometric function with a transformed argument. The whole expression can be rewritten as

$$II^0_{x_2} = -\frac{(1-x_2)^{-1-2\varepsilon}\,{}_2F_1(1,-\varepsilon,1-\varepsilon,\frac{\left(\tilde{s}+\tilde{t}\,(-1+x_2)\right)(1-x_2)}{\tilde{s}\left(1+\left(-1+\tilde{t}\right)x_2\right)})}{\tilde{s}\,\left(1+\left(-1+\tilde{t}\right)\,x_2\right)\,(1+\varepsilon)\,\varepsilon} + \tag{2.91}$$
$$\frac{(\tilde{s}(1-x_2))^{-1-\varepsilon}\,x_2^{-\varepsilon}\left(1+\left(-1+\tilde{t}\right)x_2\right)^{-1-\varepsilon}(-1-i\,\delta)^{-\varepsilon}\,{}_2F_1(-\varepsilon,-\varepsilon,1-\varepsilon,\frac{\left(\tilde{s}-\tilde{t}\,(1-x_2)\right)x_2}{(1-x_2)^2+\tilde{s}\,x_2})}{\left((1-x_2)^2+\tilde{s}\,x_2\right)^{-\varepsilon}(1+\varepsilon)\,\varepsilon}.$$

Now all the poles arise from the factors $(1-x_2)^{-1-2\varepsilon}$ and $(1-x_2)^{-1-\varepsilon}$, and one can derive the necessary subtraction term following the above procedure. I briefly mention that when $x_2 = 1$ the first hypergeometric function (in the first line of Eq. (2.91)) takes the value ${}_2F_1 = 1$. The second hypergeometric function takes the value $-\varepsilon\pi/\sin(-\varepsilon\pi)$ which, in turn, can be expanded to order ε^4 as $1+\varepsilon^2\zeta(2)+7\varepsilon^4\zeta(4)/4$. Thus, the subtraction term reads

$$II^{0,s}_{x_2} = -\frac{(1-x_2)^{-1-2\varepsilon}}{(1+\varepsilon)\varepsilon\tilde{s}\tilde{t}} + \frac{\left(1+\varepsilon^2\zeta(2)+\frac{7\varepsilon^4\zeta(4)}{4}\right)\tilde{t}^{-1-\varepsilon}\,(1-x_2)^{-1-\varepsilon}\,(-1-i\delta)^{-\varepsilon}}{(1+\varepsilon)\,\varepsilon\,\tilde{s}}. \tag{2.92}$$

Integrating this subtraction term I arrive at

$$II^{0,s} = \frac{1}{2(1+\varepsilon)\varepsilon^2\tilde{s}\tilde{t}} - \frac{\left(1+\varepsilon^2\zeta(2)+\frac{7\varepsilon^4\zeta(4)}{4}\right)\tilde{t}^{-1-\varepsilon}\,(-1-i\delta)^{-\varepsilon}}{(1+\varepsilon)\varepsilon^2\tilde{s}}, \tag{2.93}$$

which can finally be expanded up to ε^2.

Now that I have found a suitable subtraction term I can proceed with the remaining terms. I subtract from $II^0_{x_2}$ (Eq. (2.91)) the subtraction term $II^{0,s}_{x_2}$ (Eq. (2.92)). Since the result is convergent with regard to the integration over x_2, one can expand the result in terms of ε before integration which greatly simplifies the problem. One then does the last integration. The difference $II^0_{x_2} - II^{0,s}_{x_2}$ must be expanded up to third order in ε. The reason for this is that one has already one pole $\sim 1/\varepsilon$ after the x_1-integration. Therefore, in order to get results up to second order the hypergeometric functions have to be expanded to third order. The expansion for one of the hypergeometric functions is done using (2.86). For the ε-expansion of the second hypergeometric function one gets

$$\begin{aligned}{}_2F_1(-\varepsilon,-\varepsilon,1-\varepsilon,z) &= 1+\varepsilon^2\,\mathrm{Li}_2(z) \qquad (2.94)\\ &-\varepsilon^3\left(\frac{1}{2}\ln^2(1-z)\ln z+\ln(1-z)\mathrm{Li}_2(1-z)-\mathrm{Li}_3(1-z)-Li_3(z)+\zeta(3)\right).\end{aligned}$$

Using these results for the ε-expansions I expand $II^0_{x_2} - II^{0,s}_{x_2}$ up to ε^2 and integrate the resulting expression. Finally, carefully collecting all the relevant pieces, one arrives at the final result for the second four-point function. In order to reduce the length of the final result for D_2 I introduce four more abbreviations. I write

$$D \equiv m^2 s - tu, \qquad l_D \equiv \ln\frac{-D}{m^4}, \qquad l_u \equiv \ln\frac{-u}{m^2}, \qquad l_U \equiv \ln\frac{-U}{m^2} \qquad (2.95)$$

The result for the four-point box diagram D_2 reads:

$$\mathrm{Re}\,D_2^{(-2)} = 2/(st), \qquad \mathrm{Re}\,D_2^{(-1)} = -[l_s+2l_t]/(st), \qquad (2.96)$$

$$\begin{aligned}
\mathrm{Re}\,D_2^{(0)} &= [2\,l_s\,l_t - 5\,\zeta(2)]/(st),\\
\mathrm{Re}\,D_2^{(1)} &= \frac{1}{s\,t}\Bigg[\frac{1}{2}\,l_s^3+3\,l_t^3+\frac{l_x^3}{12}-l_t^2\,(4\,l_T+3\,l_x+4\,l_{z3}-l_{z4})+\frac{1}{2}\,l_x^2\,l_{z4}-l_x\,l_{z4}^2+\frac{2}{3}\,l_{z4}^3-\\
&\frac{1}{4}l_s^2\,(6l_t+3l_x+4l_{z3}+2l_{z4})-l_s\left(2l_t^2-\frac{1}{2}l_x^2-l_xl_{z3}-2l_t\,(l_x+2l_{z3}+l_{z4})\right)-\\
&l_t\left(l_T^2+\frac{1}{2}l_x^2+2l_xl_{z3}-2l_xl_{z4}-2l_T\,(l_x+l_{z3}+l_{z4})+l_{z3}^2+3l_{z4}^2\right)-(l_s-2l_t+\\
&l_x-4l_{z4})\,\zeta(2)+\zeta(3)+2\left(\mathrm{Li}_2\left(\frac{m^2}{z_5}\right)+\mathrm{Li}_2\left(\frac{-t(1-\beta)}{2\,m^2}\right)\right)(l_s-2\,l_t)-\\
&4\mathrm{Li}_2\left(\frac{T}{m^2}\right)l_t-2\mathrm{Li}_2\left(\frac{m^2x}{-T}\right)l_t-2\mathrm{Li}_2\left(\frac{T}{z_3}\right)l_t-4\mathrm{Li}_3\left(\frac{m^2}{-t}\right)+2\mathrm{Li}_3(-x)-\\
&2\mathrm{Li}_3\left(\frac{z_3}{t}\right)-2\mathrm{Li}_3\left(\frac{z_4}{t}\right)-4\mathrm{Li}_3\left(\frac{m^2}{z_5}\right)-4\mathrm{Li}_3\left(\frac{z_6}{m^2}\right)+2\mathrm{Li}_3\left(\frac{m^2(1-\beta)}{2\,z_5}\right)-\\
&2\mathrm{Li}_3\left(\frac{-t(1-\beta)}{2\,z_5}\right)+2\mathrm{Li}_3\left(\frac{2\,z_6}{m^2(1+\beta)}\right)-2\mathrm{Li}_3\left(\frac{2\,z_6}{-t(1+\beta)}\right)\Bigg],\\
\mathrm{Re}\,D_2^{(2)} &= \frac{1}{s\,t}\Bigg[-\frac{l_s^4}{48}-\frac{17}{24}l_s^3l_t-\frac{9}{2}l_s^2l_t^2-3l_sl_t^3+\frac{13}{6}l_s^3l_T+\frac{5}{2}l_s^2l_tl_T+\frac{3}{2}l_t^3l_T-\frac{1}{2}l_s^2l_T^2-\frac{7}{2}l_t^2l_T^2
\end{aligned}$$

$$-3l_tl_T^3-\frac{25}{24}l_t^4-\frac{3}{4}l_s^3l_u-\frac{13}{4}l_s^2l_tl_u+7l_sl_t^2l_u-3l_t^3l_u-\frac{5}{4}l_s^2l_Tl_u+2l_sl_tl_Tl_u-2l_t^2l_Tl_u+\frac{1}{2}l_s^2l_u^2-\frac{3}{2}l_sl_tl_u^2+l_t^2l_u^2-\frac{1}{2}l_sl_Tl_u^2+l_tl_Tl_u^2+\frac{1}{3}l_sl_u^3-\frac{1}{3}l_tl_u^3-\frac{1}{4}l_s^3l_x-\frac{23}{8}l_s^2l_tl_x+3l_sl_t^2l_x-\frac{7}{3}l_t^3l_x-\frac{7}{4}l_s^2l_Tl_x-3l_sl_tl_Tl_x-3l_t^2l_Tl_x+3l_tl_T^2l_x-\frac{1}{2}l_s^2l_ul_x+\frac{1}{2}l_sl_tl_ul_x+\frac{5}{2}l_sl_Tl_ul_x+\frac{1}{2}l_sl_u^2l_x-\frac{1}{2}l_tl_u^2l_x-\frac{1}{2}l_Tl_u^2l_x+\frac{1}{4}l_s^2l_x^2+\frac{19}{8}l_sl_tl_x^2+\frac{3}{2}l_t^2l_x^2-4l_sl_Tl_x^2-\frac{9}{2}l_tl_Tl_x^2-\frac{1}{2}l_T^2l_x^2-\frac{3}{4}l_sl_ul_x^2+\frac{3}{4}l_tl_ul_x^2-\frac{1}{4}l_Tl_ul_x^2+\frac{5}{6}l_sl_x^3+\frac{53}{24}l_tl_x^3-\frac{37}{12}l_Tl_x^3-\frac{13}{16}l_x^4-\frac{29}{24}l_s^3l_{z3}+\frac{19}{4}l_s^2l_tl_{z3}+11l_sl_t^2l_{z3}+\frac{10}{3}l_t^3l_{z3}-\frac{13}{4}l_s^2l_Tl_{z3}-10l_sl_tl_Tl_{z3}-9l_t^2l_Tl_{z3}+7l_tl_T^2l_{z3}+2l_s^2l_ul_{z3}-2l_sl_tl_ul_{z3}+\frac{5}{8}l_s^2l_xl_{z3}+\frac{11}{2}l_sl_tl_xl_{z3}+l_t^2l_xl_{z3}-\frac{5}{2}l_sl_Tl_xl_{z3}-10l_tl_Tl_xl_{z3}+\frac{19}{8}l_sl_x^2l_{z3}+\frac{7}{4}l_tl_x^2l_{z3}-\frac{5}{4}l_Tl_x^2l_{z3}-\frac{19}{24}l_x^3l_{z3}-\frac{5}{2}l_sl_tl_{z3}^2-\frac{5}{2}l_t^2l_{z3}^2+\frac{7}{2}l_sl_Tl_{z3}^2+l_tl_Tl_{z3}^2-l_sl_ul_{z3}^2+l_tl_ul_{z3}^2-\frac{1}{2}l_sl_xl_{z3}^2+\frac{7}{2}l_tl_xl_{z3}^2+\frac{3}{2}l_Tl_xl_{z3}^2-\frac{1}{2}l_x^2l_{z3}^2+\frac{1}{3}l_sl_{z3}^3+l_tl_{z3}^3-l_Tl_{z3}^3-\frac{2}{3}l_xl_{z3}^3-l_s^3l_{z4}+5l_s^2l_tl_{z4}+7l_sl_t^2l_{z4}+\frac{16}{3}l_t^3l_{z4}-\frac{5}{2}l_s^2l_Tl_{z4}-4l_sl_tl_Tl_{z4}-2l_t^2l_Tl_{z4}+8l_tl_T^2l_{z4}+3l_s^2l_ul_{z4}-3l_sl_tl_ul_{z4}-l_sl_Tl_ul_{z4}+l_sl_u^2l_{z4}-l_tl_u^2l_{z4}-l_Tl_u^2l_{z4}+4l_s^2l_xl_{z4}-3l_sl_tl_xl_{z4}+5l_t^2l_xl_{z4}+4l_tl_Tl_xl_{z4}-l_sl_ul_xl_{z4}+l_tl_ul_xl_{z4}-3l_Tl_ul_xl_{z4}+\frac{3}{2}l_sl_x^2l_{z4}+l_tl_x^2l_{z4}+\frac{5}{2}l_Tl_x^2l_{z4}+\frac{1}{6}l_x^3l_{z4}+2l_s^2l_{z3}l_{z4}-14l_sl_tl_{z3}l_{z4}-4l_t^2l_{z3}l_{z4}+6l_sl_Tl_{z3}l_{z4}+6l_tl_Tl_{z3}l_{z4}-2l_sl_ul_{z3}l_{z4}+2l_tl_ul_{z3}l_{z4}-2l_sl_xl_{z3}l_{z4}-2l_tl_xl_{z3}l_{z4}+2l_Tl_xl_{z3}l_{z4}-4l_x^2l_{z3}l_{z4}+3l_tl_{z3}^2l_{z4}-2l_Tl_{z3}^2l_{z4}-l_s^2l_{z4}^2-2l_sl_tl_{z4}^2-\frac{9}{2}l_t^2l_{z4}^2+2l_sl_Tl_{z4}^2-3l_tl_Tl_{z4}^2-2l_sl_ul_{z4}^2+2l_tl_ul_{z4}^2+l_Tl_ul_{z4}^2-l_sl_xl_{z4}^2-2l_tl_xl_{z4}^2+l_Tl_xl_{z4}^2-l_x^2l_{z4}^2+l_sl_{z3}l_{z4}^2+2l_tl_{z3}l_{z4}^2-2l_Tl_{z3}l_{z4}^2+l_xl_{z3}l_{z4}^2+\frac{1}{3}l_sl_{z4}^3+\frac{5}{3}l_tl_{z4}^3-\frac{2}{3}l_Tl_{z4}^3-\frac{1}{3}l_xl_{z4}^3-2l_s^3l_\beta-l_s^2l_tl_\beta-l_sl_t^2l_\beta-l_s^2l_Tl_\beta-\frac{3}{2}l_s^2l_xl_\beta-l_t^2l_xl_\beta-2l_sl_Tl_xl_\beta+l_sl_x^2l_\beta+l_tl_x^2l_\beta-l_Tl_x^2l_\beta-\frac{1}{2}l_x^3l_\beta+2l_s^2l_{z3}l_\beta+2l_sl_tl_{z3}l_\beta+2l_sl_xl_{z3}l_\beta+2l_tl_xl_{z3}l_\beta+2l_s^2l_{z4}l_\beta+2l_sl_xl_{z4}l_\beta-2l_sl_{z3}l_{z4}l_\beta-2l_xl_{z3}l_{z4}l_\beta-2l_s^2l_\beta^2-l_sl_tl_\beta^2-2l_sl_xl_\beta^2-l_tl_xl_\beta^2+l_sl_{z3}l_\beta^2+l_xl_{z3}l_\beta^2+l_sl_{z4}l_\beta^2+l_xl_{z4}l_\beta^2-\frac{2}{3}l_sl_\beta^3+\frac{2}{3}l_Tl_\beta^3-\frac{5}{3}l_xl_\beta^3-\frac{2}{3}l_{z3}l_\beta^3+\Big(l_s^2-\frac{19}{2}l_t^2-\frac{3}{2}l_T^2+4l_Tl_u-4l_Tl_x+5l_x^2-6l_Tl_{z3}-2l_xl_{z3}-6l_{z3}^2-2l_Tl_{z4}+2l_xl_{z4}-12l_{z3}l_{z4}-7l_{z4}^2+l_t(9l_T+21l_x+16l_{z3}+6l_{z4})+4l_Tl_\beta-4l_xl_\beta-4l_{z3}l_\beta+l_s(-11l_t-4l_T-4l_x+2l_{z3}+10l_{z4}+2l_\beta)\Big)\zeta(2)+(7l_s-8l_t+4l_T-3l_x+2l_{z3}+4l_{z4})\zeta(3)-\frac{41}{2}\zeta(4)+4\mathrm{Li}_2\left(\frac{-u}{z_4}\right)\times$$

$$l_T l_x - 2\mathrm{Li}_2(x)\left(l_s^2 - l_x l_t + l_x l_{z3} - l_s l_t + 2l_s l_T - l_s l_{z3}\right) + 2\mathrm{Li}_2^2(-x) +$$
$$\mathrm{Li}_2\left(\frac{z_3}{z_4}\right)\left(-l_s^2 + 2l_s(l_t - l_x) + 2l_x l_t + l_x^2\right) + \mathrm{Li}_2\left(\frac{m^2}{z_5}\right)\left(-2l_s^2 - 4l_t l_T - 2l_x l_T +\right.$$
$$\left. l_x^2 + 2l_s\left(2l_t - l_T - l_x\right)\right) + \mathrm{Li}_2\left(\frac{T}{m^2}\right)\left(-2\mathrm{Li}_2\left(-\frac{t}{s}\right) - 2\mathrm{Li}_2(-x) - \frac{11}{2}l_t^2 - 2l_T l_u\right.$$
$$\left. -2l_x^2 - l_{z3}^2 - 2l_{z3}l_{z4} - l_{z4}^2 - l_t l_T + 6l_t(l_{z3} + l_{z4}) - 14\zeta(2)\right) + \mathrm{Li}_2\left(\frac{m^2 s}{D}\right)\left(3l_s^2 -\right.$$
$$\left. l_t^2 - 2l_s(3l_t - l_T + l_u) + 2l_t(3l_T + l_u) + 2(l_T - l_x)l_x - 8\zeta(2)\right) +$$
$$\mathrm{Li}_2\left(-\frac{t}{s}\right)\left(3l_s^2 - 2l_s(3l_t - 2l_T + l_u) - 2\left(l_t^2 + 2l_T^2 - l_t(4l_T + l_u) + l_x^2\right) - 8\zeta(2)\right)$$
$$+\mathrm{Li}_2\left(-\frac{D}{m^2 t}\right)\left(-2\mathrm{Li}_2\left(\frac{T}{m^2}\right) - 4l_s l_t - 2l_t^2 + 6l_t l_T - l_x^2 - 4\zeta(2)\right) -$$
$$\mathrm{Li}_2\left(\frac{T}{z_3}\right)\left(2\mathrm{Li}_2(-x) + l_s^2 - 4l_s l_t + 2l_t(l_t + 3l_T - 2l_x - l_{z3} - 2l_{z4}) + 2\zeta(2)\right) +$$
$$\mathrm{Li}_2\left(\frac{m^2 x}{-T}\right)\left(-l_x^2 + 2l_t(-l_t - 3l_T - 3l_x + l_{z3} + 2l_{z4}) - 2\zeta(2)\right) +$$
$$\mathrm{Li}_2\left(\frac{-t(1-\beta)}{2\,m^2}\right)\left(-2\mathrm{Li}_2(-x) - 2l_s^2 - 4l_t l_T - 2l_x l_T + l_x^2 + 2l_s(2l_t - l_T + l_x) +\right.$$
$$\left. 8\zeta(2)\right) + \mathrm{Li}_2(-x)\left(-\frac{9}{4}l_s^2 - 8l_x l_T + \frac{7}{4}l_x^2 - 2l_x l_{z3} + 9l_t l_x - 2l_t l_{z3} + 5l_s l_t - 6l_s l_T +\right.$$
$$\left.\frac{7}{2}l_s l_x + 2l_s l_{z3} + 12\zeta(2)\right) + \mathrm{Li}_3\left(\frac{m^2}{-t}\right)(-6l_s - l_t + l_T + 2l_x - 4l_{z4}) +$$
$$\mathrm{Li}_3\left(\frac{T}{m^2}\right)(-4l_s + 8l_t + 3l_T + 2l_x - 6l_{z3} - 10l_{z4}) + 2\mathrm{Li}_3\left(\frac{z_3}{t}\right)(l_t + l_T - 4l_x -$$
$$l_{z3} - 2l_{z4}) + 2\left(\mathrm{Li}_3\left(\frac{m^2(1-\beta)}{2\,z_5}\right) - \mathrm{Li}_3\left(\frac{-t(1-\beta)}{2\,z_5}\right)\right)(l_t + 3l_T + 3l_x - l_{z3} -$$
$$2l_{z4}) + 2\mathrm{Li}_3\left(\frac{z_4}{t}\right)(l_t + l_T + 5l_x - l_{z3} - 2l_{z4}) + 2\mathrm{Li}_3\left(-\frac{u}{t}\right)(3l_s - 3l_t - 2l_T) +$$
$$2\left(\mathrm{Li}_3\left(-\frac{m^2 t}{D}\right) - \mathrm{Li}_3\left(\frac{t^2}{D}\right)\right)(2l_s + l_t - 3l_T) + 2\left(\mathrm{Li}_3\left(-\frac{m^2 u}{s z_5}\right) - 2\mathrm{Li}_3\left(\frac{z_3}{-u}\right)\right.$$
$$\left. +\mathrm{Li}_3\left(-\frac{s z_6}{m^2 u}\right) - 2\mathrm{Li}_3\left(\frac{-u}{z_4}\right)\right)(l_s - l_t - l_T) - 2\mathrm{Li}_3\left(-\frac{D}{tu}\right)l_T - 2\left(\mathrm{Li}_3\left(\frac{m^2}{z_5}\right)\right.$$
$$\left. +\mathrm{Li}_3\left(\frac{z_6}{m^2}\right)\right)(l_s - l_t + l_T) - 2\mathrm{Li}_3\left(-\frac{t}{s}\right)(l_t - l_T) - 2\mathrm{Li}_3\left(\frac{m^2 s}{D}\right)(l_t - 2l_T) +$$
$$2\mathrm{Li}_3(x)(3l_s - l_x) + 2\mathrm{Li}_3\left(\frac{s x z_5}{m^2\,t}\right)(l_s - l_x) - 2\mathrm{Li}_3\left(\frac{z_4(1-\beta)}{-2\,m^2}\right)(2l_s - l_t - l_T +$$
$$l_x) + 2\mathrm{Li}_3\left(\frac{z_3}{z_4}\right)l_x + 2\mathrm{Li}_3\left(\frac{z_6}{z_5}\right)l_s + 2\left(\mathrm{Li}_3\left(\frac{t(1-\beta)}{2\,z_3}\right) + \mathrm{Li}_3\left(\frac{z_3}{s\beta}\right) + \mathrm{Li}_3\left(\frac{z_5}{t\beta}\right)\right)$$

$$\times(l_s+l_x)+2\mathrm{Li}_3\left(\frac{s(1-\beta)}{-2\,z_3}\right)(-2l_s+l_t+l_T+l_x)-2\mathrm{Li}_3\left(\frac{1-\beta}{-2\beta}\right)(l_s+4l_T-$$
$$5l_x-4l_{z3})+2\mathrm{Li}_3\left(\frac{-x^2}{1-x^2}\right)(l_s+2l_T-2l_x-2l_{z3})+6\mathrm{Li}_3\left(\frac{1-\beta}{2}\right)(l_T-l_x-$$
$$l_{z3})-2\mathrm{Li}_3\left(\frac{2\,z_6}{m^2(1+\beta)}\right)(2l_s-l_t-3l_T+2l_x+l_{z3}+2l_{z4})+2\mathrm{Li}_3\left(\frac{-2\,z_6}{t(1+\beta)}\right)$$
$$\times(2l_s-l_t-3l_T+2l_x+l_{z3}+2l_{z4})+2\mathrm{Li}_3(-x)(8l_s+4l_t-3l_T-7l_x-2l_{z3}+$$
$$4l_{z4})-3\mathrm{Li}_4\left(\frac{m^2}{-t}\right)-6\mathrm{Li}_4\left(-\frac{t}{s}\right)+3\mathrm{Li}_4\left(\frac{T}{m^2}\right)+16\mathrm{Li}_4\left(\frac{T}{t}\right)-12\mathrm{Li}_4\left(\frac{1-\beta}{2}\right)$$
$$-12\mathrm{Li}_4\left(\frac{1+\beta}{2}\right)+L_{-++}\left(1,0,0,\frac{m^2}{-T}\right)+\frac{9}{2}L_{-++}\left(1,0,0,-\frac{t}{T}\right)-$$
$$L_{-++}\left(1,0,0,\frac{-2}{1-\beta}\right)-L_{-++}\left(1,0,0,\frac{-2}{1+\beta}\right)-5L_{-++}\left(1,0,\frac{m^2}{-T},-1\right)-$$
$$2L_{-++}\left(1,0,\frac{m^2}{-T},\frac{1}{x}\right)-2L_{-++}\left(1,0,\frac{m^2}{-T},x\right)-\frac{3}{2}L_{-++}\left(1,\frac{m^2}{-T},\frac{m^2}{-T},0\right)-$$
$$4L_{-++}\left(1,\frac{m^2}{-T},\frac{m^2}{-T},\frac{u}{t}\right)+2L_{-++}\left(1,\frac{m^2}{-T},\frac{m^2}{-T},\frac{1}{x}\right)+2L_{-++}\left(1,\frac{m^2}{-T},\frac{m^2}{-T},x\right)$$
$$-3L_{-++}\left(\frac{t}{T},0,0,\frac{s}{t}\right)+2L_{+}\left(0,0,\frac{T}{m^2},\frac{1}{x}\right)+2L_{+}\left(0,0,\frac{T}{m^2},x\right)-$$
$$2L_{+}\left(0,0,\frac{T}{t},\frac{s}{t}\right)+2L_{+}\left(0,1,-1,\frac{1}{x}\right)+2L_{+}(0,1,-1,x)+4L_{+}(0,1,\frac{t}{s},\frac{-2}{1-\beta})$$
$$+4L_{+}\left(0,1,\frac{t}{s},\frac{-2}{1+\beta}\right)+8L_{+}\left(0,1,\frac{1+\beta}{-2},-\frac{t}{T}\right)+8L_{+}\left(0,1,\frac{1-\beta}{-2},-\frac{t}{T}\right)+$$
$$2L_{+}\left(0,\frac{m^2}{-t},\frac{T}{t},\frac{m^2}{-T}\right)-2L_{+}\left(0,\frac{m^2\,t}{-D},\frac{tT}{D},\frac{m^2}{-T}\right)+2L_{+}\left(0,\frac{m^2\,t}{-D},\frac{tT}{D},\frac{1}{x}\right)+$$
$$2L_{+}\left(0,\frac{m^2\,t}{-D},\frac{t\,T}{D},x\right)+2L_{+}\left(0,-\frac{t}{s},\frac{t}{s},\frac{m^2}{-T}\right)-2L_{+}\left(0,\frac{t^2}{D},\frac{t\,T}{-D},-1\right)-$$
$$4L_{+}\left(0,\frac{T}{m^2},\frac{-T}{m^2},-1\right)+4L_{+}\left(0,\frac{T}{m^2},\frac{-T}{m^2},\frac{-2}{1-\beta}\right)+4L_{+}(0,\frac{T}{m^2},\frac{-T}{m^2},\frac{-2}{1+\beta}\right)$$
$$-6L_{+}\left(0,\frac{T}{t},-\frac{T}{t},-1\right)+4L_{+}\left(0,\frac{T}{t},-\frac{T}{t},\frac{1}{x}\right)+4L_{+}\left(0,\frac{T}{t},-\frac{T}{t},x\right)-$$
$$2L_{+}\left(0,-\frac{u}{s},-\frac{t}{s},-1\right)+2L_{+}\left(0,-\frac{u}{s},-\frac{t}{s},\frac{1}{x}\right)+2L_{+}\left(0,-\frac{u}{s},-\frac{t}{s},x\right)-$$
$$4L_{+}\left(0,\frac{T}{z_6},\frac{T(1+\beta)}{-2\,z_6},\frac{-2}{1-\beta}\right)-4L_{+}\left(0,\frac{T}{z_6},\frac{T(1+\beta)}{-2\,z_6},\frac{-2}{1+\beta}\right)+$$
$$2L_{+}\left(0,\frac{1-\beta}{2},\frac{1+\beta}{2},\frac{1}{x}\right)+2L_{+}\left(0,\frac{1-\beta}{2},\frac{1+\beta}{2},x\right)+$$

$$2L_+\left(0,\frac{m^2(1-\beta)}{2\,z_5},\frac{T(1-\beta)}{-2\,z_5},\frac{1}{x}\right)+2L_+\left(0,\frac{m^2(1-\beta)}{2\,z_5},\frac{T(1-\beta)}{-2\,z_5},x\right)+$$
$$4L_+\left(0,\frac{T(1-\beta)}{2\,z_6},\frac{T(1+\beta)}{2\,z_6},-1\right)-2L_+\left(0,\frac{T(1-\beta)}{2\,z_6},\frac{T(1+\beta)}{2\,z_6},x\right)+$$
$$2L_+\left(0,\frac{1+\beta}{2},\frac{1-\beta}{2},\frac{1}{x}\right)+2L_+\left(0,\frac{1+\beta}{2},\frac{1-\beta}{2},x\right)-2L_+\left(\frac{m^2}{-T},0,-x,x\right)$$
$$-4L_+\left(\frac{m^2}{-T},1,-1,\frac{1}{x}\right)-2L_+\left(\frac{m^2}{-T},\frac{T}{t},-\frac{T}{t},-1\right)-2L_+\left(\frac{m^2}{-T},\frac{T}{t},-\frac{T}{t},\frac{u}{t}\right)-$$
$$4L_+\left(\frac{m^2}{-T},\frac{1-\beta}{2},\frac{1+\beta}{2},\frac{u}{t}\right)-4L_+\left(\frac{m^2}{-T},\frac{1-\beta}{2},\frac{1+\beta}{2},\frac{1}{x}\right)-$$
$$4L_+\left(\frac{m^2}{-T},\frac{1+\beta}{2},\frac{1-\beta}{2},\frac{u}{t}\right)-4L_+\left(\frac{m^2}{-T},\frac{1+\beta}{2},\frac{1-\beta}{2},\frac{1}{x}\right)-$$
$$4L_+\left(\frac{1}{x},\frac{1-\beta}{2},\frac{1-\beta}{-2},-1\right)-4L_+\left(x,\frac{1+\beta}{2},\frac{1+\beta}{-2},-1\right)-$$
$$L_{+++}\left(0,0,\frac{m^2}{-T},-1\right)+2L_{+++}\left(0,0,\frac{m^2}{-T},\frac{1}{x}\right)+2L_{+++}\left(0,0,\frac{m^2}{-T},x\right)-$$
$$2L_{+++}\left(0,\frac{m^2}{-T},\frac{u}{t},-1\right)-2L_{+++}\left(0,\frac{m^2}{-T},\frac{u}{t},\frac{m^2}{-T}\right)+2L_{+++}\left(0,\frac{m^2}{-T},\frac{u}{t},\frac{1}{x}\right)+$$
$$2L_{+++}\left(0,\frac{m^2}{-T},\frac{u}{t},x\right)+2L_{+++}\left(0,\frac{m^2}{-T},\frac{1}{x},\frac{1}{x}\right)+L_{+++}\left(0,\frac{1}{x},\frac{1}{x},\frac{m^2}{-T}\right)-$$
$$L_{+++}\left(\frac{m^2}{-T},\frac{m^2}{-T},0,\frac{1}{x}\right)-L_{+++}\left(\frac{m^2}{-T},\frac{m^2}{-T},0,x\right)-L_{+++}\left(\frac{1}{x},\frac{1}{x},0,\frac{m^2}{-T}\right)\Bigg];$$

$$\begin{aligned}
\mathrm{Im}\,D_2^{(-1)} &= \pi/(st), \qquad\qquad \mathrm{Im}\,D_2^{(0)} = -2\pi l_t/(st), \qquad (2.97)\\
\mathrm{Im}\,D_2^{(1)} &= \frac{\pi}{st}\Bigg[-\frac{3}{4}l_s^2+2l_t^2-\frac{3}{4}l_x^2-l_xl_{z3}-l_tl_x-2l_tl_{z3}+l_xl_{z4}-l_{z4}^2+\\
&\quad l_s(l_t+\frac{l_x}{2}+l_{z3}+l_{z4})-2\zeta(2)-2\mathrm{Li}_2\left(\frac{m^2}{z_5}\right)-2\mathrm{Li}_2\left(\frac{t(1-\beta)}{-2\,m^2}\right)\Bigg],\\
\mathrm{Im}\,D_2^{(2)} &= \frac{\pi}{st}\Bigg[\frac{7}{12}l_s^3-\frac{l_x^3}{12}-2l_xl_tl_{z3}-l_x^2l_{z3}+l_{z4}^2-l_s\Big(2l_t^2-\frac{5}{4}l_x^2-2l_t(l_x+l_{z3})-\\
&\quad 2l_x(l_{z3}-l_{z4})-l_{z4}^2\Big)-l_s^2\left(\frac{3}{4}l_x+l_{z3}+l_{z4}\right)+4(l_t+l_x)\zeta(2)+2\mathrm{Li}_2(x)\,l_x+\\
&\quad 2\mathrm{Li}_2\left(\frac{t(1-\beta)}{-2\,m^2}\right)(l_s-l_x)+2\mathrm{Li}_2\left(\frac{z_3}{z_4}\right)l_x+2\mathrm{Li}_2\left(\frac{m^2}{z_5}\right)(l_s+l_x)-2\mathrm{Li}_3(x)-\\
&\quad 4\mathrm{Li}_3\left(\frac{z_3}{t}\right)+2\mathrm{Li}_3\left(\frac{z_3}{z_4}\right)-4\mathrm{Li}_3\left(\frac{z_4}{t}\right)-2\mathrm{Li}_3\left(\frac{z_6}{z_5}\right)\Bigg].
\end{aligned}$$

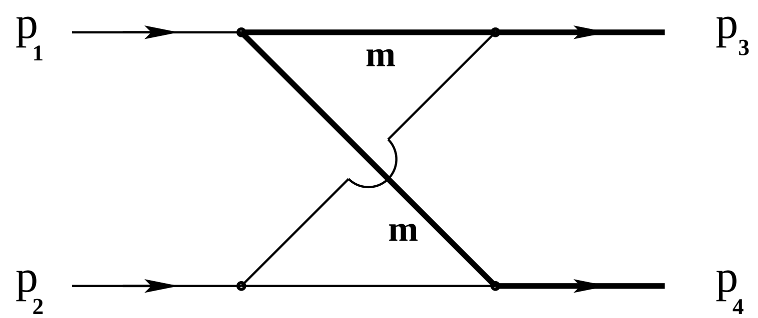

Figure 2.6: Massive box D_3 with two massive propagators. Thick and thin internal lines correspond to massive and massless propagators, respectively. Thick legs represent massive momenta *on-shell*. Thin legs represent massless momenta *on-shell*.

2.3.3 Four-point function with two massive propagators

The diagram corresponding to the third four-point function D_3 with two massive propagators is shown in Fig. 2.6 where I write

$$D_3 \equiv D(-p_2, p_4, -p_1, 0, 0, m, m)\,. \tag{2.98}$$

The kernel (2.69) for D_3 can be written as

$$\tilde{K}_D = ac\tilde{t} + bd\tilde{u} + (c+d)^2, \tag{2.99}$$

where I have introduced the positive-valued dimensionless variable

$$\tilde{u} \equiv -\frac{u}{m^2}, \qquad \tilde{u} \geq 1, \qquad \tilde{s} \geq \tilde{u}. \tag{2.100}$$

For the Feynman parameterization I choose $\{a, b, c, d\}$ as $\{x_1(1-x_2),\ 1-x_1,\ x_1x_2(1-x_3),\ x_1x_2x_3\}$, which gives the following integrand:

$$x_1^2 x_2 \left[x_1^2 x_2^2 + \tilde{t}x_1^2 x_2(1-x_2)(1-x_3) + \tilde{u}x_1(1-x_1)x_2x_3\right]^{-2-\varepsilon}.$$

The above expression never becomes negative. Therefore, the entire result for the box D_3 does not have an imaginary part. One can set $\delta = 0$ in the kernel from the very beginning.

That the box D_3 posesses no imaginary part can be seen in a less technical way by appealing to the Landau-Cutkosky cutting rules. The diagram corresponding to the box D_3 shown in Fig. 2.6 does not admit of any cuts such that the cut lines of the diagram are on their mass shell simultaneously.

As before I obtain two terms after the first integration over x_3. They are

$$I^{D_3}_{x_1x_2} = \frac{x_1^{-1-2\varepsilon}x_2^{-1-\varepsilon}\left[x_2 + \tilde{t}(1-x_2))\right]^{-1-\varepsilon}}{(1+\varepsilon)[\tilde{u}(1-x_1) - \tilde{t}x_1(1-x_2)]}, \tag{2.101}$$

$$II^{D_3}_{x_1x_2} = -\frac{x_1^{-\varepsilon}x_2^{-1-\varepsilon}\left[x_1x_2 + \tilde{u}(1-x_1)\right]^{-1-\varepsilon}}{(1+\varepsilon)[\tilde{u}(1-x_1) - \tilde{t}x_1(1-x_2)]}. \tag{2.102}$$

Note that the denominators of $I^{D_3}_{x_1x_2}$ and $II^{D_3}_{x_1x_2}$ change sign on the integration path, while the numerators stay positive (i.e. the relevant integrals have branch cuts). This can easily be seen by considering the numerators and denominators of the above integrands at two particular values of the variable x_1, for instance at $x_1 = 0$ and $x_1 = 1$. This means that although the whole D_3 box integral does not have an imaginary part, the two terms in (2.101) and (2.102) separately give rise to unphysical, spurious imaginary contributions. Of course, these are artefacts of having split the result into two terms. On the one hand, this somehow complicates things. On the other hand, the cancellation of imaginary contributions in the sum of the two terms (2.101) and (2.102) will serve as a good check for the final result. To control the imaginary contributions I do the following replacement in the denominators in (2.101) and (2.102):

$$\tilde{u}(1-x_1) - \tilde{t}x_1(1-x_2) \to \tilde{u}(1-x_1) - \tau x_1(1-x_2), \qquad \tau \equiv \tilde{t} - i\delta.$$

I start with the x_2-integration of the term $I^{D_3}_{x_1x_2}$ Eq. (2.101):

$$I^{D_3}_{x_2} = -\frac{\left[x_2\left(x_2 + \tilde{t}\,(1-x_2)\right)\right]^{-1-\varepsilon}\ {}_2F_1(1,-2\varepsilon,1-2\varepsilon,\frac{\tilde{u}+\tau-x_2\tau}{\tilde{u}})}{2\tilde{u}\,(1+\varepsilon)\,\varepsilon}. \tag{2.103}$$

The above expression is singular at the lower integration limit $x_2 = 0$ due to the term $x_2^{-1-\varepsilon}$. To find a subtraction term, one follows exactly the procedures defined after Eq. (2.85). The subtraction term reads

$$I^{D_3,s}_{x_2} = \frac{x_2^{-1-\varepsilon}\,\tilde{t}^{-1-\varepsilon}\left[-1-2\varepsilon\,\ln\left(-\frac{\tau}{\tilde{u}}\right)+4\varepsilon^2\mathrm{Li}_2\left(\frac{\tilde{u}+\tau}{\tilde{u}}\right)+8\varepsilon^3\,\mathrm{Li}_3\left(\frac{\tilde{u}+\tau}{\tilde{u}}\right)+16\varepsilon^4\mathrm{Li}_4\left(\frac{\tilde{u}+\tau}{\tilde{u}}\right)\right]}{2\tilde{u}\,(1+\varepsilon)\,\varepsilon}. \tag{2.104}$$

The above subtraction term can easily be integrated over x_2 to obtain

$$I^{D_3,s} = \frac{\tilde{t}^{-1-\varepsilon}\left[1+2\varepsilon\,\ln\left(-\frac{\tau}{\tilde{u}}\right)-4\varepsilon^2\mathrm{Li}_2\left(\frac{\tilde{u}+\tau}{\tilde{u}}\right)-8\varepsilon^3\,\mathrm{Li}_3\left(\frac{\tilde{u}+\tau}{\tilde{u}}\right)-16\varepsilon^4\mathrm{Li}_4\left(\frac{\tilde{u}+\tau}{\tilde{u}}\right)\right]}{2\,(1+\varepsilon)\,\varepsilon^2\,\tilde{u}} \tag{2.105}$$

which can readily be expanded in ε. As the difference $I^{D_3}_{x_2} - I^{D_3,s}_{x_2}$ does not contain any poles, one can expand the difference in a series in ε and perform the analytical integration over the last variable x_2.

To integrate the term (2.102), I split $II^{D_3}_{x_1x_2}$ into two contributions, i.e.

$$\begin{aligned} II^{D_3}_{x_1x_2} &= -\frac{x_2^{-1-\varepsilon}\,[x_1x_2+\tilde{u}(1-x_1)]^{-1-\varepsilon}}{(1+\varepsilon)[\tilde{u}(1-x_1)-\tau x_1(1-x_2)]} \\ &\quad -\frac{(x_1^{-\varepsilon}-1)x_2^{-1-\varepsilon}\,[x_1x_2+\tilde{u}(1-x_1)]^{-1-\varepsilon}}{(1+\varepsilon)[\tilde{u}(1-x_1)-\tau x_1(1-x_2)]}. \end{aligned} \tag{2.106}$$

Then I integrate the first term in (2.106) over x_1 to obtain two hypergeometric functions which are expanded up to ε^4. As was done previously, I then introduce a subtraction term similar to (2.104) which is integrated analytically. The finite difference of the original integral and the subtraction term is then ready to be integrated over the last integration variable.

For the second term in (2.106) I introduce a subtraction term before the last two integrations:

$$II^{D_3,s}_{x_1x_2} = -\frac{(x_1^{-\varepsilon}-1)x_2^{-1-\varepsilon}\,[\tilde{u}(1-x_1)]^{-1-\varepsilon}}{(1+\varepsilon)[\tilde{u}(1-x_1)-\tau x_1]}, \tag{2.107}$$

which is obtained from the second term in (2.106) by the substitution $x_2 = 0$ in all terms except for $x_2^{-1-\varepsilon}$. One first trivially integrates out x_2 in (2.107) and expands the resulting expression in a series of ε. Because of the factor $(x_1^{-\varepsilon}-1)$ this expansion starts at the order ε^0. Thus, the subtraction term is finite and ready for the last integration.

As the difference of the second term in (2.106) and its subtraction term (2.107) does not contain any poles, I expand it up to ε^2 and integrate over x_1. Finally, collecting all the relevant pieces, one performs the last integration. The result for the four-point box diagram D_3 reads:

$$\begin{aligned}
&\mathrm{Re}\,D_3^{(-2)} = 1/(tu), \qquad\qquad \mathrm{Re}\,D_3^{(-1)} = -[l_t+l_u]/(tu),\\
\mathrm{Re}\,D_3^{(0)} &= 2\,[l_tl_u - 2\,\zeta(2)]\,/(tu), \qquad\qquad (2.108)\\
\mathrm{Re}\,D_3^{(1)} &= \frac{1}{t\,u}\Bigg[\frac{l_D^3}{3} - l_D^2l_t - \frac{l_t^3}{3} + 2l_t^2l_u - 2l_tl_Tl_u - 4l_tl_u^2 + l_u^3 + l_s\left(l_u^2-l_t^2\right) + l_D\left(l_t^2+l_u^2\right) -\\
&2\,l_u^2l_U + l_s^2(l_t-l_u) + 2(l_D-3l_s+l_t+3l_u)\zeta(2) - 4\zeta(3) + 2\mathrm{Li}_2\left(-\frac{t}{s}\right)(l_t-l_u) +\\
&2\mathrm{Li}_2\left(\frac{m^4}{D}\right)l_u - 2\mathrm{Li}_2\left(\frac{T}{m^2}\right)l_u - 2\mathrm{Li}_2\left(\frac{U}{m^2}\right)l_u + 2\mathrm{Li}_2\left(-\frac{D}{m^2t}\right)(l_u-l_t) +\\
&2\mathrm{Li}_3\left(\frac{m^4}{D}\right) - 2\mathrm{Li}_3\left(\frac{m^2}{-t}\right) - 2\mathrm{Li}_3\left(-\frac{m^2t}{D}\right) - 2\mathrm{Li}_3\left(\frac{m^2}{-u}\right) - 2\mathrm{Li}_3\left(-\frac{m^2u}{D}\right) +\\
&2\mathrm{Li}_3\left(-\frac{u}{t}\right)\Bigg],\\
\mathrm{Re}\,D_3^{(2)} &= \frac{1}{t\,u}\Bigg[\frac{l_D^4}{3} - l_D^2l_s^2 - \frac{5}{6}\,l_s^4 - \frac{4}{3}\,l_D^3l_t + 4l_Dl_s^2l_t + \frac{1}{6}\,l_s^3l_t + 4l_D^2l_t^2 - 5l_Dl_sl_t^2 - \frac{3}{4}\,l_s^2l_t^2 -\\
&\frac{11}{3}\,l_Dl_t^3 + \frac{3}{2}\,l_sl_t^3 + \frac{17}{6}\,l_t^4 + \frac{1}{3}\,l_D^3l_T - 2l_Dl_s^2l_T + l_s^3l_T + l_D^2l_tl_T + 4l_Dl_sl_tl_T -\\
&2l_s^2l_tl_T - 4l_Dl_t^2l_T + 3l_sl_t^2l_T + l_s^2l_T^2 - 2l_sl_tl_T^2 - \frac{4}{3}\,l_D^3l_u + 2l_D^2l_sl_u - l_Dl_s^2l_u +\\
&\frac{5}{2}\,l_s^3l_u - l_D^2l_tl_u - 2l_Dl_sl_tl_u - 2l_s^2l_tl_u + 3l_Dl_t^2l_u + l_sl_t^2l_u - \frac{13}{6}\,l_t^3l_u - l_D^2l_Tl_u +\\
&2l_Dl_sl_Tl_u - 2l_sl_tl_Tl_u + l_t^2l_Tl_u - l_sl_T^2l_u - \frac{1}{2}\,l_D^2l_u^2 + l_Dl_sl_u^2 - \frac{9}{4}\,l_s^2l_u^2 + 2l_Dl_tl_u^2 +\\
&2l_sl_tl_u^2 - \frac{7}{4}\,l_t^2l_u^2 - l_Dl_Tl_u^2 - l_sl_Tl_u^2 + 5l_tl_Tl_u^2 + \frac{1}{2}\,l_T^2l_u^2 - \frac{4}{3}\,l_Dl_u^3 + \frac{1}{6}\,l_sl_u^3 +\\
&\frac{3}{2}\,l_tl_u^3 - \frac{19}{12}\,l_u^4 + 2l_D^2l_ul_U - 2l_Dl_sl_ul_U + 2l_s^2l_ul_U - 2l_Dl_tl_ul_U + 2l_t^2l_ul_U +
\end{aligned}$$

$$
2l_Dl_Tl_ul_U - 4l_tl_Tl_ul_U - 2l_sl_u^2l_U + 2l_tl_u^2l_U + l_Tl_u^2l_U + \frac{13}{3}\,l_u^3l_U + l_Dl_ul_U^2 +
l_sl_ul_U^2 - 2l_tl_ul_U^2 - l_Tl_ul_U^2 - 2l_u^2l_U^2 - \frac{2}{3}\,l_ul_U^3 + \Big(7l_D^2 - 2l_D(3l_s + 4l_t - l_T + 2l_u) -
\frac{17}{2}\,l_s^2 - \frac{3}{2}\,l_t^2 + 3l_s(7l_t + 6l_u) - 2l_t(2l_T + 7l_u) + l_T^2 - 2l_Tl_u - 7l_u^2 - 2l_ul_U\Big)\,\zeta(2) +
(-3l_s + 21\,l_t - 2l_T - 8l_u + 4l_U)\,\zeta(3) - \frac{57}{4}\,\zeta(4) + \mathrm{Li}_2\left(\frac{U}{m^2}\right)\Big(l_D^2 + 2l_s^2 - 2l_Dl_t +
3l_t^2 - 2l_tl_T + l_T^2 + 2l_tl_u + 2l_Tl_u + 3l_u^2 - 2l_s(l_t + l_u) - 2l_ul_U - 2\zeta(2)\Big) -
\mathrm{Li}_2\left(\frac{m^4}{D}\right)l_u^2 + \mathrm{Li}_2\left(-\frac{t}{s}\right)\left(2\mathrm{Li}_2\left(\frac{T}{m^2}\right) + 2\mathrm{Li}_2\left(\frac{U}{m^2}\right) - 5l_s^2 - \frac{3}{2}\,l_t^2 + 2l_tl_T -
6l_tl_u + \frac{5}{2}\,l_u^2 + 5l_s(l_t + l_u) + 4\zeta(2)\right) + \mathrm{Li}_2\left(-\frac{D}{m^2t}\right)\left(2\mathrm{Li}_2\left(\frac{T}{m^2}\right) + 2\mathrm{Li}_2\left(\frac{U}{m^2}\right) -
2l_s^2 - 2l_Dl_t + 3l_t^2 + 2l_tl_T - 2l_tl_u - 4l_u^2 + 2l_s(l_t + l_u) + 4l_ul_U + 4\zeta(2)\right) +
\mathrm{Li}_2\left(\frac{m^2s}{D}\right)\left(2\mathrm{Li}_2\left(\frac{T}{m^2}\right) + l_t^2 - l_u^2 + 2l_ul_U + 6\zeta(2)\right) + \mathrm{Li}_2\left(\frac{T}{m^2}\right)\left(\mathrm{Li}_2\left(\frac{T}{m^2}\right) +
4\mathrm{Li}_2\left(\frac{U}{m^2}\right) + 2l_D^2 + l_s^2 + 3l_t^2 - 2l_D(l_s + 2l_t) + 2l_sl_T + l_u^2 + 2l_ul_U + 6\zeta(2)\right) +
2\mathrm{Li}_3\left(\frac{m^4}{D}\right)(l_D - l_t + l_u) + 2\left(\mathrm{Li}_3\left(-\frac{m^2t}{Tu}\right) - \mathrm{Li}_3\left(\frac{m^2s}{D}\right)\right)(l_s - l_t - l_u) -
2\mathrm{Li}_3\left(-\frac{m^2t}{D}\right)(l_D - l_s + l_T + l_u) - 2\mathrm{Li}_3\left(\frac{U}{m^2}\right)(l_T - l_u) - \mathrm{Li}_3\left(-\frac{t}{s}\right)(6l_s + l_t -
7l_u) + 2\mathrm{Li}_3\left(\frac{m^2}{-u}\right)(l_D + l_s + 2l_t - 7l_T + 2l_u - 2l_U) - 2\mathrm{Li}_3\left(-\frac{m^2u}{D}\right)(l_D - l_s +
l_T) - 2\left(\mathrm{Li}_3\left(\frac{t^2}{D}\right) + \mathrm{Li}_3\left(\frac{D}{u^2}\right)\right)(l_s - l_t - l_T) - \mathrm{Li}_3\left(-\frac{u}{t}\right)(3l_s + l_t + 2l_T + 2l_u)
-2\mathrm{Li}_3\left(\frac{D}{sU}\right)l_u + 4\mathrm{Li}_3\left(-\frac{D}{tU}\right)l_u + 2\mathrm{Li}_3\left(\frac{m^4}{TU}\right)l_u - 2\mathrm{Li}_3\left(-\frac{t^2}{sT}\right)(l_s - l_t) +
2\mathrm{Li}_3\left(\frac{T}{m^2}\right)(2l_D + l_s - 3l_t - 2l_T) + 2\left(\mathrm{Li}_3\left(\frac{m^2}{-t}\right) + \mathrm{Li}_3\left(\frac{-D}{tu}\right)\right)(l_s - l_t - 2l_T)
+2\mathrm{Li}_4\left(\frac{m^2}{-t}\right) + \mathrm{Li}_4\left(-\frac{t}{s}\right) - 2\mathrm{Li}_4\left(\frac{T}{m^2}\right) + 10\,\mathrm{Li}_4\left(\frac{T}{t}\right) - 8\mathrm{Li}_4\left(\frac{m^2}{-u}\right) +
7\mathrm{Li}_4\left(-\frac{u}{s}\right) - 4\mathrm{Li}_4\left(\frac{U}{m^2}\right) - 4\mathrm{Li}_4\left(\frac{U}{u}\right) + L_{-++}\left(1, \frac{m^2}{-T}, \frac{m^2}{-T}, \frac{-U}{m^2}\right) -
2L_{-++}\left(1, \frac{m^2}{-T}, \frac{u}{t}, 0\right) + 2L_{-++}\left(1, \frac{m^2}{-T}, \frac{u}{t}, \frac{m^2}{-T}\right) - \frac{1}{2}\,L_{-++}\left(1, \frac{t}{u}, \frac{t}{u}, 0\right) -
2L_{-++}\left(1, \frac{m^2}{-T}, \frac{u}{t}, \frac{-U}{m^2}\right) + \frac{1}{2}\,L_{-++}\left(1, \frac{u}{t}, \frac{u}{t}, 0\right) - \frac{5}{2}\,L_{-++}\left(-\frac{s}{t}, 0, 0, -1\right) +
$$

$$
\begin{aligned}
& 3L_{-++}\left(\frac{t}{T},0,0,-1\right)+5L_{-++}\left(\frac{t}{T},0,0,\frac{u}{m^2}\right)+\frac{1}{2}\,L_{-++}\left(-\frac{s}{u},0,0,-1\right)- \\
& 2L_{+}\left(0,0,-\frac{t}{s},-1\right)+2L_{+}\left(0,0,\frac{T}{t},-1\right)+4L_{+}\left(0,0,\frac{T}{t},\frac{u}{m^2}\right)+ \\
& 2L_{+}\left(0,0,\frac{m^2}{-u},\frac{u}{m^2}\right)-2L_{+}\left(0,1,\frac{t}{s},-\frac{t}{T}\right)+2L_{+}\left(0,1,\frac{t}{s},\frac{u}{m^2}\right)+ \\
& 2L_{+}\left(0,\frac{t^2}{D},\frac{tT}{-D},-1\right)-2L_{+}\left(0,\frac{t^2}{D},\frac{tT}{-D},-\frac{t}{T}\right)+2L_{+}\left(0,\frac{t^2}{D},\frac{tT}{-D},\frac{u}{m^2}\right)+ \\
& 2L_{+}\left(0,\frac{T}{m^2},\frac{-T}{m^2},\frac{s}{t}\right)-2L_{+}\left(0,\frac{T}{m^2},\frac{-T}{m^2},-\frac{t}{T}\right)+2L_{+}\left(\frac{m^2}{-T},0,\frac{T}{m^2},\frac{u}{t}\right)- \\
& 2L_{+}\left(\frac{m^2}{-T},\frac{U}{u},\frac{m^2}{-u},\frac{-U}{m^2}\right)\Bigg];
\end{aligned}
$$

$$
\mathrm{Im}\, D_3^{(j)} \;=\; 0. \tag{2.109}
$$

The non-planar topological structure of the four-point function D_3 implies that D_3 has to be $(t \leftrightarrow u)$–symmetric (see Fig. 2.6). This can best be seen by exchanging the momenta $p_3 \leftrightarrow p_4$ in Fig. 2.6 followed by a twist of the r.h.s. of Fig. 2.6 [2]. The $(t \leftrightarrow u)$–symmetry provides for a check on the results for D_3. The symmetry is obviously satisfied for $\mathrm{Re}D_3^{(-2)}$, $\mathrm{Re}D_3^{(-1)}$ and $\mathrm{Re}D_3^{(0)}$ but is not manifest for $\mathrm{Re}D_3^{(1)}$ and $\mathrm{Re}D_3^{(2)}$ in (2.108). However, it is quite straightforward to verify numerically that the $(t \leftrightarrow u)$–symmetry indeed holds for all coefficient functions in (2.108).

Apart from the internal checks mentioned earlier the most important check on the four-point function results has been a comparison with numerical results provided by M.M. Weber [51] for several phase space points. Within numerical errors complete agreement was found with the results of M.M. Weber for each of the three four-point functions. It is important to emphasize that the approach of M.M. Weber to numerically evaluate the four-point functions is completely different from the approach used in this thesis [52].

[2]The $(t \leftrightarrow u)$–symmetry is not so easy to see when exchanging $p_1 \leftrightarrow p_2$ in Fig. 2.6. In this case the $(t \leftrightarrow u)$–symmetry becomes apparent only after Feynman parametrization.

Chapter 3

Simplification of expressions involving $\mathrm{Li}_2, \mathrm{Li}_3$ and Li_4

Unfortunately none of the presently existing computer algebra systems can simplify expressions involving classical polylogarithms. This is unfortunate since after integration of the scalar master integrals (see Chap. 2) a large number of different polylogarithms $\mathrm{Li}_n\,(n = 2, 3, 4)$ appears. These functions obey relations which allow one to greatly reduce the number of different classical polylogarithms[1]. In the case that the simplified expression is short, and if one has only a few of these functions, one can perform simplifications "by hand". However in the case of the $\mathcal{O}(\varepsilon^2)$ expressions treated in this thesis this task is rather difficult, because the average number of the classical polylogarithms is of the order of hundred. In this chapter I describe algorithms which were implemented with the help of the internal programming language of the computer algebra system **Mathematica** [42]. I have used these algorithms in intermediate steps in the calculation as well as for the final simplification of the results on the massive scalar master integrals presented in Chap. 2. The algorithms help to reduce the number of classical polylogarithms by a factor from eight to ten.

3.1 Definition and some properties of the classical polylogarithms

Classical polylogarithms are defined as a power series [44]

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n},\ z \in C,\ |z| < 1. \tag{3.1}$$

The number n is called the weight or the order of the classical polylogarithm. The power series (3.1) is convergent for $|z| < 1$ and can be analytically continued via the integral

[1] The properties of and the relations for classical polylogarithms presented in this chapter and their detailed derivations can be found in [44].

representation:

$$\mathrm{Li}_n(z) \equiv \int_0^z \frac{Li_{n-1}(\xi)}{\xi} d\xi, \ n \geq 2; \qquad \mathrm{Li}_1(z) \equiv -\ln(1-z). \tag{3.2}$$

There are special values of classical polylogarithms:

$$\mathrm{Li}_n(1) = \zeta(n), \tag{3.3}$$

where $\zeta(n)$ is the Riemann zeta-function defined by

$$\zeta(n) = \sum_{k=1}^{\infty} k^{-n}. \tag{3.4}$$

The classical polylogarithms have a branch cut discontinuity in the complex plane for z running from 1 to ∞. The classical polylogarithm with a real argument can be presented as

$$\mathrm{Li}_n(x \pm i\epsilon) = \mathrm{Re}\left(\mathrm{Li}_n(x)\right) \pm i\pi\theta(x-1) \times \frac{\ln^{n-1}(x)}{(n-1)!}, \ x \in R, \tag{3.5}$$

where ϵ is an infinitely small positive value. The sign "+" or "-" of the imaginary shift $i\epsilon$ defines the upper or lower edge of the branch cut for the $\mathrm{Li}_n(x \pm i\epsilon)$ function, respectively. There are three sources for the imaginary shift. One can get this imaginary shift directly from the "causal" $+i\delta$ that one always assumes in the propagators (see for example remarks after Eq. (2.19)). One can introduce this imaginary shift by itself to control the imaginary part of an expression. For example if one integrates a real function without any poles on the integration interval some spurious imaginary parts may occur after integration. Naturally the imaginary parts have to disappear analytically after simplification. In order to perform this procedure properly one can use the trick with the introduction of the artificial imaginary shift into an integrand. Finally one can use the standard definition of the classical polylogarithms on the branch cut. In this case one should take the sign "-" for the imaginary shift. This choice corresponds to the standard definition of the imaginary part of the logarithm function for negative values of the argument: $\mathrm{Im}\left(\ln(x)\right) = +i\pi$, $x < 0$.

Eq. (3.5) appears to be very useful. Normally the arguments of the classical polylogarithms for the calculations are real. It means that these can also be on the branch cut discontinuity. To control the imaginary parts one can use Eq. (3.5) to extract the imaginary parts from an expression and to handle them separately. But this procedure should be performed only after the last integration.The imaginary shift should be kept whenever it makes sense for the subsequent calculations.

3.2 Algorithms for the functions Li_2

3.2.1 Algorithm based on identities with one variable

First I consider the function $\mathrm{Li}_2(z)$ or *dilogarithm*. In order to simplify dilogarithms with different arguments one can make use of the relations

$$\begin{aligned} \mathrm{Li}_2(z) &= \zeta(2) - \ln(z)\ln(1-z) - \mathrm{Li}_2(1-z), z \in C,\ z \notin (-\infty, 0] \cup [1, +\infty)\,, \\ \mathrm{Li}_2(z) &= -\zeta(2) - 1/2\ln^2(-z) - \mathrm{Li}_2(1/z), z \in C,\ z \notin [0, +\infty)\,. \end{aligned} \tag{3.6}$$

These relations as well as their derivation are given in [44]. As one can see these relations are not valid for arguments on the branch cuts of the above functions (both logarithms and dilogarithms in the identities). However, in the calculations of the scalar master integrals the arguments of the dilogarithms occur on the branch cut. To make use of Eqs. (3.6) for these cases one can either use the trick with the imaginary shift (either "causal" or "artificial") or separate imaginary and real parts via Eq. (3.5) before simplification. When one simplifies the results after the last integration it does not make sense to keep the imaginary shift. It is more convenient to handle the imaginary and real parts separately. One can show that for the real parts the identities (3.6) hold true for the whole complex plane

$$\begin{aligned} \mathrm{Re}\left(\mathrm{Li}_2(z)\right) &= \mathrm{Re}\left(\zeta(2) - \ln(z)\ln(1-z) - \mathrm{Li}_2(1-z)\right), z \in C, \\ \mathrm{Re}\left(\mathrm{Li}_2(z)\right) &= \mathrm{Re}\left(-\zeta(2) - 1/2\ln^2(-z) - \mathrm{Li}_2(1/z)\right), z \in C. \end{aligned} \tag{3.7}$$

The relations do not depend on the sign of the imaginary shift. So one can separate real and imaginary parts, use Eqs. (3.7) for the real parts of the dilogarithms and one can forget about the imaginary parts. Below I describe an algorithm based on the identities (3.6) but exactly the same algorithms can also be worked out using identities (3.7).

Now I give a sketch for the algorithms which helps me to reduce the number of the dilogarithms in the final result. Using Eqs. (3.6) and combinations of these one can find relations between the elements of the following set of the dilogarithms:

$$\left\{\mathrm{Li}_2(z), \mathrm{Li}_2(1-z), \mathrm{Li}_2\left(\frac{1}{z}\right), \mathrm{Li}_2\left(\frac{1}{1-z}\right), \mathrm{Li}_2\left(\frac{z-1}{z}\right), \mathrm{Li}_2\left(\frac{z}{z-1}\right)\right\}. \tag{3.8}$$

All dilogarithms from the set (3.8) can be expressed via only one chosen function Li_2. This property is used for the reduction of the number of the dilogarithms. Let

$$\mathrm{DilogSet} = \left\{\mathrm{Li}_2\left(f_1(\vec{x})\right), \mathrm{Li}_2\left(f_2(\vec{x})\right), ..., \mathrm{Li}_2\left(f_n(\vec{x})\right)\right\} \tag{3.9}$$

be a set of all dilogarithms from an expression to be simplified. $\vec{x}$ is a set of independent variables. In the reaction for two initial and two final particles there are only two independent kinematic variables. In my case I have chosen to use $\tilde{s}$ and $\tilde{t}$ defined in Eqs. (2.71). i.e. $\vec{x} = \{\tilde{s}, \tilde{t}\}$. $f_i(\vec{x})$ are functions of these independent variables. The main idea is as follows: the program takes the argument of the first element of the set (3.9) $f_1(\vec{x})$ and looks for the function $f_k(\vec{x})$ (taking first $k = 2$, then $k = 3$ and so on ...) which satisfies one of the conditions

$$f_1(\vec{x}) - F_j\left(f_k(\vec{x})\right) \equiv 0\,, j = 1,..,6. \tag{3.10}$$

where the functions $F_j(z)$ are defined as

$$\begin{gathered} F_1(z) \equiv z,\; F_2(z) \equiv 1 - z,\; F_3(z) \equiv \frac{1}{z}, \\ F_4(z) \equiv \frac{1}{1-z},\; F_5(z) \equiv \frac{z-1}{z},\; F_6(z) \equiv \frac{z}{z-1}. \end{gathered} \tag{3.11}$$

It is clear, that the functions $F_i(z)$ correspond to the arguments of the dilogarithms presented in the set (3.8). I have included the function $F_1(z) \equiv z$ into the search procedure because it can happen that two functions $f_{i_1}(\vec{x})$ and $f_{i_2}(\vec{x})$ are equal to each other but this equality is not obvious. Of course when one checks the condition (3.10) with some algebra system one has to force this system to perform all possible simplifications with the difference $f_1(\vec{x}) - F_j\left(f_k(\vec{x})\right)$ (for example in the computer algebra system **Mathematica** [42] the command “FullSimplify” is suitable for these purposes). Now suppose that one has found the function $f_{k_1}(\vec{x})$ which satisfies one of the conditions (3.10). It means that $Li_2\left(f_1(\vec{x})\right)$ can be expressed via $Li_2\left(f_{k_1}(\vec{x})\right)$ and *vice versa* using Eqs. (3.6) or combinations of these. But at this step it is too early to do such transformations. There can exist other functions $f_k(\vec{x})$ satisfying one of the conditions (3.10). In the first step we should find all these functions. Then we compose the set

$$\text{DilogSmallSet} = \left\{\mathrm{Li}_2\left(f_1(\vec{x})\right), \mathrm{Li}_2\left(f_{k_1}(\vec{x})\right), ..., \mathrm{Li}_2\left(f_{k_m}(\vec{x})\right)\right\}, \tag{3.12}$$

where m is the number of the functions satisfying one of the conditions (3.10). At this step one should perform the transformations. The algebra system should ask which dilogarithms has to be chosen from the set (3.12) as a basis dilogarithm. The basis dilogarithm should obey two criteria: it should be as “simple” as possible and its argument should be away from the branch cut to avoid uncertainties with imaginary parts (if there is no dilogarithm from the set (3.12) with the argument not lying on the branch cut one can get it using one of Eqs. (3.6)). Then the system expresses the remaining dilogarithms from the set (3.12) via the basis dilogarithm using identities (3.6) and combinations of these. It means that from the initial $m+1$ functions Li_2 only **one** survives at the end. Of course it is not necessary that these functions $f_k(\vec{x})$ exist. It can happen that there is no function $f_k(\vec{x})$ obeying one of the conditions (3.10). Then $m = 0$ and there is no possible relation between $\mathrm{Li}_2\left(f_1(\vec{x})\right)$ and any other dilogarithm from the set (3.9).

After expressing all dilogarithms from the set (3.12) by a given one one eliminates all elements of the “DilogSmallSet” (3.12) from “DilogSet” (3.9) because all possible simplifications for the functions from the set (3.12) have been done and these functions should not be considered further (after checking the conditions (3.10) for all k running from 2 to n it is guaranteed that there are no more relations between the functions of the set (3.12) and the remaining dilogarithms from the set (3.9)).

Now one has a new "DilogSet" (3.9) which is shorter than the initial one (exactly by $m+1$ elements). One has to again take the first element of this new set "DilogSet" and repeat the steps with the composition of "DilogSmallSet" (3.12). One then expands the dilogarithms from "DilogSmallSet" via a given one and eliminates the elements of the set "DilogSmallSet" from the set "DilogSet". This procedure continues until the length of the set "DilogSet" will be equal to 0. One then concludes that **all possible** simplifications via the identities (3.6) and combinations of these have been performed.

Very often one would like to obtain results expressed in terms of dilogarithms which occur in other expressions. For this task we can slightly change our algorithm. At the beginning of the program one gives to the system the set of desired dilogarithms. After the program has composed the set "DilogSmallSet" (3.12) it has to check whether the dilogarithms from the set "DilogSmallSet" (3.12) relate to the one of the desired dilogarithms. If this is true then the system expresses the function from the set "DilogSmallSet" (3.12) in terms of the desired dilogarithm or else the system should ask to choose the basis dilogarithm.

In Appendix B.1 I present the program written in the internal language of the computer algebra system **Mathematica** [42]. This program realizes the algorithm described above based on Eqs. (3.7), i.e. the program reduces the number of $\mathrm{Re}[\mathrm{Li}_2(z)]$.

3.2.2 Algorithm based on the identity with two variables

In addition to the identities with one variable an identity with two variables introduced by Hill (see [44] for details) is very useful:

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(y)=\mathrm{Li}_2(xy)+\mathrm{Li}_2\left(\frac{x(1-y)}{1-xy}\right)+\mathrm{Li}_2\left(\frac{y(1-x)}{1-xy}\right)$$
$$+\ln\left(\frac{1-x}{1-xy}\right)\ln\left(\frac{1-y}{1-xy}\right),\ x,y\in C. \tag{3.13}$$

All arguments of the functions occurring in (3.13) are assumed to be away from the branch cuts. To use the identity Eq. (3.13) on the branch cuts one either uses the trick with the imaginary shift or uses the identity Eq.(3.13) only for the real parts. This identity connects five different dilogarithms. Because of its rather subtle form it is not as powerful as the identities with one variable but the identity can still provide us with some simplifications.

Let me describe the algorithm to search for dilogarithms related by Eq. (3.13). One takes any pair of n dilogarithms of the input expression. Suppose it is $\mathrm{Li}_2\left(f_\alpha(\vec{x})\right)$ and $\mathrm{Li}_2\left(f_\beta(\vec{x})\right)$. Then one uses $f_\alpha(\vec{x})$ and $f_\beta(\vec{x})$ as x and y, respectively. Hill's formula produces three new polylogarithms. This transformation makes sense if and only if these three new dilogarithms are either identical to already existing dilogarithms in the input expression or they are related to existing dilogarithms by the identities (3.6) or combinations of these. The system should then check whether this statement is true or not. If the test is positive than one uses Hill's identity in combination with (3.6) to express the five dilogarithms via four (exactly five dilogarithms are related by the Hill's formula). But there is also a very subtle remark. One could transform first the initial pair of the dilogarithms $\mathrm{Li}_2\left(f_\alpha(\vec{x})\right)$ and $\mathrm{Li}_2\left(f_\beta(\vec{x})\right)$ via (3.6) or combinations of these, obtaining new dilogarithms, and only *after that* use the arguments of these as x and y. One would also obtain three new dilogarithms from the r.h.s. of (3.13) but these *would not be related* via (3.6) or combinations of these to the three dilogarithms obtained without transformation of initial pair of the dilogarithms $\mathrm{Li}_2\left(f_\alpha(\vec{x})\right)$ and $\mathrm{Li}_2\left(f_\beta(\vec{x})\right)$. Therefore one has also to take into account the possibility of transformation of the initial pair of the dilogarithms and include this possibility into the algorithm.

Let me evaluate how many comparisons should be done for only one pair of functions Li_2 in order to find a suitable relation via Hill's identity in combination with the identities with one variable. For each initial pair of the dilogarithms from the input expression one has

$$6 \times 6 \times \frac{1}{2} = 18$$

transformations of this pair due to Eqs (3.6) and combinations of these. The number 6 corresponds to the number of possible transformations. The factor $\frac{1}{2}$ occurs here because of the fact that if one uses the Hill's identity (3.13) with $\frac{1}{x}$ and $\frac{1}{y}$ instead of x and y then one obtains the dilogarithms on the r.h.s of (3.13) related by the transformation $F_3(z) \equiv \frac{1}{z}$ to dilogarithms obtained with x and y. For each of the 36 possible transformations of the initial pair there is a partner obtained by use of the transformation $F_3(z) \equiv \frac{1}{z}$. This partner is among the 36 possible transformations. Therefore from the 36 possible transformations one should check only 18 transformations of the initial pair.

The r.h.s. of (3.13) gives three new dilogarithms. It means that one has to perform

$$3 \times (n-2) \times 6$$

analytical comparisons (n is the number of dilogarithms in the input expression). The number "$(n-2)$" corresponds to the remaining dilogarithms and "6" is the number of possible transformations of the dilogarithms from the r.h.s. of (3.13). Finally in "the worst case" for only one pair one has to do

$$18 \times 3 \times (n-2) \times 6 = 324 \times (n-2) \tag{3.14}$$

analytical comparisons! One should stop the search if one finds the combination of the transformations of the dilogarithms on the r.h.s. of (3.13), for which these dilogarithms are related to already existing dilogarithms from the input expression. It means, that one has found a relation due to Hill's identity in combination with the identities with one variable. In this case the number of the comparisons is shorter than in Eq. (3.14). But none can guarantee that the relation for exactly this initial pair exists. Therefore "the worst case" can really occur. For the systematic search the pairs are checked one after another until the relation has been found . After expressing five dilogarithms via only four one arrives at the new input expression with the number of the dilogarithms decreased by one. Then one again looks for a pair giving a positive result for the test (the positive result here means that the relation for the initial pair is found). One again expresses five dilogarithms via four and again arrives at the new input expression. The procedure goes on until all possible pairs have been checked and the test is negative for all of them. In order to be sure that there are no more simplifications due to Hill's identity with two variables one has to check

$$n(n-1)/2$$

pairs. With already a few dilogarithms in the input expression it is clear that it is rather difficult to perform all manipulations by hand. Therefore I have implemented the search algorithm based on Hill's identity in the internal language of the computer algebra system **Mathematica** [42]. Even if one uses some computer algebra system it is very difficult task for the computer to perform all manipulations analytically because the number of analytical comparisons is very large and intermediate simplifications include operations with square roots and power functions. Therefore I have decided to perform the transformations numerically. For the check of the condition

$$a-b=0 \tag{3.15}$$

I have chosen to check numerically

$$\frac{|a-b|}{|a+b|} < 10^{-m}\,, \tag{3.16}$$

where the choice of m is up to the user (it can be 2, 3, ... but should not exceed the precision of the computer algebra system). After the relation is found numerically one proofs this also analytically. The trick with the numerical check of the condition (3.15) of the equivalence increases the speed of search procedure by a few orders of magnitude. The corresponding program is presented in Appendix B.2.

There is also one special case of the Hill's identity in the case $x=y$. In this case the identity (3.13) transforms into (see [44])

$$\mathrm{Li}_2(z)+\mathrm{Li}_2(-z)=\frac{1}{2}\mathrm{Li}_2(z^2),\ z\in C, z\notin(-\infty,-1]\cup[1,+\infty). \tag{3.17}$$

To obtain the algorithm for the identity (3.17) one takes the algorithm for Hill's identity and makes some changes. Instead of the initial pair one uses only one the initial dilogarithm $\mathrm{Li}_2(z)$. One takes also into account the transformations of it via Eqs. (3.6) and combinations of these as it was with the initial pair in the case of Hill's identity. Then the system checks whether $\mathrm{Li}_2(-z)$ and $\mathrm{Li}_2(z^2)$ are equal or related via identities (3.6) or combinations of these to already existing dilogarithms from the input expression. If it is true then one has found a relation which allows one to express three dilogarithms from the input expression via only two. All other steps remain the same as it was in the case with Hill's identity.

3.3 Algorithm for the function Li_3

In this section I would like to describe the algorithms for the simplification of the functions Li_3. In the case of the functions Li_3 one has only the identities with one variable (see [44]):

$$\begin{aligned} \mathrm{Li}_3(z) &= -\zeta(2)\ln(-z) - 1/6\ln^3(-z) + \mathrm{Li}_3(1/z), z \in C, z \notin [0,+\infty)\,, \qquad (3.18)\\ \mathrm{Li}_3(z) &= \zeta(3) + \zeta(2)\ln(1-z) - \mathrm{Li}_3(1-z) - \mathrm{Li}_3\left(\frac{z}{z-1}\right), z \in C, z \notin (-\infty,0] \cup [1,+\infty). \end{aligned}$$

As one can see these relations are not valid for arguments on the branch cuts of the above functions (both logarithms and dilogarithms in the identities). To make use of Eqs. (3.18) on the real axis one can either use the trick with the imaginary shift (either "causal" or "artificial") or separate imaginary and real parts via Eq. (3.5) before simplification. One can show that for the real parts the identities (3.18) hold true for the whole complex plane

$$\begin{aligned} \mathrm{Re}\left(\mathrm{Li}_3(z)\right) &= \mathrm{Re}\left(-\zeta(2)\ln(-z) - 1/6\ln^3(-z) + \mathrm{Li}_3(1/z)\right), z \in C\,, \qquad (3.19)\\ \mathrm{Re}\left(\mathrm{Li}_3(z)\right) &= \mathrm{Re}\left(\zeta(3) + \zeta(2)\ln(1-z) - \mathrm{Li}_3(1-z) - \mathrm{Li}_3\left(\frac{z}{z-1}\right)\right), z \in C. \end{aligned}$$

The identities (3.18) (and (3.19)) are analogs of the identities (3.6) (and (3.7)) for the dilogarithms. The most important difference is that the second identity of Eq. (3.18) connects three different polylogarithms instead of two as it was in the case of Eqs. (3.6). One sees that the second identity of Eqs. (3.18) connects $\mathrm{Li}_3(z)$, $\mathrm{Li}_3(1-z)$ and $\mathrm{Li}_3\left(\frac{z}{z-1}\right)$. Combining the second identity (3.18)with the first identity (3.18) one concludes that there are relations between the functions Li_3 of the following set

$$\{\mathrm{Li}_3(z), \mathrm{Li}_3(1-z), \mathrm{Li}_3\left(\frac{1}{z}\right), \mathrm{Li}_3\left(\frac{1}{1-z}\right), \mathrm{Li}_3\left(\frac{z-1}{z}\right), \mathrm{Li}_3\left(\frac{z}{z-1}\right)\}. \qquad (3.20)$$

Due to the fact that the second identity of Eq. (3.18) connects three different polylogarithms one can express all functions of the set (3.20) via two basis polylogarithms chosen from this set. The algorithm for the reduction of the number of the functions Li_3 is very similar to the corresponding algorithm for the dilogarithms (see Sec. 3.2.1). The only difference is that after composition of the set "Li3SmallSet" of the functions Li_3 which is the direct analogue of the "DilogSmallSet" (3.12) for the dilogarithms the system expresses all functions of the set "Li3SmallSet" via **two** basis Li_3 instead of **one** basis functions as it was in the case of dilogarithms. These two basis polylogarithms should be chosen from the set of the desired functions Li_3 (like it was in the case with the dilogarithms) or they should be chosen from the set "Li3SmallSet" by the user. All other steps of the algorithm are identical to the algorithm for the dilogarithms (see Sec. 3.2.1).

In Appendix C I present the program written in the internal language of the computer algebra system **Mathematica** [42]. This program realizes the algorithm to reduce the number of the functions Li_3 based on Eqs. (3.19), i.e. the program reduces the number of $\mathrm{Re}[\mathrm{Li}_3(z)]$. Despite the fact that there is only the small difference between the algorithms for the functions Li_2 and Li_3 this difference makes the realization of the algorithm for the

Li_3 function much more complicated. This one can already see if one compares the length of the programs for the functions Li_2 and Li_3, respectively.

There is also one additional identity for the functions Li_3 [44]

$$\mathrm{Li}_3(z) + \mathrm{Li}_3(-z) = \frac{1}{4}\mathrm{Li}_3(z^2), z \in C, z \notin (-\infty, -1] \cup [1, +\infty). \tag{3.21}$$

This identity is an analogue of the identity (3.17) for the dilogarithm. One can write a separate program for the search of the relations due to Eq. (3.21). To take into account all possibilities one has to combine the identity (3.21) with the Eqs. (3.18).

3.4 Algorithms for the function Li_4

One notes that with the increase of the weight of the classical polylogarithms the possibility to use some relations to reduce the number of polilogarithms decreases. In the case of the dilogarithm one uses the identities with one variable as well as identity with two variables. In the case of the functions Li_3 one can uses only the identities with one variable and these identities are more complicated. This fact leaves fewer possibilities to reduce the number of the functions Li_3 than in the case of the dilogarithms. In the case of the functions Li_4 one has only two helpful identities [44]

$$\mathrm{Li}_4(z) = -\frac{7}{4}\zeta(4) - \frac{1}{24}\ln^4(-z) - \frac{1}{2}\zeta(2)\ln^4(-z) - \mathrm{Li}_4\left(\frac{1}{z}\right), z \in C, z \notin [0, +\infty) \tag{3.22}$$

and

$$\mathrm{Li}_4(z) + \mathrm{Li}_4(-z) = \frac{1}{8}\mathrm{Li}_4(z^2), z \in C, z \notin (-\infty, -1] \cup [1, +\infty). \tag{3.23}$$

The identity (3.22) is an analogue of the second identity of (3.6) for the dilogarithm and an analogue of the first identity of (3.18) for the function Li_3. The identity (3.23) is an analogue of identities (3.17) and (3.21) for the functions Li_2 and Li_3, respectively. Using Eqs. (3.22) one can find relation between the functions Li_4 of the following set:

$$\{\mathrm{Li}_4(z), \mathrm{Li}_4\left(\frac{1}{z}\right)\}. \tag{3.24}$$

The set (3.24) for the functions Li_4 is an reduced analogue version of the set (3.8) for the dilogarithms. Therefore in order to reduce the number of the functions Li_4 due to the identity (3.22) one can use the same algorithm as for the dilogarithms (see Sec. 3.2.1) with small changes. When checking the conditions (3.10) one has to test these only with the functions $F_1(z)$ and $F_3(z)$ from (3.11). These functions correspond to the arguments of the polylogarithms in the set (3.24). All other steps of the algorithm for the function Li_4 remain the same as it was in the case of the dilogarithm.

To find possible relations due to identity (3.23) one can use the same algorithm as for the dilogarithm (see remarks after Eq. (3.17)).

Chapter 4

Properties of the L-functions

In this chapter I discuss properties and identities involving the single- and triple-index L-functions defined in (2.46) and (2.47). There are two different categories of identities which I discuss in turn. In Sec. 4.1 I consider the simplest identities originating from symmetries related to permutations in the indices and arguments. Then in Sec. 4.2 I present further identities based on integration-by-parts techniques. All the identities of this section have been used to reduce the size of the expressions in the main part Chap. 2. Finally in Sec. 4.3 it is shown how the L-functions are related to multiple polylogarithms as they are defined in [43].

4.1 Symmetry properties

I start with the single-index function $L_{\sigma_1}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ Eq. (2.46). One notices that a change of the integration variable $y \to 1-y$ results in the identity

$$L_{\sigma_1}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = -L_{-\sigma_1}(\alpha_1 + \sigma_1, \alpha_2 + \alpha_3, -\alpha_3, -\alpha_4 - 1) \qquad (4.1)$$

which implies that L_- can always be related to L_+, and vice versa. Thus the results for the three-point (Sec. 2.2) and four-point functions (Sec. 2.3) is written only in terms of the L_+-functions.

Next I turn to the triple-index L-function Eq. (2.47). Note that $L_{\sigma_1\sigma_2\sigma_3}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is symmetric under permutations of any two pairs of indices and arguments $\{\sigma_i, \alpha_i\}$ and $\{\sigma_j, \alpha_j\}$ for $(i \neq j)$. The same change of variables as above $y \to 1-y$ results in

$$L_{\sigma_1\sigma_2\sigma_3}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = -L_{-\sigma_1-\sigma_2-\sigma_3}(\alpha_1 + \sigma_1, \alpha_2 + \sigma_2, \alpha_3 + \sigma_3, -\alpha_4 - 1). \qquad (4.2)$$

Therefore, from the eight functions L_{---}, L_{--+}, L_{-+-}, L_{+--}, L_{-++}, L_{+-+}, L_{++-} and L_{+++} only two are independent. For the results presented in Sec. 2.2 and Sec. 2.3 I consider the two independent functions L_{-++} and L_{+++}.

4.2 Integration-by-parts identities

The triple- and single-index L-functions L_{+++}, L_{-++} and L_{+} defined in Eqs. (2.46) and (2.47) have been devised such that they have neither branch cuts nor poles on the integration path $y \in [0,1]$. This also implies that the L_{+++}, L_{-++} and L_{+} functions are real. Remember that the branch cuts for the ln and Li_2 functions are $(-\infty, 0]$ and $(1, +\infty)$, respectively. The domains of the functions L_{+++}, L_{-++} and L_{+} are

$$\begin{array}{rcl} L_{+++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) & : & \alpha_1>0,\alpha_2>0,\alpha_3>0,\alpha_4<-1 \quad \text{or} \quad \alpha_4>0; \\ L_{-++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) & : & \alpha_1>1,\alpha_2>0,\alpha_3>0,\alpha_4<-1 \quad \text{or} \quad \alpha_4>0; \\ L_{+}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) & : & \alpha_1>0,\alpha_2\le 1,\alpha_2+\alpha_3\le 1,\alpha_3\neq 0,\alpha_4<-1 \quad \text{or} \quad \alpha_4>0. \end{array} \tag{4.3}$$

Looking at the definition of the triple-index L–function in (2.46) one concludes that the boundary points $\alpha_1 = 0$ and/or $\alpha_2 = 0$ and/or $\alpha_3 = 0$ can be included in the domain of definition for L_{+++}. The same holds true for $\alpha_1 = 1$ and/or $\alpha_2 = 0$ and/or $\alpha_3 = 0$ for L_{-++}. Also, from the definition of the single-index function L_{+} in (2.47) one concludes that the boundary point $\alpha_1 = 0$ can be added to its domain of definition.

The points $\alpha_4 = \{-1, 0\}$ can also be included in the domain if the values taken by other parameters α_i guarantee the convergence of the integral. In what follows I assume everywhere in this chapter that the conditions (4.3) are satisfied. Nevertheless, it is always possible to analytically continue the parameters to the complex plane.

In order to obtain integration-by-parts identities one makes use of the standard integration-by-parts formula

$$\int_0^1 UV'dy = UV\Big|_0^1 - \int_0^1 VU'dy\,. \tag{4.4}$$

4.2.1 Identities for the L_{-++}- and L_{+++}-functions

I start with the triple-index functions L_{-++} and L_{+++} defined in Eq. (2.46). Setting U in Eq. (4.4) equal to the numerator $[\ln(\alpha_1+\sigma_1 y)\ln(\alpha_2+\sigma_2 y)\ln(\alpha_3+\sigma_3 y)]$ and V' equal to the remainder $(\alpha_4+y)^{-1}$ I then arrive at

$$L_{+++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \begin{cases} \alpha_4>0: & \ln(\alpha_1+y)\ln(\alpha_2+y)\ln(\alpha_3+y)\ln(\alpha_4+y)\Big|_0^1 \\ & -L_{+++}(\alpha_4,\alpha_2,\alpha_3,\alpha_1) - L_{+++}(\alpha_1,\alpha_4,\alpha_3,\alpha_2) \\ & -L_{+++}(\alpha_1,\alpha_2,\alpha_4,\alpha_3); \\ \alpha_4<-1: & \ln(\alpha_1+y)\ln(\alpha_2+y)\ln(\alpha_3+y)\ln(-\alpha_4-y)\Big|_0^1 \\ & -L_{-++}(-\alpha_4,\alpha_2,\alpha_3,\alpha_1) - L_{-++}(-\alpha_4,\alpha_1,\alpha_3,\alpha_2) \\ & -L_{-++}(-\alpha_4,\alpha_1,\alpha_2,\alpha_3); \end{cases} \tag{4.5}$$

and

$$L_{-++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \begin{cases} \alpha_4 > 0: & \ln(\alpha_1 - y)\ln(\alpha_2 + y)\ln(\alpha_3 + y)\ln(\alpha_4 + y)\Big|_0^1 \\ & -L_{+++}(\alpha_4,\alpha_2,\alpha_3,-\alpha_1) - L_{-++}(\alpha_1,\alpha_4,\alpha_3,\alpha_2) \\ & -L_{-++}(\alpha_1,\alpha_2,\alpha_4,\alpha_3); \\ \alpha_4 < -1: & \ln(\alpha_1 - y)\ln(\alpha_2 + y)\ln(\alpha_3 + y)\ln(-\alpha_4 - y)\Big|_0^1 \\ & -L_{-++}(-\alpha_4,\alpha_2,\alpha_3,-\alpha_1) \\ & +L_{-++}(\alpha_3+1,\alpha_1-1,-\alpha_4-1,-\alpha_2-1) \\ & +L_{-++}(\alpha_2+1,\alpha_1-1,-\alpha_4-1,-\alpha_3-1). \end{cases} \tag{4.6}$$

For the second part of Eq.(4.6) I have made use of relation (4.2).

There are some special cases when some of the α_i take values on the boundary of the domain of definition where one can still make use of the identities (4.5) and (4.6) even if the conditions (4.3) are not met. For example, for the case $\{\alpha_1 = 0, \alpha_4 = -1\}$ the identitity (4.6) is still valid. There are similar special cases for further identities to be derived below.

4.2.2 Identities for the L_+-function

The integration-by-parts identities for the single-index L_+- function are more involved. I first write down the derivative of the function Li_2 in the integrand of (2.47). One has

$$\frac{d\mathrm{Li}_2(\alpha_2 + \alpha_3 y)}{dy} = -\frac{\ln(1 - \alpha_2 - \alpha_3 y)}{\frac{\alpha_2}{\alpha_3} + y}. \tag{4.7}$$

In the case of the single-index function it will prove important to consider two different choices for U in Eq. (4.4). One starts by setting the whole numerator $[\ln(\alpha_1 + \sigma_1 y)\mathrm{Li}_2(\alpha_2 + \alpha_3 y)]$ in the integrand of Eq. (2.47) to U. For V' one then has $(\alpha_4 + y)^{-1}$. One obtains

$$L_+(\alpha_1,\alpha_2,\alpha_3,\alpha_4) =$$

$$
\begin{cases}
\alpha_4 > 0, \alpha_3 > 0: & \ln(\alpha_1 + y)\mathrm{Li}_2(\alpha_2 + \alpha_3 y)\ln(\alpha_4 + y)\Big|_0^1 \\
 & -L_+(\alpha_4, \alpha_2, \alpha_3, \alpha_1) \\
 & +L_{-++}(\frac{1-\alpha_2}{\alpha 3}, \alpha_1, \alpha_4, \frac{\alpha_2}{\alpha_3}) + \ln(\alpha_3)\int\limits_0^1 \frac{\ln(\alpha_1+y)\ln(\alpha_4+y)}{\frac{\alpha_2}{\alpha_3}+y} dy; \\
\alpha_4 > 0, \alpha_3 < 0: & \ln(\alpha_1 + y)\mathrm{Li}_2(\alpha_2 + \alpha_3 y)\ln(\alpha_4 + y)\Big|_0^1 \\
 & -L_+(\alpha_4, \alpha_2, \alpha_3, \alpha_1) \\
 & +L_{+++}(\frac{\alpha_2-1}{\alpha 3}, \alpha_1, \alpha_4, \frac{\alpha_2}{\alpha_3}) + \ln(-\alpha_3)\int\limits_0^1 \frac{\ln(\alpha_1+y)\ln(\alpha_4+y)}{\frac{\alpha_2}{\alpha_3}+y} dy; \\
\alpha_4 < -1, \alpha_3 > 0: & \ln(\alpha_1 + y)\mathrm{Li}_2(\alpha_2 + \alpha_3 y)\ln(-\alpha_4 - y)\Big|_0^1 \\
 & +L_+(-\alpha_4 - 1, \alpha_2 + \alpha_3, -\alpha_3, -\alpha_1 - 1) \\
 & -L_{-++}(\alpha_1 + 1, \frac{1-\alpha_2-\alpha_3}{\alpha_3}, -\alpha_4 - 1, -\frac{\alpha_2+\alpha_3}{\alpha_3}) \\
 & +\ln(\alpha_3)\int\limits_0^1 \frac{\ln(\alpha_1+y)\ln(-\alpha_4-y)}{\frac{\alpha_2}{\alpha_3}+y} dy; \\
\alpha_4 < -1, \alpha_3 < 0: & \ln(\alpha_1 + y)\mathrm{Li}_2(\alpha_2 + \alpha_3 y)\ln(-\alpha_4 - y)\Big|_0^1 \\
 & +L_+(-\alpha_4 - 1, \alpha_2 + \alpha_3, -\alpha_3, -\alpha_1 - 1) \\
 & +L_{-++}(-\alpha_4, \alpha_1, \frac{\alpha_2-1}{\alpha 3}, \frac{\alpha_2}{\alpha_3}) + \ln(-\alpha_3)\int\limits_0^1 \frac{\ln(\alpha_1+y)\ln(-\alpha_4-y)}{\frac{\alpha_2}{\alpha_3}+y} dy.
\end{cases} \tag{4.8}
$$

An additional condition for (4.8) has to be explicated because it does not follow automatically from (4.3), namely the parameters α_2 and α_3 are restricted by

$$
\frac{\alpha_2}{\alpha_3} < -1 \quad \text{or} \quad \frac{\alpha_2}{\alpha_3} > 0. \tag{4.9}
$$

The integrals in (4.8) are simple enough to be evaluated in terms of classical polylogarithms up to Li_3. I do not provide explicit results for these integrations since they are rather lengthy and, in addition, depend on relations between the parameters.

A second choice for U in (4.4) provides further identities for L_+. In this case one sets $\mathrm{Li}_2(\alpha_2 + \alpha_3 y)$ to U and $\ln(\alpha_1 + y)/(\alpha_4 + y)$ to V'. To calculate V one has to differentiate between three cases for the set of parameters α_1 and α_4. One has

$$
V = \int \frac{\ln(\alpha_1 + y)}{\alpha_4 + y} dy = \begin{cases}
\alpha_1 < \alpha_4: & \ln(\alpha_1 + y)\ln(\frac{\alpha_4+y}{\alpha_4-\alpha_1}) + \mathrm{Li}_2(\frac{\alpha_1+y}{\alpha_1-\alpha_4}); \\
\alpha_1 > \alpha_4 > 0: & \ln(\alpha_1 - \alpha_4)\ln(\alpha_4 + y) - \mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1}); \\
\alpha_4 < -1: & \ln(\alpha_1 - \alpha_4)\ln(-\alpha_4 - y) - \mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1}).
\end{cases} \tag{4.10}
$$

Using Eq. (4.10) one then obtains

$$
L_+(\alpha_1, \alpha_2, \alpha_3, \alpha_4) =
$$

$$
\begin{cases}
\alpha_1 < \alpha_4, \alpha_3 > 0: & \mathrm{Li}_2(\alpha_2+\alpha_3 y)\Big(\ln(\alpha_1+y)\ln(\frac{\alpha_4+y}{\alpha_4-\alpha_1}) + \mathrm{Li}_2(\frac{\alpha_1+y}{\alpha_1-\alpha_4})\Big)\Big|_0^1 \\
& +L_{-++}(\frac{1-\alpha_2}{\alpha 3}, \alpha_1, \alpha_4, \frac{\alpha_2}{\alpha_3}) \\
& -L_+(\frac{1-\alpha_2-\alpha_3}{\alpha_3}, \frac{\alpha_1+1}{\alpha_1-\alpha_4}, \frac{1}{\alpha_4-\alpha_1}, -\frac{\alpha_2+\alpha_3}{\alpha_3}) + \\
& \int\limits_0^1 \frac{\ln(\alpha_3)\Big[\mathrm{Li}_2(\frac{\alpha_1+y}{\alpha_1-\alpha_4})+\ln(\alpha_1+y)\ln(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\Big]-\ln(\alpha_4-\alpha_1)\ln(\alpha_1+y)\ln(\frac{1-\alpha_2}{\alpha_3}-y)}{\frac{\alpha_2}{\alpha_3}+y}dy; \\
\alpha_1 < \alpha_4, \alpha_3 < 0: & \mathrm{Li}_2(\alpha_2+\alpha_3 y)\Big(\ln(\alpha_1+y)\ln(\frac{\alpha_4+y}{\alpha_4-\alpha_1}) + \mathrm{Li}_2(\frac{\alpha_1+y}{\alpha_1-\alpha_4})\Big)\Big|_0^1 \\
& +L_{+++}(\frac{\alpha_2-1}{\alpha 3}, \alpha_1, \alpha_4, \frac{\alpha_2}{\alpha_3}) \\
& +L_+(\frac{\alpha_2-1}{\alpha_3}, \frac{\alpha_1}{\alpha_1-\alpha_4}, \frac{1}{\alpha_1-\alpha_4}, \frac{\alpha_2}{\alpha_3}) + \\
& \int\limits_0^1 \frac{\ln(-\alpha_3)\Big[\mathrm{Li}_2(\frac{\alpha_1+y}{\alpha_1-\alpha_4})+\ln(\alpha_1+y)\ln(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\Big]-\ln(\alpha_4-\alpha_1)\ln(\alpha_1+y)\ln(\frac{\alpha_2-1}{\alpha_3}+y)}{\frac{\alpha_2}{\alpha_3}+y}dy; \\
\alpha_1 > \alpha_4 > 0, \alpha_3 > 0 \ : & \mathrm{Li}_2(\alpha_2+\alpha_3 y)\Big(\ln(\alpha_1-\alpha_4)\ln(\alpha_4+y) - \mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\Big)\Big|_0^1 \\
& +L_+(\frac{1-\alpha_2-\alpha_3}{\alpha_3}, \frac{\alpha_4+1}{\alpha_4-\alpha_1}, \frac{1}{\alpha_1-\alpha_4}, -\frac{\alpha_2+\alpha_3}{\alpha_3}) + \\
& \int\limits_0^1 \frac{-\ln(\alpha_3)\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})+\ln(\alpha_1-\alpha_4)\ln(\alpha_4+y)\ln(1-\alpha_2-\alpha_3 y)}{\frac{\alpha_2}{\alpha_3}+y}dy; \\
\alpha_4 < -1, \alpha_3 > 0 \ : & \mathrm{Li}_2(\alpha_2+\alpha_3 y)\Big(\ln(\alpha_1-\alpha_4)\ln(-\alpha_4-y) - \mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\Big)\Big|_0^1 \\
& +L_+(\frac{1-\alpha_2-\alpha_3}{\alpha_3}, \frac{\alpha_4+1}{\alpha_4-\alpha_1}, \frac{1}{\alpha_1-\alpha_4}, -\frac{\alpha_2+\alpha_3}{\alpha_3}) \\
& +\int\limits_0^1 \frac{-\ln(\alpha_3)\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})+\ln(\alpha_1-\alpha_4)\ln(-\alpha_4-y)\ln(1-\alpha_2-\alpha_3 y)}{\frac{\alpha_2}{\alpha_3}+y}dy; \\
\alpha_1 > \alpha_4 > 0, \alpha_3 < 0 \ : & \mathrm{Li}_2(\alpha_2+\alpha_3 y)\Big(\ln(\alpha_1-\alpha_4)\ln(\alpha_4+y) - \mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\Big)\Big|_0^1 \\
& -L_+(\frac{\alpha_2-1}{\alpha_3}, \frac{\alpha_4}{\alpha_4-\alpha_1}, \frac{1}{\alpha_4-\alpha_1}, \frac{\alpha_2}{\alpha_3}) + \\
& \int\limits_0^1 \frac{-\ln(-\alpha_3)\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})+\ln(\alpha_1-\alpha_4)\ln(\alpha_4+y)\ln(1-\alpha_2-\alpha_3 y)}{\frac{\alpha_2}{\alpha_3}+y}dy; \\
\alpha_4 < -1, \alpha_3 < 0 \ : & \mathrm{Li}_2(\alpha_2+\alpha_3 y)\Big(\ln(\alpha_1-\alpha_4)\ln(-\alpha_4-y) - \mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})\Big)\Big|_0^1 \\
& -L_+(\frac{\alpha_2-1}{\alpha_3}, \frac{\alpha_4}{\alpha_4-\alpha_1}, \frac{1}{\alpha_4-\alpha_1}, \frac{\alpha_2}{\alpha_3}) + \\
& \int\limits_0^1 \frac{-\ln(-\alpha_3)\mathrm{Li}_2(\frac{\alpha_4+y}{\alpha_4-\alpha_1})+\ln(\alpha_1-\alpha_4)\ln(-\alpha_4-y)\ln(1-\alpha_2-\alpha_3 y)}{\frac{\alpha_2}{\alpha_3}+y}dy.
\end{cases}
\tag{4.11}
$$

In deriving (4.11) it is important to take into account condition (4.9). As was the case in Eq. (4.8) the integrals in (4.11) can be evaluated in terms of classical polylogarithms up to Li_3.

There is also one special case of the last identity (4.11) when the first and fourth arguments of the single-index L_+ function are equal, e.g. $\alpha_1 = \alpha_4$. In this case L_+ can be expressed only in terms of the functions L_{-++} or L_{+++} as follows:

$$L_+(\alpha_1, \alpha_2, \alpha_3, \alpha_1) =$$

$$\begin{cases} \alpha_3 > 0: & \frac{1}{2}\ln^2(\alpha_1+y)\mathrm{Li}_2(\alpha_2+\alpha_3 y)\Big|_0^1 + \frac{1}{2}L_{-++}(\frac{1-\alpha_2}{\alpha_3},\alpha_1,\alpha_1,\frac{\alpha_2}{\alpha_3}) \\ & +\frac{\ln(\alpha_3)}{2}\int\limits_0^1 \frac{\ln^2(\alpha_1+y)}{\frac{\alpha_2}{\alpha_3}+y}dy, \\ \alpha_3 < 0: & \frac{1}{2}\ln^2(\alpha_1+y)\mathrm{Li}_2(\alpha_2+\alpha_3 y)\Big|_0^1 + \frac{1}{2}L_{+++}(\frac{\alpha_2-1}{\alpha_3},\alpha_1,\alpha_1,\frac{\alpha_2}{\alpha_3})+ \\ & +\frac{\ln(-\alpha_3)}{2}\int\limits_0^1 \frac{\ln^2(\alpha_1+y)}{\frac{\alpha_2}{\alpha_3}+y}dy\,. \end{cases} \tag{4.12}$$

The third and last identity for the function L_+ is obtained from the definition (2.47) without making direct use of the integration-by-parts identity Eq. (4.4). Nevertheless it can still be called an integration-by-parts identity because it makes use of the well-known identity [44]

$$\mathrm{Li}_2(z) = \zeta(2) - \ln(z)\ln(1-z) - \mathrm{Li}_2(1-z),\ z \in C,\ z \notin (-\infty, 0] \cup [1, +\infty)\,, \tag{4.13}$$

which in turn is derived from the definition of the function Li_2 (3.2) with the help of the integration-by-parts identity (4.4). After transforming $\mathrm{Li}_2(\alpha_2 + \alpha_3 y)$ according to (4.13) one gets

$$L_+(\alpha_1,\alpha_2,\alpha_3,\alpha_4) =$$

$$\begin{cases} \begin{matrix}\alpha_3 > 0 \\ 0 \le \alpha_2 \le 1, \alpha_2+\alpha_3 \le 1\end{matrix} & : -L_+(\alpha_1, 1-\alpha_2, -\alpha_2, \alpha_4) - L_{-++}(\frac{1-\alpha_2}{\alpha_3},\alpha_1,\frac{\alpha_2}{\alpha_3},\alpha_4)+ \\ & \int\limits_0^1 \frac{\ln(\alpha_1+y)}{\alpha_4+y}\Big(\zeta(2) - \ln(\alpha_3)[\ln(\alpha_2+\alpha_3 y) + \ln(\frac{1-\alpha_2}{c} - y)]\Big)\,dy; \\ \begin{matrix}\alpha_3 < 0 \\ 0 \le \alpha_2 \le 1, 0 \le \alpha_2+\alpha_3\end{matrix} & : -L_+(\alpha_1, 1-\alpha_2, -\alpha_2, \alpha_4) - L_{-++}(-\frac{\alpha_2}{\alpha_3},\alpha_1,\frac{\alpha_2-1}{\alpha_3},\alpha_4)+ \\ & \int\limits_0^1 \frac{\ln(\alpha_1+y)}{\alpha_4+y}\Big(\zeta(2) - \ln(-\alpha_3)[\ln(\alpha_2+\alpha_3 y) + \ln(\frac{\alpha_2-1}{c} + y)]\Big)\,dy. \end{cases} \tag{4.14}$$

A few final comments are appropiate. In spite of the rather complicated appearance of the identities (4.5), (4.6), (4.8), (4.11), (4.14) these turn out to be very useful to reduce the length of the results presented in Chap. 2. The first step in the chain of reductions is to write everything in terms of the functions L_+, L_{-++} and L_{+++}. In a second step one uses the identities written down in this chapter to find the set of arguments of L-functions for which the number of the functions L_+, L_{-++} and L_{+++} is minimal. I have devised several programs for the computer algebra system **Mathematica** [42] which help to find minimal sets of the single- and triple-index L-functions. With the help of these programs I have been able to greatly reduce the number of L-functions appearing in the results and I have thereby greatly reduced their length.

4.3 From the L-functions to multiple polylogarithms

It is interesting to express the L-functions by the standard set of multiple polylogarithm functions [43]. A computer code for numerical evaluation of the multiple polylogarithms based on the GiNaC library [53] has been written down by J. Vollinga and S. Weinzierl [49]. I was able to define the multiple polylogarithms in the computer algebra system **Mathematica** [42] using this external computer code.

4.3.1 Definition of the multiple polylogarithms. Special cases of the multiple polylogarithms.

In this section I shall demonstrate how the L-functions introduced in Eqs. (2.46) and (2.47) are related to multiple polylogarithms as defined in [43]. Multiple polylogarithms are defined as limits of Z-sums, e.g.

$$Li_{m_k,...,m_1}(x_k,...,x_1) = \lim_{n_1\to\infty} \sum_{n_1>n_2...>n_k>0} \frac{x_1^{n_1} x_2^{n_2} ... x_k^{n_k}}{n_1^{m_1} n_2^{m_2} ... n_k^{m_k}}. \tag{4.15}$$

The number $w = m_1 + ... + m_k$ is called the weight and k is called the depth of the multiple polylogarithm. The power series (4.15) is convergent for $|x_i| < 1$, and can be analytically continued via the iterated integral representation:

$$Li_{m_k,...,m_1}(x_k,...,x_1) = \int\limits_0^{x_1 x_2 ... x_k} \left(\frac{dt}{t}\circ\right)^{m_1-1} \frac{dt}{x_2 x_3 ... x_k - t} \circ \left(\frac{dt}{t}\circ\right)^{m_2-1} \frac{dt}{x_3 ... x_k - t} \circ ... \circ \left(\frac{dt}{t}\circ\right)^{m_k-1} \frac{dt}{1-t}, \tag{4.16}$$

where the following notation is used for the iterated integrals:

$$\int\limits_0^{\lambda} \frac{dt}{a_n - t} \circ ... \circ \frac{dt}{a_1 - t} = \int\limits_0^{\lambda} \frac{dt_n}{a_n - t_n} \int\limits_0^{t_n} \frac{dt_{n-1}}{a_{n-1} - t_{n-1}} \times ... \times \int\limits_0^{t_2} \frac{dt_1}{a_1 - t_1}. \tag{4.17}$$

The multiple polylogarithms contain a variety of other functions as subsets (see also [48]). The classical polylogarithms, defined as

$$\mathrm{Li}_n(z) \equiv \int\limits_0^z \frac{\mathrm{Li}_{n-1}(\xi)}{\xi} d\xi, \ n \geq 2; \qquad \mathrm{Li}_1(z) \equiv -\ln(1-z)\,, \tag{4.18}$$

are a subset of multiple polylogarithms with weight and depth of 1. Nielsen's generalized polylogarithms, defined by

$$S_{n,p}(x) = \frac{(-1)^{n-1+p}}{(n-1)!p!} \int\limits_0^1 dt \frac{\ln^{n-1}(t)\ln^p(1-tx)}{t}\,, \tag{4.19}$$

are related to the multiple polylogarithms by

$$S_{n,p}(x) = Li_{\underbrace{1,...,1}_{p-1},n+1}(\underbrace{1,...,1}_{p-1},x)\,. \tag{4.20}$$

The harmonic polylogarithms of Remiddi and Vermaseren [45] are defined recursively via

$$H_0(x) = \ln(x), \qquad H_1(x) = -\ln(1-x), \qquad H_{-1}(x) = \ln(1+x)\ , \tag{4.21}$$

and

$$\begin{aligned} H_{m_1+1,m_2,...,m_k} &= \int_0^x dt f_0(t) H_{m_1,m_2,...,m_k}(t), \\ H_{\pm 1,m_2,...,m_k} &= \int_0^x dt f_{\pm 1}(t) H_{m_2,...,m_k}(t), \end{aligned} \tag{4.22}$$

where the fractions $f_0(x)$, $f_1(x)$ and $f_{-1}(x)$ are given by

$$f_0(x) = \frac{1}{x}, \qquad f_1(x) = \frac{1}{1-x}, \qquad f_{-1}(x) = \frac{1}{1+x}. \tag{4.23}$$

The harmonic polylogarithms are also a subset of the multiple polylogarithms. For example for positive indices the harmonic polylogarithms are related to the multiple polylogarithms by

$$H_{m_1,...,m_k}(x) = Li_{m_k,...,m_1}(\underbrace{1,...,1}_{k-1}, x). \tag{4.24}$$

The two-dimensional harmonic polylogarithms (2dHPL) [46] are defined as an extension of the harmonic polylogarithms Eqs. (4.21),(4.22) and (4.23) by introduction of the new fractions:

$$f(z,x) = \frac{1}{z+x}, \qquad f(1-z,x) = \frac{1}{1-z-x}. \tag{4.25}$$

From the integral representation Eq. (4.16) it is clear that the 2dHPL are a subset of Goncharov's multiple polylogarithms:

$$\begin{aligned} \int_0^x dt f(z,t) \mathrm{Li}_{m_k,...,m_1}\left(x_k,...,x_2,\frac{t}{x_2...x_k}\right) &= -\mathrm{Li}_{m_k,...,m_1,1}\left(x_k,...,x_2,-\frac{z}{x_2...x_k},-\frac{x}{z}\right), \\ \int_0^x dt f(1-z,t) \mathrm{Li}_{m_k,...,m_1}\left(x_k,...,x_2,\frac{t}{x_2...x_k}\right) &= \mathrm{Li}_{m_k,...,m_1,1}\left(x_k,...,x_2,\frac{1-z}{x_2...x_k},\frac{x}{1-z}\right). \end{aligned} \tag{4.26}$$

In this section I show that all $L-$functions which occur in the results of the Sec. 2 can also be expressed in terms of the multiple polylogarithms.

4.3.2 General formula for the L_{-++}-function

I begin first with the L_{-++}-function Eq. (2.46)

$$L_{-++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\int\limits_0^1 dy\frac{\ln(\alpha_1-y)\ln(\alpha_2+y)\ln(\alpha_3+y)}{\alpha_4+y}. \tag{4.27}$$

After changing the integration variable $y=\alpha_1 t$ one gets

$$\begin{aligned}\int\limits_0^{1/\alpha_1} dt\frac{\ln(\alpha_1-\alpha_1 t)\ln(\alpha_2+\alpha_1 t)\ln(\alpha_3+\alpha_1 t)}{\frac{\alpha_4}{\alpha_1}+t} &= \int\limits_0^{1/\alpha_1} dt\frac{\ln\alpha_1\ln(\alpha_2+\alpha_1 t)\ln(\alpha_3+\alpha_1 t)}{\frac{\alpha_4}{\alpha_1}+t}\\ &+\int\limits_0^{1/\alpha_1} dt\frac{\ln(1-t)[\ln\alpha_1+\ln(\frac{\alpha_2}{\alpha_1}+t)][\ln\alpha_1+\ln(\frac{\alpha_3}{\alpha_1}+t)]}{\frac{\alpha_4}{\alpha_1}+t}=\\ &\ln\alpha_1\int\limits_0^1 dy\frac{\ln(\alpha_2+y)\ln(\alpha_3+y)}{\alpha_4+y}+\ln^2\alpha_1\int\limits_0^{1/\alpha_1} dt\frac{\ln(1-t)}{\frac{\alpha_4}{\alpha_1}+t}\\ +\ln\alpha_1\int\limits_0^{1/\alpha_1} dt\frac{\ln(1-t)\ln(\frac{\alpha_2}{\alpha_1}+t)}{\frac{\alpha_4}{\alpha_1}+t} &+\ln\alpha_1\int\limits_0^{1/\alpha_1} dt\frac{\ln(1-t)\ln(\frac{\alpha_3}{\alpha_1}+t)}{\frac{\alpha_4}{\alpha_1}+t}\\ &+\int\limits_0^{1/\alpha_1} dt\frac{\ln(1-t)\ln(\frac{\alpha_2}{\alpha_1}+t)\ln(\frac{\alpha_3}{\alpha_1}+t)}{\frac{\alpha_4}{\alpha_1}+t}.\end{aligned} \tag{4.28}$$

With the help of (4.16) the integral in the second term from the last expression can be written as

$$\int\limits_0^{1/\alpha_1} dt\frac{\ln(1-t)}{\frac{\alpha_4}{\alpha_1}+t}=\int\limits_0^{1/\alpha_1}\frac{dt_1}{-\frac{\alpha_4}{\alpha_1}-t_1}\int\limits_0^{t_1}\frac{dt_2}{1-t_2}=Li_{1,1}(-\frac{\alpha_4}{\alpha_1},-\frac{1}{\alpha_4}). \tag{4.29}$$

The third and fourth terms contain integrals of the form

$$\int\limits_0^{t_m} dt\frac{\ln(1-t)\ln(\beta_1+t)}{\beta_2+t}\,. \tag{4.30}$$

To express such integral in terms of multiple polylogarithms one proceeds as follows:

$$\begin{aligned}-Li_{1,1,1}\left(-\beta_2,\frac{\beta_1}{\beta_2},\frac{-t_m}{\beta_1}\right) &= \int\limits_0^{t_m}\frac{dt_2}{\beta_1+t_2}\int\limits_0^{t_2} dt_1\frac{\ln(1-t_1)}{\beta_2+t_1}=\int\limits_0^{t_m} dt_1\frac{\ln(1-t_1)}{\beta_2+t_1}\int\limits_{t_1}^{t_m}\frac{dt_2}{\beta_1+t_2}=\\ &\ln(\beta_1+t_m)\int\limits_0^{t_m} dt_1\frac{\ln(1-t_1)}{\beta_2+t_1}-\int\limits_0^{t_m} dt_1\frac{\ln(1-t_1)\ln(\beta_1+t_1)}{\beta_2+t_1}=\\ &\ln(\beta_1+t_m)Li_{1,1}\left(-\beta_2,\frac{-t_m}{\beta_2}\right)-\int\limits_0^{t_m} dt_1\frac{\ln(1-t_1)\ln(\beta_1+t_1)}{\beta_2+t_1},\end{aligned} \tag{4.31}$$

where in the first line I have changed the order of integration in the two-dimensional integral. I shall use this trick often in forthcoming transformations. From Eq. (4.31) one immediately concludes that

$$\int_0^{t_m} dt \frac{\ln(1-t)\ln(\beta_1+t)}{\beta_2+t} = Li_{1,1,1}\left(-\beta_2, \frac{\beta_1}{\beta_2}, \frac{-t_m}{\beta_1}\right) + \ln(\beta_1+t_m)\, Li_{1,1}\left(-\beta_2, \frac{-t_m}{\beta_2}\right). \tag{4.32}$$

Let us now turn to the more involved integral (first term of Eq. (4.28)):

$$\begin{aligned} &\int_0^1 dy \frac{\ln(\alpha_2+y)\ln(\alpha_3+y)}{\alpha_4+y} \overset{y\to -\alpha_2 t}{=} -\int_0^{-1/\alpha_2} dt \frac{[\ln\alpha_2+\ln(1-t)]\ln(\alpha_3-\alpha_2 t)}{\frac{\alpha_4}{\alpha_2}-t} = \\ &-\ln\alpha_2 \int_0^{-1/\alpha_2} dt \frac{\ln(\alpha_3-\alpha_2 t)}{\frac{\alpha_4}{\alpha_2}-t} - \int_0^{-1/\alpha_2} dt \frac{\ln(1-t)[\ln\alpha_2+\ln(\frac{\alpha_3}{\alpha_2}-t)]}{\frac{\alpha_4}{\alpha_2}-t} = \\ &+\ln\alpha_2 \int_0^1 dy \frac{\ln(\alpha_3+y)}{\alpha_4+y} - \ln\alpha_2 \int_0^{-1/\alpha_2} dt \frac{\ln(1-t)}{\frac{\alpha_4}{\alpha_2}-t} - \int_0^{-1/\alpha_2} dt \frac{\ln(1-t)\ln(\frac{\alpha_3}{\alpha_2}-t)}{\frac{\alpha_4}{\alpha_2}-t}. \end{aligned} \tag{4.33}$$

The integral in the first term can be expressed as

$$\begin{aligned} \int_0^1 dy \frac{\ln(\alpha_3+y)}{\alpha_4+y} &\overset{y\to -\alpha_3 t}{=} -\int_0^{-1/\alpha_3} dt \frac{[\ln\alpha_3+\ln(1-t)]}{\frac{\alpha_4}{\alpha_3}-t} = \\ &\ln\alpha_3 \ln\left(\frac{\alpha_4+1}{\alpha_4}\right) + Li_{1,1}\left(\frac{\alpha_4}{\alpha_3}, -\frac{1}{\alpha_3}\right). \end{aligned} \tag{4.34}$$

The integral in the second term can be written as

$$\int_0^{-1/\alpha_2} dt \frac{\ln(1-t)}{\frac{\alpha_4}{\alpha_2}-t} = -Li_{1,1}\left(\frac{\alpha_4}{\alpha_2}, -\frac{1}{\alpha_2}\right). \tag{4.35}$$

The third term from the last line of Eq. (4.33) has a form which is an analogue of the integral (4.30) and can be calculated in a similar way:

$$\int_0^{t_m} dt \frac{\ln(1-t)\ln(\beta_1-t)}{\beta_2-t} = Li_{1,1,1}\left(\beta_2, \frac{\beta_1}{\beta_2}, \frac{t_m}{\beta_1}\right) + \ln(\beta_1-t_m)\, Li_{1,1}\left(\beta_2, \frac{t_m}{\beta_2}\right). \tag{4.36}$$

Combining the Eqs. (4.34), (4.35) and (4.36) we arrive at the result for Eq. (4.33)

$$\begin{aligned} \int_0^1 dy \frac{\ln(\alpha_2+y)\ln(\alpha_3+y)}{\alpha_4+y} &= Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_2}, \frac{\alpha_3}{\alpha_4}, -\frac{1}{\alpha_3}\right) + \ln\alpha_2 Li_{1,1}\left(\frac{\alpha_4}{\alpha_3}, -\frac{1}{\alpha_4}\right) \\ &+ \ln(1+\alpha_3) Li_{1,1}\left(\frac{\alpha_4}{\alpha_2}, -\frac{1}{\alpha_4}\right) + \ln\alpha_2 \ln\alpha_3 \ln\left(\frac{\alpha_4+1}{\alpha_4}\right). \end{aligned} \tag{4.37}$$

Because the initial integrand is symmetric under the exchange of the parameters α_2 and α_3, the r.h.s. of (4.37) can be rewritten in a symmetric form if desired.

I am now left with fifth term of (4.28). It is an integral of the type

$$\int_0^{t_m} dt \frac{\ln(1-t)\ln(\gamma_1+t)\ln(\gamma_2+t)}{\gamma_3+t} . \tag{4.38}$$

In order to express such integrals in terms of polylogarithms one can perform the following chain of transformations with a polylogarithm of weight four:

$$\begin{aligned} -Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{\gamma_1}{\gamma_2}, \frac{-t_m}{\gamma_1}\right) &= \int_0^{t_m} \frac{dt_4}{\gamma_1+t_4} \int_0^{t_4} \frac{dt_3}{\gamma_2+t_3} \int_0^{t_3} \frac{dt_2}{\gamma_3+t_2} \int_0^{t_2} \frac{dt_1}{1-t_1} = \quad (4.39) \\ -\int_0^{t_m} \frac{dt_4}{\gamma_1+t_4} \int_0^{t_4} \frac{dt_3}{\gamma_2+t_3} \int_0^{t_3} dt_2 \frac{\ln(1-t_2)}{\gamma_3+t_2} &= -\int_0^{t_m} \frac{dt_4}{\gamma_1+t_4} \int_0^{t_4} dt_2 \frac{\ln(1-t_2)}{\gamma_3+t_2} \int_{t_2}^{t_4} \frac{dt_3}{\gamma_2+t_3} = \\ -\int_0^{t_m} dt_4 \frac{\ln(\gamma_2+t_4)}{\gamma_1+t_4} \int_0^{t_4} dt_2 \frac{\ln(1-t_2)}{\gamma_3+t_2} &+ \int_0^{t_m} \frac{dt_4}{\gamma_1+t_4} \int_0^{t_4} dt_2 \frac{\ln(1-t_2)\ln(\gamma_2+t_2)}{\gamma_3+t_2} = \\ -I'(t_m) &+ \int_0^{t_m} dt_2 \frac{\ln(1-t_2)\ln(\gamma_2+t_2)}{\gamma_3+t_2} \int_{t_2}^{t_m} \frac{dt_4}{\gamma_1+t_4} = \\ -I'(t_m) + I''(t_m) &- \int_0^{t_m} dt_2 \frac{\ln(\gamma_1+t_2)\ln(\gamma_2+t_2)\ln(1-t_2)}{\gamma_3+t_2}, \end{aligned}$$

where I have introduced the notation

$$\begin{aligned} I'(t_m) &= \int_0^{t_m} dt_4 \frac{\ln(\gamma_2+t_4)}{\gamma_1+t_4} \int_0^{t_4} dt_2 \frac{\ln(1-t_2)}{\gamma_3+t_2}, \\ I''(t_m) &= \ln(\gamma_1+t_m) \int_0^{t_m} dt_2 \frac{\ln(1-t_2)\ln(\gamma_2+t_2)}{\gamma_3+t_2}. \end{aligned} \tag{4.40}$$

The third term on the last line of (4.39) is exactly the integral of the required type Eq. (4.38).

The integral in $I''(t_m)$ has the form of (4.32). For the integral $I'(t_m)$ I write

$$I'(t_m) = \int_0^{t_m} dt_4 \frac{\ln(\gamma_2+t_4)}{\gamma_1+t_4} \int_0^{t_4} dt_2 \frac{\ln(1-t_2)}{\gamma_3+t_2} = \int_0^{t_m} dt_4 \frac{\ln(\gamma_2+t_4)}{\gamma_1+t_4} Li_{1,1}\left(-\gamma_3, \frac{-t_4}{\gamma_3}\right). \tag{4.41}$$

On the other hand one has

$$Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{\gamma_2}{\gamma_1}, \frac{-t_m}{\gamma_2}\right) = \int_0^{t_m} \frac{dt_2}{\gamma_2+t_2} \int_0^{t_2} \frac{dt_1}{\gamma_1+t_1} Li_{1,1}\left(-\gamma_3, \frac{-t_1}{\gamma_3}\right) = \tag{4.42}$$

$$\int_0^{t_m} \frac{dt_1}{\gamma_1+t_1} Li_{1,1}\left(-\gamma_3, \frac{-t_1}{\gamma_3}\right) \int_{t_1}^{t_m} \frac{dt_2}{\gamma_2+t_2} =$$

$$\ln(\gamma_2+t_m) \int_0^{t_m} \frac{dt_1}{\gamma_1+t_1} Li_{1,1}\left(-\gamma_3, \frac{-t_1}{\gamma_3}\right) - \int_0^{t_m} dt_1 \frac{\ln(\gamma_2+t_1)}{\gamma_1+t_1} Li_{1,1}\left(-\gamma_3, \frac{-t_1}{\gamma_3}\right) =$$

$$-\ln(\gamma_2+t_m) Li_{1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{-t_m}{\gamma_1}\right) - I'(t_m).$$

One then concludes that

$$I'(t_m) = -Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{\gamma_2}{\gamma_1}, \frac{-t_m}{\gamma_2}\right) - \ln(\gamma_2+t_m) Li_{1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{-t_m}{\gamma_1}\right). \tag{4.43}$$

Finally, substituting $I'(t_m)$ and $I''(t_m)$ into Eq. (4.39) we write down the result for the integral of the required type Eq. (4.38):

$$\begin{aligned} \int_0^{t_m} dt \frac{\ln(1-t)\ln(\gamma_1+t)\ln(\gamma_2+t)}{\gamma_3+t} &= \ln(\gamma_1+t_m)\ln(\gamma_2+t_m) Li_{1,1}\left(-\gamma_3, \frac{-t_m}{\gamma_3}\right) + \\ \ln(\gamma_2+t_m) Li_{1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{-t_m}{\gamma_1}\right) &+ \ln(\gamma_1+t_m) Li_{1,1,1}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{-t_m}{\gamma_2}\right) \\ +Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{\gamma_1}{\gamma_2}, \frac{-t_m}{\gamma_1}\right) &+ Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{\gamma_2}{\gamma_1}, \frac{-t_m}{\gamma_2}\right). \end{aligned} \tag{4.44}$$

I am now in the position to collect all required contributions to express the L_{-++}-function in terms of multiple polylogarithms. Taking into account Eqs. (4.29), (4.32), (4.37) and (4.44) I obtain

$$\begin{aligned} L_{-++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= Li_{1,1,1,1}\left(-\frac{\alpha_4}{\alpha_1}, \frac{\alpha_2}{\alpha_4}, \frac{\alpha_3}{\alpha_2}, -\frac{1}{\alpha_3}\right) \\ +Li_{1,1,1,1}\left(-\frac{\alpha_4}{\alpha_1}, \frac{\alpha_3}{\alpha_4}, \frac{\alpha_2}{\alpha_3}, -\frac{1}{\alpha_2}\right) &+ \ln\alpha_1 Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_2}, \frac{\alpha_3}{\alpha_4}, -\frac{1}{\alpha_3}\right) \\ +\ln(1+\alpha_2) Li_{1,1,1}\left(-\frac{\alpha_4}{\alpha_1}, \frac{\alpha_3}{\alpha_4}, -\frac{1}{\alpha_3}\right) &+ \ln(1+\alpha_3) Li_{1,1,1}\left(-\frac{\alpha_4}{\alpha_1}, \frac{\alpha_2}{\alpha_4}, -\frac{1}{\alpha_2}\right) \\ +\ln\alpha_1\ln\alpha_2 Li_{1,1}\left(\frac{\alpha_4}{\alpha_3}, -\frac{1}{\alpha_4}\right) &+ \ln\alpha_1\ln(1+\alpha_3) Li_{1,1}\left(\frac{\alpha_4}{\alpha_2}, -\frac{1}{\alpha_4}\right) \\ +\ln(1+\alpha_2)\ln(1+\alpha_3) Li_{1,1}\left(-\frac{\alpha_4}{\alpha_1}, -\frac{1}{\alpha_4}\right) &+ \ln\alpha_1\ln\alpha_2\ln\alpha_3\ln\left(\frac{\alpha_4+1}{\alpha_4}\right). \end{aligned} \tag{4.45}$$

Some remarks are necessary here. The final formula (4.45) contains multiple polylogarithm up to weight four. All multiple polylogarithms up to weight three can be expressed in terms of the classical polylogarithms. This option will be used later when I will reexpress the results for the massive scalar integrals in terms of multiple polylogarithms. For the variables α_i the conditions (4.3) are assumed. But in the results for the massive scalar integrals there are also cases where $\alpha_1 = 1$ and/or $\alpha_2 = 0$ and/or $\alpha_3 = 0$ and/or $\alpha_4 = \{-1, 0\}$. In such cases the general formula (4.45) is no longer valid and these cases must be studied separately.

4.3.3 Special cases for the L_{-++}-function

In the Laurent series expansion of the massive scalar one-loop integrals the following combinations of special values of the α_i appear:

4.3.3.1 $\alpha_1 = 1, \alpha_4 = 0$

In such case one can make use of Eq. (4.44). One should find the limit of the expression on the right side for $t_m = 1, \gamma_3 \to 0$. One obtains

$$\int_0^1 dt \frac{\ln(1-t)\ln(\gamma_1+t)\ln(\gamma_2+t)}{t} =$$
$$\lim_{\gamma_3\to 0}\Big\{\ln(\gamma_1+1)\ln(\gamma_2+1)\int_0^1 \frac{dt_2}{-\gamma_3-t_2}\int_0^{t_2}\frac{dt_1}{1-t_1} + \ln(\gamma_2+1)\int_0^1 \frac{dt_3}{-\gamma_1-t_3}\int_0^{t_3}\frac{dt_2}{-\gamma_3-t_2}\int_0^{t_2}\frac{dt_1}{1-t_1}$$
$$+\ln(\gamma_1+1)\int_0^1 \frac{dt_3}{-\gamma_2-t_3}\int_0^{t_3}\frac{dt_2}{-\gamma_3-t_2}\int_0^{t_2}\frac{dt_1}{1-t_1} + \int_0^1 \frac{dt_4}{-\gamma_1-t_3}\int_0^{t_4}\frac{dt_3}{-\gamma_2-t_3}\int_0^{t_3}\frac{dt_2}{-\gamma_3-t_2}\int_0^{t_2}\frac{dt_1}{1-t_1}$$
$$+\int_0^1 \frac{dt_4}{-\gamma_2-t_3}\int_0^{t_4}\frac{dt_3}{-\gamma_1-t_3}\int_0^{t_3}\frac{dt_2}{-\gamma_3-t_2}\int_0^{t_2}\frac{dt_1}{1-t_1}\Big\} = -\ln(\gamma_1+1)\ln(\gamma_2+1)\int_0^1\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_1}{1-t_1}$$
$$-\ln(\gamma_2+1)\int_0^1 \frac{dt_3}{-\gamma_1-t_3}\int_0^{t_3}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_1}{1-t_1} - \ln(\gamma_1+1)\int_0^1 \frac{dt_3}{-\gamma_2-t_3}\int_0^{t_3}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_1}{1-t_1}$$
$$-\int_0^1 \frac{dt_4}{-\gamma_1-t_3}\int_0^{t_4}\frac{dt_3}{-\gamma_2-t_3}\int_0^{t_3}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_1}{1-t_1} - \int_0^1 \frac{dt_4}{-\gamma_2-t_3}\int_0^{t_4}\frac{dt_3}{-\gamma_1-t_3}\int_0^{t_3}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_1}{1-t_1}. \tag{4.46}$$

To get the expression under the sign of the limit in Eq. (4.46) I have applied the definition (4.16) for the multiple polylogarithms in Eq. (4.44). Using the same definition for the final multidimensional integrals in (4.46) and making the change $\gamma_1 \to \alpha_2, \gamma_2 \to \alpha_3$ one finally arrives at the result for the case $\alpha_1 = 1$ and $\alpha_4 = 0$:

$$L_{-++}(1,\alpha_2,\alpha_3,0) = -\ln(\alpha_2+1)\ln(\alpha_3+1)\zeta(2)$$
$$-\ln(\alpha_3+1)Li_{2,1}\left(-\alpha_2, -\frac{1}{\alpha_2}\right) - \ln(\alpha_2+1)Li_{2,1}\left(-\alpha_3, -\frac{1}{\alpha_3}\right)$$
$$-Li_{2,1,1}\left(-\alpha_3, \frac{\alpha_2}{\alpha_3}, -\frac{1}{\alpha_2}\right) - Li_{2,1,1}\left(-\alpha_2, \frac{\alpha_3}{\alpha_2}, -\frac{1}{\alpha_3}\right). \tag{4.47}$$

4.3.3.2 $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$

For these values of the parameters α_i one has an integral of the very simple form

$$L_{-++}(1,0,0,\alpha_4) = \int_0^1 dy \frac{\ln(1-y)\ln^2 y}{\alpha_4 + y}.$$

After a change of variable $y \to 1-t$ one gets

$$\int_0^1 dt \frac{\ln t \ln^2(1-t)}{\alpha_4 + 1 - t} = -\int_0^1 dt_1 \frac{\ln^2(1-t_1)}{\alpha_4+1-t_1} \int_{t_1}^1 \frac{dt_2}{t_2} =$$

$$-\int_0^1 \frac{dt_2}{t_2} \int_0^{t_2} dt_1 \frac{\ln^2(1-t_1)}{\alpha_4+1-t_1} = -2\int_0^1 \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_1}{\alpha_4+1-t_1} \int_0^{t_1} \frac{dt_3}{1-t_3} \int_0^{t_3} \frac{dt_4}{1-t_4}. \tag{4.48}$$

Applying the definition (4.16) one obtains

$$L_{-++}(1,0,0,\alpha_4) = -2Li_{1,1,2}\left(1, \alpha_4+1, \frac{1}{\alpha_4+1}\right). \tag{4.49}$$

4.3.3.3 $\alpha_1 = 1$ and $\alpha_2 = 0$ (and $\alpha_4 = -1$)

I should find again the limit of the r.h.s. of (4.44) for $t_m = 1$ and $\gamma_1 \to 0$. The first and the third terms are equal to 0 because of the $\lim_{\gamma_1\to 0}\ln(\gamma_1+1) = 0$. The other terms transform into

$$\lim_{\gamma_1\to 0} Li_{1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{-1}{\gamma_1}\right) = -Li_{1,2}\left(-\gamma_3, -\frac{1}{\gamma_3}\right),$$

$$\lim_{\gamma_1\to 0} Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{\gamma_1}{\gamma_2}, \frac{-t_m}{\gamma_1}\right) = -Li_{1,1,2}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{-1}{\gamma_2}\right),$$

$$\lim_{\gamma_1\to 0} Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{\gamma_2}{\gamma_1}, \frac{-t_m}{\gamma_2}\right) = -Li_{1,2,1}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{-1}{\gamma_2}\right).$$

Finally I write

$$L_{-++}(1,0,\alpha_3,\alpha_4) = -Li_{1,1,2}\left(-\alpha_4, \frac{\alpha_3}{\alpha_4}, \frac{-1}{\alpha_3}\right) - Li_{1,2,1}\left(-\alpha_4, \frac{\alpha_3}{\alpha_4}, \frac{-1}{\alpha_3}\right) - \ln(\alpha_3+1) Li_{1,2}\left(-\alpha_4, -\frac{1}{\alpha_4}\right). \tag{4.50}$$

For the special case with $\alpha_4 = -1$ one gets

$$L_{-++}(1,0,\alpha_3,-1) = -Li_{1,1,2}\left(1, -\alpha_3, \frac{-1}{\alpha_3}\right) - Li_{1,2,1}\left(1, -\alpha_3, \frac{-1}{\alpha_3}\right) - \ln(\alpha_3+1)\,\zeta(3). \tag{4.51}$$

4.3.3.4 $\alpha_2 = \alpha_3 = 0$ (and $\alpha_4 = -1$)

In this case one proceeds along the following lines:

$$L_{-++}(\alpha_1, 0, 0, \alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1 - y)\ln^2 y}{\alpha_4 + y} \overset{y\to 1-t}{=} \int_0^1 dt \frac{\ln(\alpha_1 - 1 + t)\ln^2(1-t)}{\alpha_4 + 1 - t} =$$
$$-\int_0^1 dt_1 \frac{\ln^2(1-t_1)}{-\alpha_4 - 1 + t_1} \int_{-\alpha_1+2}^{t_1} \frac{dt_2}{\alpha_1 - 1 + t_2} = -\int_0^1 dt_1 \frac{\ln^2(1-t_1)}{-\alpha_4 - 1 + t_1} \left\{ \int_1^{t_1} + \int_{-\alpha_1+2}^{1} \right\} \frac{dt_2}{\alpha_1 - 1 + t_2} =$$
$$\int_0^1 \frac{dt_2}{\alpha_1 - 1 + t_2} \int_0^{t_2} dt_1 \frac{\ln^2(1-t_1)}{-\alpha_4 - 1 + t_1} - \ln\alpha_1 \int_0^1 dt_1 \frac{\ln^2(1-t_1)}{-\alpha_4 - 1 + t_1} =$$
$$2\int_0^1 \frac{dt_2}{\alpha_1 - 1 + t_2} \int_0^{t_2} \frac{dt_1}{-\alpha_4 - 1 + t_1} \int_0^{t_1} \frac{dt_3}{1 - t_3} \int_0^{t_3} \frac{dt_4}{1 - t_4} - 2\ln\alpha_1 \mathrm{Li}_3\left(-\frac{1}{\alpha_4}\right). \tag{4.52}$$

Using the definition (4.16) we arrive at the result

$$L_{-++}(\alpha_1, 0, 0, \alpha_4) = 2Li_{1,1,1,1}\left(1, \alpha_4 + 1, \frac{1-\alpha_1}{\alpha_4 + 1}, \frac{1}{1-\alpha_1}\right) - 2\ln\alpha_1 \mathrm{Li}_3\left(-\frac{1}{\alpha_4}\right). \tag{4.53}$$

For the case with $\alpha_4 = -1$ one obtains

$$L_{-++}(\alpha_1, 0, 0, -1) = -2Li_{1,2,1}\left(1, 1-\alpha_1, \frac{1}{1-\alpha_1}\right) - 2\ln\alpha_1\,\zeta(3)\,. \tag{4.54}$$

4.3.3.5 $\alpha_2 = 0$

For this integral I change the integration variable $y \to 1 - t$:

$$\int_0^1 dy \frac{\ln(\alpha_1 - y)\ln y \ln(\alpha_3 + y)}{\alpha_4 + y} = \int_0^1 dt \frac{\ln(1-t)\ln(\alpha_1 - 1 + t)\ln(\alpha_3 + 1 - t)}{\alpha_4 + 1 - t} =$$
$$\int_0^1 dt \frac{\ln(1-t)\ln(\gamma_1 + t)\ln(\gamma_2 - t)}{\gamma_3 - t}\,. \tag{4.55}$$

One notes that the last integral is an analogue of the integral in Eq. (4.44). The calculation proceeds in a similar way:

$$-Li_{1,1,1,1}\left(\gamma_3, \frac{\gamma_2}{\gamma_3}, -\frac{\gamma_1}{\gamma_2}, -\frac{1}{\gamma_1}\right) = \int_0^1 \frac{dt_4}{\gamma_1 + t_4} \int_0^{t_4} \frac{dt_3}{\gamma_2 - t_3} \int_0^{t_3} \frac{dt_2}{\gamma_3 - t_2} \int_0^{t_2} \frac{dt_1}{1 - t_1} =$$
$$-\int_0^{t_m} \frac{dt_4}{\gamma_1 + t_4} \int_0^{t_4} \frac{dt_3}{\gamma_2 - t_3} \int_0^{t_3} dt_2 \frac{\ln(1-t_2)}{\gamma_3 - t_2} = -\int_0^{t_m} \frac{dt_4}{\gamma_1 + t_4} \int_0^{t_4} dt_2 \frac{\ln(1-t_2)}{\gamma_3 - t_2} \int_{t_2}^{t_4} \frac{dt_3}{\gamma_2 - t_3} =$$

$$\int_0^1 dt_4 \frac{\ln(\gamma_2 - t_4)}{\gamma_1 + t_4} \int_0^{t_4} dt_2 \frac{\ln(1-t_2)}{\gamma_3 - t_2} - \int_0^1 \frac{dt_4}{\gamma_1 + t_4} \int_0^{t_4} dt_2 \frac{\ln(1-t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2} =$$
$$Y'(1) - \int_0^1 dt_2 \frac{\ln(1-t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2} \int_{t_2}^1 \frac{dt_4}{\gamma_1 + t_4} =$$
$$Y'(1) - \ln(\gamma_1 + 1) \int_0^1 dt_2 \frac{\ln(1-t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2} + \int_0^1 dt_2 \frac{\ln(1-t_2)\ln(\gamma_1 + t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2} =$$
$$Y'(1) - Y''(1) + \int_0^1 dt_2 \frac{\ln(1-t_2)\ln(\gamma_1 + t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2}, \tag{4.56}$$

where I have introduced the notation

$$Y'(t_m) = \int_0^{t_m} dt_4 \frac{\ln(\gamma_2 - t_4)}{\gamma_1 + t_4} \int_0^{t_4} dt_2 \frac{\ln(1-t_2)}{\gamma_3 - t_2},$$
$$Y''(t_m) = \ln(\gamma_1 + t_m) \int_0^{t_m} dt_2 \frac{\ln(1-t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2}. \tag{4.57}$$

The last term in (4.56) is the required integral. The expansion of the integral $Y'(t_m)$ in terms of multiple polylogarithms is similar to the evaluation of $I'(t_m)$ in Eq. (4.40). The result of the calculation is

$$Y'(t_m) = Li_{1,1,1,1}\left(\gamma_3, -\frac{\gamma_1}{\gamma_3}, -\frac{\gamma_2}{\gamma_1}, \frac{t_m}{\gamma_2}\right) + \ln(\gamma_2 - t_m) Li_{1,1,1}\left(\gamma_3, -\frac{\gamma_1}{\gamma_3}, -\frac{t_m}{\gamma_1}\right). \tag{4.58}$$

For the calculation of $Y''(t_m)$ one can make use of (4.36). Finally using Eqs. (4.56) and (4.58) one arrives at the result

$$\int_0^1 dt \frac{\ln(1-t)\ln(\gamma_1 + t)\ln(\gamma_2 - t)}{\gamma_3 - t} = -Li_{1,1,1,1}\left(\gamma_3, -\frac{\gamma_1}{\gamma_3}, -\frac{\gamma_2}{\gamma_1}, \frac{1}{\gamma_2}\right)$$
$$-Li_{1,1,1,1}\left(\gamma_3, \frac{\gamma_2}{\gamma_3}, -\frac{\gamma_1}{\gamma_2}, -\frac{1}{\gamma_1}\right) - \ln(\gamma_1 + 1) Li_{1,1,1}\left(\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{1}{\gamma_2}\right) \tag{4.59}$$
$$-\ln(\gamma_2 - 1) Li_{1,1,1}\left(\gamma_3, -\frac{\gamma_1}{\gamma_3}, -\frac{1}{\gamma_1}\right) - \ln(\gamma_1 + 1)\ln(\gamma_2 - 1) Li_{1,1}\left(\gamma_3, \frac{1}{\gamma_3}\right).$$

To obtain the formula for the L-function with $\alpha_2 = 0$ I must only change γ_1, γ_2 and γ_3 to $\alpha_1 - 1$, $\alpha_3 + 1$ and $\alpha_4 + 1$ according to Eq. (4.55):

$$L_{-++}(\alpha_1, 0, \alpha_3, \alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1 - y)\ln y \ln(\alpha_3 + y)}{\alpha_4 + y} =$$

$$-Li_{1,1,1,1}\left(1+\alpha_4, \frac{1-\alpha_1}{1+\alpha_4}, \frac{1+\alpha_3}{1-\alpha_1}, \frac{1}{1+\alpha_3}\right) - Li_{1,1,1,1}\left(1+\alpha_4, \frac{1+\alpha_3}{1+\alpha_4}, \frac{1-\alpha_1}{1+\alpha_3}, \frac{1}{1-\alpha_1}\right)$$
$$- \ln\alpha_1 Li_{1,1,1}\left(1+\alpha_4, \frac{1+\alpha_3}{1+\alpha_4}, \frac{1}{1+\alpha_3}\right) - \ln\alpha_3 Li_{1,1,1}\left(1+\alpha_4, \frac{1-\alpha_1}{1+\alpha_4}, \frac{1}{1-\alpha_1}\right)$$
$$- \ln\alpha_1 \ln\alpha_3 Li_{1,1}\left(1+\alpha_4, \frac{1}{1+\alpha_4}\right) . \quad (4.60)$$

For the case with $\alpha_4 = -1$ I calculate the limit of the r.h.s of (4.60) for $\alpha_4 \to 0$ and obtain

$$L_{-++}(\alpha_1, 0, \alpha_3, -1) = Li_{2,1,1}\left(1-\alpha_1, \frac{1+\alpha_3}{1-\alpha_1}, \frac{1}{1+\alpha_3}\right)$$
$$+Li_{2,1,1}\left(1+\alpha_3, \frac{1-\alpha_1}{1+\alpha_3}, \frac{1}{1-\alpha_1}\right) + \ln\alpha_1 Li_{2,1}\left(1+\alpha_3, \frac{1}{1+\alpha_3}\right)$$
$$+ \ln\alpha_3 Li_{2,1}\left(1-\alpha_1, \frac{1}{1-\alpha_1}\right) + \ln\alpha_1 \ln\alpha_3\, \zeta(2) . \quad (4.61)$$

4.3.4 General formula for the L_{+++}-function

I proceed now with the evaluation of the L_{+++}-function

$$L_{+++}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1 + y)\ln(\alpha_2 + y)\ln(\alpha_3 + y)}{\alpha_4 + y} . \quad (4.62)$$

After changing the integration variable $y = -\alpha_1 t$ we obtain:

$$- \int_0^{-1/\alpha_1} dt \frac{\ln(\alpha_1 - \alpha_1 t)\ln(\alpha_2 - \alpha_1 t)\ln(\alpha_3 - \alpha_1 t)}{\frac{\alpha_4}{\alpha_1} - t} = - \int_0^{-1/\alpha_1} dt \frac{\ln\alpha_1 \ln(\alpha_2 - \alpha_1 t)\ln(\alpha_3 + \alpha_1 t)}{\frac{\alpha_4}{\alpha_1} - t}$$
$$- \int_0^{-1/\alpha_1} dt \frac{\ln(1-t)[\ln\alpha_1 + \ln(\frac{\alpha_2}{\alpha_1} - t)][\ln\alpha_1 + \ln(\frac{\alpha_3}{\alpha_1} - t)]}{\frac{\alpha_4}{\alpha_1} - t} =$$
$$\ln\alpha_1 \int_0^1 dy \frac{\ln(\alpha_2 + y)\ln(\alpha_3 + y)}{\alpha_4 + y} - \ln^2\alpha_1 \int_0^{-1/\alpha_1} dt \frac{\ln(1-t)}{\frac{\alpha_4}{\alpha_1} - t}$$
$$- \ln\alpha_1 \int_0^{-1/\alpha_1} dt \frac{\ln(1-t)\ln(\frac{\alpha_2}{\alpha_1} - t)}{\frac{\alpha_4}{\alpha_1} - t} - \ln\alpha_1 \int_0^{-1/\alpha_1} dt \frac{\ln(1-t)\ln(\frac{\alpha_3}{\alpha_1} - t)}{\frac{\alpha_4}{\alpha_1} - t}$$
$$- \int_0^{-1/\alpha_1} dt \frac{\ln(1-t)\ln(\frac{\alpha_2}{\alpha_1} - t)\ln(\frac{\alpha_3}{\alpha_1} - t)}{\frac{\alpha_4}{\alpha_1} - t} . \quad (4.63)$$

The first integral on the r.h.s of (4.63) has been calculated in Eq. (4.37). For the second integral one makes use of the formula (4.35) (the only change is $\alpha_2 \to \alpha_1$). For the evaluation of the third and fourth integrals one uses Eq. (4.36). We are left with the most complicated fifth integral. Let us consider an integral of the type

$$\int_0^{t_m} dt \frac{\ln(1-t)\ln(\gamma_1-t)\ln(\gamma_2-t)}{\gamma_3-t}\,. \tag{4.64}$$

This integral is an analogue of the integral in Eq. (4.44). The calculation proceeds in a similar way. One obtains the result

$$\begin{aligned}\int_0^{t_m} dt \frac{\ln(1-t)\ln(\gamma_1-t)\ln(\gamma_2-t)}{\gamma_3-t} &= -\ln(\gamma_1-t_m)\ln(\gamma_2-t_m)Li_{1,1}\left(\gamma_3,\frac{t_m}{\gamma_3}\right)\\ -\ln(\gamma_2-t_m)Li_{1,1,1}\left(\gamma_3,\frac{\gamma_1}{\gamma_3},\frac{t_m}{\gamma_1}\right) &- \ln(\gamma_1-t_m)Li_{1,1,1}\left(\gamma_3,\frac{\gamma_2}{\gamma_3},\frac{t_m}{\gamma_2}\right)\\ -Li_{1,1,1,1}\left(\gamma_3,\frac{\gamma_2}{\gamma_3},\frac{\gamma_1}{\gamma_2},\frac{t_m}{\gamma_1}\right) &- Li_{1,1,1,1}\left(\gamma_3,\frac{\gamma_1}{\gamma_3},\frac{\gamma_2}{\gamma_1},\frac{t_m}{\gamma_2}\right)\,.\end{aligned} \tag{4.65}$$

Taking into account everything mentioned above for Eq. (4.63) we arrive at the final result for the L_{+++} -function:

$$\begin{aligned}L_{+++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= Li_{1,1,1,1}\left(\frac{\alpha_4}{\alpha_1},\frac{\alpha_2}{\alpha_4},\frac{\alpha_3}{\alpha_2},-\frac{1}{\alpha_3}\right)\\ +Li_{1,1,1,1}\left(\frac{\alpha_4}{\alpha_1},\frac{\alpha_3}{\alpha_4},\frac{\alpha_2}{\alpha_3},-\frac{1}{\alpha_2}\right) &+ \ln\alpha_1 Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_2},\frac{\alpha_3}{\alpha_4},-\frac{1}{\alpha_3}\right)\\ +\ln(1+\alpha_2)Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_1},\frac{\alpha_3}{\alpha_4},-\frac{1}{\alpha_3}\right) &+ \ln(1+\alpha_3)Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_1},\frac{\alpha_2}{\alpha_4},-\frac{1}{\alpha_2}\right)\\ +\ln\alpha_1\ln\alpha_2 Li_{1,1}\left(\frac{\alpha_4}{\alpha_3},-\frac{1}{\alpha_4}\right) &+ \ln\alpha_1\ln(1+\alpha_3)Li_{1,1}\left(\frac{\alpha_4}{\alpha_2},-\frac{1}{\alpha_4}\right)\\ +\ln(1+\alpha_2)\ln(1+\alpha_3)Li_{1,1}\left(\frac{\alpha_4}{\alpha_1},-\frac{1}{\alpha_4}\right) &+ \ln\alpha_1\ln\alpha_2\ln\alpha_3\ln\left(\frac{\alpha_4+1}{\alpha_4}\right)\,.\end{aligned} \tag{4.66}$$

For this equation the conditions (4.3) are assumed. I emphasize that in the results for the massive scalar integrals there is no general case for the L_{+++}-function . Despite that I present this result because it can be useful for other applications. In the results for the massive scalar integral one has only the special cases where $\alpha_1=\alpha_2$ or $\alpha_1=\alpha_3$ as well as the cases with $\alpha_1=0$ and/or $\alpha_2=0$ and/or $\alpha_3=0$ and/or $\alpha_4=\{-1,0\}$. If some α's coincide with each other Eq. (4.66) becomes simpler. For the cases with $\alpha_1=0$ and/or $\alpha_2=0$ and/or $\alpha_3=0$ and/or $\alpha_4=\{-1,0\}$ the general formula (4.66) is no longer valid and these cases must be studied separately.

4.3.5 Special cases for the L_{+++}-function

In the Laurent series expansion of the massive scalar one-loop integrals the following special cases for the α_i appear:

4.3.5.1 $\alpha_1 = \alpha_2$ or $\alpha_1 = \alpha_3$

In Sec. 4.1 it was shown that the L_{+++}-function is symmetric under the permutations α_i and α_j $(i, j = \{\overline{1,3}\}, i \neq j)$. Therefore it suffices to consider the case $\alpha_1 = \alpha_2$. I have the integral

$$L_{+++}(\alpha_1, \alpha_1, \alpha_3, \alpha_4) = \int_0^1 dy \frac{\ln^2(\alpha_1 + y)\ln(\alpha_3 + y)}{\alpha_4 + y}\,. \tag{4.67}$$

This integral can be expressed in different ways. First of all one can directly use Eq. (4.66) replacing α_2 by α_1 . The second possibility is to use symmetry properties. One takes into account the r.h.s. of Eq. (4.66) and notes that the part with multiple polylogarithms of weight four is symmetric under the exchange $\alpha_2 \leftrightarrow \alpha_3$. It allows one to reduce the number of the multiple polylogarithms from two to one. First I apply Eq. (4.66) for the case $\alpha_2 = \alpha_3$ replacing α_3 by α_2. Second I change $\alpha_1 \to \alpha_3$ and $\alpha_2 \to \alpha_1$. After these transformations one obtains the result:

$$\begin{aligned}
L_{+++}(\alpha_1, \alpha_1, \alpha_3, \alpha_4) &= \int_0^1 dy \frac{\ln^2(\alpha_1 + y)\ln(\alpha_3 + y)}{\alpha_4 + y} = \\
&+2Li_{1,1,1,1}\left(\frac{\alpha_4}{\alpha_3}, \frac{\alpha_1}{\alpha_4}, 1, -\frac{1}{\alpha_1}\right) + \ln\alpha_3\, Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_1}, \frac{\alpha_1}{\alpha_4}, -\frac{1}{\alpha_1}\right) \\
+2\ln(1+\alpha_1)Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_3}, \frac{\alpha_1}{\alpha_4}, -\frac{1}{\alpha_1}\right) &+ \ln\alpha_3\left[\ln(\alpha_1 + 1) + \ln\alpha_1\right] Li_{1,1}\left(\frac{\alpha_4}{\alpha_1}, -\frac{1}{\alpha_4}\right) \\
&+\ln^2(1+\alpha_1)Li_{1,1}\left(\frac{\alpha_4}{\alpha_3}, -\frac{1}{\alpha_4}\right) + \ln^2\alpha_1 \ln\alpha_3 \ln\left(\frac{\alpha_4 + 1}{\alpha_4}\right)\,.
\end{aligned} \tag{4.68}$$

There is also a third possibility to express $L_{+++}(\alpha_1, \alpha_1, \alpha_3, \alpha_4)$ in terms of multiple polylogarithms:

$$\begin{aligned}
&\int_0^1 dy \frac{\ln^2(\alpha_1 + y)\ln(\alpha_3 + y)}{\alpha_4 + y} \overset{y \to -\alpha_1 t}{=} -\int_0^{-1/\alpha_1} dt \frac{\ln^2(\alpha_1 - \alpha_1 t)\ln(\alpha_3 - \alpha_1 t)}{\frac{\alpha_4}{\alpha_1} - t} = \\
&-\int_0^{-1/\alpha_1} dt \frac{\left[\ln^2\alpha 1 + 2\ln\alpha_1\ln(1-t) + \ln^2(1-t)\right]\left[\ln\alpha_1 + \ln(\frac{\alpha_3}{\alpha_1} - t)\right]}{\frac{\alpha_4}{\alpha_1} - t} = \\
&+\ln^3\alpha_1 \int_0^1 \frac{dy}{\alpha_4 + y} + \ln^2\alpha_1 \int_0^1 dy \frac{\ln(\alpha_3 + y)}{\alpha_4 + y} - 2\ln^2\alpha_1 \int_0^{-1/\alpha_1} dt \frac{\ln(1-t)}{\frac{\alpha_4}{\alpha_1} - t} \\
&-\ln\alpha_1 \int_0^{-1/\alpha_1} dt \frac{\ln^2(1-t)}{\frac{\alpha_4}{\alpha_1} - t} - 2\ln\alpha_1 \int_0^{-1/\alpha_1} dt \frac{\ln(1-t)\ln(\frac{\alpha_3}{\alpha_1} - t)}{\frac{\alpha_4}{\alpha_1} - t} \\
&-\int_0^{-1/\alpha_1} dt \frac{\ln^2(1-t)\ln(\frac{\alpha_3}{\alpha_1} - t)}{\frac{\alpha_4}{\alpha_1} - t}\,.
\end{aligned} \tag{4.69}$$

The first term can be integrated immediately. For the second and third term one uses Eq. (4.34) and Eq. (4.35), respectively. The integral of the fourth term can be rewritten as

$$\int_0^{-1/\alpha_1} dt \frac{\ln^2(1-t)}{\frac{\alpha_4}{\alpha_1}-t} = 2\int_0^{-1/\alpha_1} \frac{dt_1}{\frac{\alpha_4}{\alpha_1}-t_1}\int_0^{t_1}\frac{dt_2}{1-t_2}\int_0^{t_2}\frac{dt_3}{1-t_3} = 2Li_{1,1,1}\left(1,\frac{\alpha_4}{\alpha_1},\frac{-1}{\alpha_4}\right). \quad (4.70)$$

The fifth term is calculable with Eq. (4.32). To integrate the last term one first evaluates the following integral:

$$\begin{aligned}\int_0^{t_m} dt\frac{\ln^2(1-t)\ln(\beta_1-t)}{\beta_2-t} &= -\int_0^{t_m} dt_1\frac{\ln^2(1-t_1)}{\beta_2-t_1}\left\{\int_{t_m}^{t_1}+\int_{\beta_1-1}^{t_m}\right\}\frac{dt_2}{\beta_1-t_2}\\ &\int_0^{t_m}\frac{dt_2}{\beta_1-t_2}\int_0^{t_2}dt_1\frac{\ln^2(1-t_1)}{\beta_2-t_1}+\ln(\beta_1-t_m)\int_0^{t_m}dt\frac{\ln^2(1-t)}{\beta_2-t} = \\ &2Li_{1,1,1,1}\left(1,\beta_2,\frac{\beta_1}{\beta_2},\frac{t_m}{\beta_1}\right)+2\ln(\beta_1-t_m)Li_{1,1,1}\left(1,\beta_2,\frac{t_m}{\beta_2}\right).\end{aligned} \quad (4.71)$$

Then to calculate the last term of Eq. (4.69) one only has to change β_1, β_2 and t_m by the corresponding combinations of α_i. Finally we arrive at the result for the $L_{+++}(\alpha_1, \alpha_1, \alpha_3, \alpha_4)$-function:

$$\begin{aligned}L_{+++}(\alpha_1,\alpha_1,\alpha_3,\alpha_4) = &-2Li_{1,1,1,1}\left(1,\frac{\alpha_4}{\alpha_1},\frac{\alpha_3}{\alpha_4},-\frac{1}{\alpha_3}\right)-2\ln(\alpha_3+1)Li_{1,1,1}\left(1,\frac{\alpha_4}{\alpha_1},-\frac{1}{\alpha_4}\right)\\ &+2\ln\alpha_1 Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_1},\frac{\alpha_3}{\alpha_4},-\frac{1}{\alpha_3}\right)+2\ln\alpha_1\ln(\alpha_3+1)Li_{1,1}\left(\frac{\alpha_4}{\alpha_1},-\frac{1}{\alpha_4}\right)\\ &+\ln^2\alpha_1 Li_{1,1}\left(\frac{\alpha_4}{\alpha_3},-\frac{1}{\alpha_4}\right)+\ln^2\alpha_1\ln\alpha_3\ln\left(\frac{\alpha_4+1}{\alpha_4}\right).\end{aligned} \quad (4.72)$$

This is the third possibility to express $L_{+++}(\alpha_1, \alpha_1, \alpha_3, \alpha_4)$-function in terms of multiple polylogaritms. Each of the Eqs. (4.68) and (4.72) contains only one multiple polylogarithm of weight four and they are both equally good from this point of view. One has a free choice to apply any of these equations for the required L-functions. The situation with the $L_{+++}(\alpha_1, \alpha_1, \alpha_3, \alpha_4)$-function is an example of the statement that the expansion of the L-functions in terms of multiple polylogarithms is not unique.

4.3.5.2 $\alpha_1 = 0$ (or $\alpha_2 = 0$ or $\alpha_3 = 0$)

For this integral I change the integration variable $y \to 1-t$:

$$\begin{aligned}L_{+++}(0,\alpha_2,\alpha_3,\alpha_4) = \int_0^1 dy\frac{\ln y\ln(\alpha_2+y)\ln(\alpha_3+y)}{\alpha_4+y} = \\ \int_0^1 dt\frac{\ln(1-t)\ln(\alpha_2+1-t)\ln(\alpha_3+1-t)}{\alpha_4+1-t}\end{aligned} \quad (4.73)$$

and using Eq. (4.66) we arrive at the result

$$L_{+++}(0,\alpha_2,\alpha_3,\alpha_4) = -Li_{1,1,1,1}\left(1+\alpha_4, \frac{1+\alpha_2}{1+\alpha_4}, \frac{1+\alpha_3}{1+\alpha_2}, \frac{1}{1+\alpha_3}\right)$$
$$-Li_{1,1,1,1}\left(1+\alpha_4, \frac{1+\alpha_3}{1+\alpha_4}, \frac{1+\alpha_2}{1+\alpha_3}, \frac{1}{1+\alpha_2}\right) - \ln\alpha_2 Li_{1,1,1}\left(1+\alpha_4, \frac{1+\alpha_3}{1+\alpha_4}, \frac{1}{1+\alpha_3}\right)$$
$$-\ln\alpha_3 Li_{1,1,1}\left(1+\alpha_4, \frac{1+\alpha_2}{1+\alpha_4}, \frac{1}{1+\alpha_2}\right) - \ln\alpha_2\ln\alpha_3 Li_{1,1}\left(1+\alpha_4, \frac{1}{1+\alpha_4}\right). \quad (4.74)$$

4.3.5.3 $\alpha_1 = \alpha_2 = 0$

To calculate this integral I again change the integration variable $y \to 1-t$:

$$L_{+++}(0,0,\alpha_3,\alpha_4) = \int_0^1 dy \frac{\ln^2 y \ln(\alpha_3+y)}{\alpha_4+y} =$$
$$\int_0^1 dt \frac{\ln^2(1-t)\ln(\alpha_3+1-t)}{\alpha_4+1-t}. \quad (4.75)$$

For the last integral I use Eq. (4.71). An additional simplification can be done if one notes that

$$Li_{1,1,1}\left(1, \alpha_4+1, \frac{1}{\alpha_4+1}\right) = -\mathrm{Li}_3\left(-\frac{1}{\alpha_4}\right). \quad (4.76)$$

Finally one has

$$L_{+++}(0,0,\alpha_3,\alpha_4) = 2Li_{1,1,1,1}\left(1, \alpha_4+1, \frac{\alpha_3+1}{\alpha_4+1}, \frac{1}{\alpha_3+1}\right) - 2\ln\alpha_3 \mathrm{Li}_3\left(-\frac{1}{\alpha_4}\right). \quad (4.77)$$

4.3.5.4 $\alpha_1 = \alpha_2 = 0$ and $\alpha = -1$ (or $\alpha_2 = \alpha_3 = 0$ and $\alpha_4 = -1$)

In this case one should calculate the limit of the r.h.s. of (4.77) for $t_m = 1$ and $\alpha_4 \to -1$. After this procedure one obtains

$$L_{+++}(0,0,\alpha_3,-1) = -2Li_{1,2,1}\left(1, \alpha_3+1, \frac{1}{\alpha_3+1}\right) - 2\ln\alpha_3\zeta(3). \quad (4.78)$$

For the case $\alpha_2 = \alpha_3 = 0$ and $\alpha_4 = -1$ one can use the same formula. The only change is $\alpha_3 \to \alpha_1$.

4.3.5.5 $\alpha_1 = 0$ and $\alpha_4 = -1$

To obtain the solution for these values of the α_i I have to find the limit of the r.h.s. of (4.74) for $\alpha_4 \to -1$. After taking the limit one arrives at the result

$$L_{+++}(0,\alpha_2,\alpha_3,-1) = +Li_{2,1,1}\left(1+\alpha_2, \frac{1+\alpha_3}{1+\alpha_2}, \frac{1}{1+\alpha_3}\right)$$

$$+Li_{2,1,1}\left(1+\alpha_3, \frac{1+\alpha_2}{1+\alpha_3}, \frac{1}{1+\alpha_2}\right) + \ln\alpha_2 Li_{2,1}\left(1+\alpha_3, \frac{1}{1+\alpha_3}\right) + \ln\alpha_3 Li_{2,1}\left(1+\alpha_2, \frac{1}{1+\alpha_2}\right) + \ln\alpha_2 \ln\alpha_3 \zeta(2). \quad (4.79)$$

4.3.6 General formula for the L_+-function

I start to derive the general formula for the single-index L_+-function Eq. (2.47):

$$L_+(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \int_0^1 dy \, \frac{\ln(\alpha_1 + y)}{\alpha_4 + y} \mathrm{Li}_2(\alpha_2 + \alpha_3 y). \quad (4.80)$$

After changing the integration variable $y \to (t - \alpha_2)/\alpha_3$ one gets

$$L_+ = \int_{\alpha_2}^{\alpha_2+\alpha_3} \frac{dt}{\alpha_3} \frac{\ln(\alpha_1 + \frac{t-\alpha_2}{\alpha_3})}{\alpha_4 + \frac{t-\alpha_2}{\alpha_3}} \mathrm{Li}_2(t) = \int_{\alpha_2}^{\alpha_2+\alpha_3} dt \frac{-\ln\alpha_3 + \ln(\alpha_1\alpha_3 - \alpha_2 + t)}{\alpha_3\alpha_4 - \alpha_2 + t} \mathrm{Li}_2(t). \quad (4.81)$$

The integration interval can be split into two pieces, $[\alpha_2, 0]$ and $[0, \alpha_2 + \alpha_3]$. One can then write L_+ as a sum of four terms:

$$L_+ = -\ln\alpha_3 \left\{ \int_0^{\alpha_2+\alpha_3} - \int_0^{\alpha_2} \right\} \frac{dt}{\gamma + t} \mathrm{Li}_2(t) + \left\{ \int_0^{\alpha_2+\alpha_3} - \int_0^{\alpha_2} \right\} dt \frac{\ln(\alpha + t)}{\gamma + t} \mathrm{Li}_2(t), \quad (4.82)$$

where I have introduced the notation

$$\alpha = \alpha_1\alpha_3 - \alpha_2, \qquad \gamma = \alpha_3\alpha_4 - \alpha_2. \quad (4.83)$$

Looking at Eq. (4.82) it is clear that there are only two different types of integrals to be dealt with:

$$\int_0^{t_m} \frac{dt}{\gamma + t} \mathrm{Li}_2(t) \qquad \text{and} \qquad \int_0^{t_m} dt \frac{\ln(\alpha + t)}{\gamma + t} \mathrm{Li}_2(t). \quad (4.84)$$

with upper limits $t_m = \alpha_2 + \alpha_3$ or $t_m = \alpha_2$. The first integral can be evaluated analytically in terms of standard logarithms and classical polylogarithms up to Li_3. However, the same integral can also be expressed in terms of multiple polylogarithms via the integral representation (4.16), e.g.

$$\int_0^{t_m} \frac{dt}{\gamma + t} \mathrm{Li}_2(t) = \int_0^{t_m} \frac{dt_1}{\gamma + t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_3}{1 - t_3} = -Li_{2,1}\left(-\gamma, \frac{-t_m}{\gamma}\right). \quad (4.85)$$

We now deal with the second integral in (4.84). Consider the following multiple polylogarithm of weight four:

$$Li_{2,1,1}\left(-\gamma, \frac{\alpha}{\gamma}, \frac{t_m}{-\alpha}\right) = \int_0^{t_m} \frac{dt_2}{-\alpha - t_2} \int_0^{t_2} \frac{dt_1}{-\gamma - t_1} \mathrm{Li}_2(t_1) = \int_0^{t_m} \frac{dt_1}{\gamma + t_1} \mathrm{Li}_2(t_1) \int_{t_1}^{t_m} \frac{dt_2}{\alpha + t_2} =$$

$$\int_0^{t_m} \frac{dt_1}{\gamma + t_1} \mathrm{Li}_2(t_1) \ln(\alpha + t_m) - \int_0^{t_m} \frac{dt_1}{\gamma + t_1} \mathrm{Li}_2(t_1) \ln(\alpha + t_1). \quad (4.86)$$

In the first step I have used the usual trick to change the order of integration. As already noted before (see Eq. (4.85)) the first term on the second line can be expressed through a multiple polylogarithm of weight three. Thus one has

$$\int_0^{t_m} dt \frac{\ln(\alpha+t)}{\gamma+t}\mathrm{Li}_2(t) = -Li_{2,1,1}\left(-\gamma, \frac{\alpha}{\gamma}, \frac{t_m}{-\alpha}\right) - Li_{2,1}\left(-\gamma, \frac{-t_m}{\gamma}\right)\ln(\alpha+t_m). \tag{4.87}$$

Finally, substituting Eqs. (4.85) and (4.87) into Eq. (4.82) we arrive at the desired relation

$$\begin{aligned} L_+(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= Li_{2,1,1}\left(\alpha_2-\alpha_3\alpha_4, \frac{\alpha_2-\alpha_1\alpha_3}{\alpha_2-\alpha_3\alpha_4}, \frac{\alpha_2}{\alpha_2-\alpha_1\alpha_3}\right) \\ -Li_{2,1,1}\left(\alpha_2-\alpha_3\alpha_4, \frac{\alpha_2-\alpha_1\alpha_3}{\alpha_2-\alpha_3\alpha_4}, \frac{\alpha_2+\alpha_3}{\alpha_2-\alpha_1\alpha_3}\right) &+ \ln\alpha_1 Li_{2,1}\left(\alpha_2-\alpha_3\alpha_4, \frac{\alpha_2}{\alpha_2-\alpha_3\alpha_4}\right) \\ &- \ln(\alpha_1+1)Li_{2,1}\left(\alpha_2-\alpha_3\alpha_4, \frac{\alpha_2+\alpha_3}{\alpha_2-\alpha_3\alpha_4}\right). \end{aligned} \tag{4.88}$$

I should note that, similar to Eq. (4.45), the conditions (4.3) are assumed for the variables α_i. Also, one cannot directly use Eq. (4.88) if $\alpha_2-\alpha_3\alpha_4=0$ and $\alpha_2-\alpha_1\alpha_3$. However, in the results for the massive scalar integrals these special cases appear as well as the cases where $\alpha_1=0$ and/or $\alpha_2=0$ and/or $\alpha_3=0$ and/or $\alpha_4=\{-1,0\}$. In such cases the general formula (4.88) is no longer valid and these cases must be studied separately.

4.3.7 Special cases for the L_+-function

In the Laurent series expansion of the massive scalar one-loop integrals the following special cases for the arguments of the L_+-functions appear:

4.3.7.1 $\alpha_2-\alpha_3\alpha_4=0$

In this case one has to find the limit of the r.h.s. of Eq. (4.88) for $\alpha_2\to\alpha_3\alpha_4$. First I rewrite the r.h.s of the Eq. (4.88) in terms of multidimensional integrals via the definition (4.16) and then I replace α_2 by $\alpha_3\alpha_4$ and finally I use again the definition (4.16) to obtain the result:

$$\begin{aligned} L_+(\alpha_1,\alpha_3\alpha_4,\alpha_3,\alpha_4) &= -Li_{3,1}\left(\alpha_2-\alpha_1\alpha_3, \frac{\alpha_2}{\alpha_2-\alpha_1\alpha_3}\right) \\ +Li_{3,1}\left(\alpha_2-\alpha_1\alpha_3, \frac{\alpha_2+\alpha_3}{\alpha_2-\alpha_1\alpha_3}\right) &- \ln\alpha_1\mathrm{Li}_3(\alpha_2) + \ln(\alpha_1+1)\mathrm{Li}_3(\alpha_2+\alpha_3). \end{aligned} \tag{4.89}$$

4.3.7.2 $\alpha_1=0$

Unfortunately in this case one cannot use Eq. (4.88) for $\alpha_1=0$ because one is immediately faced with the problem of a logarithmic infinity. One has to find another algorithm to express the $L_+(0,\alpha_2,\alpha_3,\alpha_4)$-function in terms of multiple polylogarithms. After changing the integration variable $y \to 1-t$ one gets

$$\int_0^1 dy\,\frac{\ln y}{\alpha_4+y}\mathrm{Li}_2(\alpha_2+\alpha_3 y) = \int_0^1 dt\,\frac{\ln(1-t)}{\alpha_4+1-t}\mathrm{Li}_2(\alpha_2+\alpha_3-\alpha_3 t) =$$

$$\int_0^1 dt_1\frac{\ln(1-t_1)}{\alpha_4+1-t_1}\int_{\alpha_2/\alpha_3+1}^{t_1} dt_2\frac{\ln(1-\alpha_2-\alpha_3+\alpha_3 t_2)}{\frac{\alpha_2}{\alpha_3}+1-t_2} =$$

$$\int_0^1 dt_1\frac{\ln(1-t_1)}{\alpha_4+1-t_1}\left\{\int_1^{t_1}+\int_{\alpha_2/\alpha_3+1}^{1}\right\}dt_2\frac{\ln(1-\alpha_2-\alpha_3+\alpha_3 t_2)}{\frac{\alpha_2}{\alpha_3}+1-t_2} = \tag{4.90}$$

$$-\int_0^1 dt_2\frac{\ln(1-\alpha_2-\alpha_3+\alpha_3 t_2)}{\frac{\alpha_2}{\alpha_3}+1-t_2}\int_0^{t_2} dt_1\frac{\ln(1-t_1)}{\alpha_4+1-t_1} - \mathrm{Li}_2(\alpha_2)Li_{1,1}\left(\alpha_4+1,\frac{1}{\alpha_4+1}\right).$$

The last integral is an analogue of $I'(t_m)$ in Eq. (4.40). First one notes that

$$\int_0^{t_2} dt_1\frac{\ln(1-t_1)}{\alpha_4+1-t_1} = -Li_{1,1}\left(\alpha_4+1,\frac{t_1}{\alpha_4+1}\right). \tag{4.91}$$

Then one considers the following chain of transformations:

$$\int_0^1 \frac{dt_2}{1-\alpha_2-\alpha_3+\alpha_3 t_2}\int_0^{t_2}\frac{dt_1}{\frac{\alpha_2}{\alpha_3}+1-t_1}Li_{1,1}\left(\alpha_4+1,\frac{t_1}{\alpha_4+1}\right) =$$

$$\int_0^1 \frac{dt_1}{\frac{\alpha_2}{\alpha_3}+1-t_1}Li_{1,1}\left(\alpha_4+1,\frac{t_1}{\alpha_4+1}\right)\int_{t_1}^1\frac{dt_2}{1-\alpha_2-\alpha_3+\alpha_3 t_2} =$$

$$\frac{1}{\alpha_3}\ln(1-\alpha_2)\int_0^1\frac{dt_1}{\frac{\alpha_2}{\alpha_3}+1-t_1}Li_{1,1}\left(\alpha_4+1,\frac{t_1}{\alpha_4+1}\right) \tag{4.92}$$

$$-\frac{1}{\alpha_3}\int_0^1 dt_1\frac{\ln(1-\alpha_2-\alpha_3+\alpha_3 t_1)}{\frac{\alpha_2}{\alpha_3}+1-t_1}Li_{1,1}\left(\alpha_4+1,\frac{t_1}{\alpha_4+1}\right)$$

Using Eq. (4.91) we see that the last integral is exactly the integral required in Eq. (4.90). The initial integral of Eq. (4.92) and the first integral of the r.h.s. of Eq. (4.92) can be expressed in terms of multiple polylogarithms due to the definition (4.16). Finally for the $L_+(0,\alpha_2,\alpha_3,\alpha_4)$-function we obtain

$$L_+(0,\alpha_2,\alpha_3,\alpha_4) = Li_{1,1,1,1}\left(\alpha_4+1,\frac{\alpha_2+\alpha_3}{\alpha_3(\alpha_4+1)},\frac{\alpha_2+\alpha_3-1}{\alpha_2+\alpha_3},\frac{\alpha_3}{\alpha_2+\alpha_3-1}\right) \tag{4.93}$$

$$+\ln(1-\alpha_2)Li_{1,1,1}\left(\alpha_4+1,\frac{\alpha_2+\alpha_3}{\alpha_3(\alpha_4+1)},\frac{\alpha_3}{\alpha_2+\alpha_3}\right) - \mathrm{Li}_2(\alpha_2)Li_{1,1}\left(\alpha_4+1,\frac{1}{\alpha_4+1}\right).$$

4.3.7.3 $\alpha_1 = 0$ and $\alpha_4 = -1$

For these values of the α_i one uses Eq. (4.93) to calculate the limit of the r.h.s. for $\alpha_4 \to -1$. One arrives at the result

$$L_+(0,\alpha_2,\alpha_3,-1) = -Li_{2,1,1}\left(\frac{\alpha_2+\alpha_3}{\alpha_3}, \frac{\alpha_2+\alpha_3-1}{\alpha_2+\alpha_3}, \frac{\alpha_3}{\alpha_2+\alpha_3-1}\right) - \ln(1-\alpha_2)Li_{2,1}\left(\frac{\alpha_2+\alpha_3}{\alpha_3}, \frac{\alpha_3}{\alpha_2+\alpha_3}\right) + \mathrm{Li}_2(\alpha_2)\zeta(2). \tag{4.94}$$

4.3.7.4 $\alpha_1 = 0$ and $\alpha_2 + \alpha_3 = 1$ (and $\alpha_4 = -1$)

If one looks at Eq. (4.93) one realizes, that there is a problem if $\alpha_2+\alpha_4 = 1$. To express the L_+-function for this configuration of the α_i the limit of the r.h.s. of (4.93) for $\alpha_2 \to 1-\alpha_3$ must be found. The result is

$$L_+(0,1-\alpha_3,\alpha_3,\alpha_4) = -Li_{1,1,2}\left(\alpha_4+1, \frac{1}{\alpha_3(\alpha_4+1)}, \alpha_3\right) + \ln\alpha_3 Li_{1,1,1}\left(\alpha_4+1, \frac{1}{\alpha_3(\alpha_4+1)}, \alpha_3\right) - \mathrm{Li}_2(1-\alpha_3)Li_{1,1}\left(\alpha_4+1, \frac{1}{\alpha_4+1}\right). \tag{4.95}$$

For the case $\alpha_1 = 0$, $\alpha_2 + \alpha_3 = 1$ and $\alpha_4 = -1$ one has to find in addition the limit for $\alpha_4 \to -1$. One arrives at the result

$$L_+(0,1-\alpha_3,\alpha_3,-1) = Li_{2,2}\left(\frac{1}{\alpha_3}, \alpha_3\right) - \ln\alpha_3 Li_{2,1}\left(\frac{1}{\alpha_3}, \alpha_3\right) + \zeta(2)\mathrm{Li}_2(1-\alpha_3). \tag{4.96}$$

4.3.7.5 $\alpha_1 = 0$ and $\alpha_2 = -\alpha_3$

To obtain the result for this case one has to calculate the limit of the r.h.s of (4.93) for $\alpha_3 \to -\alpha_2$. After taking the limit one has

$$L_+(0,\alpha_2,-\alpha_2,\alpha_4) = -Li_{1,1,2}\left(\frac{\alpha_2}{\alpha_2-1}, -\alpha_4, -\frac{1}{\alpha_4}\right) + \ln(1-\alpha_2)Li_{1,2}\left(-\alpha_4, -\frac{1}{\alpha_4}\right) + \mathrm{Li}_2(\alpha_2)\mathrm{Li}_2\left(-\frac{1}{\alpha_4}\right). \tag{4.97}$$

4.3.7.6 $\alpha_1 = 0$ and $\alpha_2 = 0$

For this case one can directly use Eq. (4.93):

$$L_+(0,0,\alpha_3,\alpha_4) = Li_{1,1,1,1}\left(\alpha_4+1, \frac{1}{\alpha_4+1}, \frac{\alpha_3-1}{\alpha_3}, \frac{\alpha_3}{\alpha_3-1}\right). \tag{4.98}$$

But there is also another very simple possibility. I change first the integration variable $y \to t/\alpha_3$:

$$\int_0^1 dy\, \frac{\ln y}{\alpha_4+y}\mathrm{Li}_2(\alpha_3 y) = \int_0^{\alpha_3} dt \frac{\ln(t/\alpha_3)}{\alpha_3\alpha_4+t}\mathrm{Li}_2(t) = \int_0^{\alpha_3} \frac{dt_1}{\alpha_3\alpha_4+t_1}\mathrm{Li}_2(t_1)\int_{\alpha_3}^{t_1}\frac{dt_2}{t_2} =$$

$$-\int_0^{\alpha_3}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_1}{\alpha_3\alpha_4+t_1}\mathrm{Li}_2(t_1)=\int_0^{\alpha_3}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_1}{-\alpha_3\alpha_4+t_1}\int_0^{t_1}\frac{dt_3}{t_3}\int_0^{t_3}\frac{dt_4}{1-t_4}. \tag{4.99}$$

Now using the definition (4.16) I obtain the result

$$L_+(0,0,\alpha_3,\alpha_4)=Li_{2,2}\left(-\alpha_3\alpha_4,\frac{-1}{\alpha_4}\right). \tag{4.100}$$

The reader has a free choice to use either formula (4.98) or (4.100). Both equations contain multiple polylogarithm of weight four. The depth of the multiple polylogarithm in Eq. (4.100) is two against four in Eq. (4.98). For $\alpha_4=-1$ Eq. (4.100) can be directly used. However, in the case of Eq. (4.98) one has to first calculate the limit for $\alpha_4\to-1$.

4.3.7.7 $\alpha_1=0$ and $\alpha_2=1$

Unfortunately in this case one cannot use Eq. (4.93) because of the term $\ln(1-\alpha_2)$. To express this L_+ function in terms of multiple polylogarithms I first make use of Eq. (4.13) for the function Li_2 under the sign of the integral:

$$\begin{aligned}
&\int_0^1 dy\frac{\ln y}{\alpha_4+y}\mathrm{Li}_2(1+\alpha_3 y)=\int_0^1 dy\frac{\ln y}{\alpha_4+y}[\zeta(2)-\ln(-\alpha_3 y)\ln(1+\alpha_3 y)-\mathrm{Li}_2(-\alpha_3 y)]=\\
&\zeta(2)\int_0^1 dy\,\frac{\ln y}{\alpha_4+y}-\int_0^1 dy\frac{\ln y}{\alpha_4+y}\mathrm{Li}_2(-\alpha_3 y)-\int_0^1 dy\frac{\ln y[\ln(-\alpha_3)+\ln y]\ln(1+\alpha_3 y)}{\alpha_4+y}=\\
&\qquad\zeta(2)\mathrm{Li}_2\left(-\frac{1}{\alpha_4}\right)-Li_{1,1,1,1}\left(\alpha_4+1,\frac{1}{\alpha_4+1},\frac{\alpha_3+1}{\alpha_3},\frac{\alpha_3}{\alpha_3+1}\right)\\
&\qquad-\ln(-\alpha_3)\int_0^1 dy\,\frac{\ln y\ln(1+\alpha_3 y)}{\alpha_4+y}-\int_0^1 dy\frac{\ln^2 y\ln(1+\alpha_3 y)}{\alpha_4+y},
\end{aligned} \tag{4.101}$$

where the $Li_{1,1,1,1}$-function was obtained with the help of Eq. (4.98). To obtain the last integral in Eq. (4.101) one proceeds as follows:

$$\begin{aligned}
&\int_0^1 dy\frac{\ln^2 y\ln(1+\alpha_3 y)}{\alpha_4+y}\overset{y\to 1-t}{=}\int_0^1 dt\frac{\ln^2(1-t)\ln(1+\alpha_3-\alpha_3 t)}{\alpha_4+1-t}=\\
&\int_0^1 dt_1\frac{\ln^2(1-t_1)}{\alpha_4+1-t_1}\int_1^{t_1}\frac{-\alpha_3 dt_2}{1+\alpha_3-\alpha_3 t_2}=\int_0^1\frac{dt_2}{\frac{1}{\alpha_3}+1-t_2}\int_0^{t_2}dt_1\frac{\ln^2(1-t_1)}{\alpha_4+1-t_1}=\\
&\qquad 2\int_0^1\frac{dt_2}{\frac{1}{\alpha_3}+1-t_2}\int_0^{t_2}\frac{dt_1}{\alpha_4+1-t_1}\int_0^{t_1}\frac{dt_3}{1-t_3}\int_0^{t_3}\frac{dt_4}{1-t_4}=\\
&\qquad 2Li_{1,1,1,1}\left(1,\alpha_4+1,\frac{\alpha_3+1}{\alpha_3(\alpha_4+1)},\frac{\alpha_3}{\alpha_3+1}\right).
\end{aligned} \tag{4.102}$$

Similarly one can evaluate the remaining integral

$$\int_0^1 dy \frac{\ln y \ln(1+\alpha_3 y)}{\alpha_4 + y} = -Li_{1,1,1}\left(\alpha_4+1, \frac{\alpha_3+1}{\alpha_3(\alpha_4+1)}, \frac{\alpha_3}{\alpha_3+1}\right). \tag{4.103}$$

Now combining Eqs. (4.101), (4.102) and (4.103) one arrives at the result

$$\begin{aligned} L_+(0,1,\alpha_3,\alpha_4) = &-2Li_{1,1,1,1}\left(1, \alpha_4+1, \frac{\alpha_3+1}{\alpha_3(\alpha_4+1)}, \frac{\alpha_3}{\alpha_3+1}\right) \\ &-Li_{1,1,1,1}\left(\alpha_4+1, \frac{1}{\alpha_4+1}, \frac{\alpha_3+1}{\alpha_3}, \frac{\alpha_3}{\alpha_3+1}\right) \\ &+\ln(-\alpha_3) Li_{1,1,1}\left(\alpha_4+1, \frac{\alpha_3+1}{\alpha_3(\alpha_4+1)}, \frac{\alpha_3}{\alpha_3+1}\right) + \zeta(2)\mathrm{Li}_2\left(-\frac{1}{\alpha_4}\right). \end{aligned} \tag{4.104}$$

4.3.7.8 $\alpha_1 = 0$ and $\alpha_2 = -\alpha_3 = 1$

For these values of the α_i I have to find the limit of the r.h.s of Eq. (4.104) for $\alpha_3 \to -1$. After taking the limit I obtain

$$\begin{aligned} L_+(0,1,-1,\alpha_4) = &Li_{1,1,2}\left(\alpha_4+1, \frac{1}{\alpha_4+1}, 1\right) \\ &+2Li_{1,1,2}\left(1, \alpha_4+1, \frac{1}{\alpha_4+1}\right) + \zeta(2)\mathrm{Li}_2\left(-\frac{1}{\alpha_4}\right). \end{aligned} \tag{4.105}$$

4.3.8 Conclusion

In this section I have derived numerous relations between the L-functions defined in Eqs. (2.46), (2.47) and the class of multiple polylogarithms. All the relations calculated in this section are needed if one wishes to present the results for the Laurent series expansion of massive scalar one-loop integrals to $\mathcal{O}(\varepsilon^2)$ in terms of multiple polylogarithms instead of the L-functions. However, despite the fact that these relations have been derived for this special task they can be used in other applications. In fact any definite integral such as

$$\int_A^B \frac{\ln(a_1+b_1x)\ln(a_2+b_2x)\ln(a_3+b_3x)dx}{a_4+b_4x} \quad \text{or} \quad \int_A^B \frac{\ln(a_1+b_1x)\mathrm{Li}_2(a_2+b_2x)dx}{a_3+b_3x}$$

can be written in terms of multiple polylogarithms with the help of the relations presented in this section. It is worthwhile to mention that all equations presented in this section have been numerically checked.

The fact that the expansion of the L-functions in terms of multiple polylogarithms is not unique (the reader can find corresponding examples in 4.3.5.1 and 4.3.7.6) should not surprise us. It is well known that even for classical polylogarithms (4.18) various identities exist (see [44] and Eqs. (3.6), (3.13), (3.17), (3.18), (3.21), (3.22) and (3.23)). In the examples given in this section the analogues of these identities for multiple polylogarithms are implicit. More information about identities between multiple polylogarithms can be found in [49].

Chapter 5

Investigation of the numerical efficiency of the results for the massive one-loop scalar integrals

5.1 Allowed kinematic region

The results for the massive one-loop scalar integrals calculated in Chap. 2 were obtained in the physical region for the reaction $g+g \to Q+\bar{Q}$ (or for any other reaction with two initial massless particles and two final massive particles with a mass m). The two-dimensional physical region is defined by

$$s \geq 4m^2,$$
$$t_{min}(s) \leq t \leq t_{max}(s), \tag{5.1}$$

where

$$t_{max}(s) = -\frac{s}{2}\left(1 - \sqrt{1 - \frac{4m^2}{s}}\right),$$
$$t_{min}(s) = -\frac{s}{2}\left(1 + \sqrt{1 - \frac{4m^2}{s}}\right). \tag{5.2}$$

$t_{max}(s)$ and $t_{min}(s)$ correspond to forward and backward scattering, respectively. The lower limit for s is given by the threshold of the reaction. The range of the variable t can be found by expressing t in terms of s and the angle between the momenta $\vec{p}_1$ and $\vec{p}_3$ in the centre of mass system of the incoming (initial) particles. Using Eq. (5.2) one obtains $t_{max}(s)$ and $t_{min}(s)$ at threshold $s = 4m^2$

$$\frac{t_{max}(4m^2)}{m^2} = \frac{t_{min}(4m^2)}{m^2} = -2. \tag{5.3}$$

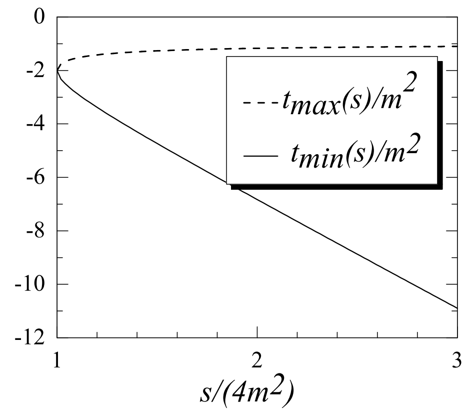

Figure 5.1: Allowed kinematic region.

For $s \to +\infty$ one has for forward scattering :

$$\lim_{s/m^2 \to +\infty} \frac{t_{max}(s)}{s} = \lim_{s/m^2 \to +\infty} -\frac{m^2}{s} = 0 \tag{5.4}$$

and for backward scattering:

$$\lim_{s/m^2 \to +\infty} \frac{t_{min}(s)}{s} = -1. \tag{5.5}$$

Graphically the functions $t_{max}(s)$ and $t_{min}(s)$ are presented in Fig. 5.1. In this chapter I investigate the numerical efficiency inside the allowed kinematic region Eq. (5.1). I pay special attention to the points close to the kinematic boundaries.

5.2 Hardware and software tools for the numerical evaluation

All numerical evaluations were performed on a PC Pentium 4 with a 2.6 GHz processor frequency and 512 MB of memory. As software I have chosen the **Mathematica** program version **5.0** [42]. All numerical results were calculated with the help of this program. Nevertheless the expressions calculated in Chap. 2 can be evaluated by any other Computer Algebra System or by any programming language. Note that use of other software may increase or decrease the efficiency of the numerical evaluation.

5.3 Results obtained in terms of logarithms and polylogarithms

The most difficult results obtained in Chap. 2 contain the new class of functions such as L_{-++}, L_{+++} and L_{+}. These functions are given by the one-fold integral representations Eqs. (2.46) and (2.47). They appear in the real parts of the second order coefficients[1] of the C_1, C_2, C_5 and D_1, D_2, D_3 scalar integrals.

Up to the first order in ε the real parts of the C_1, C_2, C_5 and D_1, D_2, D_3 scalar integrals can be presented in terms of logarithms and classical polylogaritms. The same holds true up to $\mathcal{O}(\varepsilon^2)$ for the real parts of the remaining scalar integrals, i.e. $A, B_1, B_2, B_3, B_4, B_5, C_3, C_4, C_6$ functions and for the imaginary parts of the all scalar integrals. The numerical results of these can be obtained without any numerical integration. Therefore the evaluating efficiency of these expressions are significantly higher compared to the results with the one remaining parametric integration.

I have evaluated the time and the relative error for the first order coefficient of the D_2 function as the most complicated example. For the choice[2] $s = 5m^2$ and $t = -2m^2$ (this point is well inside the allowed kinematic region) one obtains a result with a numerical relative error $\delta \simeq 3.8 \cdot 10^{-16}$ and a evaluation time of $\tau \simeq 0.01$ second. As an example of a kinematical point close to the kinematic boundaries I have chosen $s = 5m^2$ and $t = -1.382m^2$. As a measure of how close one is to the kinematic boundary, one introduces the measure

$$\eta_{max(min)} = \frac{|\, t_{max(min)}(s) - t(s)|}{t_{max}(s) - t_{min}(s)}, \tag{5.6}$$

which describes the closeness of the variable t to the limiting value $t_{max}(s)$ or $t_{min}(s)$, respectively. For the above choice of variables one has $\eta_{max} \simeq 1.5 \cdot 10^{-5}$. The relative error and the evaluation time are $\delta \simeq 8.6 \cdot 10^{-10}$ and $\tau \simeq 0.015$ second, respectively.

One can see that the closer one gets to the kinematic boundary the worse the accuracy of the result is with approximately the same evaluation time. However, the precision is still sufficiently high and the evaluation time is still relatively small. One can also calculate numerically the value for $t = t_{max}(s)$ or $t = t_{min}(s)$ as a numerical limit of the result with $t \to t_{max}(s)$ or $t \to t_{min}(s)$. There is also no problem to increase the accuracy of these calculations to any arbitrary order. The price is an increase in the evaluation time, but the situation is not so dramatic. For example if one would like to achieve a relative error of order $\sim 10^{-16}$ for the choice $s = 5m^2$ and $t = -1.382m^2$ it would cost one only $\simeq 0.13$ second evaluation time.

I have tested in some detail the efficiency of the numerical routine close to the kinematic boundary at $t_{max}(s)$ because one can expect large cancellation within our analytical expressions which may lead to a deterioration of the numerical results.

[1]When referring to the second order coefficients one understands the coefficients of the second order of the Laurent series expansion in ε, where ε is the dimension regularization parameter.

[2]For the numerical evaluation only the ratios s/m^2 and t/m^2 are of relevance.

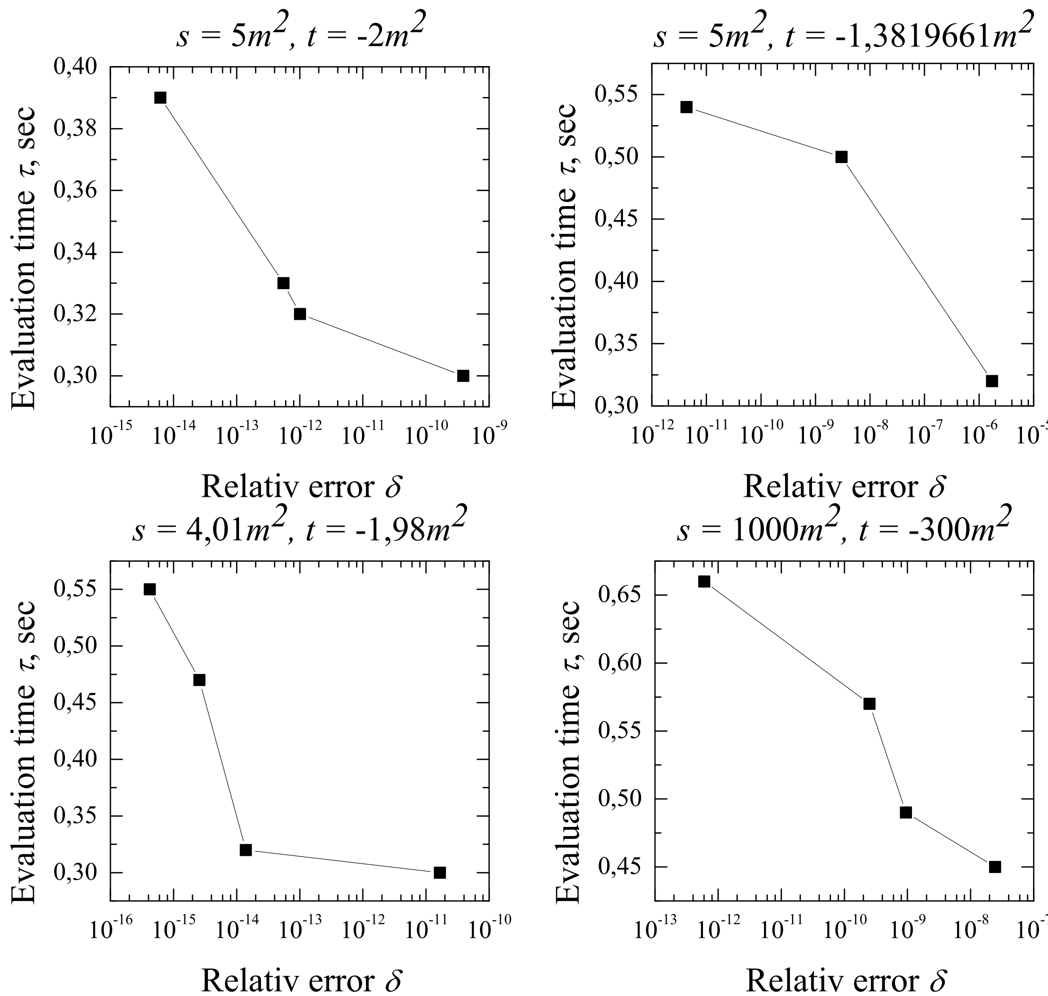

Figure 5.2: The dependence of the evaluation time on the relative error.

5.4 Results containing the L-functions.

The most difficult results for Laurent series expansion of the massive scalar one-loop integrals contain the functions L_{-++}, L_{+++} and L_{+} Eqs. (2.46) and (2.47). As described in Sec. 2.2 these functions are defined by one-fold integral representations. Since all functions L_{-++}, L_{+++} and L_{+} are well integrable one can calculate these integrals with very high accuracy. However, the need of a numerical integration and the fact that the length of the expressions containing the L-functions is larger then those without the L-functions increases the evaluation time.

For the numerical test I have chosen the real part of the second order coefficient of the D_1 function. The length of this expression and the number of L-functions in it is a fair representation of the effort needed for the numerical evaluation of similar expressions.

In Fig. 5.2 I have plotted the dependence of the evaluation time on the relative error.

The first plot in Fig. 5.2 correspond to $s = 5m^2$ and $t = -2m^2$. The shortest evaluation time $\tau \simeq 0.3$ second corresponds to the relative error $\delta \simeq 3.9 \cdot 10^{-10}$. Improvement of the accuracy of the numerical evaluation causes a increase in time. While the precision of the calculation is increased by five orders of magnitude the evaluation time is increased only by 30 percent.

The second test point is the point close to the kinematic boundary. For the variables $s = 5m^2$ and $t = -1.3819661m^2$ the coefficient η_{max} is equal to $8.9 \cdot 10^{-8}$. Comparing to the previous plot one can see that the precision of the numerical evaluation degrades by a few orders of magnitude with approximately the same evaluation time. This fact is a consequence of the closeness of the chosen point to the kinematic boundary $t_{max}(s)$. The reason is the following. The testing expression is divided into two parts: with numerical integration (containing the L-functions) and without numerical integration. Close to the kinematic boundary $t_{max}(s)$ both parts are approximately equal by magnitude but they have opposite signs. Even if the precision of the numerical results obtained separately for both of these parts is high the final relative error is worse by a few orders of magnitude due to the cancellation of large numbers. However, the situation with the accuracy is still satisfying. One is able to achieve any arbitrary accuracy of the calculations for any arbitrary small numbers η_{max} or η_{min}.

The third plot corresponds to a point close to the threshold of the reaction. The variables s and t are equal to $4.01m^2$ and $-1.98m^2$, respectively. Despite the closeness of the variable s to the threshold value $s_{min} = 4m^2$ the accuracy is quite satisfactory. Improving the accuracy by approximately five orders of magnitude the evaluation time increases only by a factor two. It also is worth noting that the numerical evaluation close to the threshold of the reaction does not reflect any special problems w.r.t. the precision of calculations.

Finally, the fourth plot correspond to the case of large colliding energy. I have chosen this kinematical point since it appears that the numerical method of M.M. Weber [51, 52] encounters difficulties when s becomes large. The variables s and t are equal to $1000m^2$ and $-300m^2$, respectively. Also in this case the dependence of the evaluation time on the desirable relative error is quite satisfactory. While the precision of the calculation increases by approximately five orders of magnitude the evaluation time increases less than by a factor two.

One concludes that despite the necessity of the additional one-dimensional numerical integration of the L-functions the results containing the L-functions can be calculated with very high accuracy. There also is no problem to increase the accuracy of the calculations to any desired order.

In recent years a number of new methods were developed for semi-numerical evaluation of general Feynman diagrams (see e.g. [52, 54, 55, 56]). First numerical tests [51] have shown that the efficiency of the results presented in Chap. 2 is better by orders of magnitude than the present implementation of the flexible approach of M.M. Weber described in [52]. The method of M.M. Weber is an all-purpose method to calculate very general one-loop amplitudes. This must be seen in comparison to my evaluation which is specific to a limited number of one-loop diagrams with special mass configurations. Further comparisons of the numerical efficiency of the two methods have to be done in the near future.

I mention that the numerical evaluation of the imaginary parts is also very fast since they are given in terms of logarithms and polylogarithms up to Li_3. First trials of the method of M.M. Weber have shown that the performance of the algorithms described in [51, 52] is not very good for the imaginary parts of the one-loop amplitudes in contrast to our representation of the imaginary parts which numerically evaluate very well.

Chapter 6

One-loop matrix elements for the hadronic heavy hadron production up to $\mathcal{O}(\varepsilon^2)$

In this chapter I present one-loop matrix elements relevant for the hadroproduction of heavy quarks contributing to that part of the NNLO corrections where the one-loop integrals appear in the loop-by-loop product (Fig. 1.1b). All results of the perturbative calculation are given in the dimensional regularization scheme up to $\mathcal{O}(\varepsilon^2)$. In dimensional regularization there are three different sources that can contribute positive ε–powers to the Laurent series of the one–loop amplitudes for the hadronic heavy hadron production. These are

- Laurent series expansion of scalar one-loop integrals
- evaluation of the spin algebra of the loop amplitudes bringing in the n–dimensional metric contraction $g_{\mu\nu}g^{\mu\nu} = n = 4 - 2\varepsilon$
- Passarino–Veltman decomposition of tensor integrals involving again the metric contraction $g_{\mu\nu}g^{\mu\nu} = n = 4 - 2\varepsilon$.

Concerning the first item the $\mathcal{O}(\varepsilon^2)$ calculation of the necessary one-, two-, three- and four-point one-loop integrals for the loop-by-loop part of NNLO QCD calculation has been presented in Chap. 2. In order to obtain the Laurent series expansion of the full one-loop amplitude for the hadronic heavy hadron production one has to combine the Laurent series expansion of the scalar one-loop integrals with the ε-expansion from the spin algebra calculation and the Passarino-Veltman decomposition. The results of this combination for the one-loop matrix elements will be presented in this chapter.

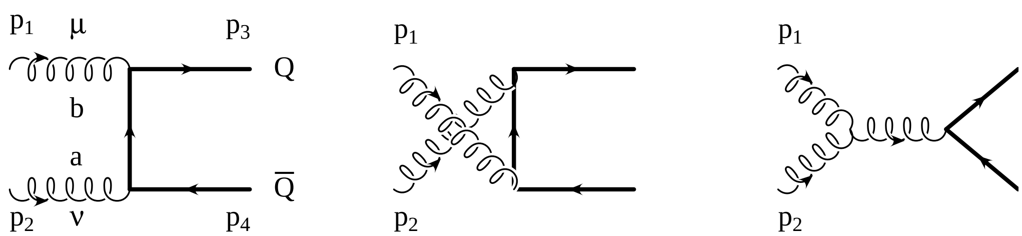

Figure 6.1: The t-, u- and s-shannel leading order (Born) graphs contributing to the gluon (curly lines) fusion amplitude. The thick solid lines correspond to heavy quarks.

6.1 Contributions of the two- and three-point functions to gluon fusion

I start with the gluon-initiated heavy quark-antiquark pair production. The Born and the one-loop contributions to the gluon fusion partonic reaction

$$g(p_1) + g(p_2) \to Q(p_3) + \overline{Q}(p_4)$$

are shown in Figs. 6.1–6.3. In this section I discuss the calculation of the self-energy and vertex graphs that contribute to the above subprocess. With the momenta $p_i\,(i = 1, ..., 4)$ as indicated in the Fig. 6.1 and with m the heavy quark mass one repeats the definition of the Mandelstam-variables (2.27):

$$s \equiv (p_1 + p_2)^2, \qquad t \equiv T - m^2 \equiv (p_1 - p_3)^2 - m^2, \\ u \equiv U - m^2 \equiv (p_2 - p_3)^2 - m^2. \tag{6.1}$$

To isolate the ultraviolet (UV) and infrared/collinear (IR/M) divergences I carry out all calculations in both conventional regularization schemes, namely the standard dimensional regularization scheme (DREG) [26, 27, 28] and the dimensional reduction scheme (DRED) [57] with the dimension of space-time being formally $n = 4 - 2\varepsilon$. In what follows, I present results for the DREG, as well as the difference Δ=DRED-DREG. A brief characterization of the two regularization schemes is the following: In DREG both tensorial structures (e.g. gamma matrices, metric tensors, etc...) and momenta are continued to $n \neq 4$, while in DRED only momenta are continued to $n \neq 4$ whereas the tensorial structures are those of $n = 4$.

First of all I note that in general the matrix elements for all the Feynman diagrams in the gluon fusion subprocess are written in the form

$$M = \epsilon_\mu(p_1)\epsilon_\nu(p_2)\bar{u}(p_3)M^{\mu\nu}v(p_4), \tag{6.2}$$

However, for the purposes of brevity, results will be presented in terms of the truncated amplitudes $M^{\mu\nu}$ omitting the polarization vectors and Dirac spinors. Of course, their

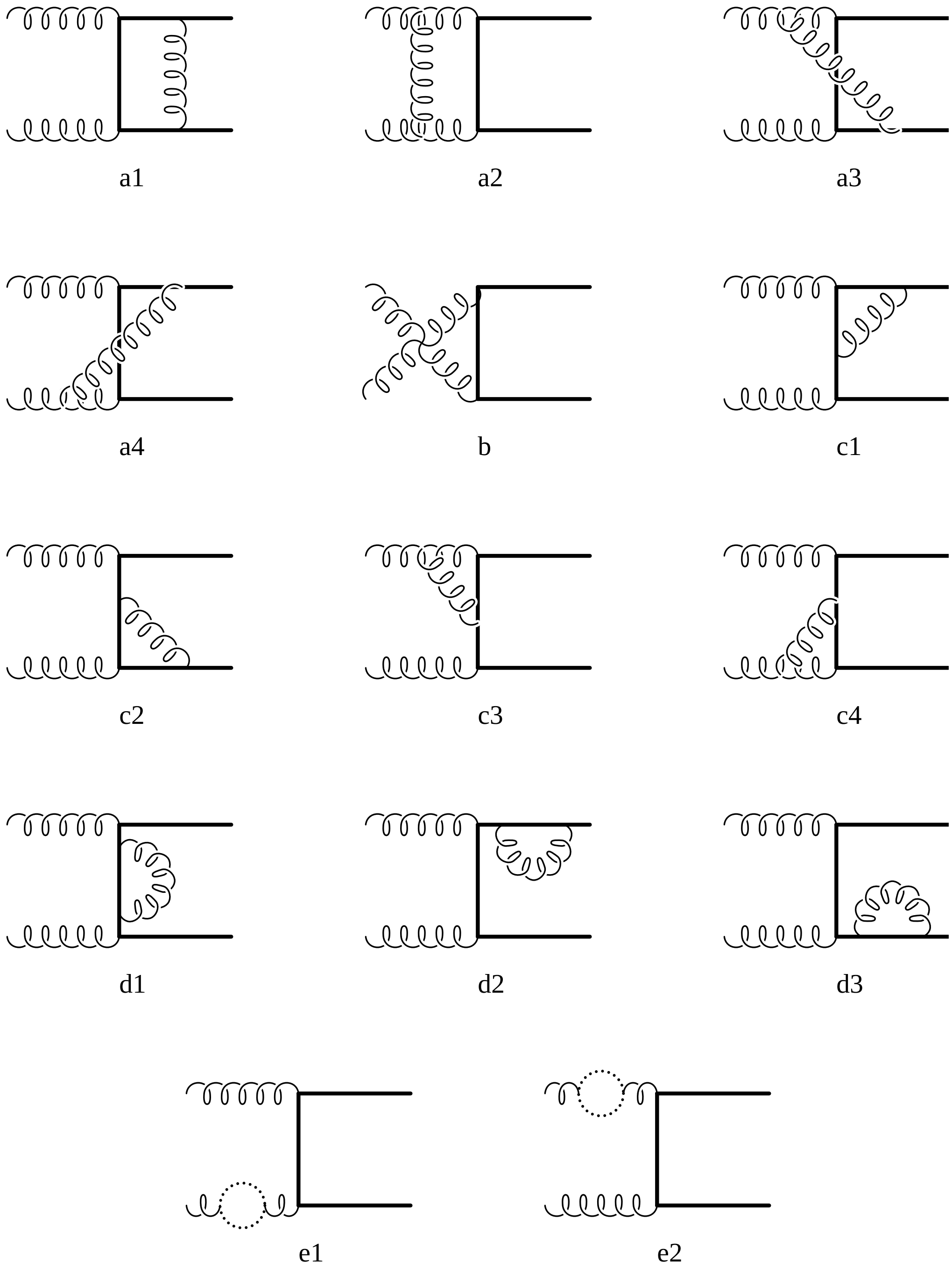

Figure 6.2: The t-channel one-loop graphs contributing to the gluon fusion amplitude. Loops with dotted lines represent gluon, ghost, light and heavy quarks.

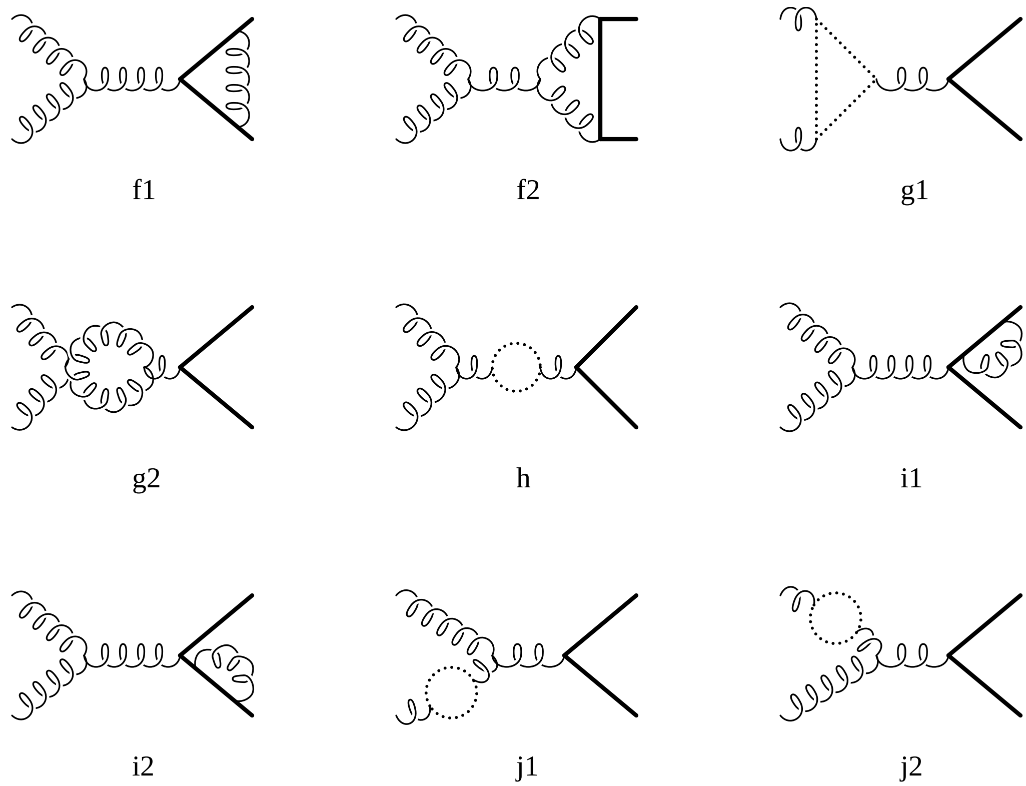

Figure 6.3: The s-channel one-loop graphs contributing to the gluon fusion amplitude. Loops with dotted lines represent gluon, ghost, light and heavy quarks.

presence is implicitly understood throughout this chapter in that the mass shell conditions $p_1^\mu \epsilon_\mu(p_1) = 0$ and $\not{p}_3 u(p_3) = m u(p_3)$ etc. are being used to simplify $M^{\mu\nu}$ [1]. Furthermore, $M^{\mu\nu}$ for all the one-loop graphs considered in this chapter contains the common factor (2.15) due to the one-loop integration

$$C_\varepsilon(m^2) \equiv \frac{\Gamma(1+\varepsilon)}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^\varepsilon . \tag{6.3}$$

I will omit from all of the one-loop $M^{\mu\nu}$ amplitudes the common factor

$$\mathcal{C} = g^4 C_\varepsilon(m^2), \tag{6.4}$$

where g is the renormalized coupling constant.

For an analysis of matrix elements it is important to describe various crossed heavy flavor production channels. One should make it clear from the outset that an additional u-channel set of graphs, that topologically differ from the t-channel ones, are obtained by interchange of bosonic lines (not momenta). In particular, for calculational purposes, I will always be relating t- and u-channel Feynman diagrams by the following procedure:

$$\mathcal{M}_t \leftrightarrow \mathcal{M}_u \equiv \{a \leftrightarrow b, \quad p_1 \leftrightarrow p_2, \quad \mu \leftrightarrow \nu\}, \tag{6.5}$$

with a, b color indices of bosons and where all three interchanges are performed simultaneously. Note that the second interchange in (6.5) implies also the interchange $t \leftrightarrow u$, but not vice versa. One case, involving two vertex diagrams, when the above transformation (6.5) does not correspond to "true" u-channel topologies, is discussed below. In general, when speaking about t-u symmetry of given amplitudes, I will imply invariance of those amplitudes under the transformations (6.5).

I start by writing down matrix elements for the leading order Born terms. For the t-channel gluon fusion subprocess (first graph in Fig. 6.1) one has:

$$B_t^{\mu\nu} = -iT^b T^a \gamma^\mu (\not{p}_3 - \not{p}_1 + m)\gamma^\nu / t,$$

where T^b and T^a are the generators of the colour group SU(3) ($T^a = \lambda^a/2$, $a = 1, ..., 8$ where the λ^a are the usual Gell-Mann matrices). These define the fundamental representation of the Lie algebra of the colour group SU(3). Analogously, for the u- and s-channels one has, respectively,

$$\begin{aligned} B_u^{\mu\nu} &= -iT^a T^b \gamma^\nu (\not{p}_3 - \not{p}_2 + m)\gamma^\mu / u, \\ B_s^{\mu\nu} &= i(T^a T^b - T^b T^a) C_3^{\mu\nu\sigma} \gamma_\sigma / s, \end{aligned}$$

where the tensor $C_3^{\mu\nu\sigma}$ is defined according to the Feynman rules for the three-gluon coupling. I have omitted a common factor g^2 in the Born amplitudes. Acting with Dirac

[1] According to the discussion in [58] this implies that, when further processing the LO and one-loop results in cross section calculations by folding in the appropriate amplitudes, one may use the Feynman gauge for the spin sums of polarization vectors. At the same time ghost contributions associated with external gluons have to be omitted.

spinors on the above Born matrix elements from the left and the right and using the effective relations $p_1^\mu = p_2^\nu = 0$, as remarked on before, we arrive at the following expressions for the leading order matrix elements:

$$\begin{aligned}
B_t^{\mu\nu} &= iT^bT^a(\gamma^\mu \not{p}_1 \gamma^\nu - 2p_3^\mu \gamma^\nu)/t; \\
B_u^{\mu\nu} &= iT^aT^b(2p_4^\mu \gamma^\nu - \gamma^\nu \not{p}_1 \gamma^\mu)/u \\
&= iT^aT^b(\gamma^\nu \not{p}_2 \gamma^\mu - 2p_3^\nu \gamma^\mu)/u; \\
B_s^{\mu\nu} &= 2i(T^aT^b - T^bT^a)(g^{\mu\nu}\not{p}_1 + p_2^\mu \gamma^\nu - p_1^\nu \gamma^\mu)/s.
\end{aligned} \tag{6.6}$$

Next I proceed with the description of the two-point contributions to the matrix element of the subprocess (1.1). But before I turn to the two-point functions one should mention that the choice of renormalization scheme will be a *fixed flavor* scheme throughout this chapter. This implies that one has a total number of flavors $n_f = n_{lf} + 1$, where n_{lf} is the number of light (e.g. massless) flavors plus one produced heavy flavor, with only n_{lf} light flavors involved/active in the β function for the running a QCD coupling α_s and in the splitting functions that determine the evolution of the structure functions. When having massless particles in the loops I am using the standard $\overline{\text{M}S}$ scheme, while the contribution of a heavy quark loop in the gluon self-energy with on-shell external legs is subtracted out entirely.

Consider first the two t-channel self-energy graphs (2d2) and (2d3) with external legs on-shell (note that in the graph numeration the first number identifies the number of the figure in the current chapter which the given diagram refers to). These graphs are very important as they determine the renormalization parameters in the quark sector. Throughout this chapter I use the so called on-shell prescription for the renormalization of heavy quarks. I describe the essential ingredients of this prescription in the following. When dealing with massive quarks one has to choose a parameter to which one renormalizes the heavy quark mass. It is natural to choose a quark pole mass for such a parameter - the only "stable" mass parameter in QCD. The condition on the renormalized heavy quark self-energy $\Sigma_r(\not{p})$ is

$$\Sigma_r(\not{p})|_{\not{p}=m} = 0, \tag{6.7}$$

which removes the singular internal propagator in these self-energy diagrams. The above condition determines the mass renormalization constant Z_m. For the wave function renormalization I have used the usual condition (see e.g. Ref. [11])

$$\frac{\partial}{\partial \not{p}} \Sigma_r(\not{p})|_{\not{p}=m} = 0, \tag{6.8}$$

which fully determines the wave function renormalization constant Z_2. Since the condition (6.8) is not mandatory in general, there is a freedom in determining the constant Z_2. Therefore, I will list the relevant expressions for these constants. In the DREG scheme I arrive at the all-order result

$$Z_m = 1 - g^2 C_F C_\varepsilon(m^2) \frac{3 - 2\varepsilon}{\varepsilon(1 - 2\varepsilon)} \tag{6.9}$$

$$
\begin{aligned}
&= 1 - g^2 C_F C_\varepsilon(m^2)\left(\frac{3}{\varepsilon} + 4 + 8\varepsilon + 16\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right), \\
Z_2 &= Z_m.
\end{aligned}
$$

And in the DRED scheme I obtain

$$
\begin{aligned}
Z_m &= 1 - g^2 C_F C_\varepsilon(m^2)\frac{3-4\varepsilon}{\varepsilon(1-2\varepsilon)(1-\varepsilon)} \\
&= 1 - g^2 C_F C_\varepsilon(m^2)\left(\frac{3}{\varepsilon} + 5 + 9\varepsilon + 17\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right), \\
Z_2 &= Z_m,
\end{aligned} \tag{6.10}
$$

where C_F=4/3 and I do not make a distinction which poles are of ultraviolet or IR/M origin as was done in the case of NLO contributions. *After the mass renormalization procedure is applied* I obtain an all-order final results for the two self-energy graphs:
for the DREG scheme

$$
\begin{aligned}
M^{\mu\nu}_{(2d2)} &= M^{\mu\nu}_{(2d3)} = -C_F B^{\mu\nu}_t \frac{3-2\varepsilon}{\varepsilon(1-2\varepsilon)} \\
&= -C_F B^{\mu\nu}_t \left(\frac{3}{\varepsilon} + 4 + 8\varepsilon + 16\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right)
\end{aligned} \tag{6.11}
$$

and for the DRED scheme

$$
\begin{aligned}
M^{\mu\nu}_{(2d2)} &= M^{\mu\nu}_{(2d3)} = -C_F B^{\mu\nu}_t \frac{3-4\varepsilon}{\varepsilon(1-2\varepsilon)(1-\varepsilon)} \\
&= -C_F B^{\mu\nu}_t \left(\frac{3}{\varepsilon} + 5 + 9\varepsilon + 17\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right).
\end{aligned} \tag{6.12}
$$

The difference between the two regularization schemes is order by order

$$
\begin{aligned}
\Delta(2d2) &= \Delta(2d3) = -C_F B^{\mu\nu}_t \frac{1}{1-\varepsilon} \\
&= -C_F B^{\mu\nu}_t \left(1 + \varepsilon + \varepsilon^2 + \ldots\right).
\end{aligned} \tag{6.13}
$$

One notices that the effect of the wave function renormalization consists of a complete removal of the quark self-energy diagrams with external legs on-shell, as is required by the second condition (6.8). One can also write the contribution of the quark self-energy with external legs off-shell, graph (2d1), after addition of the mass renormalization counterterm. Here I write down only the coefficients for the ε- and ε^2-terms:

$$
\begin{aligned}
M^{\mu\nu}_{(2d1)} &= C_F B^{\mu\nu}_t \sum_{k=1}^{2} \varepsilon^k \left(-B_1^{(k)} t/T + 4B_1^{(k)} m^2/t + B_1^{(k-1)} t/T - k16m^2/t\right) \\
&- iC_F T^b T^a m\gamma^\mu\gamma^\nu \sum_{k=1}^{2} \varepsilon^k \left(B_1^{(k)}/T + 2B_1^{(k)}/t - B_1^{(k-1)}/T - k8/t\right).
\end{aligned}
$$

The difference between the DRED and DREG results is

$$\begin{aligned}&\Delta(2\mathrm{d}1) = \\ &-C_F B_t^{\mu\nu}\left[\varepsilon\left(m^2 + B_1^{(0)}t\right) + \varepsilon^2\left(m^2 + B_1^{(1)}t\right)\right]/T \\ &+iC_F T^b T^a m\gamma^\mu\gamma^\nu\left[\varepsilon\left(1 - B_1^{(0)}\right) + \varepsilon^2\left(1 - B_1^{(1)}\right)\right]/T.\end{aligned} \tag{6.14}$$

The remaining quark self-energy diagrams (3i1) and (3i2) with external on-shell legs are derived in analogy to the ones considered above:

$$M^{\mu\nu}_{(3\mathrm{i}1)} = M^{\mu\nu}_{(3\mathrm{i}2)} = -C_F B_s^{\mu\nu}\left(8\varepsilon + 16\varepsilon^2\right), \tag{6.15}$$

$$\Delta(3\mathrm{i}1) = \Delta(3\mathrm{i}2) = -C_F B_s^{\mu\nu}\left(\varepsilon + \varepsilon^2\right). \tag{6.16}$$

Concerning the gluon self-energy graphs (2e1) and (2e2) with external legs on-shell, the only nonvanishing contribution they receive are from the loop with internal heavy quarks. It is given by

$$M^{\mu\nu}_{(2\mathrm{e}1)} = M^{\mu\nu}_{(2\mathrm{e}2)} = -B_t^{\mu\nu}\frac{1}{\varepsilon}\frac{2}{3}. \tag{6.17}$$

However, these contributions are explicitly subtracted (together with the logarithmic term $\ln(\mu^2/m^2)$ coming from the common factor $C_\varepsilon(m^2)$, see Eqs. (6.3) and (6.4), in the on-shell renormalization prescription, in order to avoid the appearance of the large mass logarithms from the gluon self-energy diagrams with off-shell external legs in the low energy limit. Therefore, due to the UV counterterm that subtracts that very same loop with heavy quarks, there are no finite contributions to the matrix element from these diagrams. However, at the same time this counterterm introduces pole terms from the light quark loop sector that are needed to cancel soft and collinear poles from the other parts of the amplitude, e.g. from the real bremsstrahlung part. This indicates that in practice it is very hard to completely disentangle UV and IR/M poles in heavy flavor production and in most cases one obtains a mixture of both instead.

For the reasons specified above it is convenient to present a gauge field renormalization constant Z_3, used for the gluon self-energy subtraction:

$$\begin{aligned}Z_3 &= 1 + \frac{g^2}{\varepsilon}\left\{(\frac{5}{3}N_C - \frac{2}{3}n_{lf})C_\varepsilon(\mu^2) - \frac{2}{3}C_\varepsilon(m^2)\right\} \\ &= 1 + \frac{g^2}{\varepsilon}\left\{(\beta_0 - 2N_C)C_\varepsilon(\mu^2) - \frac{2}{3}C_\varepsilon(m^2)\right\},\end{aligned} \tag{6.18}$$

with the QCD beta function $\beta_0 = (11N_C - 2n_{lf})/3$ containing only light quarks. $N_C = 3$ is the number of colors. Accordingly, for the coupling contant renormalization one obtains

$$Z_g = 1 - \frac{g^2}{\varepsilon}\left\{\frac{\beta_0}{2}C_\varepsilon(\mu^2) - \frac{1}{3}C_\varepsilon(m^2)\right\}. \tag{6.19}$$

Similarly to the diagrams (2e1) and (2e2), diagrams (3j1) and (3j2) also vanish due to the explicit decoupling of the heavy quarks in this subtraction prescription. However, instead of doing the renormalization separately for each Feynman diagram, one can chose to employ the renormalization group invariance of the cross section and do only a mass and coupling constant renormalization. In that case, knowing the results for gluon self-energies turns out to be useful in checking the complete cancellation of UV poles by just rescaling the coupling constant in the LO terms $g_{\rm bare} \to Z_g g$:

$$M^{\mu\nu}_{(3j1)} = M^{\mu\nu}_{(3j2)} = -B^{\mu\nu}_s \frac{1}{\varepsilon}\,\frac{2}{3}. \tag{6.20}$$

Finally I arrive at the gluon-self energy graph (3h), which contains the off-shell gluon self-energy loop that is used for the derivation of the renormalization constant Z_3. I have evaluated the internal loop in the Feynman gauge. Since it is explicitly gauge invariant, one should arrive at the same result in any other gauge. In the relevant result I show separately the gauge invariant pieces for gluon plus ghost, light quarks and a heavy quark flow inside the loop:

$$\begin{aligned} M^{\mu\nu}_{(3h)} = B^{\mu\nu}_s \Big\{ & \tfrac{B_5}{iC_\varepsilon(m^2)} \left[-N_C \tfrac{n-14+8\varepsilon}{2(3-2\varepsilon)} - n_{lf} \tfrac{2(1-\varepsilon)}{3-2\varepsilon} \right] \tfrac{C_\varepsilon(-s)}{C_\varepsilon(m^2)} \\ & - \tfrac{2}{\varepsilon(3-2\varepsilon)} \left[\tfrac{B_2}{iC_\varepsilon(m^2)} \varepsilon \left(1 - \varepsilon + \varepsilon \tfrac{2m^2}{s} \right) - \tfrac{2m^2}{s} \right] \Big\}, \end{aligned} \tag{6.21}$$

with $n = 4 - 2\varepsilon$ in the DREG scheme and $n = 4$ in the DRED scheme, B_2 and B_5 are two-point integrals which can be found in Chap. 2. Expanding (6.21) for the case of DREG in powers of ε I arrive at

$$\begin{aligned} M^{\mu\nu}_{(3h)} = \; & B^{\mu\nu}_s \Big\{ \Big[N_C \left(\frac{1}{\varepsilon}\,\frac{5}{3} + \frac{31}{9} + \varepsilon \left(\frac{188}{27} - \frac{5}{3}\zeta(2) \right) \right. \\ & \left. + \varepsilon^2 \left(\frac{1132}{81} - \frac{31}{9}\zeta(2) - \frac{10}{3}\zeta(3) \right) \right) \\ & - n_{lf} \left(\frac{1}{\varepsilon}\,\frac{2}{3} + \frac{10}{9} + \varepsilon \left(\frac{56}{27} - \frac{2}{3}\zeta(2) \right) \right. \\ & \left. + \varepsilon^2 \left(\frac{328}{81} - \frac{10}{9}\zeta(2) - \frac{4}{3}\zeta(3) \right) \right) \Big] \left(\frac{-s}{m^2} \right)^{-\varepsilon} - \frac{1}{\varepsilon}\,\frac{2}{3}\, I \Big\}, \end{aligned}$$

with

$$\begin{aligned} I = \; & 1 + \varepsilon \left[-\frac{1}{3} + B_2^{(0)} \frac{3-\beta^2}{2} \right] + \\ & + \varepsilon^2 \left[-\frac{2}{9} - \frac{1}{3} B_2^{(0)} \beta^2 + B_2^{(1)} \frac{3-\beta^2}{2} \right] \\ & + \varepsilon^3 \left[-\frac{4}{27} - \frac{2}{9} B_2^{(0)} \beta^2 - \frac{1}{3} B_2^{(1)} \beta^2 + B_2^{(2)} \frac{3-\beta^2}{2} \right]. \end{aligned} \tag{6.22}$$

In (6.22) one uses the definition from Chap. 2

$$\beta \equiv \sqrt{1 - 4m^2/s}. \tag{6.23}$$

One has to emphasize that in the last term of (6.22) the Z_3 counterterm together with the UV pole will remove also the $\ln(\mu^2/m^2)$ contribution, while $\ln(-\mu^2/s)$ from the first two terms in (6.22) will be left unsubtracted.

One notes that there is a minor problem with preserving gauge invariance when calculating the graph (3h) in DRED. It is associated with an ε-dimensional part of one of the n-dimensional metric tensors $g^n_{\mu\nu}$ that arises in every partonic loop and hampers collecting together similar terms. However, this problem appears to be an artificial one, as in this particular case it makes no difference whether one uses 4- or n-dimensional metric tensor for the evaluation of this gluon self-energy graph. For this reason in practice one would set this $g^n_{\mu\nu}$ metric tensor to be the 4-dimensional one. Or, more exactly, if one introduces a proper counterterm so that to restore gauge invariance of the gluon self-energy, then the expression in DRED would be exactly the same as in (6.22), except for the term proportional to N_C, yelding for the difference

$$\Delta(3h) = -B_s^{\mu\nu}\frac{B_5}{iC_\varepsilon(m^2)}N_C\frac{\varepsilon}{3-2\varepsilon}\left(\frac{-s}{m^2}\right)^{-\varepsilon}. \tag{6.24}$$

Concluding the discussion on the 2-point functions I remark that the matrix elements for the *additional u-channel 2-point functions* can be obtained from eqs. (6.11), (6.14) and (6.17) by the transformation (6.5).

I start by considering the t- and u-channel vertex diagrams. In this thesis I write down only ε and ε^2 terms of the relevant expansions, while the terms proportional to ε^{-2}, ε^{-1} and ε^0 can be found in [29]. I begin with the purely nonabelian graph (2b), which contains a four-point gluon vertex. The matrix element takes the following form

$$\begin{aligned}
M^{\mu\nu}_{(2b)} &= iN_C(T^bT^a\sum_{k=1}^{2}\varepsilon^k\{(2p_3^\nu\gamma^\mu + p_4^\nu\gamma^\mu - p_3^\mu\gamma^\nu - 2p_4^\mu\gamma^\nu)(B_5^{(k)} + 2C_1^{(k)}m^2 - 4k) \\
&\quad -mg^{\mu\nu}(2B_5^{(k)} + 2B_5^{(k-1)} + 4C_1^{(k)}m^2 + C_1^{(k-1)}s - 12k) \\
&\quad +3m\gamma^\mu\gamma^\nu(2B_5^{(k)} + C_1^{(k)}s - 8k)/2\}/(s\beta^2) + (a\leftrightarrow b, \mu\leftrightarrow\nu)) \\
&+ i\delta^{ab}\sum_{k=1}^{2}\varepsilon^k\{(p_3^\nu\gamma^\mu - p_4^\nu\gamma^\mu + p_3^\mu\gamma^\nu - p_4^\mu\gamma^\nu)(B_5^{(k)} + 2C_1^{(k)}m^2 - 4k)/2 \\
&\quad +mg^{\mu\nu}(B_5^{(k)} - 2B_5^{(k-1)} - 4C_1^{(k)}m^2 + 3C_1^{(k)}s/2 - C_1^{(k-1)}s)\}/(s\beta^2).
\end{aligned} \tag{6.25}$$

It is easily seen from Eq. (6.25) that the matrix element for the graph (2b) is explicitly t-u symmetric, as it follows from the geometric topology of this graph. At this order of expansion there is a difference between DREG and DRED results for this graph, e.g.

$$\begin{aligned}
\Delta(2b) &= i(N_CT^bT^a + N_CT^aT^b + \delta^{ab})mg^{\mu\nu}\sum_{k=1}^{2}\varepsilon^k \times \\
&\quad \{2B_5^{(k-1)} + C_1^{(k-1)}s - 4k\}.
\end{aligned} \tag{6.26}$$

Next I turn to graphs (2c1) and (2c2). Diagrams of this topology do not only occur in hadroproduction, but also in other processes such as photoproduction and $\gamma\gamma$ production of heavy flavors. For this reason I also present the corresponding t-channel color factors for these graphs. Then it is straightforward to separate the Dirac structure from the color coefficients and one can easily deduce results for other processes where these graphs contribute, though with different color weights. The color factor for both (2c1) and (2c2) diagrams is the same:

$$T_{\rm col}^{(2c1)} = T_{\rm col}^{(2c2)} = (C_F - \frac{N_C}{2})T^bT^a = -\frac{1}{6}T^bT^a. \tag{6.27}$$

The complete matrix elements are:

$$\begin{aligned}
M_{(2c1)}^{\mu\nu} &= B_t^{\mu\nu}\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}(6m^2/t+1) + 2B_1^{(k-1)}z_t/t + 2C_2^{(k)}m^2 + 4C_2^{(k-1)}m^2 \\
&\quad -k8(4m^2/t+1)\}/6 \\
&+ iT^bT^a(p_3^\mu\gamma^\nu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}(z_t/t+t/T) + B_1^{(k-1)}(2z_t/t - t/T) \\
&\quad +2(C_2^{(k)} + 2C_2^{(k-1)})m^2 - k8z_t/t\} \\
&+ mp_3^\mu \not{p}_1\gamma^\nu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}/T - B_1^{(k-1)}(2/t+1/T) - 2C_2^{(k-1)} + k4/t\} \\
&- m\gamma^\mu\gamma^\nu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)} + B_1^{(k-1)} + C_2^{(k-1)}t - 6k\})/(3t),
\end{aligned} \tag{6.28}$$

where one has denoted $z_t \equiv 2m^2 + t$. The difference between the corresponding results in the two n-dimensional schemes has a length comparable to the original expressions. This holds true for the majority of the vertex and all the box diagrams considered in this chapter. Therefore, in the following these differences for vertex and box diagrams will not be presented. However, the results in DRED for all the diagrams considered in this chapter can be obtained upon request.

For the graph (2c2) one obtains:

$$\begin{aligned}
M_{(2c2)}^{\mu\nu} &= B_t^{\mu\nu}\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}(6m^2/t+1) + 2B_1^{(k-1)}z_t/t + 2C_2^{(k)}m^2 + 4C_2^{(k-1)}m^2 \\
&\quad -k8(4m^2/t+1)\}/6 \\
&+ iT^bT^a(p_4^\nu\gamma^\mu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}(-2m^2/t - 3 + t/T) - B_1^{(k-1)}t/T - 2C_2^{(k)}m^2 + k8T/t\} \\
&+ mp_4^\nu(2p_3^\mu - \gamma^\mu\not{p}_1)\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}/T - B_1^{(k-1)}(2/t+1/T) - 2C_2^{(k-1)} + k4/t\} \\
&- m\gamma^\mu\gamma^\nu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)} + B_1^{(k-1)} + C_2^{(k-1)}t - 6k\})/(3t).
\end{aligned} \tag{6.29}$$

Next I write down the results for graphs (2c3) and (2c4). The color factors for both diagrams are the same:

$$T_{\rm col}^{(2c3)} = T_{\rm col}^{(2c4)} = -\frac{N_C}{2}T^bT^a = -\frac{3}{2}T^bT^a. \tag{6.30}$$

One has

$$\begin{aligned} M^{\mu\nu}_{(2c3)} &= 3B_t^{\mu\nu}\sum_{k=1}^{2}\varepsilon^k\{-3B_1^{(k)}m^2/t - C_3^{(k)}t + k4(3m^2/t+1)\} \\ &+ 3iT^bT^a(p_3^\mu\gamma^\nu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}m^2(1/T-2/t) + B_1^{(k-1)}t/T - C_3^{(k)}t + k4z_t/t\} \\ &+ 3m\gamma^\mu\gamma^\nu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}/2 - 2k\} \\ &+ mp_3^\mu \not{p}_1\gamma^\nu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}(2/t-1/T) + B_1^{(k-1)}/T - k8/t\})/t \end{aligned} \tag{6.31}$$

and

$$\begin{aligned} M^{\mu\nu}_{(2c4)} &= 3B_t^{\mu\nu}\sum_{k=1}^{2}\varepsilon^k\{-3B_1^{(k)}m^2/t - C_3^{(k)}t + k4(3m^2/t+1)\} \\ &+ 3iT^bT^a(p_4^\nu\gamma^\mu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}m^2(1/T-2/t) + B_1^{(k-1)}(t/T-2) + C_3^{(k)}t + k4(2m^2/t-1)\} \\ &+ 3m\gamma^\mu\gamma^\nu\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}/2 - 2k\} \\ &+ mp_4^\nu(2p_3^\mu - \gamma^\mu\not{p}_1)\sum_{k=1}^{2}\varepsilon^k\{B_1^{(k)}(2/t-1/T) + B_1^{(k-1)}/T - k8/t\})/t. \end{aligned} \tag{6.32}$$

The results for the matrix elements of the *additional u-channel vertex graphs* are obtained from Eqs. (6.28), (6.29), (6.31) and (6.32) by the transformation (6.5). However, there is a subtle point involved here: for the graphs (2c3) and (2c4) the transformation (6.5) transforms the t-channel result of the graph (2c3) to the u-channel result for the graph (2c4), while the t-channel result of (2c4) goes to the u-channel result for (2c3). This is important to keep in mind when dealing with reactions which involve asymmetric set of graphs as e.g. photoproduction of heavy flavors.

Next I turn to the remaining s-channel graphs shown in Fig. 6.3. For all the gluon propagators I work in Feynman gauge. Although this set of graphs is purely nonabelian for QCD type one-loop corrections, there could be also abelian (e.g. QED) virtual corrections to graph (3f1). For this reason I also give the color factor for it separately:

$$T_{\rm col}^{(3f1)} = (C_F - \frac{N_C}{2})(T^aT^b - T^bT^a) = -\frac{1}{6}(T^aT^b - T^bT^a). \tag{6.33}$$

The matrix element is

$$
\begin{aligned}
M^{\mu\nu}_{(3f1)} &= B^{\mu\nu}_s \sum_{k=1}^{2} \varepsilon^k \{3B^{(k)}_2 + 2B^{(k-1)}_2 + C^{(k)}_6 s(1+\beta^2) - 16k\}/6 \\
&+ 2i(T^aT^b - T^bT^a)m[-g^{\mu\nu}(s+2t) - 4p^\mu_3 p^\nu_4 + 4p^\mu_4 p^\nu_3] \times \\
&\times \sum_{k=1}^{2} \varepsilon^k \{B^{(k)}_2 + 2B^{(k-1)}_2 - 8k\}/(6s^2\beta^2).
\end{aligned} \tag{6.34}
$$

Graph (3f2) contributes as:

$$
\begin{aligned}
M^{\mu\nu}_{(3f2)} &= N_C B^{\mu\nu}_s \sum_{k=1}^{2} \varepsilon^k \{B^{(k)}_5(8m^2 - s) + 2C^{(k)}_1 m^2 s - k16(5m^2 - s)\}/(2s\beta^2) \\
&+ 2iN_C(T^aT^b - T^bT^a)m[-g^{\mu\nu}(s+2t) - 4p^\mu_3 p^\nu_4 + 4p^\mu_4 p^\nu_3] \times \\
&\times \sum_{k=1}^{2} \varepsilon^k \{B^{(k)}_5(8m^2 + s) - 2B^{(k-1)}_5 s + 6C^{(k)}_1 m^2 s - C^{(k-1)}_1 s^2 - k4(12m^2 - s)\}/2s^3\beta^4.
\end{aligned} \tag{6.35}
$$

I conclude the discussion of the vertex diagrams for gluon fusion with the triangle graph contribution (tri)≡(3g1)+(3g2), e.g. one sums the two graphs (3g1) and (3g2). For the case when one has gluons and ghosts inside the triangle loop one obtains:

$$
\begin{aligned}
M^{\mu\nu}_{(tri)}(g) &= -3N_C(B^{\mu\nu}_s \sum_{k=1}^{2} \varepsilon^k \{207B^{(k)}_5 + 12B^{(k-1)}_5 + 54C^{(k)}_4 s + 8k + (k-1)8\mathcal{B}^{(0)}_5\} \\
&+ 6i(T^aT^b - T^bT^a)\not{p}_1 \sum_{k=1}^{2} \varepsilon^k \{g^{\mu\nu}[9B^{(k)}_5 - 12B^{(k-1)}_5 + 9C^{(k)}_4 s - 8k - (k-1)8\mathcal{B}^{(0)}_5]/s \\
&+8p^\mu_2 p^\nu_1[3B^{(k-1)}_5 + 2k + (k-1)2\mathcal{B}^{(0)}_5]/s^2\})/324,
\end{aligned} \tag{6.36}
$$

where $\mathcal{B}^{(0)}_5 = B^{(0)}_5 - 4/3$. For the two more cases when one has light and heavy quarks inside the loop one gets

$$
\begin{aligned}
M^{\mu\nu}_{(tri)}(q) &= 6n_{lf}(B^{\mu\nu}_s \sum_{k=1}^{2} \varepsilon^k \{9B^{(k)}_5 - 3B^{(k-1)}_5 - 2k - (k-1)2\mathcal{B}^{(0)}_5\} \\
&- 3i(T^aT^b - T^bT^a)\not{p}_1[g^{\mu\nu}/s - 2p^\mu_2 p^\nu_1/s^2] \times \\
&\times \sum_{k=1}^{2} \varepsilon^k \{3B^{(k-1)}_5 + 5k + (k-1)(5\mathcal{B}^{(0)}_5 + 3)\})/81
\end{aligned} \tag{6.37}
$$

with n_{lf} number of light flavors in the triangle loop, while for the heavy flavor case one has

$$
\begin{aligned}
M^{\mu\nu}_{(tri)}(Q) &= 6(B^{\mu\nu}_s \sum_{k=1}^{2} \varepsilon^k \{6(3B^{(k)}_2 + 2B^{(k-1)}_2 + (k-1)4B^{(0)}_2/3)m^2/s + 9B^{(k)}_2 \\
&-3B^{(k-1)}_2 - 2k - 2(k-1)(B^{(0)}_2 - 4/3)\}
\end{aligned}
$$

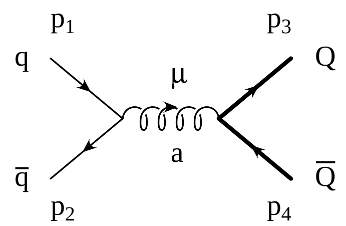

Figure 6.4: The lowest order Feynman diagram contributing to the subprocess $q\bar{q} \to Q\overline{Q}$. The thick lines correspond to heavy quarks.

$$\begin{aligned} - \;& i(T^aT^b - T^bT^a)\not{p}_1[g^{\mu\nu}/s - 2p_2^\mu p_1^\nu/s^2] \times \\ & \times \sum_{k=1}^{2} \varepsilon^k \{12(3B_2^{(k)} + 2B_2^{(k-1)} + (k-1)4B_2^{(0)}/3)m^2/s \\ & +18(C_5^{(k)} + C_5^{(k-1)} + (k-1)C_5^{(0)})m^2 + 3B_2^{(k-1)} \\ & +5k + (k-1)(5(B_2^{(0)} - 4/3) + 3)\})/81. \end{aligned} \tag{6.38}$$

The complete matrix element for the triangle is the sum of the above three expressions (6.36), (6.37) and (6.38):

$$M^{\mu\nu}_{(\mathrm{tri})} = M^{\mu\nu}_{(\mathrm{tri})}(g) + M^{\mu\nu}_{(\mathrm{tri})}(q) + M^{\mu\nu}_{(\mathrm{tri})}(Q). \tag{6.39}$$

The difference between dimensional reduction and regularization arises solely due to gluons in the loop:

$$\begin{aligned} \Delta_{(\mathrm{tri})} &= 3(B_s^{\mu\nu} \sum_{k=1}^{2} \varepsilon^k \{3B_5^{(k-1)} + 2k + (k-1)2\mathcal{B}_5^{(0)}\} \qquad (6.40) \\ &+ 6i(T^aT^b - T^bT^a)\not{p}_1[g^{\mu\nu}/s - 2p_2^\mu p_1^\nu/s^2] \sum_{k=1}^{2} \varepsilon^k \{k + (k-1)(\mathcal{B}_5^{(0)} + 1)\})/9. \end{aligned}$$

Concerning the technically most complicated four box graphs (2a1)–(2a4) calculation of the $\mathcal{O}(\varepsilon^2)$ results is still in progress. I hope that complete results on this part of the NNLO calculation will be presented in the near future [59].

6.2 Annihilation of the quark-antiquark pair

Next I turn to the calculation of the quark-initiated heavy quark-antiquark pair production The graphs contributing to this subprocess are shown in Fig. 6.4 for the leading order term and in Fig. 6.5 for the one-loop corrections. The leading order contribution proceeds only through the s-channel graph. One has:

$$B_{q\bar{q}} = iT^aT^a\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_3)\gamma_\mu v(p_4)/s. \tag{6.41}$$

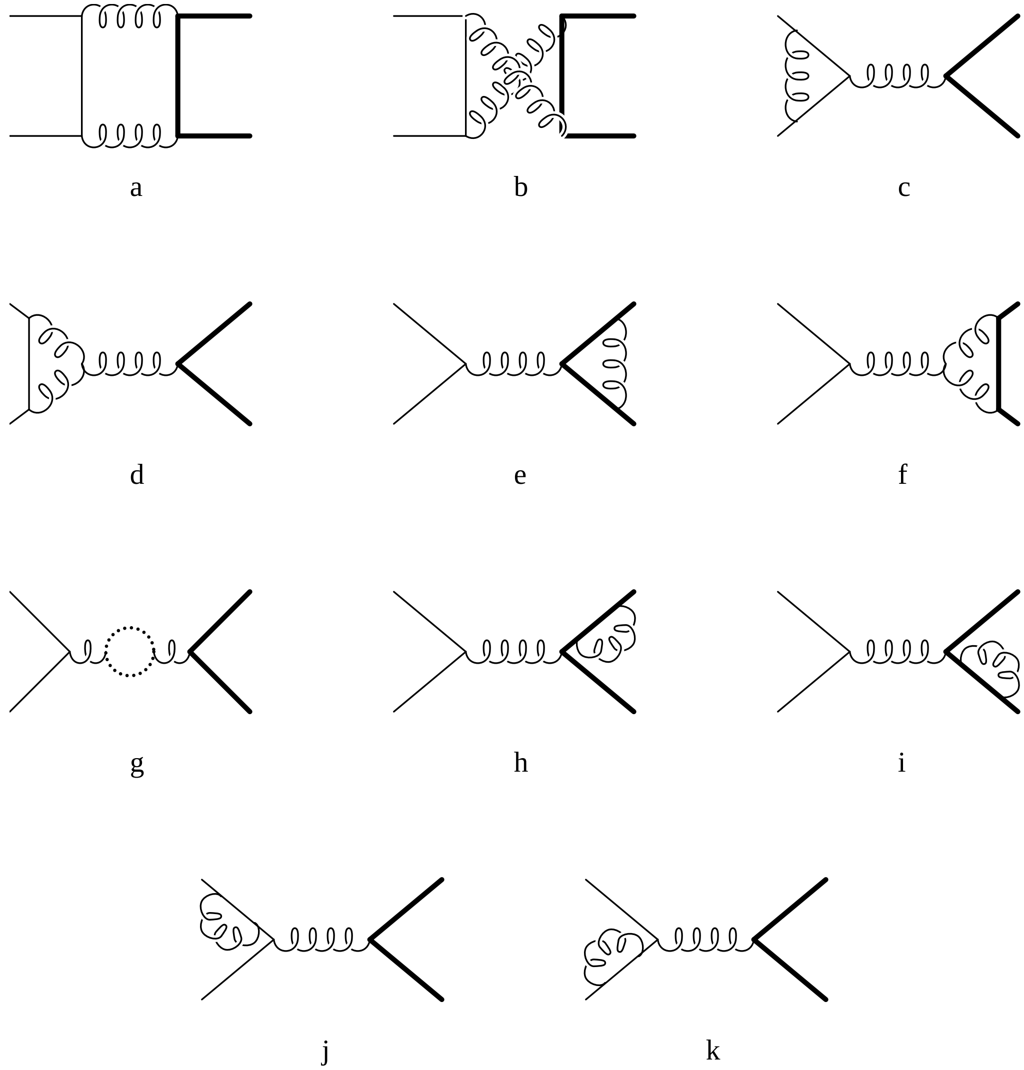

Figure 6.5: The one-loop Feynman diagrams contributing to the subprocess $q\bar{q} \rightarrow Q\overline{Q}$. The loop with dotted line represents gluon, ghost, light and heavy quarks.

Here the color matrices T^a belong to different fermion lines that are connected by the gluon having color index a. I have again left out the factor g^2 in the above equation. In the Passarino-Veltman reduction for tensor integrals I use the same scalar integrals as those appearing in the gluon fusion subprocess, with relevant shifts and interchanges of momenta as needed.

Starting again with the 2-point functions, I notice that the result for graph (5g) can be obtained from the one of (6.22) for graph (2h) in the gluon fusion subprocess by the simple replacement

$$M_{(5g)} = M^{\mu\nu}_{(2h)}\,(B^{\mu\nu}_s \to B_{q\bar{q}}), \tag{6.42}$$

and all the statements after (6.22) are equally applicable to $M_{(5g)}$.

The massless quark self-energy graphs (5j) and (5k) with external legs on-shell vanish identically:

$$M_{(5j)} = M_{(5k)} = 0. \tag{6.43}$$

The massive quark self-energy graphs (5h) and (5i) with external legs on-shell are derived analogously to the ones considered in the previous section:

$$M_{(5h)} = M_{(5i)} = -C_F B_{q\bar{q}} \frac{3-2\varepsilon}{\varepsilon(1-2\varepsilon)}, \tag{6.44}$$

and the difference between the two regularizations schemes is

$$\Delta(5h) = \Delta(5i) = -C_F B_{q\bar{q}} \frac{1}{1-\varepsilon}. \tag{6.45}$$

Results for the vertex diagrams are relatively short. Starting with graphs (5c) and (5d) one finds that they are proportional to the LO Born term:

$$M_{(5c)} = B_{q\bar{q}} \sum_{k=1}^{2} \varepsilon^k \{3B_5^{(k)} + 2B_5^{(k-1)} + 2C_4^{(k)} s\}/6 \tag{6.46}$$

and

$$M_{(5d)} = -3B_{q\bar{q}} \sum_{k=1}^{2} \varepsilon^k B_5^{(k)}/2. \tag{6.47}$$

For the other two vertex diagrams one also obtains simple expressions:

$$\begin{aligned} M_{(5e)} \;&=\; (B_{q\bar{q}} \sum_{k=1}^{2} \varepsilon^k \{3B_2^{(k)} + 2B_2^{(k-1)} + C_6^{(k)} s(1+\beta^2) - 16k\} \\ &+\; 4iT^aT^a m\bar{v}(p_2)\not{p}_3 u(p_1)\bar{u}(p_3)v(p_4) \sum_{k=1}^{2} \varepsilon^k \{B_2^{(k)} + 2B_2^{(k-1)} - 8k\}/(s^2\beta^2))/6 \end{aligned} \tag{6.48}$$

and

$$M_{(5f)} \;=\; 3(B_{q\bar{q}} \sum_{k=1}^{2} \varepsilon^k \{B_5^{(k)}(8m^2/s - 1) + 2C_1^{(k)} m^2 - k16(5m^2/s - 1)\}$$

$$
\begin{aligned}
&+ \quad 4iT^aT^a m\bar{v}(p_2)\not{p}_3 u(p_1)\bar{u}(p_3)v(p_4) \times \\
&\quad \times \sum_{k=1}^{2} \varepsilon^k \{B_5^{(k)}(8m^2/s+1) - 2B_5^{(k-1)} + 6C_1^{(k)}m^2 \\
&\quad -C_1^{(k-1)}s - k4(12m^2/s-1)\}/(s^2\beta^2))/(2\beta^2).
\end{aligned} \tag{6.49}
$$

Turning to the two box diagrams one notes that an extensive Dirac algebra manipulations lead to rather compact expressions for the matrix elements. Since for the $q\bar{q} \to Q\overline{Q}$ subprocess one has two spinor "sandwiches" one cannot have momenta with Lorentz indices, and consequently there is no expansion of matrix elements in terms of Lorentz objects. One expands the box diagrams in terms of the seven independent Dirac structures, the same set for each of the two box graphs. Then every Dirac structure is multiplied by the sums of products of a small set of analytic functions and coefficient functions. Thus, one has the following compact expansion for both box diagrams:

$$
\begin{aligned}
M &= iT_{\rm col} \sum_{k=1}^{2} \varepsilon^k \Big\{ \bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_3)\gamma_\mu v(p_4) \sum f_i^{(k)} h_i^{(0)} \\
&+ \bar{v}(p_2)\not{p}_3 u(p_1)\bar{u}(p_3)\not{p}_1 v(p_4) \sum f_i^{(k)} h_i^{(1)} \\
&+ \bar{v}(p_2)\gamma^\nu \not{p}_3 \gamma^\mu u(p_1)\bar{u}(p_3)\gamma_\mu \not{p}_1 \gamma_\nu v(p_4) \sum f_i^{(k)} h_i^{(2)} \\
&+ \bar{v}(p_2)\gamma^\nu \gamma^\alpha \gamma^\mu u(p_1)\bar{u}(p_3)\gamma_\mu \gamma_\alpha \gamma_\nu v(p_4) \sum f_i^{(k)} h_i^{(3)} \\
&+ m\bar{v}(p_2)\not{p}_3 u(p_1)\bar{u}(p_3)v(p_4) \sum f_i^{(k)} h_i^{(4)} \\
&+ m\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_3)\gamma_\mu \not{p}_1 v(p_4) \sum f_i^{(k)} h_i^{(5)} \\
&+ m\bar{v}(p_2)\gamma^\nu \not{p}_3 \gamma^\mu u(p_1)\bar{u}(p_3)\gamma_\mu \gamma_\nu v(p_4) \sum f_i^{(k)} h_i^{(6)} \Big\}.
\end{aligned} \tag{6.50}
$$

Note that the number of independent covariants in $n \neq 4$ exceeds the number of independent covariants in $n = 4$ where one has four independent covariants.

The sums over i in (6.50) run from 1 to 15 unless otherwise stated explicitly. Below I list the color factors and analytic functions for the two 4-point functions of (6.50). For the graph (5a) one gets:

$$
T_{\rm col} = (T^aT^b)(T^bT^a), \tag{6.51}
$$

where the first parentheses in (6.51) corresponds to the summation over color indices of the massless fermion line and

$$
\begin{gathered}
f_1^{(k)} = B_1^{(k)}, \qquad f_2^{(k)} = B_5^{(k)}, \\
f_3^{(k)} = C_1^{(k-1)}, \qquad f_4^{(k)} = C_1^{(k)}, \qquad f_5^{(k)} = C_3^{(k-1)}, \\
f_6^{(k)} = C_3^{(k)}, \qquad f_7^{(k)} = C_4^{(k-1)}, \qquad f_8^{(k)} = C_4^{(k)}, \\
f_9^{(k)} = D_2^{(k-1)}, \qquad f_{10}^{(k)} = D_2^{(k)}, \qquad f_{11}^{(k)} = k, \\
f_{12}^{(k)} = (k-1)C_1^{(k-2)}, \qquad f_{13}^{(k)} = (k-1)C_3^{(k-2)}, \\
f_{14}^{(k)} = (k-1)C_4^{(k-2)}, \qquad f_{15}^{(k)} = (k-1)D_2^{(k-2)}.
\end{gathered} \tag{6.52}
$$

There exist a number of universal relations among the various coefficient functions $h_i^{(j)}$ valid for any value of j:

$$\begin{gathered} h_3^{(j)} = z_t h_7^{(j)}/t, \quad h_5^{(j)} = 2th_7^{(j)}/s, \quad h_9^{(j)} = -th_7^{(j)}, \\ h_{10}^{(j)} = -th_8^{(j)}, \quad h_{12}^{(j)} = 2z_t h_7^{(j)}/t, \quad h_{13}^{(j)} = 4th_7^{(j)}/s, \\ h_{14}^{(j)} = 2h_7^{(j)}, \quad h_{15}^{(j)} = -2th_7^{(j)} \, . \end{gathered} \tag{6.53}$$

For the coefficients $h_i^{(j)}$ in Eq. (6.50) for the box diagram (5a) one has:

$$\begin{gathered} h_1^{(0)} = -2T(2/st - 1/D), \\ h_2^{(0)} = 2(1 + tz_t/\beta^2 D)/s, \\ h_4^{(0)} = (tz_t - sTz_1/D + tz_t/\beta^2)/D, \\ h_6^{(0)} = -2tT(1 + st/D)/D, \\ h_7^{(0)} = -t(s^2T/D - 2t)/D, \\ h_8^{(0)} = (m^2 s + 2t^2 + st^3/D)/D, \\ h_{11}^{(0)} = 16m^2(T/t - 2tz_t/s^2\beta^2)/D; \\[1em] h_1^{(1)} = -8T/sD, \qquad h_2^{(1)} = 8(t + m^2 z_2/s\beta^2)/sD, \\ h_4^{(1)} = -4z_t(t^2/D - (1 + 1/\beta^2)/2)/D, \\ h_6^{(1)} = 8t^2T/D^2, \qquad h_7^{(1)} = 4t(2 - t^2/D)/D, \\ h_8^{(1)} = 4stT/D^2, \qquad h_{11}^{(1)} = -64m^2 z_t/s^2\beta^2 D; \\[1em] h_1^{(2)} = z_t/tD, \qquad h_2^{(2)} = -1/D, \\ h_4^{(2)} = s(1 - st\beta^2/D)/2D, \\ h_6^{(2)} = -tz_1/D^2, \qquad h_7^{(2)} = stz_2/2D^2, \\ h_8^{(2)} = -sz_1/2D^2, \qquad h_{11}^{(2)} = -8m^2/tD; \\[1em] h_1^{(3)} = 0, \qquad h_2^{(3)} = 0, \\ h_4^{(3)} = sz_t/4D, \qquad h_6^{(3)} = t^2/2D, \\ h_7^{(3)} = st/2D, \qquad h_8^{(3)} = st/4D, \qquad h_{11}^{(3)} = 0; \\[1em] h_1^{(4)} = 4T/tD, \qquad h_2^{(4)} = 4z_t/s\beta^2 D, \\ h_4^{(4)} = 2(stz_t/D + 2m^2 z_2/s\beta^2)/D, \\ h_6^{(4)} = h_5^{(4)}, \qquad h_7^{(4)} = 2st^2/D^2, \\ h_8^{(4)} = h_7^{(4)}, \qquad h_{11}^{(4)} = -16(m^2/t - z_u/s\beta^2)/D; \end{gathered} \tag{6.54}$$

$$h_1^{(5)} = -2/D, \qquad h_2^{(5)} = -2z_2/s\beta^2 D,$$
$$h_4^{(5)} = s(z_1/D - z_2/s\beta^2)/D,$$
$$h_6^{(5)} = h_5^{(5)}, \qquad h_7^{(5)} = s^2 t/D^2,$$
$$h_8^{(5)} = h_7^{(5)} \qquad h_{11}^{(5)} = -16z_u/s\beta^2 D;$$

$$h_i^{(6)} = h_i^{(5)}/2.$$

The values for the other coefficient functions $h_i^{(j)}$ with $i = 3, 5, 9, 10, 12-14$ and arbitrary j, not written above, can be straightforwardly inferred from the relations presented in the Eq. (6.53). Next I turn to the second box graph (5b). The color factor for the graph (5b) is

$$T_{\rm col} = (T^a T^b)(T^a T^b), \tag{6.55}$$

and all the functions are obtained from the ones in (6.52) by the simple interchange $t \to u$. Two additional functions (with subscripts 16 and 17) appear, e.g.:

$$\begin{gathered}
f_1^{(k)} = B_1^{(k)}(t \to u), \qquad f_2^{(k)} = B_5^{(k)}, \\
f_3^{(k)} = C_1^{(k-1)}, \qquad f_4^{(k)} = C_1^{(k)}, \\
f_5^{(k)} = C_3^{(k-1)}(t \to u), \qquad f_6^{(k)} = C_3^{(k)}(t \to u), \\
f_7^{(k)} = C_4^{(k-1)}, \qquad f_8^{(k)} = C_4^{(k)}, \\
f_9^{(k)} = D_2^{(k-1)}(t \to u), \qquad f_{10}^{(k)} = D_2^{(k)}(t \to u), \\
f_{11}^{(k)} = k, \\
f_{12}^{(k)} = (k-1)C_1^{(k-2)}, \quad f_{13}^{(k)} = (k-1)C_3^{(k-2)}(t \to u), \\
f_{14}^{(k)} = (k-1)C_4^{(k-2)}, \quad f_{15}^{(k)} = (k-1)D_2^{(k-2)}(t \to u), \\
f_{16}^{(k)} = B_1^{(k-1)}(t \to u), \qquad f_{17}^{(k)} = B_5^{(k-1)}.
\end{gathered} \tag{6.56}$$

The last two functions appear in the expansion (6.50) only in two sums for which the superscript of the coefficients $h_i^{(j)}$ is $j = 1$ and $j = 4$ and, correspondingly, these sums run from 1 to 17.

Furthermore, relations for the various coefficient functions $h_i^{(j)}$ similar to Eq. (6.53) are valid also in this case for any value of j except for $j = 1$ and $j = 4$.

$$\begin{gathered}
h_3^{(j)} = z_u h_7^{(j)}/u, \quad h_5^{(j)} = 2u h_7^{(j)}/s, \quad h_9^{(j)} = -u h_7^{(j)}, \\
h_{10}^{(j)} = -u h_8^{(j)}, \quad h_{12}^{(j)} = 2z_u h_7^{(j)}/u, \quad h_{13}^{(j)} = 4u h_7^{(j)}/s, \\
h_{14}^{(j)} = 2h_7^{(j)}, \quad h_{15}^{(j)} = -2u h_7^{(j)}
\end{gathered} \tag{6.57}$$

These can be obtained from Eq. (6.53) by applying the $t \to u$ operation. In case of $j = 1$ and $j = 4$ one has

$$\begin{gathered}
h_5^{(j)} = 2u h_7^{(j)}/s, \quad h_9^{(j)} = -u h_7^{(j)}, \quad h_{10}^{(j)} = -u h_8^{(j)}, \\
h_{12}^{(j)} = z_u h_{14}^{(j)}/u, \quad h_{13}^{(j)} = 2u h_{14}^{(j)}/s, \\
h_{15}^{(j)} = -u h_{14}^{(j)}.
\end{gathered} \tag{6.58}$$

There exists also a partial symmetry for the box diagrams (5a) and (5b), which allows one to express most coefficients for the box graph (5b) through the ones of the box graph (5a). In particular, starting from the coefficients $h_i^{(j)}$ with superscript $j \geq 2$, I find the following general relations for most of these coefficients:

$$\begin{aligned} h_i^{(j)}[(5\text{b})] &= -h_i^{(j)}[(5\text{a})](t \leftrightarrow u), \quad j = 2; \\ h_i^{(j)}[(5\text{b})] &= h_i^{(j)}[(5\text{a})](t \leftrightarrow u), \quad j = 3, 5, 6. \end{aligned} \tag{6.59}$$

Consequently, for the graph (5b) only the coefficients $h_i^{(0)}$, $h_i^{(1)}$ and $h_i^{(4)}$ are presented here:

$$\begin{aligned} h_1^{(0)} &= 2(T/D + 2(s+U)/su), \\ h_2^{(0)} &= -2(s/D + 1/s + (2 - uz_u/D)/s\beta^2), \\ h_4^{(0)} &= 1 - (8m^2 s + 2m^2 u - s^2)/D + ut^2(t-u)/D^2 \\ &\quad -(2 - uz_u/D)/\beta^2, \\ h_6^{(0)} &= 2u(m^2 st/D + m^2 - 2u)/D, \\ h_7^{(0)} &= -u(4s + 2u - st^2/D)/D, \\ h_8^{(0)} &= -2 + s(m^2 - 2u)/D + m^2 s^2 t/D^2, \\ h_{11}^{(0)} &= -16m^2(s + U + 2tuz_u/s^2\beta^2)/uD; \end{aligned} \tag{6.60}$$

The values for the other coefficient functions $h_i^{(0)}$ with $i = 3, 5, 9, 10, 12-15$, not written above, can be straightforwardly inferred from the relations Eq. (6.57).

Next I write

$$\begin{aligned} h_1^{(1)} &= 4(2m^2 t/u - z_{2u})/sD, \\ h_2^{(1)} &= 2z_{2u}(1 + 1/\beta^2)/sD, \\ h_3^{(1)} &= -2(2m^2 s^2\beta^2 + 2utz_u + sD)/D^2, \\ h_4^{(1)} &= 2(z_{1u}(2m^2 - s)/D + 2m^2 z_{2u}/s\beta^2)/D, \\ h_6^{(1)} &= 4u(m^2 s + uz_u)/D^2, \\ h_7^{(1)} &= 2(-2ut^2/D + s + 4u)/D, \\ h_8^{(1)} &= 2s(m^2 s + uz_u)/D^2, \\ h_{11}^{(1)} &= 16m^2(3/u - 4z_u/s^2\beta^2)/D, \\ h_{14}^{(1)} &= 2u(2uz_{2u} - s^2(1 + 2\beta^2))/D^2, \\ h_{16}^{(1)} &= -4z_u/uD, \qquad h_{17}^{(1)} = 4/D. \end{aligned} \tag{6.61}$$

$$\begin{aligned} h_1^{(4)} &= -4U/uD, \qquad h_2^{(4)} = -4z_u/s\beta^2 D, \\ h_3^{(4)} &= -2z_{2u}(m^2 s/D - 1/\beta^2)/D, \\ h_4^{(4)} &= -2(suz_u/D + 2m^2 z_{2u}/s\beta^2)/D, \end{aligned}$$

$$h_6^{(4)} = -4u^3/D^2, \qquad h_7^{(4)} = 2sut/D^2,$$
$$h_8^{(4)} = -2su^2/D^2, \qquad h_{11}^{(4)} = 16m^2/uD,$$
$$h_{14}^{(4)} = -2suz_{2u}/D^2,$$
$$h_{16}^{(4)} = 4/D, \qquad h_{17}^{(4)} = 4z_{2u}/s\beta^2 D.$$

The other coefficient functions $h_i^{(j)}, j = 1, 4$ with $i = 5, 9, 10, 12, 13, 15$, not written above, can be straightforwardly inferred from the relations Eq. (6.58). One should also mention that all the one-loop matrix elements of this chapter must be multiplied by the common factor (6.4).

Chapter 7

Summary and conclusion

The most important results presented in this thesis are

- analytical results up to $\mathcal{O}(\varepsilon^2)$ for all massive scalar one-loop integrals that arise in the calculation of one-loop matrix elements in heavy flavor hadroproduction

and

- almost complete results on one-loop matrix elements in heavy flavor hadroproduction containing full spin information.

In Chap. 2 I have presented analytical results up to $\mathcal{O}(\varepsilon^2)$ for all massive scalar one-loop integrals that arise in the calculation of one-loop matrix elements in heavy flavor hadroproduction. Some of these results are new (see Table 1.1). The one-loop scalar integrals are needed for that part of the NNLO hadroproduction of heavy flavours which is obtained from the product of one-loop contributions called loop-by-loop contribution. In Chap. 6 I have presented the one-loop amplitudes for hadronic heavy hadron production themselves. Positive powers of ε (up to $\mathcal{O}(\varepsilon^2)$) results arise not only from the scalar one-loop integrals but also from the Passarino–Veltman decomposition and the spin algebra. The full one-loop amplitudes up to order ε^0 were given in [29]. The missing results for the ε- and ε^2-coefficients of the one-loop amplitudes for quark initiated reactions can be found in this thesis. The results for gluon initiated heavy hadron production are also given except for the most complicated box diagrams. This task remains to be done in the near future. The calculation of the loop-by-loop contributions in Fig. 1.1b is a necessary starting point in the evaluation of the NNLO contributions to heavy quark pair production in hadronic interactions. It is very likely that the calculation of the other three classes of diagrams in Fig. 1.1 will proof to be very difficult. This holds true in particular for the massive two-loop box contributions.

In the Laurent series expansion of the scalar one-loop integrals Chap. 2 the successive coefficient functions increase in length and complexity with each order of ε. The reason is that the ε-expansion of the integrand before the last parametric integration itself generates coefficient functions with increasing complexity with each order of ε. The most complex expressions arise from the box contributions where one encounters multiple polylogarithms up to weight and depth four at $\mathcal{O}(\varepsilon^2)$ (see Chap. 2 and Chap. 4). The algorithms to reduce the number of the classical polylogarithms described in Chap. 3 and the identities for the L-functions derived in Chap. 4 were extensively used in order to simplify the final results for the scalar one-loop integrals presented in Chap. 2.

In a numerical NNLO evaluation of heavy hadron production the various contributing pieces will have to be evaluated at many values of the kinematical variables. This requires efficiency in the numerical codes for each of the contributing pieces. I believe that I have provided for such numerical efficiency in the loop-by-loop portion of the NNLO calculation by presenting results in analytical form which are fast to evaluate numerically (see Chap. 5). All the results of Chap. 2 are available in convenient electronic form [60]. When using the formulae of Sec. 4.3 one can write all the results in terms of multiple polylogarithms. In recent years a number of new methods were developed for semi-numerical evaluation of general Feynman diagrams (see e.g. [52, 54, 55, 56]). First numerical tests [51] have shown that the efficiency of the results presented in Chap. 2 is better by orders of magnitude than the present implementation of the flexible all-purpose approach described in [52]. This is in particular true for the imaginary parts. Further comparisons of the numerical efficiency of the two methods have to be done in the near future.

The analytical results presented in this paper cover the whole kinematical domain with a single expression. They evaluate numerically very fast and efficiently. Further advantages of having the results in analytical form are that they allow one to investigate various limiting cases as well as their analyticity properties. Also, when analytical results are available the mathematical structure of the results becomes manifest which would not be visible in a purely numerical approach.

The full calculation of the NNLO corrections to heavy hadron production at hadron colliders will be a very difficult task to complete. It involves the calculation of very many Feynman diagrams of many different topologies. The problem is further complicated by the fact that heavy hadron production is a multi-scale problem with three mass scales provided by the kinematic variables s and t in the loop expressions, and the mass of the heavy quark. It is clear that an undertaking of this dimension will have to involve many theorists and cannot be done by a single group alone. In this sense the present calculation is a first step (or second step [61, 62, 63]) in the direction of obtaining NNLO results on heavy hadron production at hadron colliders. The present calculation allows one to obtain a first glimpse of the mathematical and computational complexity that is waiting for physicists in the full NNLO calculation. This complexity does in fact already reveal itself in terms of a very rich polylogarithmic and multiple polylogarithmic structure of the Laurent series expansion of the scalar one-loop integrals as shown in this thesis.

I hope that the tools for simplifying classical polylogarithms described in this thesis will be useful for other practitioners in this field. It can be expected that the detailed discussion of the properties of the L-functions and their connection to the multiple polylogarithms of Goncharov presented in this thesis will become useful also for other NNLO calculations where multiple polylogarithms can also be expected to appear.

Appendix A

Special two-point function needed for the renormalization constants

In this Appendix I evaluate a special two-point integral which is needed for the calculation of the one-loop fermion self-energy diagram insertion into the massive external fermion line. This integral is also needed for the definitions of the fermion mass and wave function renormalization constants in the on-shell renormalization scheme. In particular, one needs to evaluate the integral

$$I_1 \equiv B(p,0,m) = \mu^{2\varepsilon} \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2[(q+p)^2 - m^2]} \tag{A.1}$$

up to $\mathcal{O}(p^2 - m^2)$. One therefore Taylor expands I_1 around $p^2 = m^2$:

$$\begin{aligned} I_1 &= I_1\Big|_{p^2=m^2} + \frac{dI_1}{dp^2}\Big|_{p^2=m^2}(p^2 - m^2) + \dots \\ &\equiv E_0 + E_1(p^2 - m^2) + \dots \end{aligned} \tag{A.2}$$

Note that the expansion coefficients E_i in (A.2) are functions of ε. The first coefficient E_0 is nothing but B_3 obtained in Sec. 2.1. The second coefficient E_1 is proportional to the sum of a scalar and a vector integral obtained by differentiating I_1 w.r.t. p^μ. One obtains

$$E_1 = \frac{1}{2p_\mu}\frac{dI_1}{dp^\mu} = -I_2 - \frac{p^\mu I_{2\mu}}{m^2}, \tag{A.3}$$

where

$$I_{\{2,2\mu\}} = \mu^{2\varepsilon} \int \frac{d^n q}{(2\pi)^n} \frac{\{1, q_\mu\}}{q^2[(q+p)^2 - m^2]^2} = iC_\varepsilon(m^2)\frac{1}{m^2}\left\{\frac{1}{2\varepsilon}, \frac{p_\mu}{1-2\varepsilon}\right\}. \tag{A.4}$$

One finally has

$$I_1 = iC_\varepsilon(m^2)\frac{1}{\varepsilon(1-2\varepsilon)}\left(1 - \frac{p^2 - m^2}{2m^2}\right) + \mathcal{O}[(p^2 - m^2)^2]. \tag{A.5}$$

The result for the integral I_1 in the form (A.5) was used in [40] to evaluate external heavy quark self-energy diagrams and obtain heavy quark wave function renormalization constants in the NLO calculation.

Appendix B

Programs for the simplification of the functions Li_2

All the programs presented in App. B and App. C can be copied via "cut and paste" methods from the tex-source-file of this thesis and placed directly into **Mathematica**-files.

B.1 Program based on the identities with one variable

This program reduces the number of dilogarithms present in the input expression. The algorithm is described in Sec. 3.2.1. The user should run the program under **Mathematica** [42] using `Li2Simplify[InputExpression, SetOfDesiredArguments]`.
`InputExpression` is the expression to be simplified. It contains various `Re[PolyLog[2, z]]` terms. `SetOfDesiredArguments` is the set of the arguments of the dilogarithms which have to be chosen automatically for the expansion of the dilogarithms from the expression `InputExpression` in the case that there are relations between the dilogarithms of these arguments and dilogarithms from `InputExpression`. One collects this set from previous results to obtain new results in terms of already present dilogarithms. If one would like to ignore this option one chooses the empty set `{}` as the `SetOfDesiredArguments`. The output of the program is an expression with the reduced number of dilogarithms.
Here I present the text of the program:

```
Li2Simplify[expression_, basis_] := Module[{Li2Set, Li2SetArg, Li2SetResult,
Li2SetClue, FurtherKey, family, Finv, Fone, Finvone, Foneinv, Foneinvone,
Li2SetNew, Li2SetArgNew, expression1},

Finv[x__] = -PolyLog[2, 1/x] - Pi^2/6 - (1/2)*Log[-x]^2;
Fone[x__] = -PolyLog[2, 1 - x] + Pi^2/6 - Log[x]*Log[1 - x];
Finvone[x__] = Simplify[-(Pi^2/6) - (1/2)*Log[-x]^2
- (Pi^2/6 - Log[1 - 1/x]*Log[1/x] - PolyLog[2, 1 - 1/x])];
Foneinv[x__] = Simplify[Pi^2/6 - Log[1 - x]*Log[x]
-(-(Pi^2/6) - (1/2)*Log[-1 + x]^2 - PolyLog[2, 1/(1 - x)])];
```

```
Foneinvone[x__] = Simplify[Pi^2/3 + (1/2)*Log[-1 + x]^2 - Log[1 - x]*Log[x]
+ (1/6)*(Pi^2 - 6*Log[1/(1 - x)]*Log[x/(-1 + x)] - 6*PolyLog[2, x/(-1 + x)])];

Li2Set = Variables[Cases[expression, Re[PolyLog[2, a_]], Infinity]];
Li2SetArg = Simplify[Li2Set /. Re[PolyLog[2, a_]] -> a];
Li2SetClue = Table[True, {i, Length[Li2Set]}];
Li2SetResult = Li2Set;

  Do[Print["i=", i];
   Do[
          If[Simplify[basis[[k]] - Li2SetArg[[i]]] == 0, Li2SetClue[[i]] = False;
Li2SetResult[[i]] = Re[PolyLog[2, basis[[k]]]]];
        If[Li2SetClue[[i]], If[Simplify[basis[[k]] - 1/Li2SetArg[[i]]] == 0,
Li2SetClue[[i]] = False; Li2SetResult[[i]] = Re[Simplify[Finv[Li2SetArg[[i]]]]]]];
        If[Li2SetClue[[i]], If[Simplify[basis[[k]] - (1 - Li2SetArg[[i]])] == 0,
Li2SetClue[[i]] = False; Li2SetResult[[i]] = Re[Simplify[Fone[Li2SetArg[[i]]]]]]];
        If[Li2SetClue[[i]], If[Simplify[basis[[k]] - (1 - 1/Li2SetArg[[i]])] == 0,
Li2SetClue[[i]] = False; Li2SetResult[[i]] = Re[Simplify[Finvone[Li2SetArg[[i]]]]]]];
        If[Li2SetClue[[i]], If[Simplify[basis[[k]] - 1/(1 - Li2SetArg[[i]])] == 0,
Li2SetClue[[i]] = False; Li2SetResult[[i]] = Re[Simplify[Foneinv[Li2SetArg[[i]]]]]]];
        If[Li2SetClue[[i]],
          If[Simplify[basis[[k]] - Li2SetArg[[i]]/(Li2SetArg[[i]] - 1)] == 0,
Li2SetClue[[i]] = False; Li2SetResult[[i]] =Re[Simplify[Foneinvone[Li2SetArg[[i]]]]]]],
                      {k, Length[basis]}],
                              {i, Length[Li2Set]}];

FurtherKey = False;
Do[FurtherKey = FurtherKey || Li2SetClue[[j]], {j, Length[Li2Set]}];

If[FurtherKey,
 Print["I begin the simplification of the function Li2 that can not be
expressed in terms of Li2 from BASIS."]];
 Do[If[Li2SetClue[[i]], family = {};
  Do[If[Li2SetClue[[j]], If[Simplify[Li2SetArg[[i]] - Li2SetArg[[j]]] == 0,
       Li2SetClue[[j]] = False; family = Append[family, j]]], {j, Length[Li2Set]}];
  Do[If[Li2SetClue[[j]], If[Simplify[Li2SetArg[[i]] - 1/Li2SetArg[[j]]] == 0,
       Li2SetClue[[j]] = False; family = Append[family, j]]], {j, Length[Li2Set]}];
  Do[If[Li2SetClue[[j]], If[Simplify[Li2SetArg[[i]] - (1 - Li2SetArg[[j]])] == 0,
       Li2SetClue[[j]] = False; family = Append[family, j]]], {j, Length[Li2Set]}];
  Do[If[Li2SetClue[[j]], If[Simplify[Li2SetArg[[i]] - (1 - 1/Li2SetArg[[j]])] == 0,
       Li2SetClue[[j]] = False; family = Append[family, j]]], {j, Length[Li2Set]}];
  Do[If[Li2SetClue[[j]], If[Simplify[Li2SetArg[[i]] - 1/(1 - Li2SetArg[[j]])] == 0,
       Li2SetClue[[j]] = False; family = Append[family, j]; ]], {j, Length[Li2Set]}];
  Do[If[Li2SetClue[[j]],
         If[Simplify[Li2SetArg[[i]] - Li2SetArg[[j]]/(Li2SetArg[[j]] - 1)] == 0,
```

```
        Li2SetClue[[j]] = False; family = Append[family, j]; ]], {j, Length[Li2Set]}];

 Print["I have found DilogSmallSet ."];
 Print["Below you see the elements of the DilogSmallSet with the corresponding
 probing values from the physical region."];

 Do[Print[l, " -> ", PolyLog[2, Li2SetArg[[family[[l]]]]], " -> ",
       N[PolyLog[2, Li2SetArg[[family[[l]]]]]
          /. t -> (-s/2)*(1 - Sqrt[1 - 4*(m^2/s)]*Cos[\[Kappa]])
          /. s -> 8.1*m^2 /. \[Kappa] -> 0.2*Pi ]],
        {l, Length[family]}];
 choice = 0;
 While[choice < 1 || (choice > Length[family] ||  !IntegerQ[choice]),
  Print["You should choose one of theses Li2 as the basis function.
           All others will be expressed via it."];
  choice = Input["Type the number of chosen Li2"]];
 Do[If[Simplify[Li2SetArg[[family[[choice]]]] - Li2SetArg[[family[[i]]]]] == 0,
Li2SetResult[[family[[i]]]] = Re[PolyLog[2, Li2SetArg[[family[[choice]]]]]];
    If[Simplify[Li2SetArg[[family[[choice]]]] - 1/Li2SetArg[[family[[i]]]]] == 0,
Li2SetResult[[family[[i]]]] = Re[Simplify[Finv[Li2SetArg[[family[[i]]]]]]];
    If[Simplify[Li2SetArg[[family[[choice]]]] - (1 - Li2SetArg[[family[[i]]]])] == 0,
Li2SetResult[[family[[i]]]] = Re[Simplify[Fone[Li2SetArg[[family[[i]]]]]]];
    If[Simplify[Li2SetArg[[family[[choice]]]] - (1 - 1/Li2SetArg[[family[[i]]]])] == 0,
Li2SetResult[[family[[i]]]] = Re[Simplify[Finvone[Li2SetArg[[family[[i]]]]]]];
    If[Simplify[Li2SetArg[[family[[choice]]]] - 1/(1 - Li2SetArg[[family[[i]]]])] == 0,
Li2SetResult[[family[[i]]]] = Re[Simplify[Foneinv[Li2SetArg[[family[[i]]]]]]];
    If[Simplify[Li2SetArg[[family[[choice]]]] - Li2SetArg[[family[[i]]]]
                        /(Li2SetArg[[family[[i]]]] - 1)] == 0,
Li2SetResult[[family[[i]]]] = Re[Simplify[Foneinvone[Li2SetArg[[family[[i]]]]]]]],
                    {i, Length[family]}]],
                                                   {i, Length[Li2Set]}];

Li2SetResult = Li2SetResult //. Re[(g_) + (h_)] -> Re[g] + Re[h]
//. Re[Log[a_]^2] -> logAbs[a]^2 - Pi^2*\[Theta][-a]
//. Re[Log[a_]*Log[b_]] -> logAbs[a]*logAbs[b];
expression1 = expression;
Do[expression1 = expression1 //. Li2Set[[i]] -> Li2SetResult[[i]], {i, Length[Li2Set]}];
Li2SetNew = Variables[Cases[expression1, Re[PolyLog[2, a_]], Infinity]];
SetArgNew = Factor[Li2SetNew /. Re[PolyLog[2, a_]] -> a];
Do[expression1 = expression1 //. Li2SetNew[[i]] -> Re[PolyLog[2, SetArgNew[[i]]]],
                         {i, Length[Li2SetNew]}];
                                              expression1]
```

B.2 Program based on the identitiy with two variables

This program realizes the algorithms for the search of the dilogarithms from the input expression related by Hill's identity (3.13) in combination with the identities with one variable (3.6) and combinations of these (see Sec. 3.2.2). The user should run the program under **Mathematica** [42] using `HillSearch[SetOfArgumentsOfLi2]`. `SetOfArgumentsOfLi2` is the set of the arguments of the dilogarithms of the input expression. After one starts the program it prints some helpful information . The most important information is at the end. If the line such as
" `for a pair` *some expression* `and` *some expression* `connection has been found`"

occurs then the relation is found. Otherwise there is no relation. From the last line one gets the arguments of the dilogarithms (exactly *some expressions*) for the initial pair (see Sec. 3.2.2). The last values of `n` and `p` correspond to the transformation of the initial pair to be done. The numbers of transformations coincide with the numbers of $F_j(z)$ functions in (3.11). Then one finds the information how each dilogarithm from the r.h.s. of (3.13) is related to already existing dilogarithms. The positions of related dilogarithms on the r.h.s. of Hill's identity (3.13) (first, second or third) are given together with the numbers of the transformations which have to be performed with existing dilogarithms in order to obtain the dilogarithms from the r.h.s. of (3.13). Therefore in the output of the program all necessary information for the use of Hill's identity are presented. An exemplary end of the output is:

```
n, p 1 1
the  2'th dilogs from the r.h.s. of Hill's identity is connected by
transformation 1 to dilogs of 6'th argument.
the  3'th dilogs from the r.h.s. of Hill's identity is connected by
transformation 1 to dilogs of 8'th argument.
the  1'th dilogs from the r.h.s. of Hill's identity is connected by
transformation 1 to dilogs of 15'th argument.
for a pair -t/s and (2*m^2)/(2*m^2 - (s + t)*(1 + \[Beta])) connection
has been found
```

It means that $\mathrm{Li}_2\left(\frac{-t}{s}\right)$ and $\mathrm{Li}_2\left(\frac{2m^2}{2m^2-(s+t)(1+\beta}\right)$ are the dilogarithms for the initial pair. No transformation with the initial pair should be performed because `n` and `p` are 1 and 1, respectively. The second dilogarithm from the r.h.s. of (3.13) is related to the dilogarithm with the argument which is equal to the sixth element of the set `SetOfArgumentsOfLi2`. One has to perform the transformation number "1" with this dilogarithm to obtain the second dilogarithm from the r.h.s. of (3.13). The same information can be found for the remaining two dilogarithms from the r.h.s. of (3.13).
Here I present the text of the program:

```
HillSearch[set_] := Module[{probe, x, y, logic, Key, Poss},

  Poss[arg1_, arg2_] := Abs[2*((arg1 - arg2)/(arg1 + arg2))
```

```
/. \[Beta] -> Sqrt[1 - 4*(m^2/s)]
/. t -> (-s/2)*(1 - Sqrt[1 - 4*(m^2/s)]*Cos[\[Tau]])
/. u -> (-s/2)*(1 + Sqrt[1 - 4*(m^2/s)]*Cos[\[Tau]])
/. s -> 8.5*m^2 /.\[Tau]-> 0.4*Pi] < 0.01;

Key = True;
  Do[If[Key, Print[ i, " ", j];

      x = Simplify[set[[i]] /. a_ -> {a,1 - a, 1/a,1/(1 - a),1 - 1/a, a/(a - 1)}];
      Print["x", x];
      y = Simplify[set[[j]] /. a_ -> {a,1 - a, 1/a,1/(1 - a),1 - 1/a, a/(a - 1)}];
      Print["y", y];

      Do[If[Key,
          Print["n, p ", n, " ", p]; logic = {False, False, False};
          probe = Simplify[{x[[n]]*y[[p]], (x[[n]]*(-1 + y[[p]]))/(-1 + x[[n]]*y[[p]]),
                          ((-1 + x[[n]])*y[[p]])/(-1 + x[[n]]*y[[p]])}];
Do[If[ !logic[[k]], If[Poss[probe[[k]], set[[l]]], logic[[k]] = True;
      Print["the  ", k,"'th dilogs from the r.h.s. of Hill's identity is
connected by transformation 1 to dilogs of ", l,"'th argument."]]];
   If[ !logic[[k]], If[Poss[probe[[k]], 1/set[[l]]] == 0, logic[[k]] = True;
      Print["the  ", k ,"'th dilogs from the r.h.s. of Hill's identity is
connected by transformation 1 to dilogs of ", l,  "'th argument."]]];
   If[ !logic[[k]], If[Poss[probe[[k]], 1 - set[[l]]] == 0, logic[[k]] = True;
      Print["the  ", k, "'th dilogs from the r.h.s. of Hill's identity is
connected by transformation 3 to dilogs of ", l, "'th argument."]]];
   If[ !logic[[k]], If[Poss[probe[[k]], 1/(1 - set[[l]])] == 0, logic[[k]] = True;
      Print["the  ", k,"'th dilogs from the r.h.s. of Hill's identity is
connected by transformation 4 to dilogs of ", l, "'th argument."]]];
   If[ !logic[[k]], If[Poss[probe[[k]], 1 - 1/set[[l]]] == 0, logic[[k]] = True;
      Print["the  ", k,"'th dilogs from the r.h.s. of Hill's identity is
connected by transformation 5 to dilogs of ", l,  "'th argument."   ]]];
   If[ !logic[[k]], If[Poss[probe[[k]] - set[[l]]/(set[[l]] - 1)] == 0,
      logic[[k]] = True;
      Print["the  ", k, " 'th dilogs from the r.h.s. of Hill's identity is
connected by transformation 6 to dilogs of ",  l, "'th argument.  "]]],
                                        {l, Length[set]}, {k, Length[logic]}];

If[logic[[1]] && logic[[2]] && logic[[3]],
           Key = False; Print["for a pair ", set[[i]], " and ", set[[j]],
            "connection has been found"]]],
                       {n, 6}, {p, 6}]],
                       {i, Length[set]},{j, i + 1, Length[set]}]]
```

Appendix C

Program for the simplification of the functions Li_3

This program reduces the number of functions Li_3 present in the input expression. The algorithm is described in Sec. 3.3. The user should run the program under **Mathematica** [42] using `Li3Simplify[InputExpression, SetOfDesiredArguments]`.
`InputExpression` is the expression to be simplified. It contains various `Re[PolyLog[3, z]]` terms. `SetOfDesiredArguments` is the set of the arguments of the dilogarithms which have to be chosen automatically for the expansion of the dilogarithms from the expression `InputExpression` in the case that there are relations between the dilogarithms of these arguments and dilogarithms from `InputExpression`. One collects this set from previous results to obtain new results in terms of already present dilogarithms. If one would like to ignore this option one chooses the empty set `{}` as the `SetOfDesiredArguments`. The output of the program is an expression with the reduced number of function Li_3.
Here I present the text of the program:

```
Li3Simplify[expression_, basis_] :=
Module[{Li3Set, Li3SetArg, Li3SetResult, Li3SetClue, FurtherKey, family,
  forfamily, choice1, choice},
Li3Set = Variables[Cases[expression, Re[PolyLog[3, a_]], Infinity]];
Li3SetArg = Simplify[Li3Set /. Re[PolyLog[3, a_]] -> a];
Li3SetClue = Table[True, {i, Length[Li3Set]}];
Li3SetResult = Li3Set;

Finv3[x__] = PolyLog[3, 1/x] - (Pi^2/6)*Log[-x] - (1/6)*Log[-x]^3;
Fbig3a[x__] = Simplify[-PolyLog[3, 1 - x] - PolyLog[3, -x/(1 - x)] + PolyLog[3, 1]
+ (Pi^2/6)*Log[1 - x] - (1/2)*Log[x]*Log[1 - x]^2 + (1/6)*Log[1 - x]^3];
Fbig3b[x__] = Simplify[-PolyLog[3, 1 - x] - (1/6)*((-Pi^2)*Log[x/(1 - x)]
- Log[x/(1 - x)]^3 + 6*PolyLog[3, (-1 + x)/x]) + PolyLog[3, 1]
+(Pi^2/6)*Log[1 - x] - (1/2)*Log[x]*Log[1 - x]^2 + (1/6)*Log[1 - x]^3];
Fbig3c[x__] = Simplify[(-(1/6))*((-Pi^2)*Log[-1 + x] - Log[-1 + x]^3
+ 6*PolyLog[3, 1/(1 - x)]) - PolyLog[3, -x/(1 - x)] + PolyLog[3, 1]
+ (Pi^2/6)*Log[1 - x] - (1/2)*Log[x]*Log[1 - x]^2 + (1/6)*Log[1 - x]^3];
```

```
Fbig3d[x__] = Simplify[(-(1/6))*((-Pi^2)*Log[-1 + x] - Log[-1 + x]^3
+ 6*PolyLog[3, 1/(1 - x)]) - (1/6)*((-Pi^2)*Log[x/(1 - x)] - Log[x/(1 - x)]^3
+ 6*PolyLog[3, (-1 + x)/x]) + PolyLog[3, 1] + (Pi^2/6)*Log[1 - x]
- (1/2)*Log[x]*Log[1 - x]^2 + (1/6)*Log[1 - x]^3];

Do[If[Li3SetClue[[i]],
 family = {}; forfamily = {};
 Do[If[Li3SetClue[[j]], If[Simplify[Li3SetArg[[i]] - Li3SetArg[[j]]] == 0,
   Li3SetClue[[j]] = False; family = Append[family, j]]], {j, Length[Li3Set]}];
 Do[If[Li3SetClue[[j]], If[Simplify[Li3SetArg[[i]] - 1/Li3SetArg[[j]]] == 0,
   Li3SetClue[[j]] = False; family = Append[family, j]]], {j, Length[Li3Set]}];
 Do[If[Li3SetClue[[j]], If[Simplify[Li3SetArg[[i]] - (1 - Li3SetArg[[j]])] == 0,
   Li3SetClue[[j]] = False; family = Append[family, j]]], {j, Length[Li3Set]}];
 Do[If[Li3SetClue[[j]], If[Simplify[Li3SetArg[[i]] - (1 - 1/Li3SetArg[[j]])] == 0,
   Li3SetClue[[j]] = False; family = Append[family, j]]], {j, Length[Li3Set]}];
 Do[If[Li3SetClue[[j]], If[Simplify[Li3SetArg[[i]] - 1/(1 - Li3SetArg[[j]])] == 0,
   Li3SetClue[[j]] = False; family = Append[family, j]]], {j, Length[Li3Set]}];
 Do[If[Li3SetClue[[j]],
   If[Simplify[Li3SetArg[[i]] - Li3SetArg[[j]]/(Li3SetArg[[j]] - 1)] == 0,
   Li3SetClue[[j]] = False; family = Append[family, j]]], {j, Length[Li3Set]}];

 Print["ThreelogSmallSet is composed"];
 Print["Below you can see the elements of ThreelogSmallSet with the corresponding
   probe values from the physical region."];

 Do[Print[l, " -> ", PolyLog[3, Li3SetArg[[family[[l]]]]], " -> ",
      N[PolyLog[3, Li3SetArg[[family[[l]]]]] /. \[Beta] -> Sqrt[1 - 4*(m^2/s)] /.
      t -> (-s/2)*(1 - Sqrt[1 - 4*(m^2/s)]*Cos[\[Kappa]]) /. s -> 8.1*m^2
      /. \[Kappa] -> 0.2*Pi]], {l, Length[family]}];

 Do[If[Simplify[Li3SetArg[[family[[1]]]] - basis[[i]]] == 0,
   forfamily = Append[forfamily, basis[[i]]]], {i, Length[basis]}];
 Do[If[Simplify[1/Li3SetArg[[family[[1]]]] - basis[[i]]] == 0,
   forfamily = Append[forfamily, basis[[i]]]], {i, Length[basis]}];
 Do[If[Simplify[(1 - Li3SetArg[[family[[1]]]]) - basis[[i]]] == 0,
   forfamily = Append[forfamily, basis[[i]]]], {i, Length[basis]}];
 Do[If[Simplify[(1 - 1/Li3SetArg[[family[[1]]]]) - basis[[i]]] == 0,
   forfamily = Append[forfamily, basis[[i]]]], {i, Length[basis]}];
 Do[If[Simplify[1/(1 - Li3SetArg[[family[[1]]]]) - basis[[i]]] == 0,
   forfamily = Append[forfamily, basis[[i]]]], {i, Length[basis]}];
 Do[If[Simplify[Li3SetArg[[family[[1]]]]/(Li3SetArg[[family[[1]]]] - 1)
        - basis[[i]]] == 0,
   forfamily = Append[forfamily, basis[[i]]]], {i, Length[basis]}];

 familyClue = Table[True, {i, Length[family]}];
```

```
If[Length[family] == 1,

  If[Length[forfamily] == 0, familyClue[[1]] = False;
    Print["Please choose the option:"];
    Print[" 1 - leave the Li3 without transformation."];
    Print[" 2 - Transform Li3 using inversion of argument."]; choice = 0;
    While[choice < 1 || (choice > 2 ||  !IntegerQ[choice]),
       choice = Input["Type option for transformation"]];
    If[choice == 2, Li3SetResult[[family[[1]]]] =
       Re[Simplify[Finv3[Li3SetArg[[family[[1]]]]]]]]];

  If[Length[forfamily] == 1,

   If[Simplify[Li3SetArg[[family[[1]]]] - forfamily[[1]]] == 0,
       familyClue[[1]] = False; Li3SetResult[[family[[1]]]] =
        Re[PolyLog[3, forfamily[[1]]]]];

   If[familyClue[[1]],
    If[Simplify[Li3SetArg[[family[[1]]]] - 1/forfamily[[1]]] == 0,
     familyClue[[1]] = False;
         Li3SetResult[[family[[1]]]] = Re[Simplify[Finv3[Li3SetArg[[family[[1]]]]]]]]];

   If[familyClue[[1]], familyClue[[1]] = False; Print["Please choose the option:"];
    Print[" 1 - leave the Li3 without transformation."];
    Print[" 2 - Transform Li3 using inversion of argument."];
    choice = 0;
    While[choice < 1 || (choice > 2 ||  !IntegerQ[choice]),
               choice = Input["Type option for transformation"]];
    If[choice == 2,
       Li3SetResult[[family[[1]]]] = Re[Simplify[Finv3[Li3SetArg[[family[[1]]]]]]]]]];

  If[Length[forfamily] == 2,

   If[Simplify[Li3SetArg[[family[[1]]]] - forfamily[[1]]] == 0, familyClue[[1]] = False;
     Li3SetResult[[family[[1]]]] = Re[PolyLog[3, forfamily[[1]]]]];
   If[familyClue[[1]], If[Simplify[Li3SetArg[[family[[1]]]] - forfamily[[2]]] == 0,
     familyClue[[1]] = False;
     Li3SetResult[[family[[1]]]] = Re[PolyLog[3, forfamily[[2]]]]]];
   If[familyClue[[1]], If[Simplify[Li3SetArg[[family[[1]]]] - 1/forfamily[[1]]] == 0,
     familyClue[[1]] = False;
     Li3SetResult[[family[[1]]]] = Re[Simplify[Finv3[Li3SetArg[[family[[1]]]]]]]]];
   If[familyClue[[1]], If[Simplify[Li3SetArg[[family[[1]]]] - 1/forfamily[[2]]] == 0,
     familyClue[[1]] = False;
     Li3SetResult[[family[[1]]]] = Re[Simplify[Finv3[Li3SetArg[[family[[1]]]]]]]]];
```

```
  If[familyClue[[1]], familyClue[[1]] = False;
    Print["Please choose the option:"];
    Print[" 1 - leave the Li3 without transformation."];
    Print[" 2 - Transform Li3 using inversion of argument."]; choice = 0;
    While[choice < 1 || (choice > 2 ||  !IntegerQ[choice]),
               choice = Input["Type option for transformation"]];
    If[choice == 2,
     Li3SetResult[[family[[1]]]] = Re[Simplify[Finv3[Li3SetArg[[family[[1]]]]]]]]]];

If[Length[family] != 1,
 If[Length[forfamily] == 2,
  Print["I express the elements of the ThreelogSmallSet in terms of ",
           PolyLog[3, forfamily[[1]]], " and ", PolyLog[3, forfamily[[2]]],
           " from the set of the desired functions Li3"]];

 If[Length[forfamily] == 1,
  Print["I have found only one polylog ", PolyLog[3, forfamily[[1]]],
        "from the set of the desired functions Li3"];
  Print["For the functions Li3 I need two Li3 to express all others elements
        from the ThreelogSmallSet"];
  Print["Please choose the second Li3 from ThreelogSmallSet"];
  choice = 0;
  While[choice < 1 || (choice > Length[family] ||  !IntegerQ[choice]),
           choice = Input["Type the number of chosen Li3"];
           If[choice >= 1 && (choice <= Length[family] && IntegerQ[choice]),
            If[Simplify[Li3SetArg[[family[[choice]]]] - forfamily[[1]]] == 0,
             choice = 0]]];
  forfamily = Append[forfamily, Li3SetArg[[family[[choice]]]]]];

 If[Length[forfamily] == 0,
  Print["There are NO Li3 from the set of the desired functions Li3 for
        this ThreelogSmallSet"];
  Print["For the functions Li3 I need two Li3 to express all others
        elements from the ThreelogSmallSet"];
  Print["Please choose two Li3 from the hreelogSmallSet"];
  choice = 0;
  While[choice < 1 || (choice > Length[family] ||  !IntegerQ[choice]),
         choice = Input["Type the number of the first chosen Li3"]];
  forfamily = Append[forfamily, Li3SetArg[[family[[choice]]]]];
  choice1 = 0;
  While[choice1 < 1 || (choice1 > Length[family] ||  !IntegerQ[choice1])
  || choice1 == choice,
         choice1 = Input["Type the number of the second chosen Li3"]];
  forfamily = Append[forfamily, Li3SetArg[[family[[choice1]]]]]]];
```

```
Print["forfamily -> ", forfamily, "family", family];

Do[If[familyClue[[i]] == True,
    If[Simplify[Li3SetArg[[family[[i]]]] - forfamily[[1]]] == 0,
     Li3SetResult[[family[[i]]]] = Re[PolyLog[3, forfamily[[1]]]];
      familyClue[[i]] = False]];

   If[familyClue[[i]] == True,
    If[Simplify[Li3SetArg[[family[[i]]]] - forfamily[[2]]] == 0,
     Li3SetResult[[family[[i]]]] = Re[PolyLog[3, forfamily[[2]]]];
      familyClue[[i]] = False]];

   If[familyClue[[i]] == True,
    If[Simplify[Li3SetArg[[family[[i]]]] - 1/forfamily[[1]]] == 0,
     Li3SetResult[[family[[i]]]] = Re[Simplify[Finv3[Li3SetArg[[family[[i]]]]]]];
      familyClue[[i]] = False]];

   If[familyClue[[i]] == True,
    If[Simplify[Li3SetArg[[family[[i]]]] - 1/forfamily[[2]]] == 0,
     Li3SetResult[[family[[i]]]] = Re[Simplify[Finv3[Li3SetArg[[family[[i]]]]]]];
      familyClue[[i]] = False]];

   If[familyClue[[i]] == True,
    If[Simplify[forfamily[[1]] - (1 - Li3SetArg[[family[[i]]]])] == 0,
     familyClue[[i]] = False;
     If[Simplify[forfamily[[2]] - Li3SetArg[[family[[i]]]]
         /(Li3SetArg[[family[[i]]]] - 1)] == 0,
        Print["First"];
        Li3SetResult[[family[[i]]]] = Re[Simplify[Fbig3a[Li3SetArg[[family[[i]]]]]]]];
     If[Simplify[forfamily[[2]] - (1 - 1/Li3SetArg[[family[[i]]]])] == 0,
        Print["Second"];
        Li3SetResult[[family[[i]]]] = Re[Simplify[Fbig3b[Li3SetArg[[family[[i]]]]]]]]]];

   If[familyClue[[i]] == True,
    If[Simplify[forfamily[[2]] - (1 - Li3SetArg[[family[[i]]]])] == 0,
     familyClue[[i]] = False;
     If[Simplify[forfamily[[1]] - Li3SetArg[[family[[i]]]]
        /(Li3SetArg[[family[[i]]]] - 1)] == 0,
        Print["First"];
        Li3SetResult[[family[[i]]]] = Re[Simplify[Fbig3a[Li3SetArg[[family[[i]]]]]]]];
     If[Simplify[forfamily[[1]] - (1 - 1/Li3SetArg[[family[[i]]]])] == 0,
        Print["Second"];
     Li3SetResult[[family[[i]]]] = Re[Simplify[Fbig3b[Li3SetArg[[family[[i]]]]]]]]]];

   If[familyClue[[i]] == True,
```

```
     If[Simplify[forfamily[[1]] - 1/(1 - Li3SetArg[[family[[i]]]])] == 0,
      familyClue[[i]] = False;
      If[Simplify[forfamily[[2]] - Li3SetArg[[family[[i]]]]
         /(Li3SetArg[[family[[i]]]] - 1)] == 0,
         Print["Third"];
         Li3SetResult[[family[[i]]]] = Re[Simplify[Fbig3c[Li3SetArg[[family[[i]]]]]]]];
      If[Simplify[forfamily[[2]] - (1 - 1/Li3SetArg[[family[[i]]]])] == 0,
         Print["Forth"];
         Li3SetResult[[family[[i]]]] = Re[Simplify[Fbig3d[Li3SetArg[[family[[i]]]]]]]]]];

    If[familyClue[[i]] == True,
     If[Simplify[forfamily[[2]] - 1/(1 - Li3SetArg[[family[[i]]]])] == 0,
      familyClue[[i]] = False;
      If[Simplify[forfamily[[1]] - Li3SetArg[[family[[i]]]]
         /(Li3SetArg[[family[[i]]]] - 1)] == 0,
         Print["Third"];
         Li3SetResult[[family[[i]]]] = Re[Simplify[Fbig3c[Li3SetArg[[family[[i]]]]]]]];
      If[Simplify[forfamily[[1]] - (1 - 1/Li3SetArg[[family[[i]]]])] == 0,
         Print["Forth"];
         Li3SetResult[[family[[i]]]] = Re[Simplify[Fbig3d[Li3SetArg[[family[[i]]]]]]]]]];

    Print["It was -> ", Li3SetArg[[family[[i]]]]];
    Print["Result -> ", Li3SetResult[[family[[i]]]]];
    Print["-------------------------------------------------"],

                                                      {i, Length[family]}]],

                                                                {i, Length[Li3Set]}];

Li3SetResult = Simplify[Li3SetResult //. Re[(g_) + (h_)] -> Re[g] + Re[h]
 //. Re[Log[\[Xi]__]^3] -> logAbs[\[Xi]]^3 - 1*3*Pi^2*logAbs[\[Xi]]*\[Theta][-\[Xi]]^2
 //.Re[Log[\[Alpha]__]*Log[\[Beta]__]^2] ->
 logAbs[\[Alpha]]*logAbs[\[Beta]]^2 - 1*Pi^2*logAbs[\[Alpha]]*\[Theta][-\[Beta]]^2
 //. Re[Log[\[Zeta]__]] -> logAbs[\[Zeta]]];

expression1 = expression;
Do[expression1 = expression1 //. Li3Set[[i]] -> Li3SetResult[[i]], {i, Length[Li3Set]}];

Li3SetNew = Variables[Cases[expression1, Re[PolyLog[3, a_]], Infinity]];
SetArgNew = Factor[Li3SetNew /. Re[PolyLog[3, a_]] -> a];
Do[expression1 = expression1
 //. Li3SetNew[[i]] -> Re[PolyLog[3, SetArgNew[[i]]]], {i, Length[Li3SetNew]}];

expression1                                                                   ]
```

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