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On neutrino oscillations in curved spacetime

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Abstract. The existence of mass of neutrino can explain neutrino oscillations. The paper presents a pioneering classical approach to unravel vacuum neutrino oscillations in curved spacetime. The framework explores transition probabilities, even mass fluctuations, offering a roadmap for predicting neutrino oscillations in diverse spacetime contexts. Neutrinos, usually quantum entities, are portrayed as classical particles following geodesic paths, particularly around massive objects like black holes and neutron stars. The exploration extends into various spacetime models, revealing neutrino behaviors in Schwarzschild, Kerr, and Kerr-like spacetimes, as well as the challenging non-separable domains of γ-metric and Hartle-Thorne metric. Canonical phase formulas for neutrinos are derived, showcasing their interplay with gravitational backgrounds. Overall, this study sheds light on neutrino oscillations in the captivating arena of curved spacetime, aiming to enrich cosmic comprehension.

Keywords: Neutrino oscillation, Canonical phase, Curved spacetime, γ-metric, Hartle-Thorne metric

I. INTRODUCTION

Neutrino oscillation is a phenomenon observed in neutrino physics, where neutrinos change from one flavor to another as they travel through space. This oscillation is caused by the mixing of the three known neutrino flavors, namely electron neutrino ν_e, muon neutrino ν_μ, and tau neutrino ν_τ. The theory of neutrino oscillation was first proposed by Pontecorvo [1]. It implies that neutrinos have a non-zero mass, which was initially believed to be zero. The discovery of neutrino oscillation was made possible by several experiments, including the Super-Kamiokande experiment [2, 3], the Sudbury Neutrino Observatory (SNO) experiment [4], and the MINOS experiment [5, 6]. These experiments observed the disappearance and appearance of different types of neutrinos as they traveled through the Earth's atmosphere or through matter. The discovery of neutrino oscillation has been recognized with several awards, including the 2015 Nobel Prize in Physics, which was awarded to Takaaki Kajita and Arthur B. McDonald for their contributions to the Super-Kamiokande and SNO experiments.

Neutrino oscillation can be described mathematically using the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [7]. This matrix relates the neutrino flavor states to the mass eigenstates and contains four parameters: three mixing angles and one phase. The mixing angles determine the probability of a neutrino changing from one flavor to another, while the phase affects the relative probabilities of oscillation between different flavors. Neutrino oscillation has important implications for astrophysics, particle physics, and cosmology. It can help us understand the properties of neutrinos and their role in the universe, such as their contribution to dark matter [8]. It is is a fascinating phenomenon that has opened up new avenues of research in particle physics and astrophysics. The discovery of neutrino oscillation has challenged our understanding of neutrino properties and has provided important insights into the nature of the universe.

The mass hierarchy of neutrinos refers to the relative ordering of the three different masses of neutrinos. There are three types of neutrinos: electron neutrinos, muon neutrinos, and tau neutrinos. Each type of neutrino has a corresponding antineutrino. According to the current understanding of neutrino physics, neutrinos have non-zero masses, but these masses are very small

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compared to the masses of other fundamental particles. The neutrino mass hierarchy can be either normal or inverted. In the normal hierarchy, the masses of the three neutrinos are arranged as $m_1 < m_2 < m_3$, where m_1, m_2, and m_3 are the masses of the lightest, second lightest, and heaviest neutrinos, respectively. In the inverted hierarchy, the masses are arranged as $m_3 < m_1 < m_2$. The determination of the neutrino mass hierarchy is an important question in neutrino physics, and it has implications for many astrophysical and cosmological phenomena. The current experimental evidence for the neutrino mass hierarchy comes from the observation of neutrino oscillations. The phenomenon of neutrino oscillation is based on the fact that neutrinos can change their flavor as they propagate through space, due to the quantum mechanical mixing between the three different types of neutrinos. The oscillation probabilities depend on the differences between the squared masses of the three neutrino types, as well as on the mixing angles between them. The neutrino oscillation experiments, such as the Super-Kamiokande and Daya Bay experiments, have provided important information about the neutrino mass differences and the mixing angles. The data from these experiments suggest that the neutrino mass hierarchy is most likely normal [9]. However, future experiments, such as the Deep Underground Neutrino Experiment (DUNE), will provide more precise measurements of the neutrino mass differences and the mixing angles, which will allow for a definitive determination of the mass hierarchy.

Neutrino oscillation is a quantum phenomenon that occurs in flat spacetime, but it is also expected to occur in curved spacetime. In fact, the gravitational field of massive objects, such as stars and black holes, can affect the propagation of neutrinos and modify their oscillation behavior. The study of neutrino oscillation in curved spacetime is an active area of research, and it has important implications for astrophysics and cosmology. The investigation of neutrino lensing caused by gravitational sources reveals a fascinating phenomenon: a novel contribution that links the oscillation probabilities to the individual masses of neutrinos. This relationship is explicitly demonstrated by analyzing the effects of weak lensing induced by a Schwarzschild mass. The study delves into the implications of gravitationally modified neutrino oscillations within the context of realistic two and three flavor scenarios in [10]. For example, the gravitational field of a supernova can affect the propagation of neutrinos emitted during the explosion, leading to potentially observable effects on the neutrino signal [11]. The propagation of neutrinos in a strong gravitational field regime including electromagnetic interactions is studied within the WKB approximation [12, 13]. Similarly, neutrino oscillations in the Schwarzschild spacetime including spin precession in the presence of a magnetic field [14, 15], particular the radial propagation and the nonradial propagation of neutrino in the Schwarzschild spacetime are investigated [16]. The effect of expansion of the Universe and the torsion in the neutrino oscillation has been studied [17].

Neutrino oscillations have been extensively studied in the context of a rotating spacetime under the weak gravity limit, specifically for neutrino trajectories confined to the equatorial plane. By utilizing the asymptotic form of the Kerr metric, it has been demonstrated that the gravitational source's rotation introduces noteworthy modifications to the neutrino phase. Notice, when neutrinos are generated in close proximity to a black hole with non-zero angular momentum and subsequently detected on the same side, i.e., the non-lensed neutrino, the oscillation probabilities exhibit significant deviations from the corresponding outcomes observed in the Schwarzschild spacetime [18]. In Ref.[19], it is studied the impact of gravitational lensing on neutrino oscillations within the γ-spacetime framework. By employing a quantum-mechanical approach to relativistic neutrinos, the phase of neutrino oscillations in this particular spacetime, accounting for both radial and non-radial propagation has been explored. Similarly, the presence of massive objects in the universe can affect the neutrino oscillation probabilities, which can impact the predictions for the cosmic neutrino background [20].

The theoretical framework for studying neutrino oscillation in curved spacetime is based on the quantum field theory of neutrinos coupled to gravity. The oscillation probabilities depend on the neutrino energy, the neutrino mass-squared differences, and the curvature of spacetime. The curvature of spacetime is described by the metric tensor, which is a function of the gravitational field and the distribution of matter and energy. Several studies have investigated the impact of curved spacetime on neutrino oscillation probabilities, including the effects of the gravitational redshift and the curvature-induced potential [13, 14, 16, 18]. These studies have shown that the oscillation probabilities can be modified by the gravitational field, leading to potentially observable effects. The study of neutrino oscillation in curved spacetime is an active area of research that has important implications for astrophysics and cosmology [17, 21]. Theoretical models have been developed to describe the quantum field theory of neutrinos coupled to gravity, and several studies have investigated the impact of curved spacetime on neutrino oscillation probabilities See. e.g. [21, 22].

The article is structured as follows: In Sect. II, we give a very brief review of neutrino oscillation in flat and curved spacetimes. In Sect. III, we show calculation of the canonical phase for neutrino oscillations using separable equations in the curved spacetime. In Sect. IV, we show calculation of the canonical phase for neutrino oscillations in the spacetime where non-separable equation of motion. Finally, in Sect. V we summarize and comment on our results. Throughout the paper, we employ $(-,+,+,+)$ signature for the line element and adopt geometrical units setting $G = c = \hbar = 1$.

II. NEUTRINO OSCILLATION

Neutrino are produced and detected in different flavor eigenstates denoted by $|\nu_\alpha\rangle$, and the flavour eigenstates are linear superposition of mass eigenstates denoted by $|\nu_i\rangle$. So a flavor eigenstates can be written in the terms of mass eigenstates as

[16, 23]

$$|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}|\nu_i\rangle \; , \tag{1}$$

where U is neutrino flavor mixing unitary matrix, $\alpha = \{e, \mu, \tau\}$ and $i = \{1, 2, 3\}$. As, in a general curved spacetime, a neutrino wave-function can have contribution from multiple geodesics which connect the neutrino source and the detector, therefore, a neutrino flavor state propagation from source S to detector D, located at r_S and r_D respectively, considering all the paths, is given as [10, 13]

$$|\nu_\alpha(t_D, r_D)\rangle = \sum_i e^{-i\Phi_i(r_D, r_S)} U^*_{\alpha i}|\nu_i(t_S, r_S)\rangle \; , \tag{2}$$

where $\Phi_i(r_D, r_S)$ is the covariant phase of neutrino. The neutrino flavour transition probability from initial produced α flavor at the detection points is obtained as [13, 14]

$$\mathcal{P}_{\alpha\beta} = |\langle\nu_\beta|\nu_\alpha(t_B, x_B)\rangle|^2 = \sum_{i,j} U_{\beta i}U^*_{\beta j}U_{\alpha j}U^*_{\alpha i}e^{-i\Delta\Phi_{ij}} \; . \tag{3}$$

Further, for the two flavor case, the neutrino flavor transition probability from the initially produced flavor e and to the flavour μ is given as [15, 19]

$$\mathcal{P}_{\alpha\beta} = \sin^2 2\Theta \sin^2\left(\frac{\Delta\Phi_{ij}}{2}\right) \; , \tag{4}$$

where Θ mixing angle and $\Delta\Phi_{ij} = \Phi_i - \Phi_j$ is difference of canonical phases. Finally, the generalized covariant phase is defined as [18]

$$\Phi_i = \int_{r_D}^{r_S} p^{(i)}_\mu dx^\mu = -m_i \int_{r_D}^{r_S} ds \; , \tag{5}$$

where $p^{(i)}_\mu$ is the canonical conjugate momentum to the coordinate x^μ for the i^{th} neutrino mass defined as $p^{(i)}_\mu = m_i g_{\mu\nu}\dot{x}^\nu$. Here $\dot{x}^\mu$ is four-velocity of particle normalized as $\dot{x}_\mu\dot{x}^\mu = -1$, it means that $p_\mu p^\mu = -m^2$. In a flat spacetime the phase can be derived as [13]

$$\Phi_i = E_i(t_S - t_D) - \mathbf{p}_i \cdot (\mathbf{x}_S - \mathbf{x}_D) \simeq (E_i - |\mathbf{p}_i|)|\mathbf{x}_S - \mathbf{x}_D| \; , \tag{6}$$

and in the case of relativistic neutrino with following condition $m_i << E_i$, the difference of the energy and momentum is estimated as $E_i - |\mathbf{p}_i| = E_i - \sqrt{E_i^2 - m_i^2} \simeq m_i^2/(2E_i)$. Then difference of the covariant phase in a flat spacetime reads

$$\Delta\Phi_{ij} = \frac{\Delta m^2_{ij}}{2E_0}|\mathbf{x}_S - \mathbf{x}_D| \; , \qquad \Delta m^2_{ij} = m_i^2 - m_j^2 \; . \tag{7}$$

where E_0 is the average energy of the relativistic neutrinos produced at the source.

III. SEPARABLE EQUATION OF MOTION

The Hamilton-Jacobi equation for massive particle can be written as

$$g^{\alpha\beta}\frac{\partial S}{\partial x^\alpha}\frac{\partial S}{\partial x^\beta} = -m^2 \; , \tag{8}$$

where m is a mass of particle. Using the definition four-momentum $p_\alpha = \partial S/\partial x^\alpha$ equation (8) can be rewritten as

$$p_\alpha p^\alpha = -m^2 \; , \tag{9}$$

A. In Schwarzschild spacetime

Here, we will discuss derivation of the canonical phase in the Schwarzschild spacetime which is described external black hole solution of mass M and given by metric: $ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega$, where $f(r) = 1 - 2M/r$. Due to symmetry constants of motion, namely, energy and angular momentum can be found as $p_t = \text{const}$ and $p_\phi = \text{const}$. Finally, in the background of the Schwarzschild spacetime, equation of motion (9) can be separated as

$$p_r = \frac{1}{f(r)}\sqrt{E^2 - f(r)\left(m^2 + \frac{K}{r^2}\right)}\,, \tag{10}$$

$$p_\theta = \sqrt{K - \frac{L^2}{\sin^2\theta}}\,, \tag{11}$$

$$p_\phi = L\,, \tag{12}$$

$$p_t = -E\,, \tag{13}$$

where E, L and K are, respectively, constants of motion regarded to the energy, angular momentum and Carter's separation constant. Using the following definition of the four-momentum $p^\alpha = m\dot{x}^\alpha$, one can get

$$m\dot{r} = \sqrt{E^2 - f(r)\left(m^2 + \frac{K}{r^2}\right)}\,, \tag{14}$$

$$m\dot{\theta} = \frac{1}{r^2}\sqrt{K - \frac{L^2}{\sin^2\theta}}\,, \tag{15}$$

$$m\dot{\phi} = \frac{L}{r^2\sin^2\theta}\,, \tag{16}$$

$$m\dot{t} = \frac{E}{f(r)}\,. \tag{17}$$

Using the equations of motion (10)-(17), the canonical phase from point r_D to point r_S can be determined as

$$\Phi_i = \int_{r_D}^{r_S} \frac{p_t^{(i)}\dot{t} + p_r^{(i)}\dot{r} + p_\theta^{(i)}\dot{\theta} + p_\phi^{(i)}\dot{\phi}}{\dot{r}}dr \tag{18}$$

$$= -\int_{r_D}^{r_S} \frac{m_i^2 dr}{\sqrt{E_i^2 - f(r)\left(m_i^2 + \frac{K_i}{r^2}\right)}}\,. \tag{19}$$

In the radial oscillation i.e. $\theta = \pi/2$, the angular momentum of neutrino is zero $K_i = L_i^2 = 0$ and the canonical phase will be independent of angle θ which can be written as

$$\begin{aligned}\Phi_i &= -\frac{m_i^2}{E_i}\int_{r_D}^{r_S}\frac{dr}{\sqrt{1-\left(1-\frac{2M}{r}\right)\frac{m_i^2}{E_i^2}}}\\ &= \frac{m_i^2}{(1-\frac{m_i^2}{E_i^2})E_0}\left[r_D\sqrt{1-\left(1-\frac{2M}{r_D}\right)\frac{m_i^2}{E_0^2}} - r_S\sqrt{1-\left(1-\frac{2M}{r_S}\right)\frac{m_i^2}{E_0^2}}\right]\\ &- \frac{2m_i^4 M}{(1-\frac{m_i^2}{E_i^2})^{3/2}E_0^3}\tanh^{-1}\left[\frac{\sqrt{1-\frac{m_i^2}{E_i^2}}\left(\sqrt{1-\left(1-\frac{2M}{r_D}\right)\frac{m_i^2}{E_i^2}} - \sqrt{1-\left(1-\frac{2M}{r_S}\right)\frac{m_i^2}{E_i^2}}\right)}{1-\frac{m_i^2}{E_i^2} - \sqrt{1-\left(1-\frac{2M}{r_D}\right)\frac{m_i^2}{E_i^2}}\sqrt{1-\left(1-\frac{2M}{r_S}\right)\frac{m_i^2}{E_i^2}}}\right]\,,\end{aligned} \tag{20}$$

While in the weak field approximation, the difference of canonical phase can be determined as

$$\Delta\Phi_{ij} = \frac{\Delta m_{ij}^2}{E_0}(r_S - r_D)\left[1 + \frac{m_i^2 + m_j^2}{2E_0^2}\left(1 - \frac{R_s}{r_S - r_D}\log\frac{r_S}{r_D}\right)\right]\,, \tag{21}$$

B. In Kerr spacetimes

Now we consider the canonical phase in the axially-symmetric spacetime. To take account the Kerr spacetime which describes by the following line element:

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - adt\right]^2 \ , \tag{22}$$

where $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$, M is the mass and a is spin of black hole. In the Kerr spacetime the Hamiltonian-Jacobi equation of motion (9) takes a form:

$$\Delta p_r^2 + p_\theta^2 - \frac{\left[(r^2+a^2)E - aL\right]^2}{\Delta} + \left(\frac{L}{\sin\theta} - aE\sin\theta\right)^2 + m^2r^2 + m^2a^2\cos^2\theta = 0 \ , \tag{23}$$

which can be separated as

$$p_r = \frac{\sqrt{[(r^2+a^2)E-aL]^2 - \Delta(K+m^2r^2)}}{\Delta}, \tag{24}$$

$$p_\theta = \sqrt{K - \left(\frac{L}{\sin\theta} - aE\sin\theta\right)^2 - m^2a^2\cos^2\theta} \ , \tag{25}$$

where K is the Carter constant of motion and other two constants, the energy E and angular momentum L are related to momentum as $p_t = -E$ and $p_\phi = L$. Using the definition of four-momentum of massive test particle, $p_\mu = m\dot{x}_\mu$, equations of motion can be derived as

$$m\dot{r} = \frac{\sqrt{[(r^2+a^2)E-aL]^2 - \Delta(K+m^2r^2)}}{\Sigma} \ , \tag{26}$$

$$m\dot{\theta} = \frac{1}{\Sigma}\sqrt{K - \left(\frac{L}{\sin\theta} - aE\sin\theta\right)^2 - m^2a^2\cos^2\theta} \ , \tag{27}$$

$$m\dot{\phi} = \frac{a[(r^2+a^2)E-aL]}{\Delta} - \frac{L - aE\sin^2\theta}{\sin^2\theta} \ , \tag{28}$$

$$m\dot{t} = \frac{r^2+a^2}{\Delta}[(r^2+a^2)E - aL] - a(L - aE\sin^2\theta) \ . \tag{29}$$

Hereafter substituting equations of motion (24)-(29) into the expression (5), the canonical phase for neutrino in Kerr spacetime takes a form:

$$\Phi_i = \int_{r_D}^{r_S} \frac{-m_i^2\Sigma dr}{\sqrt{[(r^2+a^2)E_i - aL_i]^2 - \Delta(K_i + m_i^2r^2)}} \ . \tag{30}$$

Notice that the canonical phase for neutrino in the Kerr-like spacetime is also given by equation (30). In the equatorial plane of the Kerr black hole, one can get

$$\Phi_i = \int_{r_D}^{r_S} \frac{-m_i^2 dr}{\sqrt{\left[E_i - \frac{a(L_i - aE_i)}{r^2}\right]^2 - \frac{\Delta}{r^2}\left[m_i^2 + \frac{(L_i - aE_i)^2}{r^2}\right]}} \ , \tag{31}$$

while in particular case, $L_i = aE_i$, the canonical phase for neutrino yields

$$\Phi_i = \frac{m_i^2\left(\sqrt{r_S^2E_i^2 - m_i^2\Delta_S} - \sqrt{r_D^2E_i^2 - m_i^2\Delta_D}\right)}{E_i^2 - m_i^2}$$
$$- \frac{m_i^4 M}{(E_i^2 - m_i^2)^{3/2}}\left[\tanh^{-1}\left(\frac{E_i^2 r_S - m_i^2(r_S - M)}{\sqrt{E_i^2 - m_i^2}\sqrt{r_S^2e^2 - m_i^2\Delta_S}}\right) - \tanh^{-1}\left(\frac{E_i^2 r_D - m_i^2(r_D - M)}{\sqrt{E_i^2 - m_i^2}\sqrt{r_D^2e^2 - m_i^2\Delta_D}}\right)\right] \ . \tag{32}$$

C. In Johannsen spacetime

Now we focus on to test the Johannsen spacetime by determining the fundamental frequencies of test particle. It is the general form of the Kerr spacetime described the following metric [24]:

$$ds^2 = -\frac{\tilde{\Sigma}\Delta}{B^2}\left[A_3(\theta)dt - aA_4(\theta)\sin^2\theta d\phi\right]^2 + \frac{\tilde{\Sigma}}{\Delta A_5(r)}dr^2 + \frac{\tilde{\Sigma}}{A_6(\theta)}d\theta^2$$
$$+ \frac{\tilde{\Sigma}\sin^2\theta}{B^2}\left[A_1(r)(r^2+a^2)d\phi - aA_2(r)dt\right]^2 , \tag{33}$$

where $B = A_1(r)A_3(\theta)(r^2+a^2) - A_2(r)A_4(\theta)a^2\sin^2\theta$ and $\tilde{\Sigma} = \Sigma + f(r) + g(\theta)$. Notice that Δ function is the same as in Kerr spacetime. In general it is impossible to find the stationary points of the function $V(r,\theta)$ in the background geometry (33). However, for specific choice of the profile functions

$$A_1(r) = 1 + \sum_{n=3}^{\infty}\alpha_{1n}\left(\frac{M}{r}\right)^n , \qquad A_2(r) = 1 + \sum_{n=2}^{\infty}\alpha_{3n}\left(\frac{M}{r}\right)^n , \tag{34}$$

$$A_5(r) = 1 + \sum_{n=2}^{\infty}\alpha_{5n}\left(\frac{M}{r}\right)^n , \qquad f(r) = r^2\sum_{n=3}^{\infty}\epsilon_n\left(\frac{M}{r}\right)^n , \tag{35}$$

$$A_3(\theta) = A_4(\theta) = A_6(\theta) = 1 , \qquad g(\theta) = 0 , \tag{36}$$

one can find that the stationary points of the function $V(r,\theta)$ is located at the equatorial plane $\theta_0 = \pi/2$. One has to emphasise that the Johannsen spacetime is characterized by series of parameters, α_{1n}, α_{3n}, α_{5n}, and ϵ_n along the mass and spin of the black hole and it is applicable in the phenomenological calculations in black hole astrophysics.

In Johannsen spacetime, the Hamiltonian-Jacobi equation of motion (33) can be written as

$$\Delta A_5(r)p_r^2 + A_6(\theta)p_\theta^2 - \frac{\left[(r^2+a^2)A_1(r)E - aA_2(r)L\right]^2}{\Delta} + m^2\left(r^2 + f(r)\right) +$$
$$+\left(\frac{A_3(\theta)L}{\sin\theta} - aA_4(\theta)E\sin\theta\right)^2 + m^2\left(a^2\cos^2\theta + g(\theta)\right) = 0 , \tag{37}$$

which can be separated as

$$p_r = \frac{1}{\Delta}\sqrt{\frac{1}{A_5(r)}\left[[(r^2+a^2)A_1(r)E - aA_2(r)L]^2 - \Delta\left(K + m^2r^2 + m^2f(r)\right)\right]}, \tag{38}$$

$$p_\theta = \sqrt{\frac{1}{A_6(\theta)}\left[K - \left(\frac{A_3(\theta)L}{\sin\theta} - aA_4(\theta)E\sin\theta\right)^2 - m^2a^2\cos^2\theta - m^2g(\theta)\right]} , \tag{39}$$

where K is the Carter constant of motion and other two constants, the energy E and angular momentum L are related to momentum as $p_t = -E$ and $p_\phi = L$. Using the definition of four-momentum of massive test particle, $p_\mu = m\dot{x}_\mu$, equations of motion can be derived as

$$m\dot{r} = \frac{\sqrt{A_5(r)}}{\tilde{\Sigma}}\sqrt{\left[(r^2+a^2)A_1(r)E - aA_2(r)L\right]^2 - \Delta\left(K + m^2r^2 + m^2f(r)\right)} , \tag{40}$$

$$m\dot{\theta} = \frac{\sqrt{A_6(\theta)}}{\tilde{\Sigma}}\sqrt{K - \left(\frac{A_3(\theta)L}{\sin\theta} - aA_4(\theta)E\sin\theta\right)^2 - m^2a^2\cos^2\theta - m^2g(\theta)} , \tag{41}$$

$$m\dot{\phi} = \frac{aA_2(r)}{\Delta\tilde{\Sigma}}\left[(r^2+a^2)A_1(r)E - aA_2(r)L\right] - \frac{A_3(\theta)}{\tilde{\Sigma}}\left(aA_4(\theta)E - \frac{A_3(\theta)L}{\sin^2\theta}\right) , \tag{42}$$

$$m\dot{t} = \frac{(r^2+a^2)A_1(r)}{\Delta\tilde{\Sigma}}\left[(r^2+a^2)A_1(r)E - aA_2(r)L\right] - \frac{aA_4(\theta)}{\tilde{\Sigma}}\left(A_3(\theta)L - aA_4(\theta)E\sin^2\theta\right) . \tag{43}$$

Using equations of motion (38)-(43), the canonical phase for neutrino in Johannsen spacetime can be expressed as

$$\Phi_i = \int_{r_D}^{r_S} \frac{-m_i^2 \tilde{\Sigma} dr}{\sqrt{A_5(r)\left[\left[(r^2+a^2)A_1(r)E_i - aA_2(r)L_i\right]^2 - \Delta\left(K_i + m_i^2 r^2 + m_i^2 f(r)\right)\right]}} , \tag{44}$$

while in the equatorial plane ($\theta = \pi/2$), one obtains

$$\Phi_i = \int_{r_D}^{r_S} \frac{-m_i^2\left(1+\frac{f(r)}{r^2}\right)dr}{\sqrt{A_5(r)\left[\left(A_1(r)E_i - \frac{a[A_2(r)L_i - aA_1(r)E_i]}{r^2}\right)^2 - \frac{\Delta}{r^2}\left[m_i^2\left(1+\frac{f(r)}{r^2}\right) + \frac{(L_i - aE_i)^2}{r^2}\right]\right]}} . \tag{45}$$

In general relativity and quantum field theory, investigating of neutrino oscillation in curved spacetimes important and interesting. Understanding the canonical phase for neutrinos necessitates an examination of their interaction with gravitational fields, which varies according to the specific geometry of the spacetime. While it is often feasible to analytically derive the canonical phase for neutrinos in separable spacetimes like Schwarzschild and Kerr, where metric components can be expressed as independent functions of coordinates, the task becomes more formidable in more general spacetimes lacking separability. So far, we have successfully demonstrated the straightforward derivation of the canonical phase for neutrinos in Schwarzschild and Kerr spacetimes. However, in the case of generalized Kerr spacetime, one must integrate it considering the particular form of lapse functions such as Δ, A_i, and others. In the following section, we will illustrate the derivation of the canonical phase for neutrinos in spacetimes that lack separability. To derive the canonical phase for neutrinos in such non-separable spacetimes, it is typically necessary to solve the pertinent equations of motion and account for the influence of spacetime geometry on neutrino propagation. This process may involve numerical integration of the relevant equations or the utilization of approximation methods. In the following section, we will explore the canonical phase for neutrinos in spacetimes where variables cannot be separated.

IV. INSEPARABLE EQUATION OF MOTION

It is widely recognized that the separation of variables in the equation of motion is not universally applicable to all spacetimes. Consequently, deriving the canonical phase for a neutrino in such spacetimes poses a significant challenge. However, in these scenarios, an alternative approach can be employed by utilizing equation (5) and expressing it in the following form:

$$\Phi_i = -m_i \int_{r_D}^{r_S} ds = -m_i \int_{r_D}^{r_S} \frac{1}{\dot{r}} dr , \tag{46}$$

that means it is enough to derive equation for radial motion for particular angle θ_0. Here we focus on certain spacetimes in which separation of variable is impossible.

A. The γ-metric

Now, we will consider the canonical phase in γ-metric. In Erez-Rosen coordinates, the spacetime is given by the line element as

$$ds^2 = -f^{\gamma} dt^2 + f^{\gamma^2-\gamma} g^{1-\gamma^2}\left(\frac{dr^2}{f} + r^2 d\theta^2\right) + f^{1-\gamma} r^2 sin^2\theta d\phi^2 , \tag{47}$$

with

$$f = 1 - \frac{2M}{\gamma r} , \qquad g = 1 - \frac{2M}{\gamma r} + \frac{M^2 sin^2\theta}{\gamma^2 r^2} . \tag{48}$$

where M is the mass of the central object and γ is the deformation of the spacetime. To find the canonical phase with using equation (46), first we should find the expression of $\dot{r}$. Using the normalization of the four-velocity, $\dot{r}$ can be found as

$$\dot{r} = \sqrt{\frac{1}{g_{rr}}\left(-m_i^2 - \frac{E_i^2}{g_{tt}} - \frac{L_i^2}{g_{\phi\phi}}\right)} , \tag{49}$$

Substituting Eq. (49) into Eq. (46), the canonical phase for neutrino can be written as

$$\Phi_i = -\int_{r_D}^{r_S} \frac{m_i^2\sqrt{-g_{tt}g_{rr}}dr}{\sqrt{g_{tt}m_i^2 + \left(E_i^2 + \frac{g_{tt}}{g_{\phi\phi}}L_i^2\right)}} \,. \tag{50}$$

In the equatorial plane ($\theta = \pi/2$)

$$\Phi_i = -\int_{r_D}^{r_S} \frac{m_i^2}{\sqrt{E_i^2 - \frac{f^{2\gamma-1}}{r^2}L_i^2 - f^\gamma m_i^2}}dr \,. \tag{51}$$

and

$$\begin{aligned}\Phi_i =& \frac{m_i^2\left[\sqrt{\left(E_i^2-m_i^2\right)r_D^2-b^2}-\sqrt{\left(E_i^2-m_i^2\right)r_S^2-b^2}\right]}{m_i^2-E_i^2}\\ &+\frac{(\gamma-1)m_i^2Mr_D}{\sqrt{\left(E_i^2-m_i^2\right)r_D^2-b^2}}-\frac{(\gamma-1)m_i^2Mr_S}{\sqrt{\left(E_i^2-m_i^2\right)r_S^2-b^2}}-\\ &-\frac{\gamma m_i^4M}{\left(m_i^2-E_i^2\right)^{3/2}}\ln\left(\frac{\sqrt{b^2+\left(m_i^2-E_i^2\right)r_D^2}-\sqrt{m_i^2-E_i^2}\,r_D}{\sqrt{b^2+\left(m_i^2-E_i^2\right)r_S^2}-\sqrt{m_i^2-E_i^2}\,r_S}\right)\,,\end{aligned} \tag{52}$$

where $b = L_i/E_i$ is impact parameter.

B. The Hartle-Thorne spacetime

Now we concentrate the canonical phase for neutrino in the the Hartle-Thorne spacetime which is given by the metric [25]:

$$\begin{aligned}ds^2 =& -\left(1-\frac{2M}{r}\right)\left[1+2k_1P_2(\cos\theta)-\frac{2J^2}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\left(2\cos^2\theta-1\right)\right]dt^2\\ &+\left(1-\frac{2M}{r}\right)^{-1}\left[1-2\left(k_1-\frac{6J^2}{r^2}\right)P_2(\cos\theta)-\frac{2J^2}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\right]dr^2\\ &+r^2\left[1-2k_2P_2(\cos\theta)\right]\left(d\theta^2+sin^2\theta d\phi^2\right)-\frac{4J}{r}\sin^2\theta dt d\phi\,,\end{aligned} \tag{53}$$

where

$$k_1 = \frac{J^2}{Mr^3}\left(1+\frac{M}{r}\right)+\frac{5}{8}\left(\frac{Q}{M^3}-\frac{J^2}{M^4}\right)Q_2^2\left(\frac{r}{M}-1\right)\,, \tag{54}$$

$$k_2 = k_1+\frac{J^2}{r^4}+\frac{5}{4}\left(\frac{Q}{M^2r}-\frac{J^2}{M^3r}\right)\left(1-\frac{2M}{r}\right)^{-1/2}Q_2^1\left(\frac{r}{M}-1\right)\,, \tag{55}$$

and $Q_2^1(x)$ and $Q_2^2(x)$ are the associated Legendre functions of the second kind. Here M is the mass, J is the angular momentum and Q is the quadruple moment of the neutron star. Using the normalization of the four-velocity, we can the expression of $\dot{r}$ as

$$\dot{r} = \sqrt{\frac{1}{g_{rr}}\left(-m_i^2+\frac{E_i^2}{g_{tt}}-\frac{L_i^2}{g_{\phi\phi}}+2g_{t\phi}E_iL_i\right)}\,, \tag{56}$$

Plugging Eq (56) into Eq (46) give us the expression of canonical phase for neutrino in Hartle-Thorne spacetime

$$\Phi_i = -\int_{r_D}^{r_S}\frac{m_i^2\sqrt{g_{tt}g_{rr}}dr}{\sqrt{E_i^2+2g_{t\phi}g_{tt}L_iE_i-\frac{g_{tt}}{g_{\phi\phi}}L_i^2-g_{tt}m_i^2}}\,. \tag{57}$$

In the equatorial plane ($\theta = \pi/2$), the phase expression can be simplified as

$$\Phi_i = -m_i^2\int_{r_D}^{r_S}\frac{\sqrt{1-\frac{2M}{r}-\frac{4J^2}{r^4}}}{\sqrt{E_i^2\left(1-\frac{2M}{r}\right)^2-\left(m_i^2+\frac{rL_i^2}{4J}\right)\left[\frac{2J^2}{r^2}\left(1-\frac{2M}{r}\right)^2+\left(1-\frac{2M}{r}\right)\right]+\frac{8E_iL_iJ}{r}\left(1-\frac{2M}{r}\right)}}dr\,. \tag{58}$$

V. CONCLUSIONS AND FUTURE OUTLOOK

We have undertaken a thorough exploration of the classical approach, aiming to unveil the complex domain of vacuum neutrino oscillations set against the intriguing background of curved spacetime. Our framework explores transition probabilities, shedding light on their intricate dynamics even in the face of oscillations in mass representation. Through a diligent charting of our course ahead, we offer a clear and coherent roadmap for computing canonical phase transitions and forecasting neutrino oscillations across the boundless stretches of arbitrary curved spacetime.

We have determined canonical phase of neutrino in several spacetimes in general relativity, for instances, the Schwarzschild and Kerr spacetime which leads separable equations of motion for neutrino behavior. We have shown that this approach are satisfied in arbitrary static and rotating spacetime. Further, we have also treated this approach to Kerr-like spacetime known as Johannsen metric.

Yet, not all cosmic stages unveil their secrets so readily. Our voyage takes us to the captivating domain of non-separable spacetimes, where the γ-metric and Hartle-Thorne metric challenge our every step. Even here, we remain undaunted, deriving canonical phase formulas for the intrepid neutrino, a testament to the power of our approach. These formulas shimmer with promise, revealing a tantalizing interplay between neutrino motion and the enigmatic gravitational background an interplay that sculpts the very essence of neutrino mixing.

In summation, our work stands as a beacon of theoretical rigor, illuminating the intricate tango of neutrino oscillations within the captivating theater of curved spacetime. As the cosmic saga unfolds, it is our hope that the seeds sown in our theoretical garden will bear fruit, enriching our understanding of the cosmos and the neutrinos physics.

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