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RESEARCH

Entropy as a Clock: Foundations and Parametrizations of Emergent Time

José Weberszpil[1] · Oscar Sotolongo-Costa[2,3]

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Abstract
In canonical quantum gravity, time does not appear as a fundamental coordinate, posing the longstanding problem of how dynamical evolution arises in a fundamentally timeless universe. In this work, we propose that entropy—interpreted as a coarse-grained, monotonically increasing measure of system complexity—can serve as an emergent internal clock. We unify three complementary mechanisms underpinning this idea: (i) the monotonic growth of entanglement entropy under unitary dynamics, (ii) thermal modular flow associated with Kubo-Martin-Schwinger states, and (iii) relational time from the Page–Wootters framework. These mechanisms jointly define a physical arrow and parametrization of time grounded in the informational structure of the quantum state. From this foundation, we derive explicit entropic time laws of the form $\tau(\Delta S) = (\Delta S/\lambda)^{1/\gamma}$, showing how the parameters $(q, N_0, \lambda, \gamma)$ emerge from microscopic statistical properties such as non-extensive correlations, phase space growth, and entropy production rates. We apply this framework to cosmological epochs, identifying entropy increase across inflation, radiation and matter domination as a natural proxy for internal time progression. This entropic approach provides a unified view linking quantum foundations, thermodynamic irreversibility, and cosmological evolution. We also discuss interpretational subtleties, clarifying how entropic time differs from coordinate time and under what conditions it defines a meaningful temporal structure. We emphasize that the entropic time, τ, provides an arrow and parametrization of change in relational regimes (Wheeler–DeWitt,Page–Wootters, KMS/modular frameworks), but it is not proposed as a universal bijective substitute for *coordinate time, t.*

Keywords Deformed derivatives · Deformed Pesin relations · Entropic time · Thermodynamic time paradigm · Cosmological time

✉ José Weberszpil
josewebe@gmail.com

Oscar Sotolongo-Costa
osotolongo@gmail.com

1 Universidade Federal Rural do Rio de Janeiro, UFRRJ-DEFIS/ICE, BR-465, Km 7, 23897-000 Seropédica-Rio de Janeiro, Brazil

2 Centro de Investigación en Ciencias, Universidad Autónoma del Estado de Morelos, Av. Universidad 1001, 62209 Cuernavaca, Morelos, México

3 Fac. Física, Universidad de La Habana, 10400 Habana, Cuba

1 Introduction

The concept of time is traditionally treated as an external, linear dimension in physical theories. However, developments in quantum gravity, statistical mechanics, and cosmology have prompted alternative formulations in which time is emergent from more fundamental entities. This paper investigates such a formulation, expressing cosmic *entropic time* τ as a function of entropy change ΔS, incorporating concepts from the thermal time hypothesis [1], geometric thermodynamics [2], and non-extensive statistical mechanics [3].

In our framework, cosmological entropic time τ is modeled as an emergent quantity, functionally dependent on entropy change: $\tau = \tau(\Delta S)$. This approach treats time as emerging from thermodynamic evolution, rather than as an externally imposed background variable. While this model is particularly suited for describing cosmological epochs characterized by global entropy production (e.g., inflation, recombination, and the far future). Its extension to fully static spacetimes or to quantum regimes and equilibrium regions—such as black hole horizons, or quantum gravity regimes without classical entropy flow (e.g., solutions of the Wheeler– DeWitt equation [4, 5]) would require a further more generalized treatment of informational entropy.

Analyzing the cosmological possibility of time-dependence of entropy involves exploring how entropy changes with time in the context of the universe's evolution. This touches thermodynamics, statistical mechanics, and cosmology, particularly in the study of the arrow of time and the second law of thermodynamics in a relativistic or quantum cosmological framework.

If $\Delta S = 0$, then there is no entropy change and consequently no time τ progression. If entropy (ΔS) increases smoothly, *entropic time (τ)* increases smoothly as well. The form is compatible with non-extensive thermodynamics, where entropy is not additive (reflecting gravitational systems or the early universe).

In standard cosmology, entropy is not conserved but rather increases with time, consistent with the second law of thermodynamics. The universe evolves from low-entropy initial conditions (e.g., the Big Bang) to increasingly disordered states (e.g., the far-future heat death scenario).

These considerations suggest that the flow of time is not a fundamental entity, but rather a manifestation of entropy increase or information growth, which defines its direction and progression. The direction of time is set by the monotonic growth of entropy, which provides a natural arrow of time. Cosmological time can be measured or inferred from the state of entropy.

The deformed derivative (DD) approach serves as the mathematical framework for developing our paradigm. For the sake of brevity, we refer the reader to Ref. [6] and the references therein for further information on DD and the deformed Pesin relations that underpin our discussion.

Scope and Assumptions

Scope The entropic clock proposed here applies to *timeless or relational frameworks* such as the Wheeler–DeWitt equation, the Page–Wootters mechanism, and the modular (Kubo–Martin–Schwinger, KMS) flow of algebraic quantum statistical mechanics, where no external coordinate time is defined.

Clock Property The entropic time provides a *partial order*–a monotonic arrow, not a bijection and is required only to increase monotonically on typical physical histories.

Equilibrium When entropy production vanishes ($\dot{S} = 0$), the entropic clock stalls: the thermodynamic arrow ceases, although microscopic reversible dynamics (unitary evolution, fluctuations) may persist.

2 Entropy as an Internal Clock in Timeless Physics

A. Timeless Quantum Gravity and the Problem of Time

In canonical approaches to quantum gravity, such as the Wheeler–DeWitt equation [5], time does not appear as a fundamental coordinate. This leads to the "problem of time." A possible resolution, proposed by Page and Wootters[7] and later developed by Rovelli [8, 9], is to recover time relationally via internal quantum correlations (*internal clock*), a degree of freedom within the system that can play the role of time. This relational approach, as proposed in [7], evolution is not with respect to an external time variable, but with respect to correlations between subsystems. The dynamics of the universe are governed by the timeless constraint:

$$\hat{H}\Psi[h_{ij}, \phi] = 0, \tag{1}$$

where $\Psi[h_{ij}, \phi]$ is the wavefunction of the universe defined over superspace (the space of 3-metrics h_{ij} and matter fields ϕ), and $\hat{H}$ is the Hamiltonian constraint operator. This formulation leads to the so-called "problem of time," wherein coordinate time evolution is frozen, raising the question: how can we recover dynamical evolution?

B. Entropy as a Candidate Clock Variable

In our framework, we propose that entropy, understood as a coarse-grained, monotonic function of the system's microstate complexity, can serve as such an internal clock. As the system evolves, the growth of entropy provides a natural ordering parameter.

Following the Page-Wootters construction, the probability of observing a subsystem ϕ, conditioned on an internal clock variable (in our case, entropy ΔS) is given by:

$$P(\phi \mid \Delta S) = \frac{|\langle \phi, \Delta S|\Psi\rangle|^2}{\sum_\phi |\langle \phi, \Delta S|\Psi\rangle|^2}, \tag{2}$$

where $|\Psi\rangle$ is the total state of the universe. The entropy ΔS here acts as an internal parameter labeling "when" observations occur, even though there is no fundamental background time. The dynamics of the subsystem ϕ are thus relationally tracked by the growth of entropy.

C. Temporal Ordering from Monotonic Entropy

Entropy is assumed to increase monotonically under coarse-grained evolution. This assumption implies a natural partial ordering of physical states:

$$\Delta S_1 < \Delta S_2 \quad \Rightarrow \quad \text{State}_1 \text{ precedes State}_2. \tag{3}$$

This ordering defines a causal-like structure among states based solely on entropic progression, without invoking any external time coordinate. In this sense, *time emerges* not as an independent background parameter, but as a relational attribute of evolving states, measurable via entropy growth.

Terminology Note Throughout this work, the term "evolution" is used in a structural sense—denoting progression along the entropy-induced ordering of states—rather than as implying the existence of an external, pre-established time variable.

D. Connection to Thermal Time Hypothesis

Rovelli's thermal time hypothesis further supports this approach. It proposes that, given a statistical state ρ, time evolution is governed by the modular flow [1, 9]:

$$\sigma_t(\mathcal{O}) = \rho^{-it}\mathcal{O}\rho^{it}, \tag{4}$$

where $\mathcal{O}$ is an observable and t is the emergent "thermal" time parameter. The modular generator plays the role of a physical Hamiltonian derived from the informational content of the state ρ.

Our formulation is conceptually parallel: entropy plays an analogous role to the modular Hamiltonian, driving a thermodynamic flow that defines an effective emergent time. The function $t(\Delta S)$, introduced in later sections, is not an arbitrary inversion but a phenomenological realization of this deeper principle. It captures the thermodynamic arrow of time in settings where no fundamental time exists.

Summary

In this view, entropy serves as an internal clock, grounded in the informational structure of physical states. The passage of time reflects not external parameterization, but irreversible thermodynamic evolution. The expressions derived in this paper, including the entropic time function, should thus be interpreted as effective coarse-grained reconstructions of temporal structure, consistent with modern approaches to quantum gravity and statistical mechanics.

E. Scope and Domain of Validity

Before proceeding to cosmological applications, we clarify the scope and limitations of the entropic time framework to address fundamental questions about its conceptual foundations.

1. When is Entropic Time Applicable? The entropic clock is not proposed as a universal replacement for all notions of time, but rather as a *complementary* framework particularly suited to specific physical regimes:

Primary Domain of Applicability

- **Quantum cosmology:**In canonical approaches to quantum gravity, the Wheeler–DeWitt equation $\hat{H}\Psi[h_{ij}, \phi] = 0$ contains no external time parameter. Time must be reconstructed from internal degrees of freedom, and entropy provides one such reconstruction.
- **Non-equilibrium thermodynamic systems:** During epochs of significant irreversible entropy production (cosmic inflation, reheating, structure formation, black hole formation), entropy growth provides a natural parameterization of system evolution.
- **Semiclassical regimes:** When quantum coherence is sufficiently suppressed and coarse-graining is well-defined, entropic ordering provides an effective temporal structure.

- **Systems with absent or ill-defined coordinate time:** Near black hole horizons, in emergent gravity scenarios, or in regions where spacetime itself is dynamical and not a fixed background.

Regimes Where Entropic Time is Insufficient

- **Perfect thermal equilibrium:** When $\mathrm{d}S/\mathrm{d}t = 0$, the thermodynamic arrow vanishes and entropic time ceases to provide temporal ordering. However, this does not imply all dynamics cease—microscopic reversible evolution, modular flow in the sense of Kubo–Martin–Schwinger (KMS) states, and Page–Wootters relational time between subsystems may still be well-defined.
- **Fully coherent quantum systems:** Before decoherence and coarse-graining, unitary quantum evolution is reversible and entropy (in the von Neumann sense for pure states) remains zero. Entropic time emerges only after adopting a coarse-grained description.
- **Static spacetimes without matter:** In solutions such as empty Minkowski or anti-de Sitter space, there is no entropy production and thus no entropic flow.

2. Relationship to Other Time Concepts Table 1 provides a comparative overview of different time frameworks and their domains of applicability.

3. Philosophical Status: Primitive vs. Derived A fundamental question raised by this framework concerns the conceptual hierarchy: *Is time primitive and entropy derived, or vice versa?*

Our position is that this question has no universal answer—the choice depends on the theoretical context:

- **In standard quantum mechanics:** Coordinate time t is primitive (appearing in the Schrödinger equation $i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle$), and entropy is derived from the density matrix.
- **In canonical quantum gravity:** The Hamiltonian constraint $\hat{H}\Psi = 0$ admits no external time. Here, entropy (or other internal observables) must be taken as primitive to reconstruct temporal ordering.

Table 1 Comparison of different time concepts across physical contexts

Context	Coord. Time	Thermal Time	Entropic Time	Notes
Quantum Mechanics	Yes (Fundamental)	Not applicable	Optional	Schrödinger evolution
QFT	Yes (Fundamental)	Yes (In QFT vacua)	Yes (For horizons)	Modular structure
Wheeler–DeWitt	No	Yes (Via KMS)	Yes (Via entanglement)	Time reconstructed
Thermal Equilibrium	Yes (External)	Yes (Modular flow)	No (No arrow)	No arrow
FLRW Cosmology	Yes	Yes (Implicit)	Yes (Explicit)	Multiple frames
Black Hole	Singular	Yes	Yes	Converging pictures

KMS = Kubo–Martin–Schwinger; FLRW = Friedmann–Lema tre–Robertson–Walker; QFT = Quantum Field Theory

- **In our framework:** We adopt the conditional stance that *when coordinate time is absent or inaccessible*, entropy provides a physically meaningful alternative ordering parameter.

This is analogous to choosing coordinates in General Relativity: there is no unique "correct" choice, but different coordinates are better suited to different problems. Similarly, different time concepts are suited to different physical regimes.

Crucially, we do not claim one-to-one correspondence between entropic time and coordinate time. They parameterize different aspects of evolution:

- Coordinate time t parameterizes *reversible microscopic dynamics*
- Entropic time $\tau(\Delta S)$ parameterizes *irreversible macroscopic evolution*

In non-equilibrium regimes where both are defined, they are typically aligned but not identical. In equilibrium, irreversible evolution ceases (entropic time stops flowing) while reversible dynamics continue (coordinate time persists).

In quantum gravity, where coordinate time may not exist, entropic ordering provides one possible internal structure for recovering temporal ordering—not the only one, but a physically motivated one grounded in quantum information and thermodynamics.

F. Philosophical and Ontological Basis for Considering Entropy as a Primitive Concept

In any scientific framework, certain notions are regarded as *primitive*: they serve as the foundational entities from which other quantities are defined. Primitive concepts are not reducible to simpler terms within the same theory; their meaning is established axiomatically through their relations to other primitives. Derived concepts, by contrast, depend on these primitives for their definition and operational meaning. The selection of what is treated as primitive is a methodological decision aimed at ensuring logical coherence and avoiding circular definitions [10, 11].

In most physical theories, time is postulated as a primitive variable—fundamental, intuitive, and universally applicable. However, in formulations where the coordinate time loses operational significance, such as canonical quantum gravity, covariant statistical mechanics, or modular (KMS) thermodynamics [1, 9], it becomes legitimate to search for alternative primitives capable of restoring an ordering structure. Entropy naturally provides such a candidate.

Philosophical Motivation

Several arguments support the consideration of entropy as a primitive concept in these contexts:

- **Universality across scales.**Entropy characterizes irreversibility in every known physical regime, from microscopic fluctuations to cosmological evolution. This ubiquity suggests that entropy captures a more fundamental aspect of nature than the coordinate parameter used to describe change [3, 12].
- **Origin of the temporal arrow.** Entropy quantifies the directionality of natural processes—the thermodynamic arrow of time—and thereby explains temporal asymmetry

rather than presupposing it. In this sense, it provides the structural basis from which temporal order can emerge [1, 9, 13].

- **Irreducibility of definition.** Neither the Clausius nor the Boltzmann formulation of entropy can be meaningfully expressed through simpler quantities [10, 11]. Entropy synthesizes energetic, statistical, and temporal aspects into a single invariant, resisting further reduction.
- **Cross-disciplinary unification.** Entropy serves as a unifying measure of order and complexity in physics, information theory, and biological systems. Its capacity to quantify organization and transformation across domains reinforces its foundational status [3, 12].

Ontological Arguments

From an ontological standpoint, three complementary perspectives strengthen this interpretation:

- **Physical fundamentality.** Entropy reflects intrinsic features of the universe associated with dissipation, energy exchange, and irreversibility [13, 14]. In this view, it represents an elemental property of physical reality rather than a derived construct.
- **Generative role.** Entropy acts not only as a measure of disorder but also as a creative principle driving the emergence of structure and complexity [12]. Its dual role— degrading gradients while generating new patterns—assigns it an active ontological significance.
- **Epistemic foundation.** Logically, entropy constitutes the minimal informational quantity required to construct consistent theories of change and causality. It provides the epistemic foundation from which temporality and evolution acquire measurable meaning [1, 3].

Limits and Scientific Validity

The proposal of using entropy as a primitive variable is not universal. In systems where entropy remains constant or the thermodynamic arrow is absent, the entropic notion of time becomes inoperative, while coordinate or modular time retains relevance. Nevertheless, the approach remains scientifically valid under Popper's criterion of falsifiability: it predicts empirical correlations between entropy production and observable temporal phenomena, which can be confirmed or contradicted by experiment. Empirical verification in non-equilibrium and gravitational systems would therefore provide a critical test of this framework [13, 14].

In this formulation, treating entropy as primitive and time as derived is a contextual hypothesis rather than a metaphysical claim. It becomes meaningful precisely in regimes where coordinate time loses operational definition and an internal ordering variable must be introduced. By grounding temporal succession in the statistical structure of nature, entropy provides a physically motivated bridge between irreversibility, information, and the emergence of time itself [3, 9, 14].

3 Monotonic Entropy Increase and Its Implications for the Arrow of Time in Cosmology

Suppose entropy increases monotonically (as in many cosmological models). If you know the entropy function: Using your earlier expression: Then, time is constructed from entropy

—a radical but consistent idea in non-equilibrium cosmology. One immediate Implication is that in systems where entropy is constant (perfect equilibrium), time "freezes" in this interpretation. The arrow of time: Since entropy increases, the direction of time is always forward.

The Freezing of Time in Equilibrium

We wish to clarify a potentially misleading interpretation of our statement that "in perfect equilibrium, time freezes." This phrase refers specifically to the entropic time framework we adopt, wherein the flow of time is parametrized by entropy production:

$$\frac{dS}{dt} > 0 \quad \Rightarrow \quad \text{entropic time flows.} \tag{5}$$

In a perfect thermodynamic equilibrium state, entropy ceases to increase ($dS/dt = 0$), and therefore the entropic arrow of time vanishes. Within this framework, time as reconstructed from entropy becomes undefined or static.

However, this does not imply that all physical phenomena cease in equilibrium. Systems in equilibrium still:

- Obey dynamical equations (e.g., unitary quantum evolution),
- Exhibit stationary states with well-defined observables,
- Maintain fluctuations, correlations, and spectra characteristic of equilibrium distributions (e.g., Planck radiation, Fermi-Dirac statistics).

The "freezing" of time in our context refers solely to the absence of irreversible thermodynamic change, and not to the absence of dynamical structure or observables. Microscopic time parameters may still exist in the form of reversible or cyclic dynamics, but the entropic flow that underpins temporal succession and causality no longer provides a direction.

Thus, equilibrium in this model corresponds to a fixed point of the entropic time function, not to the cessation of all physics.

So, we emphasize that equilibrium $\neq$ inert; it means no entropy gradient = no thermodynamic arrow.

Equilibrium Remark In strict equilibrium, the production of coarse-grained (or entanglement) entropy vanishes, so the entropic time τ becomes constant. This does not imply that microscopic unitary dynamics or stationary correlations disappear; only that the *thermodynamic arrow* of evolution, as measured by entropy change, ceases to advance.

A. Entropic Time in Equilibrium: Clarifications and Subtleties

A central conceptual challenge concerns the behavior of entropic time in thermodynamic equilibrium, where $\mathrm{d}S/\mathrm{d}t = 0$. This requires careful clarification to avoid misinterpretation.

1. What Does "Time Freezes" Mean?
When we state that "entropic time freezes in equilibrium," this phrase requires precise interpretation:
What we do mean is that the *thermodynamic arrow of time vanishes*—there is no preferred direction of irreversible macroscopic change; the entropic ordering relation $\prec_{\text{ent}}$ ceases to provide new ordering, as all accessible states have the same coarse-grained entropy

and that entropy-based parameterizations like $\tau(\Delta S) = (\Delta S/\lambda)^{1/\gamma}$ become undefined or constant since $\Delta S = 0$.

It is important to emphasize that this statement does not imply that all physical processes cease or that quantum dynamics stop evolving, nor that coordinate time—when defined—loses its meaning. Rather, only the *entropic* clock ceases to advance; the universe does not become "frozen" in any absolute sense.

2. Dynamics in Equilibrium

Systems in perfect thermodynamic equilibrium retain rich dynamical structure:

1. **Unitary Quantum Evolution:** For a closed quantum system described by a density matrix $\rho_{\text{eq}} = e^{-\beta H}/Z$, observables evolve via:

$$\langle \hat{O}(t) \rangle = \text{Tr}(\rho_{\text{eq}}\, e^{iHt/\hbar}\, \hat{O}\, e^{-iHt/\hbar}). \tag{6}$$

   This evolution is reversible and does not produce net entropy.
2. **Modular Flow (Thermal Time):** The Kubo–Martin–Schwinger (KMS) condition characterizes equilibrium states [15]. For a thermal state $\rho_\beta = e^{-\beta H}/Z$, the modular Hamiltonian is:

$$K = -\log \rho_\beta = \beta H + \log Z. \tag{7}$$

   This generates *modular flow*:

$$\sigma_{\tau_{\text{th}}}(\hat{O}) = e^{i\tau_{\text{th}} K}\, \hat{O}\, e^{-i\tau_{\text{th}} K}, \tag{8}$$

   where τ_{th} is the thermal time parameter [9]. This provides an intrinsic notion of time evolution even in equilibrium, but it is *not* entropic time.
3. **Microscopic Fluctuations:** In statistical mechanics, equilibrium corresponds to maximum entropy at the macroscopic level, but microscopic configurations continually fluctuate:

$$\langle (\Delta \hat{O})^2 \rangle = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2 > 0. \tag{9}$$

   These fluctuations exhibit temporal correlations (e.g., correlation functions $C(t) = \langle \hat{O}(t)\hat{O}(0) \rangle$) that define timescales.
4. **Page–Wootters Relational Time:** Even in a globally static state $|\Psi_{\text{total}}\rangle$ satisfying $\hat{H}|\Psi_{\text{total}}\rangle = 0$, subsystems remain entangled. Conditional states $|\psi(S)\rangle = \langle S|\Psi_{\text{total}}\rangle$ can still exhibit correlations, providing relational temporal structure [7].

3. Distinguishing Entropic Time from Other Time Concepts

The resolution to the equilibrium puzzle is that **entropic time measures only the irreversible, coarse-grained aspect of evolution**. It is not the *only* temporal structure, but rather one particular aspect. In the Table (2), some aspects are shown.

This multiplicity of time concepts is not a weakness but reflects the multi-faceted nature of temporal structure in physics. Different notions capture different physical aspects.

Table 2 Distinguishing entropic time

Time Concept	What It Measures	Defined in Equilibrium?
Coordinate time t	External parameter	Yes (if spacetime exists)
Thermal time τ_{th}	Modular flow of KMS states	Yes
Entropic time $\tau(\Delta S)$	Irreversible coarse-grained evolution	No ($\Delta S = 0$)
Page–Wootters relational time	Entanglement correlations	Yes (if entangled)

4. Cosmological Implications
In cosmological contexts:

- **Far-future heat death:** If the universe approaches a de Sitter vacuum state with $\Delta S \to 0$, entropic time effectively stops. However, quantum fluctuations persist, Hawking radiation continues, and de Sitter symmetries define alternative temporal structures.
- **Equilibrium subsystems:** Regions that have locally thermalized (e.g., inside galaxy clusters after virialization) cease to contribute to the cosmic entropic arrow. Global cosmic time continues due to expansion and ongoing structure formation in other regions.
- **Black hole interiors:** Near the singularity, spacetime curvature diverges and coordinate time becomes ill-defined. Bekenstein– Hawking entropy $S_{\text{BH}} = k_B A/(4\ell_P^2)$ may provide the only viable temporal parameter, blending entropic and thermal time concepts.

5. Philosophical Perspective
From a foundational viewpoint, the equilibrium objection reveals that entropic time is a *conditional* and *contextual* notion:

Entropic time is the appropriate temporal framework for non-equilibrium, irreversible processes in semiclassical regimes where coordinate time is absent or inaccessible. It is not a universal replacement for all temporal structures.

Just as temperature is meaningful only for systems in or near equilibrium, entropic time is meaningful only for systems far from equilibrium. Neither concept loses physical validity due to having a limited domain of applicability.

Consistency with Equilibrium Regimes The interpretation of entropy as a primitive concept remains compatible with the equilibrium limit already discussed.

In such regimes, the entropic ordering variable τ becomes stationary because entropy production vanishes, but microscopic, modular, and relational dynamics persist.

This emphasizes that the primitive status of entropy applies to regimes where irreversibility and information loss define temporal order, while equilibrium merely represents the boundary where this ordering ceases to evolve.

4 Discussion: Entropic Time Paradigm

A. Rovelli's Thermal Time Hypothesis

Rovelli's thermal time hypothesis [1] proposes that time is not fundamental but emerges from the state of a system via statistical considerations. Our framework, wherein ΔS drives time progression, embodies this idea. The entropy-based formulation defines a time flow dependent on thermodynamic evolution rather than external metrics, which aligns well with modular flow structures in quantum statistical mechanics. In Carlo Rovelli's relational quantum mechanics, time is not fundamental. Instead, time emerges from correlations between physical systems. Entropy provides an internal "clock" : systems evolve when entropy changes. In our framework, time is not fundamental but emerges from entropy variation: the passage of time reflects the progression of entropy. We interpret this as time passing. Time is generated by the modular flow of the state (entropy).

B. Geometric Thermodynamics and Temporal Trajectories

Geometric thermodynamics treats thermodynamic processes as trajectories on curved manifolds, where entropy defines the structure of space [2, 16, 17]. In this context, each cosmological epoch corresponds to a segment of a geodesic along an entropy manifold. Inflation is associated with a steep entropic gradient, while the far future suggests an asymptotic approach to thermodynamic equilibrium. Also, In some approaches (e.g., entropic gravity), gravity and space-time geometry emerge from thermodynamic or informational principles. Time, in this view, is not fundamental.

C. Non-Additive Tsallis Entropy and Cosmic Systems

The Tsallis entropy formalism [3] generalizes Boltzmann–Gibbs statistics and is particularly applicable to systems with long-range interactions, such as gravitational systems. The parameter q controls the deviation from extensivity, allowing for tailored entropy-time relations in different epochs. This is especially useful for modeling non-equilibrium conditions in the early universe. In this *entropic time hypothesis* (statistical approach), entropy increases as entropic time passes $\tau = \tau(\Delta S)$, that implies that entropic time emerges from, or can be measured by, changes in entropy—where 'change' specifically implies an increase in entropy. If entropy is the driver of dynamics (as in entropic gravity or Tsallis thermodynamics), then time can be defined as a monotonic function of entropy, as in (21). In generalized statistical mechanics (non-extensive entropy, conformal Tsallis Cosmology), time can be recast as a function of entropy, see Ref.([6]), particularly in cosmological models where power-law expansion or fractal-like dynamics dominate.

D. Definition of the Entropic Parameter

Definition: Entropic monotonicity without external time Let $\{\Phi_\alpha\}_{\alpha\in\mathbb{R}}$ denote a directed family of physically admissible coarse-grainings, represented by Completely Positive Trace-Preserving (CPTP) maps on the density operator, $\Phi_\alpha : \rho \mapsto \Phi_\alpha(\rho)$. Alternatively, α may label the *modular flow* $\{\sigma_\alpha\}$ of a Kubo-Martin-Schwinger (KMS) state [28] , or an internal correlation parameter arising in the Page–Wootters construction of relational dynamics.

An entropy functional S^* is *monotone along the flow* if

$$S^*(\Phi_{\alpha_2}(\rho)) \geq S^*(\Phi_{\alpha_1}(\rho)) \quad \text{whenever} \quad \alpha_2 > \alpha_1. \tag{10}$$

This defines a partial order on physical states without invoking an external time variable. An *entropic time parameter* τ is then any smooth, strictly increasing reparametrization $\tau = f(S^*)$ along this monotone trajectory.

Proposition (Clock without Bijection) If an entropy functional S^* is monotone along a directed physical flow, then any smooth reparametrization $\tau = f(S^*)$ defines a valid internal clock (arrow and parametrization), even if different states share the same entropy value. Bijectivity is unnecessary; monotonicity on typical histories suffices.

E. Entropic Ordering Without External Time

To avoid conceptual circularity, we must formulate the notion of "monotonically increasing entropy" without presupposing an external time parameter. This can be achieved through an order-theoretic approach.

1. Entropic Ordering Relation

Definition Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space, and let $\{\psi_\alpha\}$ denote a family of pure states of a closed quantum system. Define the *entropic ordering relation* $\prec_{\text{ent}}$ as:

$$\psi_\alpha \prec_{\text{ent}} \psi_\beta \quad \Leftrightarrow \quad S(\rho_\alpha^A) < S(\rho_\beta^A), \tag{11}$$

where $\rho^A = \text{Tr}_B|\psi\rangle\langle\psi|$ is the reduced density matrix for subsystem A, and $S(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy.
This definition makes no reference to external time. Entropy values $S(\psi_\alpha)$ and $S(\psi_\beta)$ are computed *statically* as properties of the quantum states themselves. The ordering $\prec_{\text{ent}}$ is a structural relation on the state space.

2. Mathematical Properties For generic entangling Hamiltonians and initially separable states, the relation $\prec_{\text{ent}}$ defines a *partial order* on the state space, satisfying:

- **Irreflexivity:** $\psi \not\prec_{\text{ent}} \psi$ (a state is not ordered before itself)
- **Transitivity:** If $\psi_1 \prec_{\text{ent}} \psi_2$ and $\psi_2 \prec_{\text{ent}} \psi_3$, then $\psi_1 \prec_{\text{ent}} \psi_3$
- **Antisymmetry (in thermodynamic limit):** If $\psi_\alpha \prec_{\text{ent}} \psi_\beta$ and $\psi_\beta \prec_{\text{ent}} \psi_\alpha$, then $\psi_\alpha = \psi_\beta$

This partial ordering defines a causal-like structure among states based solely on their informational content, without invoking any external temporal coordinate.
3. Physical Justification: Entanglement Growth Theorem The physical basis for this ordering comes from quantum dynamics:

Theorem (Entanglement Monotonicity) *Let $|\Psi(0)\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ be an initially separable pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$. Under unitary evolution generated by a generic entangling Hamiltonian H, the entanglement entropy $S_{\text{ent}}(\tau) = -\text{Tr}(\rho_A(\tau) \log \rho_A(\tau))$ increases monotonically until saturation at thermal equilibrium.*
Here, τ is merely a label parameterizing the unitary orbit $|\Psi(\tau)\rangle = e^{-iH\tau/\hbar}|\Psi(0)\rangle$. The theorem establishes that for "most" Hamiltonians, states generated later in the orbit have higher entropy than states generated earlier.

Crucially, this theorem does not presuppose external time. It states that the unitary orbit has a natural direction defined by increasing entropy. This direction can then be *used to define* an internal temporal structure.
4. Connection to Page–Wootters Formalism
The Page–Wootters mechanism [7] provides a concrete realization of this idea. Consider the timeless state:

$$|\Psi_{\text{total}}\rangle = \sum_S |\psi(S)\rangle_{\text{sys}} \otimes |S\rangle_{\text{clock}}, \tag{12}$$

where the "clock" is indexed by entropy S rather than coordinate time t. The conditional state:

$$|\psi(S)\rangle = \langle S|\Psi_{\text{total}}\rangle \tag{13}$$

depends on the entropic parameter S. The "evolution" is the correlation structure within the global state $|\Psi_{\text{total}}\rangle$, which itself satisfies a timeless constraint:

$$(\hat{H}_{\text{sys}} + \hat{H}_{\text{clock}})|\Psi_{\text{total}}\rangle = 0. \tag{14}$$

In this framework, temporal ordering emerges from *relational correlations between the system and the entropic clock*, not from evolution in external time.

5. Reformulated Statement of Temporal Arrow

With this foundation, we can now state the arrow of time without circularity:

The entropic ordering $\prec_{\text{ent}}$ defines a partial order on the space of coarse-grained quantum states. This ordering provides a relational notion of "earlier" and "later" based purely on informational structure, without reference to external temporal coordinates.

When coordinate time t exists and is well-defined (e.g., in semiclassical cosmology), the entropic ordering typically aligns with it:

$$t(\psi_\alpha) < t(\psi_\beta) \quad \Longrightarrow \quad \psi_\alpha \prec_{\text{ent}} \psi_\beta, \tag{15}$$

but the entropic structure can be defined independently, and survives in regimes where coordinate time is absent (such as solutions to the Wheeler– DeWitt equation).

This completes the conceptual foundation for using entropy as an internal clock without presupposing external time.

F. On Primitive and Derived Concepts in Physical Theories

1. Theory-Dependence of Foundational Concepts

A recurring question in foundational physics concerns which concepts should be treated as *primitive* (undefined, taken as given) versus *derived* (constructed from other concepts). This question has ancient roots—Euclid's *Elements* took point, line, and plane as primitive geometrical concepts, defining all else in terms of them.

However, a crucial insight from modern physics is that **the choice of primitive concepts is not unique and depends on the theoretical framework**:

- **In Newtonian mechanics:** Absolute space and absolute time are primitive; forces and accelerations are derived.
- **In General Relativity:** Spacetime geometry is dynamical (not primitive); the metric $g_{\mu\nu}$ satisfies Einstein's equations. Time is no longer absolute but part of a curved manifold structure.
- **In Quantum Mechanics:** The Hilbert space $\mathcal{H}$, inner product, and time-evolution operator $e^{-iHt/\hbar}$ are primitive; probabilities are derived via the Born rule.
- **In Canonical Quantum Gravity:** The Hamiltonian constraint $\hat{H}\Psi = 0$ has no external time parameter. Evolution must be reconstructed from internal observables.

2. Entropy vs. Time: Context Matters

In our framework, we confront the question: *Can entropy be primitive and time derived?* The answer depends critically on context:

Case 1: Standard Quantum Mechanics

In the Schrödinger formulation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle, \tag{16}$$

time t appears as a primitive parameter. Entropy is derived via:

$$S(\rho) = -k_B \mathrm{Tr}(\rho \log \rho), \quad \rho(t) = |\psi(t)\rangle\langle\psi(t)|. \tag{17}$$

Here, time is clearly more fundamental than entropy.

Case 2: Wheeler–DeWitt Quantum Cosmology

The fundamental equation is:

$$\hat{H}\Psi[h_{ij}, \phi] = 0, \tag{18}$$

where h_{ij} is the spatial 3-metric and ϕ represents matter fields. This equation is *manifestly timeless*—there is no $\partial/\partial t$ term.

In this context:

- Coordinate time t does not exist as a primitive concept
- To recover dynamics, one must identify an internal "clock" variable
- Entropy (or entanglement entropy of subsystems) provides one such candidate

Here, it is natural to treat entropy as primitive and reconstruct time from it.

Case 3: Thermal Time Hypothesis

Rovelli and Connes [8, 9] propose that for a system in a state ρ, time is defined by the modular flow:

$$\sigma_{\tau_{\mathrm{th}}}(\hat{O}) = \rho^{-i\tau_{\mathrm{th}}} \hat{O} \rho^{i\tau_{\mathrm{th}}}, \tag{19}$$

where the modular Hamiltonian $K = -\log \rho$ encodes the thermodynamic properties of the state. Time τ_{th} emerges from the statistical state itself.

This is analogous to our entropic time, but uses the modular structure rather than entropy production directly.

3. Our Position: Conditional Foundational Status

We adopt the following conditional stance:

1. **When coordinate time exists and is well-defined** (e.g., Minkowski spacetime, Friedmann–Lema tre–Robertson–Walker (FLRW) cosmology), it is natural to treat t as primitive and entropy as derived.
2. **When coordinate time is absent, ill-defined, or inaccessible** (quantum gravity, black hole singularities, Planck-scale physics), entropy (or related informational measures) can serve as a primitive ordering parameter from which temporal structure is reconstructed.
3. **In intermediate regimes** (semiclassical gravity, non-equilibrium thermodynamics), multiple valid temporal parameterizations may coexist, each capturing different physical aspects.

4. Analogy: Coordinate Systems in General Relativity

This situation is analogous to the choice of coordinate systems in General Relativity. The question "Which coordinate system is fundamental?" has no unique answer:

- For Schwarzschild black holes, Schwarzschild coordinates (t, r, θ, ϕ) are natural far from the horizon
- Near the horizon, Eddington–Finkelstein coordinates are better behaved
- For studying Hawking radiation, Kruskal–Szekeres coordinates are essential

Similarly, the question "Which temporal parameter is fundamental?" depends on the physical problem:

- For laboratory quantum mechanics: coordinate time t
- For black hole thermodynamics: thermal time τ_{th} or Hawking temperature
- For cosmological evolution: cosmic time or entropic time $\tau(\Delta S)$

5. Addressing the One-to-One Correspondence Objection
A critical objection states: *"In equilibrium the entropy does not vary but this does not mean that time stopped. Therefore, time and entropy are not in one-to-one correspondence."*
This objection is valid and important. Our response is threefold:

1. **We do not claim one-to-one correspondence.** Entropic time and coordinate time (when both exist) parameterize *different aspects* of physical evolution:
    - Coordinate time t: reversible microscopic dynamics
    - Entropic time $\tau(\Delta S)$: irreversible macroscopic evolution
2. **The absence of correspondence in equilibrium is a feature, not a bug**. It correctly reflects the physics: in equilibrium, the thermodynamic arrow vanishes while microscopic dynamics persist.
3. **Analogy to other non-bijective parameterizations:** In thermodynamics, temperature T and inverse temperature $\beta = 1/(k_B T)$ are both valid but not defined at $T = 0$. Similarly, entropic time is valid in non-equilibrium but undefined in equilibrium.

6. Conclusion: Pluralism About Time
The framework presented here advocates for a **pluralistic view of temporal structure**: there is no single "correct" notion of time, but rather multiple complementary notions suited to different physical contexts.
This pluralism is consistent with the historical development of physics:

- Newton: absolute time
- Einstein (Special Relativity): relative time, spacetime
- Einstein (General Relativity): dynamical time as part of geometry
- Quantum mechanics: external time parameter
- Quantum gravity: emergent/relational time

Our contribution is to identify entropy as one viable internal structure for temporal ordering in regimes where traditional time concepts become inapplicable or insufficient.

G. Toward a Thermodynamic Time Paradigm

These approaches converge toward a paradigm where time is not an independent coordinate but a relational construct, emerging from entropy and state evolution. Our results illustrate this visually and quantitatively, suggesting that cosmological time can be reinterpreted as a manifestation of thermodynamic flow.

We propose a model in which time interval t is considered as a function of entropy S. The key to the deformed Pesin relations was obtained by considering in the case of a fractal phase space, intimately related to non additive statistics

$$N(\varGamma) \sim \varGamma^q, \tag{20}$$

where Γ is the phase-space volume.

We proceed by inverting the generalized Pesin relation, (58) from Ref. [6], which relates entropy production to time, $\triangle S = \frac{\lambda t^{\gamma}}{N_0^{\frac{(1-q)}{q}}}$. We obtain:

$$t = \left(\frac{\Delta S \cdot N_0^{\frac{1-q}{q}}}{\lambda} \right)^{\frac{1}{\gamma}}, \tag{21}$$

where $S(t)$ is the physical entropy. We define N_0 as the initial number of accessible microstates in phase space at the beginning of the cosmological timeline, serving as the normalization scale for entropy production in our model. As in Ref. [6]), q is the Tsallis entropic index, λ a constant, and γ governs the entropy-time scaling. This model is then used to analyze time evolution across cosmological epochs. This function defines how much time has passed, not by clock ticks, but by how much entropy has been produced.

This form of (21) comes from the use of deformed derivative approach to obtain deformed Pesin-like relations ([6]) and suggests that entropy S increases with time t as a power law. The parameter γ controls the growth rate. The q-entropic parameter might relate to non-extensive statistical mechanics (Tsallis statistics), used in systems with long-range interactions (like gravity). The parameter λ is a scaling parameter, tied to cosmological constants or system-specific energy scales.

Connection with Deformed Derivatives and Pesin Relations The use of deformed or conformable derivatives provides a rigorous local formulation for generalized entropy production, bridging microscopic dynamics and coarse-grained entropy flow. Recent results, on a seminal article [18], derived these operators from disorder-averaged Ginzburg–Landau kinetics and demonstrated that power-law memory kernels $K(\tau) \sim \tau^{\mu-1}$ lead, in the adiabatic limit, to a local evolution law of the form $T^{1-\mu} d\psi/dT$; T is the the temperature. This structure ensures thermodynamic consistency and naturally yields generalized Pesin-type relations, where the deformation parameter μ (or equivalently the nonextensivity index q) encodes the strength of memory and spatial heterogeneity. In the entropic-time framework, this same conformable operator governs the rate of change of entropy itself, providing the differential form underlying the entropy-production-Lyapunov connection. For more details, the reader is invited to consult the Ref. [18].

Alternative expressions can be derived, for instance, by isolating t in (59) of the referenced work, $\triangle S = N_0^{\frac{1-q}{q}} \, ln_q[e^{\frac{\lambda t^{\alpha}}{\alpha q}}]$. [6]:

$$\boxed{t = \left[\frac{\alpha q}{\lambda(1-q)} \ln \left(1 + \frac{\Delta S(1-q)}{N_0^{\frac{1-q}{q}}} \right) \right]^{\frac{1}{\alpha}},} \tag{22}$$

or isolating t in (50) in the same reference [6], $\triangle S = N_0^{\frac{1-q}{q}} \, ln_q[E_{\alpha}(\lambda^{\alpha} t^{\alpha\gamma})]^{\frac{1}{q}}$:

$$t = \left(\frac{E_{\alpha}^{-1} \left(\exp_q \left(\left(\frac{\Delta S}{N_0^{\frac{1-q}{q}}} \right)^q \right) \right)}{\lambda^{\alpha}} \right)^{\frac{1}{\alpha}}. \tag{23}$$

Meaning of the Mittag–Leffler Index $\boldsymbol{\alpha}$ In the fractional formulation, $\alpha \in (0, 1]$ is the memory (or anomaly) index of the kernel $K(t) \propto t^{\alpha-1}$ that yields the Riemann–Liouville operator in the kinetic equation for accessible states $N(t)$. Subdiffusive (long-memory) regimes correspond to $\alpha < 1$, normal Markovian kinetics to $\alpha = 1$. Consequently, entropy production inherits the same memory class, and the corresponding entropic-time law involves $E_\alpha(\cdot)$ (or its inverse). This α is determined by the physics of the kernel (memory/fractality) rather than chosen ad hoc. See Ref. [6] for the derivation from $K(t) \propto t^{\alpha-1}$ and the ensuing fractional Pesin relations.

G. On the Multiplicity of Time Functions and Physical Interpretability

One potential concern with the entropic time formulation lies in the apparent multiplicity of expressions for time derived from entropy relations, such as (21), (22) and (23). Since each of these equations involves time as a function of entropy, one might question whether this leads to arbitrariness in defining emergent time, that is, can any entropy-time equation simply be inverted to yield a notion of time?

5 On the Physical Non-Equivalence of Entropic Time Forms

Although (21), (22) and (23) all express a relation between time and entropy change ΔS, they are not physically equivalent. Each arises from a different model of entropy production, grounded in distinct mathematical frameworks and physical assumptions. Equation (21) is based on a power-law scaling between entropy and time, typically associated with systems exhibiting fractal phase space structure and governed by non-extensive dynamics. This form assumes a simple monotonic algebraic relationship, representing minimal deformation and constant memory exponent.

In contrast, (22) emerges from a generalized q-logarithmic formalism, introducing nonlinear saturation effects into the entropy-time relationship. This structure captures scenarios in which entropy production saturates over finite timescales, such as during cosmological phase transitions or bounded evolution in closed systems.

Equation (23) is rooted in fractional dynamics, where time and entropy are linked through an inverse Mittag-Leffler function, E_α^{-1}. Such functional forms naturally describe systems with long memory, anomalous transport, or nonlocal relaxation processes, commonly found in glassy systems, early universe thermodynamics, or black hole entropy flows.

These formulations are not interchangeable parameterizations of the same process. Rather, they represent distinct physical mechanisms and classes of entropic evolution: memoryless and algebraic, saturating and bounded, or memory-retaining and nonlocal. Their inversions yield fundamentally different notions of internal time evolution, adapted to the regimes they model.

Among these, (21) occupies a privileged position within the framework. It is derived from the minimal power-law relation, invoking no special functions or memory kernels, and is analytically tractable in closed form. Its simplicity also aligns well with large-scale cosmological entropy trends, which grow superlinearly across epochs. This makes (21) a natural choice for a lowest-order approximation to emergent time, especially when the goal is to isolate the essential behavior without introducing unnecessary complexity.

More sophisticated expressions like (22) and (23) are essential in more constrained or memory-sensitive systems, but they should be seen as refinements or corrections to the leading-order behavior captured by (21). In this sense, the diversity of entropic time parametrizations is not a weakness of the framework, but rather a reflection of its adaptability to a range of physically distinct dynamical scenarios.

3. The Physical Role of Inversion
While it is true that inverting entropy, time relations produces formal expressions for time, this inversion does not imply that the model is tautological. The entropy functions used are themselves derived from physical principles, such as deformed Pesin identities, non-extensive statistics, or fractional calculus, which encode the microscopic dynamics of complex systems.
Therefore, the emergence of time is not an artifact of inversion per se, but a reflection of the fact that entropy production serves as a consistent, observer-independent ordering parameter. It is this thermodynamic evolution, and not coordinate time, that defines the causal structure of the universe in this formulation.

E. Parameter Justification and Physical Constraints

A key challenge in any emergent-time formulation is to assign clear physical meaning and possible observational constraints to the parameters entering the proposed relations. In (21), we define an internal parameter τ as a function of entropy variation, representing the progression along the entropic trajectory, explicitly identified as the entropic-time variable. The entropic parameter τ introduced here does not represent an external coordinate time but a monotonic ordering variable generated by entropy production. This definition is introduced to distinguish τ from the external coordinate time t, which merely labels an observer's frame when such a coordinate exists. To reinforce the difference, we claim thatcoordinate time t, when present, merely parametrizes the observer's frame and may coincide with τ only under stationary thermodynamic flow. In contrast, entropic time emerges from internal irreversible processes and measures progression along the causal chain of entropy increase. Hence, a system can evolve, in the entropic sense, even when the coordinate time is frozen in a given foliation, consistent with timeless or covariant formulations of dynamics. Using (21), the entropic time, τ, can be written as

$$\tau = \left(\frac{\Delta S \, N_0^{\frac{1-q}{q}}}{\lambda} \right)^{1/\gamma}, \tag{24}$$

where ΔS denotes the entropy change, N_0 is the initial number of accessible microstates, q is the Tsallis entropic index, λ is a characteristic scaling parameter, and γ governs the power-law scaling of entropy with time.

Possible correlation between entropic and coordinate time.

The entropic parameter τ is constructed as an internal ordering variable, derived from the cumulative entropy variation ΔS, according to (24). Although τ is defined without reference to any geometric background, a functional correlation with the coordinate time t can be established once the entropy production rate $\dot{S}(t)$ is known. By differentiating $\tau(\Delta S) = (\Delta S/\lambda)^{1/\gamma}$, one obtains

$$\frac{d\tau}{dt} = \frac{1}{\gamma \, \lambda^{1/\gamma}} \big(S(t) - S_0 \big)^{\frac{1}{\gamma}-1} \dot{S}(t), \tag{25}$$

which expresses the local rate at which the entropic clock advances with respect to the coordinate time. Integrating (25) gives the general mapping

$$\tau(t) = \frac{1}{\gamma\,\lambda^{1/\gamma}} \int_{t_0}^{t} \left[S(t') - S_0\right]^{\frac{1}{\gamma}-1} \dot{S}(t')\,dt'. \tag{26}$$

This provides an operational bridge between the informational and geometric parametrizations of evolution. In regimes of steady entropy production, $\dot{S}(t) = \text{const.}$, (26) reduces to $\tau \propto t^{1/\gamma}$, implying a monotonic but generally nonlinear correlation between the two temporal parameters. When $\gamma = 1$, the relation becomes linear and the entropic clock coincides with the coordinate clock up to a scale factor. In equilibrium, where $\dot{S} = 0$, one finds $d\tau/dt = 0$: the entropic clock halts, although microscopic or reversible dynamics in t continue to exist. This expresses the precise sense in which the thermodynamic arrow ceases while geometric time persists.

Therefore, the correlation between τ and t is not universal but contextual: it depends on the entropy-production model that characterizes the physical system. In non-equilibrium or dissipative regimes, $\tau(t)$ accelerates relative to t, signaling a faster internal ordering of states, while near equilibrium the relation tends toward linearity. This flexibility ensures that entropic time complements, rather than replaces, coordinate time, acting as an emergent temporal variable whose "speed" reflects the degree of irreversibility encoded in the system's entropy growth.

A concise comparison between the coordinate time t and the entropic time τ, including their respective physical roles and limiting behaviors, is summarized in Table 3.

On Dimensions and Non-Circularity

Equation (24) inverts a generalized Pesin-type scaling $\Delta S = \lambda\,\tau^{\gamma} N_0^{(1-q)/q}$ (see Ref. [6]) to define an internal parameter $\tau(\Delta S)$. Here, λ is not "per unit of *coordinate* time" ; rather, it is a scale that balances units as $[\lambda] = [S]\,[\tau]^{-\gamma}$. The clock variable τ is the internal parameter recovered from monotone entropy production; thus no premise assumes external time. Coordinate time, when present, may correlate with τ but remains conceptually distinct, thereby avoiding any circular reasoning.

To avoid arbitrariness and address concerns of model dependence, we now provide a physically motivated interpretation of these parameters and suggest potential ways to constrain

Table 3 Comparison between coordinate time t and entropic time τ

Concept / Quantity	Physical Interpretation
τ grows with entropy	Measures informational progression; increases with entropy production.
t (coordinate time)	Measures geometric progression; parameterizes spacetime or the observer's frame.
$\frac{d\tau}{dt}$	Acts as a thermodynamic clock rate: faster for irreversible processes, vanishes at equilibrium.
Reversible or equilibrium limit	t continues to flow, but τ becomes stationary; the entropic arrow ceases.
Mapping $\tau(t)$	Describes how informational and geometric descriptions of evolution synchronize or diverge.

The correlation $d\tau/dt$ quantifies how informational and geometric descriptions of evolution diverge or synchronize them. Table 4 highlights the need to link each parameter to either fundamental theoretical constructs (e.g., horizon entropy, gravitational entropy bounds) or observational cosmological signatures. The goal is to ensure that the entropic time function does not remain arbitrary, but rather emerges from, and is testable against, known features of cosmic thermodynamic evolution. Also, the parameters appearing in (24) are summarized in the same Table 4.

In gravitationally interacting systems—such as dark matter halos, intergalactic gas, black-hole horizons, and large-scale structure– long-range correlations and fractal spatial distributions naturally give rise to nonextensive thermodynamics. Within Tsallis statistics, the entropic index q quantifies the strength of these gravitational correlations. Recent analyses of black-hole thermodynamics within this framework [14] show that the q-dependent entropy– area relation leads to modified horizon temperatures and specific heats, recovering the standard Bekenstein– Hawking limit as $q \to 1$. Together with earlier studies on gravitational clustering and cosmic thermodynamics [13], these results support the interpretation of $q \neq 1$ as a measurable signature of long-range gravitational coupling. Our illustrative examples therefore align naturally with this class of gravitationally induced nonextensivity.

H. Parameter Scaling and the Entropy Scale Problem

A key concern raised regarding (24) is the scale and justification of the parameters used in numerical illustrations, particularly the choice $N_0 = 10^{23}$, the supposed "initial" number of accessible microstates. While this is numerically convenient and of the order of Avogadro's number, it is clearly insufficient for cosmological applications.

1. *Cosmological Entropy Scales* Entropy estimates for key cosmological structures suggest that realistic values for the number of microstates should be many orders of magnitude larger:

   - **Cosmic Microwave Background (CMB):** Photon entropy today is estimated at $S_{\text{CMB}} \sim 10^{88}\, k_B$.
   - **Supermassive Black Holes (total):** Dominant contributor to present-day entropy, with total entropy $S_{\text{BH}} \sim 10^{103}$ to 10^{105}.
   - **Maximal entropy in far future (cosmic horizon):** Estimates suggest $S \sim 10^{120}$ in de Sitter or heat death scenarios.

   Thus, for entropies in the range $S \sim 10^{88} - 10^{120}\, k_B$, the total number of accessible microstates is

$$N \sim e^{S/k_B} = 10^{S/(k_B \ln 10)}, \tag{27}$$

Table 4 Interpretation and suggested constraints for the free parameters in the entropic–time expression

Parameter	Physical Meaning	Suggested Constraint
q	Tsallis non-extensivity index	Constrained by non-additivity in early-universe statistics.
N_0	Initial number of microstates	Related to phase-space volume at inflation onset.
λ	Entropy-production rate	Generalized entropy-production scale (units chosen so that $[\lambda] = [S][\tau]^{-\gamma}$), where τ denotes an internal parametrization (not necessarily the coordinate time).
γ	Scaling exponent	Fractal dimension of accessible phase space.

so that, in order-of-magnitude terms,

$$\log_{10} N \sim 10^{88} \text{ to } 10^{120}, \tag{28}$$

i.e. $N \sim 10^{10^{88}}$ to $10^{10^{120}}$ up to factors of order unity in the exponent. A realistic initial value N_0 should therefore lie somewhere within this astronomically large range.

2. *Interpretation of Parameters* We clarify that the parameter values:

$$q = 0.5, \quad N_0 = 10^{23}, \quad \lambda = 10^{-9}, \quad \gamma = 2, \tag{29}$$

were not intended to be physically definitive, but to produce qualitatively illustrative curves of entropic time growth across epochs. These values are orders of magnitude smaller than required for a precise cosmological match.

3. *Toward Physically Grounded Constraints*
The parameters appearing in the entropic time law, such as N_0, λ, and γ, are not arbitrary from a physical standpoint. Future investigations will aim to constrain these quantities by connecting them with empirical data and theoretical models. One promising strategy involves comparing the predicted entropic time functions $t(\Delta S)$ to established cosmological timelines and known stages of entropy accumulation throughout the universe's history. The initial entropy scale N_0 could be matched to the horizon-scale entropy at the end of inflation, estimated using the Gibbons– Hawking entropy associated with the inflationary patch. The parameter λ, which governs the growth rate of entropy, may be inferred from the entropy production rates characteristic of key epochs such as inflation, reheating, recombination, and star formation. Finally, the deformation exponent γ might be extracted from nonextensive scaling behaviors, particularly within Tsallis cosmology or generalized thermodynamic frameworks.
We fully acknowledge that the specific values used for these parameters in our illustrative figures are not derived from observational constraints or microphysical derivations. Rather, they were chosen to highlight the qualitative behavior of entropic time emergence across different scenarios. Establishing a quantitatively predictive model of entropic cosmology, with parameters grounded in fundamental physics and observational data, remains an important direction for future research.

F. Connecting Entropic Time to Physical Models and Observables

To improve the physical credibility of the entropic time framework and reduce model-dependence, it is essential to link the parameters and entropy change ΔS in (24) to actual cosmological processes and observationally accessible data. Rather than using abstract or purely illustrative entropy curves, we can estimate entropy production across known epochs of cosmic history.

1. **Cosmological Epochs and Entropy Sources**
Each cosmological epoch contributes differently to the total entropy content of the universe. Table 5 summarizes major epochs and their dominant entropy-producing mechanisms, alongside estimated entropy magnitudes derived from established literature.
2. **Estimating Physical Priors on Model Parameters**
The entropy estimates presented in Table 5 allow us to suggest physically motivated priors for the parameters appearing in the entropic time law, (24). The entropy variation ΔS can be interpreted as the cumulative entropy produced up to a given cosmological

Table 5 Major cosmological epochs and their dominant entropy sources, with estimated entropy magnitudes

Epoch	Dominant Source of Entropy	Estimated Entropy ΔS
Big Bang (initial state)	Quantum vacuum / inflationary horizon	~ 0
Inflation Reheating	Particle production, rapid expansion	$\sim 10^{26}$
Recombination (CMB decoupling)	Photon entropy in CMB	$\sim 10^{88}$
Galaxy and Star Formation	Dissipation, structure formation	$\sim 10^{90}$
Black Hole Formation (present epoch)	Bekenstein-Hawking entropy of SMBHs	$\sim 10^{103}$
Far Future	Hawking radiation, heat death	$\sim 10^{120}$

–time expression. [19–24]

epoch. This provides a natural dynamical variable for indexing time evolution across the thermal history of the universe.

The initial number of accessible microstates N_0 may be associated with the number of quantum degrees of freedom enclosed within the inflationary horizon. Estimates suggest this value could plausibly range from 10^{23} to 10^{30}, depending on the specific model and energy scale of inflation. The non-extensivity parameter q, which reflects deviations from Boltzmann– Gibbs behavior, can be adjusted to account for gravitational clustering, long-range correlations, or other nonlocal effects. In cosmological contexts, such features often lead to values of $q \lesssim 1$, consistent with the illustrative choice of $q = 0.5$ used in earlier plots.

The entropy production rate λ is another key parameter that can be normalized by comparing entropy accumulation between well-characterized epochs, such as from recombination to the present day, or by relating it to astrophysical processes like the cosmic star formation rate. Finally, the scaling exponent γ captures the nonlinearity in entropy accumulation over time. Empirically, values near $\gamma = 2$ seem to describe power-law growth regimes observed across various cosmological eras, including inflationary reheating and structure formation.

By grounding these parameters in known physical processes, the entropic time framework becomes increasingly testable and constrained, offering a pathway from abstract entropy parametrizations to concrete cosmological modeling.

3. **Observational Anchoring of the Entropic Clock**

   The Cosmic Microwave Background (CMB) provides one of the most precise entropy measurements to date, with entropy per photon roughly conserved since recombination. Similarly, black hole thermodynamics provides a robust observational handle on gravitational entropy.

   One may calibrate the entropic time function by matching time estimates derived from (24) to known cosmological ages (e.g., recombination at $t \sim 370{,}000$ years, present age $t \sim 13.8$ Gyr) using the entropy values of Table 5. This approach would render the emergent time scale predictive rather than arbitrary.

4. **Toward an Observationally Constrained Entropic Cosmology**

   By aligning the entropy inputs ΔS with observational cosmology, and constraining model parameters with theoretical priors and known thermodynamic processes, we establish a bridge between the entropic time hypothesis and measurable physical evolution. Such anchoring allows the entropic formulation to function not only as a conceptual reinterpretation of cosmic time, but as a predictive and testable tool within early-universe cosmology, gravitational thermodynamics, and high-entropy scenarios such as black hole mergers or cosmic horizon evolution.

Parameter Values Used for the Entropic Time Model

The figures shown in this work are based on the entropic time relation (there called time, for simplicity):

$$\tau = \left(\frac{\Delta S \cdot N_0^{\frac{1-q}{q}}}{\lambda} \right)^{1/\gamma}, \tag{30}$$

with the following parameters:

- $q = 0.5$ (Tsallis non-extensive entropic index)
- $N_0 = 10^{23}$ (initial number of accessible states)
- $\lambda = 10^{-9}$ (entropy production rate coefficient)
- $\gamma = 2$ (scaling exponent)

The plots values were chosen to generate a realistic entropic trajectory that qualitatively matches major cosmological epochs, such as inflation, recombination, and the present era, that is, to qualitatively reproduce the expected shape of cosmological time evolution across entropy growth, particularly highlighting early sensitivity and late-time saturation. It illustrates the nonlinear progression of time as a function of entropy, both in linear and logarithmic scales.

The resulting plots are not direct fits to observational data but serve to illustrate the functional behavior implied by the model.

Entropic Time And Cosmological Epochs

To avoid going into too much detail, we focus our attention only to (21).

To visualize the implications of the entropy-based time relation, we numerically evaluate (21) across a wide range of entropy changes, ΔS. This allows us to explore how time unfolds as entropy increases, under physically reasonable parameter values for q, λ, N_0, and γ. The resulting plots provide a macroscopic representation of cosmological time evolution, with key epochs, such as the Big Bang, inflation, and recombination, marked along the entropy axis.

The evolution of time as a function of entropy across cosmological epochs, Fig. 1, presents the relation in linear scale, emphasizing the early-time sensitivity to entropy growth. Key events such as the Big Bang, cosmic inflation, recombination, the present era, and the far future are highlighted. The rate of entropy change provides an intuitive perspective on the emergence of time, emphasizing its dependency on thermodynamic progression. The figure is showing cosmological time as a function of entropy. It illustrates how time can be modeled as emerging from entropy increase aligning with the idea that the passage of time is driven by entropic change.

In contrast, Fig. 2 adopts a logarithmic time scale and presents the same relationship between t and ΔS. It reveals the asymptotic behavior and long-term flattening, which reflects the saturation of entropy production as the universe approaches thermodynamic equilibrium. It better capture the extremely compressed times scales of the early universe. This approach allows clearer visualization of rapid transitions during the universe's earliest moments—such as inflation—suggesting that significant entropy variations occur over very short temporal intervals.

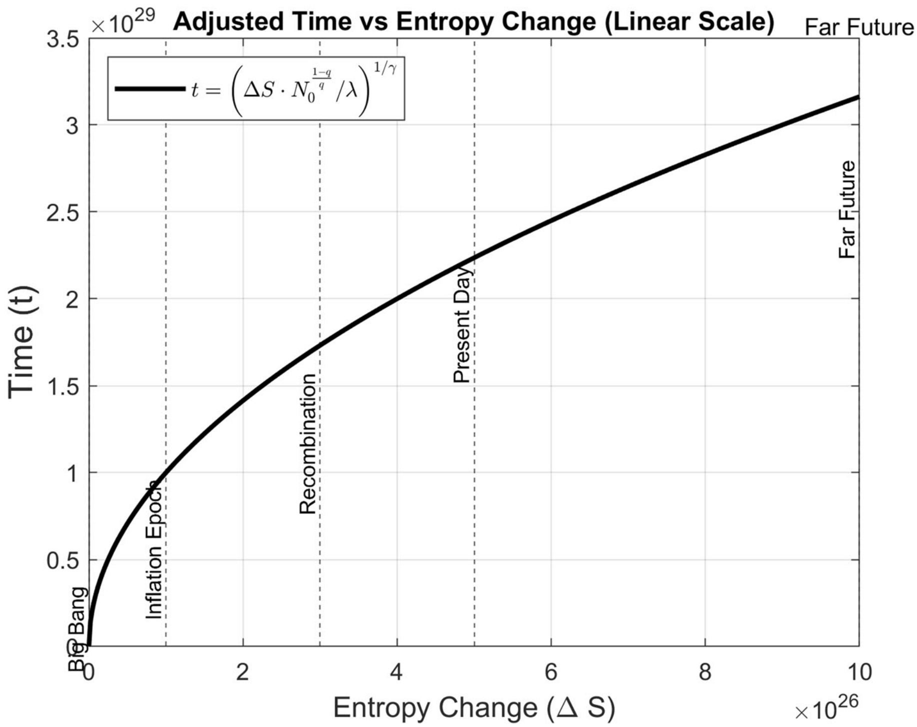

Fig. 1 Time as a function of entropy change with cosmological epochs highlighted. See (21)

At the Big Bang, entropy is nearly zero, and time lacks a meaningful definition. These epochs, such as the Big Bang, inflation, and recombination, are characterized by distinct entropy transitions. By the end of inflation, a sharp increase in entropy corresponds to the rapid emergence of time. During recombination, the universe continues to develop structure, accompanied by further entropy growth. In the present day, entropy is high due to the formation of stars, galaxies, and black holes. Looking toward the far future, the universe is expected to approach a state of maximum entropy—often referred to as heat death—where thermodynamic equilibrium prevails and the emergence of new structure ceases.

1. Clarifying the Notion of Entropic Time Compression during Inflation

A careful semantic clarification is required concerning statements like: "By the end of inflation, a sharp increase in entropy corresponds to the rapid emergence of time" (Fig. 1). Since our framework posits that time itself is emergent from entropy growth, it is conceptually inconsistent to use the term "rapid", which presupposes an external time parameter, to describe the very process by which time is reconstructed.

What we mean, more precisely, is that the entropy increases steeply in this epoch, and thus, under our entropic time relation $t \sim \Delta S^{1/\gamma}$, the early stages of entropy production account for a compressed segment of entropic time. The majority of time "emerges" only after entropy passes certain thresholds.

In this sense, temporal intervals are not defined externally, but are reconstructed from intervals in the entropy domain. The steep slope in S implies that large portions of entropy space are traversed over small entropic-time intervals.

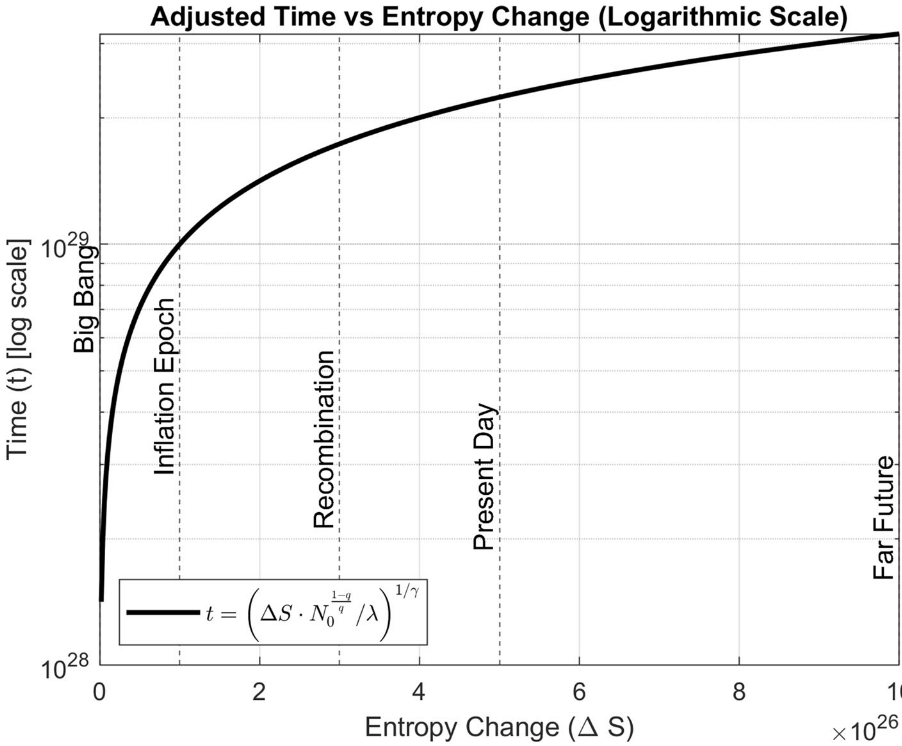

Fig. 2 Same as Fig. 1 but with a logarithmic time scale to emphasize early universe dynamics

Thus, in place of "rapid emergence of time," we propose the more accurate formulation:

"The inflationary era corresponds to a steep segment in the entropy growth curve, implying a compressed interval of emergent time as defined by the entropic time function."

The entropy values associated with each cosmological epoch are summarized in Table 6. Additional conceptually significant considerations are:

Reversed View Entropy as Fundamental, Time as Emergent. This reiterates the central thesis introduced earlier: the progression of time is directly linked to entropy growth, that is, there is an direct connection of ΔS and time t progression.

Table 6 Entropy values at key cosmological epochs, represented in arbitrary units consistent with the entropic time model

Epoch	Entropy Change ($\boldsymbol{\Delta S}$)
Big Bang	0
Inflation Epoch	1×10^{26}
Recombination	3×10^{26}
Present Day	5×10^{26}
Far Future	1×10^{27}

Conceptual Consequences **1.** Time is an emergent, derived concept. **2.** Entropy is the true measure of "evolution" or "becoming" in the universe. **3.** The direction of time is entropic, not fundamental. **4.** In quantum gravity or a final theory, time may not appear explicitly—only entropy, geometry, and correlations.

Implications for Cosmology Early Universe Entropy was low, with rapid production during inflation, baryogenesis, and recombination. Current Epoch: Entropy production continues through star formation, supernovae, and black holes. Far Future: Heat death scenario (maximum entropy), or perhaps entropy oscillates in cyclic cosmologies.

Arrow of Time as an Emergent Property The arrow of time exists because entropy increases. If $\dot{S} = 0$, the entropic arrow stalls; this does not imply that microscopic, reversible dynamics cease to exist.

Big Bang and Low Entropy The Big Bang marks a boundary condition of very low entropy. Time may be undefined or meaningless before this point if entropy was constant or undefined.

Black Holes and Time Near Singularities In high-gravity systems (e.g., black holes), entropy dominates. Time may "slow down" or "freeze" as entropy gradients vanish or become unobservable.

Possible Experimental / Observational Hints Holographic principle: Time near black hole horizons may be best described by entropy on the boundary. Cosmic evolution: Entropy of the cosmic microwave background, large-scale structure, and black holes might serve as a "cosmic clock." Quantum cosmology: Wheeler– DeWitt equation [5] lacks time—evolution could instead be parametrized by entropy.

Emergent Time in Quantum Gravity In theories without a fundamental time variable (e.g., the Wheeler-DeWitt equation), time is "reconstructed" by identifying a clock variable—which could be entropy, especially in cosmological models.

Conceptual Clarification: Entropy-Driven Time Is Not Mere Inversion

A key conceptual concern that may arise is whether the entropic time framework merely amounts to a formal algebraic inversion, i.e., that if entropy grows nonlinearly with time, then expressing time as a function of entropy adds nothing substantive. This critique, while mathematically valid in a narrow sense, overlooks the physical motivation and conceptual shift underlying our proposal.

We do not assert that any invertible relation between entropy and time constitutes a meaningful notion of emergent time. Rather, the claim is that under specific physical conditions, such as in systems where coordinate time is absent, inaccessible, or undefined, entropy growth provides a natural and operationally meaningful replacement for external time parameters. The guiding assumption is that entropy, or its rate of production, serves as the fundamental dynamical variable, and that time can be reconstructed as an emergent parameter that tracks the irreversible evolution of the system.

This is particularly relevant in regimes such as canonical quantum gravity, where the Wheeler-DeWitt equation imposes a timeless constraint on the universe's wavefunction, or in thermal field theory where time evolution emerges from the modular flow associated with a KMS state. In these contexts, entropy becomes a physically grounded ordering variable, enabling one to define a consistent temporal structure even in the absence of a background time coordinate.

The expression for entropic time, such as the power-law form in (24), is not introduced as a mathematical curiosity. It arises from well-motivated physical models, including non-extensive statistical mechanics (via Tsallis entropy), memory-dependent dynamics (through deformed Pesin relations), and entropy accumulation processes in cosmological settings. These functions encode distinct dynamical mechanisms, such as anomalous diffusion, fractal evolution, and phase-space deformation, each yielding a different entropic clock.

The novelty, therefore, lies not in the mathematical inversion itself, but in the physical reinterpretation of what constitutes the temporal variable. In contrast to standard formulations where time is taken as fundamental and entropy is a derived quantity, the present framework reverses this logic: entropy is treated as the primitive observable, and time is reconstructed from its evolution. This reversal has significant implications for our understanding of cosmological history, quantum evolution, and the foundational role of thermodynamic irreversibility.

In summary, the entropic time approach represents a substantive shift in perspective. It is not a trivial reformulation of standard dynamics, but a physically motivated redefinition of time grounded in the informational and statistical structure of the universe.

6 Conclusion and Outlook

In this work we have developed a systematic framework in which time emerges as a relational and state-dependent quantity, parametrized by entropy growth rather than assumed as a primitive background coordinate. The central proposal is inherently contextual: entropic time serves as a physically meaningful temporal parameter only in regimes where conventional notions of coordinate time are absent, ill-defined, or insufficient to describe the dynamics. This includes settings such as canonical quantum gravity, where solutions of the Wheeler–DeWitt equation lack an explicit temporal variable; non-equilibrium cosmological epochs, such as inflation, reheating, and structure formation, characterized by irreversible entropy production; black-hole thermodynamics, where the Bekenstein–Hawking entropy provides the dominant thermodynamic signature near horizons; and semiclassical regimes in which quantum coherence is suppressed and coarse-graining becomes well defined.

Conversely, the framework naturally loses applicability in perfectly equilibrated systems with vanishing entropy production, in fully coherent quantum states prior to decoherence, and in static vacuum spacetimes devoid of matter, where no entropy gradients or coarse-graining processes are available. In such cases, the entropic clock ceases to advance, though other forms of microscopic or reversible dynamics may still occur. Thus, the proposed construction does not imply that physical processes or quantum evolution stop, but only that the thermodynamic arrow becomes stationary.

The theoretical foundation developed here is rooted in well-established formulations of timeless physics, including the Wheeler–DeWitt equation in canonical quantum gravity, the thermal time hypothesis based on Kubo–Martin–Schwinger (KMS) modular flow, and the Page–Wootters approach to relational quantum dynamics. In each of these frameworks the absence of an external temporal coordinate calls for an intrinsic ordering variable capable of capturing evolution. Entropy, as a statistical measure of complexity and information loss under coarse-graining, provides precisely such a structure.

The main technical contribution of this paper is the derivation of explicit entropic-time functions, such as

$$\tau(\Delta S) = \left(\frac{\Delta S}{\lambda}\right)^{1/\gamma},$$

which arise from generalized Pesin relations and entanglement monotonicity theorems. Contrary to the view that such expressions merely invert the entropy-time relationship, their functional form follows from physically motivated models of entropy production and statistical geometry. The parameters involved acquire physical meaning from nonextensive statistical mechanics, entropy scaling laws, and cosmological entropy estimates. In this sense, the temporal structure is not externally imposed but self-consistently constructed from the system's internal thermodynamic evolution.

A key conceptual clarification follows from this formulation. We use "causal ordering among macrostates" in an order–theoretic sense: along typical coarse–grained histories, entropy is monotone and thus induces a partial order $\prec_{ent}$ on states. This structure suffices to define an arrow and a parametrization $\tau(\Delta S)$ without postulating an external metric time. In FLRW cosmology, one may model expansions by rewriting background equations in terms of τ wherever irreversible entropy production dominates, and subsequently verify that τ correlates with the standard cosmic time t in the appropriate regimes. In the local (Minkowskian) limit with negligible entropy production, $\dot{S} \to 0$, the entropic clock stalls and τ becomes uninformative—yet laboratory coordinate time and modular (thermal) time remain well defined, and microscopic dynamics persist. Hence, entropic time complements rather than replaces coordinate time: it is predictive in non-equilibrium, semiclassical regimes, and purposely silent in equilibrium domains.

Importantly, this framework concerns the evolution of entropy in state space, not its spatial distribution; it therefore avoids conflating temporal and spatial thermodynamic gradients, such as those associated with black-hole entropy profiles.

From a conceptual standpoint, the analysis suggests that time need not be regarded as a universal primitive but as an emergent, coarse-grained construct that acquires meaning through entropy production. Different physical contexts may thus exhibit distinct temporal structures-coordinate, modular, or entropic, each capturing complementary aspects of the same underlying reality. Adopting this pluralistic view allows for a unified description of thermodynamic, gravitational, and quantum processes, particularly in regimes where traditional time parameters lose their operational basis.

The present results open several promising directions for further investigation. One line of inquiry involves applying the entropic-time formalism to concrete cosmological models of inflation, reheating, or recombination, allowing comparison between predicted and observed entropy production rates. Another extension concerns dynamical reformulations in which equations of motion evolve along entropic rather than coordinate time, offering potential insights into non-equilibrium behavior in gravitational and quantum systems. Further work should also examine whether entropy-based clocks can yield observational signatures, for example through constraints derived from black-hole thermodynamics, cosmic microwave background anisotropies, or large-scale structure correlations. Finally, the role of entropy in shaping the subjective and informational perception of time, particularly within coarse-grained or observer-dependent frameworks, deserves deeper exploration.

In conclusion, we have proposed that time – commonly treated as a fundamental backdrop for physical phenomena – may instead emerge as a derived property encoded in the growth of entropy. By grounding this proposal in statistical mechanics, quantum theory, and cosmology, the present work reframes the longstanding problem of temporal origin in terms of observable, quantifiable informational structures, thereby linking the arrow of time to the deeper architecture of physical law.

Appendix A: From Entropic Pesin Relations to Emergent Time

1. **Generalized Pesin Identity in Deformed Dynamics**
   In classical chaotic systems, the Pesin identity relates the Kolmogorov-Sinai (KS) entropy rate h_{KS} to the sum of positive Lyapunov exponents λ_i. For systems with non-standard dynamics, such as those with long-range correlations, fractal phase spaces, or memory effects, the Pesin identity can be generalized using the tools of non-extensive statistical mechanics.
   In such contexts, the entropy production rate is deformed, and a generalized Pesin-like relation can be written as:
   $$\Delta S = \lambda \cdot t^{\gamma} \cdot N_0^{\frac{1-q}{q}}, \tag{A1}$$
   where:
   - ΔS is the entropy increase over time,
   - λ is a generalized entropy production rate,
   - γ is a scaling exponent linked to the deformation,
   - N_0 is the number of accessible microstates at $t = 0$,
   - q is the Tsallis entropic index, characterizing the degree of non-additivity.

   This formulation reflects fractal or anomalous dynamics where entropy increases sub-linearly or super-linearly with time, and the scaling factor depends on the entropic geometry of the system.
2. **Emergent Time from Entropy Production**
   The central postulate of this work is that entropy change ΔS is a more fundamental measure of dynamical evolution than coordinate-time t. That is, time should be treated as a parameterized function of entropy growth:
   $$\tau = \tau(\Delta S). \tag{A2}$$
   Assuming monotonic entropy growth (i.e., $dS/dt > 0$), the relation (A1) can be inverted to express time in terms of entropy:
   $$\tau = \left(\frac{\Delta S}{\lambda \cdot N_0^{\frac{1-q}{q}}} \right)^{1/\gamma}. \tag{A3}$$
   This is exactly (24) of the main text, and it defines time as an emergent quantity that measures the accumulated entropy production of a system. Instead of a clock, the "*flow of time*" is captured by the production of entropy.
3. **Interpretation and Domain of Validity**
   The entropic time expression (A3) is valid under the following conditions:
   - The entropy production follows a power-law scaling, typical of systems with memory or non-Markovian dynamics.
   - The system evolves irreversibly with $\Delta S > 0$, consistent with the second law of thermodynamics.

- The initial state of the system is specified by N_0, capturing the available phase space volume or microstate multiplicity at $\tau = 0$.

The appearance of q connects this formulation to Tsallis statistics, widely used to describe gravitational systems, critical phenomena, and complex networks. The generalized Pesin identity is not a purely dynamical law but rather a phenomenological relation inferred from non-equilibrium statistical behavior.

4. **Dimensional Consistency: λ as a generalized entropy production rate**
We consider the entropic-time relation proposed in this work:

$$\tau = \left(\frac{\Delta S \cdot N_0^{\frac{1-q}{q}}}{\lambda} \right)^{1/\gamma}, \tag{A4}$$

where ΔS is the entropy change, N_0 is the initial number of accessible states (dimensionless), q is the entropic index, λ is a scaling parameter, and γ is a dimensionless exponent.
To ensure dimensional consistency, we analyze the units of each term. Entropy ΔS has dimensions of energy per temperature:

$$[\Delta S] = \mathrm{J/K} = \frac{\mathrm{ML}^2}{\mathrm{T}^2\mathrm{K}}. \tag{A5}$$

Since N_0 is dimensionless and q, γ are pure numbers, the units of time arise solely from the balance between ΔS and λ. To ensure that the expression yields a time dimension on the left-hand side, we require:

$$[\lambda] = [\Delta S] \cdot [\tau]^{-\gamma} = \frac{\mathrm{ML}^2}{\mathrm{T}^{2+\gamma}\mathrm{K}}. \tag{A6}$$

This implies that λ can be interpreted as a generalized entropy production rate, scaled appropriately with time. Therefore, the proposed entropic time relation is dimensionally consistent, provided λ carries the required composite units.
To better illustrate the functional behavior of (24) and its implications across cosmological epochs, we now evaluate the model numerically using physically motivated parameters. The resulting plots offer a visual representation of entropic time and its alignment with known stages in cosmic evolution.
Thus, the entropic time relation is dimensionally consistent provided that λ absorbs the necessary units of entropy production rate per unit τ^γ.

5. **Conclusion**
We have shown that under the assumption of generalized entropy production following Pesin-like deformed scaling, it is possible to define an emergent time variable $t(\Delta S)$ based on entropy accumulation. This framework replaces coordinate time as a primitive variable with a thermodynamic evolution parameter, and aligns conceptually with the thermal time hypothesis and informational geometry approaches to cosmology.

Appendix B: Unified Mechanisms for Entropic Time Emergence

1. **Timeless Quantum Gravity and the Problem of Time**

In canonical quantum cosmology, time does not appear as a fundamental parameter. The Wheeler– DeWitt equation governs the evolution of the universe's wavefunction:

$$\hat{H}\Psi[\mathbf{h}_{ij}, \phi] = 0, \tag{B1}$$

where $\mathbf{h}_{ij}$ is the spatial 3-metric, ϕ represents matter fields, and $\hat{H}$ is the Hamiltonian constraint. This equation is timeless, there is no coordinate t. The universe's state appears "frozen", leading to the well-known "problem of time" in quantum gravity.
However, since we observe a universe with temporal evolution, it is widely conjectured that time must be an *emergent, relational* property. Several mechanisms have been proposed to reconstruct temporal order from within the system's internal structure. We explore three such mechanisms that converge in support of an entropy-based time framework.

2. **Mechanism I: Entanglement Entropy as a Temporal Arrow**

   a. Monotonic Entanglement Growth
   Consider a bipartite quantum system $|\Psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$. The entanglement entropy is given by:

$$S_{\text{ent}}(A) = -\text{Tr}\left(\rho_A \log \rho_A\right), \quad \rho_A = \text{Tr}_B |\Psi\rangle\langle\Psi|. \tag{B2}$$

For a wide class of generic Hamiltonians and initial separable states, the entanglement entropy grows monotonically over time, defining an internal direction of evolution.

Theorem: Entanglement Monotonicity

Statement Let $\mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space, and consider a closed quantum system evolving unitarily under a generic entangling Hamiltonian H. If the system starts in a separable pure state, then the entanglement entropy of subsystem A,

$$S_{\text{ent}}(\tau) = -\text{Tr}(\rho_A(\tau) \log \rho_A(\tau)),$$

increases monotonically in time τ until it saturates at thermal equilibrium.

Sketch of Proof

1. **Initial separable state.** The total system begins in a product state:

$$|\Psi(0)\rangle = |\psi_A\rangle \otimes |\psi_B\rangle,$$

so that the initial entanglement entropy is zero: $S_{\text{ent}}(0) = 0$.
2. **Unitary evolution.** The system evolves according to:

$$|\Psi(\tau)\rangle = e^{-iH\tau/\hbar}|\Psi(0)\rangle,$$

where H is a generic Hamiltonian acting non-trivially on both subsystems.
3. **Entanglement growth.** For typical interacting Hamiltonians, the unitary evolution generates increasing entanglement between A and B. Thus, for $\tau_1 < \tau_2$,

$$S_{\text{ent}}(\tau_1) < S_{\text{ent}}(\tau_2),$$

until saturation.
4. **Saturation.** At equilibrium (or thermalization), entanglement entropy reaches its maximum:

$$S_{\text{ent}}^{\max} = \log \min\left(\dim \mathcal{H}_A, \dim \mathcal{H}_B\right),$$

consistent with the Page curve for random pure states [25] and supported by rigorous results on quantum thermalization in closed systems [26].

Temporal Ordering from Entanglement Because $S_{\text{ent}}(\tau)$ is monotonic under unitary evolution for generic H, it defines a relational ordering among quantum states:

$$\tau_1 < \tau_2 \quad \Leftrightarrow \quad S_{\text{ent}}(\tau_1) < S_{\text{ent}}(\tau_2).$$

This allows entropy to serve as an internal clock variable, establishing a coarse-grained direction of time within a closed, timeless quantum system.

Physical Interpretation The theorem suggests that the entanglement entropy of subsystems can act as a reliable and universal parameter of progression, even in the absence of external coordinate time, thereby providing the basis for emergent temporal structure from purely quantum correlations.

3. **Mechanism II: Thermal Time from Modular Flow**
In the conventional formulation of quantum mechanics, time is an external parameter driving evolution. However, in generally covariant systems such as gravity, no preferred time coordinate exists. The thermal time hypothesis, introduced by Rovelli [9], proposes that time can instead emerge from the statistical state of a system, especially one in equilibrium.

A. KMS Condition and Modular Flow
A system in thermal equilibrium is described by a Kubo-Martin-Schwinger (KMS) condition [15], which characterizes thermal equilibrium states in algebraic quantum statistical mechanics.

$$\rho_\beta = \frac{e^{-\beta H}}{Z}, \qquad Z = \text{Tr}(e^{-\beta H}), \qquad \beta = \frac{1}{k_B T}, \tag{B3}$$

where H is the Hamiltonian and β is the inverse temperature.
One can define the *modular Hamiltonian* associated with the state:

$$K = -\log \rho_\beta = \beta H + \log Z. \tag{B4}$$

This modular Hamiltonian generates a flow on the space of observables via:

$$\mathcal{O}(\tau_{\text{th}}) = e^{i\tau_{\text{th}} K} \mathcal{O}\, e^{-i\tau_{\text{th}} K}, \tag{B5}$$

where τ_{th} is known as *thermal time*. This flow defines an intrinsic evolution, not with respect to external clocks, but as a property of the statistical state itself.
The modular Hamiltonian not only generates internal time flow but also acquires geometrical meaning in quantum field theory near horizons. For instance, in Rindler space, the modular flow corresponds to Lorentz boosts, reproducing the thermal spectrum of the Unruh effect [15]. Lashkari et al. (2014) [27] further connect modular Hamiltonians to local energy densities in curved spacetime, reinforcing the idea that thermal time and entropic evolution are linked to causal structure.
B. Entropy as a Clock Variable
In this context, a subsystem whose entropy increases monotonically under modular flow can be used as a physical clock:

$$\frac{dS_{\text{clock}}}{d\tau_{\text{th}}} > 0. \tag{B6}$$

This condition defines an emergent direction of time from thermodynamic irreversibility. Entropy thus plays a dual role: it quantifies coarse-graining and simultaneously parametrizes the arrow of time.

4. **Mechanism III: Page–Wootters Correlational Time**
While the thermal time hypothesis relies on equilibrium states, a complementary picture arises from the Page– Wootters mechanism [7], which treats the universe as a closed, static quantum system.

A. Timeless Global State
In this framework, the total state of the universe, including both the physical system and a quantum clock, is globally static:

$$|\Psi_{\text{total}}\rangle = \sum_t |\psi(t)\rangle_{\text{sys}} \otimes |t\rangle_{\text{clock}}, \tag{B7}$$

and satisfies the global constraint:

$$(H_{\text{sys}} + H_{\text{clock}})|\Psi_{\text{total}}\rangle = 0. \tag{B8}$$

Although the full system does not evolve, correlations between the system and clock subsystems can encode effective dynamics.
B. Conditional Evolution and Emergent Schrödinger Equation
By projecting onto a definite clock state $|t\rangle$, the reduced system state is:

$$|\psi(t)\rangle = \langle t|\Psi_{\text{total}}\rangle, \tag{B9}$$

which satisfies the standard Schrödinger equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H_{\text{sys}}|\psi(t)\rangle. \tag{B10}$$

Thus, time emerges not from absolute evolution, but from conditional correlations between entangled subsystems.
C. Entropy as a Clock Basis
One can extend this idea by choosing the clock observable to be entropy:

$$\hat{S}|S\rangle = S|S\rangle, \tag{B11}$$

and define conditional states:

$$|\psi(S)\rangle = \langle S|\Psi_{\text{total}}\rangle. \tag{B12}$$

If the entropy of the clock subsystem increases monotonically, this provides a natural ordering of quantum states, again realizing time as a relational, entropic variable.
This construction complements the modular thermal flow in the previous section. In both cases, entropy functions as an internal clock, grounding the concept of time in informational and relational properties of physical states.

5. **Unified Semiclassical Parametrization: Derivation of Entropic Time**
In the semiclassical limit, the above mechanisms converge and allow construction of a coarse-grained entropy-based time, see also Ref. [18]. Assume:

- Total accessible phase space volume grows as $\Gamma(\tau) = \Gamma_0 + \alpha \left(\frac{\tau}{\tau_0}\right)^{\gamma}$
- Non-extensive statistics apply: $N(\Gamma) \sim \Gamma^q$

Then, entropy becomes:

$$\Delta S = k_B q \alpha \Gamma_0^{-1} \left(\frac{\tau}{\tau_0} \right)^{\gamma}. \tag{B13}$$

Define the constant:

$$\lambda = \frac{k_B q \alpha}{\Gamma_0 \tau_0^{\gamma}}, \tag{B14}$$

so that:

$$\Delta S = \lambda \tau^{\gamma} \quad \Rightarrow \quad \tau = \left(\frac{\Delta S}{\lambda} \right)^{1/\gamma}. \tag{B15}$$

Including the normalization for non-extensive entropy:

$$\tau = \left(\frac{\Delta S \cdot N_0^{\frac{1-q}{q}}}{\lambda} \right)^{1/\gamma}. \tag{B16}$$

This is the same entropic time parametrization presented in (24) of the main text. Here,

- N_0: Initial number of accessible microstates, set by early-universe entropy.
- q: Degree of non-extensivity due to long-range correlations; relates to effective dimensionality.
- γ: Dominant growth exponent; reflects whether evolution is entanglement-driven, thermal, or classical.
- λ: Effective entropy production rate constant.

6. **Regime of Validity**
   The entropic time parametrization is valid when:

   - Quantum coherence is suppressed (semiclassical limit),
   - Local thermal equilibrium is established,
   - Microscopic fluctuations are coarse-grained,
   - Entropy increases monotonically.

   This unifies fundamental mechanisms of time emergence with practical parametrization, grounded in informational and thermodynamic foundations.

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Declaration of generative AI and AI-assisted technologies in the writing process During the preparation of this work the author(s) used ChatGPT in order to improve the english. After using this tool/service, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the publication. AI was used as a consultation tool for drafting and conceptualizing the final content.

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