Thermodynamic phase transition based on the nonsingular temperature*

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The Hawking temperature for the Schwarzschild black hole is divergent when the mass of the black hole vanishes; however, the corresponding geometry becomes the Minkowski spacetime whose intrinsic temperature is zero. In connection with this issue, we construct a nonsingular temperature which follows the Hawking temperature for the large black hole, while it vanishes when the black hole is completely evaporated. For the thermodynamic significances of this modified temperature, we calculate thermodynamic quantities and study phase transitions. It turns out that even the small black hole can be stable below a certain temperature, and the hot flat space is always metastable so that it decays into the stable small black hole or the stable large black hole.

Hawking has shown that there is radiation from the black hole through the analysis for the origin of the entropy from the point of view of quantum field theory.[1] The Hawking temperature could be defined generically as $T_{\rm H} = \hbar\kappa_H/(2\pi)$, where κ_H is the surface gravity at the horizon. However, it shows that the temperature is proportional to the inverse of the mass, and it is divergent when the mass of the black hole vanishes, although the black hole disappears and its metric becomes the Minkowski spacetime. In this work, we would like to present a nonsingular temperature without resort to the cutoff in the UV region. And then, in order for studying phase transition based on the newly defined temperature, the free energies will be considered for the hot flat space, the small black hole, and the large black hole, respectively. Then, we find a Hawking-Page-type phase transition[2–4] between the small black hole and the large black hole.

Let us assume that the Hawking temperature can be modified in such a way that the temperature of the black hole vanishes when the mass of the black hole goes to zero while it follows the behavior of the well-known Hawking temperature for the large black hole.

$$T = \frac{1}{8\pi GM} \frac{\sum_{i=0}^{n} a_i \left(\frac{M}{M_{\rm P}}\right)^{1+\alpha_i}}{\sum_{i=0}^{n} b_i \left(\frac{M}{M_{\rm P}}\right)^{1+\beta_i} + C}, \tag{1}$$

where $T \to 0$ for $M \to 0$ and $T \to 1/(8\pi GM)$ for $M \to \infty$, and α_i, β_i, and C are positive constants with $\alpha_n = \beta_n$. Additionally, $\alpha_i < \alpha_j$, $\beta_i < \beta_j$ for $i < j$, and $a_n = b_n$.

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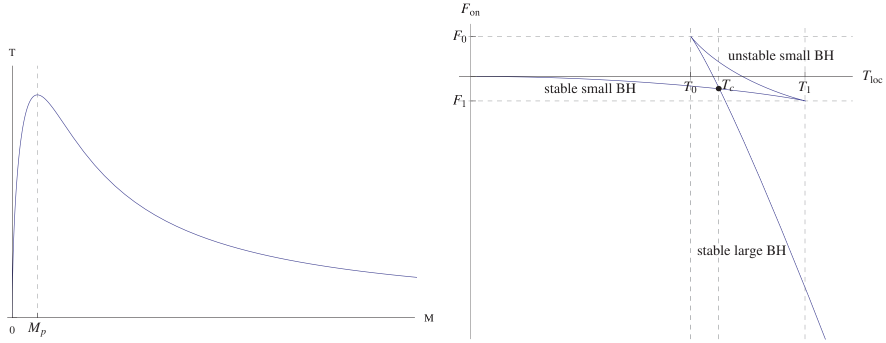

(a) The modified temperature

(b) Free energy based on the modified temperature

Fig. 1. The constants of both graphs are chosen as $G = 1$, $\alpha = 1/2$, and, in Fig. 1(b), the cavaty size is set as $r = 10$. The temperature in Fig. 1(a) follows the Hawking temperature asymptotically and vanishes at $M \to 0$. The maximum temperature appears at the Planck mass. The phase transition appears between the hot flat space and the stable small black hole, and it happens between the stable small black hole and the stable large black hole in Fig. 1(b).

Then, the entropy calculated from the first law of thermodynamics still respects the Bekenstein-Hawking entropy for the large black hole as $S \sim 4\pi G M^2$. On the other hand, the entropy for the small mass can be calculated as

$$S = \frac{8\pi G M_{\rm P} C}{a_0} \int \left(\frac{M_{\rm P}}{M}\right)^{\alpha_0} dM \sim \begin{cases} -\frac{1}{M^{\alpha_0 - 1}} & \text{for } \alpha_0 > 1, \\ \ln M & \text{for } \alpha_0 = 1, \\ M^{1-\alpha_0} & \text{for } 0 < \alpha_0 < 1, \end{cases} \tag{2}$$

where we neglected the subleading terms. Note that the entropy is negative divergent for $\alpha_0 \geq 1$, and it vanishes for $0 < \alpha_0 < 1$ when $M \to 0$. As a result, we obtain the additional condition of $0 < \alpha_0 < 1$ in order for the positive entropy.

The most simple form of the modified temperature (1) without loss of generality corresponds to $n = 0$, which is written as

$$T = \frac{1}{8\pi G M}\left[1 + \frac{1}{\alpha}\left(\frac{M_{\rm P}}{M}\right)^{1+\alpha}\right]^{-1}, \tag{3}$$

where, all constraints are simplified as $0 < \alpha < 1$. The behavior of the temperature (3) is illustrated in Fig. 1(a). The entropy corresponding to the modified temperature (3) is calculated as

$$S = 4\pi G M^2 + \frac{8\pi}{\alpha(1-\alpha)}\left(\frac{M}{M_{\rm P}}\right)^{1-\alpha}. \tag{4}$$

As a result, the temperature and entropy vanish at the end state of the black hole.

In connection with the modified temperature (3), we are going to investigate thermodynamic phase transitions. Let us consider the cavity as a boundary with a radius r to study quasilocal thermodynamics along the line of the procedure.[4] Then, the local temperature measured at the boundary is given as

$$T_{\rm loc} = \frac{1}{8\pi GM\sqrt{1-2GM/r}}\left[1+\frac{1}{\alpha}\left(\frac{M_{\rm P}}{M}\right)^{1+\alpha}\right]^{-1}. \quad (5)$$

The free energy of the black hole is calculated in order to study phase transition between the black holes and the hot flat space as,

$$F_{\rm on}^{\rm bh} = \frac{r}{G} - \frac{r}{G}\sqrt{1-\frac{2GM}{r}} - \frac{4\pi GM^2 + \frac{8\pi GM_{\rm P}^2}{\alpha(1-\alpha)}\left(\frac{M}{M_{\rm P}}\right)^{1-\alpha}}{8\pi GM\sqrt{1-\frac{2GM}{r}}\left(1+\frac{1}{\alpha}\left(\frac{M_{\rm P}}{M}\right)^{1+\alpha}\right)}. \quad (6)$$

Note that for the limit of $M_{\rm P}/M \to 0$, Eq. (6) is reduced to the well-known free energy of the black hole in the conventional thermodynamics of the Schwarzschild black hole.[4]

As seen in Fig. 1, it turns out that there is a single state of the stable small black hole for $T < T_0$ or the stable large black hole for $T > T_1$. For $T_0 < T < T_1$, there are three black hole states which consist of the two small black holes and one large black hole. Apart from the existence of the stable small black hole, the most interesting thing to be distinguished from the standard thermodynamics is that the flat space is no longer a stable state thermodynamically in any temperature since it should always decay into the stable small black hole or the stable large black hole, so that the final state becomes a black hole state.

Finally, the method for the present regular temperature requires an explanation about the limitations of the present approach since we assumed a certain modification of the Hawking temperature as a function of the black hole mass but did not discuss the origin of such a modification. First of all, the temperature (3) derived from the polynomial expansion of the black hole mass is, indeed, not unique even in spite of the plausibility of reproducing the conventional Hawking temperature; for example, another type of temperature such as $T = 1/(8\pi GM)(1 - e^{-k(M/M_p)^{1+\alpha}})$, where k is an arbitrary positive constant also satisfies the two boundary conditions mentioned. To fix the physically meaningful temperature uniquely and figure out what happens at the end state of evaporation of the black hole, the complete theory of quantum gravity covering the trans-Planckian regime should be defined. The second limitation of our approach is that we employed the classical metric for the local Tolman temperature as seen from Eq. (5), which is a temporary expedient. In particular, one can expect that such a modified temperature (3) comes from a change in the spacetime geometry, so that the local temperature (5) changes accordingly. Note that modifications in the geometry could have a nontrivial effect on our analysis and an impact on the physics of small black holes. Using the one-to-one correspondence between the GUP and the GUP temperature, a corresponding modified uncertainty relation, which is written as $\Delta x\Delta p + (2\ell_p/\alpha)(2\ell_p/\Delta x)^\alpha \Delta p \ge 1$, can also be derived straightforwardly from Eq. (3). This modified uncertainty relation will modify the classical geometry so that the local temperature will be changed somehow near the horizon and, consequently, the thermodynamic behaviors of small black holes may be different from the the present results. This deserves further study, which we hope will appear in the future.

References

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