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Shape Dynamics of the $T\bar{T}$ Deformation

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Abstract: I will show how the flow triggered by deforming two-dimensional conformal field theories on a torus by the $T\bar{T}$ operator is identical to the evolution generated by the (radial) quantum Shape Hamiltonian in 2 + 1 dimensions. I will discuss how the gauge invariances of the Shape Dynamics, i.e., volume-preserving conformal invariance and diffeomorphism invariance along slices of constant radius are realized as Ward identities of the deformed quantum field theory. I will also comment about the relationship between the reduction to shape space on the gravity side and the solvability of the irrelevant operator deformation of the conformal field theory

Keywords: gauge-gravity duality; renormalization group; quantum gravity in low dimensions

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1. Introduction

Shape Dynamics [1] trades the gauge invariance of re-foliation invariance of general relativity for volume preserving Weyl invariance of space. In effect, it is a theory of the dynamics of the conformal geometry of space. Local classical gravitational physics described by shape dynamics is identical to what general relativity describes and in this sense the two theories are dual to each other. There are significant and interesting global differences between the two theories, however, which is most clearly evident in the novel physics discovered in the shape dynamical approach to modeling the universe's origins [2].

The expectation is that the two theories arise as classical limits of different quantum theories. Specifically, the criterion for physical states in canonical quantum general relativity is that said states satisfy the Wheeler de Witt equation in addition to the spatial momentum constraint. In Shape Dynamics, the physical states satisfy the constraint that encodes local volume preserving Weyl invariance in addition to the spatial diffeomorphism constraint. Additionally, there is a global reparameterization constraint physical states in shape dynamics must satisfy. The classical equivalence, from this perspective is the local matching matching of a certain special class of states of either theory that exhibit semi-classical behavior. In this article, I will report on a peculiar connection between quantum shape dynamics in 2 + 1 dimensions and irrelevant operator deformations of two-dimensional conformal field theories. The connection to quantum general relativity in fact arises from the classical equivalence of shape dynamics with general relativity.

I will begin by describing the flow on the space of two-dimensional quantum field theories triggered by deforming a conformal field theory by the $T\bar{T}$ operator [3,4]. Specifically, I will describe the flow of the energy levels of the theory on a cylinder under the deformation. Then, I will show how the resulting equation defines a flow equation for the partition sum or the torus partition function. I will then relate this flow equation to the global constraint equation that the physical wavefunction of quantum shape dynamics has to satisfy. This is a refinement to the original holographic correspondance between 3D general relativity in a space of negative cosmological constant and the $T\bar{T}$ deformation of the dual conformal field theory that now inhabits a finite radial cutoff surface put forward in [5], which was a refinement of the observation made earlier in [6]. It does sharpen the correspondence proposed in [7] relating the renormalization group of quantum field theories in one lower dimension and shape dynamics.

The connection to three-dimensional gravity is just one facet of the $T\bar{T}$ deformation. Others include (but are not limited to) the relationship between it and two-dimensional quantum field theories coupled to random geometry [8], Jackiw–Teitelboim gravity [9,10], ghost free massive gravity [11], and non critical string theory [12]. A more detailed review of the literature around the $T\bar{T}$ deformation and its cousins can be found in [13].

Summary of Main Results

The key point of this article is the following: when we identify shape dynamics in 2 + 1 dimensions as dual to the $T\bar{T}$ deformation of a CFT, we can explain the latter's solvability. In other words, our ability to write down simple differential equations for the energy levels and the torus partition function and solve them exactly can be seen as arising from the imposition of the volume preserving conformal constraint of the dual quantum 2 + 1 shape dynamics theory. This constraint freezes the inhomogenous modes of the metric in the partition function and renders it a function only of the zero modes of the metric components. This condition *cannot* be imposed as a constraint when we do the Dirac quantization of general relativity fixed in Constant Mean Curvature gauge.

1.1. Radial Quantization in Conformal Field Theory

Let us consider a Euclidean Conformal Field Theory on a plane $\mathbb{R}^2$, on which we consider the following coordinates:

$$ds^2 = dR^2 + R^2 d\theta^2. \quad (1)$$

This line element describes the foliation of the plane by circles of radii R. The dilatation operator $\hat{D}$ generates radial development $R \to R + \delta R$ that expands the circles uniformly. We will be interested in the eigenstates of this operator. Much like how eigenstates of the Hamiltonian are defined on surfaces of constand time, the eigenstates of the dilatation operator are defined on the circles we are foliating the plane by:

$$\hat{D}|\Delta\rangle = i\Delta|\Delta\rangle. \quad (2)$$

The main point idea behind the state operator map is to take the analogy between the dilatation operator and a Hamiltonian very literally.

In particular, note that the plane can be related to the cylinder through the following Weyl transformation:

$$dr^2 + r^2 d\theta^2 = e^{2\tau}(d\tau^2 + d\theta^2) \quad (3)$$

where $\tau = \log r$. Another consequence of conformal symmetry is that the dynamics of the CFT are not sensitive to the difference between (3) and (1). Note that, under the coordinate transformation we consider, the radial development on the plane maps to the development along the height of the cylinder, which we interpret as time—in other words, it describes the evolution of the state of a CFT on a spatial circle. Therefore, the Hamiltonian for the conformal field theory quantized on a circle ($\times$ time) is the dilatation operator. In other words, the evolution operator on the cylinder is given by

$$\hat{U} = e^{i\hat{H}\tau} = e^{\hat{D}\tau}. \tag{4}$$

The eigenstates we identified before evolve according to

$$\hat{U}|\Delta\rangle = e^{-\tau\Delta}|\Delta\rangle = R^{-\Delta}|\Delta\rangle. \tag{5}$$

On the plane, we can define the vacuum state that we define on a circle of vanishing radius—i.e., the origin. The other states of the theory are defined by acting with operators $\mathcal{O}_\delta$ on the vacuum [14]:

$$\mathcal{O}_\Delta(0)|0\rangle = |\Delta\rangle. \tag{6}$$

We can then associate this state to an eigenstate of the generator of time translations along the cylinder. This is the state-operator map.

In a unitary conformal field theory, the spectrum of dilatation eigenvalues, or conformal weights, is given by discrete, positive real numbers Δ_n. The formula for the energy associated with the nth weight is given by:

$$E_n = \frac{1}{R}\left(\Delta_n - \frac{c}{12}\right). \tag{7}$$

The constant c is known as the central charge of the confromal field theory, and this negative contribution to the energy is the Casimir energy born through putting the conformal field theory on a circle. Note that the Casimir energy accounts entirely for the energy of the ground state:

$$E_o = -\frac{c}{12R}. \tag{8}$$

In the following section, we will study what happens to the energy spectrum of the CFT on a cylinder under the $T\bar{T}$ deformation.

1.2. The $T\bar{T}$ Operator

The $T\bar{T}$ operator is the following composite operator formed from the energy momentum tensor $T^{\mu\nu}$:

$$T\bar{T}(x) := \frac{1}{8}\left(g_{\mu(\alpha}g_{\beta)\nu}T^{\mu\nu}(x)T^{\alpha\beta}(x) - (T^\sigma_\sigma(x))^2\right). \tag{9}$$

This operator evaluated in a conformal field theory is the product of the holomorphic and anti-holomorphic stress tensor components $T_{zz} = T, T_{\bar{z}\bar{z}} = \bar{T}$, and thus the name. I will continue to use $T\bar{T}$ as a stand in for the right-hand side of (9) throughout this article.

Note that this operator is irrelevant in the renormalization group sense: the energy momentum tensor has scaling dimension 2. On account of being a conserved current, its scaling dimension is protected from quantum corrections. Furthermore, Zamolodchkov showed that the OPE between the stress tensor components in (9) defines a local operator with definite scaling dimension 4. However, we can in fact follow the energy levels of the CFT under the flow triggered by this irrelevant operator. This is the sense in which it is a "solvable" irrelevant operator deformation.

The main property we will exploit is that the expectation value of the $T\bar{T}$ operator on spaces with translation invariance factorized when computed in translation invariant states. The proof of this property was first found in [15]. In other words:

$$\langle n|T\bar{T}|n\rangle = \frac{1}{8}\Big(\langle n|T^{\mu\nu}|n\rangle\langle n|T_{\mu\nu}|n\rangle - (\langle n|T^{\mu}_{\mu}|n\rangle)^2\Big) \tag{10}$$

Here, $|n\rangle$ denotes translation invariant, which on the cylinder are just the energy eigenstates. The reason why the above factorization formula is a simplification is that the expectation value of the $T\bar{T}$ operator can now be computed from just knowing the diagonal matrix elements of the stress tensor $T^{\mu\nu}$ in energy eigenstates.

1.3. The Deformed Energy Spectrum

To see why being able to calculate the expectation value of $T\bar{T}$ matters, note that the definition of the $T\bar{T}$ flow is that one parameter family of two-dimensional field theories for which the following equation holds:

$$\partial_\mu \log Z = \langle T\bar{T}(x)\rangle. \tag{11}$$

Here, μ is the deformation parameter which carries dimensions of length squared, and $\log Z$ denotes the partition function of the $T\bar{T}$ deformed theory. The boundary condition for this differential equation is the specification of the undeformed theory:

$$\log Z(\mu = 0) = \log Z_{Seed} \tag{12}$$

In all that is to follow, I will focus on the case where $\log Z_{Seed} = \log Z_{CFT}$.

It follows from (11) that the energy levels of the deformed theory on the cylinder satisfy:

$$\begin{aligned}\partial_\mu E_n(\mu, R) &= \langle n|T\bar{T}|n\rangle \\ &= \langle n|T^{\mu\nu}|n\rangle\langle n|T_{\mu\nu}|n\rangle - (\langle n|T^{\mu}_{\mu}|n\rangle)^2.\end{aligned} \tag{13}$$

The matrix elements of various stress tensor components on the cylinder are given in terms of the energy $E_n = -R\langle n|T^{00}|n\rangle$, momentum $j_n = -i\langle n|T^{01}|n\rangle$ and pressure $\partial_R E_n = \langle n|T_{11}|n\rangle$. Thus, we can readily write the right-hand side of the above equation entirely in terms of these quantities:

$$\partial_\mu E_n = E_n \partial_R E_n + \frac{1}{R^3} j_n^2. \tag{14}$$

Note that the deformation preserves Poincare invariance on flat backgrounds, but, more generally, the covariant conservation of the energy momentum tensor. When we look at the theory defined on the cylinder, this implies that it is only the energy E_n that gains μ dependence while the momenta j_n will not flow under the deformation.

Equation (14) has the structure of the Burgers' equation of hydrodynamics without the viscosity term (see for instance [16]). This was noticed in [3,4].

Thus far, we have not specified what the seed theory is that we are deforming. I will now specialize to the case where we deform a conformal field theory, whose spectrum is given by:

$$E_n(\mu = 0, R) = \frac{1}{R}\Big(\Delta_n - \frac{c}{12}\Big) \tag{15}$$

and $j_n = \frac{1}{R} J_n$, where J_n is an integer.

The solution to the Burgers' equation reads:

$$E_n(R, \mu) = \frac{R}{2\mu}\left(-1 \pm \sqrt{1 + \frac{4\mu}{R^2}\Big(\Delta_n - \frac{c}{12}\Big) + \frac{4\mu^2 J_n^2}{R^4}}\right) \tag{16}$$

The two signs correspond to whether we choose to deform the theory in the positive or negative μ direction. Note that this formula depends on μ only though the dimensionless ratio $\lambda = \mu/R^2$.

Let us take the example of a collection of D free massless bosons in two dimensions as the seed CFT. The spectrum in such a theory is given by:

$$E_n^{CFT} = \frac{1}{R}\left(n + \bar{n} - \frac{D}{12}\right), \ \ J_n = n - \bar{n}, \ \ n, \bar{n} \in \mathbb{Z}_+. \tag{17}$$

The deformed spectrum is now given by:

$$E_n(\mu, R) = \frac{R}{2\mu}\left(-1 + \sqrt{1 + \frac{4\mu}{R^2}\left(n + \bar{n} - \frac{D}{12}\right) + \frac{4\mu^2}{R^4}(n - \bar{n})^2}\right). \tag{18}$$

This is the energy spectrum that arises from the Nambu Goto action for a string propagating in dimension $D - 2$ when taken in static gauge [17]. We see here that the $T\bar{T}$ deformation bridges the simple quantum field theory of D free bosons to the theory of the string world-sheet.

In fact, we can cast the entire exercise in terms of dimensionless quantities. Let me define:

$$\mathcal{E}_n\left(\lambda = \frac{\mu}{R^2}\right) = RE_n(\mu, R). \tag{19}$$

Then, we see that the Burgers' equation implies that $\mathcal{E}_n$ satisfies:

$$\partial_\lambda \mathcal{E}_n = -2\lambda \mathcal{E}_n \partial_\lambda \mathcal{E}_n - \mathcal{E}_n^2 + J_n^2. \tag{20}$$

1.4. Flow Equation for the Partition Function

The partition sum is the following quantity:

$$Z(\beta, \omega) = \mathrm{Tr} \exp(-\beta H + i\omega J) \tag{21}$$

Here, β is the inverse temperature and ω is the chemical potential conjugate to the angular momentum. We interpret this quantity as describing a system living on a torus whose modulus is given by $\tau = \frac{1}{R}(\omega + i\beta)$. In other words, the real and imaginary parts of the torus modulus are given by $\tau_1 = \omega/R, \tau_2 = \beta/R$. The partition function can therefore be written as

$$Z(\tau_1, \tau_2) = \mathrm{Tr} \exp\left(- \tau_2 RH + i\tau_1 RJ\right) = \sum_n e^{-\tau_2 \mathcal{E}_n + i\tau_1 J_n} \tag{22}$$

Now, the idea is to obtain the flow equation that determines the λ dependence of the partition function

$$Z(\tau_1, \tau_2, \lambda) = \sum_n e^{-\tau_2 \mathcal{E}_n(\lambda) + i\tau_1 J_n} \tag{23}$$

with initial condition $Z(\lambda = 0, \tau_1, \tau_2) = Z_{CFT}(\tau_1, \tau_2)$ knowing the Burgers' Equation (20). This is a straightforward algebraic exercise, which results in the equation:

$$\left(-\frac{\tau_2}{2}\left(\partial_{\tau_1}^2 + \partial_{\tau_2}^2\right) + \left(\frac{1}{2} + \frac{\lambda}{\tau_2}\right)\partial_\lambda - \lambda \partial_\lambda \partial_{\tau_2}\right) Z(\lambda, \tau_1, \tau_2) = 0. \tag{24}$$

This equation provides the starting point for the rest of the analysis to follow in this article. It was first written down in [18].

1.5. Introducing the Volume

The line element on the torus on which the two-dimensional $T\bar{T}$ deformed CFT lives is given by:

$$ds^2 = g^{\mathbb{T}^2}_{\mu\nu} dx^\mu dx^\nu = |dx + \tau dy|^2 \tag{25}$$

The coordinates x, y are periodic with period $2\pi R$. The volume of the torus is given by

$$V = \int d^2x \sqrt{g^{\mathbb{T}^2}} = 4\pi^2 R^2 \tau_2. \tag{26}$$

Let us now measure the radius in units of the deformation parameter μ. In these terms, the volume is given by:

$$V = 4\pi^2 \mu \frac{\tau_2}{\lambda}. \tag{27}$$

We can now trade the λ dependence of the partition function for V dependence and re-write the flow equation:

$$-\frac{\tau_2^2}{2}\left(\partial^2_{\tau_1} + \partial^2_{\tau_2}\right) Z(V, \tau_1, \tau_2) - \frac{V^2}{8\pi^2\mu}\left(\partial_V - 4\pi^2\mu\partial^2_V\right) Z(V, \tau_1, \tau_2) = 0. \tag{28}$$

The introduction of this variable can be seen merely as a means to decouple λ and τ_2 in (24). Alternatively, we can think of the deformation either in terms of varying μ and keeping the volume of the cylinder fixed, or alternatively, by keeping μ fixed and varying the volume. To vary the volume and ask how the partition function responds is a more general means to characterize the RG flow. In fact, (28) can be seen as the Callan–Symanzik equation for partition function (i.e., the 0 point function). To see this, we rewrite (28) as:

$$\int d^2x \sqrt{g} \langle T^\mu_\mu \rangle = \frac{1}{Z} V \partial_V Z = \frac{4\pi^2\mu}{Z}\left(\frac{\tau_2^2}{V}\left(\partial^2_{\tau_1} + \partial^2_{\tau_2}\right) Z - \partial^2_V Z\right). \tag{29}$$

In a pure conformal field theory, the trace of the energy momentum tensor on curved spaces is given by the conformal anomaly. On a torus, this would be zero. If we deform the CFT, then, in addition to the conformal anomaly, the expectation value of the deforming operator itself will contribute to $\langle T^\mu_\mu \rangle$. In other words, we have the general equation:

$$\int d^2x \langle T^\mu_\mu \rangle = \int d^2x \beta^I \langle O_I \rangle + \mathcal{A} \tag{30}$$

Here, β^I is the beta function for the coupling J^I of the deforming operator O_I, and to leading order this is proportional to the source itself $\beta^I = (d - \Delta_O) J^I + \cdots$. Here, δ_O is the scaling dimension of the deforming operator. Interpreting the Callan–Symanzik equation this way is due to Osborn [19], and the approach is known as the local renormalization group.

This is exactly what happens for the $T\bar{T}$ deformation on the torus:

$$\int d^2x \langle T\bar{T} \rangle = \frac{4\pi^2}{Z}\left(\frac{\tau_2^2}{V}(\partial^2_{\tau_1} + \partial^2_{\tau_2}) Z - \partial^2_V Z\right). \tag{31}$$

In other words, the integrated trace of the energy momentum tensor captures the system's response to changes of scale, and the above equation shows what happens to a conformal field theory when it is deformed by the $T\bar{T}$ operator.

Furthermore, by multiplying Z by a phase, we can define:

$$\psi(V, \tau_1, \tau_2) = e^{-\frac{V}{8\pi^2\mu}} Z(V, \tau_1, \tau_2) \tag{32}$$

which satisfies the equation

$$\tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)\psi(V,\tau_1,\tau_2) - V^2\partial_V^2\psi(V,\tau_1,\tau_2) + \frac{V^2}{64\pi^4\mu^2}\psi(V,\tau_1,\tau_2) = 0. \quad (33)$$

The above change of variables was first worked out in [20]. The expression (33) is the quantization of the global constraint equation of shape dynamics in 2 + 1 dimensions. I will elaborate on this point in the following section.

2. Shape Dynamics in 2 + 1 Dimensions

In this section, I will present the theory of shape dynamics in 2 + 1 dimensions following [21]. Specifically, I will focus on the case of Euclidean signature, and take the two geometries to have the topology of the sphere. The phase space is spanned by two sets of conjugate variables, the first pair is the conformal metric

$$\bar{g}_{\mu\nu} = \begin{pmatrix} \frac{1}{\tau_2} & \frac{\tau_1}{\tau_2} \\ \frac{\tau_1}{\tau_2} & \frac{\tau_1^2+\tau_2^2}{\tau_2} \end{pmatrix}, \quad (34)$$

and its conjugate momentum:

$$\sigma^{\mu\nu} = \frac{\sqrt{\bar{g}}}{2V}\begin{pmatrix} (\tau_1^2 - \tau_2^2)p_{\tau_2} - 2\tau_1\tau_2 p_{\tau_1} & p_{\tau_1}\tau_2 - p_{\tau_2}\tau_1 \\ p_{\tau_1}\tau_2 - p_{\tau_2}\tau_1 & p_{\tau_2}. \end{pmatrix} \quad (35)$$

Here, the p_{τ_i} are conjugate to the real and imaginary parts of the torus modulus τ_i.

The second pair of conjugate variables are the volume V and the York time variable $T = \mathrm{tr}K$, where K is the extrinsic curvature.

It helps to introduce the momentum $\pi^{\mu\nu}$ conjugate to the full metric $g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu}$:

$$\pi^{\mu\nu} = e^{-2\phi}\left(\sigma^{\mu\nu} + \frac{1}{2}\mathrm{tr}\pi\bar{g}^{\mu\nu} + \sqrt{\bar{g}}(\bar{\nabla}^\mu Y^\nu + \bar{\nabla}^\nu Y^\mu - \bar{g}^{\mu\nu}\bar{\nabla}_\gamma Y^\gamma)\right) \quad (36)$$

The reason for doing introducing this quantity is to be able to express the following constraint:

$$\mathrm{tr}\pi(x) - \sqrt{g}(x)\frac{\int d^2x\,\mathrm{tr}\pi}{V} = 0. \quad (37)$$

This constraint encodes the invariance of the theory under Weyl transformations that preserve the total volume of the torus $\mathbb{T}^2$. Imposing this constraint is equivalent to imposing the constant mean curvature condition

$$\frac{\mathrm{tr}\pi}{\sqrt{g}} = \mathrm{tr}K = T,\ \partial_\mu T = 0. \quad (38)$$

Additionally, the invariance of the theory under diffeomoprhisms of the torus is captured by the momentum constraint:

$$H_\mu = \nabla_\mu \pi^\mu_\nu = 0 \quad (39)$$

The dynamics of the theory are dictated by the following totally constrained Hamiltonian:

$$H_{tot} = NH_{SD} + C(\rho) + H_i(\xi^i), \quad (40)$$

$$H_{SD} = \frac{\tau_2^2}{2V}\left(p_{\tau_1}^2 + p_{\tau_2}^2\right) + \frac{V}{2}\left(T^2 - 4\Lambda\right), \quad (41)$$

$$H_\mu(\xi^\mu) = \int d^2x\xi^\nu\nabla_\mu\pi^\mu_\nu, \quad (42)$$

$$C(\rho) = \int d^2x\rho(\mathrm{tr}\pi - T\sqrt{g}) = 0. \quad (43)$$

The first constraint generates global radial reparameterizations, the second generates diffeomoprhisms along the torus and the third, as discussed already, expresses the volume preserving conformal invariance of the theory.

2.1. Quantization

Notice that the phase space is really just spanned by the constant variables $(\tau_i, p_{\tau_i}; V, T)$. The statement that the reduced phase space involves only V, i.e., just the zero mode of the conformal factor $\phi(x)$, we have met the requirement of the constraint encoding volume preserving conformal invariance.

The quantization of the theory is straightforward to perform, seeing as how the reduced phase space is finite dimensional. We will focus on the polarization where the wavefunction depends on (V, τ_i), on which the momenta act according to:

$$\hat{T}\psi(V, \tau_1, \tau_2) = -\partial_V \psi(V, \tau_1, \tau_2), \tag{44}$$

$$\hat{p}_{\tau_i}\psi(V, \tau_1, \tau_2) = -\partial_{\tau_i}\psi(V, \tau_1, \tau_2). \tag{45}$$

The lack of is in the above expressions reflect the fact that we are doing radial quantization of the Euclidean theory.

The quantization of the reparameterization constraint is given by:

$$\left(\tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2) - V^2\left(\partial_V^2 - 4\Lambda\right)\right)\psi(V, \tau_1, \tau_2). \tag{46}$$

Notice that this matches (33) if we identify $\Lambda = \frac{1}{256\pi^2\mu^2}$. In addition, note that the ordering of the derivative operators in the above constraint is fixed by picking the ordering that recovers the $T\bar{T}$ flow equation when we make a change of variables from V back to λ. The flow equation's form is entirely fixed by the requirement that the energy levels in the partition sum should satisfy the Burgers' equation whose form in turn follows from the definition of the $T\bar{T}$ operator on the plane, which is unambiguous.

2.2. Constraints and Ward Identities

We see that the flow equation of the torus partition function maps to the global reparameterization constraint on the shape dynamics side. We would also like to get a sense for how the other constraints are represented on the quantum field theory side. The short answer is that they are all manifestations of Ward Identities.

Let us take the diffeomorphism constraint, which when written locally in the quantum theory and in terms of $Z = e^{\frac{V}{8\pi\mu}}\psi$ takes the form:

$$\nabla_\mu \frac{\delta Z}{\delta g^{\mu\nu}} = 0 = \langle \nabla_\mu T^{\mu\nu} \rangle Z = 0. \tag{47}$$

This is nothing but the covariant conservation of the stress tensor on the background specified by the metric $g_{\mu\nu}$. In the case of interest, this statement reduces to the conservation of energy and momentum, i.e.,

$$\partial_{x_o}\mathcal{E}_n = 0 = \partial_{x_o} J_n. \tag{48}$$

The constraint $C(\rho)$ that encodes volume preserving conformal invariance, however, is harder to interpret. In some sense, it is similar to the trace Ward identity one would have in a conformal field theory. Namely, on flat space,

$$\frac{\delta \log Z_{CFT}}{\delta\phi(x)} \propto \langle T_\mu^\mu(x) \rangle_{CFT} = 0. \tag{49}$$

However, in our case, we have all but the zero mode part of this condition holding. In other words, we have

$$\langle T^\mu_\mu(x)\rangle = \frac{\sqrt{g}(x)}{V}\int d^2y\sqrt{g}\langle T^\mu_\mu(y)\rangle. \quad (50)$$

This is a slightly peculiar Ward identity. In correlation functions between the trace of $T^{\mu\nu}$ and other operators, it basically tells us that we should replace every instance of T^μ_μ with its zero mode part, or in momentum space, it is telling us that the only kinds of correlations functions of the trace of the stress tensor that are allowed are the zero momentum ones. Given that the other components of the stress tensor are also spatially constant (i.e., the energy, pressure, and momentum), we see that in fact the only allowable correlation functions of *any* stress tensor components in this theory when taken on $\mathbb{T}^2$ are the ones at zero momentum. This fact is crucial for the solvability of the deformed theory as we saw in the previous section. In the following section, we see why this Ward identity is also what distinguishes quantum shape dynamics and the Dirac quantization of general relativity in the CMC gauge.

3. Discussion: Shape Dynamics vs. General Relativity in CMC Gauge

Thus far, we have discussed the transition from the classical theory to the quantum theory in terms of the quantization of shape dynamics. We could alternatively interpret the system evolving on the $(V, \tau_i; T, p_{\tau_i})$ phase space as the reduced phase space one lands on after fixing CMC gauge in General Relativity and partially solving the constraints. The point made in this section was emphasized previously in Section 4.1 of [21].

However, when we discuss quantization, we see the following difference: the *Dirac* quantization of shape dynamics directly leads to the wavefunction $\psi(V, \tau_i)$ satisfying Equation (46). As discussed at the end of the last section, this is because of the volume preserving conformal constraint, which I will rewrite as:

$$\left(\frac{\delta}{\delta\phi(x)} - \frac{\sqrt{g}}{V}\int d^2y\frac{\delta}{\delta\phi(y)}\right)\psi = 0. \quad (51)$$

This condition implies that the wavefunction ψ depends on ϕ through only its zero mode V, which then immediately simplifies the rest of the analysis, and to re-iterate the point made several times in this article, renders the $T\bar{T}$ deformation solvable.

However, in general relativity, the condition $C(\rho) = 0$ is not a constraint, but a gauge fixing condition. This in turn means that we ought not to impose it as (51), which means that the wavefunction depends a priori on $\phi(x)$, and not just its zero mode V. In what follows, I follow the treatment of Dirac quantization presented in [22]. In particular, this implies that, when we decompose the momentum conjugate to $g_{\mu\nu}$,

$$\pi^{\mu\nu} = e^{-2\phi}\left(\sigma^{\mu\nu} + \frac{1}{2}\mathrm{tr}\pi\bar{g}^{\mu\nu} + \sqrt{\bar{g}}(\bar{\nabla}^\mu Y^\nu + \bar{\nabla}^\nu Y^\mu - \bar{g}^{\mu\nu}\bar{\nabla}_\gamma Y^\gamma)\right). \quad (52)$$

The vector field Y^μ does not decouple and acts on the wave-function as:

$$\bar{Y}_\mu\psi = -\frac{1}{4}\bar{\Delta}^{-1}\left(e^{2\phi}\bar{\nabla}_\mu\left(e^{-2\phi}\frac{\delta}{\delta\phi}\right)\right)\psi = 0. \quad (53)$$

Plugging this back into the Wheeler–de Witt equation leads to non local terms, making it intractable to solve. Thus, the Dirac quantization of GR in CMC gauge does not lead to a quantum gravitational system that can be mapped to the $T\bar{T}$ deformation of a CFT, whereas the Dirac quantization of shape dynamics does.

Furthermore, it was noted in [21] that any solution ψ of the quantum shape dynamics constraint equations satisfies the following property:

$$-V\partial_V\psi_\pm(V, \tau_i) = \pm 2\sqrt{\Lambda}V\psi_\pm(V, \tau_i), \text{ as } V \to \infty. \quad (54)$$

This is a sheer reflection of the fact that, as $V \to \infty$, the deformation parameter $\lambda \to 0$ and

$$Z(\lambda = 0, \tau_i) = Z_{CFT}(\tau_i). \tag{55}$$

Since

$$Z = e^{\pm 2\sqrt{\Lambda} V}\psi,$$

the above condition implies that, as $\lambda \to 0$, the only dependence of $\psi(V, \tau_i)$ on the volume is through the phase $\exp \pm 2\sqrt{\Lambda}V$ and thus follows the scaling property (54). The sign in front of the exponent comes from noting that $\sqrt{\Lambda} \propto \frac{1}{\mu}$ and μ can either be positive or negative. Demanding that V be positive implies that we redefine μ so that it is always positive as well and change the overall sign in front of wherever it appears accordingly.

Thus, we conclude that (54) anticipated that there is a connection between the solution to the quantum SD constraint equations at large volume and the partition function of a CFT.

4. Speculation

We saw how quantum shape dynamics in 2 + 1 dimensions is related by a change of variables to the $T\bar{T}$ deformation of a CFT on a torus. In order to notice this equivalence, we needed to know beforehand what the flow equation was for the torus partition function of a $T\bar{T}$ deformed CFT. We saw that, to derive this equation, we needed to know the Burgers' equation that implicitly required the factorization property of the expectation value of $T\bar{T}$ on the cylinder. On the side, the flow equation was mapped to the residual radial constraint equation, which in principle we can derive on any Riemann surface (provided we know how to solve the Lichnerowicz York equation). However, in the case of Riemann surfaces of higher genus, we cannot utilize the factorization property of the $T\bar{T}$ operator on the cylinder in any useful way to derive a flow equation. As such, we do not know what the flow equation should be for a $T\bar{T}$ deformed CFT on a higher genus Riemann surface at all. This is where Shape Dynamics can help.

If we take to heart the equivalence between $T\bar{T}$ deformed CFTs and Shape dynamics, then we conjecture that the following equivalence holds on any Riemann surface:

$$\psi(V, m_i) = \exp -\frac{V}{8\pi^2\mu} Z(V, m_i), \tag{56}$$

where m_i denotes the moduli of the higher genus Riemann surface; then, we know that the shape dynamics Hamiltonian on the Riemann surface is given by [21]:

$$H_{SD} = -\frac{V}{2}\Big(T^2 - 4\Lambda\Big) - \bar{R} + \frac{1}{V^2}\int d^2x e^{-2\mu}\bar{g}_{\mu\rho}\bar{g}_{\nu\gamma}\sigma^{\mu\nu}(m_i)\sigma^{\rho\gamma}(m_i). \tag{57}$$

Here, the curvature is given by

$$\bar{R} = -8\pi(g-1), \tag{58}$$

and λ solves a modified version of the Lichnerowicz equation. Note that this is still a somewhat implicit definition of the Hamiltonian, and therefore we do not expect to be able to solve the constraint equations as we can in the case of the torus.

Now, if we proceed to quantize this Hamiltonian, we obtain the following equation for $\psi(V, m_i)$:

$$-\left(\frac{V}{2}\partial_V^2 - 4\Lambda\right)\psi(V, m_i) + 8\pi(g-1)\psi(V, m_i) + \frac{1}{V^2}\int d^2x\, e^{-2\lambda}\bar{g}_{\mu\rho}\bar{g}_{\nu\rho}\frac{\delta^2\psi(V, m_i)}{\delta\bar{g}_{\mu\nu}(m_i)\delta\bar{g}_{\rho\sigma}(m_i)} = 0. \tag{59}$$

Then, the equation for the trace of the energy momentum tensor on the $T\bar{T}$ side of the duality is given by:

$$V\partial_V Z(V, m_i) = 4\pi^2\mu\left(-\partial_V^2 Z + \int d^2x e^{-2\lambda}\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}\frac{\delta^2 Z}{\delta\bar{g}_{\mu\nu}(m_i)\delta\bar{g}_{\rho\sigma}(m_i)}\right). \tag{60}$$

Despite the fact that this equation also requires us to solve the Lichnerowicz equation, and as such, it is not solvable in the same way that the torus partition function is, but what we have now that we did not before is the definition of the expectation value of the $T\bar{T}$ operator on a higher genus Riemann surface:

$$\langle T\bar{T}\rangle = \frac{4\pi^2\mu}{Z}\left(-\partial_V^2 Z + \int d^2x\, e^{-2\lambda}\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}\frac{\delta^2 Z}{\delta\bar{g}_{\mu\nu}(m_i)\delta\bar{g}_{\rho\sigma}(m_i)}\right). \tag{61}$$

What we can say is that this operator also depends only on the zero momentum part of the stress tensor two point function. Note that this method of defining the expectation value of $T\bar{T}$ on curved spaces is different from alternatives in the literature like in [23,24].

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