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GROUP REPRESENTATION IN A CONTINUOUS BASIS - AN EXAMPLE*

C. Itzykson†
Stanford Linear Accelerator Center
Stanford University, Stanford, California

*Work supported by the U. S. Atomic Energy Commission

†Permanent address: Service de Physique Theorique, CEN Saclay, BP No. 2, 91, Gif sur Yvette, France.

Given an irreducible unitary representation of a noncompact group, what happens if one tries to diagonalize one of the noncompact generators? We study some aspects of this question on an example.

I. INTRODUCTION

Classical Fourier analysis is the standard example of diagonalization of a noncompact generator (in this case, the generator of translations along the real line). Some interesting properties arise when such an abelian noncompact group is imbedded in a larger structure. This occurs, for instance, when one studies the group $G \equiv SL(2R)$, of two-by-two real unimodular matrices. Let us first recall elementary properties of this group that will be used. An arbitrary element $g \in G$ is of the form $z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a, b, c, d real, $ad - bc = 1$. The Lie algebra of this simple group is realized as traceless real two-by-two matrices, a basis of which is

$$t_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad t_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad r = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

satisfying the commutation rules

$$[r, t_1] = t_2 \qquad [r, t_2] = -t_1 \qquad [t_1, t_2] = -r \tag{2}$$

These generators are such that given a unitary representation of G, their representatives are i times self-adjoint operators. Let us assume that we are given such a representation,and let us denote by T_1, T_2 and R these representatives. Since R generates a compact subgroup, its spectrum, though unbounded, is discrete. From the commutation rule $\left[R, \left(T_1 \pm iT_2\right)\right] = \pm i\left(T_1 \pm iT_2\right)$, one sees that $T_1 \pm iT_2$ play the role of raising and lowering operators. On the other hand, suppose we diagonalize T_1. Its spectrum will be continuous of the form $i\lambda$ (λ real).

The commutation rule $\left[T_1, (R \mp T_2)\right] = \pm (R \mp T_2)$ seems to indicate that acting with $R \mp T_2$ on some "improper states" $|\lambda\rangle$ corresponding to the spectral value λ of $-iT_1$, will lead to the "improper state" $|\lambda \mp i\rangle$. We intend to discuss more precisely this question.

To do this, we shall specifically study one irreducible representation of G which we choose to be one of the discrete series.[1] In Section II, we describe this representation following Ref. 2. Section III is devoted to the diagonalization of T_1 through a Mellin transform. It turns out that we are naturally lead to study some properties of a set of orthogonal polynomials, of a type introduced by Pollaczek.[3] Finally, in Section IV, we consider the representation of G in this new basis.[4]

It will be understood in the following that when a real positive number x is taken to complex power y, $\operatorname{Arg} x = 0$. The complex conjugate of z will be denoted z^*.

II. A REPRESENTATION OF THE DISCRETE SERIES

As in Ref. 2, let us consider the vector space $\mathscr{D}$ of analytic function such that if $f \in \mathscr{D}$:

(1) $f(z)$ is analytic for $\operatorname{Im} z > 0$, and continuous with all its derivatives in $\operatorname{Im} z \geq 0$

(2) $\hat{f}(z) = \frac{1}{z} f\left(-\frac{1}{z}\right)$ is also continuous with all its derivatives in $\operatorname{Im} z \geq 0$

As a result of (1) and (2), one can define a norm on $\mathscr{D}$ through

$$\|f\|^2 = \frac{1}{\pi} \int_{-\infty}^{+\infty} dx \, |f(x)|^2 \tag{3}$$

Equipped with this norm, $\mathscr{D}$ is not complete. Its completion is a Hilbert space $\mathscr{H}$ of analytic functions in the upper half plane. Indeed, if $f \in \mathscr{D}$, its value at a given point z ($\mathrm{Im}\, z > 0$) is such that

$$|f(z)| = \frac{1}{2\pi}\left|\int_{-\infty}^{+\infty} f(x)\frac{1}{x-z}\,dx\right| \leq \frac{1}{2\sqrt{\pi}}\,\|f\|\left[\int_{-\infty}^{+\infty}\frac{dx}{x^2+\mathrm{Im}\, z^2}\right]^{\frac{1}{2}}$$

or

$$\mathrm{Im}\, z > 0: \ |f(z)| \leq \frac{\|f\|}{2\sqrt{\mathrm{Im}\, z}} \tag{4}$$

Equation (4) shows that a Cauchy sequence in $\mathscr{D}$ with respect to the norm (3) will converge uniformly, in the ordinary sense, on any compact set of the upper half z-plane, to an analytic function. $\mathscr{D}$ is dense in $\mathscr{H}$, but it is clear that it is not all of $\mathscr{H}$ as shown by the example of $\frac{1}{z+i}\log\frac{1-iz}{2}$ which belongs to $\mathscr{H}$ but not to $\mathscr{D}$; it is, however, the limit of

$$\sum_{1}^{N}\frac{1}{p}(z-i)^{p}(z+i)^{-p-1}$$

which belongs to $\mathscr{D}$.

For an alternative description of $\mathscr{H}$, we introduce the functions $f_n(z) \in \mathscr{D}$:

$$f_n(z) = (z-i)^n (z+i)^{-n-1} \qquad n = 0, 1 \ldots . \tag{5}$$

$$(f_n|f_m) = \delta_{nm}$$

Let us show that the system $\{f_n\}$ is complete. It is sufficient to prove that if $f \in \mathscr{D}$ and $(f_n|f) = 0$ for all n, then f = 0. Indeed, an explicit computation leads to $\left(\frac{d}{dz}\right)^n (z+i)^n f(z)\big|_{z=i} = 0$, for n = 0, 1, ... By recurrence, all derivatives of f vanish at z = i and since f is analytic, f = 0. As a consequence, the elements of $\mathscr{H}$ are characterized by sequences of complex numbers $\{a_n\}$, n non-negative integer such that $\sum |a_n|^2 < \infty$; the analytic function itself is obtained as $\sum_0^\infty a_n f_n(z)$. This series converges uniformly in any compact domain of the upper half plane, since such a domain can be enclosed in a circle $\left|\frac{z-i}{z+1}\right| \leq \rho < 1$ where one has

$$\left|\sum_0^N a_n f_n(z)\right| \leq \sqrt{\frac{\sum_0^\infty |a_n|^2}{1-\rho^2}}$$

Let $\begin{pmatrix} ab \\ cd \end{pmatrix} \equiv g \in G$ and $f \in \mathscr{D}$, the set of transformations

$$f \longrightarrow U(g)f \qquad U(g)f(z) = \frac{1}{bz+d}\, f\left(\frac{az+c}{bz+d}\right) \tag{6}$$

leave $\mathscr{D}$ invariant and can be extended to a unitary representation of G in $\mathscr{H}$. This representation belongs to the discrete series,[1] it is irreducible and will be studied in the following. From the global form (6), we can derive the representatives of the generators T_1, T_2 and R defined in the introduction. They are the differential operators:

$$\begin{aligned} T_1 &= \frac{1}{2}\left(1 + 2z\frac{d}{dz}\right) \\ T_2 &= -\frac{1}{2}\left(z + \left(z^2 - 1\right)\frac{d}{dz}\right) \\ R &= \frac{1}{2}\left(z + \left(z^2 + 1\right)\frac{d}{dz}\right) \end{aligned} \tag{7}$$

The complete set $\{f_n\}$ satisfies:

$$\begin{aligned} Rf_n &= i\left(n + \frac{1}{2}\right) f_n \\ \left(T_1 + iT_2\right) f_n &= n f_{n-1} \\ \left(T_1 - iT_2\right) f_n &= -(n+1) f_{n+1} \end{aligned} \tag{8}$$

In other words, in this basis R is diagonal with eigenvalues of the form $i\left(n+\frac{1}{2}\right)$, n non-negative integer.

III. DIAGONALIZATION OF A NONCOMPACT GENERATOR

Our aim is now to diagonalize a noncompact generator, T_1, say. An eigen-function of the corresponding differential operator (7) is a homogeneous function z^a. For no value of the exponent does such a function belong to $\mathcal{H}$. This is to be expected: T_1 has no eigenvalue (in the sense that they would correspond to normalizable eigenstates) but we expect its spectrum to be purely imaginary, or $\left(izI - T_1\right)^{-1}$ to exist as a bounded operator for $\mathrm{Im}\, z \neq 0$.

We shall obtain this diagonal form by studying the following Mellin transform. Let $f \in \mathcal{H}$, we introduce the function of the real variable λ, $F(\lambda)$ by

$$f \longrightarrow F \qquad F(\lambda) = i\,\frac{\mathrm{ch}\pi\lambda}{\pi}\int_0^\infty d\rho\, f(i\rho)\,\rho^{-\frac{1}{2}-i\lambda} \tag{9}$$

It is clear that the integral converges in the ordinary sense for $f \in \mathcal{D}$. We shall extend it with the help of the transforms $\{F_n\}$ of the basic functions $\{f_n\}$ introduced in Section II.

$$f_n \longrightarrow F_n \qquad F_n(\lambda) = \frac{\mathrm{ch}\pi\lambda}{\pi}\int_0^\infty d\rho\,(\rho-1)^n(\rho+1)^{-n-1}\rho^{-\frac{1}{2}-i\lambda} \tag{10}$$

A convenient way of performing the integral (10) is to observe that the series

$$\sum_0^\infty t^n(\rho-1)^n(\rho+1)^{-n-1} = \frac{1}{(\rho+1)-t(\rho-1)} \qquad |t|<1 \quad 0\leq\rho$$

converges absolutely and uniformly in $\left[|t| \leq 1-\epsilon\right]\times\left[0 \leq \rho < \infty\right]$. Hence we obtain for the set $\{F_n\}$ the generating function:

$$|t| < 1 \quad \sum_0^\infty t^n F_n(\lambda) = \frac{\operatorname{ch}\pi\lambda}{\pi} \int_0^\infty d\rho \frac{\rho^{-\frac{1}{2}-i\lambda}}{\rho(1-t)+(1+t)}$$

$$= (1+t)^{-\frac{1}{2}-i\lambda} (1-t)^{-\frac{1}{2}+i\lambda} \tag{11}$$

where we have used the fact that

$$F_0(\lambda) = \frac{\operatorname{ch}\pi\lambda}{\pi} \int_0^\infty d\rho \frac{\rho^{-\frac{1}{2}-i\lambda}}{\rho+1}$$

$$= \lim_{\epsilon \to +0} \frac{e^{-\pi\lambda}}{2\pi} \left[\int_{-\infty-i\epsilon}^{-i\epsilon} + \int_{i\epsilon}^{\infty} \right] \frac{d\rho}{\rho+1} \rho^{-\frac{1}{2}-i\lambda} = 1$$

by Cauchy's theorem.

In formula (11), the phases of $(1+t)$ and $(1-t)$ are zero for $-1 < t < +1$.

We summarize elementary properties of the functions $F_n(\lambda)$ in the following:

<u>Proposition 1</u>

(1) $F_n(\lambda)$ is polynomial in λ of precise degree n and $F_n\left(\frac{i}{2}\right) = 1$.

(2) $(-)^n F_n^*(\lambda^*) = (-1)^n F_n(-\lambda) = F_n(\lambda) = F\left(-n, \frac{1}{2} + i\lambda; 1; 2\right)$ (12)

(3) for $|t| < 1 \quad \sum_0^\infty t^n F_n(\lambda) = (1+t)^{-\frac{1}{2}-i\lambda} (1-t)^{-\frac{1}{2}+i\lambda}$

(4) $\int_{-\infty}^{+\infty} F_m(\lambda)^* F_n(\lambda) \frac{d\lambda}{\operatorname{ch}\pi\lambda} = \delta_{nm}$ (13)

(5) $2i\lambda F_n(\lambda) = n F_{n-1}(\lambda) - (n+1) F_{n+1}(\lambda)$.

Proof. Proposition 1 asserts that the $F_n(\lambda)$ forms an orthonormal set of polynomials in the Hilbert space of functions $F(\lambda)$ such that

$$\|F\|^2 = \int_{-\infty}^{+\infty} \frac{d\lambda}{\mathrm{ch}\pi\lambda} \left|F(\lambda)\right|^2 < \infty .$$

We have already proved (3) from which (1) and (2) easily follow. Indeed $F_n(\lambda)$ appears equal to the polynomial of degree n:

$$F_n(\lambda) = \sum_0^n \frac{\left(-\frac{1}{2} - i\lambda\right)_{n-p} \left(-\frac{1}{2} + i\lambda\right)_p (-)^p}{(n-p)!\ p!} \tag{14}$$

where $(x)_p = \frac{\Gamma(x+p)}{\Gamma(x)}$. Hence the coefficient of λ^n in $F_n(\lambda)$ is

$$(-i)^n \sum_0^n \frac{1}{p!\ n-p!} = \frac{(-2i)^n}{n!} \neq 0 .$$

From the integral representation (10), we obtain the expression (12) of F_n in terms of the hypergeometric function from which the value $F_n\left(\frac{i}{2}\right) = 1$ follows.

To establish the orthogonality relation, we make again use of the generating function. From the equality $\left(\left|\mathrm{Im}\,x\right| < \frac{\pi}{2}\right)$

$$\int_{-\infty}^{+\infty} e^{2i\lambda x} \frac{d\lambda}{\mathrm{ch}\pi\lambda} = \frac{1}{\mathrm{ch}\,x} ,$$

we derive for $-1 < u, t < 1$:

$$\sum_{m,n=0}^{\infty} u^m t^n \int_{-\infty}^{+\infty} P_m(\lambda)^* P_n(\lambda) \frac{d\lambda}{\mathrm{ch}\pi\lambda} = \frac{1}{\sqrt{(1-u^2)(1-t^2)}} \int_{-\infty}^{+\infty} \left[\frac{(1-t)(1+u)}{(1+t)(1-u)}\right]^{i\lambda} \frac{d\lambda}{\mathrm{ch}\pi\lambda} =$$

$$= \frac{1}{1-tu} = \sum_{m,n=\infty}^{\infty} \delta_{mn} u^m t^n$$

Since we are dealing with an analytic function of t and u in $|t| < 1$, $|u| < 1$, we can identify the coefficients of its Taylor expansion and thus arrive at the desired orthogonality property.

Finally, the recurrence relation is an immediate consequence of the representation of the Lie algebra of G. Indeed, from equations (6) and (8), we have:

$$\frac{1}{2}\left[n F_{n-1}(\lambda) - (n+1) F_{n+1}(\lambda)\right] = \frac{ch\pi\lambda}{\pi} \int_{-\infty}^{+\infty} d\rho \rho^{-\frac{1}{2}-i\lambda} \frac{d}{d\alpha}\left[e^{\alpha T_1} f_n(i\rho)\right]_{\alpha=0}$$

$$= \frac{ch\pi\lambda}{\pi} \int_{-\infty}^{+\infty} d\rho \rho^{-\frac{1}{2}-i\lambda} \frac{d}{d\alpha}\left[e^{\frac{\alpha}{2}} f_n(i\rho e^{\alpha})\right]_{\alpha=0}$$

In the last integral, interchange of the order of integration and derivation is allowed. As a result:

$$\frac{1}{2}\left[n F_{n-1}(\lambda) - (n+1) F_{n+1}(\lambda)\right] = \frac{d}{d\alpha}\left[e^{i\alpha\lambda} F_n(\lambda)\right]_{\alpha=0} = i\lambda F_n(\lambda) \quad .$$

The polynomials F_n belong to a class which has been studied by F. Pollaczek.[3] We denote by H the Hilbert space of square integrable functions on the real line with measure $\frac{d\lambda}{ch\pi\lambda}$. As usual, two functions which differ on a set of measure zero are identified.

Proposition 2. The polynomials F_n form a complete orthonormal basis in H.

Proof. In view of proposition 1, it is sufficient to prove that the functions λ^n, n non-negative integer form a complete set in H. Let $F \in H$ be orthogonal to all λ^n. Consider the function

$$g(s) = \int_{-\infty}^{+\infty} \frac{d\lambda}{ch\pi\lambda} e^{is\lambda} F(\lambda) \quad .$$

It is analytic in the strip $\left|\mathrm{Im}\,\lambda\right| < \frac{\pi}{2}$ and all its derivatives vanish at the origin. As a result, $g(s) = 0$ and $F(\lambda)$ vanishes almost everywhere. This proves that the system $\{F_n\}$ is complete.

We can recover f_n from F_n through an inverse Mellin transform

$$f_n(z) = \frac{1}{2i}\int_{-\infty}^{+\infty} F_n(\lambda)\,(-iz)^{-\frac{1}{2}+i\lambda}\,\frac{d\lambda}{\mathrm{ch}\,\pi\lambda} \qquad -\frac{\pi}{2} < \mathrm{Arg}(-iz) < \frac{\pi}{2} \tag{15}$$

With these results, we can return to the integral transformation (9). First let $f \in \mathscr{D}$, then the integral in (9) is absolutely convergent. Moreover, one readily shows that for real λ $\left|F(\lambda)\right| e^{\frac{-\pi}{2}|\lambda|} \longrightarrow 0$ as $|\lambda| \longrightarrow \infty$ faster than any power of $|\lambda|$ and hence $F \in H$. Using (15), one finds that if $f = \sum_0^\infty a_n f_n$, then $F = \sum_0^\infty a_n F_n$. In other words, the Mellin transform M is an isometric mapping from $\mathscr{D} \subset \mathscr{H}$ in a dense subset of H which will be denoted D. By continuity, it is then uniquely extended to a one-to-one isometric mapping M from $\mathscr{H}$ to H.

We close this section by mentioning some properties of D. Let $f^{(n)}$ stand for the n-th derivative of $f \in \mathscr{D}$, and $\hat{f}(z) = \frac{1}{z} f\left(-\frac{1}{z}\right)$, then:

Proposition 3. Any $F = Mf \in D$ can be extended as an entire function in the complex λ plane. Moreover:

$$\left.\begin{aligned} F\left[-i\left(\frac{1}{2}+n\right)\right] &= i^{(1-n)}\,\frac{f^{(n)}}{n!}(0) \\ F\left[\;i\left(\frac{1}{2}+n\right)\right] &= i^{(2-n)}\,\frac{\hat{f}^{(n)}}{n!}(0) \end{aligned}\right\} \quad , n = 0, 1, \ldots \tag{16}$$

and

$$\hat{F}(\lambda) = M\hat{f}(\lambda) = -i\,F(-\lambda)$$

Proof. Note that one can write

$$F(\lambda) = \phi_1(\lambda) + i\,\phi_2(\lambda)$$

where

$$\phi_1(\lambda) = i\,\frac{\mathrm{ch}\pi\lambda}{\pi}\int_0^1 f(i\rho)\,\rho^{-\frac{1}{2}-i\lambda}$$

$$\phi_2(\lambda) = i\,\frac{\mathrm{ch}\pi\lambda}{\pi}\int_0^1 \hat{f}(i\rho)\,\rho^{-\frac{1}{2}+i\lambda}$$

ϕ_2 is deduced from ϕ_1 by changing $\lambda \longrightarrow -\lambda$ and $f \longrightarrow \hat{f}$. Thus it is sufficient to consider ϕ_1. At first it is analytic for $\mathrm{Im}\,\lambda > -\frac{1}{2}$, vanishing at the points $i\left[\frac{1}{2}+n\right]$, for non-negative integer n. Furthermore, integration by parts, gives for an arbitrary positive integer p, and $\mathrm{Im}\,\lambda > -\frac{1}{2}$:

$$\phi_1(\lambda) = \frac{i\,\mathrm{ch}\,\pi\lambda}{\left(\frac{1}{2}-i\lambda\right)_p \pi}\left\{\sum_{r=1}^{p}(-)^{r-1}\left(\frac{d}{d\rho}\right)^{r-1} f(i\rho)\frac{d^{p-r}}{d\rho}\,\rho^{-\frac{1}{2}-i\lambda+p}\Bigg|_{\rho=1} + \right.$$

$$\left. + (-)^p\int_0^1 d\rho\frac{d^p}{d\rho}\,f(i\rho)\,\rho^{-\frac{1}{2}-\lambda+p}\right\}$$

The zeros of the $\left(\frac{1}{2}-i\lambda\right)_p$ are just cancelled by those of $\mathrm{ch}\,\pi\,\lambda$. We can then analytically continue this formula to $\mathrm{Im}\,\lambda > -\frac{1}{2}-p$. Since p is arbitrary, $\phi_1(\lambda)$ is an entire function of λ. If we set $p = n+1$ and $\lambda = -i\left(\frac{1}{2}+n\right)$, n non-negative integer in the above expression, we get:

$$\phi_1\left[-i\left(\frac{1}{2}+n\right)\right] = \frac{i(-)^n}{n!}\,\frac{d^n}{d\rho}\,f(i\rho)\Bigg|_{\rho=0} = \frac{i^{(1-n)}}{n!}\,f^{(n)}(0) \quad .$$

Combining these results with similar ones for ϕ_2, we arrive at formula (16).

IV. REPRESENTATION OF G

The isometric operator M of the preceding section enables one to carry the representation U of G, defined in $\mathcal{H}$, to an equivalent representation V, defined in H through $V(g) = M\,U(g)\,M^{-1}$. The inverse transformation V^{-1} was already indicated in (15). Hence, for V(g), we obtain the following expression:

$$F \longrightarrow V(g)F \qquad V(g)F(\lambda) = \frac{\mathrm{ch}\pi\lambda}{2\pi}\int_0^\infty d\rho\, \rho^{-\frac{1}{2}-i\lambda} \int_{-\infty}^{+\infty} (ib\rho+d)^{-\frac{1}{2}-i\mu} (a\rho-ic)^{-\frac{1}{2}+i\mu} F(\mu)\frac{d\mu}{\mathrm{ch}\pi\mu}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{with } \rho > 0, \; -\frac{\pi}{2} < \mathrm{Arg}\left(\frac{a\rho - ic}{ib\rho + d}\right) < \frac{\pi}{2} \tag{17}$$

At first this formula is defined when $F \in D$. It is then extended by the unitarity property to all H. Assume $F \in D$ and none of the real numbers a, b, c, d to vanish. The interchange of the order of integration is allowed in (17). Let us, therefore, compute the kernel

$$K_g(\lambda,\mu) = \frac{\mathrm{ch}\pi\lambda}{2\pi}\int_0^\infty d\rho\, \rho^{-\frac{1}{2}-i\lambda} (ib\rho+d)^{-\frac{1}{2}-i\mu} (a\rho-ic)^{-\frac{1}{2}+i\mu} \tag{18}$$

Let G_1 denote the subgroup of elements of the form $g(\alpha) = e^{\alpha t_1} = \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix}$. The manifold S of elements g in G, such that $a\,b\,c\,d \neq 0$, is invariant under right and left translation by G_1. Moreover

$$K_{g(\alpha)gg(\beta)}(\lambda,\mu) = e^{i\lambda\alpha}\, K_g(\lambda\mu)\, e^{i\mu\beta} \tag{19}$$

which enables one to compute K only for representatives of each type of double coset $G_1\backslash G/G_1$.

These fall into four classes. We select representatives of the form

$$(A) = e^{2At_2} = \begin{pmatrix} \mathrm{ch}\,A & \mathrm{sh}\,A \\ \mathrm{sh}\,A & \mathrm{ch}\,A \end{pmatrix} ; \qquad (B) = e^{2Br} = \begin{pmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{pmatrix}$$

$$(C) = \begin{pmatrix} \mathrm{sh}\,C & \mathrm{ch}\,C \\ -\mathrm{ch}\,C & -\mathrm{sh}\,C \end{pmatrix} \qquad (D) = \begin{pmatrix} \mathrm{sh}\,D & -\mathrm{ch}\,D \\ \mathrm{ch}\,D & -\mathrm{sh}\,D \end{pmatrix} \tag{20}$$

where the parameters A, B, C and D are all different from zero. The last two classes are taken into account by remarking that they can be obtained from the first one by left multiplication by $g_o = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$, or right multiplication by g_o^{-1}, and that one has

$$K_{g_o g}(\lambda_1 \mu) = -i K_g(-\lambda_1 \mu) \qquad K_{gg_o^{-1}}(\lambda_1 \mu) = i K_g(\lambda_1 - \mu) \tag{21}$$

a fact which is readily related to the mapping $F \longrightarrow \hat{F}$. Let us, therefore, compute:

$$K_{(A)} \equiv K_{e^{2At_2}} \quad \text{and} \quad K_{(B)} = K_{e^{2Br}} \;.$$

By transforming the integral in (18) into a path integral around the origin, rotating the contour to enclose the two singular points $\rho_1 = \frac{ic}{a}$ and $\rho_2 = \frac{id}{b}$, one recovers a classical representation for the hypergeometric function. The final result for K_A is:

$$A \neq 0 \quad K_{(A)}(\lambda,\mu) = \frac{e^{-\epsilon\pi\frac{(\lambda-\mu)}{2}}}{2} [\mathrm{ch}\,A]^{i(\lambda+\mu)} [|\mathrm{sh}\,A|]^{-1-i(\lambda+\mu)} F\left(\frac{1}{2}+i\lambda, \frac{1}{2}+i\mu; 1; \frac{-1}{\mathrm{sh}^2 A}\right)$$

$$\epsilon = \frac{A}{|A|} \tag{22}$$

For Class (B), we make again use of relation (21), which allows one to restrict

B to $0 < B < \frac{\pi}{2}$, and obtain:

$$K_{(B)}(\lambda,\mu) = \frac{1}{2} e^{\frac{\pi}{2}(\mu-\lambda)} \frac{\Gamma\left(\frac{1}{2}+i\lambda\right)}{\Gamma\left(\frac{1}{2}-i\mu\right)\Gamma\left(1+i(\lambda+\mu)\right)} (\sin B)^{i(\mu-\lambda)} (\cos B)^{i(\lambda+\mu)}$$

$$F\left(\frac{1}{2}+i\mu, \frac{1}{2}+i\mu; 1+i(\lambda+\mu); \cos^2 B\right) + i\frac{1}{2} e^{\frac{\pi(\lambda+\mu)}{2}} \frac{\Gamma\left(\frac{1}{2}+i\lambda\right)}{\Gamma\left(\frac{1}{2}+i\mu\right)\Gamma\left(1+i(\lambda-\mu)\right)}$$

$$(\sin B)^{i(\lambda-\mu)} (\cos B)^{-i(\lambda+\mu)} F\left(\frac{1}{2}-i\mu, \frac{1}{2}-i\mu, 1+i(\lambda+\mu), \sin^2 B\right) \qquad (23)$$

Expression (22) can be brought to a form similar to (23) using transformation properties of the hypergeometric function.

On the manifold G-S (which contains the subgroup G_1), the kernel is singular. Of particular interest is the representation of the subgroup G_1. It follows from (17) that:

$$V\left(e^{\alpha t_1}\right) F(\lambda) = e^{i\alpha\lambda} F(\lambda) \qquad (24)$$

In other words, in this basis the representation of this subgroup is diagonal. Our calculation of the kernel K_g is not very well suited to obtain the other generators, but they can be readily recovered using for instance the Pollaczek polynomials of the preceding section. Indeed we have

$$\text{(a)}\quad T_1 F_n(\lambda) = \frac{1}{2}\left[n F_{n-1}(\lambda) - (n+1)F_{n+1}(\lambda)\right] = i\lambda F_n(\lambda)$$

$$\text{(b)}\quad RF_n(\lambda) = i\left(\frac{1}{2}+n\right) F_n(\lambda) = \frac{i}{2}\left[\left(\frac{1}{2}+i\lambda\right) F_n(\lambda-i) + \left(\frac{1}{2}-i\lambda\right) F_n(\lambda+i)\right] \qquad (25)$$

$$\text{(c)}\quad T_2 F_n(\lambda) = -\frac{i}{2}\left[n F_{n-1}(\lambda) - (n+1)F_{n+1}(\lambda)\right] = \frac{i}{2}\left[\left(\frac{1}{2}+i\lambda\right)F_n(\lambda-i) - \left(\frac{1}{2}-i\lambda\right) F_n(\lambda+i)\right]$$

The first equation is the recurrence relation already proved in Section III, and only reflect the fact that T_1 is diagonal.

The two others are derived, using the generating function (11). For instance:

$$\sum t^n R F_n(\lambda) = i\sum_0^\infty \left(n+\frac{1}{2}\right)t^n F_n(\lambda) = i\left(\frac{1}{2}+t\frac{d}{dt}\right)(1+t)^{-\frac{1}{2}-i\lambda}(1-t)^{-\frac{1}{2}+i\lambda}$$

$$= \frac{i}{2}\left[\left(\frac{1}{2}+i\lambda\right)(1+t)^{-\frac{3}{2}-i\lambda}(1-t)^{\frac{1}{2}+i\lambda} + \left(\frac{1}{2}-i\lambda\right)(1+t)^{\frac{1}{2}-i\lambda}(1-t)^{-\frac{3}{2}+i\lambda}\right]$$

$$= \frac{i}{2}\sum_0^\infty t^n \left[\left(\frac{1}{2}+i\lambda\right) F_n(\lambda-i) + \left(\frac{1}{2}-i\lambda\right) F_n(\lambda+i)\right] \quad ,$$

and similarly for T_2. As a result, wherever they are defined (and at least on D), the generators are expressed in the Hilbert Space H as difference operators by the formulas

$$T_1 F(\lambda) = i\lambda F(\lambda)$$

$$T_2 F(\lambda) = \frac{i}{2}\left[\left(\frac{1}{2}+i\lambda\right) F(\lambda-i) - \left(\frac{1}{2}-i\lambda\right) F(\lambda+i)\right] \tag{26}$$

$$R F(\lambda) = \frac{i}{2}\left[\left(\frac{1}{2}+i\lambda\right) F(\lambda-i) + \left(\frac{1}{2}-i\lambda\right) F(\lambda+i)\right]$$

It is easily verified that T_1, T_2 and R satisfy the correct commutation rules and are antisymmetric on D. For instance, one can directly show that for any two F and G in D $\left(G\middle|\left[T_2+R\right]F\right)+\left(\left[T_2+R\right]G\middle|F\right)=0$. Indeed, the left-hand side can be written as a contour integral:

$$i\int_C \frac{d\lambda}{\mathrm{ch}\pi\lambda}\,\tilde{G}(\lambda)\left(\frac{1}{2}+i\lambda\right)F(\lambda-i) \ ,$$

where $\tilde{G}(\lambda) = G^*(\lambda^*)$, and the contour C consists of the lines $\mathrm{Im}\,\lambda = 0$, $\mathrm{Im}\,\lambda = i$, and two infinitely remote segments joining these two lines on $\mathrm{Re}\,\lambda = \pm A$, $A \to \infty$. The integrand is nowhere singular inside this contour since the zero of $\mathrm{ch}\,\pi\lambda$ for $\lambda = \frac{i}{2}$ is cancelled by the factor $\left(\frac{1}{2}+i\lambda\right)$ and as a result, the integral vanishes as expected.

The relations (26) give a precise meaning to the remarks made in the introduction concerning the representatives of the other generators in the basis where T_1 is diagonal. When the generators are realized as differential operators in a Hilbert space of functions, we require the existence of an adequate supply of infinitely differentiable functions, though the whole Hilbert space need not contain only differentiable functions. In very much the same way, we are led in the present case to the existence of a sufficient number of entire functions to be able to exponentiate the generators.

Similar considerations can be extended to other representations of G or, more generally, to those of semi-simple noncompact groups.

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4. While completing this note, we received a preprint from A. O. Barut and E. C. Phillips, "Matrix elements of representations of noncompact groups in a continuous basis," University of Colorado (1967), which deals with a similar subject.