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MONOMIAL FACTORIZATION OF SYMPLECTIC MAPS

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A method is proposed for replacing a Lie transformation generated by a polynomial in phase-space variables with a composition of explicit symplectic maps that can be made to agree with the Lie transformation to an arbitrary order in phase-space variables. By dividing the Lie transformation into steps the method can be made as accurate as desired, even at a fixed order in phase-space variables. The results can be used to symplectify a truncated power series map, and may be useful in the construction of explicit symplectic integrators.

KEY WORDS: Particle Dynamics, Symplectic Maps

1 INTRODUCTION

In the study of Hamiltonian systems where the Hamiltonian is a periodic function of time, it is often computationally convenient to replace the continuous motion in time by a sequence of discrete symplectic maps that relate conditions on the trajectory separated by a finite, and generally not small, time step. One convenient choice of the time step, for example, is the period of the Hamiltonian, in which case one symplectic map is sufficient. Even when the underlying differential equations of motion are known, however, it is in general not possible to write down an explicit expression for the final conditions in terms of the initial ones (final and initial refer to the end and the beginning of the time step, respectively). In one of the approximation schemes used to treat this problem, one expresses the final conditions as a power series in the initial conditions, and then calculates a finite number of terms in the series. While this method can often give a very accurate description of the map, the truncation of the power series generally leads to the violation of the symplectic condition, which in turn gives rise to spurious effects if the map is iterated many times.

A map that consists of a truncated power series can be symplectified by replacing the series either by an implicit[1,2] or an explicit expression that is exactly symplectic and that agrees with the series through the order of truncation. Explicit expressions, to which we confine our attention in this paper, can be obtained by adding higher-order terms to the truncated power series such that the modified series satisfies the symplectic condition exactly. Ref. 3 demonstrates that only a finite number of higher-order terms are necessary. While the procedure of Ref. 3 can be implemented with relative ease in particle tracking codes, the symplectified map often adequately reproduces the nonsymplectic map only in a small region of phase space. On the other hand, there exist algorithms, based on Lie transformations generated by polynomials in phase-space variables, that turn the truncated power series into an infinite series that is exactly symplectic. Two examples of such algorithms are the Dragt-Finn transformation[4,5] and the single Lie transformation generated by a polynomial of one order higher than the truncated power series.[5] A limitation of these procedures is that in general they require the computation and summation of an infinite number of terms. Except in simple cases, this cannot be done exactly. In this paper we propose a method to evaluate explicitly, to a desired accuracy, the Lie transformations generated by polynomials. First we show that the Lie transformations generated by monomials in phase-space variables can be computed exactly in closed form. We then use this result to demonstrate that any Lie transformation generated by polynomials can be approximated to any desired accuracy by a composition of explicit, exactly symplectic maps.

The organization of the paper is as follows. In Section 2 we introduce the notation and present the method. Section 3 is devoted to a simple example that illustrates the use and some limitations of the method, and Section 4 contains a summary and additional remarks.

2 METHOD

2.1 *Preliminaries*

We consider a 2d-dimensional phase space and in it a transformation

$$z_i^f = M_{ij} z_j^i + N_{ijk}^{(2)} z_j^i z_k^i + N_{ijkl}^{(3)} z_j^i z_k^i z_l^i + \dots \, . \tag{1}$$

Here z is a 2d-dimensional vector with coordinates $z_i = q_i$, $z_{i+d} = p_i$, $1 \le i \le d$; q_i and p_i form a canonically conjugate pair; M and $N^{(k)}$ are coefficient tensors; and the superscripts i and f denote initial and final coordinates. The summation over repeated lower indices is assumed, and we have chosen the coordinates such that the origin is mapped into itself. Usually Equation (1) represents the Taylor series. For the discussion that follows, however, that is not necessary.

Now we let the right side of Equation (1) be truncated at order $N - 1$; that is, we neglect terms of order N. (Through Section 2.1 we use the word order to refer only to the order in z^i.) Then, in general, the transformation only preserves the symplectic condition through order $N-2$. One way to symplectify the transformation is to write z^f as $z^f = \mathcal{M} z^i$, where

$$\mathcal{M} = M e^{:h_N:} \, . \tag{2}$$

M denotes the linear transformation, h_N is a polynomial in z^i containing orders 3 through N, and $: h_N :$ is the Lie generator associated with h_N,: $h_N :\equiv J_{ij} \frac{\partial h_N}{\partial z_i} \frac{\partial}{\partial z_j}$. J is a $2d \times 2d$ matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} , \tag{3}$$

where I is the $d \times d$ identity matrix. The exponential of the generator $: h_N :$ is defined by the usual infinite series

$$e^{:h_N:} = \sum_{s=0}^{\infty} \frac{: h_N :^s}{s!} . \tag{4}$$

The coefficients of M and h_N are chosen in such a way that terms through order $N-1$ that arise from the expansion of $\mathcal{M}z^i$ agree with the ones of the same order in Equation (1) truncated at order $N-1$.

In this paper we do not discuss the construction of M and h_N. Instead, for a tutorial on the procedure we refer the reader to Ref. 4 and 5. Those contemplating the construction of these quantities in actual accelerator problems are also urged to learn about the efficient numerical procedures contained in automatic differentiation packages.[6–8]

We now introduce the assumption that in the region of interest, $\mathcal{M}$ gives a good approximation to the map of Equation (1) truncated at order $N-1$. By "good" we mean that terms of order N that originate from the expansion of $\mathcal{M}z^i$, and whose sole role is to make the transformation symplectic, do not appreciably affect the dynamics, which is specified by terms of order 1 through $N-1$. We emphasize, however, that while this assumption simplifies calculations, the method described below is also applicable in cases where it is advantageous to take a Lie-algebraic transformation that is of a different form than the one assumed here.

Since the linear part of the transformation $\mathcal{M}$ is simply a multiplication of a vector by a matrix, the main task to be accomplished is to evaluate explicitly $e^{:h_N:}z$. (From now on we drop the superscript i on z.) We show how this can be done such that the explicit and exactly symplectic map agrees with $e^{:h_N:}z$ through terms of order $N-1$ in a series expansion. (For $d = 1$ and $N \leq 4$ another method is given in Ref. 9.)

2.2 *Monomial factorization*

The first step is to use the Baker-Campbell-Hausdorff (BCH) formula to rewrite $e^{:h_N:}$ as a product of Lie transformations generated by *monomials* in z. The required manipulations are standard[4], and to order N in Lie generators we rewrite $e^{:h_N:}$ as

$$\mathcal{M}_N = \prod_{r=3}^{N} \prod_{j=1}^{\binom{r+2d-1}{r}} e^{:\alpha_j^{(r)} P_j^{(r)}:} . \tag{5}$$

Here $P_j^{(r)}$ are monomials in z of order r, $P_j^{(r)} = q_1^{rj1} q_2^{rj2} \dots p_d^{rj2d}$, $r_{j1} + r_{j2} + \dots + r_{j2d} = r$, and $\alpha_j^{(r)}$ are constants. The number of monomials of order r is $\binom{r+2d-1}{r}$. This decomposition of $e^{:h_N:}$ can be performed numerically in an efficient way, again by the use of automatic differentiation packages.[6–8] To proceed further we note that for any two functions $g(z)$ and $f(z)$ the following holds: $\exp(: f :)g(z) = g(\exp(: f :)z)$.[5] Therefore, in order to evaluate explicitly the action of $\mathcal{M}_N$ on z, it is necessary and sufficient to evaluate $\exp(: \alpha_j^{(r)} P_j^{(r)} :)z$.

Two remarks are in order here. First, given $e^{:h_N:}$, the corresponding $\mathcal{M}_N$ is not unique, since the ordering of Lie transformations generated by monomials is not specified. We have tested numerically the agreement between $e^{:h_N:}$ and various monomial orderings for several 2-dimensional maps. No significant differences were found between differently ordered monomial maps. When applying the prescription given below to a specific problem, however, the reader may want to examine which ordering is optimal for his/her task. And second, while h_N contains only terms through order N, one can compute its monomial factorization to an order higher than N (higher-order monomials cancel the commutators arising from lower-order monomials). Since the difference in Lie generators between $\mathcal{M}_{N+k}$ and $e^{:h_N:}$ scales as z^{N+k+1} as $z \to 0$, computing the monomial factorization to an order higher than N improves the agreement between the two maps in the vicinity of the origin. It is not unequivocally true, however, that $\mathcal{M}_{N+k}z$ with $k \geq 1$ approximates $e^{:h_N:}z$ better than $\mathcal{M}_N z$ does at values of z that are not very small. We will return to this question in the example treated later in the paper.

Returning to the evaluation, we need to compute $\exp(: \alpha_j^{(r)} P_j^{(r)} :)q_i$ and $\exp(: \alpha_j^{(r)} P_j^{(r)} :)p_i$, where i labels the degree of freedom. We write $P_j^{(r)}$ as $P_j^{(r)} = q_i^n p_i^m \tilde{P}_j^{(r)}$, where the factor $\tilde{P}$ does not depend on q_i and p_i, and then notice that

$$: P_j^{(r)} :^s (q_i \,\text{or}\, p_i) = (\tilde{P}_j^{(r)})^s : q_i^n p_i^m :^s (q_i \,\text{or}\, p_i). \tag{6}$$

This relation is easily proven by induction on s. Therefore, when acting on q_i or p_i, the factor $\tilde{P}_j^{(r)}$ can be combined with the constant $\alpha_j^{(r)}$. For simplicity we now drop the subscript i, and set $a \equiv \alpha_j^{(r)} \tilde{P}_j^{(r)}$. Hence, we need to evaluate

$$e^{a:q^n p^m:}(q \,\text{or}\, p). \tag{7}$$

First we consider the case $m \neq n$, and outline the derivation for q. The procedure for p is analogous, and we merely quote the result. By carrying out the required Poisson brackets explicitly, it can be shown, for example by induction on s of Equation (8b), that

$$e^{a:q^n p^m:}q = q - amq^n p^{m-1} + mna^2 \left[\frac{1}{2} q^{2n-1} p^{2m-2} + qR \right], \tag{8a}$$

where

$$R=\sum_{s=3}^{\infty}\frac{\rho^s}{s!}\frac{[a(m-n)]^{s-2}}{\Gamma(\frac{m-2n}{m-n})}\Gamma(s-2+\frac{m-2n}{m-n}), \tag{8b}$$

and

$$\rho=q^{n-1}p^{m-1}\,. \tag{8c}$$

We differentiate R with respect to ρ twice, introduce $k=s-2$, and use a standard relation between gamma functions, $\Gamma\left(k+\frac{m-2n}{m-n}\right)/\Gamma\left(\frac{m-2n}{m-n}\right)=(-1)^k\Gamma\left(1-\frac{m-2n}{m-n}\right)/\Gamma\left(1-k-\frac{m-2n}{m-n}\right)$, to get

$$\frac{d^2R}{d\rho^2}=\sum_{k=1}^{\infty}(a(n-m)\rho)^k\frac{\Gamma\left(1-\frac{m-2n}{m-n}\right)}{\Gamma(k+1)\Gamma\left(1-k-\frac{m-2n}{m-n}\right)}. \tag{9}$$

This series is convergent, and can be summed using the binomial theorem, provided

$$|a(n-m)\rho|<1. \tag{10}$$

This, then, gives us directly the radius of convergence in ρ of the Lie transformation of Equation (7) (the condition is the same for p). We can also find the radius of convergence from the location of the singular point of the final expressions given below.

By performing the summation in $d^2R/d\rho^2$, integrating the result twice, and substituting the expression into Equation (8a), we obtain for the final answer

$$e^{a:q^np^m:}q=q[1+(n-m)aq^{n-1}p^{m-1}]^{\frac{m}{m-n}}\,. \tag{11}$$

Analogously, the result for p is

$$e^{a:q^np^m:}p=p[1+(n-m)aq^{n-1}p^{m-1}]^{\frac{n}{n-m}}\,. \tag{12}$$

For the case $m=n$ the derivations are very simple. From Equation (7) we get directly the exponential series, with the result

$$e^{a:q^mp^m:}q=qe^{-amq^{m-1}p^{m-1}}, \tag{13}$$

and

$$e^{a:q^mp^m:}p=pe^{amq^{m-1}p^{m-1}}. \tag{14}$$

Alternatively, we can obtain these expressions by taking the limit $n\to m$ in Equations (11) and (12). Equations (13) and (14) are valid for all values of $amq^{m-1}p^{m-1}$. In order to verify directly that the symplectic condition is satisfied, we take the Poisson bracket of the right sides of Equations (11) and (12), and also (13) and (14). The result is 1, as it should be. Evidently, the use of expressions (11–14) in numerical computations should be straightforward. (Note: After the completion of this work, it was brought to my attention that in Ref. 10, Equations (11–14) were derived from a Hamiltonian.)

2.3 *Subdivision of the time interval*

We have expressed $e^{:h_N:}z$ through order $N-1$ in z as a composition of explicit maps that are exactly symplectic. Since $e^{:h_N:}$ itself is computed from a power series truncated at order $N-1$, the result given here is a consistent symplectification of the original power series. In many problems of interest, however, the value of z can be sufficiently large that the agreement between $e^{:h_N:}z$ and $\mathcal{M}_N z$ through order $N-1$ does not make the two transformations as close in magnitude as one would like. In addition, the requirement of Equation (10) can be too limiting for the region of interest. This is especially true if the right sides of Equations (11) and (12) possess branch points. In that case the natural analytic continuation of Equation (8a,b,c) by the right side of Equation (11) (and analogously for p) may require that p and q take on complex values — an obviously complicating feature. We now outline a procedure, based on generalized Trotter's formula,[11,12] that can be used to make $e^{:h_N:}z$ and its monomial factorization as close as desired and to make the condition of Equation (10) less stringent. (Of course, progress towards these goals comes at the expense of increased computation.) Since we have assumed that $Me^{:h_N:}z$ is a good, but formal, symplectification of the truncated series in Equation (1), the procedure enables us to obtain an explicit, and also good, symplectification of that power series.

We begin by writing $e^{:h_N:}$ as

$$e^{:h_N:} = (e^{\frac{1}{L}:h_N:})^L, \tag{15}$$

where L is an integer. This relation clearly holds since $\frac{1}{L}:h_N:$ commutes with itself. Next, following the procedure that yields Equation (5), we decompose $e^{\frac{1}{L}:h_N:}$ into Lie transformations generated by monomials through order N. We denote this result by $\mathcal{M}_N^{(L)}$. Then, upon taking the composition of $\mathcal{M}_N^{(L)}$ L times, we obtain an explicit map whose difference from $e^{:h_N:}$ in Lie generators is suppressed by a factor of at least $\frac{1}{L}$ compared with $\mathcal{M}_N$. Furthermore, in the evaluation of Lie transformations that comprise $\mathcal{M}_N^{(L)}$, the left side of inequality (10) is multiplied by at least $\frac{1}{L}$.

To demonstrate the truth of these assertions, we note that each monomial in $\mathcal{M}_N^{(L)}$ carries a factor of not less than $\frac{1}{L}$ (monomials of order in z higher than 3 can be multiplied by factors that contain higher powers of $\frac{1}{L}$) Hence the less restrictive condition on convergence (the equivalent of Equation (10)). By using the BCH formula we can recombine the exponents in $\mathcal{M}_N^{(L)}$ into a single exponent yielding

$$\mathcal{M}_N^{(L)} = \exp\left[: \frac{1}{L}h_N + O(z^{N+1}) :\right]. \tag{16}$$

Terms of order $N+1$ in z in the exponent on the right side of Equation (16) arise only from commutators of lower-order monomials in $\mathcal{M}_N^{(L)}$. Therefore, in $\frac{1}{L}$, they must be at least of order 2. Upon raising the right side of Equation (16) to the Lth power, we see that terms of order $N+1$ in z remain suppressed by at least a factor of

$\frac{1}{L}$. Therefore, by choosing L large enough, we can make $(\mathcal{M}_N^{(L)})^L$ approach $e^{:h_N:}$ as closely as desired. (For z in a bounded domain the procedure is convergent. See Ref. 12.)

The power of $\frac{1}{L}$ suppressing spurious terms can be raised. The most straightforward approach is to compute, and adjoin to $\mathcal{M}_N^{(L)}$, all Lie transformations whose generators carry a factor of $(\frac{1}{L})^2$, that is those arising from a single commutator of monomials present in $\frac{1}{L}h_N$ which are not included in $\mathcal{M}_N^{(L)}$. We call this result $\mathcal{R}_N^{(L)}$. The difference in Lie generators between $\mathcal{R}_N^{(L)}$ and $e^{\frac{1}{L}:h_N:}$ is of order $\left(\frac{1}{L}\right)^3$ and so $\mathcal{R}_N$ and $e^{:h_N:}$ differ by terms of order $\left(\frac{1}{L}\right)^2 z^{N+1}$. At the expense of increasing the number of Lie transformations that make up $\mathcal{R}_N^{(L)}$, the suppression factor $\frac{1}{L}$ can obviously be raised to any power desired.

Another approach for raising the power of $\frac{1}{L}$ is based on judicious ordering of Lie transformations that make up $\mathcal{M}_N^{(L)}$. The ordering is chosen such that an application of the BCH formula to bring the transformation into the form of Equation (16) leads to the vanishing of commutators through a certain order in $\frac{1}{L}$. The most common ordering scheme is symmetrization.[13–16] An elementary application of this technique is given in the example that follows. As in the case when additional commutators are computed, the price paid for increasing the power of $\frac{1}{L}$ in the ordering schemes is an increase in the number of Lie transformations compared with $\mathcal{M}_N^{(L)}$.

The method described above which uses powers of $\frac{1}{L}$ to suppress unwanted terms is similar to the construction of explicit symplectic integrators, with $\frac{1}{L}$ playing the role of the time step. The two approaches differ in that in the present one z is the primary small parameter (for example, in $\mathcal{M}_N^{(L)}$ terms that are multiplied by $\left(\frac{1}{L}\right)^2$ but have the power of z less than $N+1$ are kept); powers of $\frac{1}{L}$ provide additional suppression to terms that are already of higher order in z. It is worth noting that monomial factorization can provide an additional tool for the construction of symplectic integrators: Instead of relying on symmetrization to remove the commutators that cannot be explicitly evaluated, one can use Equations (11–14) to explicitly evaluate all commutators for Lie transformations generated by polynomials. There are two caveats. First, the computation of commutators does not in general require less work or lead to fewer Lie transformations than symmetrization. And second, in contrast to symmetrization, the difference between the exact and approximate map can be of any order in $\frac{1}{L}$, even or odd, and can contain both even and odd powers of $\frac{1}{L}$. Hence replacing $\frac{1}{L}$ by $-\frac{1}{L}$ in general yields the inverse of the approximate map only through the order of approximation in $\frac{1}{L}$.

3 EXAMPLE

To illustrate the use of monomial factorization, we apply the procedures described above to the map

$$\mathcal{M} = \exp(:h_6:), \tag{17a}$$

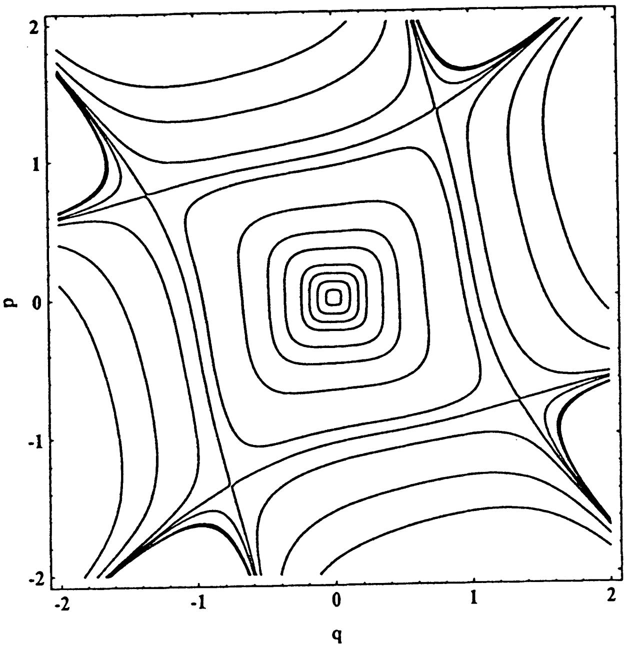

FIGURE 1: Contours of constant values of h_6 as given by Equation (17b). Each of the first seven rings contains a point that is chosen as an initial condition for the iterations of the map $\mathcal{M}_6'$ displayed in Figure 2.

where

$$h_6 = \frac{1}{4}q^4 + \frac{1}{4}p^4 + \frac{1}{5}q^5 p - \frac{1}{5}qp^5 \ . \tag{17b}$$

The linear transformation is the identity . The phase-space contour of $\mathcal{M}$ is shown in Figure 1.

It is interesting to write $\mathcal{M}$ in terms of action and angle variables, $J = \frac{1}{2}(q^2 + p^2), \phi = \arctan(\frac{q}{p})$. $\mathcal{M}$ then reads

$$\mathcal{M} = \exp(: h_2 :) \exp(: h_6 :), \tag{18a}$$

where

$$h_2 = 2\pi n J \quad (n \, \text{integer}), \tag{18b}$$

and

$$h_6 = \frac{3}{4}J^2 + \frac{1}{4}J^2 \cos(4\phi) - \frac{2}{5}J^3 \sin(4\phi). \tag{18c}$$

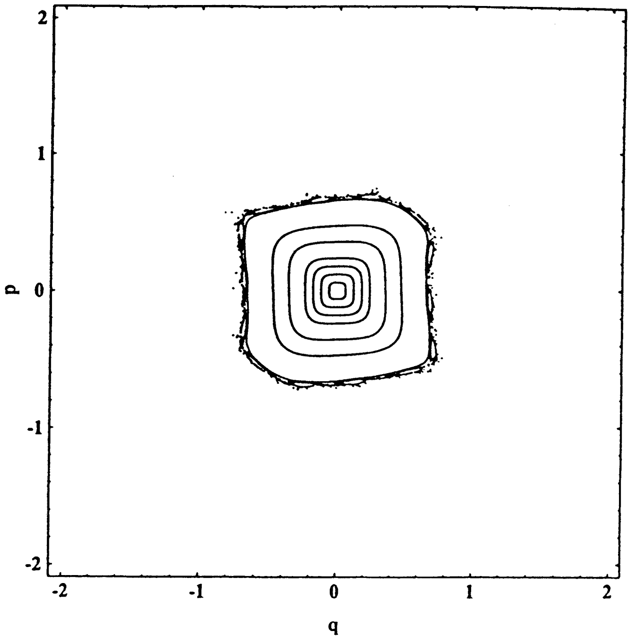

FIGURE 2: 3000 iterations of the map $\mathcal{M}_6'$ for each of the initial conditions along the $p=-q$ axis: $q=$ 0.05, 0.1, 0.15, 0.2, 0.3, 0.4, 0.56, 0.57.

We have explicitly displayed the identity transformation as a linear transformations with an integer unperturbed tune. The first term in h_6 represents anharmonicity (tune shift with amplitude). If the unperturbed tune were not an integer, then the second and third terms would drive the resonance at the shifted tune of $\frac{m}{4}$ (m integer). The choice of the identity for the linear transformation stems from the ease of drawing the phase-space portrait of $\mathcal{M}$ in this case, as well as from the fact that monomial factorization is not required for the linear part of the map.

Returning to the variables q and p, $\mathcal{M}$ can be factorized through sixth order into transformations generated by monomials as follows:

$$\mathcal{M}_6' = e^{\frac{1}{4}:q^4:} e^{\frac{1}{4}:p^4:} e^{\frac{1}{5}:q^5 p:} e^{-\frac{1}{2}:q^3p^3:} e^{-\frac{1}{5}:qp^5:}. \tag{19}$$

(The prime is affixed to $\mathcal{M}_6$ to distinguish it from maps considered later.) The third and fifth transformations have finite domains of absolute convergence,

$$q < \left(\frac{5}{4}\right)^{1/4} \; ; \quad p < \left(\frac{5}{4}\right)^{1/4} . \tag{20}$$

(Of course, neither of these transformations acts on initial conditions.) Figure 2 shows 3000 iterations of $\mathcal{M}_6'$ for eight initial conditions along the $p = -q$ axis. Evidently, $\mathcal{M}_6'$

reproduces the behavior of $\mathcal{M}$ well for small values of z, but fails to capture the entire stable region around the origin.

As discussed above, we can subdivide $\mathcal{M}$ into L steps and monomial-factorize each one of them. This procedure yields

$$\mathcal{M}_6'' = \left(e^{\frac{1}{4L}:q^4:}e^{\frac{1}{4L}:p^4:}e^{\frac{1}{5L}:q^5p:}e^{-\frac{1}{2L^2}:q^3p^3:}e^{-\frac{1}{5L}qp^5:}\right)^L. \tag{21}$$

The difference between $\mathcal{M}_6''$ and $\mathcal{M}$ in Lie generators is of order $\frac{1}{L}z^8$. We can also produce a symmetrized version of this decomposition:[13–16]

$$\mathcal{M}_6''' = (e^{\frac{1}{8L}:q^4:}e^{\frac{1}{8L}:p^4:}e^{\frac{1}{10L}:q^5p:}e^{-\frac{1}{5L}:qp^5:}e^{\frac{1}{10L}:q^5p:}e^{\frac{1}{8L}:p^4:}e^{\frac{1}{8L}:q^4:})^L. \tag{22}$$

When the Lie transformations that make up the Lth root of $\mathcal{M}_6'''$ are brought into a single exponent, the lowest-order commutators between the sixth-order terms, between the sixth- and fourth-order terms, and between the fourth-order terms vanish because of symmetrization (the reader may easily check this). Thus the difference in Lie generators between $\mathcal{M}$ and $\mathcal{M}_6'''$ is of order $\frac{1}{L^2}z^8$ (these terms come from double commutators between fourth-order terms). We do not employ here the procedure for constructing a map that agrees with $e^{\frac{1}{L}:h_6:}$ through order $\left(\frac{1}{L}\right)^2$ in Lie generators by computing the relevant commutators. Such a computation leads to 9 Lie transformations, compared with 7 in Equation (22).

Since $L \geq 1$, the analogues of Equation (20) for $\mathcal{M}_6''$ and $\mathcal{M}_6'''$ are less stringent. This fact is especially useful if repeated applications of the Lth root of these maps do not take the trajectory to infinity (that is, if the motion is bounded).

In monomial-factorizing $\mathcal{M}$ or $\mathcal{M}^{1/L}$ we can also include corrections of order z^8. The result is

$$\begin{aligned}\mathcal{M}_8 =&(e^{\frac{1}{4L}:q^4:}e^{\frac{1}{4L}:p^4:}e^{\frac{1}{5L}:q^5p:}e^{-\frac{1}{2L^2}:q^3p^3:}e^{-\frac{1}{5L}:qp^5:}\times\\ &e^{-\frac{1}{10L^2}:q^8:}e^{\frac{1}{2L^3}:q^6p^2:}e^{\frac{1}{L^2}:q^4p^4:}e^{-\frac{1}{L^3}:q^2p^6:}e^{-\frac{1}{10L^2}:p^8:})^L.\end{aligned} \tag{23}$$

The difference between $\mathcal{M}_8$ and $\mathcal{M}$ in Lie generators is of order $\frac{1}{L}z^{10}$.

For $L = 1$ and close to the origin $\mathcal{M}_8$ produces rings that are closer to the exact result than those of Figure 2. Neither $\mathcal{M}_8$ nor $\mathcal{M}_6'''$, however, captures the bounded trajectories that are present in Figure 1, but not in Figure 2. This fact is displayed in Figure 3 whose description is given below.

In many applications it may be of interest to find the last bounded trajectory (boundary curve) for some $\mathcal{M}$. In general $\mathcal{M}$ contains a linear transformation that is different from the identity, in which case the simple drawing of contours that we have used here cannot be done. Thus it is useful to know how well the monomial-factorized maps capture this aspect of the exact map. As $L \to \infty, \mathcal{M}_8, \mathcal{M}_6''$, and $\mathcal{M}_6'''$ all approach $\mathcal{M}$. We now examine how fast they reproduce the stable region of $\mathcal{M}$ with increasing L.

The value of h_6 on the boundary curve for maps $\mathcal{M}$ is approximately 0.30. On the $p = -q$ axis this value of h_6 corresponds to the value of q of 0.88. For the maps $\mathcal{M}_6'', \mathcal{M}_6'''$, and $\mathcal{M}_8$ we select a value of L, and vary q along the $p = -q$ axis until

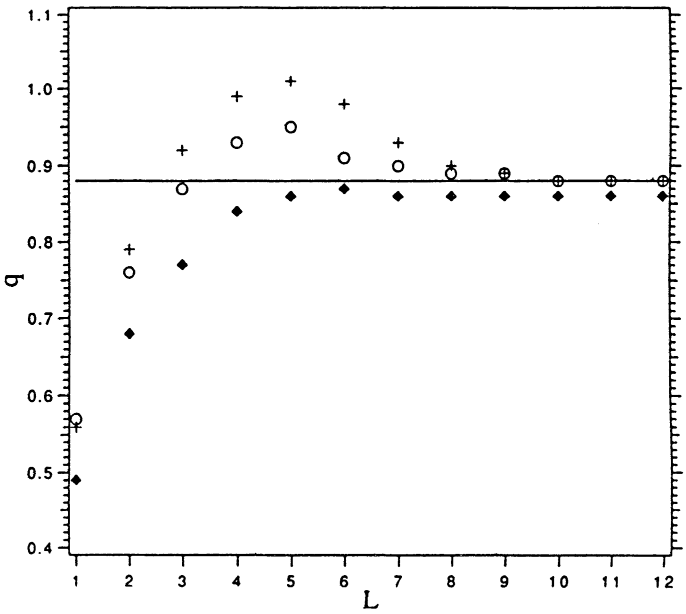

FIGURE 3: Location of boundary curves for $\mathcal{M}_6''$ (crosses), $\mathcal{M}_6'''$ (open circles), and $\mathcal{M}_8$ (solid diamonds) vs. L. The vertical axis labels the values of q along the $p = -q$ axis where the boundary curve lies. The solid line is the result for $\mathcal{M}$.

we find the largest value of q that gives a closed curve. This value of q is then plotted versus L for the three maps.

The results are displayed in Figure 3. The steps in q are 0.01, and for each initial condition the maps are iterated between 3000 and 50 000 times, depending on the resolution necessary to establish that a closed curve exists. When $L = 1$ $\mathcal{M}_8$ is appreciably less accurate than $\mathcal{M}_6''$ or $\mathcal{M}_6'''$ in locating the boundary curve (although, of course, as $z \to 0$ $\mathcal{M}_8 z$ approaches $\mathcal{M}z$ more rapidly than the other two transformations). We thus see that computing the monomial factorization of $\mathcal{M}$ to higher order in z than the highest one contained in h_N does not necessarily lead to an improved approximation at appreciable values of z. With increasing L, however, the locations of the boundary curves for $\mathcal{M}_8$ and $\mathcal{M}_6'''$ approach the one for the exact map more rapidly than does the location of the boundary curve for $\mathcal{M}_6''$. This does not mean, though, that $\mathcal{M}_6'''$ or $\mathcal{M}_8$ is more economical in terms of the number of Lie transformations required: to reach the accuracy in q of 0.02, $\mathcal{M}_8$ needs 50 Lie transformations ($L = 5$), $\mathcal{M}_6'''$ needs $49(L = 7)$, and $\mathcal{M}_6''$ needs $40(L = 8)$.It would be interesting to see if comparable statements hold for maps of physical interest. Finally we note that even at $L = 12$, the boundary curve for $\mathcal{M}_8$ is at $q = 0.86$. We have verified that this curve does move to the value of q of 0.88 when L is substantially increased (to the order of 1000).

4 SUMMARY

We have expressed an arbitrary nonlinear symplectic map given to a finite order in Lie generators as a composition of explicit maps that are exactly symplectic. We have also shown that this procedure can be used to evaluate the initial map explicitly to arbitrary accuracy. While we had assumed that the symplectic map was given as a single Lie transformation and then used the BCH formula to get the monomial factorization, we could have also extracted this factorization directly from the truncated power series (without the need for first computing the single Lie transformation). Efficient numerical techniques for performing this calculation are contained, for example, in automatic differentiation packages.[6–8] Furthermore, the prescription of Equation (15) can also be used directly on power series by taking the Lth root of the series. Whether this one or the procedure in the paper is adopted may depend on the radius of convergence of the operation on the series, as well as the computational effort required by each method.

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