

Quantum Grothendieck rings as quantum cluster algebras

Léa Bittmann

ABSTRACT

We define and construct a quantum Grothendieck ring for a certain monoidal subcategory of the category \mathcal{O} of representations of the quantum loop algebra introduced by Hernandez–Jimbo. We use the cluster algebra structure of the Grothendieck ring of this category to define the quantum Grothendieck ring as a quantum cluster algebra. When the underlying simple Lie algebra is of type A , we prove that this quantum Grothendieck ring contains the quantum Grothendieck ring of the category of finite-dimensional representations of the associated quantum affine algebra. In type A_1 , we identify remarkable relations in this quantum Grothendieck ring.

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1. Introduction

Let \mathfrak{g} be a simple Lie algebra of Dynkin type A , D or E (also called simply laced types), and let $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$ be the loop algebra of \mathfrak{g} . For q a generic complex number, Drinfeld [17] introduced a q -deformation of the universal enveloping algebra $U(L\mathfrak{g})$ of $L\mathfrak{g}$ called the *quantum loop algebra* $\mathcal{U}_q(L\mathfrak{g})$. It is a Hopf algebra over \mathbb{C} and therefore the category \mathcal{C} of its finite-dimensional representations is monoidal. The category \mathcal{C} was studied extensively, in particular to build solutions to the quantum Yang–Baxter equation with spectral parameter (see [1, 12–14, 24, 27, 38, 42, 46] to name but a few).

Using the so-called ‘Drinfeld–Jimbo’ presentation of the quantum loop algebra, one can define a *quantum Borel subalgebra* $\mathcal{U}_q(\mathfrak{b})$, which is a Hopf subalgebra of $\mathcal{U}_q(L\mathfrak{g})$. We are here interested in studying a category \mathcal{O} of representations of $\mathcal{U}_q(\mathfrak{b})$ introduced by Hernandez–Jimbo [32]. The category \mathcal{O} is a monoidal category which contains all finite-dimensional $\mathcal{U}_q(\mathfrak{b})$ -modules, as well as some infinite-dimensional representations, however, with finite-dimensional weight spaces. In particular, this category \mathcal{O} contains the *prefundamental representations*. These are a family of infinite-dimensional simple $\mathcal{U}_q(\mathfrak{b})$ -modules, which first appeared in the work of Bazhanov, Lukyanov, Zamolodchikov [3] for $\mathfrak{g} = \mathfrak{sl}_2$ under the name *q -oscillator representations*.

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These prefundamental representations were also used by Frenkel–Hernandez [22] to prove Frenkel–Reshetikhin’s conjecture on the spectra of quantum integrable systems [25]. More precisely, quantum integrable systems are studied via a partition function Z , which in turns can be scaled down to the study of the eigenvalues λ_j of the transfer matrix T . For the 6-vertex (and 8-vertex) models, [2] showed that the eigenvalues of T have the following remarkable form:

$$\lambda_j = A(z) \frac{Q_j(zq^{-2})}{Q_j(z)} + D(z) \frac{Q_j(zq^2)}{Q_j(z)}, \quad (1.1)$$

where q and z are parameters of the model, the rational functions $A(z)$, $D(z)$ are universal and Q_j is a polynomial. This relation is called the *Baxter relation*. In the context of representation theory, relation (1.1) can be categorified as a relation in the Grothendieck ring of the category \mathcal{O} . For $\mathfrak{g} = \mathfrak{sl}_2$, if V is the two-dimensional simple representation of $\mathcal{U}_q(L\mathfrak{g})$ of highest loop-weight $Y_{aq^{-1}}$, then, in the Grothendieck ring $K_0(\mathcal{O})$,

$$[V \otimes L_a^+] = [\omega_1][L_{aq^{-2}}^+] + [-\omega_1][L_{aq^2}^+], \quad (1.2)$$

where $[\pm\omega_1]$ are one-dimensional representations of weight $\pm\omega_1$ and L_a^+ denotes the positive prefundamental representation of quantum parameter a .

Frenkel–Reshetikhin’s conjecture stated that for more general quantum integrable systems, constructed via finite-dimensional representations of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ (of which the quantum loop algebra is a quotient) the spectra had a similar form as relation (1.1).

Let t be an indeterminate. The Grothendieck ring of the category \mathcal{C} has an interesting t -deformation called the *quantum Grothendieck ring*, which is contained in some quantum torus \mathcal{Y}_t . The quantum Grothendieck ring was first studied by Nakajima [42] and Varagnolo–Vasserot [45] in relation with quiver varieties. Inside this ring, one can define for all simple modules L classes $[L]_t$, called (q, t) -characters. Using these classes, and the knowledge of characters of fundamental modules, Nakajima was able to compute the characters of all simple modules L , thanks to a Kazhdan–Lusztig type algorithm.

One would want to extend these results to the context of the category \mathcal{O} , with the ultimate goal of (algorithmically) computing characters of all simple modules in \mathcal{O} . In order to do that, one first needs to build a quantum Grothendieck ring $K_t(\mathcal{O})$ inside which the classes $[L]_t$ can be defined.

Another interesting approach to this category \mathcal{O} is its *cluster algebra* structure (see below). Hernandez–Leclerc [34] first noted that the Grothendieck ring of a certain monoidal subcategory \mathcal{C}_1 of the category \mathcal{C} of finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$ -modules had the structure of a cluster algebra. Then, they proved [36] that the Grothendieck ring of a certain monoidal subcategory $\mathcal{O}_{\mathbb{Z}}^+$ of the category \mathcal{O} had a cluster algebra structure, of infinite rank, for which one can take as initial seed the classes of the positive prefundamental representations (the category $\mathcal{O}_{\mathbb{Z}}^+$ contains the finite-dimensional representations and the positive prefundamental representations whose spectral parameter satisfy an integrality condition). Moreover, some exchange relations coming from cluster mutations appear naturally. For example, the Baxter relation (1.2) is an exchange relation in this cluster algebra.

In order to construct of quantum Grothendieck ring for the category \mathcal{O} , the approaches used previously are not applicable anymore. The geometrical approach of Nakajima and Varagnolo–Vasserot (in which the t -graduation naturally comes from the graduation of cohomological complexes) requires a geometric interpretation of the objects in the category \mathcal{O} , which has not yet been found. The more algebraic approach consisting of realizing the (quantum) Grothendieck ring as an invariant under a sort of Weyl symmetry, which allowed Hernandez to define a quantum Grothendieck ring of finite-dimensional representations in non-simply laced types, is again not relevant for the category \mathcal{O} . Only the cluster algebra approach yields results in this context.

In this paper, we propose to build the quantum Grothendieck of the category $\mathcal{O}_{\mathbb{Z}}^+$ as a *quantum cluster algebra*. Quantum cluster algebras are non-commutative versions of cluster algebras, they live inside a quantum torus, generated by the initial variables, together with t -commutation relations:

$$X_i * X_j = t^{\Lambda_{ij}} X_j * X_i. \quad (1.3)$$

First of all, one has to build such a quantum torus \mathcal{T}_t , and check that it contains the quantum torus \mathcal{Y}_t of the quantum Grothendieck ring of the category \mathcal{C} . This is proven as the first result of this paper (see Theorem 5.2.1) :

THEOREM. *There is an injective homomorphism of $\mathbb{Z}(t)$ -algebras :*

$$\mathcal{J} : \mathcal{T}_t \rightarrow \mathcal{Y}_t.$$

Next, one has to show that this quantum torus is compatible with a quantum cluster algebra structure based on the same quiver as the cluster algebra structure of the Grothendieck ring $K_0(\mathcal{O}_{\mathbb{Z}}^+)$. In order to do that, we exhibit a *compatible pair* (see Proposition 6.2.5) :

PROPOSITION. *The quiver appearing in the cluster algebra structure of $K_0(\mathcal{O}_{\mathbb{Z}}^+)$ and the quantum torus \mathcal{T}_t form a compatible pair, in the sense of quantum cluster algebras.*

From then, the quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ is defined as the quantum cluster algebra defined from this compatible pair.

We then conjecture (Conjecture 7.2.1) that this quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ contains the quantum Grothendieck ring $K_t(\mathcal{C}_{\mathbb{Z}})$. We propose to demonstrate this conjecture by proving that $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ contains the (q, t) -characters of the fundamental representations $[L(Y_{i, q^r})]_t$, as they generate $K_t(\mathcal{C}_{\mathbb{Z}})$. We state in Conjecture 7.2.6 that these objects can be obtained in $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ as quantum cluster variables, by following the same finite sequences of mutations used in the classical cluster algebra $K_0(\mathcal{O}_{\mathbb{Z}}^+)$ to obtain the $[L(Y_{i, q^r})]$. Naturally, Conjecture 7.2.6 implies Conjecture 7.2.1. Finally, we prove Conjecture 7.2.6 (and thus Conjecture 7.2.1) in the case where the underlying simple Lie algebra \mathfrak{g} is of type A (see Theorem 8.1.1).

THEOREM. *When the underlying simple Lie algebra is of type A , the constructed quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ contains the quantum Grothendieck ring $K_t(\mathcal{C}_{\mathbb{Z}})$.*

The proof is based on the thinness property of the fundamental representations in this case. When $\mathfrak{g} = \mathfrak{sl}_2$, some explicit computations are possible. For example, we give a quantum version of the Baxter relation (8.5), for all $r \in \mathbb{Z}$,

$$[V_{q^{2r-1}}]_t * [L_{1, q^{2r}}^+]_t = t^{-1/2} [\omega_1]_t [L_{1, q^{2r-2}}^+]_t + t^{1/2} [-\omega_1]_t [L_{1, q^{2r+2}}^+]_t.$$

Additionally, we realize a part of the quantum cluster algebra we built as a quotient of the Drinfeld double of the full quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. This is a reminiscence of the result of Qin [43] who constructed $\mathcal{U}_q(\mathfrak{g})$ as a quotient of the Grothendieck ring arising from certain cyclic quiver varieties.

The paper is organized as follows. The first three sections are mostly reminders. In Section 2 we recall some background on cluster algebras and quantum cluster algebras, including some recent and important results, such as the positivity theorem in Section 2.6, which we require later on. In Section 3 we introduce some notations, the usual notations for the Cartan data associated to a simple Lie algebra, as well as what we call *quantum Cartan data*, which is related to the quantum Cartan matrix and its inverse. In Section 4 we review some results for the category \mathcal{O} , its subcategories \mathcal{O}^\pm and $\mathcal{O}_{\mathbb{Z}}^\pm$ and their Grothendieck rings. In Section 5, after recalling the definition of the quantum torus \mathcal{Y}_t , we define the quantum torus \mathcal{T}_t in which

$K_t(\mathcal{O}_{\mathbb{Z}}^+)$ will be constructed and we prove the inclusion of the quantum tori. In Section 6 we prove that we have all the elements to build a quantum cluster algebra and we define the quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$. In the concluding Section 7, we state some properties of the quantum Grothendieck ring. We present the two conjectures regarding the inclusion of the quantum Grothendieck rings in Section 7.2. Finally, in Section 8 we prove these conjectures in type A , and we prove finer properties specific to the case when $\mathfrak{g} = \mathfrak{sl}_2$.

2. Cluster algebras and quantum cluster algebras

Cluster algebras were defined by Fomin and Zelevinsky in the 2000s in a series of fundamental papers [4, 18, 20, 21]. They were first introduced to study total positivity and canonical bases in quantum groups but soon applications to many different fields of mathematics were found.

In [6], Berenstein and Zelevinsky introduced natural non-commutative deformations of cluster algebras called *quantum cluster algebras*.

In this section, we recall the definitions of these objects. The interested reader may refer to the aforementioned papers for more details, or to surveys such as [19].

2.1. Cluster algebras

Let $m \geq n$ be two positive integers and let \mathcal{F} be the field of rational functions over \mathbb{Q} in m independent commuting variables. Fix of subset $\mathbf{ex} \subset \llbracket 1, m \rrbracket$ of cardinal n .

In what follows, we use the usual notation: $[x]_+ = \max(x, 0)$.

DEFINITION 2.1.1. A seed in \mathcal{F} is a pair $(\tilde{\mathbf{x}}, \tilde{B})$, where

- $\tilde{\mathbf{x}} = \{x_1, \dots, x_m\}$ is an algebraically independent subset of \mathcal{F} which generates \mathcal{F} .
- $\tilde{B} = (b_{i,j})$ of \tilde{B} is a $m \times n$ integer matrix with rows labeled by $\llbracket 1, m \rrbracket$ and columns labeled by \mathbf{ex} such that
 - (1) the $n \times n$ submatrix $B = (b_{i,j})_{i,j \in \mathbf{ex}}$ is skew-symmetrizable.
 - (2) \tilde{B} has full rank n .

The matrix B is called the *principal part* of \tilde{B} , $\mathbf{x} = \{x_j \mid j \in \mathbf{ex}\} \subset \tilde{\mathbf{x}}$ is the *cluster* of the seed $(\tilde{\mathbf{x}}, \tilde{B})$, \mathbf{ex} are the *exchangeable indices* and $\mathbf{c} = \tilde{\mathbf{x}} \setminus \mathbf{x}$ is the set of *frozen variables*.

For all $k \in \mathbf{ex}$, define the *seed mutation* in direction k as the transformation from $(\tilde{\mathbf{x}}, \tilde{B})$ to $\mu_k(\tilde{\mathbf{x}}, \tilde{B}) = (\tilde{\mathbf{x}}', \tilde{B}')$, with

- $\tilde{B}' = \mu_k(\tilde{B})$ is the $m \times n$ matrix whose entries are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik} [b_{kj}]_+ & \text{otherwise.} \end{cases} \quad (2.1)$$

This operation is called *matrix mutation* in direction k . This matrix can also be obtained via the operation

$$\tilde{B}' = \mu_k(\tilde{B}) = E_k \tilde{B} F_k, \quad (2.2)$$

where E_k and F_k are the $m \times m$ and $n \times n$ matrices with entries

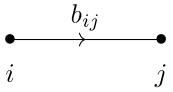
$$(E_k)_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k, \\ -1 & \text{if } j = i = k, \\ [-b_{ik}]_+ & \text{if } i \neq j = k, \end{cases} \quad (F_k)_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k, \\ -1 & \text{if } j = i = k, \\ [b_{kj}]_+ & \text{if } i = k \neq j. \end{cases} \quad (2.3)$$

- $\tilde{\mathbf{x}}' = (\tilde{\mathbf{x}} \setminus \{x_k\}) \cup \{x'_k\}$, where $x'_k \in \mathcal{F}$ is determined by the exchange relation

$$x_k x'_k = \prod_{\substack{i \in [1, m] \\ b_{ik} > 0}} x_i^{b_{ik}} + \prod_{\substack{i \in [1, m] \\ b_{ik} < 0}} x_i^{-b_{ik}}. \quad (2.4)$$

REMARK 2.1.2. $(\tilde{\mathbf{x}}', \tilde{B}')$ is also a seed in \mathcal{F} and the seed mutation operation is involutive: $\mu_k(\tilde{\mathbf{x}}', \tilde{B}') = (\tilde{\mathbf{x}}, \tilde{B})$. Thus, we have a equivalence relation: $(\tilde{\mathbf{x}}, \tilde{B})$ is mutation-equivalent to $(\tilde{\mathbf{x}}', \tilde{B}')$, denoted by $(\tilde{\mathbf{x}}, \tilde{B}) \sim (\tilde{\mathbf{x}}', \tilde{B}')$, if $(\tilde{\mathbf{x}}', \tilde{B}')$ can be obtained from $(\tilde{\mathbf{x}}, \tilde{B})$ by a finite sequence of seed mutations.

Graphically, if the matrix \tilde{B} is skew-symmetric, it can be represented by a quiver and the matrix mutation by a simple operation on the quiver. Fix \tilde{B} a skew-symmetric matrix. Define the quiver Q whose set of vertices is $\llbracket 1, m \rrbracket$, where the vertices corresponding to \mathbf{c} are usually denoted by a square \square and called frozen vertices. For all $i \in \llbracket 1, m \rrbracket$, $j \in \mathbf{ex}$, b_{ij} is the number of arrows from i to j (can be negative if the arrows are from j to i).



In this context, the operation of matrix mutation can be translated naturally to an operation on the quiver Q . For $k \in \mathbf{ex}$, the quiver $Q' = \mu_k(Q)$ is obtained from Q by the following operations:

- For each pair of arrows $i \rightarrow k \rightarrow j$ in Q , create an arrow from i to j .
- Invert all arrows adjacent to k .
- Remove all 2-cycles that were possibly created.

DEFINITION 2.1.3. Let \mathcal{S} be a mutation-equivalence class of seeds in \mathcal{F} . The *cluster algebra* $\mathcal{A}(\mathcal{S})$ associated to \mathcal{S} is the $\mathbb{Z}[\mathbf{c}^\pm]$ -subalgebra of \mathcal{F} generated by all the clusters of all the seeds in \mathcal{S} .

2.2. Compatible pairs

A quantum cluster algebra is a non-commutative version of a cluster algebra. Cluster variables will not commute anymore, but, if they are in the same cluster, commute up to some power of an indeterminate t . These powers can be encoded in a skew-symmetric matrix Λ . In order for the quantum cluster algebra to be well defined, one needs to check that these t -commutation relations behave well with the exchange relations. This is made explicit via the notion of compatible pairs.

DEFINITION 2.2.1. Let \tilde{B} be a $m \times n$ integer matrix, with rows labeled by $\llbracket 1, m \rrbracket$ and columns labeled by \mathbf{ex} . Let $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq m}$ be a skew-symmetric $m \times m$ integer matrix. We say that (Λ, \tilde{B}) forms a *compatible pair* if, for all $i \in \mathbf{ex}$ and $1 \leq j \leq m$, we have

$$\sum_{k=1}^m b_{ki} \lambda_{kj} = \delta_{i,j} d_i, \quad (2.5)$$

with $(d_i)_{i \in \mathbf{ex}}$ some integers, all positive or negative. Relation (2.5) is equivalent to saying that, up to reordering, if $\mathbf{ex} = \llbracket 1, n \rrbracket$, the matrix $\tilde{B}^T \Lambda$ consists of two blocks, a diagonal $n \times n$ block, and a $n \times (m - n)$ zero block:

$$\left(\begin{array}{c|c} d_1 & \\ d_2 & \\ \vdots & \\ d_n & \end{array} \right) \quad (0)$$

Fix a compatible pair (Λ, \tilde{B}) and fix $k \in \mathbf{ex}$. Define, in a similar way as in (2.2),

$$\Lambda' = \mu_k(\Lambda) := E_k^T \Lambda E_k, \quad (2.6)$$

with E_k from (2.3).

PROPOSITION 2.2.2 [6]. *The pair (Λ', \tilde{B}') is compatible.*

We say that (Λ', \tilde{B}') is the *mutation* in direction k of the pair (Λ, \tilde{B}) , and we will use the notation:

$$\mu_k(\Lambda, \tilde{B}) := \left(\mu_k(\Lambda), \mu_k(\tilde{B}) \right) = (\Lambda', \tilde{B}'). \quad (2.7)$$

PROPOSITION 2.2.3 [6]. *The mutation of a compatible pair is involutive. For any compatible pair (Λ, \tilde{B}) and any mutation direction $k \in \mathbf{ex}$, $\mu_k(\mu_k(\Lambda, \tilde{B})) = (\Lambda, \tilde{B})$.*

2.3. Definition of quantum cluster algebras

We now introduce the last notions we need in order to define quantum cluster algebras.

Let t be a formal variable. Consider $\mathbb{Z}[t^{\pm 1/2}]$, the ring of Laurent polynomials in the variable $t^{1/2}$.

Recall that any skew-symmetric integer matrix Λ of size $m \times m$ determines a skew-symmetric \mathbb{Z} -bilinear form on \mathbb{Z}^m , which will also be denoted by Λ :

$$\Lambda(\mathbf{e}_i, \mathbf{e}_j) := \lambda_{i,j}, \quad \forall i, j \in \llbracket 1, m \rrbracket, \quad (2.8)$$

where $\{\mathbf{e}_i \mid 1 \leq i \leq m\}$ is the standard basis of \mathbb{Z}^m .

DEFINITION 2.3.1. The (based) *quantum torus* $\mathcal{T} = (\mathcal{T}(\Lambda), *)$ associated with the skew-symmetric bilinear form Λ is the $\mathbb{Z}[t^{\pm 1/2}]$ -algebra generated by the $\{X^{\mathbf{e}} \mid \mathbf{e} \in \mathbb{Z}^m\}$, together with the t -commuting relations:

$$X^{\mathbf{e}} * X^{\mathbf{f}} = t^{\Lambda(\mathbf{e}, \mathbf{f})/2} X^{\mathbf{e} + \mathbf{f}} = t^{\Lambda(\mathbf{e}, \mathbf{f})} X^{\mathbf{f}} * X^{\mathbf{e}}, \quad \forall \mathbf{e}, \mathbf{f} \in \mathbb{Z}^m. \quad (2.9)$$

The quantum torus $\mathcal{T}(\Lambda)$ is an Ore domain (see details in [6]), thus it is contained in its skew-field a fractions $\mathcal{F} = (\mathcal{F}, *)$. The field \mathcal{F} is a $\mathbb{Q}(t^{1/2})$ -algebra.

DEFINITION 2.3.2. A *toric frame* in \mathcal{F} is a map $M : \mathbb{Z}^m \rightarrow \mathcal{F} \setminus \{0\}$ of the form

$$M(c) = \phi(X^{\eta(c)}), \quad \forall c \in \mathbb{Z}^m, \quad (2.10)$$

where $\phi : \mathcal{F} \rightarrow \mathcal{F}$ is a $\mathbb{Q}(t^{1/2})$ -algebra automorphism and $\eta : \mathbb{Z}^m \rightarrow \mathbb{Z}^m$ is an isomorphism of \mathbb{Z} -modules.

For any toric frame M , define $\Lambda_M : \mathbb{Z}^m \times \mathbb{Z}^m \rightarrow \mathbb{Z}$, a skew-symmetric bilinear form, by

$$\Lambda_M(\mathbf{e}, \mathbf{f}) = \Lambda(\eta(\mathbf{e}), \eta(\mathbf{f})), \quad \forall \mathbf{e}, \mathbf{f} \in \mathbb{Z}^m. \quad (2.11)$$

Then,

$$M(\mathbf{e}) * M(\mathbf{f}) = t^{\Lambda_M(\mathbf{e}, \mathbf{f})/2} M(\mathbf{e} + \mathbf{f}) = t^{\Lambda_M(\mathbf{e}, \mathbf{f})} M(\mathbf{f}) * M(\mathbf{e}). \quad (2.12)$$

DEFINITION 2.3.3. A *quantum seed* in \mathcal{F} is a pair (M, \tilde{B}) , where

- M is a toric frame in \mathcal{F} ;
- \tilde{B} is an $m \times \mathbf{ex}$ integer matrix;
- the pair (Λ_M, \tilde{B}) is compatible, as in Definition 2.2.1.

Next, we need to define mutations of quantum seeds. Let (M, \tilde{B}) be a quantum seed, and fix $k \in \mathbf{ex}$. Define $M' : \mathbb{Z}^m \rightarrow \mathcal{F} \setminus \{0\}$ by setting

$$M'(\mathbf{f}) = \begin{cases} \sum_{p=0}^k \binom{f_k}{p} t^{d_k/2} M(E_k \mathbf{f} + pb^k) & \text{if } f_k \geq 0, \\ M'(-\mathbf{f})^{-1} & \text{otherwise,} \end{cases}$$

where E_k is the matrix from (2.3), $b^k \in \mathbb{Z}^m$ is the k th column of \tilde{B} and the t -binomial coefficient is defined by

$$\begin{bmatrix} r \\ p \end{bmatrix}_t := \frac{(t^r - t^{-r})(t^{r-1} - t^{-r+1}) \cdots (t^{r-p+1} - t^{-r+p-1})}{(t^p - t^{-p})(t^{p-1} - t^{-p+1}) \cdots (t - t^{-1})}, \quad \forall 0 \leq p \leq r. \quad (2.13)$$

Recall the definition of the mutated matrix $\tilde{B}' = \mu_k(\tilde{B})$ from Section 2.1. Then the *mutation* in direction k of the quantum seed (M, \tilde{B}) is the pair $\mu_k(M, \tilde{B}) = (M', \tilde{B}')$

PROPOSITION 2.3.4 [6]. (1) *The pair (M', \tilde{B}') is a quantum seed.*

(2) *The mutation in direction k of the compatible pair (Λ_M, \tilde{B}) is the pair $(\Lambda_{M'}, \tilde{B}')$.*

For a quantum seed (M, \tilde{B}) , let $\tilde{\mathbf{X}} = \{X_1, \dots, X_m\}$ be the free generating set of \mathcal{F} , given by $X_i := M(\mathbf{e}_i)$. Let $\mathbf{X} = \{X_i \mid i \in \mathbf{ex}\}$, we call it the *cluster* of the quantum seed (M, \tilde{B}) , and let $\mathbf{C} = \tilde{\mathbf{X}} \setminus \mathbf{X}$.

For all $k \in \mathbf{ex}$, if $(M', \tilde{B}') = \mu_k(M, \tilde{B})$, then the $X'_i = M'(\mathbf{e}_i)$ are obtained by:

$$X'_i = \begin{cases} X_i & \text{if } i \neq k, \\ M(-\mathbf{e}_k + \sum_{b_{ik} > 0} b_{ik} \mathbf{e}_i) + M(-\mathbf{e}_k - \sum_{b_{ik} < 0} b_{ik} \mathbf{e}_i) & \text{if } i = k. \end{cases} \quad (2.14)$$

The mutation of quantum seeds, as the mutation of compatible pairs, is an involutive process: $\mu_k(M', \tilde{B}') = (M, \tilde{B})$. Thus, as before, we have an equivalence relation: two quantum seeds (M_1, \tilde{B}_1) and (M_2, \tilde{B}_2) are *mutation equivalent* if (M_2, \tilde{B}_2) can be obtained from (M_1, \tilde{B}_1) by a sequence of quantum seed mutations. From (2.14), the set \mathbf{C} only depends on the mutation-equivalence class of the quantum seed. The variables in \mathbf{C} , $(X_i)_{i \notin \mathbf{ex}}$, are called the *frozen variable* of the mutation-equivalence class.

DEFINITION 2.3.5. Let \mathcal{S} be a mutation-equivalence class of quantum seeds in \mathcal{F} and \mathbf{C} the set of its frozen variables. The *quantum cluster algebra* $\mathcal{A}(\mathcal{S})$ associated with \mathcal{S} is the $\mathbb{Z}[t^{\pm 1/2}]$ -subalgebra of the skew-field \mathcal{F} generated by the union of all clusters in all seeds in \mathcal{S} , together with the elements of \mathbf{C} and their inverses.

2.4. *Laurent phenomenon and quantum Laurent phenomenon*

One of the main properties of cluster algebras is the so-called *Laurent phenomenon* which was formulated in [4]. Quantum cluster algebras present a counterpart to this result called the *quantum Laurent phenomenon*.

Here, we follow [6, Section 5]. In order to state this result, one needs the notion of *upper cluster algebras*.

Fix (M, \tilde{B}) a quantum seed, and $\tilde{\mathbf{X}} = \{X_1, \dots, X_m\}$ given by $X_k = M(\mathbf{e}_k)$. Let $\mathbb{Z}\mathbb{P}[\mathbf{X}^{\pm 1}]$ denote the based quantum torus generated by the $(X_k)_{1 \leq k \leq m}$; it is a $\mathbb{Z}[t^{\pm 1/2}]$ -subalgebra of \mathcal{F} with basis $\{M(c) \mid c \in \mathbb{Z}^m\}$, such that the ground ring $\mathbb{Z}\mathbb{P}$ is the ring of integer Laurent polynomials in the variables $t^{1/2}$ and $(X_j)_{j \notin \mathbf{ex}}$. For $k \in \mathbf{ex}$, let (M_k, \tilde{B}_k) be the quantum

seed obtained from (M, \tilde{B}) by mutation in direction k , and let \mathbf{X}_k denote its cluster, thus:

$$\mathbf{X}_k = (\mathbf{X} \setminus \{X_k\}) \cup \{X'_k\}.$$

Define the quantum upper cluster algebra as the $\mathbb{Z}\mathbb{P}$ -subalgebra of \mathcal{F} given by

$$\mathcal{U}(M, \tilde{B}) := \mathbb{Z}\mathbb{P}[\mathbf{X}^{\pm 1}] \cap \bigcap_{k \in \mathbf{ex}} \mathbb{Z}\mathbb{P}[\mathbf{X}_k^{\pm 1}]. \quad (2.15)$$

THEOREM 2.4.1 [6, Theorem 5.1]. *The quantum upper algebra $\mathcal{U}(M, \tilde{B})$ depends only on the mutation-equivalence class of the quantum seed (M, \tilde{B}) .*

Thus we use the notation: $\mathcal{U}(M, \tilde{B}) = \mathcal{U}(\mathcal{S})$, where \mathcal{S} is the mutation-equivalence class of (M, \tilde{B}) , one has

$$\mathcal{U}(\mathcal{S}) = \bigcap_{(M, \tilde{B}) \in \mathcal{S}} \mathbb{Z}\mathbb{P}[\mathbf{X}^{\pm 1}]. \quad (2.16)$$

Theorem 2.4.1 has the following important corollary, which we refer to as the *quantum Laurent phenomenon*.

COROLLARY 2.4.2 [6, Corollary 5.2]. *The cluster algebra $\mathcal{A}(\mathcal{S})$ is contained in $\mathcal{U}(\mathcal{S})$. Equivalently, $\mathcal{A}(\mathcal{S})$ is contained in the quantum torus $\mathbb{Z}\mathbb{P}[\mathbf{X}^{\pm 1}]$ for every quantum seed $(M, \tilde{B}) \in \mathcal{S}$ of cluster \mathbf{X} .*

2.5. Specializations of quantum cluster algebras

Fix a quantum seed (M, \tilde{B}) and \mathbf{X} its cluster. Using notations from Section 2.1, the based quantum torus $\mathbb{Z}\mathbb{P}[\mathbf{X}^{\pm 1}]$ specializes naturally at $t = 1$, via the ring morphism:

$$\pi : \mathbb{Z}\mathbb{P}[\mathbf{X}^{\pm 1}] \rightarrow \mathbb{Z}[\tilde{\mathbf{X}}^{\pm 1}], \quad (2.17)$$

such that

$$\begin{aligned} \pi(X_k) &= x_k, \quad (1 \leq k \leq m) \\ \pi(t^{\pm 1/2}) &= 1. \end{aligned}$$

If we restrict this morphism to the quantum cluster algebra $\mathcal{A}(\mathcal{S})$, it is not clear that we recover the (classical) cluster algebra $\mathcal{A}(\tilde{B})$. This question was tackled in a recent paper by Geiss, Leclerc and Schröer [26].

REMARK 2.5.1. For a combinatorial point of view, the cluster algebras $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\tilde{B})$ are constructed on the same quiver \tilde{B} , and the mutations have the same effect on the quiver. Assume the initial seeds are fixed and identified, via the morphism (2.17). Then, each quantum cluster variable in $\mathcal{A}(\mathcal{S})$ is identified to a cluster variable in $\mathcal{A}(\tilde{B})$.

PROPOSITION 2.5.2 [26, Lemma 3.3]. *The restriction of π to $\mathcal{A}(\mathcal{S})$ is surjective on $\mathcal{A}(\tilde{B})$, and quantum cluster variables are sent to the corresponding cluster variables.*

They also conjectured that the specialization at $t = 1$ of the quantum cluster algebra is isomorphic to the classical cluster algebra, and gave a proof under some assumptions on the initial seed.

Nevertheless, by applying Proposition 2.5.2 to different seeds (while keeping the identification (2.17) of the initial seeds), one gets

COROLLARY 2.5.3. *The evaluation morphism π sends all quantum cluster monomials to the corresponding cluster monomials.*

2.6. Positivity

Let us state a last general result on quantum cluster algebras: Davison's positivity theorem [15].

We have recalled in Section 2.4 that each (quantum) cluster variable can be written as a Laurent polynomial in the initial (quantum) cluster variables (and $t^{1/2}$). For classical cluster algebras, Fomin–Zelevinski conjectured that these Laurent polynomials have positive coefficients. The so-called *positivity conjecture* was proven by Lee–Schiffler in [40].

For quantum cluster algebras, the result is the following.

THEOREM 2.6.1 [15, Theorem 2.4]. *Let \mathcal{A} be a quantum cluster algebra defined by a compatible pair (Λ, \tilde{B}) . For a mutated toric frame M' and a quantum cluster monomial Y , let us write:*

$$Y = \sum_{\mathbf{e} \in \mathbb{Z}^m} a_{\mathbf{e}}(t^{1/2}) M'(\mathbf{e}), \quad (2.18)$$

with $a_{\mathbf{e}} \in \mathbb{Z}[t^{\pm 1/2}]$. Then $a_{\mathbf{e}} = t^{-\deg(b_{\mathbf{e}})/2} b_{\mathbf{e}}$ for some $b_{\mathbf{e}} \in \mathbb{N}[t]$.

3. Cartan data and quantum Cartan data

We fix here some notations for the rest of the paper.

3.1. Root data

Let the \mathfrak{g} be a simple finite-dimensional Lie algebra of type A, D or E , and let $I := \{1, \dots, n\}$ be the indexed set of its Dynkin diagram.

Fix simple coroots $(\alpha_i^\vee)_{i \in I}$ of \mathfrak{g} and let $(\alpha_i)_{i \in I}$, $(\omega_i)_{i \in I}$ be the corresponding sets of simple roots (respectively, fundamental weights). We will use the usual lattices $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$, $Q^+ = \bigoplus_{i \in I} \mathbb{N} \alpha_i$ and $P = \bigoplus_{i \in I} \mathbb{Z} \omega_i$. Let $P_{\mathbb{Q}} = P \otimes \mathbb{Q}$, endowed with the partial ordering : $\omega \leq \omega'$ if and only if $\omega' - \omega \in Q^+$.

The Dynkin diagram of \mathfrak{g} is numbered as in [39], and let a_1, a_2, \dots, a_n be the Kac labels ($a_0 = 1$).

The *Cartan matrix* of \mathfrak{g} is the $n \times n$ matrix C such that $C_{i,j} = \alpha_j(\alpha_i^\vee)$. As \mathfrak{g} is of simply laced type:

$$C_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \sim j \quad (i \text{ and } j \text{ are adjacent in the Dynkin diagram of } \mathfrak{g}), \\ 0 & \text{otherwise.} \end{cases}$$

3.2. Quantum Cartan matrix

Let z be an indeterminate.

DEFINITION 3.2.1. *The quantum Cartan matrix of \mathfrak{g} is the matrix $C(z)$ with entries,*

$$C_{ij}(z) = \begin{cases} z + z^{-1} & \text{if } i = j, \\ -1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 3.2.2. The evaluation $C(1)$ is the Cartan matrix of \mathfrak{g} . As $\det(C) \neq 0$, then $\det(C(z)) \neq 0$ and $\det(C(z))$ is invertible as a formal Laurent series with degree bounded from below. Thus we can define $\tilde{C}(z)$, the inverse of the matrix $C(z)$. The entries of the matrix $\tilde{C}(z)$ belong to $\mathbb{Q}((z))$.

One can writexc

$$C(z) = (z + z^{-1}) \text{Id} - A,$$

where A is the adjacency matrix of the Dynkin diagram of \mathfrak{g} . Hence,

$$\tilde{C}(z) = \sum_{m=0}^{+\infty} (z + z^{-1})^{-m-1} A^m.$$

Therefore, we can write the entries of $\tilde{C}(z)$ as power series in z . For all $i, j \in I$,

$$\tilde{C}_{ij}(z) = \sum_{m=1}^{+\infty} \tilde{C}_{i,j}^{(m)} z^m \in \mathbb{Z}[[z]]. \quad (3.1)$$

EXAMPLE 3.2.3. (i) For $\mathfrak{g} = \mathfrak{sl}_2$, one has

$$\tilde{C}_{11} = \sum_{n=0}^{+\infty} (-1)^n z^{2n+1} = z - z^3 + z^5 - z^7 + z^9 - z^{11} + \dots \quad (3.2)$$

(ii) For $\mathfrak{g} = \mathfrak{sl}_3$, one has

$$\tilde{C}_{ii} = z - z^5 + z^7 - z^{11} + z^{13} + \dots, \quad 1 \leq i \leq 2$$

$$\tilde{C}_{ij} = z^2 - z^4 + z^8 - z^{10} + z^{14} + \dots, \quad 1 \leq i \neq j \leq 2.$$

We will need the following lemma:

LEMMA 3.2.4. For all $(i, j) \in I^2$,

$$\tilde{C}_{ij}^{(m-1)} + \tilde{C}_{ij}^{(m+1)} - \sum_{k \sim j} \tilde{C}_{ik}^{(m)} = 0, \quad \forall m \geq 1,$$

$$\tilde{C}_{ij}^{(1)} = \delta_{i,j}.$$

Proof. By definition of \tilde{C} , one has

$$\tilde{C}(z) \cdot C(z) = \text{Id} \in \mathcal{M}_n(\mathbb{Q}(z)). \quad (3.3)$$

By writing $\tilde{C}(z)$ as a formal power series, and using the definition of $C(z)$, we obtain, for all $(i, j) \in I^2$,

$$\sum_{m=0}^{+\infty} \left(\tilde{C}_{ij}^{(m)} (z^{m+1} + z^{m-1}) - \sum_{k \sim j} \tilde{C}_{ik}^{(m)} z^m \right) = \delta_{i,j} \in \mathbb{C}[[z]]. \quad (3.4)$$

This is equivalent to

$$\tilde{C}_{ij}^{(m-1)} + \tilde{C}_{ij}^{(m+1)} - \sum_{k \sim j} \tilde{C}_{ik}^{(m)} = 0, \quad \forall m \geq 1,$$

$$\tilde{C}_{ij}^{(1)} - \sum_{k \sim j} \tilde{C}_{ij}^{(0)} = \delta_{i,j},$$

$$\tilde{C}_{ij}^{(0)} = 0. \quad \square$$

3.3. Infinite quiver

Next, let us define an infinite quiver Γ as in [37]. Let $\tilde{\Gamma}$ be the quiver with vertex set $I \times \mathbb{Z}$ and arrows

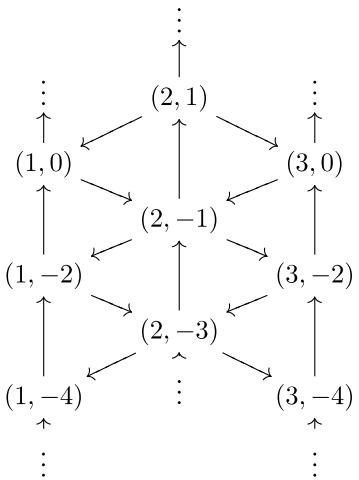
$$((i, r) \rightarrow (j, s)) \iff (C_{i,j} \neq 0 \text{ and } s = r + C_{i,j}). \quad (3.5)$$

This quiver has two isomorphic connected components (see [37]). Let Γ be one of them, and let \hat{I} be the set of its vertices.

EXAMPLE 3.3.1. For $\mathfrak{g} = \mathfrak{sl}_4$, fix \hat{I} to be

$$\hat{I} := \{(1, 2p) \mid p \in \mathbb{Z}\} \cup \{(2, 2p+1) \mid p \in \mathbb{Z}\} \cup \{(3, 2p) \mid p \in \mathbb{Z}\},$$

and Γ is the following:



4. Category \mathcal{O} of representations of quantum loop algebras

We now start with the more representation theoretic notions of this paper. We first recall the definitions of the quantum loop algebra and its Borel subalgebra, before introducing the Hernandez–Jimbo category \mathcal{O} of representations, as well as some known results on the subject. We will sporadically use concepts and notations from the two previous sections.

4.1. Quantum loop algebra and Borel subalgebra

Fix a non-zero complex number q , which is not a root of unity, and $h \in \mathbb{C}$ such that $q = e^h$. Then for all $r \in \mathbb{Q}$, $q^r := e^{rh}$ is well defined. Since q is not a root of unity, for $r, s \in \mathbb{Q}$, we have $q^r = q^s$ if and only if $r = s$.

We will use the following standard notations:

$$[m]_z = \frac{z^m - z^{-m}}{z - z^{-1}}, \quad [m]_z! = \prod_{j=1}^m [j]_z.$$

The z -binomial being already defined in Section 2.3:

$$\begin{bmatrix} r \\ s \end{bmatrix}_z = \frac{[r]_z!}{[s]_z! [r-s]_z!}.$$

DEFINITION 4.1.1. The quantum loop algebra $\mathcal{U}_q(L\mathfrak{g})$ corresponding to \mathfrak{g} is the associative \mathbb{C} -algebra generated by $e_i, f_i, k_i^{\pm 1}, 0 \leq i \leq n$, subject to the relations, for $0 \leq i, j \leq n$:

$$\begin{aligned}
 k_i k_j &= k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_0^{a_0} k_1^{a_1} \cdots k_n^{a_n} = 1, \\
 [e_i, f_j] &= \delta_{i,j} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\
 k_i e_j k_i^{-1} &= q^{C_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-C_{ij}} f_j, \\
 \sum_{r=0}^{1-C_{ij}} (-1)^r e_i^{(1-C_{ij}-r)} e_j e_i^{(r)} &= 0, \quad (i \neq j), \\
 \sum_{r=0}^{1-C_{ij}} (-1)^r f_i^{(1-C_{ij}-r)} f_j f_i^{(r)} &= 0, \quad (i \neq j),
 \end{aligned} \tag{4.1}$$

where $x_i^{(r)} = x_i^r / [r]_q!$, $(x_i = e_i, f_i)$.

DEFINITION 4.1.2. The Borel algebra $\mathcal{U}_q(\mathfrak{b})$ is the subalgebra of $\mathcal{U}_q(L\mathfrak{g})$ generated by the $e_i, k_i^{\pm 1}$, for $0 \leq i \leq n$.

Both the quantum loop algebra and its Borel subalgebra are Hopf algebras.

From now on, except when explicitly stated otherwise, we are going to consider representations of the Borel algebra $\mathcal{U}_q(\mathfrak{b})$. Particularly, we consider the action of the ℓ -Cartan subalgebra $\mathcal{U}_q(\mathfrak{b})^0$: a commutative subalgebra of $\mathcal{U}_q(\mathfrak{b})$ generated by the so-called Drinfeld generators (generators appearing in the Drinfeld presentation [17] of $\mathcal{U}_q(L\mathfrak{g})$):

$$\mathcal{U}_q(\mathfrak{b})^0 := \langle k_i^{\pm 1}, \phi_{i,r}^+ \rangle_{i \in I, r > 0}.$$

4.2. Highest ℓ -weight modules

Let V be a $\mathcal{U}_q(\mathfrak{b})$ -module and $\omega \in P_{\mathbb{Q}}$ a weight. One defines the weight space of V of weight ω by

$$V_{\omega} := \{v \in V \mid k_i v = q^{\omega(\alpha_i^{\vee})} v, 1 \leq i \leq n\}.$$

The vector space V is said to be *Cartan diagonalizable* if $V = \bigoplus_{\omega \in P_{\mathbb{Q}}} V_{\omega}$.

DEFINITION 4.2.1. A series $\Psi = (\psi_{i,m})_{i \in I, m \geq 0}$ of complex numbers, such that $\psi_{i,m} \in q^{\mathbb{Q}}$ for all $i \in I$ is called an ℓ -weight. The set of ℓ -weights is denoted by P_{ℓ} . One identifies the ℓ -weight Ψ to its generating series :

$$\Psi = (\psi_i(z))_{i \in I}, \quad \psi_i(z) = \sum_{m \geq 0} \psi_{i,m} z^m.$$

Let us define some particular ℓ -weights which are important in our context.

For any $\omega \in P_{\mathbb{Q}}$, define $[\omega] \in P_{\ell}$ by

$$[\omega]_i(z) = q^{\omega(\alpha_i^{\vee})}, 1 \leq i \leq n. \tag{4.2}$$

Furthermore, for all $i \in I$, $a \in \mathbb{C}^{\times}$, define $\Psi_{i,a}, Y_{i,a} \in P_{\ell}$ as

$$(\Psi_{i,a})_j(z) = \begin{cases} 1 - az & \text{if } j = i \\ 1 & \text{if } j \neq i \end{cases} \tag{4.3}$$

Then, define $Y_{i,a} \in P_{\ell}$ as

$$Y_{i,a} := [\omega_i] \Psi_{i,aq^{-1}} (\Psi_{i,aq})^{-1}. \tag{4.4}$$

The set P_ℓ is a group with respect to the component-wise multiplication of formal power series, as the components Ψ_i of ℓ -weights satisfy $\psi_i(0) \in q^\mathbb{Q}$. Moreover, for all $\Psi \in P_\ell$, if one writes $\psi_i(0) = q^{b_i}$, for all $i \in I$, then the element $\omega = \sum_{i \in I} b_i \omega_i \in P_\mathbb{Q}$ satisfy $\psi_i(0) = q^{\omega(\alpha_i^\vee)}$. Let ϖ be the surjective group morphism $\varpi : P_\ell \rightarrow P_\mathbb{Q}$ such that $\psi_i(0) = q^{\varpi(\Psi)(\alpha_i^\vee)}$, for all $\Psi \in P_\ell$ and all $i \in I$.

Let V be $\mathcal{U}_q(\mathfrak{b})$ -module and $\Psi \in P_\ell$ an ℓ -weight. One defines the ℓ -weight space of V of ℓ -weight Ψ by

$$V_\Psi := \{v \in V \mid \exists p \geq 0, \forall i \in I, \forall m \geq 0, (\phi_{i,m}^+ - \psi_{i,m})^p v = 0\}.$$

REMARK 4.2.2. With the usual convention $\phi_{i,0}^+ = k_i$, one has $V_\Psi \subset V_{\varpi(\Psi)}$.

DEFINITION 4.2.3. Let V be a $\mathcal{U}_q(\mathfrak{b})$ -module. It is said to be of highest ℓ -weight $\Psi \in P_\ell$ if there is $v \in V$ such that $V = \mathcal{U}_q(\mathfrak{b})v$,

$$e_i v = 0, \forall i \in I \quad \text{and} \quad \phi_{i,m}^+ v = \psi_{i,m} v, \quad \forall i \in I, m \geq 0.$$

In that case, the ℓ -weight Ψ is entirely determined by V , it is called the ℓ -weight of V , and v is the highest ℓ -weight vector of V .

PROPOSITION 4.2.4 [32]. For all $\Psi \in P_\ell$ there is, up to isomorphism, a unique simple highest ℓ -weight module of ℓ -weight Ψ , denoted by $L(\Psi)$.

EXAMPLE 4.2.5. For $\omega \in P_\mathbb{Q}$, $L([\omega])$ is a one-dimensional representation of weight ω . We also denote it by $[\omega]$ (tensoring by this representation is equivalent to shifting the weights by ω).

4.3. Definition of the category \mathcal{O}

As explained in the Introduction, our focus here is a category \mathcal{O} of representations of the Borel algebra, which was first defined in [32], mimicking the usual definition of the BGG category \mathcal{O} for Kac–Moody algebras. Here, we are going to use the definition in [36], which is slightly different.

For all $\lambda \in P_\mathbb{Q}$, define $D(\lambda) := \{\omega \in P_\mathbb{Q} \mid \omega \leq \lambda\}$.

DEFINITION 4.3.1. A $\mathcal{U}_q(\mathfrak{b})$ -module V is in the category \mathcal{O} if

- (1) V is Cartan diagonalizable;
- (2) for all $\omega \in P_\mathbb{Q}$, one has $\dim(V_\omega) < \infty$;
- (3) there is a finite number of $\lambda_1, \dots, \lambda_s \in P_\mathbb{Q}$ such that all the weights that appear in V are in the cone $\bigcup_{j=1}^s D(\lambda_j)$.

The category \mathcal{O} is a monoidal category.

EXAMPLE 4.3.2. All finite-dimensional $\mathcal{U}_q(\mathfrak{b})$ -modules are in the category \mathcal{O} .

Let P_ℓ^r be the set of ℓ -weights Ψ such that, for all $i \in I$, $\Psi_i(z)$ is rational. We will use the following result.

THEOREM 4.3.3 [32]. Let $\Psi \in P_\ell$. Simple objects in the category \mathcal{O} are highest ℓ -weight modules. The simple module $L(\Psi)$ is in the category \mathcal{O} if and only if $\Psi \in P_\ell^r$. Moreover, if V is in the category \mathcal{O} and $V_\Psi \neq 0$, then $\Psi \in P_\ell^r$.

EXAMPLE 4.3.4. For all $i \in I$ and $a \in \mathbb{C}^\times$, define the *prefundamental representations* $L_{i,a}^\pm$ as

$$L_{i,a}^\pm := L(\Psi_{i,a}^{\pm 1}), \quad (4.5)$$

for $\Psi_{i,a}$ defined in (4.3). Then from Theorem 4.3.3, the prefundamental representations belong to the category \mathcal{O} .

4.4. Connection to finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$ -modules

Throughout this paper, we will use results already known for finite-dimensional representations of the quantum loop algebra $\mathcal{U}_q(L\mathfrak{g})$ with the purpose of generalizing some of them to the context of the category \mathcal{O} of representations of the Borel subalgebra $\mathcal{U}_q(\mathfrak{b})$. Let us first see why this approach is valid, as the representation theories of these two algebras are similar.

Let \mathcal{C} be the category of all (type 1) finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$ -modules.

PROPOSITION 4.4.1 [8; 11, Proposition 2.7]. *Let V be a simple finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$ -module. Then V is simple as a $\mathcal{U}_q(\mathfrak{b})$ -module.*

Using this result and the classification of finite-dimensional simple module of quantum loop algebras in [12], one has

PROPOSITION 4.4.2. *For all $i \in I$, let $P_i(z) \in \mathbb{C}[z]$ be a polynomial with constant term 1. Let $\Psi = (\Psi_i)_{i \in I}$ be the ℓ -weight such that*

$$\Psi_i(z) = q^{\deg(P_i)} \frac{P_i(zq^{-1})}{P_i(zq)}, \quad \forall i \in I. \quad (4.6)$$

Then $L(\Psi)$ is finite-dimensional.

Moreover the action of $\mathcal{U}_q(\mathfrak{b})$ can be uniquely extended to an action of $\mathcal{U}_q(L\mathfrak{g})$, and any simple object in the category \mathcal{C} is of this form.

Hence, the category \mathcal{C} is a subcategory of the category \mathcal{O} and the inclusion functor preserves simple objects.

EXAMPLE 4.4.3. For all $i \in I$ and $a \in \mathbb{C}^\times$, consider the simple $\mathcal{U}_q(\mathfrak{b})$ -module $L(\Psi)$ of highest ℓ -weight $Y_{i,a}$, as in (4.4), then by Proposition 4.4.2, $L(Y_{i,a})$ is finite-dimensional. This module is called a *fundamental representation* and will be denoted by

$$V_{i,a} := L(Y_{i,a}). \quad (4.7)$$

In general, simple modules in \mathcal{C} are indexed by monomials in the variables $(Y_{i,a})_{i \in I, a \in \mathbb{C}^\times}$, called *dominant monomials*. Frenkel–Reshetikhin [25] defined a q -character morphism χ_q (see Section 4.8) on the Grothendieck ring of \mathcal{C} . It is an injective ring morphism

$$\chi_q : K_0(\mathcal{C}) \rightarrow \hat{\mathcal{Y}} := \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}. \quad (4.8)$$

EXAMPLE 4.4.4. In the notation of Example 4.4.3 we have, for $\mathfrak{g} = \mathfrak{sl}_2$ and for all $a \in \mathbb{C}^\times$,

$$\chi_q(L(Y_{1,a})) = Y_{1,a} + Y_{1,aq^2}^{-1}. \quad (4.9)$$

4.5. Categories \mathcal{O}^\pm

Let us now recall the definitions of some subcategories of the category \mathcal{O} , introduced in [36]. These categories are interesting to study for different reasons; here we use in particular the cluster algebra structure of their Grothendieck rings.

DEFINITION 4.5.1. Consider the submonoids P_ℓ^\pm of P_ℓ^r generated by the $[\omega]$, $\omega \in P_\mathbb{Q}$, the $Y_{i,a}$ and the $\Psi_{i,a}^{\pm 1}$, $i \in I$, $a \in \mathbb{C}^\times$. An ℓ -weight of P_ℓ^+ (respectively, P_ℓ^-) is said to be *positive* (respectively, *negative*).

DEFINITION 4.5.2. The category \mathcal{O}^\pm is the full subcategory of \mathcal{O} whose objects are modules whose simple constituents are $L(\Psi)$ with $\Psi \in P_\ell^\pm$.

REMARK 4.5.3. As for the classical category \mathcal{O} the multiplicity of a simple module in a module is well defined, see Section 4.7.

The category \mathcal{O}^+ (respectively, \mathcal{O}^-) contains the category of finite-dimensional representations, as well as the positive (respectively, negative) prefundamental representations $L_{i,a}^+$ (respectively, $L_{i,a}^-$), defined in (4.3.4), for all $i \in I$, $a \in \mathbb{C}^\times$.

THEOREM 4.5.4 [36]. The categories \mathcal{O}^+ and \mathcal{O}^- are monoidal categories.

4.6. The category $\mathcal{O}_\mathbb{Z}^+$

Recall the infinite quiver Γ from Section 3.3 and its set of vertices \hat{I} .

In [33, Section 3.7], Hernandez and Leclerc defined a subcategory $\mathcal{C}_\mathbb{Z}$ of the category \mathcal{C} . Let $\mathbf{Y}_\mathbb{Z}$ be the submonoid of P_ℓ generated by the $Y_{i,q^{r+1}}$ for $(i,r) \in \hat{I}$. Then $\mathcal{C}_\mathbb{Z}$ is the full subcategory of \mathcal{C} whose objects have simple constituents $L(\Psi)$ such that $\Psi \in \mathbf{Y}_\mathbb{Z}$.

This subcategory is interesting to study because each simple object in \mathcal{C} can be written as a tensor product of simple objects which are essentially in $\mathcal{C}_\mathbb{Z}$ (see [33, Section 3.7]). Thus, the study of simple modules in \mathcal{C} is equivalent to the study of simple modules in $\mathcal{C}_\mathbb{Z}$.

Consider the same type of restriction on the category \mathcal{O} . Let $P_{\ell,\mathbb{Z}}^\pm$ be the submonoids of P_ℓ generated by the $\Psi_{i,r}^{\pm 1}$ for $(i,r) \in \hat{I}$, respectively.

DEFINITION 4.6.1. Define $\mathcal{O}_\mathbb{Z}^\pm$ as the subcategories of representations of \mathcal{O} whose simple constituents have a highest ℓ -weight Ψ belonging to the monoids $P_{\ell,\mathbb{Z}}^\pm$.

Let us note that our definition of $\mathcal{C}_\mathbb{Z}$ is slightly different from that of [33]. As the $Y_{i,a}$ are expressed in terms of $\Psi_{i,aq^{\pm 1}}^{\pm 1}$, the definition of $\mathcal{C}_\mathbb{Z}$ is shifted for it to be a subcategory of $\mathcal{O}_\mathbb{Z}^\pm$. From now on, we will consider the Ψ_{i,q^r} and the $Y_{i,q^{r+1}}$, for $(i,r) \in \hat{I}$.

4.7. The Grothendieck ring $K_0(\mathcal{O})$

Hernandez and Leclerc showed that the Grothendieck rings of the categories $\mathcal{O}_\mathbb{Z}^\pm$ have some interesting cluster algebra structures.

First of all, define \mathcal{E} as the additive group of maps $c : P_\mathbb{Q} \rightarrow \mathbb{Z}$ whose support is contained in a finite union of sets of the form $D(\mu)$. For any $\omega \in P_\mathbb{Q}$, define $[\omega] \in \mathcal{E}$ as the δ -function at ω (this is compatible with the notation in Example 4.2.5). The elements of \mathcal{E} can be written as formal sums

$$c = \sum_{\omega \in \text{supp}(c)} c(\omega) [\omega]. \quad (4.10)$$

\mathcal{E} can be endowed with a ring structure, where the product is defined by

$$[\omega] \cdot [\omega'] = [\omega + \omega'], \quad \forall \omega, \omega' \in P_\mathbb{Q}.$$

This product is well defined because the supports of the maps c in \mathcal{E} are contained in a finite union of cones.

If $(c_k)_{k \in \mathbb{N}}$ is a countable family of elements of \mathcal{E} such that for any $\omega \in P_{\mathbb{Q}}$, $c_k(\omega) = 0$ except for finitely many $k \in \mathbb{N}$, then $\sum_{k \in \mathbb{N}} c_k$ is a well-defined map from $P_{\mathbb{Q}}$ to \mathbb{Z} . In that case, we say that $\sum_{k \in \mathbb{N}} c_k$ is a *countable sum of elements* in \mathcal{E} .

The Grothendieck ring of the category \mathcal{O} can be viewed as a ring extension of \mathcal{E} . Similar to the case of representations of a simple Lie algebra (see [39, Section 9.6]), every object in this category \mathcal{O} has a (generalized) Jordan–Hölder series, thus the multiplicity of an irreducible representation in a given representation of the category \mathcal{O} is well defined. The Grothendieck ring of the category \mathcal{O} is formed of formal sums

$$\sum_{\Psi \in P_{\ell}^{\mathbb{r}}} \lambda_{\Psi} [L(\Psi)], \quad (4.11)$$

such that the $\lambda_{\Psi} \in \mathbb{Z}$ satisfy

$$\sum_{\Psi \in P_{\ell}^{\mathbb{r}}, \omega \in P_{\mathbb{Q}}} |\lambda_{\Psi}| \dim(L(\Psi)_{\omega}) [\omega] \in \mathcal{E}.$$

In this context, \mathcal{E} is identified with the Grothendieck ring of the category of representations of \mathcal{O} with constant ℓ -weight.

A notion of countable sum of elements in $K_0(\mathcal{O})$ is defined exactly as for \mathcal{E} .

Now consider the cluster algebra $\mathcal{A}(\Gamma)$ defined by the infinite quiver Γ of Section 3.3, with infinite set of coordinates denoted by

$$\mathbf{z} = \left\{ z_{i,r} \mid (i, r) \in \hat{I} \right\}. \quad (4.12)$$

By the *Laurent phenomenon* (see 2.4.2), $\mathcal{A}(\Gamma)$ is contained in $\mathbb{Z}[z_{i,r}^{\pm 1}]_{(i,r) \in \hat{I}}$. Define $\chi : \mathbb{Z}[z_{i,r}^{\pm 1}] \otimes_{\mathbb{Z}} \mathcal{E} \rightarrow \mathcal{E}$, the \mathcal{E} -algebra homomorphism by

$$\chi(z_{i,r}^{\pm 1}) = \left[\mp \frac{r}{2} \omega_i \right], \quad ((i, r) \in \hat{I}). \quad (4.13)$$

In particular, the map χ is defined on $\mathcal{A}(\Gamma) \otimes_{\mathbb{Z}} \mathcal{E}$, and for each $A \in \mathcal{A}(\Gamma) \otimes_{\mathbb{Z}} \mathcal{E}$, one can write $\chi(A) = \sum_{\omega \in P_{\mathbb{Q}}} A_{\omega} [\omega] \in \mathcal{E}$. Define $|\chi|$ by $|\chi|(A) = \sum_{\omega \in P_{\mathbb{Q}}} |A_{\omega}| [\omega]$.

Consider the completed tensor product

$$\mathcal{A}(\Gamma) \hat{\otimes}_{\mathbb{Z}} \mathcal{E}, \quad (4.14)$$

of countable sums $\sum_{k \in \mathbb{N}} A_k$ of elements $A_k \in \mathcal{A}(\Gamma) \otimes_{\mathbb{Z}} \mathcal{E}$, such that $\sum_{k \in \mathbb{N}} |\chi|(A_k)$ is a countable sum of elements of \mathcal{E} , as defined above.

THEOREM 4.7.1 [36, Theorem 4.2]. *The category $\mathcal{O}_{\mathbb{Z}}^+$ is monoidal, and the identification*

$$z_{i,r} \otimes \left[\frac{r}{2} \omega_i \right] \equiv [L_{i,q^r}^+], \quad ((i, r) \in \hat{I}), \quad (4.15)$$

defines an isomorphism of \mathcal{E} -algebras

$$\mathcal{A}(\Gamma) \hat{\otimes}_{\mathbb{Z}} \mathcal{E} \simeq K_0(\mathcal{O}_{\mathbb{Z}}^+). \quad (4.16)$$

EXAMPLE 4.7.2. We mentioned in the introduction that the Baxter relation (1.2) was an exchange relation for this cluster algebra structure, let us detail this. For $\mathfrak{g} = \mathfrak{sl}_2$, the quiver Γ is

$$\begin{array}{ccc}
 \cdot & & \text{If we mutate at the node } (1,0), \text{ the new cluster variable obtained is} \\
 \uparrow & & \\
 (1,2) & & z'_{1,0} = \frac{1}{z_{1,0}} (z_{1,2} + z_{1-2}). \\
 \uparrow & & \\
 (1,0) & \text{Thus, via the identification (4.15),} & \\
 \uparrow & & z'_{1,0} = \frac{1}{[L_{1,1}^+]} \left([-\omega_1][L_{1,q^2}^+] + [\omega_1][L_{1,q^{-2}}^+] \right). \\
 \uparrow & \text{We indeed recognize the Baxter relation.} & \\
 \cdot & &
 \end{array}$$

Moreover, the new cluster variable $z'_{1,0}$ identifies to a fundamental representation:

$$z'_{1,0} = [L(Y_{1,q^{-1}})]. \quad (4.17)$$

REMARK 4.7.3. An analog theorem could be written for $K_0(\mathcal{O}_{\mathbb{Z}}^-)$, as these are isomorphic as \mathcal{E} -algebras [36, Theorem 5.17].

4.8. The q -character morphism

Here we detail the notion of q -character on the category \mathcal{O} . This notion extends the q -character morphism on the category of finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$ -modules mentioned in Section 4.4.

Similar to Section 4.7, consider \mathcal{E}_ℓ , the additive group of maps $c : P_\ell^r \rightarrow \mathbb{Z}$ such that the image by ϖ of its support is contained in a finite union of sets of the form $D(\mu)$, and for any $\omega \in P_\mathbb{Q}$, the set $\text{supp}(c) \cap \varpi^{-1}(\{\omega\})$ is finite. The map ϖ extends naturally to a surjective morphism $\varpi : \mathcal{E}_\ell \rightarrow \mathcal{E}$. For $\Psi \in P_\ell^r$, define the delta function $[\Psi] = \delta_\Psi \in \mathcal{E}_\ell$.

The elements of \mathcal{E}_ℓ can be written as formal sums

$$c = \sum_{\Psi \in P_\ell^r} c(\Psi)[\Psi]. \quad (4.18)$$

Endow \mathcal{E}_ℓ with a ring structure given by

$$(c \cdot d)(\Psi) = \sum_{\Psi' \Psi'' = \Psi} c(\Psi')d(\Psi''), \quad (c, d \in \mathcal{E}_\ell, \Psi \in P_\ell^r). \quad (4.19)$$

In particular, for $\Psi, \Psi' \in P_\ell^r$,

$$[\Psi] \cdot [\Psi'] = [\Psi\Psi']. \quad (4.20)$$

As in Section 4.7, this multiplication is well defined thanks to the support condition on \mathcal{E}_ℓ .

For V a module in the category \mathcal{O} , define the q -character of V as in [25, 32]:

$$\chi_q(V) := \sum_{\Psi \in P_\ell^r} \dim(V_\Psi)[\Psi]. \quad (4.21)$$

By definition of the category \mathcal{O} , $\chi_q(V)$ is an object of the ring \mathcal{E}_ℓ .

The following result extends the one from [25] to the context of the category \mathcal{O} .

PROPOSITION 4.8.1 [32]. *The q -character map*

$$\begin{aligned}
 \chi_q : \quad K_0(\mathcal{O}) &\rightarrow \mathcal{E}_\ell \\
 [V] &\mapsto \chi_q(V),
 \end{aligned} \quad (4.22)$$

is an injective ring morphism.

EXAMPLE 4.8.2. For any $a \in \mathbb{C}^\times$, $i \in I$, one has [22, 32],

$$\chi_q(L_{i,a}^+) = [\Psi_{i,a}]\chi_i, \quad (4.23)$$

where $\chi_i = \chi(L_{i,a}^+) \in \mathcal{E}$ does not depend on a .

For example, if $\mathfrak{g} = \mathfrak{sl}_2$,

$$\chi_1 = \chi = \sum_{r \geq 0} [-2r\omega_1]. \quad (4.24)$$

5. Quantum tori

Let t be an indeterminate. The aim of this section is to build a non-commutative quantum torus \mathcal{T}_t which will contain the quantum Grothendieck ring for the category \mathcal{O} . For the category \mathcal{C} of finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$ -modules, such a quantum torus already exists, denoted by \mathcal{Y}_t here. Thus one natural condition on \mathcal{T}_t is for it to contain \mathcal{Y}_t . We show it is the case in Theorem 5.2.1.

We start this section by recalling the definition and some properties of \mathcal{Y}_t . Here we use the same quantum torus as in [31], which is slightly different from the one used in [42, 45].

5.1. The torus \mathcal{Y}_t

In this section, we consider $\mathcal{U}_q(L\mathfrak{g})$ -modules and no longer $\mathcal{U}_q(\mathfrak{b})$ -modules. We have seen in Section 4.4 that for finite-dimensional representations, these settings were not too different.

As seen in (4.8), the Grothendieck ring of \mathcal{C} can be seen as a subring of a ring of Laurent polynomials

$$K_0(\mathcal{C}) \subseteq \hat{\mathcal{Y}} = \mathbb{Z}[Y_{i,a} \mid (i, r) \in \hat{I}].$$

In order to define a t -deformed non-commutative version of this Grothendieck ring, one first needs a non-commutative, t -deformed version of $\hat{\mathcal{Y}}$, denoted by \mathcal{Y}_t .

Following [31], we define

$$\mathcal{Y} := \mathbb{Z}[Y_{i,q^r}^{\pm 1} \mid (i, r) \in \hat{I}], \quad (5.1)$$

the Laurent polynomial ring generated by the commuting variables Y_{i,q^r} .

Let $(\mathcal{Y}_t, *)$ be the $\mathbb{Z}(t^{1/2})$ -algebra generated by the $(Y_{i,q^r}^{\pm})_{(i,r) \in \hat{I}}$, with the t -commutations relations:

$$Y_{i,q^r} * Y_{j,q^s} = t^{\mathcal{N}_{i,j}(r-s)} Y_{j,q^s} * Y_{i,q^r}, \quad (5.2)$$

where $\mathcal{N}_{i,j} : \mathbb{Z} \rightarrow \mathbb{Z}$ is the antisymmetrical map, defined by

$$\mathcal{N}_{i,j}(m) = \tilde{C}_{i,j}^{(m+1)} - \tilde{C}_{i,j}^{(m-1)}, \quad \forall m \geq 0, \quad (5.3)$$

using the notations from Section 3.2.

EXAMPLE 5.1.1. If we continue Example 3.2.3, for $\mathfrak{g} = \mathfrak{sl}_2$, in this case, $\hat{I} = (1, 2\mathbb{Z})$, for $r \in \mathbb{Z}$, one has

$$Y_{1,2r} * Y_{1,2s} = t^{2(-1)^{s-r}} Y_{1,2s} * Y_{1,2r}, \quad \forall s > r > 0. \quad (5.4)$$

The $\mathbb{Z}(t^{1/2})$ -algebra \mathcal{Y}_t is viewed as a quantum torus of infinite rank.

Consider the bar-involution $\overline{}$, the antiautomorphism of \mathcal{Y}_t defined by

$$\overline{t^{1/2}} = t^{-1/2}, \quad \overline{Y_{i,q^{r+1}}} = Y_{i,q^{r+1}}, \quad \left((i, r) \in \hat{I} \right). \quad (5.5)$$

For a family of integers with finitely many non-zero components $(u_{i,r})_{(i,r) \in \hat{I}}$, one can consider the *bar-invariant monomial* $\prod_{(i,r) \in \hat{I}} Y_{i,q^r}^{u_{i,r}}$, invariant under the bar-involution:

$$\prod_{(i,r) \in \hat{I}} Y_{i,q^r}^{u_{i,r}} := t^{\frac{1}{2} \sum_{(i,r) < (j,s)} u_{i,r} u_{j,s} \mathcal{N}_{i,j}(r,s)} \overrightarrow{*}_{(i,r) \in \hat{I}} Y_{i,q^r}^{u_{i,r}}, \quad (5.6)$$

where on the right-hand side an order on \hat{I} is chosen so as to give meaning to the sum, and the product $*$ is ordered by it (note that the result does not depend on the order chosen).

The bar-invariant monomials form a basis of the free $\mathbb{Z}(t^{1/2})$ -module \mathcal{Y}_t .

5.2. The torus \mathcal{T}_t

We now want to extend the quantum torus \mathcal{Y}_t to a larger non-commutative algebra \mathcal{T}_t which would contain at least all the ℓ -weights, and possibly all the candidates for the (q,t) -characters of the modules in the category $\mathcal{O}_{\mathbb{Z}}^+$.

In particular, \mathcal{T}_t contains the Ψ_{i,q^r} , for $(i,r) \in \hat{I}$, and these t -commutes with a relation compatible with the t -commutation relation between the $Y_{i,q^{r+1}}$ (5.2).

We start as in Section 5.1. First of all, define

$$\mathcal{T} := \mathbb{Z}\left[z_{i,r}^{\pm 1} \mid (i,r) \in \hat{I}\right], \quad (5.7)$$

the Laurent polynomial ring generated by the commuting variables $z_{i,r}$.

Then, build a t -deformation \mathcal{T}_t of \mathcal{T} , as the $\mathbb{Z}[t^{\pm 1}]$ -algebra generated by the $z_{i,r}^{\pm 1}$, for $(i,r) \in \hat{I}$, with a non-commutative product $*$, and the t -commutations relations

$$z_{i,r} * z_{j,s} = t^{\mathcal{F}_{ij}(s-r)} z_{j,s} * z_{i,r}, \quad ((i,r), (j,s) \in \hat{I}), \quad (5.8)$$

where, for all $i,j \in I$, $\mathcal{F}_{ij} : \mathbb{Z} \rightarrow \mathbb{Z}$ is an antisymmetrical map such that, for all $m \geq 0$,

$$\mathcal{F}_{ij}(m) = - \sum_{\substack{k \geq 1 \\ m \geq 2k-1}} \tilde{O}_{ij}^{(m-2k+1)}. \quad (5.9)$$

Now, let

$$\mathcal{T}_t := \mathbb{Z}[t^{\pm 1/2}] \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathcal{T}. \quad (5.10)$$

This based quantum torus will be enough to define a structure of quantum cluster algebra, but for it to contain the quantum Grothendieck ring of the category $\mathcal{O}_{\mathbb{Z}}^+$, one needs to extend it. In order to do that, we draw inspiration from Section 4.7. Recall the definition of χ from (4.13). We extend it to the \mathcal{E} -algebra morphism $\chi : \mathcal{T}_t \otimes_{\mathbb{Z}} \mathcal{E} \rightarrow \mathcal{E}$ defined by imposing $\chi(t^{\pm 1/2}) = 1$, as well as

$$\chi(z_{i,r}^{\pm 1}) = \left[\left(\frac{\mp r}{2} \right) \omega_i \right], \quad ((i,r) \in \hat{I}).$$

As before, for $z \in \mathcal{T}_t \otimes_{\mathbb{Z}} \mathcal{E}$, one writes $\chi(z) = \sum_{\omega \in P_{\mathbb{Q}}} z_{\omega}[\omega]$ and $|\chi|(z) = \sum_{\omega \in P_{\mathbb{Q}}} |z_{\omega}|[\omega]$.

Define the completed tensor product

$$\mathcal{T}_t := \mathcal{T}_t \hat{\otimes}_{\mathbb{Z}[t^{\pm 1/2}]} \mathcal{E}, \quad (5.11)$$

of countable sums $\sum_{k \in \mathbb{N}} z_k$ of elements $z_k \in \mathcal{T}_t \otimes_{\mathbb{Z}} \mathcal{E}$, such that $\sum_{k \in \mathbb{N}} |\chi|(z_k)$ is a countable sum of \mathcal{E} , as in Section 4.7.

The bar involution defined on \mathcal{Y}_t has a counterpart on the larger quantum torus \mathcal{T}_t .

There is unique \mathcal{E} -algebra antiautomorphism of \mathcal{T}_t such that

$$\overline{t^{1/2}} = t^{-1/2}, \quad \overline{z_{i,r}} = z_{i,r}, \quad \text{and} \quad \overline{[\omega_i]} = [\omega_i], \quad ((i,r) \in \hat{I}).$$

Similarly, we define the bar-invariant monomials in \mathcal{T}_t as

$$\prod_{(i,r) \in \hat{I}} z_{i,r}^{v_{i,r}} := t^{\frac{1}{2} \sum_{(i,r) < (j,s)} v_{i,r} v_{j,s} \mathcal{F}_{i,j}(r,s)} \vec{\ast}_{(i,r) \in \hat{I}} z_{i,r}^{v_{i,r}}. \quad (5.12)$$

Consistently with the identification (4.15), and the character of the $z_{i,r}^{\pm 1}$, we use the following notation, for $(i,r) \in \hat{I}$,

$$[\Psi_{i,q^r}^{\pm 1}] := z_{i,r}^{\pm 1} \left[\frac{\pm r}{2} \omega_i \right] \in \mathcal{T}_t. \quad (5.13)$$

THEOREM 5.2.1. *The assignments*

$$Y_{i,q^{r+1}} \mapsto t^{-\frac{1}{2}} z_{i,r} * z_{i,r+2}^{-1} = t^{-\frac{1}{2}} [\omega_i] [\Psi_{i,q^r}] * [\Psi_{i,q^{r+2}}^{-1}], \quad (5.14)$$

where the products on the right-hand side are bar-invariant, defines an injective homomorphism $\mathcal{J} : \mathcal{Y}_t \rightarrow \mathcal{T}_t$ of $\mathbb{Z}(t)$ -algebras.

Proof. One needs to check that the images of the $Y_{i,q^{r+1}}$ satisfy (5.2). Thus, we need to show that, for all $(i,r), (j,s) \in \hat{I}$,

$$(z_{i,r} z_{i,r+2}^{-1}) * (z_{j,s} z_{j,s+2}^{-1}) = t^{\mathcal{N}_{i,j}(s-r)} (z_{j,s} z_{j,s+2}^{-1}) * (z_{i,r} z_{i,r+2}^{-1}),$$

which is equivalent to checking that

$$2\mathcal{F}_{i,j}(s-r) - \mathcal{F}_{i,j}(s-r+2) - \mathcal{F}_{i,j}(s-r-2) = \mathcal{N}_{i,j}(s-r). \quad (5.15)$$

Suppose $s \geq r+2$, let $m = s-r$.

$$\begin{aligned} 2\mathcal{F}_{i,j}(m) - \mathcal{F}_{i,j}(m+2) - \mathcal{F}_{i,j}(m-2) &= - \sum_{\substack{k \geq 1 \\ m \geq 2k-1}} \tilde{C}_{ij}^{(m-2k+1)} \\ &+ \sum_{\substack{k \geq 0 \\ m \geq 2k-1}} \tilde{C}_{ij}^{(m-2k+1)} + \sum_{\substack{k \geq 2 \\ m \geq 2k-1}} \tilde{C}_{ij}^{(m-2k+1)} = -\tilde{C}_{ij}^{(m-1)} + \tilde{C}_{ij}^{(m+1)}. \end{aligned}$$

Thus $2\mathcal{F}_{i,j}(m) - \mathcal{F}_{i,j}(m+2) - \mathcal{F}_{i,j}(m-2) = \mathcal{N}_{i,j}(m)$, using (5.3).

If $s = r+1$, the left-hand side of (5.15) is equal to

$$3\mathcal{F}_{i,j}(1) - \mathcal{F}_{i,j}(3) = \tilde{C}_{ij}^{(2)} = \mathcal{N}_{i,j}(1). \quad \square$$

EXAMPLE 5.2.2. Let us continue Examples 3.2.3 and 5.1.1. For all $r \in \mathbb{Z}$. One has

$$z_{1,2r} * z_{1,2s} = t^{f(s-r)} z_{1,2s} * z_{1,2r}, \quad \forall r, s \in \mathbb{Z}, \quad (5.16)$$

where $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is antisymmetric and defined by

$$f|_{\mathbb{N}} : m \mapsto \frac{(-1)^m - 1}{2}. \quad (5.17)$$

And this is compatible with the relations (5.4).

DEFINITION 5.2.3. Define the evaluation at $t = 1$ as the \mathcal{E} -morphism

$$\pi : \mathcal{T}_t \rightarrow \mathcal{E}_\ell, \quad (5.18)$$

such that

$$\pi(z_{i,r}) = \left[\frac{-r}{2} \omega_i \right] [\Psi_{i,q^r}],$$

$$\pi(t^{\pm 1/2}) = 1.$$

REMARK 5.2.4. The identification (4.15) is between the element $z_{i,r}[r\omega_i/2]$ and the class of the prefundamental representation $[L_{i,q^r}^+]$. But this identification is not compatible with the character χ defined in (4.13), as the character of L_{i,q^r}^+ is χ_i , as in (4.23). Here, we choose to identify the variables $z_{i,r}$ with the highest ℓ -weights of the prefundamental representations (up to a shift of weight), in particular, this identification is compatible with the character morphism χ .

6. Quantum Grothendieck rings

The aim of this section is to build $K_t(\mathcal{O}_{\mathbb{Z}}^+)$, a t -deformed version of the Grothendieck ring of the category $\mathcal{O}_{\mathbb{Z}}^+$. This ring will be built inside the quantum torus \mathcal{T}_t , as a quantum cluster algebra.

Let us summarize the existing objects in this context in a diagram:

$$\begin{array}{ccc}
 \mathcal{C}_{\mathbb{Z}} & \subset & \mathcal{O}_{\mathbb{Z}}^{\pm} \\
 \downarrow & & \downarrow \\
 K_0(\mathcal{C}_{\mathbb{Z}}) & \subset & K_0(\mathcal{O}_{\mathbb{Z}}^{\pm}) \simeq \mathcal{A}(\Gamma) \hat{\otimes}_{\mathbb{Z}} \mathcal{E} \\
 \swarrow & & \searrow \\
 K_t(\mathcal{C}_{\mathbb{Z}}) & \subset & ?
 \end{array}$$

A natural idea to build a t -deformation of the Grothendieck ring $K_0(\mathcal{O}_{\mathbb{Z}}^+)$ is to use its cluster algebra structure and define a t -deformed quantum cluster algebra, as in Section 2.3, with the same basis quiver. One has to make sure that the resulting object is indeed a subalgebra of the quantum torus \mathcal{T}_t .

6.1. The finite-dimensional case

We start this section with some reminders regarding the quantum Grothendieck ring of the category of finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$ -modules.

This object was first discussed by Nakajima [42] and Varagnolo–Vasserot [45] in the study of perverse sheaves. Then Hernandez gave a more algebraic definition, using t -analogs of screening operators [30, 31]. This is the version we consider here, with the restriction to some specific tensor subcategory $\mathcal{C}_{\mathbb{Z}}$, as in [35].

6.1.1. *Definition of the quantum Grothendieck ring.* As in Section 4.6, consider $\mathcal{C}_{\mathbb{Z}}$ the full subcategory of \mathcal{C} whose simple components have highest ℓ -weights which are monomials in the Y_{i,q^r} , with $(i, r) \in \hat{I}$.

For $(i, r-1) \in \hat{I}$, define the bar-invariant monomials

$$A_{i,r} := Y_{i,q^{r+1}} Y_{i,q^{r-1}} \prod_{j \sim i} Y_{j,q^r}^{-1} \in \mathcal{Y}_t. \quad (6.1)$$

For all $i \in I$, let $K_{i,t}(\mathcal{C}_{\mathbb{Z}})$ be the $\mathbb{Z}(t^{1/2})$ -subalgebra of \mathcal{Y}_t generated by the

$$Y_{i,q^r}(1 + A_{i,r+1}^{-1}), \quad Y_{j,q^s} \quad \left((i, r), (j, s) \in \hat{I}, j \neq i \right). \quad (6.2)$$

Finally, as in [31], define

$$K_t(\mathcal{C}_{\mathbb{Z}}) := \bigcap_{i \in I} K_{i,t}(\mathcal{C}_{\mathbb{Z}}). \quad (6.3)$$

REMARK 6.1.1. Frenkel–Mukhin’s algorithm [23] allows for the computation of certain q -characters, in particular those of the fundamental representations. In [31], Hernandez

introduced a t -deformed version of this algorithm to compute the (q, t) -characters of the fundamental representations, and thus to characterize the quantum Grothendieck ring as the subring of \mathcal{Y}_t generated for those (q, t) -characters:

$$K_t(\mathcal{C}_{\mathbb{Z}}) = \left\langle [L(Y_{i, q^r})]_t \mid (i, r) \in \hat{I} \right\rangle. \quad (6.4)$$

6.1.2. (q, t) -Characters in $K_t(\mathcal{C}_{\mathbb{Z}})$. Let us recall some more detailed results about the theory of (q, t) -characters for the modules in the category $\mathcal{C}_{\mathbb{Z}}$.

Let \mathcal{M} be the set of monomials in the variables $(Y_{i, q^{r+1}})_{(i, r) \in \hat{I}}$, also called *dominant monomials*. From [31] we know that for all dominant monomial m , there is a unique element $F_t(m)$ in $K_t(\mathcal{C}_{\mathbb{Z}})$ such that m occurs in $F_t(m)$ with multiplicity 1, and no other dominant monomial occurs in $F_t(m)$. These $F_t(m)$ form a $\mathbb{C}(t^{1/2})$ -basis of $K_t(\mathcal{C}_{\mathbb{Z}})$.

For all dominant monomial $m = \prod_{(i, r) \in \hat{I}} Y_{i, q^{r+1}}^{u_{i, r}(m)} \in \mathcal{M}$, define

$$[M(m)]_t := t^{\alpha(m)} \overleftarrow{\star}_{r \in \mathbb{Z}} F_t \left(\prod_{i \in I} Y_{i, q^{r+1}}^{u_{i, r}(m)} \right) \in K_t(\mathcal{C}_{\mathbb{Z}}), \quad (6.5)$$

where $\alpha(m) \in \frac{1}{2}\mathbb{Z}$ is fixed such that m appears with coefficient 1 in the expansion of $[M(m)]_t$ on the basis of the bar-invariant monomials. The specialization at $t = 1$ of $[M(m)]_t$ recovers the q -character $\chi_q(M(m))$ of the standard module $M(m)$.

REMARK 6.1.2. By definition, the subring $K_t(\mathcal{C}_{\mathbb{Z}})$ is invariant under the bar-involution defined in Section 5.

Using Lusztig's lemma [41, 7.10] and the triangularity of the basis $([M(m)]_t)_{m \in \mathcal{M}}$ with respect to the bar-involution, there is a unique family $\{[L(m)]_t \in K_t(\mathcal{C}_{\mathbb{Z}}) \mid m \in \mathcal{M}\}$ such that

(i)

$$\overline{[L(m)]_t} = [L(m)]_t, \quad (6.6)$$

(ii)

$$[L(m)]_t \in [M(m)]_t + \sum_{m' < m} t^{-1} \mathbb{Z}[t^{-1}] [M(m')]_t, \quad (6.7)$$

where $m' \leq m$ means that $m(m')^{-1}$ is a product of $A_{i, r}$ (see [42]).

Lastly, we recall this result from Nakajima, proven using the geometry of quiver varieties.

THEOREM 6.1.3 [42]. For all dominant monomial $m \in \mathcal{M}$, the specialization at $t = 1$ of $[L(m)]_t$ is equal to $\chi_q(L(m))$.

Moreover, the coefficients of the expansion of $[L(m)]_t$ as a linear combination of products of $Y_{i, r}^{\pm 1}$ belong to $\mathbb{N}[t^{\pm 1}]$.

Thus to all simple modules $L(\Psi)$ in $\mathcal{C}_{\mathbb{Z}}$ is associated an object $[L(m)]_t \in K_t(\mathcal{C}_{\mathbb{Z}})$, called the (q, t) -character. It is compatible with the q -character of the representation.

REMARK 6.1.4. With the cluster algebra approach, we shed a new light on this last positivity result. We interpret the (q, t) -characters of the fundamental modules (and actually all simple modules which are realized as cluster variables in $K_0(\mathcal{O}_{\mathbb{Z}}^+)$) as quantum

cluster variables (Conjecture 7.2.6). Thus using Theorem 2.6.1, we recover the fact that the coefficients of their expansion on the commutative monomials in the $(Y_{i,r}^{\pm 1})$ belong to $\mathbb{N}[t^{\pm 1}]$.

REMARK 6.1.5. In order to fully extended this picture to the context of the category \mathcal{O} , and implement a Kazhdan–Lusztig type algorithm to compute the (q, t) -characters of all simple modules, one would need an equivalent of the standard modules in this category. These do not exist in general. This question was tackled by the author in another paper [7], in which equivalent of standard modules where defined when $\mathfrak{g} = \mathfrak{sl}_2$.

6.2. Compatible pairs

We now begin the construction of $K_t(\mathcal{O}_{\mathbb{Z}}^+)$.

First of all, to define a quantum cluster algebra, one needs a *compatible pair*, as in Section 2.2. The basis quiver we consider here is the same quiver Γ as before (see Section 3.3).

Explicitly, the corresponding exchange matrix is the $\hat{I} \times \hat{I}$ skew-symmetric matrix \tilde{B} such that, for all $((i, r), (j, s)) \in \hat{I}^2$,

$$\tilde{B}_{((i,r),(j,s))} = \begin{cases} 1 & \text{if } \begin{array}{l} i = j \text{ and } s = r + 2 \\ \text{or } i \sim j \text{ and } s = r - 1, \end{array} \\ -1 & \text{if } \begin{array}{l} i = j \text{ and } s = r - 2 \\ \text{or } i \sim j \text{ and } s = r + 1, \end{array} \\ 0 & \text{otherwise.} \end{cases} \quad (6.8)$$

Let Λ be the $\hat{I} \times \hat{I}$ skew-symmetric infinite matrix encoding the t -commutation relations (5.8). Precisely, for $((i, r), (j, s)) \in \hat{I}^2$ such that $s > r$,

$$\Lambda_{(i,r),(j,s)} = \mathcal{F}_{i,j}(s-r) = - \sum_{\substack{k \geq 1 \\ m \geq 2k-1}} \tilde{C}_{ij}^{(m-2k+1)}. \quad (6.9)$$

REMARK 6.2.1. In [36], it is noted that one can use sufficiently large finite subseed of Γ instead of an infinite rank cluster algebra. For our purpose, the same statement stays true, but one has to check that the subquiver still forms a compatible pair with the torus structure. Hence, we have to give a more precise framework for the restriction to finite subseeds.

For all $N \in \mathbb{Z}_{>0}$, define Γ_N , which is a finite slice of Γ of length $2N + 1$, containing an upper and lower row of frozen vertices. More precisely, define \hat{I}_N and \tilde{I}_N as

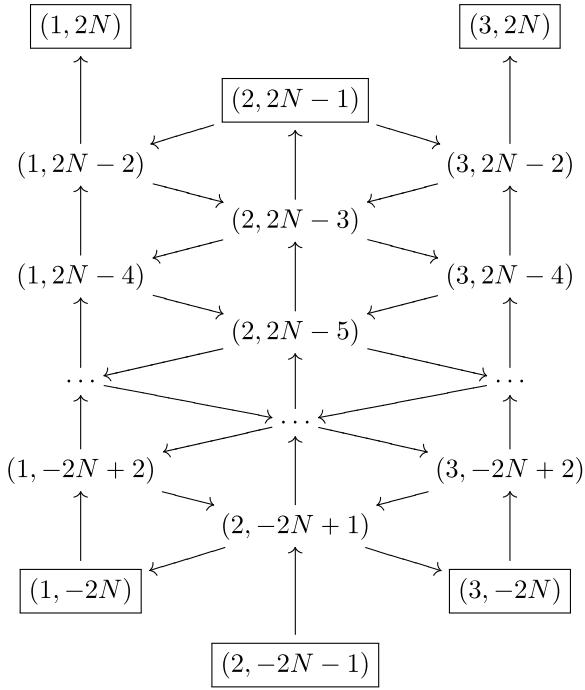
$$\hat{I}_N := \{(i, r) \in \hat{I} \mid -2N + 1 \leq r < 2N - 1\}, \quad (6.10)$$

$$\tilde{I}_N := \{(i, r) \in \hat{I} \mid -2N - 1 \leq r < 2N + 1\}. \quad (6.11)$$

Then Γ_N is the subquiver of Γ with set of vertices \tilde{I}_N , where the vertices in $\tilde{I}_N \setminus \hat{I}_N$ are frozen (thus the vertices in \hat{I}_N are the exchangeable vertices).

This way, all cluster variables of $\mathcal{A}(\Gamma)$ obtained from the initial seed after a finite sequence of mutations are cluster variables of the finite rank cluster algebra $\mathcal{A}(\Gamma_N)$, for N large enough. With the same index restrict on \tilde{B} , we will be able to define a size increasing family of finite rank quantum cluster algebras.

EXAMPLE 6.2.2. Recall from Example 3.3.1 the infinite quiver Γ when $\mathfrak{g} = \mathfrak{sl}_4$. Then the quiver Γ_N is the following:

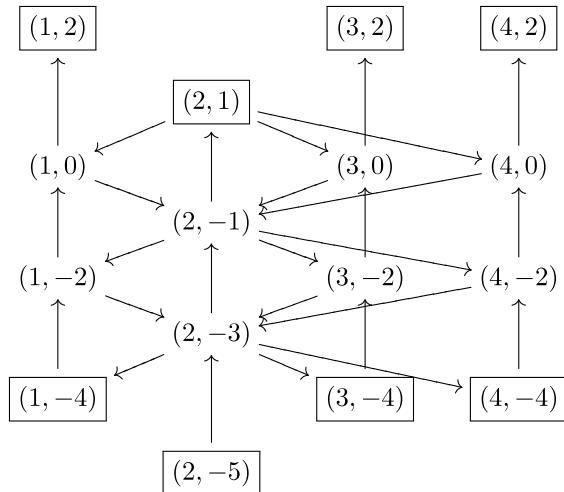


where the boxed vertices are frozen.

For $N \in \mathbb{Z}_{>0}$, let \tilde{B}_N be the corresponding exchange matrix. It is the $\tilde{I}_N \times \hat{I}_N$ submatrix of \tilde{B} , thus its coefficients are as in (6.8).

For all $N \in \mathbb{Z}_{>0}$, let Λ_N be the $\tilde{I}_N \times \tilde{I}_N$ submatrix of Λ . It is a finite $(n(2N+1))^2$ skew-symmetric matrix, where n is the rank of the simple Lie algebra \mathfrak{g} .

EXAMPLE 6.2.3. For \mathfrak{g} of type D_4 , let us exhibit a finite slice of Γ of length 4, containing an upper and lower row of frozen vertices (which is thus not Γ_1 , of length 3, nor Γ_2 , of length 5):



If the set $\tilde{I} = \{(i, r) \in \hat{I} \mid i \in \llbracket 1, 4 \rrbracket, -5 \leq r \leq 2\}$ is ordered lexicographically by r then i (reading order), the quiver is represented by the following exchange matrix:

$$\tilde{B} := \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (6.12)$$

The principal part B of \tilde{B} is the square submatrix obtained by omitting the first four rows and the last four rows. One notes that B is skew-symmetric.

Moreover, using Formula (6.9), one can compute the corresponding matrix Λ . We get the following 16×16 skew-symmetric matrix (with the same order of \tilde{I} as before):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 & 2 & 2 & 2 & 4 & \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 3 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ -2 & -2 & -2 & -3 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & -1 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -2 & -2 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & -2 & -4 & -2 & -2 & -2 & -3 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (6.13)$$

From here, it is easy to check that the product $\tilde{B}^T \Lambda$ is of the form:

$$\tilde{B}^T \Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (6.14)$$

Thus, (Λ, \tilde{B}) is a compatible pair.

REMARK 6.2.4. Note the coefficients appearing of the diagonal here are all negative. From the definition of compatible pair from [6], for (Λ, \tilde{B}) to be a compatible pair, these should be positive. That is why in Definition 2.2.1, the coefficients are of constant sign (the properties of quantum cluster algebras are still satisfied by taking negative coefficients in the compatibility condition). This allows for our t -commutation to be coherent with that of the quantum torus \mathcal{Y}_t .

We show that this result is true in general. Furthermore, the specific form we obtain in Equation (6.14) is what we get in general.

PROPOSITION 6.2.5. We have

$$\begin{aligned}\tilde{B}^T \Lambda &= -2 \text{Id}_{\hat{I}}, \\ \tilde{B}_N^T \Lambda_N &= -2 \left(\begin{array}{c|c|c} (0) & \text{Id}_{\hat{I}_N} & (0) \end{array} \right).\end{aligned}$$

In particular, (Λ, \tilde{B}) and $((\Lambda_N, \tilde{B}_N))_{N>0}$ are compatible pairs, in the sense of structure condition for quantum cluster algebras.

As in Definition 2.2.1, up to reordering of the variables (placing the mutable variable first), the resulting matrix is of the form $[D \ 0]$, with D diagonal with sign coherent entries.

Proof. Let $((i, r), (j, s)) \in \hat{I}^2$. Let us compute:

$$\left(\tilde{B}^T \Lambda \right)_{(i, r), (j, s)} = \sum_{(k, u) \in \hat{I}} b_{(k, u), (i, r)} \lambda_{(k, u), (j, s)}. \quad (6.15)$$

This is a finite sum, as each vertex in Γ is adjacent to a finite number of other vertices.

Suppose first that $r \neq s$. Without loss of generality, we can assume that $r < s$. Then, using the definition of the matrix Λ in (6.9) and the coefficients of \tilde{B} in (6.8), we obtain

$$\left(\tilde{B}^T \Lambda \right)_{(i, r), (j, s)} = -\tilde{C}_{ij}^{(s-r-1)} - \tilde{C}_{ij}^{(s-r+1)} + \sum_{k \sim i} \tilde{C}_{kj}^{(s-r)}. \quad (6.16)$$

By Lemma 3.2.4, for all $(i, j) \in I^2$,

$$\tilde{C}_{ij}^{(m-1)} + \tilde{C}_{ij}^{(m+1)} - \sum_{k \sim i} \tilde{C}_{kj}^{(m)} = 0, \quad \forall m \geq 1.$$

Thus, for all $(i, j) \in I^2$ and $r < s$, equation (6.16) gives

$$\left(\tilde{B}^T \Lambda \right)_{(i, r), (j, s)} = 0. \quad (6.17)$$

Suppose now that $r = s$. In that case,

$$\left(\tilde{B}^T \Lambda \right)_{(i, r), (j, r)} = -2\tilde{C}_{ij}^{(1)} = -2\delta_{i,j},$$

using the other result from Lemma 3.2.4. Thus,

$$\tilde{B}^T \Lambda = -2 \text{Id}_{\hat{I}}. \quad (6.18)$$

Now, for all $N \in \mathbb{Z}_{>0}$, let $(i, r) \in \hat{I}_N$ and $(j, s) \in \tilde{I}_N$. Let us write:

$$\left(\tilde{B}^T \Lambda \right)_{(i, r), (j, s)} = \sum_{(k, u) \in \tilde{I}_N} b_{(k, u), (i, r)} \lambda_{(k, u), (j, s)}. \quad (6.19)$$

As $(i, r) \in \hat{I}_N$ is not a frozen variable, the $(j, s) \in \hat{I}$ such that $b_{(k, u), (i, r)} \neq 0$ are all in \tilde{I}_N . Hence the rest of the reasoning is still valid, and the result follows. \square

6.3. Definition of $K_t(\mathcal{O}_{\mathbb{Z}}^+)$

Everything is now in place to define $K_t(\mathcal{O}_{\mathbb{Z}}^+)$. Recall the based quantum torus \mathcal{T}_t , defined in Section 5.2. By construction, the associated skew-symmetric bilinear form Λ identifies with the infinite skew-symmetric $\hat{I} \times \hat{I}$ -matrix from the previous section:

$$\Lambda(\mathbf{e}_{(i, r)}, \mathbf{e}_{(j, s)}) = \Lambda_{(i, r), (j, s)} = \mathcal{F}_{ij}(s - r), \quad (i, r), (j, s) \in \hat{I}, \quad (6.20)$$

where $(\mathbf{e}_{(i, r)})_{(i, r) \in \hat{I}}$ is the standard basis of $\mathbb{Z}^{(\hat{I})}$.

Let \mathcal{F} be the skew-field of fractions of \mathcal{T}_t . We define the toric frame $M : \mathbb{Z}^{(\hat{I})} \rightarrow \mathcal{F} \setminus \{0\}$ by setting

$$M(\mathbf{e}_{(i, r)}) = z_{i, r} \in \mathcal{F}, \quad \forall (i, r) \in \hat{I}. \quad (6.21)$$

Then the infinite rank matrix Λ_M satisfies

$$\Lambda_M = \Lambda. \quad (6.22)$$

REMARK 6.3.1. From the result of Proposition 6.2.5, $\mathcal{S} = (M, \tilde{B})$ is an infinite rank quantum seed, and one would want to define our quantum Grothendieck ring as the corresponding quantum cluster algebra. However, the definition of quantum cluster algebra recalled in Section 2 does not typically cover infinite rank quantum cluster algebras. Nonetheless, in [29], Grabowski–Gratz gave a construction of certain infinite rank quantum cluster algebra as (co)limits of sequences of finite rank quantum cluster algebras. We give here an explicit construction of this result.

Fix $N \in \mathbb{Z}_{>0}$. Let $m = (2N + 1) \times n$, where n is the rank of the simple Lie algebra \mathfrak{g} .

Consider L_N , the sublattice of \mathcal{T}_t generated by the $z_{i, r}$, with $(i, r) \in \tilde{I}_N$ (recall the definition of \tilde{I}_N in (6.11)). L_N is of rank m . Consider the toric frame M_N which is the restriction of M to L_N . In that case,

$$\Lambda_{M_N} = \Lambda_N, \quad \text{from the previous section.}$$

Thus, from the result of Proposition 6.2.5

$$\mathcal{S}_N := (M_N, \tilde{B}_N) \quad (6.23)$$

is a quantum seed.

DEFINITION 6.3.2. We define $\mathcal{A}_t(\Gamma_N)$ to be the quantum cluster algebra associated to the mutation-equivalence class of the quantum seed \mathcal{S}_N .

Then, let us define $\mathcal{A}_t(\Gamma)$ to be the quantum cluster algebra associated to the mutation-equivalence class of the infinite rank quantum seed \mathcal{S} . As the mutation sequences are finite, one can always assume we are working in the quantum cluster algebra $\mathcal{A}_t(\Gamma_N)$, with N large enough.

DEFINITION 6.3.3. Define

$$K_t(\mathcal{O}_{\mathbb{Z}}^+) := \mathcal{A}_t(\Gamma) \hat{\otimes} \mathcal{E}, \quad (6.24)$$

where the tensor product is completed as in (5.11). The ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ is a $\mathcal{E}[t^{\pm 1/2}]$ -subalgebra of \mathcal{T}_t .

For $N \in \mathbb{Z}_{>0}$, with the same completion of the tensor product, define

$$K_t(\mathcal{O}_{\mathbb{Z},N}^+) := \mathcal{A}_t(\Gamma_N) \hat{\otimes} \mathcal{E}. \quad (6.25)$$

7. Properties of $K_t(\mathcal{O}_{\mathbb{Z}}^+)$

7.1. The bar involution

Recall the bar-involution maps defined on \mathcal{Y}_t and \mathcal{T}_t in Section 5. We have seen in Section 6.1.2 that the (q,t) -character of simple modules in $\mathcal{C}_{\mathbb{Z}}$ are bar-invariant by definition. Thus it is natural for (q,t) -characters of simple modules in $\mathcal{O}_{\mathbb{Z}}^+$ to also be bar-invariant.

What is crucial to note here is that the definition of the bar-involution on \mathcal{T}_t is compatible with the bar-involution defined in general on the quantum torus of any quantum cluster algebra (see [6, Section 6]). However, this latter bar-involution has an important property: all cluster variables are invariant under the bar involution.

PROPOSITION 7.1.1. *All elements of $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ of the form $\chi_t \otimes 1$, where $\chi_t \in \mathcal{A}_t(\Gamma)$ is a cluster variable, are invariant under the bar-involution on \mathcal{T}_t .*

7.2. Inclusion of quantum Grothendieck rings

As stated earlier, one natural property we would want to be satisfied by the quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ is to include the already-existing quantum Grothendieck $K_t(\mathcal{C}_{\mathbb{Z}})$ of the category $\mathcal{C}_{\mathbb{Z}}$.

Note that those rings are contained in quantum tori, which are included in one another by the injective morphism \mathcal{J} from Theorem 5.2.1:

$$\begin{array}{ccc} K_t(\mathcal{C}_{\mathbb{Z}}) & \subset & \mathcal{Y}_t \\ \downarrow & & \downarrow \mathcal{J} \\ K_t(\mathcal{O}_{\mathbb{Z}}^+) & \subset & \mathcal{T}_t. \end{array}$$

Thus it is natural to formulate the following conjecture:

CONJECTURE 7.2.1. The injective morphism \mathcal{J} restricts to an inclusion of the quantum Grothendieck rings

$$\mathcal{J} : K_t(\mathcal{C}_{\mathbb{Z}}) \subset K_t(\mathcal{O}_{\mathbb{Z}}^+). \quad (7.1)$$

Recall that the quantum Grothendieck ring $K_t(\mathcal{C}_{\mathbb{Z}})$ is generated by the classes of the fundamental representations $[L(Y_{i,q^{r+1}})]_t$, for $(i,r) \in \hat{I}$ (see Section 6.1.1). Hence, in order to prove Conjecture 7.2.1, it is enough to show that the images of these $[L(Y_{i,q^{r+1}})]_t$ belong to $K_t(\mathcal{O}_{\mathbb{Z}}^+)$.

In Example 4.7.2 we saw how, when the $\mathfrak{g} = \mathfrak{sl}_2$, the class of the fundamental representation $[L(Y_{1,q^{-1}})]$ could be obtained as a cluster variable in $\mathcal{A}(\Gamma)$ after one mutation in direction $(1,0)$.

This fact is actually true in more generality, as seen in [36], in the proof of Proposition 6.1. Let us recall this process precisely.

Fix $(i, r) \in \hat{I}$. We first define a specific sequence of vertices in Γ , as in [37]. Recall the definition of the dual Coxeter number h^\vee .

\mathfrak{g}	A_n	D_n	E_6	E_7	E_8
h^\vee	$n+1$	$2n-2$	12	18	30

Let $h' = \lceil h^\vee/2 \rceil$. Fix an ordering (j_1, \dots, j_n) of the vertices of the Dynkin diagram of \mathfrak{g} by taking first $j_1 = i$, then all vertices which appear with the same oddity as i in \hat{I} (the j such that $(j, r) \in \hat{I}$), then the vertices which appear with a different oddity $((j, r+1) \in \hat{I})$. For all $k \in \{2, \dots, h'\}$, $j \in \{1, \dots, n\}$, define the sequence $S_{j,k}$ of k vertices of the column j of Γ in decreasing order:

$$S_{j,k} = (j, r+2h'-\epsilon), (j, r+2h'-\epsilon-2), \dots, (j, r+2h'-\epsilon-2k+2), \quad (7.2)$$

where $\epsilon \in \{0, 1\}$, depending of the oddity. Then define

$$S_k = \bigcup_j S_{j,k}, \quad (7.3)$$

with the order defined before. Finally, let

$$S = S_{h'} \cdots S_2 (i, r+2h'),$$

by reading left to right and adding one last $(i, r+2h')$ at the end.

EXAMPLE 7.2.2. For \mathfrak{g} of type D_4 , and $(i, r) = (1, 0)$, the sequence S is

$$\begin{aligned} S = & (1, 6) (1, 4) (1, 2) (3, 6) (3, 4) (3, 2) \\ & (4, 6) (4, 4) (4, 2) (2, 5) (2, 3) (2, 1) \\ & (1, 6) (1, 4) (3, 6) (3, 4) (4, 6) (4, 4) \\ & (2, 5) (2, 3) (1, 6). \end{aligned}$$

Using [37, Theorem 3.1] and elements from the proof of [36, Proposition 6.1], one gets the following result.

PROPOSITION 7.2.3. Let $\chi_{i,r}$ be the cluster variable of $\mathcal{A}(\Gamma)$ obtained at the vertex $(i, r+2h')$ after following the sequence of mutations S , then, via the identification (4.15)

$$\chi_{i,r} \equiv [L(Y_{i,q^{r+1}})]. \quad (7.4)$$

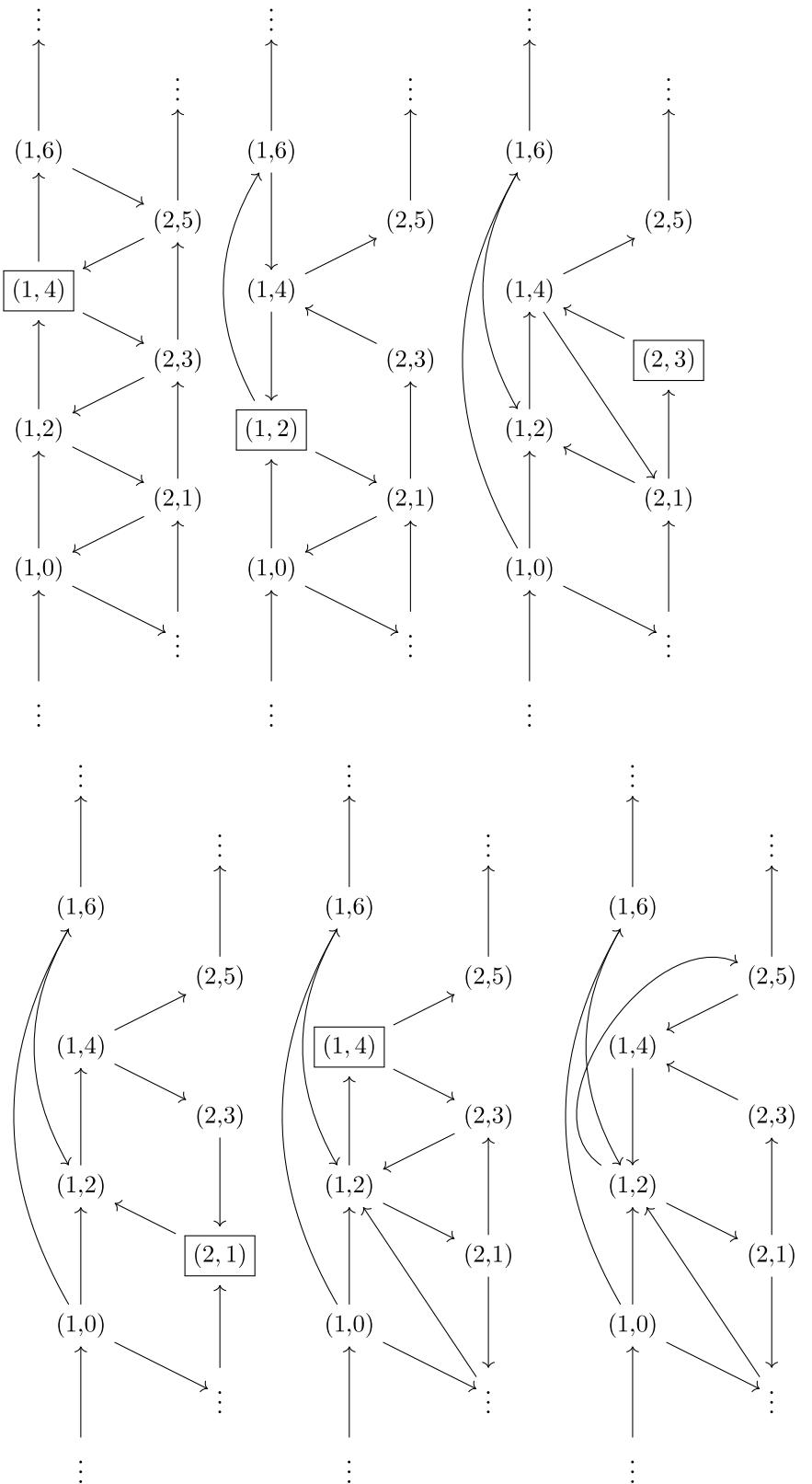
To see this result differently, if one writes $\chi_{i,r}$ as a Laurent polynomial in the variables $(z_{j,s})$, then $\chi_{i,r}$ is in the image of \mathcal{J} , and

$$\chi_{i,r} = \mathcal{J}(\chi_q(L(Y_{i,q^{r+1}}))). \quad (7.5)$$

EXAMPLE 7.2.4. Let $\mathfrak{g} = \mathfrak{sl}_3$ and $(i, r) = (1, 0)$. The sequence of vertices S is

$$S = (1, 4) (1, 2) (2, 3) (2, 1) (1, 4). \quad (7.6)$$

Let us compute the sequence of mutations S :



The associated cluster variables are

$$\begin{aligned} z_{1,4}^{(1)} &= z_{1,2}z_{1,4}^{-1}z_{2,5} + z_{1,4}^{-1}z_{1,6}z_{2,3}, \\ z_{1,2}^{(1)} &= z_{1,0}z_{1,4}^{-1}z_{2,5} + z_{1,0}z_{1,2}^{-1}z_{1,4}^{-1}z_{1,6}z_{2,3} + z_{1,2}^{-1}z_{1,6}z_{2,-1}, \\ z_{2,3}^{(1)} &= z_{2,1}z_{2,3}^{-1} + z_{1,2}z_{1,4}^{-1}z_{2,5}z_{2,3}^{-1} + z_{1,4}^{-1}z_{1,6}, \\ z_{1,4}^{(2)} &= z_{1,0}z_{1,2}^{-1} + z_{1,2}^{-1}z_{1,4}z_{2,1}z_{2,3}^{-1} + z_{2,3}^{-1}z_{2,5}. \end{aligned}$$

Thus, $\chi_{1,0} = z_{1,4}^{(2)}$ is in the image of \mathcal{J} , and

$$\chi_{1,0} = \mathcal{J}(Y_{1,q} + Y_{1,q^3}^{-1}Y_{2,q^2} + Y_{2,q^4}^{-1}) = \mathcal{J}(\chi_q(L(Y_{1,q}))). \quad (7.7)$$

Note also that $z_{2,3}^{(1)}$ was already in the image of \mathcal{J} and that $z_{2,3}^{(1)} = \mathcal{J}(\chi_q(L(Y_{2,q^2})))$.

Thus, for each $(i, r) \in \hat{I}$, consider the quantum cluster variables $\tilde{\chi}_{i,r} \in K_t(\mathcal{O}_{\mathbb{Z}}^+)$ obtained from the initial quantum seed (\mathbf{z}, Λ) via the sequence of mutations S .

EXAMPLE 7.2.5. Suppose $\mathfrak{g} = \mathfrak{sl}_2$. Consider the quiver Γ_1 a well as the skew-symmetric matrix Λ_1 ,

$$\Gamma_1 = \begin{array}{c} \boxed{(1,2)} \\ \uparrow \\ (1,0) \\ \uparrow \\ \boxed{(1,-2)} \end{array}, \quad \Lambda_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

As seen in Example 4.7.2 (with a shift of quantum parameters), the fundamental representation $[L(Y_{1,q^{-1}})]$ is obtained in $K_0(\mathcal{O}_{\mathbb{Z}}^+)$ after one mutation at $(1,0)$ (here $S = (1,0)$).

The quantum cluster variable obtained after a quantum mutation at $(1,0)$, written with bar-invariant monomials, is

$$\begin{aligned} \tilde{\chi}_{1,-2} &= z_{1,-2}z_{1,0}^{-1} + z_{1,2}z_{1,0}^{-1} = \mathcal{J}(Y_{1,q^{-1}} + Y_{1,q}^{-1}) = \mathcal{J}([L(Y_{1,q^{-1}})]_t), \\ &= \mathcal{J}(Y_{1,q^{-1}}(1 + A_{1,1}^{-1})) \in \mathcal{J}(K_t(\mathcal{C}_{\mathbb{Z}})), \end{aligned}$$

Thus, we note that in this particular case, the quantum cluster variable $\tilde{\chi}_{1,-2}$ recovers the (q,t) -character $[L(Y_{1,q^{-1}})]_t$ of the fundamental representation $L(Y_{1,q^{-1}})$.

In particular, Conjecture 7.2.1 is satisfied in this case.

This example incites us to formulate another conjecture.

CONJECTURE 7.2.6. For all $(i, r) \in \hat{I}$, the quantum cluster variable $\tilde{\chi}_{i,r}$ recovers, via the morphism \mathcal{J} , the (q,t) -character of the fundamental representation $L(Y_{i,q^{r+1}})$:

$$\tilde{\chi}_{i,r} = \mathcal{J}([L(Y_{i,q^{r+1}})]_t). \quad (7.8)$$

The author plans to prove this conjecture for all simply laced types in a follow-up work.

REMARK 7.2.7. Note that Conjecture 7.2.6 implies Conjecture 7.2.1, and that Conjecture 7.2.6 is already proven when $\mathfrak{g} = \mathfrak{sl}_2$, from Example 7.2.5.

What can be said, in general, of the quantum cluster variables $\tilde{\chi}_{i,r}$?

PROPOSITION 7.2.8. *For all $(i, r) \in \hat{I}$, the quantum cluster variable $\tilde{\chi}_{i,r}$ satisfies the following properties:*

(i) *invariant under the bar involution:*

$$\overline{\tilde{\chi}_{i,r}} = \tilde{\chi}_{i,r}; \quad (7.9)$$

(ii) *the coefficients of its expansion as a Laurent polynomial in the initial quantum cluster variables $\{z_{i,r}\}$ are Laurent polynomials in $t^{1/2}$ with non-negative integers coefficients:*

$$\tilde{\chi}_{i,r} \in \bigoplus_{\mathbf{u} = u_{i,r} \in \mathbb{Z}^{(\hat{I})}} \mathbb{N}[t^{\pm 1/2}] \mathbf{z}^{\mathbf{u}} \quad (7.10)$$

with $\mathbf{z}^{\mathbf{u}} = \prod_{(i,r) \in \hat{I}} z_{i,r}^{u_{i,r}}$ denoting the bar-invariant monomial;

(iii) *its evaluation at $t = 1$ (as seen in (5.18)), recovers the q -character of the fundamental representation $L(Y_{i,q^{r+1}})$:*

$$\pi(\tilde{\chi}_{i,r}) = \chi_q(L(Y_{i,q^{r+1}})). \quad (7.11)$$

Proof. The first property is a direct consequence of Proposition 7.1.1 and the second is a direct consequence of the positivity result of Theorem 2.6.1.

For the third property, note we have used two evaluation maps so far, with the same notation.

- The evaluation map defined in (2.17) on the bases quantum torus of a quantum cluster algebra:

$$\pi : \mathcal{A}_t(M, \tilde{B}) \rightarrow \mathbb{Z}[\tilde{\mathbf{X}}^{\pm 1}],$$

- The evaluation map defined in (5.18) on \mathcal{T}_t :

$$\pi : \mathcal{T}_t \rightarrow \mathcal{E}_\ell.$$

These notations are coherent because the map π from (5.18) is the evaluation map defined on a based quantum torus (of infinite rank) of a quantum cluster algebra, extended to a \mathcal{E} -morphism on \mathcal{T}_t . In this case, the Laurent polynomial ring $\mathbb{Z}[\tilde{\mathbf{X}}^{\pm 1}]$ is $\mathbb{Z}[z_{i,r}^{\pm 1} \mid (i, r) \in \hat{I}]$, which becomes $\mathcal{E}[\Psi_{i,r}^{\pm 1}]$ after extension to a \mathcal{E} -morphism and via the identification (4.15).

Thus we can apply Corollary 2.5.3 to this map π . As $\tilde{\chi}_{i,r}$ is a quantum cluster variable, its evaluation by π is the cluster variable $\chi_{i,r}$, which is obtained from the initial seed \mathbf{z} , via the same sequence of mutations S (the initial seed and quantum seeds are fixed and identified by the evaluation π on the quantum torus \mathcal{T}_t). By Proposition 7.2.3,

$$\pi(\tilde{\chi}_{i,r}) = \chi_{i,r} = \chi_q(L(Y_{i,q^{r+1}})). \quad (7.12)$$

□

These two properties imply that the $\tilde{\chi}_{i,r}$ are good candidates for the (q, t) -characters of the fundamental representations, as stated in Conjecture 7.2.1.

7.3. (q, t) -Characters for positive prefundamental representations

Recall the q -characters of the positive prefundamental representations in (4.23), for all $i \in I, a \in \mathbb{C}^\times$,

$$\chi_q(L_{i,a}^+) = [\Psi_{i,a}] \chi_i,$$

where $\chi_i \in \mathcal{E}$ is the (classical) character of $L_{i,a}^+$.

DEFINITION 7.3.1. For $(i, r) \in \hat{I}$, define

$$[L_{i, q^r}^+]_t := [\Psi_{i, q^r}] \otimes \chi_i \in K_t(\mathcal{O}_{\mathbb{Z}}^+), \quad (7.13)$$

using the notation from (5.13).

REMARK 7.3.2. It is the quantum cluster variable obtained from the initial quantum seed, via the same sequence of mutations used to obtain $[L_{i, q^r}^+]$ in $K_0(\mathcal{O}_{\mathbb{Z}}^+)$, which in this case, is no mutation at all.

In particular, the evaluation of $[L_{i, q^r}^+]_t$ recovers the q -character of $[L_{i, q^r}^+]$:

$$\pi([L_{i, q^r}^+]_t) = [\Psi_{i, q^r}] \otimes \chi_i = \chi_q(L_{i, a}^+) \in \mathcal{E}_\ell. \quad (7.14)$$

8. Results in type A

Throughout this section we suppose that \mathfrak{g} is of type A .

8.1. Proof of the conjectures

In this case, the situation of Example 7.2.5 generalizes.

THEOREM 8.1.1. Conjecture 7.2.6 holds for \mathfrak{g} of type A .

In this case, the key ingredient is the following well-known result (see, for example, [24, Section 11], and references therein).

THEOREM 8.1.2. When \mathfrak{g} is of type A , all ℓ -weight spaces of all fundamental representations $L(Y_{i, a})$ are of dimension 1.

Proof. Fix $(i, r) \in \hat{I}$. From the second property of Proposition 7.2.8, we know that $\tilde{\chi}_{i, r}$ can be written as

$$\tilde{\chi}_{i, r} = \sum_{\mathbf{u} \in \mathbb{Z}^{(\hat{I})}} P_{\mathbf{u}}(t^{1/2}) \mathbf{z}^{\mathbf{u}}, \quad (8.1)$$

where the $P_{\mathbf{u}}(t^{1/2})$ are Laurent polynomials with non-negative integer coefficients. Using the third property of Proposition 7.2.8, we deduce the evaluation at $t = 1$ of equality (8.1):

$$\chi_q(L(Y_{i, q^{r-1}})) = \sum_{\mathbf{u} \in \mathbb{Z}^{(\hat{I})}} P_{\mathbf{u}}(1) \prod_{(i, r) \in \hat{I}} ([\Psi_{i, q^r}] [-r\omega_i/2])^{u_{i, r}} \in \mathcal{E}_\ell. \quad (8.2)$$

From the above theorem, this decomposition is multiplicity-free. Thus, the non-zero coefficients $P_{\mathbf{u}}(t^{1/2})$ are of the form $t^{k/2}$, with $k \in \mathbb{Z}$. Finally, as $\chi_{i, r}$ is bar-invariant, from the first property of Proposition 7.2.8, and the $\mathbf{z}^{\mathbf{u}}$ are also bar-invariant, we know that the Laurent polynomials $P_{\mathbf{u}}(t^{1/2})$ are even functions:

$$P_{\mathbf{u}}(-t^{1/2}) = P_{\mathbf{u}}(t^{1/2}). \quad (8.3)$$

Thus the variable $t^{1/2}$ does not explicitly appear in the decomposition (8.1), and so:

$$\begin{aligned} \chi_{i, r} &= \sum_{\mathbf{u} \in \mathbb{Z}^{(\hat{I})}} P_{\mathbf{u}}(1) \mathbf{z}^{\mathbf{u}}, \\ &= \mathcal{J}(\chi_q(L(Y_{i, q^{r-1}}))). \end{aligned}$$

Moreover, with the same arguments, as $[L(Y_{i,q^{r-1}})]_t$ is bar-invariant by definition,

$$[L(Y_{i,q^{r-1}})]_t = \chi_q(L(Y_{i,q^{r-1}})), \quad (8.4)$$

written in the basis of the bar-invariant monomials.

Hence we recover the fact that the quantum cluster variable $\chi_{i,r}$ is equal, via the inclusion map \mathcal{J} , to the (q,t) -character of $L(Y_{i,q^{r-1}})$ and Conjecture 7.2.6 is satisfied. \square

8.2. A remarkable subalgebra in type A_1

When $\mathfrak{g} = \mathfrak{sl}_2$, we can make explicit computations. Retain the notation of Example 5.2.2.

For all $r \in \mathbb{Z}$, the (q,t) -character of the prefundamental representation $L_{1,q^{2r}}^+$ defined in (7.13) is

$$[L_{1,q^{2r}}^+]_t = [\Psi_{1q^{2r}}] \chi_1.$$

PROPOSITION 8.2.1. *With these (q,t) -characters, we can write a t -deformed version of the Baxter relation (1.2), for all $r \in \mathbb{Z}$,*

$$[L(Y_{1,q^{2r-1}})]_t * [L_{1,q^{2r}}^+]_t = t^{-1/2} [\omega_1] [L_{1,q^{2r-2}}^+]_t + t^{1/2} [-\omega_1] [L_{1,q^{2r+2}}^+]_t. \quad (8.5)$$

We call this relation the *quantized Baxter relation*.

REMARK 8.2.2. If we identify the variables $Y_{1,q^{2r}}$ and their images through the injection \mathcal{J} , this relation is actually the exchange relation related to the quantum mutation in Example 7.2.5 (for a generic quantum parameter q^{2r}).

Now consider the quantum cluster algebra $\mathcal{A}(\Lambda_1, \Gamma_1)$, with notations from Section 6.2 (Λ_1 and Γ_1 are given explicitly in Example 7.2.5).

It is a quantum cluster algebra of finite type (if we remove the frozen vertices from the quiver, we get just one vertex, which is a quiver of type A_1). It has two quantum clusters, containing the two frozen variables $z_{1,2}, z_{1,-2}$ and the mutable variables $z_{1,0}$ and $z_{1,0}^{(1)}$, respectively. Thus, it is generated as a $\mathbb{C}(t^{1/2})$ -algebra by

$$\begin{aligned} E &:= [L(Y_{1,q^{-1}})]_t (= z_{1,0}^{(1)}), & F &:= [L_{1,1}^+]_t (= z_{1,0}), \\ K &:= [\omega_1] [L_{1,q^{-2}}^+]_t (= z_{1,-2}), & K' &:= [-\omega_1] [L_{1,q^2}^+]_t (= z_{1,2}). \end{aligned} \quad (8.6)$$

This algebra is a quotient of a well-known $\mathbb{C}(t^{1/2})$ -algebra.

Let q be a formal parameter. The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ can be seen as the quotient

$$\mathcal{U}_q(\mathfrak{sl}_2) = \mathfrak{D}_2 / \langle KK' = 1 \rangle, \quad (8.7)$$

where \mathfrak{D}_2 is the $\mathbb{C}(q)$ -algebra with generators E, F, K, K' and relations:

$$\begin{aligned} KE &= q^2 EK, & K'E &= q^{-2} EK' \\ KF &= q^{-2} FK, & K'F &= q^2 FK' \\ KK' &= K'K, & \text{and } [E, F] &= (q - q^{-1})(K - K'). \end{aligned} \quad (8.8)$$

REMARK 8.2.3. • As in [44, Remark 3.1], note that the last relation in (8.8) is not the usual relation

$$[e, f] = \frac{K - K'}{q - q^{-1}}.$$

But both presentations are equivalent, given the change of variables

$$E = (q - q^{-1})e, \quad F = (q - q^{-1})f.$$

The presentation (8.8) also appeared in the work of Bridgeland [9], as it is more natural from the point of view of Hall algebras than the usual presentation.

- \mathfrak{D}_2 is the Drinfeld double [16] of the Borel subalgebra of $\mathcal{U}_q(\mathfrak{sl}_2)$ (the subalgebra generated by K, E), see also [5, equation (1.2)].

PROPOSITION 8.2.4. *The $\mathbb{C}(t^{1/2})$ -algebra $\mathcal{A}(\Lambda_1, \Gamma_1)$ is isomorphic to the quotient of the Drinfeld double \mathfrak{D}_2 of parameter $-t^{1/2}$,*

$$\mathcal{A}(\Lambda_1, \Gamma_1) \xrightarrow{\sim} \mathfrak{D}_2/C_{-t^{1/2}}, \quad (8.9)$$

where $C_{-t^{1/2}}$ is the quantized Casimir element:

$$C_{-t^{1/2}} := EF - t^{-1/2}K - t^{1/2}K'. \quad (8.10)$$

Proof. One has, in $\mathcal{A}(\Lambda_1, \Gamma_1)$,

$$E * F = t^{-1/2}K + t^{1/2}K'. \quad (8.11)$$

This is the quantized Baxter relation (8.5). Thus,

$$[E, F] = (-t^{1/2} + t^{-1/2})(K - K').$$

We check that the other relations in (8.8) are also satisfied using the structure of the quantum torus \mathcal{T}_t (which is given explicitly in Example 5.2.2).

Hence the map

$$\mathcal{A}(\Lambda_1, \Gamma_1) \xrightarrow{\theta} \mathfrak{D}_2,$$

sending generators to generators is well defined and descends onto the quotient

$$\mathcal{A}(\Lambda_1, \Gamma_1) \rightarrow \mathfrak{D}_2/C_{-t^{1/2}}.$$

Moreover, from [10], the cluster monomials in a given cluster in a cluster algebra are linearly independent. In this case, the quantum cluster algebra $\mathcal{A}(\Lambda_1, \Gamma_1)$ is of type A_1 (without frozen variables), thus of finite-type. It has two (quantum) clusters : (E, K, K') and (F, K, K') . Thus, the set of bar-invariant quantum cluster monomials

$$\{E^\alpha K^\beta K'^\gamma \mid \alpha, \beta, \gamma \in \mathbb{Z}\} \cup \{K^\beta K'^\gamma F^\alpha \mid \alpha, \beta, \gamma \in \mathbb{Z}\}, \quad (8.12)$$

forms a $\mathbb{C}(t^{1/2})$ -basis of $\mathcal{A}(\Lambda_1, \Gamma_1)$.

Consider the PBW basis of \mathfrak{D}_2 :

$$\{E^\alpha K^\beta K'^\gamma F^\delta \mid \alpha, \beta, \gamma, \delta \in \mathbb{Z}\}. \quad (8.13)$$

From the expression of the Casimir element $C_{-t^{1/2}}$ (8.10), we deduce a $\mathbb{C}(t^{1/2})$ -basis of $\mathfrak{D}_2/C_{-t^{1/2}}$, of the same form as (8.12):

$$\{E^\alpha K^\beta K'^\gamma \mid \alpha, \beta, \gamma \in \mathbb{Z}\} \cup \{K^\beta K'^\gamma F^\alpha \mid \alpha, \beta, \gamma \in \mathbb{Z}\}. \quad (8.14)$$

Hence, the map θ sends a basis to a basis, thus it is isomorphic. \square

REMARK 8.2.5. The basis obtained in (8.14) is related to the double canonical basis of $\mathcal{U}_q(\mathfrak{sl}_2)$ introduced by Berenstein–Greenstein [5]. Their work also adopts the presentation (8.8), moreover it uses crucially the quantum Heisenberg algebra, which is also related to Bridgeland’s Hall algebra (see [28, Remark 4.8]).

This result should be compared with the recent work of Schrader and Shapiro [44], in which they recognize the same structure of \mathfrak{D}_2 in an algebra built on a quiver, with some quantum \mathcal{X} -cluster algebra structure. In their work, they generalized this result in type A (Theorem 4.4).

Ultimately, they obtain an embedding of the whole quantum group $\mathcal{U}_q(\mathfrak{sl}_n)$ into a quantum cluster algebra. The result of Proposition 8.2.4, together with their results, gives hope that one could find a realization of the quantum group $\mathcal{U}_q(\mathfrak{g})$ as a quantum cluster algebra, related to the representation theory of $\mathcal{U}_q(L\mathfrak{g})$.

Furthermore, define in this case \mathcal{O}_1^+ , the subcategory of $\mathcal{O}_{\mathbb{Z}}^+$ of objects whose image in the Grothendieck ring $K_0(\mathcal{O}_{\mathbb{Z}}^+)$ belongs to the subring generated by $[L_{1,q^{-2}}^+], [L_{1,1}^+], [L_{1,q^2}^+]$ and $[L(Y_{1,q^{-1}})]$. Then \mathcal{O}_1^+ is a monoidal category.

From the classification of simple modules when $\mathfrak{g} = \mathfrak{sl}_2$ in [36, Section 7], we know that the only prime simple modules in \mathcal{O}_1^+ are

$$L_{1,q^{-2}}^+, L_{1,1}^+, L_{1,q^2}^+, L(Y_{1,q^{-1}}). \quad (8.15)$$

Moreover, a tensor product of those modules is simple if and only if it does not contain both a factor $L_{1,1}^+$ and a factor $L(Y_{1,q^{-1}})$ (the others are in so-called pairwise general position). Thus, in this situation, the simple modules are in bijection with the cluster monomials:

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{simple modules} \\ \text{in } \mathcal{O}_1^+ \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{bar-invariant quantum cluster} \\ \text{monomials in } \mathcal{A}(\Lambda_1, \Gamma_1) \end{array} \right\} \\ \left(L_{1,q^{-2}}^+ \right)^{\otimes \alpha} \otimes \left(L_{1,1}^+ \right)^{\otimes \beta} \otimes \left(L_{1,q^2}^+ \right)^{\otimes \gamma} & \mapsto & K^\alpha F^\beta K'^\gamma, \\ \left(L_{1,q^{-2}}^+ \right)^{\otimes \alpha'} \otimes \left(L_{1,q^2}^+ \right)^{\otimes \beta'} \otimes L(Y_{1,q^{-1}})^{\otimes \gamma'} & \mapsto & K^{\alpha'} K'^{\beta'} E^{\gamma'}. \end{array}$$

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Léa Bittmann
 Faculty of Mathematics
 Universität Wien
 Oskar-Morgenstern-Platz 1
 Wien 1090
 Austria

lea.bittmann@univie.ac.at