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Statistical Formulation of Background Independence in Manifestly-Covariant Quantum Gravity Theory

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Abstract: The notion of background independence is a distinguished feature that should characterize the conceptual foundation of any physically-acceptable theory of quantum gravity. It states that the structure of the space-time continuum described by classical General Relativity should possess an emergent character, namely, that it should arise from the quantum-dynamical gravitational field. In this paper, the above issue is addressed in the framework of manifestly-covariant quantum gravity theory. Accordingly, a statistical formulation of background independence is provided, consistent with the principle of manifest covariance. In particular, it is shown that the classical background metric tensor determining the geometric properties of space-time can be expressed consistently in terms of a suitable statistical average of the stochastic quantum gravitational field tensor. As an application, a particular realization of background independence is shown to hold for analytical Gaussian solutions of the quantum probability density function.

Keywords: quantum gravity; background independence; emergent space-time; stochastic quantum gravity; manifest covariance

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1. Introduction

The notions of background independence and diffeomorphism invariance, that is, General Covariance or General Relativity frame independence, are often claimed to represent distinctive physical properties and compelling requirements for models of Quantum Gravity (QG) theories. In the literature, they have been investigated in terms of either exact or approximate mathematical methods, including the case of perturbative renormalization schemes [1–5]. In fact, it is believed that the ultimate theory of quantum gravity should actually be able to predict the emergence of the classical background structure of the space-time continuum from quantum degrees of freedom of some sort [6,7]. The latter, however, should be selected on physical grounds, i.e., suitable boundary conditions. This means also that they should be (possibly non-uniquely) rooted in the corresponding classical Hamiltonian representations on which the same QG theory must rely [8–10].

1.1. The Two Routes to the Hamiltonian Representation of GR and QG

Two possible routes are available to establish a Hamiltonian representation of classical, and consequently quantum, gravity theories. These are respectively based on two possible variational settings, i.e., either the constrained $\{g(r)\}_C$ or corresponding unconstrained $\{g(r)\}_U$ setting [10,11]. The two settings can also be referred to as "Multi-verse space-time" and respectively "Uni-verse or background space-time" since they correspond to two different possible choices of space-time.

The first choice is adopted in Quantum Geometrodynamics (QGD), i.e., either the Wheeler–DeWitt (WDW) equations [12,13] of QGD or the Ashtekar-variable representation [14], as well as the Loop Quantum Gravity (LQG) approach [15]. In this case, the generic classical tensor $g(r) \in \{g(r) \equiv g_{\mu\nu}(r)\}$ is itself identified with a metric tensor. This means that each virtual variational tensor $g(r)$ underlying the theory generates its own, and in principle different, space-time. The structure of the corresponding classical space-time is therefore identified with the differential manifold

$$\left\{\mathbf{Q}^4, g(r)\right\}. \tag{1}$$

This implies that the counter and covariant components of the symmetric $g(r)$, namely, $\{g_{\mu\nu}(r)\}$ and $\{g^{\mu\nu}(r)\}$, are necessarily constrained. Namely, they are subject to the orthogonality conditions

$$g_{\mu\nu}(r)g^{\alpha\nu}(r) = \delta_\mu^\alpha, \tag{2}$$

which must hold identically for any $g(r)$ evaluated at arbitrary $4-$positions $r \equiv \{r^\mu\}$ spanning $\{\mathbf{Q}^4, g(r)\}$ and for arbitrary indices $\mu, \alpha = 0, 3$. The consequent implication is that the functional setting $\{g(r)\}$ is represented by constrained tensor functions subject to the orthogonality constraints (2), i.e., a suitably-defined constrained-function space with $\{g(r)\} = \{g(r)\}_C$ (see [10]).

The unconstrained variational setting $\{g(r)\}_U$ is instead adopted in the context of the so-called manifestly-covariant QG theory (CQG-theory) [16]. For this purpose, the assumption is introduced that there exists a suitable and possibly non-unique quantum background metric field tensor $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$, with $\widehat{g}(r)$ itself belonging to an appropriate functional set $\{\widehat{g}(r)\}$. The metric tensor $\widehat{g}(r)$ is allowed to be determined consistently by the same quantum gravity theory, that is, to represent an emergent space-time metric tensor. In this case, $\widehat{g}(r)$ has to be suitably determined such that it coincides with an appropriate (and yet to be prescribed) quantum expectation value of underlying quantum variables or degrees of freedom. Correspondingly, the associated background "quantum" space-time is introduced as

$$\left\{\mathbf{Q}^4, \widehat{g}(r)\right\}. \tag{3}$$

In the unconstrained setting, the generic classical variational $g(r)$ introduced above is not required to be a metric tensor, which means that it remains unconstrained. However, its tensorial properties remain defined unambiguously. In fact, the same $g(r)$ is assumed to be a $4-$tensor with respect to the space-time $\widehat{g}(r)$.

As a remark, we recall that for the quantum treatment the validity of the unconstrained setting is implied by the corresponding synchronous unconstrained variational principle holding in classical General Relativity [16,17], which realizes a deDonder–Weyl representation of Lagrangian variational theory for the continuum gravitational field [18–20]. In fact, the synchronous variational approach is expressed by a $4-$scalar Lagrangian function depending on both variables $g_{\mu\nu}$ and $\widehat{g}_{\mu\nu}$. According to the language of variational theory, this implements a superabundant variable approach. In this setting, the variational tensor $g \equiv \{g_{\mu\nu}\}$ remains distinguished from the background metric tensor $\widehat{g} \equiv \{\widehat{g}_{\mu\nu}\}$, which is to be regarded as non-variational. This latter is a fundamental element of the theory, as its inclusion permits the definition of the covariance properties of the theory and ultimately the proper definition of background independence holding for the quantum theory, as displayed below. The tensor $\widehat{g}$ is assumed to be determined a posteriori by the extremal (and possibly quantum-modified) Einstein field equations. Hence, the metric tensor $\widehat{g}$ is endowed with a geometric connotation, that is, it satisfies the orthogonality condition $\widehat{g}_{\mu\nu}\widehat{g}^{\mu k} = \delta_\nu^k$. Because of this, $\widehat{g}$ can be used to raise or lower tensor indices, satisfies the so-called metric compatibility condition of vanishing covariant derivative $\widehat{\nabla}_\alpha \widehat{g}_{\mu\nu} = 0$, and as such permits the definition of the standard Christoffel connections and the curvature tensors of space-time. On the contrary, in the unconstrained framework the variational tensor g is such that $g_{\mu\nu}g^{\mu k} \neq \delta_\nu^k$. It must be noted that the distinction between g and $\widehat{g}$ is true only within the variational principle, as the identity $g = \widehat{g}$ is restored in the extremal field equations identified with the classical Einstein field equations. The same distinction applies to the definition of the invariant 4-volume element as well, which takes the "synchronous" form $d\widehat{\Omega} = d^4 r\sqrt{-|\widehat{g}|}$, such that its variation vanishes and does not contribute to the variational calculus. This volume-preserving property permits us to establish an interesting formal connection between the synchronous setting and certain relevant approaches in the literature sharing a similar conceptual viewpoint. More precisely, we refer here to non-metric volume forms or modified measures, defined for example in [21,22]. Additionally, it is worth mentioning the so-called non-Riemannian space-time volume elements [23], which instead suggest variational models for the GR equations in which the volume elements of integration in the action principles are assumed to be metric independent. This requirement is met by demanding that the latter ones be determined dynamically through additional degrees of freedom, which are, for example, associated with the inclusion of additional scalar fields.

1.2. The Concept of Background Space-Time

In classical gravity the concept of background space-time, i.e., a Riemannian differential manifold formed by the couple $\{\mathbf{Q}^4, \widehat{g}(r)\}$, and respectively that of background metric tensor $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\} \equiv \{\widehat{g}^{\mu\nu}(r)\}$, is related to the Einstein Field Equations (EFE) [24]. Here, $\widehat{g}_{\mu\nu}(r)$ and $\widehat{g}^{\mu\nu}(r)$ denote the covariant and counter-variant components of $\widehat{g}(r)$, which by construction satisfy the orthogonality conditions $\widehat{g}_{\mu\alpha}(r)\widehat{g}^{\mu\beta}(r) = \delta_{\alpha}^{\beta}$. More precisely, the background space-time metric tensor $\widehat{g}(r)$ is an in principle arbitrary and particular solution of EFE [25].

However, in QG, $\widehat{g}(r)$ should more generally identify a suitable quantum-modified metric tensor. In this regard, the question is whether, in the context of QG,

- $\widehat{g}(r)$ can actually acquire the meaning of a classical metric field tensor and can be identified with the quantum expectation value of a suitable quantum observable, i.e., correspondingly acquiring, in a sense, a kind of "statistical interpretation".
- $\widehat{g}(r)$ can be ultimately determined by a suitable "dynamical equation" having a well-defined Hamiltonian character, to be identified with quantum-modified (tensor) EFE following from the quantum-gravity wave equation.
- The same tensor field $\widehat{g}(r)$ fulfills the property of background independence, i.e., its representation is not assigned, and it is rather determined a posteriori as a solution of the same quantum-modified EFE.

Accordingly, in order to be acceptable a QG theory should not depend on the particular realization of $\widehat{g}(r)$, to be identified, for example with a continuum classical metric tensor provided by the solution of the said quantum-modified EFE. This means that QG should not be formulated in a given space-time to be imposed a priori. Rather, QG itself should be able to generate the background space-time through gravitational quantum dynamics.

From the outset, however, two crucial difficulties arise, i.e., the identification/construction of the quantum-modified EFE, and consequently the definition of background space-time.

In fact, in the framework of QGD there is, first, no obvious way to identify and cast in explicit $4-$tensor form the said quantum-modified EFE; second, the very definition of the concept of background space-time remains to be defined. In fact, in the literature a precise mathematical formulation of the concept of background may actually depend on the choice of the setting implemented for the development of QG [26]. The same notion nevertheless places stringent constraints on the validity of a given QG theory, requiring the establishment of a precise relationship between the structure and geometrical interpretation of the quantum gravitational field and its classical limit provided by the four-dimensional continuum space-time of GR [27].

In contrast, the approach adopted in the context of CQG-theory appears quite different. The reasons for this are as follows. First, one can show that the quantum-modified EFE emerges from the quantum-wave equation which determines the evolution of the quantum wave-function (a hyperbolic first-order PDE) [28,29]. Second, the background space-time $\widehat{g}(r)$ acquires an "emergent character", in the sense that it can be determined as the quantum expectation value of a suitable quantum (second order symmetric) tensor field. As a consequence, in such a context the concept of background independence holding in QG becomes intrinsically related with that of emergent space-time applying to the classical field. Reversing the perspective, this means that the structure of the continuum space-time described by classical GR should possess an emergent character, that is, it should arise from the quantum-dynamical gravitational field [30,31].

1.3. Background Independence

In the context of both GR and QG, a possible viewpoint is that background independence might/should be identified with the property of the metric tensor of space-time to be the general solution (i.e., an arbitrary particular solution) of a "dynamical equation" of some sort, i.e., actually a PDE subject to prescribed boundary conditions. Such an equation should be related to the existence of a suitable Hamiltonian structure, both classical and quantum, underlying an abstract dynamical field description of GR and QG. Therefore, in order to qualify the property of background independence, the preliminary requirement remains that of determining a prescription of the same Hamiltonian structure. This restricts the selection of candidate QG theories to those that truly possess the requisite Hamiltonian character [32]. In the setting of CQG-theory, such an equation has a natural 4−tensor character and is represented by the quantum-modified EFE.

With these premises, in this paper the meaning of background independence and its mathematical formulation are addressed for the non-perturbative manifestly-covariant quantum gravity theory (CQG-theory [33,34]). This framework in fact admits a well-defined Hamiltonian structure and is consistent with the Principle of Manifest Covariance (PMC), namely, the property of dynamical equations and physical observables of being cast in 4−tensor form [35]. Specifically, in the following, we contend that in the context of CQG-theory the property of background independence requires the existence of a classical background space-time with respect to which the notion of covariance can be defined. On the other hand, the background metric field tensor ($\widehat{g}$) associated with such a space-time and determining its geometric properties does not possess an independent character; rather, it can be prescribed in terms of the quantum expectation value of a suitable stochastic quantum observable. Accordingly, a statistical formulation of background independence is proved to hold for CQG-theory. To reach the target, the mathematical setting identified with the so-called generalized Lagrangian-path (GLP) theory of the quantum-wave equation is implemented, which relies on an appropriate statistical formulation of the Bohmian trajectory-based representation peculiar of CQG-theory [28].

The scheme of the investigation is structured as follows. First, the fundamental elements of CQG-theory, the validity of quantum hydrodynamic equations and their Hamiltonian character are recalled. Second, the theoretical formulation of statistical trajectory-based representation of CQG-theory in terms of GLP approach is revised. The composite framework established in this way permits a non-perturbative formulation of statistical background independence holding for CQG-theory to be addressed and the main physical properties to be pointed out. The final part of the research is dedicated to the study of a particular realization of the background independence principle, which concerns the case of analytical Gaussian solutions of the quantum probability density function consistent with the Principle of Entropy Maximization.

2. Quantum Hydrodynamic Equations

The starting point of CQG-theory is provided by the manifestly-covariant 4−scalar quantum-gravity wave equation (CQG-wave equation) determined in [34,36]. Consistent with the unitarity principle for the quantum wave-function, this equation is formally equivalent to the Schrödinger equation of quantum mechanics, and takes the form of the Eulerian and hyperbolic PDE given by

$$i\hbar \frac{d}{ds}\psi(s) = H_R^{(q)}\psi(s). \tag{4}$$

Here, s belongs to the time axis $I \equiv \mathbb{R}$ and plays the role of the evolution-time parameter, so that $\frac{d}{ds} = \frac{d}{ds}\Big|_s + \frac{\partial}{\partial s}$ denotes the covariant $s-$derivative in Eulerian form. More precisely, s represents the proper time along a massive graviton geodetics, which, according to the prediction of CQG-theory [34], is expressed on the background metric tensor $\widehat{g}(r)$ through the differential identity $ds^2 = \widehat{g}_{\mu\nu} dr^\mu dr^\nu$. Furthermore, $\psi(s)$ identifies (in principle for arbitrary s) the $4-$scalar quantum wave function associated with a graviton particle, while $H_R^{(q)}$ is a suitable self-adjoint quantum Hamiltonian operator for which the definition can be found in [34]. In particular, the functional dependence included in $\psi(s)$ is of the type $\psi(s) \equiv \psi(g, \widehat{g}, r, s)$, with $r = r(s)$ being a geodesic trajectory and $g(r) = \{g_{\mu\nu}(r)\}$ being the generalized-coordinate field (continuum Lagrangian coordinates) spanning the quantum configurations space, namely, the $10-$dimensional real vector space $U_g \subseteq \mathbb{R}^{10}$ on which the quantum probability density function $\rho(s) = |\psi(s)|^2$ (quantum PDF) is normalized.

Notice that here, without possible ambiguities, the distinction is made between the tensor $g_{\mu\nu}$, which identifies the continuum Lagrangian coordinate associated with the quantum gravitational field, and the background metric tensor $\widehat{g}_{\mu\nu}$, which takes into account the geometric properties of the background space-time. Thus, by construction, the tensor $g(r)$ belongs to $\{g(r)\}_U$ while the orthogonality conditions $\widehat{g}^{\mu\nu}\widehat{g}_{\beta\nu} = \delta^\mu_\beta$ apply only to the background field. Accordingly, the quantum field $g_{\mu\nu}$ is distinguished from $\widehat{g}_{\mu\nu}$, meaning that it is subject to a quantum dynamical evolution and is characterized by a non-vanishing quantum momentum. Finally, we remark that at this stage only formal algebraic properties of $\widehat{g}_{\mu\nu}$ are imposed, which means that the representation of the background tensor $\widehat{g}_{\mu\nu}$ remains arbitrary and undetermined, and has yet to be properly assigned in agreement with the background independence principle. On the other hand, the role of $\widehat{g}_{\mu\nu}$ is fundamental, as it permits the raising and lowering of tensor indices; therefore, it enters the definitions of $4-$scalar quantities, realizing a formalism consistent with PMC. In addition, we stress that in order to warrant the uniqueness of the solution, the quantum wave Equation (4) needs to be supplemented with initial boundary conditions of the type

$$\psi(g, s_0) = \Psi(g, \widehat{g}, r(s_0)), \tag{5}$$
$$\psi(\partial g, s) = \psi(\partial g, \widehat{g}, r(s), s) \equiv 0, \tag{6}$$

where $\psi(\partial g, s_0)$ denotes the value of $\psi(g, s)$ on the boundary ∂U_g (i.e., the improper domain on U_g) and $\Psi(g, \widehat{g}, r(s_0))$ denotes an appropriate and smooth initial wave function.

In full analogy with quantum mechanics [37], the CQG-wave Equation (4) together with the associated initial boundary conditions (5) and (6) is equivalent to an appropriate set of quantum hydrodynamics equations. These are obtained upon introducing the so-called Madelung representation for the wave function $\psi(s)$, i.e., a complex $4-$scalar field which in exponential form can be written as

$$\psi(g, \widehat{g}, r, s) = \sqrt{\rho} \exp\left\{ \frac{i}{\hbar} \mathcal{S}^{(q)} \right\}. \tag{7}$$

Here, $\left\{\rho, S^{(q)}\right\} \equiv \left\{\rho(g, \widehat{g}, r, s), \mathcal{S}^{(q)}(g, \widehat{g}, r, s)\right\}$ identify two $4-$scalar quantum fluid fields, respectively the quantum PDF and the quantum phase-function. As a result, it follows that the same quantum fluid fields satisfy the set of GR-quantum hydrodynamic equations which are identified with the continuity and quantum Hamilton–Jacobi (H-J) equations, namely,

$$\frac{d\rho}{ds} + \frac{\partial}{\partial g_{\mu\nu}}(\rho V_{\mu\nu}) = 0, \tag{8}$$
$$\frac{d\mathcal{S}^{(q)}}{ds} + H^{(q)} = 0. \tag{9}$$

Here, $V_{\mu\nu} \equiv \frac{1}{\alpha L}\frac{\partial \mathcal{S}^{(q)}}{\partial g^{\mu\nu}}$ is the tensor "velocity" field, while one can show that the dimensional 4−scalar constant αL is related to the graviton mass estimate provided in [34]. In addition, here $H^{(q)}$ identifies the effective quantum Hamiltonian density in the absence of classical sources,

$$H^{(q)} = \frac{1}{2\alpha L}\frac{\partial \mathcal{S}^{(q)}}{\partial g^{\mu\nu}}\frac{\partial \mathcal{S}^{(q)}}{\partial g_{\mu\nu}} + V_{QM} + V_o, \tag{10}$$

so that V_o and V_{QM} respectively denote the vacuum effective potential and quantum Bohm interaction potential [38] given by

$$V_o = \kappa\left(2 - \frac{1}{4}g^{\mu\nu}g_{\mu\nu}\right)g^{\alpha\beta}\widehat{R}_{\alpha\beta}, \tag{11}$$

$$V_{QM} \equiv \frac{\hbar^2}{8\alpha L}\frac{\partial \ln\rho}{\partial g^{\mu\nu}}\frac{\partial \ln\rho}{\partial g_{\mu\nu}} - \frac{\hbar^2}{4\alpha L}\frac{\partial^2\rho}{\rho\partial g_{\mu\nu}\partial g^{\mu\nu}}, \tag{12}$$

with $\widehat{R}_{\alpha\beta}$ being the Ricci tensor evaluated in terms of the background metric tensor $\widehat{g}_{\alpha\beta}$ and κ now identifying the 4−scalar dimensional constant $\kappa = 8\pi G/c^4$.

Based on the quantum Hamilton–Jacobi Equation (9), and again in analogy to quantum mechanics [37], a quantum Hamiltonian structure can be established for the quantum hydrodynamic state as well. This is represented by the set $\left\{x, H^{(q)}\right\}$, where the 4−tensor canonical state $x(s) \equiv (g_{\mu\nu}(s), \Pi^{\mu\nu}(s))$ identifies the Hamiltonian (quantum) hydrodynamic state, with $\Pi^{\mu\nu} = \frac{\partial \mathcal{S}^{(q)}}{\partial g_{\mu\nu}}$ and $H^{(q)}$ respectively denoting the canonical momentum and the effective quantum Hamiltonian density (see Equation (10) above). This implies that Equation (9) can be equally represented in terms of a set of manifestly-covariant quantum Hamilton equations, taking the form of evolution equations with respect to the proper-time invariant parameter s introduced above. Thus, in vacuum these equations are expressed as

$$\frac{d}{ds}g^{\mu\nu} = \frac{\Pi^{\mu\nu}}{\alpha L}, \tag{13}$$

$$\frac{d}{ds}\Pi_{\mu\nu} = -\frac{\partial}{\partial g^{\mu\nu}}(V_o + V_{QM}), \tag{14}$$

with $x(s) \equiv (g_{\mu\nu}(s), \Pi^{\mu\nu}(s))$ being subject to generic initial conditions of the type $x(s_o) = x_o \equiv \left(g^{\mu\nu}_{(o)} \equiv g^{\mu\nu}(s_o), \Pi_{(o)\mu\nu} \equiv \Pi_{\mu\nu}(s_o)\right)$. The existence of a manifestly-covariant Hamiltonian structure governing the quantum hydrodynamic equations for the quantum-gravity state represents a feature that distinguishes CQG-theory from alternative approaches to the problem in the literature. At the same time, it provides a consistency property that establishes the connection with the corresponding Hamiltonian structure underlying the Einstein field equations and that can be recovered by means of an appropriate semi-classical limit (see proof in [29,34]).

3. Statistical Background Independence

In order to proceed, we need to obtain a trajectory-based (i.e., Lagrangian) representation of the quantum hydrodynamic equations. The framework implemented here is provided by the generalized Lagrangian-path (GLP) parametrization (sometimes called GLP-theory) of CQG-theory, which realizes a statistical formulation of the Bohmian representation first proposed in quantum mechanics [37]. The physical principles motivating the GLP theory can be found in [28,37]. For completeness, here we report the main steps that are needed to establish the mathematical formulation of statistical background independence. Thus, in the context of CQG-theory it is reasonable to assume that the Lagrangian trajectories spanning the 10-dimensional quantum configuration space $U_g \subseteq \mathbb{R}^{10}$ cannot be deterministic, and as such should have a stochastic character. The GLP parametrization provides an explicit realization of this type based on the introduction of a suitable set of stochastic trajectories, which are referred to as generalized Lagrangian paths (GLPs). The GLP-formalism is achieved by first introducing the decomposition

$$g(r,s) = \widehat{g}(r) + \delta g(r,s), \tag{15}$$

assumed to apply for the quantum-gravity Lagrangian coordinate $g(r,s) \equiv \{g_{\mu\nu}(r,s)\}$, with $\delta g(r,s) \equiv \{\delta g_{\mu\nu}(r,s)\}$ denoting the corresponding quantum displacement fluctuation. We then introduce the GLP tensor $G(s) \equiv \{G_{\mu\nu}(s)\}$ satisfying the decomposition

$$G(s) = \widehat{g}(r(s)) + \delta G(s), \tag{16}$$

with $\delta G(s) \equiv \{\delta G_{\mu\nu}(r(s),s)\}$ denoting the GLP-displacement tensor field. Each GLP trajectory is then parameterized in terms of the displacement field, to be regarded as a stochastic field tensor of the type

$$\Delta g = g(s) - G(s) = \delta g(s) - \delta G(s). \tag{17}$$

Here, $\Delta g \equiv \{\Delta g_{\mu\nu}\}$ identifies a second-order tensor field referred to as the stochastic displacement field tensor, which by construction is such that identically

$$\frac{D}{Ds}\Delta g_{\mu\nu} = \frac{D}{Ds}\Delta g^{\mu\nu} = 0, \tag{18}$$

where $\frac{D}{Ds}$ is the Lagrangian derivative operator

$$\frac{D}{Ds} \equiv \left.\frac{d}{ds}\right|_{\delta g_{\mu\nu}(s)} + V_{\mu\nu}(g(s),s)\frac{\partial}{\partial \delta g_{\mu\nu}(s)}. \tag{19}$$

Because $\Delta g_{\mu\nu}$ and $\Delta g^{\mu\nu}$ are related via the background metric tensor $\widehat{g}(r(s))$, they remain generally $s-$dependent, as the components $\widehat{g}(r(s))$ are non-constant as well. However, we may require that the mixed co- and counter-variant components $\Delta g_{\nu}^{\mu}(s) = \Delta g_{\nu\beta}(s)\widehat{g}^{\beta\mu}(r(s))$ be constant, i.e., independent of s, letting (at arbitrary proper-times s and s_o)

$$\Delta g_{\nu}^{\mu}(s) = \Delta g_{\nu}^{\mu}(s_o) \equiv \Delta g_{\nu}^{\mu}. \tag{20}$$

As a consequence, provided each GLP trajectory is represented in terms of its mixed (co- and counter-variant) tensor components, it can actually be represented by a configuration-space curve of the type

$$\left\{G(s) \equiv G_{\nu}^{\mu}(s), s \in I\right\} \equiv \left\{\widehat{g}_{\nu}^{\mu}(r(s)) + \delta g_{\nu}^{\mu}(r(s)) - \Delta g_{\nu}^{\mu}, s \in I\right\}, \tag{21}$$

meaning that under variation of the stochastic displacement field tensor Δg_{ν}^{μ} it actually gives rise to a statistical family of trajectories. Nevertheless, in terms of the mixed-component tensor $G_{\nu}^{\mu}(s)$ the corresponding co- and counter-variant components remain defined as well, because

$$G_{\alpha\nu}(s) = G_{\nu}^{\mu}(s)\widehat{g}_{\mu\alpha}(r(s)), \tag{22}$$

$$G^{\alpha\mu}(s) = G_{\nu}^{\mu}(s)\widehat{g}^{\nu\alpha}(r(s)). \tag{23}$$

Therefore, the stochastic character of CQG-theory in this representation emerges as a natural consequence of Equation (21). Its meaning is that for each (deterministic) Lagrangian (or Bohmian) trajectory $\left\{\delta g(s) \equiv \delta g_{\nu}^{\mu}(s), s \in I\right\}$ there are infinite stochastic GLPs, each one obtained for a different value of Δg_{ν}^{μ}.

From the condition (20), it additionally follows that $\frac{D}{Ds}\,\delta g_{\mu\nu}(s) = \frac{D}{Ds}\delta G_{\mu\nu}(s) \equiv V_{\mu\nu}(G(s),\Delta g,s)$, with $V_{\mu\nu}(G(s),\Delta g,s)$ denoting the tensor "velocity" field in the GLP-representation:

$$V_{\mu\nu}(G(s),\Delta g,s) = \frac{1}{\alpha L}\frac{\partial S^{(q)}(G(s),\Delta g,s)}{\partial \delta g^{\mu\nu}(s)}. \tag{24}$$

In consequence of these definitions, for all $s \in I$ the GLP-map $s \to \delta G(s)$ is established such that for each realization of the stochastic displacement Δg expressed in arbitrary tensor components, $G(s,\Delta g) \equiv G(s)$ belongs to a well-defined curve $\{G(s), \forall s \in I\}$ identifying a GLP which spans the quantum configuration space U_g. More precisely, a generic GLP curve $\{G(s), \forall s \in I\}$ is determined by integration of the GLP-initial-value problem

$$\begin{cases} \frac{D}{Ds}\delta G_{\mu\nu}(s) = V_{\mu\nu}(G(s),\Delta g,s), \\ \delta G_{\mu\nu}(s_o) = \delta g^{(o)}_{\mu\nu} - \Delta g_{\mu\nu}. \end{cases} \tag{25}$$

Here, it can be observed that the map $G(s_o) \Leftrightarrow G(s)$ defines a classical dynamical system having a Jacobian determinant

$$\left|\frac{\partial G(s)}{\partial G(s_o)}\right| = \exp\left\{\int_{s_o}^{s} ds' \frac{\partial V_{\mu\nu}(G(s'),\Delta g,s')}{\partial g_{\mu\nu}(s')}\right\}. \tag{26}$$

Therefore, the ensemble of integral curves $\{G(s), \forall s \in I\}$ obtained by varying Δg in U_g identifies an infinite set of GLP which depend on the tensor velocity field $V_{\mu\nu}(G(s),\Delta g,s)$. However, it must be stressed that, actually,

$$V_{\mu\nu}(G(s),\Delta g,s) = V_{\mu\nu}(g(s),s), \tag{27}$$

which means that the same infinite set of GLP remains always associated with the same local value of the tensor velocity field $V_{\mu\nu}(g(s),s)$. This means that the non-uniqueness characteristic of the GLP is only induced by the property of the stochastic tensor Δg.

The next step requires us to specify the GLP-representation of CQG-theory, namely, the type of parametrization of the CQG wave-function $\psi(g,r,s)$ and the corresponding quantum fluid fields, in terms of the GLP-displacement $\delta G(s) = \delta g(s) - \Delta g$. In analogy with the GLP-approach to non-relativistic quantum mechanics [37], a general parametrization of relevant functions that includes the explicit dependence on the stochastic displacement tensor field $\Delta g \equiv \{\Delta g_{\mu\nu}\}$ is allowed. Therefore, this amounts to introducing the composed mapping $\psi(g,r,s) \to \psi(G(s),\Delta g,\widehat{g},r,s)$, in which $\psi(G(s),\Delta g,\widehat{g},r,s)$ denotes the GLP-parametrized quantum wave-function. Similarly, the corresponding GLP-parametrization of the quantum fluid fields becomes

$$\left\{\rho, S^{(q)}\right\}_{(s)} \equiv \left\{\rho(G(s),\Delta g,s), S^{(q)}(G(s),\Delta g,s)\right\}. \tag{28}$$

Nevertheless, the quantum hydrodynamic Equations (8) and (9) remain formally analogous when expressed in the GLP-parametrization, meaning that they continue to determine the map

$$\left\{\rho, S^{(q)}\right\}_{(s_o)} \equiv \left\{\rho_o, S^{(q)}_o\right\} \to \left\{\rho, S^{(q)}\right\}_{(s)}, \tag{29}$$

with $\left\{\rho_o, S^{(q)}_o\right\}$ identifying the initial quantum fluid fields to be assumed, for consistency, of the type

$$\left\{\rho_o, S^{(q)}_o\right\} \equiv \left\{\rho_o(G(s_o),\Delta g), S^{(q)}_o(G(s_o),\Delta g)\right\}. \tag{30}$$

As a result, in the GLP-representation, the quantum hydrodynamic equations are realized by the PDEs

$$\left\{ \begin{array}{c} \frac{D}{Ds}\rho(G(s),\Delta g,s) = -\rho(G(s),\Delta g,s)\frac{\partial V_{\mu\nu}(G(s),\Delta g,s)}{\partial \delta g_{\mu\nu}(s)}, \\ \frac{D}{Ds}S^{(q)}(G(s),\Delta g,s) = K^{(q)}(G(s),\Delta g,s), \end{array} \right. \tag{31}$$

which are respectively denoted as the GLP-parametrized quantum continuity and H-J equations, where

$$K^{(q)}(G(s),\Delta g,s) = V_{\mu\nu}(G(s),\Delta g,s)\frac{\partial S^{(q)}(G(s),\Delta g,s)}{\partial \delta g_{\mu\nu}(s)} - H^{(q)}(G(s),\Delta g,s). \tag{32}$$

Here, $H^{(q)}(G(s),\Delta g,s)$ identifies the effective quantum Hamiltonian density (10) expressed in terms of the GLP-parametrization. The continuity equation in (31) can equivalently be written as

$$\frac{D}{Ds}\ln \rho(G(s),\Delta g,s) = -\frac{\partial V_{\mu\nu}(G(s),\Delta g,s)}{\partial \delta g_{\mu\nu}(s)}, \tag{33}$$

which allows it to be formally integrated. This yields the map $\rho(G(s_o),\Delta g,s_o) \to \rho(G(s),\Delta g,s)$, with $\rho(G(s),\Delta g,s)$ denoting the proper-time evolved quantum PDF, namely,

$$\rho(G(s),\Delta g,s) = \rho(G(s_o),\Delta g,s_o)\exp\left\{ -\int_{s_o}^{s} ds' \frac{\partial V_{\mu\nu}(G(s'),\Delta g,s')}{\partial \delta g_{\mu\nu}(s')} \right\}. \tag{34}$$

Notice that the integration on the rhs is performed along the GLP-trajectory $\{G(s), \forall s \in I\}$, i.e., for a prescribed stochastic displacement 4−tensor Δg, while $\rho_o \equiv \rho(G(s_o),\Delta g,s_o)$ identifies the initial, and in principle still arbitrary, PDF.

The stochastic character of $\Delta g_{\mu\nu}$ demands that it must be endowed with a stochastic PDF as well, which is denoted by f and at this stage remains to be prescribed. Thus, because Δg is an independent stochastic variable, we assume that the same PDF is a stationary and spatially uniform probability distribution, that is, a function independent of r, s as well as $\delta g_L(s)$, but which is allowed to depend in principle on the background metric tensor $\widehat{g}_{\mu\nu}(r)$. This means that f must be represented in terms of a smoothly differentiable and strictly positive function of the form

$$f = f(\Delta g, \widehat{g}). \tag{35}$$

The consequence is that, for arbitrary smooth functions $X(\Delta g,r,s)$, the corresponding notion of the stochastic average over Δg is defined by the weighted integral on the configuration space U_g as

$$\langle X(\Delta g,r,s)\rangle_{stoch} \equiv \int_{U_g} d(\Delta g) X(\Delta g,r,s) f(\Delta g,\widehat{g}). \tag{36}$$

In order to be acceptable as a physical theory, the GLP approach must warrant the ontological equivalence of the GLP-parametrization for the quantum state ψ with the "standard" Eulerian representation of the same quantum wave-function. This in turn demands that the prescription of the stochastic PDF $f(\Delta g,\widehat{g})$ should be possible, leaving the axioms of CQG-theory unchanged. In fact, for CQG-theory and similarly for Quantum Mechanics (see [37]), the independent prescription of $f(\Delta g,\widehat{g})$ may potentially give rise to additional conceptual difficulties related to the notions of quantum measurement and quantum expectation values. In order to prevent such an inconvenience and support the ontological equivalence mentioned here, the PDF $f(\Delta g, \widehat{g})$ is required to coincide with the initial quantum PDF ρ_o, assuming the identity

$$f(\Delta g, \widehat{g}) \equiv \rho(G(s_o), \Delta g, s_o). \tag{37}$$

Having established the conceptual framework for the validity of CQG-theory and its statistical trajectory-based representation in terms of stochastic GLP formalism, we can now address the main target of the investigation, namely, the formulation of statistical background independence and the proof of its validity for CQG-theory. The first consideration concerns the analysis of the quantum hydrodynamic Equation (31) parametrized in GLP variables. Consistent with PMC, these are expressed as $4-$scalar equations for the two $4-$scalar quantum fluid fields $\left\{\rho, S^{(q)}\right\}_{(s)}$. The $4-$scalar character is warranted by the existence of the background space-time, and is effectively realized by index saturation of tensor quantities through the background space-time metric tensor $\widehat{g}$. Therefore, the dependence on $\widehat{g}$ in the quantum hydrodynamic equations is twofold, through possible direct functional dependence and through $4-$scalar products. At this stage, while the background tensor $\widehat{g}$ is included in the formalism, and its algebraic properties are specified, it continues to represents an unknown quantity to be properly determined. Its prescription, however, cannot come from the solution of the quantum hydrodynamic equations, as $\widehat{g}$ is neither a quantum functional dependence nor a dynamical variable of the theory that evolves according to the same equations. Necessarily, the representation of $\widehat{g}$ must be assigned in agreement with the foundational principles of GQC-theory and the GLP approach. We then require that the background tensor $\widehat{g}$ be generated by the quantum gravitational field, and in particular from its stochastic fluctuations. This amounts to introducing the identity

$$\widehat{g}_{\mu\nu} \equiv \langle \Delta g_{\mu\nu} \rangle_{stoch}, \tag{38}$$

where the stochastic average is expressed by Equation (36), while from the vanishing of its covariant derivative require $\widehat{g}_{\mu\nu}$ to be subject to the orthogonality condition $\widehat{g}_{\alpha\nu}\widehat{g}^{\beta\nu} = \delta_\alpha^\beta$. As a result, the validity of the following relationship applies:

$$\widehat{g}_{\mu\nu} \equiv \langle \Delta g_{\mu\nu} \rangle_{stoch} = \int_{U_g} d(\Delta g) \Delta g_{\mu\nu} f(\Delta g, \widehat{g}), \tag{39}$$

which expresses the mathematical content of the concept of statistical background independence holding for CQG-theory. The implications of such a realization are as follows. First, the classical background space-time metric tensor arises consistently from the quantum nature of the gravitational field, and in particular from its intrinsic stochastic behavior. This is evident from the fact that the integral explicitly contains the stochastic tensor $\Delta g_{\mu\nu}$ as well as the PDF $f(\Delta g, \widehat{g})$, which is related to the initial quantum PDF ρ_o by the identity (37). Second, this result appears unique to CQG-theory, as it is a consequence of its Hamiltonian character and the consequent admitted trajectory-based representation of the quantum-wave equation. Third, it establishes a connection between the stochastic properties of quantum trajectories obtained in GLP theory and the stochastic character of the quantum gravitational field with its averaged expression, in turn identified with the classical metric tensor. Fourth, Equation (39) realizes the emergent character of the classical field from its quantum degrees of freedom. Finally, thanks to Equation (39), the quantum hydrodynamic equations become a set of integro-differential equations that consistently solve the quantum fluid fields. In particular, the form of the quantum PDF $\rho(G(s), \Delta g, s)$ must be determined by the continuity equation under the assumption (39) and the suitable prescription of the initial PDF ρ_o.

4. Gaussian Probability Density and Emergent Gravity

In order to illustrate the validity of the theoretical result obtained above, we are now in position to discuss an application of the principle of background independence, considering the particular case in which the quantum PDF admits an analytical representation in terms of Gaussian distribution. This problem arises in connection with the determination of the stochastic PDF for Δg and of the quantum PDF. The goal here is the proof that, consistent with the principle of entropy maximization [39], the initial quantum PDF $f(\Delta g, \widehat{g})$ can be effectively realized by a shifted Gaussian PDF. In turn, this permits us to display explicitly, in terms of the analytical solution, the validity of the emergent gravity picture of quantum gravity associated with the background independence character of the theory.

To start with, we invoke the prescription (35) and require additionally that $f(\Delta g, \widehat{g})$ should fulfill the following stochastic averages:

$$\begin{cases} \langle 1 \rangle_{stoch} \equiv \int\limits_{U_g} d(\Delta g) f(\Delta g, \widehat{g}) = 1, \\ \left\langle \Delta g^{\mu}_{\nu} \right\rangle_{stoch} \equiv \int\limits_{U_g} d(\Delta g) \Delta g^{\mu}_{\nu} f(\Delta g, \widehat{g})) = \pm \widehat{g}^{\mu}_{\nu} \equiv \pm \delta^{\mu}_{\nu}, \\ \sigma^2_{\Delta g} \equiv \left\langle \left(\Delta g - \langle \Delta g \rangle_{stoch} \right)^2 \right\rangle_{stoch} \equiv \\ \int\limits_{U_g} d(\Delta g) (\Delta g - \langle \Delta g \rangle_{stoch})^2 f(\Delta g, \widehat{g}) = r^2_{th}, \end{cases} \tag{40}$$

with

$$(\Delta g - \langle \Delta g \rangle_{stoch})^2 \equiv \left[\Delta g^{\mu}_{\nu} - \left\langle \Delta g^{\mu}_{\nu} \right\rangle_{stoch} \right] \left[\Delta g^{\nu}_{\mu} - \left\langle \Delta g^{\nu}_{\mu} \right\rangle_{stoch} \right] \tag{41}$$

and $\sigma_{\Delta g}$ denoting the standard deviation of Δg to be identified with the dimensionless $4-$scalar parameter $r^2_{th} > 0$, assumed to be independent of both (r, s). Then, we assume that, among the admissible choices for the initial PDF $f(\Delta g, \widehat{g})$, the latter takes the representation

$$f(\Delta g, \widehat{g}) \equiv \rho_o(\Delta g \pm \widehat{g}(r_o)), \tag{42}$$

which additionally fulfills the constraint conditions indicated above in Equation (40).

As to the validity of the identification (42), the constraints (40) then prescribe the form of the initial PDF $\rho_o(\Delta g \pm \widehat{g}(r_o))$. In fact, upon introducing the Boltzmann–Shannon entropy associated with the same initial PDF and provided by the functional

$$S(\rho_o(\Delta g \pm \widehat{g}(r_o))) = -\int_{U_g} d(\Delta g) \rho_o(\Delta g \pm \widehat{g}(r_o)) \ln \rho_o(\Delta g \pm \widehat{g}(r_o)), \tag{43}$$

one can show that the PDF $\rho_o(\Delta g \pm \widehat{g}(r_o))$ which fulfills the principle of entropy maximization and maximizes $S(\rho_o(\Delta g \pm \widehat{g}(r_o)))$ when subject to the same constraints (40) is unique. In detail, we find that in the configuration domain U_g this PDF reads

$$\begin{aligned} \rho_o(\Delta g \pm \widehat{g}(r_o)) &= \frac{1}{\pi^5 r^{10}_{th}} \exp\left\{ -\frac{(\Delta g \pm \widehat{g}(r_o))^2}{r^2_{th}} \right\} \\ &\equiv \rho_G(\Delta g \pm \widehat{g}(r_o)), \end{aligned} \tag{44}$$

with $\rho_G(\Delta g \pm \widehat{g}(r_o))$ denoting a shifted Gaussian PDF in which both r^2_{th} and $(\Delta g \pm \widehat{g}(r_o))^2$ are $4-$scalars, with

$$(\Delta g \pm \widehat{g}(r_o))^2 \equiv (\Delta g \pm \widehat{g}(r_o))^{\nu}_{\mu} (\Delta g \pm \widehat{g}(r_o))^{\mu}_{\nu}. \tag{45}$$

Thus, thanks to Equation (20),

$$(\Delta g \pm \widehat{g}(r_o))^2 = (\Delta g(s) \pm \widehat{g}(r))^2, \tag{46}$$

where $r_o = r(s_o)$ and $r = r(s)$. Hence, the Gaussian PDF (44) realizes the most likely PDF, i.e., the one which, when subject to the constraints (40), maximizes the Boltzmann–Shannon entropy $S(\rho_o(\Delta g \pm \widehat{g}(r_o)))$ in Equation (43).

More generally, denoting by

$$\rho_G(\Delta g \pm \widehat{g}(r)) = \frac{1}{\pi^5 r_{th}^{10}} \exp\left\{ -\frac{(\Delta g \pm \widehat{g}(r))^2}{r_{th}^2} \right\} \tag{47}$$

the Gaussian PDF (44) evaluated for a generic 4−position $r(s) \neq r_o \equiv r(s_o)$, it is then possible to prove that a formal solution $\rho(G_L(s), \Delta g, s)$ of the quantum continuity equation can be realized in terms of the function

$$\rho(G(s), \Delta g, s) = \rho_G(\Delta g \pm \widehat{g}(r)) \exp\left\{ -\int_{s_o}^{s} ds' \frac{\partial V_{\mu\nu}(G(s'), \Delta g, s')}{\partial \delta g_{\mu\nu}(s')} \right\}. \tag{48}$$

The main consequence of this analysis is that an exact analytical solution for the quantum PDF which solves the continuity equation has been obtained. The same solution additionally realizes the emergent gravity relationship of space-time. In fact, from the second of Equation (40) and the shifted Gaussian solution (47), selecting the root $\rho_G(\Delta g - \widehat{g}(r))$ yields

$$\widehat{g}_{\mu\nu} = \int_{U_g} d(\Delta g) \Delta g_{\mu\nu} \rho_G(\Delta g - \widehat{g}(r)), \tag{49}$$

meaning that $\langle \Delta g_{\mu\nu} \rangle_{stoch} \equiv \widehat{g}_{\mu\nu}$ remains satisfied for any arbitrary proper-time s, as we can notice that, under suitable assumptions, the 4−scalar $\frac{\partial V_{\mu\nu}}{\partial \delta g_{\mu\nu}}$ in Equation (48) does not depend explicitly on the displacement tensor Δg.

The explicit Gaussian realization of the PDF provides a different, but still admissible and eventually analogous, point of view to the mathematical problem associated with the solution of the quantum hydrodynamic equations. In fact, in the general case discussed previously, the representation of the background metric tensor is assigned through Equation (39) and this is replaced in Equation (31), which is then solved for the two 4−scalar fluid fields. Instead, in the analytical realization considered here for Gaussian PDF, the mathematical problem is translated into that of determining the 4−scalar phase function $S^{(q)}$ through the second quantum hydrodynamic equation and the still-unknown background tensor $\widehat{g}_{\mu\nu}$. In fact, according to Equation (48), the solution of the continuity equation for the quantum PDF $\rho(G(s), \Delta g, s)$ is analytically known. The background independence principle is automatically satisfied by construction through the shifted Gaussian solution.

In such a setting, the metric tensor $\widehat{g}(r)$ is obtained a posteriori as the solution of the quantum-modified Einstein field equations that follow from the quantum Hamilton Equations (13) and (14). In the case of vacuum (i.e., absence of external sources), they take the general expression

$$\widehat{R}_{\mu\nu} - \frac{1}{2}\widehat{R}\widehat{g}_{\mu\nu}(r,s) = B_{\mu\nu}(r,s), \tag{50}$$

where the source term $B_{\mu\nu}(r,s)$ is due to quantum gravity and carries the contributions generated by both potential and quantum momentum terms, e.g., the non-linear Bohm interaction and $\Pi^{\mu\nu} \neq 0$ in Equation (13), respectively. Examples of implementations of these occurrences can be found in [29,40]. In particular, in the case of potential origin, the source term $B_{\mu\nu}(r,s)$, evaluated at $\Delta g_{\mu\nu} = 0$, takes the form

$$B_{\mu\nu}(r,s) = -\frac{1}{\kappa} \frac{\partial}{\partial g^{\mu\nu}} V_{QM} \Big|_{\Delta g_{\mu\nu}=0}. \tag{51}$$

As shown in [29], $B_{\mu\nu}(r,s)$ is therefore produced by an intrinsic quantum gravity effect, e.g., the vacuum non-linear Bohm interaction, for which the explicit form depends on the precise (i.e., not necessarily Gaussian) realization of the quantum PDF $\rho(G(s), \Delta g, s)$. We stress that the tensor equation (50) holds for arbitrary boundary conditions. Thus, its solution $\widehat{g}(r)$ can be interpreted as its general solution, i.e., identified with an arbitrary particular solution of the same equation, thereby realizing the property of background independence in the context of QG.

5. Conclusions

In the context of quantum gravity (QG), the intuitive notion of background independence refers to the property that the corresponding QG theory is independent of the particular realization of space-time structure on which the same theoretical model holds. This means that the QG theory, which might possibly be a non-unique one, should apply to any background space-time. However, the precise meaning of this notion needs to be specified on mathematical grounds, while the realization of this feature, and consequently the prediction of its possible physical existence, is not assured a priori for any QG theory or any quantum theoretical framework. On the other hand, the same QG theory should satisfy a number of physical requirements. The first refers, indeed, to the prescription of the same background space-time, which should be emergent in character. This means that it should be determined self-consistently by the same QG theory under suitable initial/boundary conditions. The effective realization of this property identifies the so-called emergent gravity phenomenon. A further basic requirement, however, refers to the property of General Covariance. This implies that the QG theory should be frame-independent, that is, it should hold for arbitrary choices of coordinates. Notably, such a property is fulfilled if the same theory is set in a 4−tensor form with respect to the same aforementioned background space-time, i.e., it acquires a so-called manifestly-covariant form.

Among past and more recent approaches to QG in the literature, one model theory exhibiting all these properties at the same time, namely, background independence, emergent character, and frame-independence, is the so-called covariant theory of QG (CQG-theory). In this paper, we have displayed its basic mathematical structure, and in particular, have shown that:

- CQG-theory, in addition to being manifestly covariant by construction, exhibits the characteristics of both emergent gravity and background independence.
- The latter two are a consequence of the explicit stochastic configuration-space representation of the quantum wave equation achieved by means of an appropriate trajectory-based formalism. This is realized by the generalized Lagrangian-path trajectories of CQG-theory, denoted as stochastic GLP-representation.
- The same quantum wave equation, with its intrinsic unitary character and Hamiltonian structure, permits a representation of the quantum state in terms of manifestly-covariant quantum fluid fields that are peculiar for the same CQG-theory.
- Finally, the emergent gravity phenomenon and the prescription of the background space-time are found to be accomplished.

A comparison with earlier works on CQG-theory is instrumental in order to clarify the novelty reported here. Two main results have been achieved. First, the background independence of QG theory has been formulated in general form based on the requisite validity of the principle of emergent gravity relating the quantum and continuum metric classical tensors of the gravitational field. The same concepts of background independence and emergent gravity are exclusively established in terms of a statistical relationship holding for a generic quantum probability density function. In this way, the theory proposed here provides a mathematical proof of the possible realization of a statistical description underlying CQG-theory and its classical limit counterpart, and as such is consistent with the principle of manifest covariance. This picture yields a comprehensive conceptual framework for the theory of stochastic trajectory-based description developed in previous work [28] for the representation of the quantum-gravity hydrodynamic equations in terms of statistical ensemble of generalized Lagrangian-path (GLP) formalism. Second, it has been shown that an explicit realization of emergent gravity can be reached in terms of a Gaussian solution for the quantum PDF. This outcome extends the theoretical results established in [29]. In fact, the target here is not only that of determining or prescribing the analytical form of the solution for the quantum PDF; the true goal is to highlight the statistical connotation of such a Gaussian solution in reference to the validity of the Principle of Entropy Maximization (PEM).

Altogether, these theoretical achievements permit the setting up of a convenient framework for the realization of the novel notion of statistical background independence proposed here for the classical and quantum theories of gravitational fields. In fact, as a characteristic feature of CQG-theory, background independence and emergent gravity acquire a statistical meaning expressed in terms of averages performed over stochastic quantum degrees of freedom. In this way, the proposed model relies uniquely on the axioms of quantum mechanics and its probabilistic interpretation as well as on the foundations of relativistic statistical mechanics. More precisely, the distinguished feature of the theory proposed in this research is the construction of a statistical theory describing the stochastic ensemble of Lagrangian-path trajectories, which is in turn built consistently over the existing CQG-theory. Remarkably, this model leaves unchanged the fundamental axioms and probabilistic interpretation of the same quantum theory. This represents an advance for CQG-theory and its mathematical structure, as well as for the investigation of the connection between quantum and classical gravitational fields and prediction of measurable quantum effects. As illustrated in the paper, an example of the application of such statistical formalism is provided by the determination of the quantum cosmological constant and its role in the corresponding quantum-modified Einstein field equations for the Gaussian solution of the quantum PDF. Accordingly, the two settings of classical and quantum gravity retain their independent character, while the information associated with the microscopic quantum degrees of freedom can be consistently transferred to the continuum macroscopic classical domain through appropriate integral relationships (i.e., statistical averages) among observable quantities.

In conclusion, the conceptual results determined in this paper appear to have promising features for the analytical study of quantum-gravity field dynamics and related stochastic behavior, as well as its semi-classical limit and the connection with the continuum description of space-time emerging in General Relativity.

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