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An Invitation to the Algebra of Canonical Commutation Relations

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Contents

Preface

This volume contains the written notes of a series of lectures delivered at the University of Leuven in the academic year 1988/89. The aim of the lectures was to give an introduction to the rigorous treatment of the canonical commutation relation and to demonstrate that the method of operator algebras in a suitable tool for this purpose. These notes cover quasifree states, the KMS-condition, central limit theorems and the equivalence of Fock states. Fundamentals of operator algebras and functional analysis are required to follow the presentation. Physical motivation and relevance are slightly commented.

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Chapter 1

Representations of canonical commutation relations

In the Hilbert space formulation of quantum mechanics one considers the abstract selfadjoint operators $q_1, q_2, \ldots, q_n$, $p_1, p_2, \ldots, p_n$ acting on a Hilbert space $\mathcal{H}$ and satisfying the Heisenberg commutation relations for n degrees of freedom :

$$\begin{aligned} [q_i, q_j] &= [p_i, p_j] = 0 \\ [q_i, p_j] &= \mathrm{i}\,\delta(i,j)\, I \qquad (i,j = 1,2,\ldots,n) \end{aligned} \tag{1.1}$$

where for any pair of operators A and B on $\mathcal{H}$ the symbol $[A, B]$ stands for the commutator $AB - BA$; I is the identity operator. There is no essential difference between the case $n = 1$ and the general n, therefore we restrict ourself for one degree of freedom. Then q is associated to the position and p to the momentum of a single particle. Since the early days of quantum mechanics it has been a basic problem to find concrete operators satisfying the Heisenberg commutation relations.

If one takes the complex Hilbert space $L^2(\mathbb{R})$, q as the multiplication operator by the variable, i.e.

$$(q_S f)(t) = t\, f(t) \tag{1.2}$$

and p the differentiation, i.e.

$$(p_S f)(t) = -i\, f'(t) \tag{1.3}$$

then these operators form a representation of the Heisenberg commutation relation on $L^2(\mathbb{R})$. (Formally q_S and p_S may be defined on some dense set of functions, for example the C^∞ ones with compact support.) This representation is called the Schrödinger representation.

A problem to solve in the matrix mechanics of Heisenberg is to find selfadjoint matrices q and p satisfying the relation

$$[q,p] = i\,I\,. \tag{1.4}$$

If the Schrödinger representation is unique in a suitable sense, then all results of the matrix theory must be equivalent to those of the Schrödinger theory. If p and q were finite matrices satisfying (1.4) then the trace of the left hand side would be zero in contradiction with the nonzero trace of the right hand side. Hence Heisenberg did not find a solution within the set of finite matrices but found a solution in the form of infinite matrices :

$$q_H = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \\ \dots & & & & \end{bmatrix} \tag{1.5}$$

$$p_H = \frac{-i}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & \dots \\ \dots & & & & \end{bmatrix} \tag{1.6}$$

This representation (q_H, p_H) is called the Heisenberg representation.

The Schrödinger and Heisenberg solutions of the representation problem coincide. One can find an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$ such that the matrix of q_S (p_S) in the given basis is q_H (p_H). For any real number t one deduces from equation (1.4) that

$$\begin{aligned} e^{itp}\,q\,e^{-itp} &= q + i\,t[p,q] + \frac{(it)^2}{2!}[p,[p,q]] + \dots \\ &= q + t\,I \end{aligned} \tag{1.7}$$

For $U(a) = \exp(iap)$ and $V(b) = \exp(ibq)$ $(a, b \in \mathbb{R})$ we have

$$U(a)\,V(b)\,U(-a) \quad = \quad U(a) \sum_{n=0}^{\infty} \frac{(ib)^n}{n!} q^n\,U(-a) =$$

$$\sum_{n=0}^{\infty} \frac{(ib)^n}{n!}(q + a\,I)^n \quad = \quad \exp(ib(q + aI)) = e^{iab}V(b).$$

Therefore,

$$U(a)\,V(b) = e^{iab}V(b)\,U(a) \qquad (a, b \in \mathbb{R}) \tag{1.8}$$

which is called the Weyl form of the canonical commutation relation. (1.8) contains only unitary operators and hence it will be suitable for a C^*-algebraic approach.

It was von Neumann's idea to compress the two families of operators, U and V, in a single one. If one takes

$$S(a, b) = \exp\left(-\frac{1}{2}iab\right)U(a)\,V(b) \qquad (a, b \in \mathbb{R})$$

then U and V may be obviously recovered from S and (1.8) is equivalent to the following relation

$$S(a, b)\,S(c, d) = \exp\left(\frac{1}{2}i(ad - bc)\right)S(a + c, b + d) \tag{1.9}$$

Let $\mathcal{H}$ be a Hilbert space and assume that $S(a, b) \in B(\mathcal{H})$. We say that S is a representation of the canonical commutation relation if the following conditions hold.

(i) $S(a, b)$ is a unitary $(a, b \in \mathbb{R})$.

(ii) $S(a, b)^* = S(-a, -b)$ $(a, b \in \mathbb{R})$.

(iii) $S(0, 0)$ is the identity.

(iv) The relation (1.9) is satisfied for all $a, b, c, d \in \mathbb{R}$.

(v) The mapping $(a, b) \mapsto S(a, b)$ is continuous in the weak operator topology.

Remember that on the group of unitaries the weak and strong operator topologies coincide. So (v) may be formulated also another way.

From the Schrödinger representation one gets a representation of the canonical commutation relation (CCR) in the above sense on $L^2(\mathbb{R})$.

Since U and V are given by exponentiation of selfadjoint operators, they are strongly continuous. So is their product.

A representation of the CCR on a Hilbert space $\mathcal{H}$ is called irreducible if only the trivial closed subspaces of $\mathcal{H}$ are invariant under all $S(a, b)$.

Proposition 1.1 *The Schrödinger representation of the CCR is irreducible.*

p_S and q_S are related by the Fourier transform. If $\mathcal{F}f$ denotes the Fourier transform of $f \in L^2(\mathbb{R})$ defined as

$$\mathcal{F}f(t) = \frac{1}{\sqrt{2\pi}} \int e^{-its} f(s) ds$$

then $\mathcal{F} q_S \mathcal{F}^* = p_S$.

We show that if $f \in L^2(\mathbb{R})$ is a nonzero vector and $\langle g, S(a, b)f \rangle = 0$ for every $(a, b) \in \mathbb{R}^2$ then necessarily $g = 0$. This gives that the smallest invariant closed subspace containing f is $L^2(\mathbb{R})$ itself.

Since the first factor in

$$\langle g, S(a, b)f \rangle = e^{\frac{1}{2}iab} \langle e^{-iap} g, e^{ibq} f \rangle$$

is nonzero, we concentrate on the second one. By direct computation we have

$$2\pi \langle e^{-iap} g, e^{ibq} f \rangle = \int e^{-ita} \int e^{ibu} \int e^{-its} e^{itu} g(s) \overline{f(u)} ds\, du\, dt$$

Assuming that this is 0 we may refer to the uniqueness of the Fourier transform and conclude that

$$e^{-itu} \overline{f(u)} \int e^{-its} g(s) ds = 0$$

for almost all $t, u \in \mathbb{R}$. As $f \neq 0$ we arrive at

$$\int e^{-its} g(s) ds = 0$$

for almost all $t \in \mathbb{R}$. So g must be 0.

The uniqueness theorem of von Neumann asserts that up to a unitary equivalence the Schrödinger representation is the only irreducible (continuous) representation of the CCR.

Theorem 1.2 *If* $\mathbb{R}^2 \ni (a,b) \mapsto S(a,b) \in B(\mathcal{H})$ *and* $\mathbb{R}^2 \ni (a,b) \mapsto S'(a,b) \in B(\mathcal{K})$ *are irreducible (continuous) representations of the CCR on the Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$*, respectively, then there exists a unitary* $U : \mathcal{H} \to \mathcal{K}$ *such that*

$$S(a,b) = U^* \, S'(a,b) \, U \qquad (a, b \in \mathbb{R}).$$

For $h \in L^1(\mathbb{R}^2)$ the formula

$$s(f,g) = \int h(a,b) \langle S(a,b) f, g \rangle da \, db \qquad (f, g \in \mathcal{H})$$

sets a bounded sesquilinear form and there is an operator $S(h) \in B(\mathcal{H})$ such that

$$\langle S(h) \, f, g \rangle = s(f,g) \qquad (f, g \in \mathcal{H})$$

and $\|S(h)\| \le \|h\|_1$. We write simply

$$S(h) = \int h(a,b) \, S(a,b) \, da \, db.$$

It is clear that for a real valued h the operator $S(h)$ is selfadjoint. We show that the linear correspondence $h \mapsto S(h)$ is an injection.

Assume that $S(h) = 0$ and compute

$$\langle S(-a,-b) S(h) S(a,b) g, f \rangle = \int\!\!\int e^{-iav} e^{ibu} h(u,v) \langle S(u,v) g, f \rangle du \, dv.$$

By our hypothesis this vanishes for every $f, g \in \mathcal{H}$ and every $(a,b) \in \mathbb{R}^2$. Hence

$$h(u,v) \langle S(u,v) g, f \rangle = 0 \tag{1.10}$$

for almost all $u, v \in \mathbb{R}$ and for every $f, g \in \mathcal{H}$. (If you are really pedantic you should argue by the separability of $\mathcal{H}$ at this point.) (1.10) readily gives $h = 0$.

Now set a bounded selfadjoint operator A on $\mathcal{H}$ as

$$A = \int\!\!\int \exp\left(-\frac{1}{4} (|a|^2 + |b|^2) \right) \, S(a,b) \, da \, db \tag{1.11}$$

and one checks that

$$A \; S(a,b) A = 2\pi \; e^{-\frac{1}{4}(|a|^2 + |b|^2)} A \, . \tag{1.12}$$

In particular, $P = \frac{1}{2\pi} A$ is a projection and it can not be 0. It is a projection of rank one. Let f and $g \neq 0$ be orthogonal vectors such that $Pf = f$ and $Pg = g$. Due to the relation (1.12) we have

$$\langle S(a,b)g, f\rangle = \langle P\,S(a,b)\,P\,g, f\rangle = \text{constant } \langle g, f\rangle$$

and the irreducibility of the representation $(a,b) \mapsto S(a,b)$ yields that $f = 0$. P is a rank one projection indeed and we may assume that $\|g\| = 1$.

We construct similarly a vector $\eta \in \mathcal{K}$ starting from the irreducible representation $(a,b) \mapsto S'(a,b)$. Set

$$U\,S(a,b)g = S'(a,b)\eta \qquad (a, b \in \mathbb{R}) \tag{1.13}$$

and extend U by linearity to the linear span of $\{S(a,b)g : a, b \in \mathbb{R}\}$. So U will be an inner product preserving densely defined operator with dense range. It follows immediately from (1.13) that U intertwines between the representations S and S'.

Von Neumann's uniqueness theorem is true in any finite dimension and the above proof with small alteration will go through. (The interested reader may consult with the nicely written original paper [**vN**].)

Let $C(\mathbb{R})$ stand for the C^*-algebra of bounded continuous function on $\mathbb{R}$ (endowed with the sup norm) It contains the characters $\chi_\lambda(x) = e^{i\lambda x}$ $(\lambda \in \mathbb{R})$. Set $A(\mathbb{R})$ as the closure of the linear span of the characters. $A(\mathbb{R})$ is called the algebra of almost periodic functions. It possesses a natural inner product

$$\langle f, g\rangle = \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} f(x)\,\overline{g(x)}\,dx$$

and its completion is a Hilbert space. $\{\chi_\lambda : \lambda \in \mathbb{R}\}$ forms an orthonormal basis in $\mathcal{H}$ and in particular, $\mathcal{H}$ is not separable. Define the unitary operators

$$\begin{array}{ll} V_0(a)\chi_\lambda = \chi_{a+\lambda} & (a \in \mathbb{R}) \\ U_0(b)\chi_\lambda = \exp(i\,b\,\lambda)\chi_\lambda & (b \in \mathbb{R}). \end{array}$$

Then $U_0(a)$ and $V_0(b)$ satisfy the Weyl commutation relation

$$U_0(a)V_0(b) = \exp(i\,a\,b)V_0(b)U_0(a) \qquad (a, b \in \mathbb{R}).$$

This representation of the Weyl commutation relation is irreducible and can not be unitarily equivalent to the Schrödinger representation since $\mathcal{H}$ is not separable. It is easy to see that

$$\| (V_0(b) - I)\, \chi_\lambda \| = \begin{cases} \sqrt{2} & \text{if} \quad b \neq \lambda \\ 0 & \text{if} \quad b = \lambda \end{cases}$$

hence the strong continuity of the group $V_0(b)$ fails completely.

Let $(a, b) \mapsto S(a, b)$ be a representation of the CCR. The linear span of $\{S(a, b) \quad a, b \in \mathbb{R}\}$ is a *-algebra. (It is closed under the product operation due to (1.9).) Its norm closure is a C^*-algebra, call it for a while $\mathcal{A}(S)$. It follows from the uniqueness theorem that for two irreducible representations S and S' there is a C^*-algebra isomorphism $\alpha : \mathcal{A}(S') \to \mathcal{A}(S)$ such that $\alpha(S'(a, b)) = S(a, b)$ $(a, b \in \mathbb{R})$. The main subject of the next chapter is to show that the isomorphism exists independently of the irreducibility of the representations (and even finite dimensional "testfunction space" will not be required). The C^*-algebraic point of view prefers the Weyl form of the canonical commutation relation. To show that the question of unicity for the Heisenberg form is more subtle we describe an example of Fuglede ([**Fu**]).

As a common dense domain $\mathcal{D}$ for the operators q and p he takes the span of the functions

$$\left\{ x^n \exp(-ax^2 + bx) \ : \ n \in \mathbb{Z}^+ , \quad a > 0 , \quad b \in \mathbb{C} \right\}$$

All these functions are entire analytic and for $f \in \mathcal{D}$ and $z \in \mathbb{C}$ $f(z)$ makes sense. Denote by T and M the following operators

$$\begin{aligned} (Tf)(x) &= f(x + i\sqrt{2\pi}) \\ (Mf)(x) &= \exp(\sqrt{2\pi}\, x) f(x) \end{aligned}$$

and define

$$\begin{aligned} q_F &= q_S + T \\ p_F &= p_S + M \end{aligned}$$

(Here (q_S, p_S) is the Schrödinger representation.) Fuglede proved that

(i) $\langle p_F f, q_F g \rangle - \langle q_F f, p_F g \rangle = -i \langle f, g \rangle \qquad (f, g \in \mathcal{D})$

(ii) The closure of p_F and q_F is selfadjoint and the pair $(\overline{p_F}, \overline{q_F})$ is not unitarily equivalent to any direct sum of Schrödinger representations.

Under additional hypothesis the uniqueness of the Schrödinger representation may be proven in the frame of selfadjoint operators. This subject is out of the scope of the present lecture and we refer to the monograph [**Pu**].

In the selection of the material in this chapter, we benefited from the unpublished notes [**Ve**]. æ

Chapter 2

The C^*-algebra of the canonical commutation relation

Let H be a real linear space. A bilinear form σ is called symplectic form if $\sigma(x, y) = -\sigma(y, x)$ for every $x, y \in H$. σ is nondegenerate if $\sigma(x, y) = 0$ for every $y \in H$ implies that $x = 0$. The pair (H, σ) will be refered to as a symplectic space. If it is not stated explicitly otherwise, all symplectic spaces will be assumed to be nondegenerate.

In Chapter 1 we met already a symplectic form. The bilinear form

$$((a, b), (c, d)) \mapsto \frac{1}{2}(ad - bc)$$

appearing in (1.9) is a nondegenerate symplectic form on $\mathbb{R}^2$. More generally, if $\mathcal{H}$ is a complex Hilbert space then $\sigma(f, g) = Im\langle f, g\rangle$ is a nondegenerate symplectic form on the real linear space $\mathcal{H}$. In finite dimension this is the typical way to define a symplectic space. (Hence the dimension of a nondegenerate symplectic space is even or infinite.)

Let (H, σ) be a symplectic space. The C^*-algebra of the canonical commutation relation over (H, σ), written as $CCR(H, \sigma)$, is by definition a C^*-algebra generated by elements $\{W(f) : f \in H\}$ such that

(i) $W(-f) = W(f)^* \qquad (f \in H)$

(ii) $W(f)W(g) = \exp(i\,\sigma(f, g))W(f + g) \qquad (f, g \in H)$

Condition (ii) tells us that $W(f)W(0) = W(0)W(f) = W(f)$. Hence $W(0)$ is the unit of the algebra and it follows that $W(f)$ is a unitary for every $f \in H$. Comparing with the previous definition for a (continuous) representation of the CCR you see that the continuity assumption (v) is missing here. Weak and strong continuity do not make sense here (and norm continuity would be really too much as it turns out later.)

Theorem 2.1 *For any nondegenerate symplectic space (H, σ) the C^*-algebra $CCR(H, \sigma)$ exists and unique up to isomorphism.*

To establish the existence will be easier than proof of the uniqueness. Consider H as a discrete abelian group (with the vectorspace addition).

$$l^2(H) = \left\{ F : H \to \mathbb{C} : \sum_{x \in H} |F(x)|^2 < +\infty \right\}$$

is a Hilbert space. (Any element of $l^2(H)$ is a function with countable support.) Setting

$$(R(x)F)(y) = \exp(i\,\sigma(y,x))F(x+y) \qquad (x, y \in H) \tag{2.1}$$

we get a unitary $R(x)$ on $l^2(H)$ and one may check that

$$R(x_1)R(x_2) = \exp(i\,\sigma(x_1, x_2)R(x_1 + x_2).$$

The norm closure of the set

$$\left\{ \sum_{i=1}^{n} \lambda_i\, R(x_i) \quad : \quad \lambda_i \in \mathbb{C},\ \ 1 \le i \le n\,,\ \ n \in \mathbb{N},\ \ x_i \in H \right\}$$

in $B(l^2(H))$ is a C^*-algebra fulfulling the requirements (i) and (ii). Let us denote this C^*-algebra by $\mathcal{A}$.

Assume that $\mathcal{B} \subset B(\mathcal{H})$ is another C^*-algebra generated by elements $W(x)$ $(x \in H)$ satisfying (i) and (ii). We have to show an isomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ such that $\alpha(R(x)) = W(x)$ $(x \in H)$. α will be constructed in several steps.

We shall need the Hilbert space

$$l^2(H, \mathcal{H}) = \left\{ A : H \to \mathcal{H} : \sum_{x \in H} \|A(x)\|^2 < +\infty \right\}.$$

Set $x \otimes f$ for $x \in H$ and $f \in \mathcal{H}$ as

$$(x \otimes f)(y) = \begin{cases} f & x = y \\ 0 & x \neq y\,. \end{cases}$$

(Note that $l^2(H,\mathcal{H})$ is isomorphic to $l^2(H) \otimes \mathcal{H}$.) The application

$$y \mapsto \pi(y) \qquad \pi(y)(x \otimes f) = (x - y) \otimes W(y)f$$

is a representation of the CCR on the Hilbert space $l^2(H,\mathcal{H})$. π is equivalent to R. If a unitary $U : l^2(H,\mathcal{H}) \to l^2(H,\mathcal{H})$ is defined as

$$U(x \otimes f) = x \otimes W(x)f$$

then

$$U\,\pi(y) = (R(y) \otimes id)U \qquad (y \in H).$$

To prove our claim it is sufficient to find an isomorphism between $\mathcal{B}$ and the C^*-algebra generated by $\{\pi(y) : y \in H\}$. We show that for any finite linear combination

$$\left\|\sum \lambda_i\, W(y_i)\right\| = \left\|\sum \lambda_i\, \pi(y_i)\right\| \tag{2.2}$$

holds.

Let $\hat{H}$ stand for the dual group of the discrete group H. $\hat{H}$ consists of characters of H and endowed by the topology of pointwise convergence forms a compact topological group. We consider the normalized Haar measure on $\hat{H}$. The spaces $l^2(H)$ and $L^2(\hat{H})$ are isomorphic by the Fourier transformation, which establishes the unitary equivalence between the above π and $\hat{\pi}$ defined below.

$$\hat{\pi}(y)\hat{A}(\chi) = \chi(y)W(y)\hat{A}(\chi) \qquad (y \in H\,,\ \ \chi \in \hat{H}\,,\ \ \hat{A} \in L^2(\hat{H},\mathcal{H}))$$

Hence

$$\left\|\sum \lambda_i\, \pi(y_i)\right\| = \left\|\sum \lambda_i\, \hat{\pi}(y_i)\right\|. \tag{2.3}$$

A closer look at the definition of $\hat{\pi}$ gives that $\hat{\pi}(y)$ is essentially a multiplication operator (by $\chi(y)W(y)$) and its norm is the sup norm. That is,

$$\left\|\sum \lambda_i\, \hat{\pi}(y_i)\right\| = \sup\left\{\left\|\sum \lambda_i\, \chi(y_i)W(y_i)\right\| \,:\, \chi \in \hat{H}\right\}. \tag{2.4}$$

Since the right hand side is the sup of a continuous function over $\hat{H}$, this sup may be taken over any dense set.

Let us set

$$G = \{\exp(2i\,\sigma(x,\cdot)\,:\,x \in H\}.$$

Clearly, $G \subset \hat{H}$ is a subgroup. The following result is at our disposal (see (23.26) of [**H-R**]).

If $K \subset \hat{H}$ is a proper closed subgroup then there exists $0 \neq h \in H$ such that $k(h) = 1$ for every $k \in K$.

Assume that $\exp(2i\,\sigma(x,y)) = 1$ for every $x \in H$. Then for every $t \in \mathbb{R}$ there exists an integer $l \in \mathbb{Z}$ such that $t\,\sigma(x,y) = l\,\pi$. This is possible if $\sigma(x,y) = 0$ (for every $x \in H$) and y must be 0. According to the above cited result of harmonic analysis the closure of G must be the whole $\hat{H}$.

Now we are in a position to complete the proof. For

$$\chi(\cdot) = \exp(2i\,\sigma(x,\cdot)) \in G$$

we have

$$\begin{aligned} \Big\|\sum \lambda_i\,\chi(y_i)W(y_i)\Big\| &= \Big\|W(x)\sum \lambda_i\,W(y_i)W(-x)\Big\| = \\ &= \Big\|\sum \lambda_i\,W(y_i)\Big\| \end{aligned}$$

and this is the supremum in (2.4). Through (2.4) we arrive at (2.2).

The previous theorem is due to Slawny [**Sl**]. We learnt from the proof that $CCR(H,\sigma)$ has a representation on $l^2(H)$ given by (2.1). The subalgebra

$$\left\{\sum_{x\in H} \lambda(x)R(x)\,:\,\lambda\,:\,H \to \mathbb{C} \text{ has finite support}\right\}$$

is dense in $CCR(H,\sigma)$ and there exists a state τ on $CCR(H,\sigma)$ such that

$$\tau\left(\sum \lambda(x)R(x)\right) = \lambda(0)\,. \tag{2.5}$$

It is simple to verify that $\tau(ab) = \tau(ba)$. Therefore, τ is called the tracial state of $CCR(H,\sigma)$. We can use τ to prove the following.

Proposition 2.2 *If $f, g \in H$ are different then*

$$\|W(f) - W(g)\| \geq \sqrt{2}.$$

For $h_1 \neq h_2$, we have $\tau(W(h_1)W(-h_2)) = 0$.
Hence $\|W(f) - W(g)\|^2 \geq \tau((W(f) - W(g)^*(W(f) - W(g))) = 2$.

It follows from the Proposition that the unitary group $t \mapsto W(tf)$ is never normcontinuous and the C^*-algebra $CCR(H, \sigma)$ can not be separable.

Slawny's theorem has also a few important consequences. Clearly for $(H_1, \sigma_1) \subset (H_2, \sigma_2)$ the inclusion $CCR(H_1, \sigma_1) \subset CCR(H_2, \sigma_2)$ must hold. (If H_1 is a proper subspace of H_2 then $CCR(H_1, \sigma_2)$ is a proper subalgebra of $CCR(H_2, \sigma_2)$.) If $T : H \to H$ is an invertible linear mapping such that

$$\sigma(f, g) = \sigma(Tf, Tg) \tag{2.6}$$

then it may be lifted into a *-automorphism of $CCR(H, \sigma)$. Namely, there exists an automorphism γ_T of $CCR(H, \sigma)$ such that

$$\gamma_T(W(f)) = W(Tf) \tag{2.7}$$

A simple example is the parity automorphism

$$\pi(W(f)) = W(-f) \qquad (f \in H)\,. \tag{2.8}$$

Let (H, σ) be a symplectic space. A real linear mapping $J : H \to H$ is called a complex structure if

(i) $J^2 = -id$

(ii) $\sigma(Jf, f) \leq 0 \qquad (f \in H)$

(iii) $\sigma(f, g) = \sigma(Jf, Jg) \qquad (f, g \in H)\,.$

If a complex structure J is given then H may be considered as a complex vectorspace setting

$$(t + is)f = tf + sJf \qquad (s, t \in \mathbb{R}, \quad f \in H) \tag{2.9}$$

The definition

$$\langle f, g \rangle = \sigma(f, Jg) + i\,\sigma(f, g) \tag{2.10}$$

supplies us (a complex) inner product. So to have a symplectic space (over the reals) with a complex structure is equivalent to being given a complex inner product space.

Let J be a complex structure over (H, σ). The gauge automorphism

$$\gamma_\alpha(W(f)) = W(\cos\alpha\, f + J\,\sin\alpha\, f) \qquad (\alpha \in [0, 2\pi]\,,\quad f \in H) \tag{2.11}$$

is another example for lifting of a mapping into an automorphism.

We shall restrict ourselves mainly to C^*-algebras associated to a nondegenerate symplectic space but degeneracy of the symplectic form appears in certain cases. Now this possibility will be discussed following the paper [**M-S-T-V**].

Let σ be (a possible degenerate) symplectic form on H. We write $\Delta(H, \sigma)$ for the free vectorspace generated by the symbols $\{W(h) : h \in H\}$. So $\Delta(H, \sigma)$ consists of formal finite linear combinations like

$$\sum \lambda_i\, W(h_i)\,.$$

We may endow $\Delta(H, \sigma)$ by a *-algebra structure by setting

$$W(h)^* = W(-h) \qquad (h \in H) \tag{2.12}$$

and

$$W(h)W(g) = \exp(i\,\sigma(x, y))W(h + y) \qquad (h, y \in H) \tag{2.13}$$

On the *-algebra $\Delta(H, \sigma)$ we shall consider the so-called minimal regular norm (cf. [**Na**], Ch. IV §18.3). We take all *-representations π of $\Delta(H, \sigma)$ by bounded Hilbert space operators and define

$$\|a\| = \sup\{\|\pi(a)\| \,:\, \pi \text{ is a representation}\} \qquad (a \in \Delta(H, \sigma)) \tag{2.14}$$

Another possibility is to take all positive normalized functionals (that is, states) φ on $\Delta(H, \sigma)$ and to introduce the norm

$$\|a\| = \sup\{\varphi(a^* a)^{1/2} \,:\, \varphi \text{ is a state}\} \qquad (a \in \Delta(H, \sigma)) \tag{2.15}$$

One can see that (2.14) and (2.15) determine the same norm, called the minimal regular norm. The completion of $\Delta(H,\sigma)$ with respect to $\|\cdot\|$ will be a C^*-algebra and it is $CCR(H,\sigma)$ by definition. It follows from Slawny's theorem that for nondegenerate σ the previous and the latter definitions coincide.

Now we study the extreme case when $\sigma \equiv 0$. Then $\Delta(H,\sigma)$ is commutative and a state φ of it corresponds to a positive-definite function F on the discrete abelian group H. We have

$$\varphi\left(\sum \overline{\lambda_i}\, W(h_i)^* \sum \lambda_j\, W(h_j)\right) \geq 0$$

for every $\lambda_i \in \mathbb{C}$ and $h_i \in H$ if and only if the function

$$F \,:\, h \mapsto \varphi(W(h)) \qquad (h \in H)$$

is positive-definite. Due to Bochner's theorem ([**H-R**], 33.1) there is a probability measure μ on the compact dual group $\hat{H}$ such that

$$F(h) = \int \chi(h)\, d\mu(\chi) \qquad (h \in H)\,.$$

Hence

$$\sup\left\{\varphi(a^*a)^{1/2} \,:\, \varphi \text{ is a state}\right\} = \sup\left\{\chi(a^*a)^{1/2} \,:\, \chi \in \hat{H}\right\}$$

where for $a = \sum \lambda_i\, W(h_i) \in \Delta(H,\sigma)$ $\quad \chi(a)$ (or $a(\chi)$) is defined as

$$\sum \lambda_i\, \chi(h_i)\,.$$

In this way every element a of $\Delta(H,\sigma)$ may be viewed to be a continuous function on $\hat{H}$ and

$$\|a\| = \sup\left\{|a(\chi)| \,:\, \chi \in \hat{H}\right\} \qquad (a \in \Delta(H,\sigma))\,.$$

$\Delta(H,\sigma)$ evidently separates the points of $\hat{H}$ and the Stone-Weierstrass theorem tells us that $CCR(H,\sigma)$ is isomorphic to the C^*-algebra of all continuous functions on the compact space $\hat{H}$.

The case of a vanishing symplectic form does not occur frequently, however, it may happen that $H = H_0 \oplus H_1$ and

$$\sigma\left(h_0 \oplus h_1\ ,\ h_0' \oplus h_1'\right) = \sigma_1\left(h_1 \oplus h_1'\right)$$

with a nondegenerate symplectic form σ_1 on H_1. Then the *-algebra $\Delta(H,\sigma)$ is the algebraic tensor product of $\Delta(H_0,0)$ and $\Delta(H_1,\sigma_1)$ and $CCR(H,\sigma)$ will be

$$CCR(H_0,0)\otimes CCR(H_1,\sigma_1)\,. \tag{2.16}$$

(Note that since $CCR(H_0,0)$ is commutative, the C^*-norm on the tensor product is unique.)

Now we review briefly the general case. For a degenerate symplectic form σ we set

$$H_0=\{x\in H:\quad \sigma(x,y)=0\quad \text{for every}\quad y\in H\}$$

for the kernel of σ. $\Delta(H_0,0)$ is the center of the *-algebra $\Delta(H,\sigma)$ and there exists a natural projection E given by

$$E\left(\sum_{x\in H}\lambda(x)W(x)\right)=\sum_{x\in H_0}\lambda(x)W(x) \tag{2.17}$$

and mapping $\Delta(H,\sigma)$ onto $\Delta(H_0,0)$. Having introduced the minimal regular norm we observe that $CCR(H_0,0)$ is the center of $CCR(H,\sigma)$ and E is a conditional expectation. The maximal *-ideals of $CCR(H,\sigma)$ are in one-to-one correspondence with those of $CCR(H_0,0)$. In particular, $CCR(H,\sigma)$ is simple if and only if $H_0=\{0\}$, that is, σ is nondegenerate. Concerning the details we refer to [**M-S-T-V**].

For a nondegenerate symplectic form Slawny's theorem provides readily that $CCR(H,\sigma)$ is simple. æ

Chapter 3

States and fields

Let (H, σ) be a symplectic space and $\pi : CCR(H, \sigma) \to B(\mathcal{H})$ a representation. If

$$t \mapsto \langle \pi(W(tf))\xi, \eta \rangle \qquad (\xi, \eta \in \mathcal{H})$$

is continuous then there is a selfadjoint $B_\pi(f)$ due to the Stone theorem such that

$$\pi(W(tf)) = \exp(it\, B_\pi(f)) \qquad (t \in \mathbb{R}) \tag{3.1}$$

$B_\pi(f)$ is called field operator. Since the unitary group $\pi(W(tf))$ is not normcontinuous (cfr. Proposition 2.2), $B_\pi(f)$ is unbounded. From physical point of view the field operators are more relevant in may cases than elements of $CCR(H, \sigma)$. The representation π is called regular if the field $B(f)$ exists for every f in the test function space.

Let us consider the representation R given by (2.1). One computes that

$$\langle R(tx)\delta_y, \delta_y \rangle = \begin{cases} 0 & t \neq 0 \\ 1 & t = 0 \end{cases} \qquad (x, y \in H)\,.$$

So it is not regular. We saw another non-regular representation in Chapter 1 on the Hilbert space of almost periodic functions. On the other hand, the Schrödinger representation is regular.

A linear functional φ of a C^*-algebra $\mathcal{A}$ is called state if $\varphi(I) = 1$ and $\varphi(x^*x) \geq 0$ for every $x \in \mathcal{A}$. (The latter condition may be replaced by $\|\varphi\| = 1$.) Every state gives rise to a representation through the GNS construction. A state φ of $CCR(H, \sigma)$ is called regular if the corresponding GNS-representation is regular.

Let X be an arbitrary (nonempty) set. A function $F : X \times X \to \mathbb{C}$ is called a positive definite kernel if and only if

$$\sum_{j,k=1}^{n} c_j \, \overline{c}_k \, \varphi(x_j, x_k) \geq 0$$

for all $n \in \mathbb{N}$, $\{x_1, x_2, \ldots, x_n\} \subset X$ and $\{c_1, c_2, \ldots, c_n\} \subset \mathbb{C}$.

Proposition 3.1 *Let (H, σ) be a symplectic space and $G : H \to \mathbb{C}$ a function. There exists a state φ on $CCR(H, \sigma)$ such that*

$$\varphi(W(f)) = G(f) \qquad (f \in H)$$

if and only if $G(0) = 1$ and the kernel

$$(f, g) \mapsto G(f - g) \, \exp(-i \, \sigma(f, g))$$

is positive definite.

For $x = \sum c_j \, W(f_j)$ we have

$$x \, x^* = \sum c_j \, \overline{c}_k \, W(f_j - f_k) e^{-i\sigma(f_j, f_k)} .$$

Since for a state φ $\varphi(x^* x) \geq 0$ we see that the positivity condition is necessary.

On the other hand, the positivity condition allows us to define a positive functional on the linear hull of the Weyl operators and a continuous extension to $CCR(H, \sigma)$ supplies a state.

Lemma 3.2 *Let (H, σ) be a symplectic space. (It might be degenerate.) If $\alpha(\cdot, \cdot)$ is a positive symmetric bilinear form on H then the following conditions are equivalent.*

(i) *The kernel $(f, g) \mapsto \alpha(f, g) - i \, \sigma(f, g)$ is positive definite.*

(ii) $\alpha(z, z) \, \alpha(x, x) \geq \sigma(z, x)^2$ *for every $x, z \in H$.*

Both condition (i) and (ii) hold on H if and only if they hold on all finite dimensional subspaces. Hence we may assume that H is of finite dimension.

If $\alpha(x,x)=0$ then both condition (i) and (ii) imply that $\sigma(x,y)=0$ for every $y \in H$. Due to possible factorization we may assume that α is strictly positive and it will be viewed as an inner product.

There is an operator Q such that

$$\sigma(x,y)=\alpha(Qx,y) \qquad (x,y \in H)\,.$$

and $Q^* = -Q$ follows from $\sigma(x,y) = -\sigma(y,x)$. According to linear algebra in a certain basis the matrix of Q has a diagonal form $\mathrm{Diag}(A_1, A_2, \dots, A_k)$ where A_i is a 1×1 0-matrix or

$$A_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}.$$

(The first possibility occurs only if σ is degenerate.) It is easy to see that condition (i) is equivalent to $|a_i| \le 1$ and so is condition (ii).

Lemma 3.3 *If $A=(a_{ij})$ and $B=(b_{ij})$ are $n \times n$ positive definite matrices then the matrix $C=(a_{ij}b_{ij})_{i,j}$ is positive definite, too.*

The entrywise product may be defined for any two matrices of the same type (and it is called frequently Hadamard product). One can check that

$$\langle C\xi, \eta\rangle = \sum_k \langle B(\xi * t_k), (\eta * t_k)\rangle \tag{3.2}$$

where $*$ stands for the Hadamard product and t_k is the k^{th} column of the matrix $A^{1/2}$. From (3.2) the Lemma follows.

Theorem 3.4 *Let (H,σ) be a symplectic space and $\alpha : H \times H \to \mathbb{R}$ a symmetric positive bilinear form such that*

$$\sigma(f,g)^2 \le \alpha(f,f)\alpha(g,g) \qquad (f,g \in H)\,. \tag{3.3}$$

Then there exists a state φ on $CCR(H,\sigma)$ such that

$$\varphi(W(f)) = \exp\left(-\frac{1}{2}\alpha(f,f)\right) \qquad (f \in H)\,. \tag{3.4}$$

We are going to apply Proposition 3.1. Due to the positivity condition

$$
\begin{aligned}
&\sum c_j \overline{c_k} \exp\left(-\frac{1}{2}\alpha(f_j - f_k, f_j - f_k) - i\sigma(f_j, f_k)\right) \\
&\quad = \sum \left(c_j \exp\left(-\frac{1}{2}\alpha(f_j, f_j)\right)\right)\left(\overline{c}_k \exp\left(-\frac{1}{2}\alpha(f_k, f_k)\right)\right) \\
&\quad \times \exp(\alpha(f_j, f_k) - i\sigma(f_j, f_k)) \\
&\quad = \sum b_j \overline{b}_k \exp(\alpha(f_j, f_k) - i\sigma(f_j, f_k))
\end{aligned}
$$

should be shown to be nonnegative. According to Lemma 3.2

$$(\alpha(f_j, f_k) - i\sigma(f_j, f_k))_{j;k}$$

is positive definite and entrywise exponentiation preserves this property as it follows from Lemma 3.3.

A state φ on $CCR(H, \sigma)$ determined in the form (3.4) is called quasifree. A quasifree state is regular.

Proposition 3.5 *Let φ be a state on $CCR(H, \sigma)$. If*

$$\lim_{t \to 0} \varphi(W(tf)) = 1 \qquad (f \in H)$$

then φ is regular.

We set $G(f) = \varphi(W(f))$ $(f \in H)$. According to Proposition 3.1 the matrix

$$
\begin{pmatrix}
1 & G(-f_1) & G(-f_2) \\
G(f_1) & 1 & G(f_1 - f_2)\exp(-i\sigma(f_1, f_2)) \\
G(f_2) & G(f_2 - f_1)\exp(i\sigma(f_1, f_2)) & 1
\end{pmatrix}
$$

is positive definite. From this we obtain

$$|G(f_2) - G(f_1)| \leq 4|1 - G(f_2 - f_1)\exp(-i\sigma(f_2, f_1))|\,. \tag{3.5}$$

Combining (3.5) with the hypothesis we arrive at the continuity of the function

$$t \mapsto G(tf + g) \qquad (t \in \mathbb{R})$$

for every $f, g \in H$. Let $(\pi_\varphi, \mathcal{H}_\varphi, \Phi)$ stand for the GNS-triple. We verify by computation the continuity of the function

$$t \mapsto \langle \pi_\varphi(tf)\pi_\varphi(g_1)\Phi, \pi_\varphi(g_2)\Phi\rangle \qquad (t \in \mathbb{R})$$

and the regularity of φ is proven.

A state φ on $CCR(H, \sigma)$ is said to be analytic if the numerical function

$$t \mapsto \varphi(W(tf)) \qquad (t \in \mathbb{R})$$

is analytic. Quasifree states are obviously analytic.

Assume that π is a regular representation of $CCR(H, \sigma)$. The field operator $B(g)$ is obtained by differentiation of the function

$$t \mapsto \pi(W(tg))\eta = \exp(itB(g))\eta \qquad (t \in \mathbb{R})\,. \tag{3.6}$$

More precisely, if (3.6) is weakly differentiable at $t = 0$ and the derivative is $\xi \in \mathcal{H}_\varphi$ then η is in the domain of $B(g)$ and

$$iB(g)\eta = \xi\,.$$

Proposition 3.6 *Let φ be an analytic state on $CCR(H, \sigma)$ with GNS triple $(\pi_\varphi, \mathcal{H}_\varphi, \Phi)$. Then $\pi_\varphi(W(g))\Phi$ is in the domain of*

$$B(f_n)B(f_{n-1})\dots B(f_1)$$

for every $g, f_1, f_2, \dots, f_n \in H$ and $n \in \mathbb{N}$.

We apply induction and suppose that

$$\eta = B(f_{n-1})\dots B(f_1)\pi_\varphi(W(g))\Phi$$

makes sense. For the sake of simpler notation we omit π_φ in the proof.

It suffices to show that

$$\lim_{t\to 0} t^{-1}\langle (W(f_n) - I)\eta, \xi\rangle = F(\xi) \tag{3.7}$$

exists if ξ is in a dense subset $\mathcal{D}_0$ of $\mathcal{H}_\varphi$ and $|F(\xi)| \le C\|\xi\|$. This ensures that

$$t \mapsto W(tf_n)\eta$$

is differentiable in the weak sense. Since for $\xi = W(h)\Phi$ the limit in (3.7) equals to

$$(-i)^n \frac{\partial}{\partial t} \frac{\partial^{n-1}}{\partial t_{n-1} \partial t_{n-2} \dots \partial t_1} \varphi(W(-h)W(tf_n) \dots W(t_1 f_1) W(g))$$

at the point $t = t_{n-1} = t_{n-2} = \dots = t_1 = 0$, the function F is defined on the linear hull $\mathcal{D}_W$ of the vectors

$$\{W(h)\Phi : h \in H\}.$$

F is linear on $\mathcal{D}_W$ and by differentiation one can see that

$$C = \lim_{t \to 0} \frac{1}{t} \|(W(tf_n) - I)\eta\|$$

exists and it fulfils $|F(\xi)| \le C\|\xi\|$ for $\xi \in \mathcal{D}_W$.

Although $B(f) \notin CCR(H, \sigma)$ it will be rather convenient to write $\varphi(B(f_n)B(f_{n-1}) \dots B(f_1))$ instead of $\langle B(f_n)B(f_{n-1}) \dots B(f_1)\Phi, \Phi\rangle$. We shall keep also the notation $\mathcal{D}_W$ from the above proof. Remember that $\mathcal{D}_W$ as well as the superset Hilbert space $\mathcal{H}_\varphi$ depend on the state φ even if it is excluded from the notation.

Proposition 3.7 *Let φ be an analytic state on $CCR(H, \sigma)$. Then for $f, g \in H$ and $t \in \mathbb{R}$ the following relations hold on $\mathcal{D}_W$.*

(i) $B(tf) = tB(f)$

(ii) $B(f + g) = B(f) + B(g)$

(iii) $[B(f), W(g)] = 2\sigma(f, g)W(g)$

(iv) $[B(f), B(g)] = -2i\sigma(f, g)$

(i)-(iv) are deduced by derivation from (the Weyl form of) the CCR. One gets similarly that

$$\varphi(B(f)B(g)) = \alpha(f, g) - i\sigma(f, g) \tag{3.8}$$

if φ is a quasifree state given by (3.4).

Since

$$\varphi(\sum \overline{c}_k B(f_k) \sum c_i B(f_i)) \ge 0$$

the kernel $(f,g) \mapsto \alpha(f,g) - i\sigma(f,g)$ is positive definite. Therefore, (3.3) in Theorem 3.4 is not only sufficient but also necessary for the existence of a quasifree state.

Proposition 3.8 *Let φ be a quasifree state on $CCR(H,\sigma)$ given by (3.4) and $f_1, f_2, \dots, f_n \in H$. Then*

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = 0$$

if n is odd. For an even n we have

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = \sum \prod_{m=1}^{n/2} (\alpha(f_{k_m}, f_{j_m}) - i\sigma(f_{k_m}, f_{j_m}))$$

where the summation is over all partitions $\{H_1, H_2, \dots, H_{n/2}\}$ of $\{1, 2, \dots, n\}$ such that $H_m = \{j_m, k_m\}$ with $j_m < k_m$ ($m = 1, 2, \dots, n/2$).

We benefit from the formula

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = (-i)^n \frac{\partial^n}{\partial_n \dots \partial_1} \varphi(W(t_n f_n)\dots W(t_1 f_1)) .$$

Since we have

$$W(t_n f_n)W(t_{n-1}f_{n-1})\dots W(t_1 f_1) = W(f_n f_n + t_{n-1}f_{n-1} + \dots + t_1 f_1) \times \exp i \left(\sum_{l>k} t_l t_k \sigma(f_l, f_k) \right)$$

(3.4) yields

$$\varphi(W(t_n f_n)\dots W(f_1 f_1)) = \exp\left(-\frac{1}{2} \sum_{m=1}^{n} t_m^2 \alpha(f_m, f_m) \right) \exp\left(\sum_{l>k} t_l t_k (-\alpha(f_l, f_k) + i\sigma(f_l, f_k)) \right) . \tag{3.9}$$

What we need is the coefficient of $t_1 t_2 \dots t_n$ in the power series expansion. Such term comes only from the second factor of (3.9) and only in the case of an even n. In the claim it is described exactly the possibilities for getting $t_1t_2\dots t_n$ as a product of factors t_lt_k $(l > k)$.

By means of (3.8) we have also

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = \sum\prod \varphi\left(B(f_{k_m})B(f_{j_m})\right) \qquad (3.10)$$

were summation and product are similar to those in Proposition 3.8. The expression (3.10) makes clear that the value of a quasifree state φ on any polynomial of field operators is completely determined by the two-point-functions $\varphi(B(f)B(g))$ $(f, g \in H)$.

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and let σ be a nondegenerate symplectic form on H such that

$$|\sigma(f,g)|^2 \le (f,f)(g,g) \qquad (f,g \in H) \qquad (3.11)$$

holds. There exists a contraction D on H such that

$$\sigma(f,g) = (Df,g) \qquad (f,g \in H)\,. \qquad (3.12)$$

Evidently $D^* = -D$. If $Df = 0$ then due to the nondegeneracy of σ $f = 0$ and hence D is invertible. Consider the polar decomposition

$$D = J|D|\,. \qquad (3.13)$$

The property $D^* = -D$ gives that

$$J|D|J^* = -J^2|D|$$

and the uniqueness of the polar decomposition (applied for the positive operator $J|D|J^*$) guarantees that

$$-J^2 = I \quad \text{and} \quad J|D| = |D|J\,. \qquad (3.14)$$

The state space of a C^*-algebra is a compact convex subset of the dual space if it is endowed with the weak topology. A state is called pure if it is an extremal point of the state space.

Proposition 3.9 *Let φ be a (quasifree) state on $CCR(H,\sigma)$ so that*

$$\varphi(W(h)) = \exp\left(-\frac{1}{2}(h,h)\right)\,. \qquad (h \in H)$$

If φ is pure then $|D|$ (given by (3.13)) is the identity.

We shall argue by contradiction. Assume that there exists $f \in H$ such that

$$(|D|f, f) = 1 \quad \text{and} \quad (|D|^{-1} - I)^{1/2} f \neq 0\,. \tag{3.15}$$

Set $L = |D|^{1/2}(|D|^{-1} - I)^{1/2}$ and note that L is a contraction. We define a symmetric bilinear form as

$$S(g_1, g_2) = (g_1, g_2) - (Lg_1, |D|^{1/2} f) \cdot (Lg_2, |D|^{1/2} f) \frac{(Lf, |D|^{1/2} f)^2}{(Lf, Lf)}$$

and show that

$$S(g, g) \geq (|D|g, g) \qquad (g \in H) \tag{3.16}$$

or equivalently

$$((I - |D|)g, g)(Lf, Lf) \geq (Lg, |D|^{1/2} f)^2 (Lf, |D|^{1/2} f)^2\,. \tag{3.17}$$

(3.17) is a consequence of the Schwarz inequality :

$$\begin{aligned} (Lf, |D|^{1/2} f)^2 &\leq (Lf, Lf)(|D|f, f) = (Lf, Lf) \\ (Lg, |D|^{1/2} f)^2 &\leq (L^2 g, g)(|D|f, f) = ((I - |D|)g, g)\,. \end{aligned}$$

By means of (3.12) and (3.13) we infer from (3.16) that

$$|\sigma(h, g)|^2 = (J|D|h, g)^2 \leq (|D|h, h)(|D|g, g) \leq S(h, h) S(g, g)\,.$$

Now Theorem 3.4 tells us that there is a (quasifree) state ω on $CCR(H, \sigma)$ such that

$$\omega(W(h)) = \exp\left(-\frac{1}{2} S(h, h)\right)\,. \qquad (h \in H)$$

We can see from Proposition 3.1 that if ω is any state of $CCR(H, \sigma)$ and F is a linear functional on H then there exists a state ω_F such that

$$\omega_F(W(h)) = \omega(W(h)) \exp(iF(h))\,.$$

Writing a for $(Lf, |D|^{1/2} f)(Lf, Lf)^{-1/2}$ we set a state ω_λ for $\lambda \in \mathbb{R}$ as follows.

$$\omega_\lambda(W(h)) = \exp\left(-\frac{1}{2} S(h, h) + i\lambda(Lh, |D|^{1/2} f) a\right) \quad (h \in H)\,.$$

With the shorthand notation b for $(Lh, |D|^{1/2}f)$ we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-\lambda^2/2}}{(2\pi)^{1/2}} \omega_\lambda(W(h))d\lambda &= e^{-\frac{1}{2}(h,h)}(2\pi)^{-1/2} \\ &\quad \times \int_{-\infty}^{\infty} \exp(-\lambda^2/2 + b^2a^2/2 + iba)d\lambda \\ &= e^{-\frac{1}{2}(h,h)}(2\pi)^{-1/2} \\ &\quad \times \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(\lambda + iab)^2\right) d\lambda \\ &= e^{-\frac{1}{2}(h,h)}\end{aligned}$$

and this means that

$$\varphi = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-\lambda^2/2)\omega_\lambda d\lambda \,.$$

(3.18) is in contradiction with the starting assumption on φ. Hence the proof has been completed.

The unitary operator J in the polar decomposition (3.13) of D is a complex structure. (2.9) and (2.10) tell us that by means of J the space$H$ may be regarded as a complex inner product space. Then

$$-\sigma(f,g) = Im\langle f,g\rangle \quad \text{and} \quad (|Q|f,g) = Re\langle f,g\rangle \,.$$

Assume that a complex structure J is given on a Hilbert space $\mathcal{H}$ the creation $B^+(f)$ and annihilation operator $B^-(f)$ are defined for $f \in H$ by the formula

$$B^\pm(f) = \frac{1}{2}(B(f) \mp iB(Jf)) \,. \tag{3.19}$$

Since

$$B^+(if) = B^+(Jf) = \frac{1}{2}(B(Jf) - iB(-f)) = iB^+(f)$$

the creation operator $B^+(f)$ is a linear function of $f \in H$. From Proposition 3.7 we can deduce the commutation relations of the annihilation and creation operators.

$$[B^+(f), B^+(g)] = [B^-(f), B^-(g)] = 0 \tag{3.20}$$

$$[B^-(f), B^+(g)] = \sigma(Jf, g) - i\sigma(f, g)\,. \tag{3.21}$$

Proposition 3.6 tells us that if the GNS representation of an analytic state is considered then $B^{\pm}(f)$ is defined on the dense domain $\mathcal{D}_W$ and on this the relations (3.20) and (3.21) hold.

Since both sides of (3.10) are multilinear in $B(f_i)$, the formula remains valid if some or all the $B(f_i)$'s are replaced by $B^+(f_i)$ or $B^-(f_i)$.

Let φ be as the quasifree state in Proposition 3.9. It is called a Fock state if $|D|$ given in (3.13) is the identity. Fock states will be treated in the next chapter in detail.

Among several sources the book [**Ho2**] and [**M-V**] were used to write this chapter.

Chapter 4

Fock states

Let φ be a quasifree state on $CCR(H, \sigma)$ given by the formula

$$\varphi(W(h)) = \exp\left(-\frac{1}{2}\alpha(f,f)\right) \qquad (f \in H)\,.$$

φ is said to be a Fock state if H is a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ such that

$$\sigma(f,g) = Im\langle f,g\rangle\,,\ \alpha(f,g) = Re\langle f,g\rangle \qquad (f,g \in H)\,.$$

(It is assumed that the real vectorspace structure of the symplectic space is the same as that of the Hilbert space.) This definition is obviously equivalent to the previous one given at the end of Chapter 3.

The GNS representation corresponding to a Fock state is called Fock representation and the cyclic vector is frequently referred as vacuum vector. Field, creation and annihilation operators will be considered in this chapter in the Fock representation. From (3.21) we have the basic commutation relation :

$$[B^-(f), B^+(g)] = \langle g, f\rangle \qquad (f,g \in H)\,. \tag{4.1}$$

Lemma 4.1 $B^-(f)\Phi = 0$.

We use (3.12-14) to compute that

$$\|B^-(f)\Phi\|^2 = \frac{1}{2} Re\langle (I - |D|)f, f\rangle\,.$$

Now we see that $B^-_\varphi(f)\Phi = 0$ if and only if $|D| = I$, that is, φ is a Fock state. This is slightly more than the lemma.

Lemma 4.2 *For $k \in \mathbb{Z}$ we have*

$$B^-(f)B^+(f)^k\Phi = k\|f\|^2 B^+(f)^{k-1}\Phi \qquad (f \in H)\,.$$

We apply induction. The case $k = 0$ is contained in the previous lemma. Due to the commutation relation (4.1) we have

$$\begin{aligned} B^-(f)B^+(f)^{k+1}\Phi &= (B^+(f)B^-(f) + \langle f, f\rangle)B^+(f)^k\Phi \\ &= (k-1)\|f\|^2 B^+(f)^k\Phi + \|f\|^2 B^+(f)^k\Phi\,. \end{aligned}$$

One obtains by induction again the following.

Proposition 4.3 *If $n, k \in \mathbb{N}$ and $f \in H$ then*

$$B^-(f)^n B^+(f)^k\Phi = \begin{cases} 0 & \text{if} \quad n > k \\ \frac{k!}{(k-n)!}\|f\|^{2n}B^+(f)^{k-n}\Phi & \text{if} \quad n \le k\,. \end{cases}$$

(3.8) gives that

$$\varphi(B(f)B(g)) = \langle f, g\rangle\,.$$

Therefore

$$\varphi(B^\pm(f)B^\pm(g)) = 0$$

if $f \perp g$. If the sequence $f_1, f_2, \dots, f_n$ in H has the property that any two vectors are orthogonal or identical then in the expansion (3.10) of

$$\varphi(B^\pm(f_n)B^\pm(f_{n-1})\dots B^\pm(f_1))$$

we may have a nonzero term if always identical vectors are paired together. We benefit from this observation in the next proposition.

Proposition 4.4 *Assume that $g_1, g_2, \dots, g_k$ are pairwise orthogonal vectors in H. Then*

$$B^+(g_1)^{m_1}B^+(g_2)^{m_2}\dots B^+(g_k)^{m_k}\Phi$$

and

$$B^+(g_1)^{n_1}B^+(g_2)^{n_2}\dots B^+(g_k)^{n_k}\Phi$$

are orthogonal whenever $m_j \ne n_j$ for at least one $1 \le j \le k$.

Suppose that $m_1 \neq n_1$ and $n_1 > m_1$. The inner product of the above vectors is given by

$$\varphi\left(B^-(g_k)^{n_k}\dots B^-(g_1)^{n_1}B^+(g_1)^{m_1}\dots B^+(g_k)^{m_k}\right)$$

and equals to

$$\varphi\left(B^-(g_1)^{n_1}B^+(g_2)^{m_1}\right)\varphi\left(B^-(g_k)^{n_k}\dots B^-(g_2)^{n_2}B^+(g_2)^{m_2}\dots B^+(g_k)^{m_k}\right).$$

Here the first factor vanishes due to $n_1 > m_1$.

We are able to conclude also the formula

$$\|B^+(g_1)^{n_1}B^+(g_2)^{n_2}\dots B^+(g_k)^{n_k}\Phi\|^2 = n_1!n_2!\dots n_k! \tag{4.2}$$

provided that $\|g_1\| = \|g_2\| = \dots = \|g_k\| = 1$.

Lemma 4.5 *For $g_1, g_2, \dots, g_n, f \in H$ with $\|f\| = 1$ we have*

$$\|B(f)B(g_1)B(g_2)\dots B(g_n)\Phi\| \le 2\sqrt{n+1}\|B(g_1)\dots B(g_n)\Phi\|.$$

We consider the linear subspace spanned by the vectors $f, g_1, g_2, \dots, g_n$ and take an orthonormal basis $f_1 = f, f_2, \dots, f_k$. We may express $B(g_i)$ by creation and annihilation operators corresponding to the basis vectors and get

$$\eta = B(g_1)\dots B(g_n)\Phi = \sum \lambda(n_1,\dots,n_k)B^+(f_1)^{n_1}\dots B^+(f_k)^{n_k}\Phi$$

(Here the summation is over the k-triples $(n_1,\dots,n_k)$ such that $n_i \in \mathbb{Z}_+$ and $\sum n_i \le n$.) Since

$$\|B(f)\eta\| \le \|B^+(f_1)\eta\| + \|B^-(f_1)\eta\|$$

it suffices to show that

$$\|B^\pm(f_1)\eta\|^2 \le (n+1)\|\eta\|^2.$$

Now we estimate as follows.

$$\begin{aligned}
&\|B^+(f_1)\eta\|^2\\
&= \|\sum \lambda(n_1,\dots,n_k)B^+(f_1)^{n_1+1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2\\
&= \sum \|\lambda(n_1,\dots,n_k)B^+(f_1)^{n_1+1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2\\
&= \sum (n_1+1)\|\lambda(n_1,\dots,n_k)B^+(f_1)^{n_1}\dots B^+(f_k)^{n_k}\Phi\|^2\\
&\le (n+1)\|B(g_1)\dots B(g_n)\Phi\|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|B^-(f_1)\eta\|^2 \\
&= \|\sum_{n_1\ge 1}\lambda(n_1,\dots,n_k)n_1 B^+(f_1)^{n_1-1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2 \\
&= \sum n_1^2|\lambda(n_1,\dots,n_k)|^2\|B^+(f_1)^{n_1-1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2 \\
&= \sum n_1\|\lambda(n_1,\dots,n_k)B^+(f_1)^{n_1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2 \\
&\le n\sum \|\lambda(n_1,\dots,n_k)B^+(f_1)^{n_1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2 = n\|\eta\|^2 .
\end{aligned}$$

Lemma 4.2 and the explicit norm expression (4.2) have been used.

Let the Fock representation act on a Hilbert space $\mathcal{H}$ containing the vacuum vector Φ. The linear span of the vectors $B(g_1)B(g_2)\dots B(g_n)\Phi$ $(g_1, g_2, \dots, g_n \in H,\ n\in\mathbb{N})$ and $B^+(g_1)B^+(g_2)\dots B^+(g_n)\Phi$ $(g_1, g_2, \dots, g_n \in H,\ n\in\mathbb{N})$ coincide. It will be denoted by $\mathcal{D}_B$. So far it is not clear whether $\mathcal{D}_B$ is complete. This is what we are going to show.

Let A be a linear operator on a Hilbert space $\mathcal{K}$. A vector $\xi\in\mathcal{K}$ is called entire analytic (for A) if ξ is in the domain of A^n for every $n\in\mathbb{N}$ and

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}\|A^k\xi\| < +\infty$$

for every $t>0$. If ξ is an entire analytic vector then $\exp(zA)\xi$ makes sense for every $z\in\mathbb{C}$ and it is an entire analytic function of z.

Theorem 4.6 *$\mathcal{D}_B$ consists of entire analytic vectors for $B(f)$ $(f\in H)$.*

Let $\xi = B(f_1)B(f_2)\dots B(f_n)\Phi\in\mathcal{D}_B$. By a repeated application of Lemma 4.5 we have

$$\|B(f)^k\xi\| \le 2^k\sqrt{\frac{(n+k)!}{n!}}\|\xi\|$$

and it is straightforward to check that the power series

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}\|B(f)^k\xi\|$$

converges for every t. Since the entire analytic vectors form a linear subspace, the proof is complete.

Due to Theorem 4.6 every vector $W(f)\Phi = \exp(iB(f))\Phi$ can be approximated (through the power series expansion of the exponential function) by elements of $\mathcal{D}_B$. This yields, immediately that $\mathcal{D}_B$ is dense in $\mathcal{H}$. According to Nelson's theorem on analytic vectors (see [**R-S**], X.39), $\mathcal{D}_B$ is a core for $B(f)$ $(f \in H)$, in other words, $B(f)$ is the closure of its restriction to $\mathcal{D}_B$. It follows also that $\mathcal{D}_B$ is core for $B^{\pm}(f)$ and $B^-(f)^* = B^+(f)$.

Assume now for a while that H is of one dimension (over $\mathbb{C}$). Fix a unit (basis) vector η in H and set

$$f_n = \frac{1}{\sqrt{n!}} B^+(\eta)^n \Phi \qquad (n \in \mathbb{Z}_+). \tag{4.3}$$

Then $\{f_0, f_1, \dots\}$ is an orthonormal basis in $\mathcal{H}$. If we write a^+ for $B^+(\eta)$ and a for $B^-(\eta)$ then

$$a^+ f_n = \sqrt{n+1} f_{n+1} \qquad a f_n = \begin{cases} \sqrt{n} f_{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases}$$

and

$$[a, a^+] = 1.$$

With the choice

$$q = \frac{1}{\sqrt{2}}(a + a^+) \qquad p = \frac{i}{\sqrt{2}}(a^+ - a)$$

the Heisenberg commutation relation (1.4) is satisfied and in the basis (f_n) the matrices of q and p are given by (1.5) and (1.6), respectively.

The vector f_n is called n-particle vector in the physics literature. Transforming f_n into f_{n+1} the operator a^+ increases the number of particles. This is the origin of the term "creation operator". The operator a annihilates in the similar sense.

Let $\{\eta_i : i \in I\}$ be an orthonormal basis in the complex Hilbert space H. We set

$$\left|\eta_{i_1}^{n_1}; \eta_{i_2}^{n_2}; \dots; \eta_{i_k}^{n_k}\right\rangle = \frac{1}{\sqrt{n_1! \dots n_k!}} B^+(f_{i_1})^{n_1} \dots B^+(f_{i_k})^{n_k} \Phi. \tag{4.4}$$

So for every choice of different indices $i_1, i_2, \ldots, i_k$ in I and $n_1, n_2, \ldots, n_k \in \mathbb{N}$ we get to a unit vector in $\mathcal{H}$. The vectors

$$\left|\eta_{i_1}^{n_1}; \ldots, \eta_{i_k}^{n_k}\right\rangle \quad \text{and} \quad \left|\eta_{j_1}^{m_1}; \ldots, \eta_{j_l}^{m_l}\right\rangle$$

are different if $((n_1, i_1), \ldots, (n_k, i_k))$ is not a permutation of $((m_1, j_1), \ldots, (m_l, j_l))$ and in this case they are orthogonal. All such vectors form a canonical orthonormal basis in $\mathcal{H}$.

Theorem 4.7 *The Fock representation is irreducible.*

We have to show that for any $0 \neq \eta \in \mathcal{H}$ the closed linear subspace $\mathcal{H}_1$ generated by $\{W(f)\eta : f \in H\}$ is $\mathcal{H}$ itself. Let $\mathcal{M}$ be the von Neumann algebra generated by the unitaries $\{W(f) : f \in H\}$ in $B(\mathcal{H})$. Clearly, $\mathcal{M}\eta \subset \mathcal{H}_1$.

We consider a canonical basis in $\mathcal{H}$ consisting of vectors (4.4). $\eta \in \mathcal{H}$ has an expansion as (countable) linear combinations of basis vectors. Assume that a vector

$$|f_1^{n_1}; f_2^{n_2}; \ldots; f_k^{n_k}\rangle \tag{4.5}$$

has a nonzero coefficient.

The operator

$$B^+(f_1)B^-(f_1)\ldots B^+(f_k)B^-(f_k) \tag{4.6}$$

is selfadjoint and (4.5) is its eigenvector with eigenvalue $n_1+n_2+\ldots+n_k$. Since (4.6) is affiliated with $\mathcal{M}$, its spectral projections are in $\mathcal{M}$. In this way we conclude that the vector (4.5) lies in $\mathcal{H}_1$.

It is easy to see that

$$B^{\pm}(f)\mathcal{H}_1 \subset \mathcal{H}_1$$

for every $f \in H$. By application of the annihilation operators $B^-(f_i)$ $(1 \leq i \leq k)$ we obtain that the cyclic (vacuum) vector Φ is in $\mathcal{H}_1$. Therefore, $\mathcal{H}_1 = \mathcal{H}$ must hold.

Corollaries 4.8 *Let the quasifree state φ defined on $CCR(H,\sigma)$ be given by a complete inner product $\alpha(\cdot,\cdot)$ as*

$$\varphi(W(h)) = \exp\left(-\frac{1}{2}\alpha(h,h)\right).$$

Then φ is pure if and only if it is a Fock state.

This corollary makes Proposition 3.9 more complete. Remember that a state on a C^*-algebra is pure if and only if the corresponding GNS representation is irreducible (see [**B-R**], Thm. 2.3.19). Theorem 4.7 tells us that Fock states are pure and Proposition 3.9 yields that the other states are not so.

Now we are going to see that every quasifree state is a restriction of a Fock state of a bigger CCR-algebra. Let $(H, (\cdot,\cdot))$ be a real Hilbert space and

$$\sigma(f,g) = (Df,g) \qquad (f,g \in H)$$

a nondegenerate symplectic form on H such that $\|D\| \le 1$. Then there exists a quasifree state φ on $CCR(H,\sigma)$ such that

$$\varphi(W(f)) = \exp\left(-\frac{1}{2}\|f\|^2\right) \tag{4.7}$$

Let $D = J|D|$ be the polar decomposition of D.

Theorem 4.9 *Let $H_2 = H \oplus H$ be the direct sum Hilbert space and set a contraction D_2 of H_2 by the matrix*

$$D_2 = \begin{pmatrix} D & J\sqrt{I+D^2} \\ J\sqrt{I+D^2} & -D \end{pmatrix}. \tag{4.8}$$

Then the bilinear form

$$\sigma_2(W(f_2)) = \exp\left(-\frac{1}{2}\|f_2\|^2\right) \qquad (f_2, g_2 \in H_2)$$

is a symplectic form and the quasifree state

$$\varphi_2(W(f_2)) = \exp\left(-\frac{1}{2}\|f_2\|^2\right) \qquad (f_2 \in H_2) \tag{4.9}$$

on $CCR(H_2, \sigma_2)$ is a Fock state.

The proof is rather straightforward. We recall the relations

$$JD = DJ\,,\ D^* = -D\,,\ J = J^*\,,\ J^2 = -id$$

These give that

$$D_2^* = -D_2 \quad \text{and} \quad D_2^2 = -id\,,$$

in other words, D_2 is a skewadjoint unitary. Hence σ_2 is an antisymmetric form and (4.9) defines a quasifree state. By the definition at the end of Chapter 3, φ_2 is a Fock state.

Since

$$\sigma_2(f \oplus 0, f' \oplus 0) = (D_2(f \oplus 0), f' \oplus 0) = (Df, f') = \sigma(f, f')$$

the mapping

$$W(f) \mapsto W(f \oplus 0) \qquad (f \in H)$$

gives rise to an embedding of $CCR(H, \sigma)$ into $CCR(H_2, \sigma_2)$. Fock states are pure and that is the reason why the procedure described in Theorem 4.9 is called purification. Due to the direct sum $H_2 = H \oplus H$, doubling is another used term.

Purification is a standard way to reduce assertions on arbitrary quasifree states to those on Fock states. For example, we have

Corollaries 4.10 *For an arbitrary quasifree state φ the linear manifold $\mathcal{D}_B^\varphi$ is dense in the GNS Hilbert space $\mathcal{H}_\varphi$ and consists of entire analytic vectors for every field operator $B_\varphi(f)$ $(f \in H)$.*

Returning to the Fock representation we introduce some vectors of special importance by means of the Weyl operators. For $f \in H$ set

$$e(f) = \exp\left(\frac{1}{2}\|f\|^2\right) W(f)\Phi\,, \tag{4.10}$$

which is called exponential vector. One may compute that

$$\langle e(f), e(g)\rangle = \exp\langle g, f\rangle \qquad (f, g \in H) \tag{4.11}$$

Proposition 4.11 *$\{e(f) : f \in H\}$ is a linearly independent complete subset of $\mathcal{H}$.*

We use the fact that the family $\{e^{tx} : x \in \mathbb{R}\}$ of exponential functions is linearly independent.

Let $f_1, f_2, \dots, f_n \in H$ be a sequence of different vectors and assume that $\sum \lambda_i e(f_i) = 0$. We choose a vector $g \in H$ such that the numbers

$$\mu_i = \langle f_i, g\rangle \qquad (1 \le i \le n)$$

are distinct. For any $t \in \mathbb{R}$ we have

$$0 = \langle e(tg), \sum \lambda_i e(f_i)\rangle = \sum \lambda_i \exp(t\langle f_i, g\rangle)$$

and we may conclude that $\lambda_i = 0$ for every $1 \le i \le n$.

Due to the cyclicity of the vacuum vector Φ the set $\{e(f) : f \in H\}$ is complete. A little bit more is true. The norm expression

$$\|e(f) - e(g)\|^2 = \exp(\|f\|^2) + \exp\|g\|^2 - 2Re\ \exp\langle f, g\rangle \tag{4.12}$$

tells us that the mapping $f \mapsto e(f)$ is norm continuous. Therefore $\{e(f) : f \in S\}$ is complete whenever S is a dense subset of H.

Consider again the example of the one dimensional testfunction space $H = \mathbb{C}$. We compute the coordinates of the exponential vectors in the basis $\{f_n : n \in \mathbb{Z}\}$ given in (4.3).

$$\begin{aligned}
\langle W(z)\Phi, f_n\rangle &= \langle \exp(iB^+(z) + iB^-(z))\Phi, f_n\rangle \\
&= \exp\left(-\frac{1}{2}[iB^+(z), iB^-(z)]\right)\langle \exp(iB^+(z))\Phi, f_n\rangle \\
&= \exp\left(-\frac{1}{2}|z|^2\right)\sum \frac{(iz)^m}{m!}\langle \sqrt{m!} f_m, f_n\rangle \\
&= \exp\left(-\frac{1}{2}|z|^2\right)\frac{(iz)^n}{\sqrt{n!}}
\end{aligned}$$

Hence for any $z \in \mathbb{C}$ the associated exponential vector $e(z)$ is the sequence

$$\left(1, iz, \frac{(iz)^2}{\sqrt{z!}}, \dots, \frac{(iz)^n}{\sqrt{n!}}, \dots\right) \tag{4.13}$$

Chapter 5

Fluctuations and central limit

In Chapter 2 the C^*-algebra of the canonical commutation relation was introduced as a C^*-algebra containing a representation of the Weyl commutation relation. A probabilisticly natural way leading up to the CCR-algebra and its quasifree states goes through a central limit theorem. Let $\mathcal{A}$ be an arbitrary C^*-algebra with a fixed faithful state φ. Setting

$$\langle a, b\rangle_\varphi = \varphi(b^*a) \qquad (a, b \in \mathcal{A})$$

(i.e., the inner product of the GNS-representation) we possess a complex inner product space $(\mathcal{A}, \langle , \rangle_\varphi)$.
Let

$$\sigma(a, b) = Im\langle a, b\rangle_\varphi \quad \text{and} \quad \alpha(a, b) = Re\langle a, b\rangle_\varphi \quad (a, b \in \mathcal{A})\,.$$

Then on the C^*-algebra $CCR(\mathcal{A}, \sigma)$ there exists a quasifree state ρ such that

$$\rho(W(a)) = \exp\left(-\frac{1}{2}\alpha(a, a)\right) \qquad (a \in \mathcal{A}) \tag{5.1}$$

We know that ρ is a Fock state and we denote by $B(a)$ $(a \in \mathcal{A})$ the field operators in the corresponding representation.

Consider the infinite tensorproduct C^*-algebra

$$\mathcal{B} = \otimes_{i=1}^{\infty}\mathcal{A}_i$$

where $\mathcal{A}_i$'s are copies of $\mathcal{A}$. Each $x \in \mathcal{A}$ will be indentified with

$$x \otimes I \otimes I \otimes \ldots$$

and so $\mathcal{A}$ becomes a subalgebra of $\mathcal{B}$. The right shift endomorphism γ of $\mathcal{B}$ is determined by the property

$$\gamma : x_1 \otimes x_2 \otimes \ldots \otimes x_n \otimes I \otimes I \otimes \ldots \mapsto I \otimes x_1 \otimes \ldots \otimes x_n \otimes I \otimes \ldots$$

On the language of algebraic probability $\mathcal{B}$ with the state $\psi = \varphi \otimes \varphi \otimes \ldots$ forms a probability space and $\mathcal{A} \subset \mathcal{B}$ corresponds to a randon variable. Speaking this language

$$\mathcal{A}, \gamma(\mathcal{A}), \gamma^2(\mathcal{A}), \ldots$$

is a sequence of identically distributed independent randon variables, that is, a Bernoulli process (cf. [**Kü**]). For $x \in \mathcal{A}$

$$F_k(x) = \frac{1}{\sqrt{k}} \sum_{i=0}^{k-1} (\gamma^i(x) - \psi(x)) \tag{5.2}$$

is called the k^{th} fluctuation of x.

Theorem 5.1 *Let* $a_1, a_2, \ldots, a_k \in \mathcal{A}^{sa}$. *Then*

$$\lim_{n\to\infty} \psi(F_n(a_1)F_n(a_2)\ldots F_n(a_k)) = \rho(B(a_1 - \varphi(a_1))\ldots B(a_k - \varphi(a_k)) .$$

The proof consists of an enjoyable combinatorical argument. First of all, we may assume that $\varphi(a_i) = 0$ for every $1 \le i \le k$. The idea is to group the terms in the product

$$s_n(a_1)s_n(a_2)\ldots s_n(a_k) \tag{5.3}$$

in a certain way where $s_n(a_i)$ stands for $\sqrt{n}F_n(a_i)$. The general term in the expansion of (5.3) is of the form

$$\gamma^{n_1}(a_{m_1})\gamma^{n_2}(a_{m_2})\ldots\gamma^{n_k}(a_{m_k}) \tag{5.4}$$

Since $\gamma^i(a)$ commutes with $\gamma^j(b)$ if $i \ne j$ and $a, b \in \mathcal{A}$, we may reorder the monomial (5.4) as

$$\gamma^0(a_{i^0(1)}a_{i^0(2)}\ldots a_{i^0(m(0))})\ldots\gamma^{n-1}(a_{i^{n-1}(1)}\ldots a_{i^{n-1}(m(n-1))}) \tag{5.5}$$

where

$$\{i^0(1),\dots,i^0(m(0))\},\dots,\{i^{n-1}(1),\dots,i^{n-1}(m(n-1))\} \tag{5.6}$$

form a partition of the set $\{1,2,\dots,k\}$. Since n is going to be very big, many of the sets in (5.6) will be empty. We denote by l the number of the non-empty ones. The possible values of l are $1,2,\dots,k$. At the tensor (5.4) ψ takes the value

$$\varphi\left(a_{j^1(1)}\dots a_{j^1(p(1))}\right)\dots\varphi\left(a_{j^l(1)}\dots a_{j^l(p(l))}\right) \tag{5.7}$$

where the empty sets of the pariton (6) of $\{1,2,\dots,k\}$ are not counted anymore and therefore the number of factors is exactly l. The multiplicity of (5.7) in the expression of

$$\psi(s_n(a_1)\dots s_n(a_k))$$

is $n(n-1)(n-2)\dots(n-l+1)$. (You may arrive at this number by choosing l different values for an exponent of γ from the set $\{0,1,\dots,n-1\}$.) Therefore we have

$$\begin{aligned}\psi(F_n(a_1)\dots F_n(a_k)) &= \sum_{l=0}^{k}\frac{n(n-1)\dots(n-l+1)}{n^{k/2}}\\ &\times \sum\varphi(a_{j^1(1)}\dots a_{j^1(p(2))})\dots\varphi(a_{j^l(1)}\dots a_{j^l(p(l))})\end{aligned}$$

where the second summation is over all partition of $\{1,2,\dots,k\}$ into l sets

$$\{j^1(1),j^1(2),\dots,j^1(p(1))\},\dots,\{j^l(1),j^l(2),\dots,j^l(p(l))\}\,.$$

(Note that in each product

$$a_{j^t(1)}a_{j^t(2)}\dots a_{j^t(p(t))}$$

the subscripts are ordered increasingly.) If a partition contains a singleton then due to the assumption $\varphi(a_i)=0$ the contribution of the corresponding term vanishes. If $l>\frac{k}{2}$ then any partition into l nonempty sets must contain at least one singleton. Hence we may neglect these values of l. On the other hand for $l<k/2$ the coefficient

$$\frac{n(n-1)(n-2)\dots(n-l+1)}{n^{k/2}}$$

tends to 0 as $n \to \infty$ while

$$\left|\sum \varphi(a_{j^1(1)} \dots a_{j^1(p(1))}) \dots \varphi(a_{j^l(1)} \dots a_{j^l(p(l))})\right|$$

remains bounded. The only l that may contribute to the limit is $k/2$ provided that k is even. So for an odd k

$$\lim_{n\to\infty} \psi(F_n(a_1)\dots F_n(a_k)) = 0$$

and for an even $k = 2l$ we have

$$\lim_{n\to\infty} \psi(F_n(a_1)\dots F_n(a_k)) = \sum \varphi(a_{j^1(1)}a_{j^1(2)}) \dots \varphi(a_{j^l(1)}a_{j^l(2)})$$

where the summation is over all partition of $\{1, 2, \dots, k\}$ into $\{j^1(1) < j^1(2)\}, \{j^2(1) < j^2(2)\}, \dots, \{j^l(1) < j^l(2)\}$. Since

$$\begin{aligned} \varphi(a_i a_j) &= \langle a_j, a_i\rangle_\varphi = Re\langle a_i, a_j\rangle_\varphi - i\, Im\langle a_i, a_j\rangle_\varphi \\ &= \alpha(a_i, a_j) - i\sigma(a_i, a_j) \end{aligned}$$

reference to Proposition 3.8 makes the proof complete.

What we proved in an example of convergence "in distribution" or "in law". The sequence of fluctuations converge to Bose fields in the sense that all correlation functions converge.

In probability theory it is a frequently used fact that convergence in distribution is equivalent to the convergence of characteristic functions ([**Fe**],XV.3). For the moment there is no similar theorem in quantum probability. So we treat independently the characteristic function version of the central limit theorem and show other methods.

For bounded selfadjoint operators a and b we introduce the notation

$$L(a, b) = \exp(ia)\exp(ib) - \exp(ia + ib)\exp\left(-\frac{1}{2}[a, b]\right) .$$

Inserting the power series expansion of the exponential function we find that only monomials (of a and b) with degree greater than 2 are present. Therefore the following lemma is straightforward.

Lemma 5.2 *There exists a constant $C > 0$ such that for $\varepsilon > 0$ small enough*

$$\|L(a,b)\| \le C\,\varepsilon^3$$

provided that $\|a\| < \varepsilon$ and $\|b\| < \varepsilon$.

Lemma 5.3 *If $[a_i, b_j] = [a_i, a_j] = [b_i, b_j] = 0$ for $i \neq j$ then*

$$\|L(a_1 + a_2, b_1 + b_2)\| \le \|L(a_1, b_1)\| + \|L(a_2, b_2)\| \exp\left(\frac{1}{2}\|[a_1, b_1]\|\right).$$

Under the hypothesis the equality

$$\begin{aligned} L(a_1 + a_2, b_1 + b_2) &= L(a_1, b_1)\exp(ia_2)\exp(ib_2) \\ &+ \exp(ia_1 + ib_1)\exp\left(-\frac{1}{2}[a_1, b_1]\right) L(a_2, b_2) \end{aligned}$$

holds and provides clearly the estimate of the norm.

Recall that for $a \in \mathcal{A}$ and $n \in \mathbb{Z}_+$ we write $s_n(a)$ for

$$a + \gamma(a) + \cdots + \gamma^{n-1}(a).$$

Proposition 5.4 *Let $a, b \in \mathcal{A}^{sa}$. Then*

$$\left\| L\left(\frac{1}{\sqrt{n}} s_n(a), \frac{1}{\sqrt{n}} s_n(b)\right)\right\| \to 0 \quad \textit{and} \quad n \to \infty.$$

We represent n in the form $n = k \cdot m + l$ where $\log n \le m < 1 + \log n$ (i.e., m is the integer part of $\log n$) and $0 \le l < m$. Application of Lemma 5.3 yields

$$\begin{aligned} \left\| L\left(\frac{1}{\sqrt{n}} s_n(a), \frac{1}{\sqrt{n}} s_n(b)\right)\right\| &\le \left\| L\left(\frac{1}{\sqrt{n}} s_{km}(a), \frac{1}{\sqrt{n}} s_{lm}(b)\right)\right\| \\ &+ \left\| L\left(\frac{1}{\sqrt{n}} s_l(a), \frac{1}{\sqrt{n}} s_l(b)\right)\right\| \cdot \exp\left(\frac{km\|a\|\|b\|}{n}\right) \end{aligned}$$

First we concentrate on the second term. Since

$$\left\|\frac{1}{\sqrt{n}} s_l(a)\right\| \le \frac{l\|a\|}{\sqrt{n}} \le \frac{1 + \log n}{\sqrt{n}}\|a\|$$

(and similarly with b) the norm continuity of $L(\cdot,\cdot)$ tells us that the first factor converges to 0 while the second remains bounded. Hence the second term tends to 0 and we have to show that so does the first one.

A repeated use of Lemma 5.3 gives that

$$\left\| L\left(\frac{1}{\sqrt{n}}s_{km}(a), \frac{1}{\sqrt{n}}s_{km}(b)\right)\right\| \\ \leq \left\| L\left(\frac{1}{\sqrt{n}}s_m(a), \frac{1}{\sqrt{n}}s_m(b)\right)\right\| \cdot \left(1 + C_n + \cdots + C_n^{k-1}\right)$$

where

$$C_n = \exp(m^2\|a\|\|b\|n^{-1})\,.$$

One can easily see that

$$1 + C_n + \cdots + C_n^{k-1} = O(k)\,.$$

According to Lemma 5.2

$$\left\| L\left(\frac{1}{\sqrt{n}}s_m(a), \frac{1}{\sqrt{n}}s_m(a)\right)\right\| \leq C\frac{m^3}{n\sqrt{n}}$$

if n is big enough. Since

$$0 \leq \frac{km^3}{n\sqrt{n}} \leq \frac{(1+\log n)^2}{\sqrt{n}} \to 0$$

we arrive at the end of the proof.

After the preparation we are ready to state another version of the central limit theorem.

Theorem 5.5 *For $a_1, a_2, \ldots, a_k \in \mathcal{A}^{sa}$ we have*

$$\lim_{n\to\infty} \psi(\exp(iF_n(a_1))\exp(iF_n(a_2))\ldots\exp(iF_n(a_k))) \\ = \rho(W(a_1 - \varphi(a_1))W(a_2 - \varphi(a_2))\ldots W(a_k - \varphi(a_k)))\,.$$

We may assume that $\varphi(a_i) = 0$ $(1 \le i \le k)$ and apply induction by k. Since ψ is a product state we have

$$\psi(\exp(iF_n(a)) = \varphi\left(\exp\frac{ia}{\sqrt{n}}\right)^n = \left(1 + \frac{1}{n}\left(n\varphi\left(\exp\frac{ia}{\sqrt{n}}\right) - n\right)\right)^n .$$

It follows from the power series expansion of the exponential function that

$$n\varphi\left(\exp\frac{ia}{\sqrt{n}}\right) - n = -\frac{\varphi(a^2)}{2} + O(n^{-1/2}) .$$

Consequently,

$$\psi(\exp(iF_n(a)) \to \exp\left(-\frac{1}{2}\varphi(a^2)\right)$$

which is the value of $\rho(W(a))$. (Remember that $\varphi(a) = 0$ was assumed.)

Now we are going to carry out the induction step. Proposition 5.4 tells us that

$$\exp(iF_n(a_k))\exp(iF_n(a_{k+1})) - \exp(iF_n(a_k + a_{k+1})\exp\left(\frac{1}{2n}s_n(x)\right) \tag{5.8}$$

converges to 0 in norm if x abbreviates $[a_k, a_{k+1}]$.

Let us represent $\mathcal{B}$ on the GNS Hilbert space $\mathcal{H}$ with cyclic vector Ψ. We apply von Neumann's statistical ergodic theorem for the isometry $V : \mathcal{H} \to \mathcal{H}$ defined by the formula

$$Vb\Psi = \gamma(b)\psi \qquad (b \in \mathcal{B}) .$$

Set E for the projection onto the fixed point space of V. We claim that E is of rank one. It is easy to see that for local elements $a, b \in \mathcal{B}$ the relation

$$\psi\left(\frac{1}{n}s_n(a^*)b\right) \to \psi(a^*)\psi(b)$$

holds. Equivalently,

$$\langle b\Psi, \frac{1}{n}(I + V + \cdots V^{n-1})a\Psi\rangle \to \langle\Psi, a\Psi\rangle\langle b\Psi, \Psi\rangle$$

Therefore we have

$$\langle b\Psi, Ea\Psi\rangle = \langle\Psi, a\Psi\rangle\langle b\Psi, \Psi\rangle$$

for every $a, b \in \mathcal{B}$. Being ψ cyclic

$$Ea\Psi = \langle a\Psi, \Psi\rangle\Psi = \psi(a)\Psi$$

must hold and E is really of rank one.

Let $a \in \mathcal{A}$ and $(b_n) \subset \mathcal{B}$ be a bounded sequence. We show that

$$\lim_{n\to\infty} \psi\left(b_n \exp\left(\frac{1}{n}s_n(a)\right)\right) = \exp\varphi(a) \lim_{n\to\infty}\psi(b_n) \tag{5.9}$$

whenever the right hand side makes sense. Indeed,

$$\begin{aligned}&\langle b_n \exp(n^{-1}s_n(a) - \varphi(a))\Psi, \Psi\rangle \\ &= \langle b_n\Psi, \Psi\rangle + \langle n^{-1}s_n(a)\Psi - \varphi(a)\Psi, \sum_{k=0}^{\infty} \frac{(n^{-1}s_n(a^*) - \varphi(a^*))^k}{(k+1)!} b_n^*\Psi\rangle\end{aligned}$$

where the second term converges to 0 obviously. Benefiting from the induction hypothesis and (5.9) we obtain

$$\begin{aligned}&\lim_{n\to\infty} \psi(\exp(iF_n(a_1))\dots\exp(iF_n(a_k))\exp(iF_n(a_{k+1})) \\ &\quad= \lim_{n\to\infty} \psi(\exp(iF_n(a_1))\dots\exp(iF_n(a_{k-1})) \\ &\quad\times \exp(iF_n(a_k + a_{k+1}))\exp\left(\frac{1}{2n}s_n(x)\right) \\ &\quad= \lim_{n\to\infty} \psi(\exp(iF_n(a_1))\dots\exp(iF_n(a_{k-1})) \\ &\quad\times \exp(iF_n(a_k + a_{k+1})))\exp(-i\sigma(a_{k_1}a_{k+1})) \\ &\quad= \rho(W(a_1)\dots W(a_{k-1})W(a_k + a_{k+1}))\exp(-i\sigma(a_k, a_{k+1})) \\ &\quad= \rho(W(a_1)\dots W(a_k)W(a_{k+1})\,.\end{aligned}$$

It was used that

$$\psi([a_k, a_{k+1}]) = -i\sigma(x_k, x_{k+1})\,.$$

It is worthwhile to point out that the above proof of Theorem 5.4 has two critical points.

(i) $\displaystyle\lim_{n\to\infty} \psi(\exp iF_n(a)) = \exp\left(-\frac{1}{2}\alpha(a,a)\right)$ for every $x \in \mathcal{A}^{sa}$ with $\psi(a) = 0$

(ii) $\displaystyle\lim_{n\to\infty} \frac{1}{n}(I + V + \cdots V^{n-1})x\Psi = \psi(x)\Psi$ for every $x \in \mathcal{A}$.

In the language of ergodic theory (ii) means that the isometry V is ergodic. The proof works with a nonproduct state ψ if conditions (i) and (ii) are satisfied.

Let Λ be a parameter set. A process (indexed by Λ) means that for every $a \in \Lambda$ an element $\eta(a)$ of an algebra $\mathcal{B}$ is given and a state ψ of $\mathcal{B}$ is fixed. It is said that the sequence

$$(\eta_n, \mathcal{B}_n, \psi_n)$$

of process converges in law to a process $(\eta, \mathcal{B}, \psi)$ if for every $a_1, a_2, \dots, a_k \in \Lambda$ and $k \in \mathbb{N}$ the limit relation

$$\lim_{n\to\infty} \psi_n(\eta_n(a_1)\eta_n(a_2)\dots\eta_n(a_k)) = \psi(\eta(a_1)\eta(a_2)\dots\eta(a_k))$$

holds. Due to Theorem 5.4 the process

$$(a \mapsto \exp(iF_n(a)), \mathcal{B}, \psi)$$

converges to the Weyl process

$$(a \mapsto W(a), CCR(\mathcal{A}^{sa}, \sigma), \rho)$$

where ρ is a certain quasifree state and

$$\Lambda = \{a \in \mathcal{A}^{sa} : \varphi(a) = 0\}.$$

Quasifree states are analogues of Gaussian distributions. (The central state is not quasifree in the sense of our definition. It would correspond to a degenerate normal distribution in probability theory.)

The presentation of this chapter benefited from [**G-vW**], [**A-B**] and [**G-V-V**].

Chapter 6

The KMS condition

Let $\mathcal{A}$ be a C^*-algebra and $t \mapsto \alpha_t$ a homomorphism of the additive group of reals into the automorphisms of $\mathcal{A}$. A state φ of $\mathcal{A}$ satisfies the Kubo-Martin-Schwinger, or KMS, condition if for every $a, b \in \mathcal{A}$ there is a function $F_{a,b}$ such that

(i) $F_{a,b}$ is defined on the strip $\{z \in \mathbb{C} : 0 \leq Imz \leq 1\}$.

(ii) $F_{a,b}$ is continuous and bounded.

(iii) $F_{a,b}$ is analytic on the open strip $\{z \in \mathbb{C} : 0 < Imz < 1\}$.

(iv) $F_{a,b}(t) = \varphi(a\alpha_t(b))$ for every $t \in \mathbb{R}$.

(v) $F_{a,b}(i + t) = \varphi(\alpha_t(b)a)$ for every $t \in \mathbb{R}$.

On the boundary $F_{a,b}$ is fixed by φ and (α_t). Sometimes we shall call $F_{a,b}$ KMS-function. In mathematical physics the KMS-condition describes the equilibrium states for the dynamics (α_t). In particular, a KMS-state is time invariant. We can see this by choosing $a = I$. The function $F_{I,b}$ takes the same value at t and $t + i$ and may be continued periodically to the whole complex plane into an entire analytic function which is bounded. Liouville theorem says that such a function must be constant. So $\varphi(\alpha_t(b))$ is independent of $t \in \mathbb{R}$.

In this chapter we see quasifree states on the CCR-algebra satisfying the KMS-condition. Let $\mathcal{H}$ be a complex Hilbert space and L a selfadjoint operator on $\mathcal{H}$. We consider the C^*-algebra $CCR(\mathcal{H}, \sigma)$

where $\sigma(f,g) = Im\langle f,g\rangle$ $(f,g \in \mathcal{H})$. The automorphism α_t is given by the Bogoliubov transformation $\exp(-2itL)$, that is,

$$\alpha_t(W(f)) = W(\exp(-2itL)f) \quad (t \in \mathbb{R},\ f \in \mathcal{H})\,. \tag{6.1}$$

Theorem 6.1 *If there is an $\epsilon > 0$, such that $L \geq \epsilon I$ then a quasifree state φ given by*

$$\varphi(W(f)) = \exp\left(-\frac{1}{2}Re\langle \coth L f, f\rangle\right) \quad (f \in H) \tag{6.2}$$

satisfies the KMS condition for (α_t). Moreover, φ is the only KMS state such that $f \mapsto \varphi(W(f))$ is continuous.

First we note that $\coth L \geq I$ is a bounded selfadjoint operator and for the positive form

$$\alpha(f,g) = Re\langle \coth L f, g\rangle$$

the condition (3.3) holds. Therefore (6.2) determines a state φ.

Denote by $\mathcal{D}$ the set of all entire analytic vectors of L. For $h \in \mathcal{D}$ the function

$$z \mapsto \sum_{n=0}^{\infty} \frac{z^n}{n!}\|L^n h\| \tag{6.3}$$

is analytic on the whole complex plain.

Lemma 6.2 *Let $\eta \in \mathcal{H}$ and $h \in \mathcal{D}$. The function*

$$F(t) = \alpha(\eta, \exp(-2itL)h) \qquad (t \in \mathbb{R})$$

admits an entire analytic extension and

$$\begin{aligned} F(i/2) &= \alpha(\eta, \cosh Lh) - i\alpha(\eta, i \sinh Lh) \\ F(-i/2) &= \alpha(\eta, \cosh Lh) + i\alpha(\eta, i \sinh Lh)\,. \end{aligned}$$

We work with the real linear structure of $\mathcal{H}$. Since α is bounded the function

$$\sum_{n=0}^{\infty} \frac{(-2z)^n}{n!}\alpha(\eta, i^n L^n h) \tag{6.4}$$

is an entire analytic function and supplies the extension of F. Furthermore,

$$\begin{aligned} F(-i/2) &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \alpha(\eta, i^n L^n h) \\ &= \sum_{n \text{ even}}^{\infty} \frac{1}{n!}(\eta, L^n h) + i \sum_{n \text{ odd}} \frac{1}{n!} \alpha(\eta, i L^n h) \\ &= \alpha(\eta, \cosh L h) + i\alpha(\eta, i \sinh L h)\,. \end{aligned}$$

The computation for $F(i/2)$ is similar.

Let us write T_t for $\exp(-2itL)$. Since $\mathcal{D}$ is stable under T_t we obtain from Lemma 6.2 that

$$F(i/2+S) = \alpha(\eta, \cosh LT_S h) - i\alpha(\eta, i \sinh LT_S h) \tag{6.5}$$

$$F(-i/2+S) = \alpha(\eta, \cosh LT_S h) + i\alpha(\eta, i \sinh LT_S h) \tag{6.6}$$

for every $S \in \mathbb{R}$.

Set for $t \in \mathbb{R}$

$$G_1(t) = \varphi(W(h)W(T_t h')) \quad \text{and} \quad G_2(t) = \varphi(W(T_t h')W(h))$$

where $h, h' \in \mathcal{H}$ are fixed. By simple computation we have

$$\begin{aligned} G_1(t) &= \exp\left(-\frac{1}{2}\|h\|^2 - \frac{1}{2}\|h'\|^2 - \alpha(h, T_t h') + i\alpha(\text{tgh}\, Lh, iT_t h')\right) \\ G_2(t) &= \exp\left(-\frac{1}{2}\|h\|^2 - \frac{1}{2}\|h'\|^2 - \alpha(h, T_t h') - i\alpha(\text{tgh}\, Lh, iT_t h')\right) \end{aligned}$$

In the light of Lemma 5.2 the function G_1 and G_2 admit entire analytic extensions. From (6.5) and (6.6) we obtain

$$G_1(i/2+s) = \exp\left(-\frac{1}{2}\|h\|^2 - \frac{1}{2}\|h'\|^2 + A + C + D + B\right) \tag{6.7}$$

and

$$G_2(-i/2+s) = \exp\left(-\frac{1}{2}\|h\|^2 - \frac{1}{2}\|h'\|^2 + A - C - D + B\right) \tag{6.8}$$

where

$$A = -\alpha(h, \cosh LT_s h') \quad B = \sigma(h, i \sinh LT_s h')$$
$$C = i\alpha(h, i \sinh LT_s h') \quad D = i\sigma(h, \cosh LT_t h')\,.$$

Since $C + D = 0$ we arrive at the equality

$$G_1(i/2 + s) = G_2(-i/2 + s)$$

for every $s \in \mathbb{R}$. It follows from the uniqueness of analytic continuations that $G_2(z) = G_1(z + i)$ for every $z \in \mathbb{C}$.

For $a = W(h)$ and $b = W(h')$ we have succeeded in the construction of the KMS-function. (In fact, our $F_{a,b}$ is an entire analytic function.) The next task is to obtain the KMS-function for arbitrary $a, b \in CCR(\mathcal{H}, \sigma)$ by approximation.

When a and b are linear combinations of Weyl operators the KMS-function $F_{a,b}$ is at our disposal.

We carry out the GNS construction with φ and get the triple $(\mathcal{H}_\varphi, \pi_\varphi, \phi)$. The representation π_φ is injective since $CCR(\mathcal{H}, \sigma)$ is simple and we simply omit π_φ from the formulas.

Lemma 6.3 *Let $\xi_n, \xi \in \mathcal{H}$ and assume that $\xi_n \to \xi$ in the norm of $\mathcal{H}$. Then $W(\xi_n) \to W(\xi)$ in the strong operator topology.*

On the unitaries the strong and weak operator topologies coincide. It is sufficient to check that

$$\langle W(\xi_n) \sum \lambda(g) W(g)\Phi\,,\ \sum \lambda(f) W(f)\Phi\rangle$$

converges to

$$\langle W(\xi) \sum \lambda(g) W(g)\Phi\,,\ \sum \lambda(f) W(f)\Phi\rangle$$

for arbitrary linear combinations $\sum \lambda(g) W(g)$ and $\sum \lambda(f) W(f)$. This is, however, clear from the explicite expressions.

For a while we denote by $\mathcal{A}_0$ the linear hull of the set $\{W(h) : h \in \mathcal{D}\}$. Kaplansky's density theorem tells us that for $a, b \in CCR(\mathcal{H}, \sigma)$

(or rather $a, b \in \pi_\varphi(CCR(\mathcal{H}, \sigma))''$) there are sets (a_i) and (b_i) in $\mathcal{A}_0$ such that the following conditions hold.

(i) $\|a_i\| \le \|a\|\,,\ \|b_i\| \le \|b\|$

(ii) $a_i \to a$ and $a_i^* \to a^*$ in the strong operator topology.

(iii) $b_i \to b$ and $b_i^* \to b^*$ in the strong operator topology.

Let $F_i = F_{a_i, b_i}$ be the KMS-function for a_i and b_i. We prove that $\lim F_i(z) = F(z)$ exists of $0 \le Im z \le 1$ and F is the KMS-function of a and b.

One can see that there exists a unitary u_t on $\mathcal{H}_\varphi$ for every $t \in \mathbb{R}$ such that

$$u_t W(h)\Phi = W(T_t h)\Phi \qquad (h \in \mathcal{H})\,.$$

To prove that F_i converges uniformly on the strip we estimate on the boundary as follows.

$$\begin{aligned} F_i(t) - F_j(t) &= \varphi(a_i \alpha_t(b_i)) - \varphi(a_j \alpha_t(b_j)) \\ &= \langle u_t b_i \Phi, a_i^* \Phi\rangle - \langle u_t b_j \Phi, a_j^* \Phi\rangle \\ &= \langle (b_i\Phi - b_j\Phi), u_t^* a_i^* \Phi\rangle + \langle b_j \Phi, u_t^*(a_n^* - a_n)\Phi\rangle \end{aligned}$$

Hence

$$|F_i(t) - F_j(t)| \le \|(b_i - b_j)\Phi\|\|a\| + \|(a_i^* - a_j^*)\Phi\|\|b\|$$

and similarly,

$$|F_k(i+t) - F_j(i+t)| \le \|(a_k - a_j)\Phi\|\|b\| + \|(b_k^* - b_j^*)\Phi\|\|a\|\,.$$

$F_i - F_j$ is a bounded analytic function which for big i and j is small on the boundary. Therefore, it must be small everywhere and F_i converges uniformly on the strip to a bounded analytic function F. Since

$$\varphi(a_i\alpha_t(b_i)) = \langle u_t b_i \Phi, a_i^*\Phi\rangle \to \langle u_t b\Phi, a^*\Phi\rangle = \varphi(a\alpha_t(b))$$

and similarly

$$\varphi(\alpha_t(b_i)a_i) \to \varphi(\alpha_t(b)a)\,,$$

the function F satisfies the necessary boundary conditions (iv) and (v).

We have not checked the boundedness of the KMS-functions yet. It is enough to do this in the case $a = W(h)$ and $b = W(h')$ with $h' \in \mathcal{D}$. Returning to (6.7) we have

$$\begin{aligned}\left|G_1\left(\frac{\delta i}{2}+s\right)\right| &= \exp\left(-\frac{1}{2}\|h\|^2-\frac{1}{2}\|h'\|^2\right)-\alpha(h,\cosh\delta LT_s h')\\ &+ \sigma(h, i\,\sinh\delta LT_s h')\end{aligned}$$

We estimate in terms of the inner product.

$$\begin{aligned}|\alpha(h,\cosh\delta LT_S h')| &\le |\langle\coth Lh,\cosh\delta LT_S h'\rangle|\\ &\le \|\coth Lh\|\|\cosh\delta Lh'\|\\ |\sigma(h,i\sinh\delta LT_S h')| &\le |\langle h, i\sinh\delta LT_S h'\rangle|\\ &\le \|h\|\|\sinh\delta Lh'\|\end{aligned}$$

Since $h' \in \mathcal{D}$ the norm $\|e^{\delta L}h'\|$ is bounded if δ is in any compact interval. It follows that $\|\cosh\delta Lh'\|$ and $\|\sinh\delta Lh'\|$ are bounded as well. It is worthwhile to note that we showed a bit more than the boundedness of the KMS-function. Namely, we obtained that

$$e^{-K} \le |G_1(z)| \le e^K \qquad (0 \le Re z \le 1) \tag{6.9}$$

for certain $K > 0$.

So far we have proven that (6.2) really gives a KMS-state and now we turn to the uniqueness.

Suppose that ω is another KMS-state. We consider for $f, g \in \mathcal{H}$ the KMS-functions

$$F^\omega(f,g) \quad \text{and} \quad F^\varphi(f,g)$$

corresponding to the operators $a = W(f)$ and $b = W(g)$. Set

$$v_{f,g}(z) = \frac{F^\omega(f,g)(z)}{F^\varphi(f,g)(z)}$$

Due to the estimate (6.9) for $g \in \mathcal{D}$ the function $v_{f,g}$ is bounded. It is also clear that it is analytic on the open strip and continuous on the closed strip. On the one hand we have for $t \in \mathbb{R}$

$$v_{f,g}(t) = \frac{\omega(W(f)W(T_t g))}{\varphi(W(f)W(T_t g))} = \frac{\omega(W(f+T_t g))}{\varphi(W(f+T_t g))}$$

and on the other hand

$$v_{f,g}(i+t) = \frac{\omega(W(T_t g)W(f))}{\varphi(W(T_t g)W(f))} = \frac{\omega(W(f+T_t g))}{\varphi(W(f+T_t g))}$$

Since $v_{f,g}(i+t) = v_{f,g}(t)$ we may apply the same trick used in the proof of the invariance of the KMS-state. $v_{f,g}$ has an entire bounded analytic continuation which should be constant. If

$$u(f) = \frac{\omega(W(f))}{\varphi(W(f))} = \omega(W(f)) \exp\left(\frac{1}{2}\alpha(f,f)\right)$$

then we have

$$u(f+g) = u(f+T_t g) \qquad (t \in \mathbb{R}) \tag{6.10}$$

We have to show that under the continuity condition on $h \mapsto \omega(W(h))$ the function u is identically 1.

First of all, u is continuous on $\mathcal{H}$. A repeated application of (6.10) yields that

$$u(g) = u\left(\frac{1}{n}\sum_{i=0}^{n-1} T_t^i g\right) \qquad (g \in \mathcal{H}\,,\ t \in \mathbb{R})\,. \tag{6.11}$$

So

$$u(g) = u\left(\frac{1}{n}\sum_{i=0}^{n-1} T_{1/n}^i g\right)$$

and letting $n \to \infty$ we obtain

$$u(g) = u\left(\int_0^1 T_t g dt\right)$$

Using (6.11) once more we have

$$u(g) = u\left(\frac{1}{n}\sum_{i=0}^{n-1} T_1^i \left(\int_0^1 T_t g dt\right)\right) = u\left(\frac{1}{n}\int_0^n T_t g dt\right)\,. \tag{6.12}$$

The ergodic theorem tells us that

$$\frac{1}{n}\int_0^n T_t g dt \to Eg$$

strongly where E is the orthogonal projection onto the kernel of L. Since L is injective and u is continuous from (6.12) we conclude that $u(g) = u(0)$ for every $g \in \mathcal{H}$.

Now we try to follow the lines of the above proof without the assumption $L \geq \varepsilon$. Assume that L is a positive selfadjoint operator on the Hilbert space $\mathcal{H}$. Let H be the set of all entire analytic vectors for L. Then we have

(i) $T_t H \subset H \qquad (t \in \mathbb{R})$

(ii) $tanh\, LH \subset H$

since $\tanh L$ is a bounded operator. We denote $\tanh L(H)$ by H_0 and set

$$\alpha_0(f_0, g) = Re\langle \coth L f_0, g\rangle \quad (f_0 \in H_0\,,\ g \in H) \tag{6.13}$$

Theorem 6.4 *If there exists a positive symmetric nondegenerate bilinear form $\alpha : H \times H \to \mathbb{R}$ which is the extension of α_0 given by (6.13), then the quasifree state*

$$\varphi(W(f)) = \exp\left(-\frac{1}{2}\alpha(f,f)\right) \qquad (f \in H)$$

is defined on $CCR(H, \sigma)$ and satisfies the KMS-condition for the group

$$\alpha_t(W(f)) = W(\exp(-2itL)) \qquad (t \in \mathbb{R},\ f \in H)\,.$$

Lemma 6.5 $|\sigma(f,g)|^2 \leq \alpha(f,f)\alpha(g,g) \qquad (f, g \in H)$

Let $D = -i \tanh L|H$ and $D|H_0 = D_0$. Simple verification shows that

$$\begin{aligned} \alpha(Df,g) &= \sigma(f,g) && (f,g \in H) \\ \alpha(Df,g) &= -\alpha(f,Dg) && (f,g \in H)\,. \end{aligned} \tag{6.14}$$

Since

$$\alpha(D_0 f_0, D_0 f_0) = \langle f_0, \tanh L f_0\rangle \leq \langle \coth L f_0, f_0\rangle = \alpha(f_0, f_0)$$

for every $f_0 \in H_0$, we get that D_0 is a contraction for α. Forming the completion $\overline{H}$ of H with respect to α we obtain a real Hilbert space and we denote by P the orthogonal projection onto the closure of H_0. Let D' be $\overline{D}_0 P$ where $\overline{D}_0$ stands for the closure of D_0 (defined on $\overline{H}_0$). For $f, g \in H$ we have

$$\begin{aligned}\alpha(D'g, Df) &= \alpha(\overline{D}_0 Pg, Df) = -\alpha(Pg, \overline{D}_0 Df) \\ &= -\alpha(g, P\overline{D}_0 Df) = -\alpha(g, \overline{D}_0 Df) = -\alpha(g, D^2 f) \\ &= \alpha(Dg, Df)\end{aligned}$$

that is,

$$\alpha(D'g - Dg, Df) = 0 \qquad (f, g \in H)\,.$$

This gives that $D'g - Dg \perp H_0$. But $D'g - Dg \in \overline{H}_0$, therefore $D'g = Dg$ for every $g \in H$. D' is a contraction on $\overline{H}$ and the Schwarz inequality yields

$$\begin{aligned}|\sigma(f,g)|^2 &= |\alpha(Df, g)|^2 = |\alpha(D'f, g)|^2 \\ &\leq \alpha(D'f, D'g)\alpha(g,g) \leq \alpha(f,f)\alpha(g,g)\end{aligned}$$

for every $f, g \in H$.

Lemma 6.5 provides the existence of the quasifree state φ on the algebra $CCR(H, \sigma)$. The rest of the proof of Theorem 6.1 works without modification. In the framework of Theorem 6.5 the description of all "reasonable" KMS states seems to be a delicate problem.

KMS states for the free evolution are treated in [**R-S-T**]. The chapter uses several ideas of that paper, which, however, contains some gaps.

Chapter 7

The duality theorem

In axiomatic field theory one associates with every region O in space-time a von Neumann algebra $\mathcal{M}(O)$. Let O' be the causal complement of O. Then duality means that the commutant of $\mathcal{M}(O)$ is equal to $\mathcal{M}(O')$. Without entering field theory in the present chapter we show an abstract duality theorem in the frame of CCR-algebras. First, however, we treat some elements of the Tomita-Takesaki theory.

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. For a set $\mathcal{D} \subset B(\mathcal{H})$ we denote by $\mathcal{D}'$ its commutant $\{a \in B(\mathcal{H}) : [a, d] = 0$ for every $d \in \mathcal{D}\}$. Von Neumann's bicommutant theorem tells us that for a von Neumann algebra $\mathcal{M}$ we have $\mathcal{M} = \mathcal{M}''$ (see [**S-Zs**], 3.2). The main point of the Tomita-Takesaki theory is to establish a deep connection between a von Neumann algebra $\mathcal{M}$ and its commutant $\mathcal{M}'$ provided that $\mathcal{M}$ admits a cyclic and separating vector.

A vector $\Omega \in \mathcal{H}$ is called cyclic for $\mathcal{M}$ if $\mathcal{M}\Omega$ is dense in $\mathcal{H}$. The vector Ω is said to be separating for $\mathcal{M}$ if $a \in \Omega$ and $a\Omega = 0$ imply that $a = 0$. By means of the bicommutant theorem one can easily verify that $\Omega \in \mathcal{H}$ is cyclic for a von Neumann algebra $\mathcal{M}$ if and only if it is separating for its commutant $\mathcal{M}'$. Below we assume that Ω is a cyclic and separating vector for $\mathcal{M}$.

Lemma 7.1 *The conjugate linear operator given by the formula*

$$S_0 \,:\, a\Omega \mapsto a^*\Omega \qquad (a \in \mathcal{M})$$

is densely defined and closable.

S_0 is densely defined because Ω is cyclic for $\mathcal{M}$. Set another conjugate linear operator as

$$F_0 : a'\Omega \mapsto a'^*\Omega \qquad (a' \in \mathcal{M}')$$

which is densely defined, too. One sees that $S_0 \subset F_0^*$ and therefore S_0 must be closable.

Denote by S the closure of S_0 and consider the polar decomposition

$$S = J\Delta^{1/2} \tag{7.1}$$

where $\Delta = S^*S$ is a positive selfadjoint operator. It follows from $S = S^{-1}$ that J is an antiunitary, i.e. conjugate linear and $J^{-1} = J^*$. The relation

$$J^2\Delta^{1/2} = J\Delta^{-1/2}J$$

combined with the uniqueness of the polar decomposition gives that

$$J^2 = id \quad \text{and} \quad \Delta^{1/2} = J\Delta^{-1/2}J\,. \tag{7.2}$$

Let us consider the example of a finite dimensional von Neumann algebra $\mathcal{N}$. Such an algebra always possesses a faithful tracial state τ. We perform the GNS-construction with τ. So we get an inner product

$$\langle a, b\rangle = \tau(b^*a) \qquad (a, b \in \mathcal{N})$$

and a representation

$$L_a b = ab \qquad (a, b \in \mathcal{N})$$

of the von Neumann algebra $\mathcal{N}$ on the Hilbert space $\mathcal{N}$. If ω is any state of $\mathcal{N}$ then it may be written in the form

$$\omega(a) = \tau(\rho a) \qquad (a \in \mathcal{N})$$

by means of a density operator $\rho \in \mathcal{N}$. Suppose that ρ is invertible. Then the vector $\Omega = \rho^{1/2}$ is cyclic for the von Neumann algebra

$$\mathcal{M} = \{L_a : a \in \mathcal{N}\}\,.$$

It is a simple computation that

$$\mathcal{M}' = \{R_a \,:\, a \in \mathcal{N}\}$$

where R_a is defined as

$$R_a b = ba\,. \qquad b \in \mathcal{N})$$

In our concrete example

$$S \,:\, a\rho^{1/2} \mapsto a^*\rho^{1/2} \qquad (a \in \mathcal{N})$$

and we find that

$$Ja = a^* \quad \text{and} \quad \Delta = L_\rho R_{\rho^{-1}}\,. \tag{7.3}$$

By computation

$$\Delta^{it} L_a \Delta^{-it} b = L_{\rho^{it} a \rho^{-it}} b\,. \qquad (a, b \in \mathcal{N})$$

Hence

$$\Delta^{it} \mathcal{M} \Delta^{-it} \subset \mathcal{M}\,. \tag{7.4}$$

On the other hand

$$JL_a J b = ba^* = R_{a^*} b \qquad (a, b \in \mathcal{N})$$

yields

$$J\mathcal{M}J \subset \mathcal{M}'\,. \tag{7.5}$$

Tomita's theorem states that (7.4) and (7.5) hold generally.

Theorem 7.2 *Let $\Omega \in \mathcal{H}$ be a cyclic and separating vector for a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$. Then the relations (7.4) and (7.5) hold if the modular conjugation operator J and the modular operator Δ are given by (7.1).*

We shall sketch the proof in the case where $\mathcal{M}$ may be approximated by finite dimensional subalgebras (cf. [**Lo**]). We assume that there exists a sequence $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}$ of subalgebras such that each $\mathcal{M}_n$ is finite dimensional and $\cup\{\mathcal{M}_n \,:\, n \in \mathbb{N}\}$ is strongly dense in $\mathcal{M}$. Let $\mathcal{H}_n = \mathcal{M}_n\Omega$ and $\tilde{\mathcal{M}}_n = \mathcal{M}_n|\mathcal{H}_n$. Then $\mathcal{M}_n$ and $\tilde{\mathcal{M}}_n$ are algebraically isomorphic and for the triplet $(\mathcal{H}_n, \tilde{\mathcal{M}}_n, \Omega)$ Tomita's theorem has been proven. Let $\tilde{J}_n$ and $\tilde{\Delta}_n$ be the corresponding conjugation and modular operators, P_n the orthogonal projection onto $\mathcal{H}_n$. Due to our hypothesis we have $P_n \to I$ strongly.

Lemma 7.3 *Set* $\Delta_n = \tilde{\Delta}_n P_n + P_n^\perp$ $(n \in \mathbb{N})$. *Then* $\Delta_n \to \Delta$ *strongly in the resolvent sense where* Δ *denotes the modular operator corresponding to* $(\mathcal{M}, \mathcal{H}, \Omega)$.

Let $\mathcal{H}'$ be the domain of the closure S of the operator $a\Omega \mapsto a^*\Omega$ $(a \in \mathcal{M})$. So $\mathcal{H}'$ becomes a Hilbert space with the scalar product

$$\langle \xi, \eta \rangle_S = \langle \xi, \eta \rangle + \langle S\eta, S\xi \rangle \qquad (\xi, \eta \in \mathcal{H}')\,.$$

We show that $\cup\{\mathcal{H}_n : n \in \mathbb{N}\}$ is a core for S, or equivalently, it is dense in $\mathcal{H}'$ with respect to the graph norm $\|\cdot\|_S$. Fix $a \in \mathcal{M}$. According to to Kaplansky's density theorem ([**S-Zs**], 3.10) we can find nets (a_j) and (b_j) from the selfadjoint part of $\cup\{\mathcal{M}_n : n \in \mathbb{N}\}$ such that

$$a_j \to \frac{1}{2}(a + a^*) \quad \text{and} \quad b_j \to \frac{i}{2}(a^* - a)$$

strongly. Then

$$(a_j + ib_j)\Omega \to a\Omega \quad \text{and} \quad (a_j - ib_j)\Omega \to a^*\Omega\,.$$

This means that $\cup\{\mathcal{H}_n : n \in \mathbb{N}\}$ is really a core for S.

Let Q_n be the orthogonal projection of $\mathcal{H}'$ onto $\mathcal{H}_n$ with respect to the inner product $\langle \cdot, \cdot \rangle_S$. There exists a contraction T on $\mathcal{H}'$ such that

$$\langle \xi, \eta \rangle = \langle T\xi, \eta \rangle_S \qquad (\xi, \eta \in \mathcal{H}')\,.$$

We have

$$(I + \Delta)^{-1}\xi = T\xi \qquad (\xi \in \mathcal{H}')$$

and

$$(I + \Delta_n)^{-1}\xi = (A_n T Q_n + Q_n^\perp)\xi \qquad (\xi \in \mathcal{H}_k,\ n \geq k)\,.$$

Since

$$Q_n T Q_n + Q_n^\perp \to T$$

strongly in the graph norm we obtain

$$(I + \Delta_n)^{-1}\xi \to (I + \Delta)^{-1}\xi \qquad (\xi \in U\{\mathcal{H}_n : n \in \mathbb{N}\})$$

in the norm of $\mathcal{H}$. Being $\cup\mathcal{H}_n$ total and the sequence $(I + \Delta_n)^{-1}$ bounded we have completed the proof of the Lemma.

According to a simple topological argument it is sufficient to prove (7.4) for $a \in \mathcal{M}_k$ ($k \in \mathbb{N}$). It follows from Lemma 7.3 that $\Delta_n^{it} \to \Delta^{it}$ strongly for every $t \in \mathbb{R}$ (cf. [**R-S**], VIII.7). Therefore

$$\Delta_n^{it}\, a\, \Delta_n^{-it} \to \Delta^{it}\, a\, \Delta^{it} \qquad (t \in \mathbb{R})$$

strongly. However

$$\Delta_n^{it}\, a\, \Delta_n^{-it} P_n \in \mathcal{M} P_n \qquad (n \geq k)$$

and (7.4) has been shown.

Lemma 7.4 *If J is the modular conjugation operator corresponding to $(\mathcal{M}, \mathcal{H}, \Omega)$ then*

$$\tilde{J}_n P_n \to J$$

strongly.

Since the strong convergence is equivalent to weak convergence and convergence of the norms, we shall prove weak convergence only. For $\xi, \eta \in \mathcal{H}_k$ and $n \geq k$ we have

$$\begin{aligned} \langle \tilde{J}_n P_n \xi, \eta \rangle &= \langle \tilde{J}_n \xi, \eta \rangle = \langle \Delta_n^{1/2} S\xi, \eta \rangle \\ &= \frac{2}{\pi} \int_0^\infty \langle \Delta_n (I + t^2 \Delta_n)^{-1} S\xi, \eta \rangle dt\,. \end{aligned}$$

Lemma 7.3 tells us that

$$\langle \Delta_n (I + t^2 \Delta_n)^{-1} S\xi, \eta \rangle \to \langle \Delta (I + t^2 \Delta)^{-1} S\xi, \eta \rangle$$

and using Lebesgue's theorem on the convergence of integrals we get

$$\langle \tilde{J}_n P_n \xi, \eta \rangle \to \frac{2}{\pi} \int_0^\infty \langle \Delta (I + t^2 \Delta)^{-1} S\xi, \eta \rangle dt\,.$$

The limit is just $\langle J\xi, \eta \rangle$ and the proof is complete.

In order to complete the proof of Theorem 7.2 we show (7.5). To do this it suffices to prove that

$$[J a_k J, a_l] = 0 \tag{7.6}$$

for $a_k \in \mathcal{M}_k$ and $a_l \in \mathcal{M}_l$. Since

$$J_{a_k} J_{a_l} = \lim_{n\to\infty} \tilde{J}_n a_k \tilde{J}_n a_l P_n$$

and

$$a_l J a_k J = \lim_{n\to\infty} a_l \tilde{J}_n a_k \tilde{J}_n P_n$$

as a consequence of Lemma 7.4, (7.6) follows from its finite dimensional version.

It is worthwhile to emphasize that we proved the continuity of the main ingredients, the modular operator and the modular conjugation, of the Tomita-Takesaki theory with respect to an increasing sequence of subalgebras.

Let $\mathcal{K}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Setting

$$\sigma(f,g) = -Im\langle f,g\rangle \qquad (f,g \in \mathcal{K})$$

the C^*-algebra $CCR(\mathcal{K},\sigma)$ and its Fock state

$$\varphi(W(f)) = \exp\left(-\frac{1}{2}\langle f,f\rangle\right) \qquad (f \in \mathcal{K})$$

are at our disposal. If $K \subset \mathcal{K}$ is a subspace then we write $CCR(K)$ for the subalgebra of $CCR(\mathcal{K}) \equiv CCR(\mathcal{K},\sigma)$ generated by the unitaries $\{W(f) : f \in K\}$ and we assume that all these algebras act on the Fock space $\mathcal{H}$ containing the cyclic vector Φ.

Lemma 7.5 *The mapping*

$$(f_1, f_2, \dots, f_k) \mapsto B^+(f_1)B^+(f_2)\dots B^+(f_k)\Phi$$

is normcontinuous from $\mathcal{K}\times\mathcal{K}\times\dots\times\mathcal{K}$ into $\mathcal{H}$ for every $k \in \mathbb{N}$.

To show the Lemma it is enough to see that

$$(g_1, g_2, \dots, g_n) \to \varphi(B(g_1)B(g_2)\dots B(f_n))$$

is normcontinuous. This is obvious from Proposition 3.8.

Lemma 7.6 *Let $K \subset \mathcal{K}$ be a real subspace and set $\mathcal{H}_0$ for the closure of $CCR(K)\Phi$. Then*

$$B^+(f_1)B^+(f_2)\dots B^+(f_k)\Phi \in \mathcal{H}_0$$

whenever $f_1, f_2, \dots, f_k \in K$ and $k \in \mathbb{N}$.

We apply induction on k. For $k = 0$ the statement is trivial. In order to carry out the induction step we write

$$B^+(f_{k+1})B^+(f_k)\dots B(f_1)\Phi =$$

$$B(f_{k+1})B^+(f_k)\dots B(f_1)\phi - B^-(f_{k+1})B^+(f_k)\dots B^+(f_1)\Phi\,.$$

Due to the induction hypothesis

$$B^+(f_k)B^+(f_{k-1})\dots B^1(f_1)\Phi \in \mathcal{H}_0$$

and differentiating

$$t \mapsto W(tf_{k+1})B^+(f_k)\dots B^+(f_1)\Phi$$

we obtain that

$$B(f_{k+1})B^+(f_k)B^+(f_{k-1})\dots B^+(f_1)\Phi \in \mathcal{H}_0\,.$$

If we use the commutation relation

$$[B^-(f), B^+(g)] = \langle g, f\rangle$$

repeatedly, we find that the vector

$$B^-(f_{k+1})B^+(f_k)B^+(f_{k-1})\dots B^+(f_1)\Phi$$

is a linear combination of the vectors

$$B^+(f_k)B^+(f_{k-1})\dots B^+(f_{l+1})B^+(f_{l-1})\dots B^+(f_1)\Phi$$

where $1 \le l \le k$. The latter vectors lie in $\mathcal{H}_0$ by the induction hypothesis.

Proposition 7.7 *If $K \subset \mathcal{K}$ is a real subspace such that $K + iK$ is dense in $\mathcal{K}$ then Φ is cyclic for the algebra $CCR(K)$.*

Denote by $\mathcal{H}_0$ the closure of the subspace $CCR(K)\Phi$. We know from Lemma 7.6 that

$$B^+(f_k)B^+(f_{k-1})\dots B^+(f_1)\Phi \in \mathcal{H}_0 \tag{7.7}$$

for every $f_k, f_{k-1}, \dots, f_1 \in K$. Since $B^+(f)$ is complex linear in f (7.7) holds also for $f_k, f_{k-1}, \dots, f_1 \in K+iK$. If $K+iK$ dense then reference to Lemma 7.5 yields that (7.7) is true for every $f_k, f_{k-1}, \dots, f_1 \in \mathcal{K}$ and $k \in \mathbb{N}$. Hence $\mathcal{D}_B \subset \mathcal{H}_0$ and $\mathcal{H}_0 = \mathcal{H}$ follows.

For a real linear subspace $K \subset \mathcal{K}$ let us denote by $\mathcal{M}(K)$ the von Neumann algebra generated by $CCR(K)$. According to Proposition 7.7 Φ is cyclic for $\mathcal{M}(K)$ provided that $K+iK$ is dense in $\mathcal{K}$.
Set

$$L = \{f \in \mathcal{K} \,:\, \sigma(h,f) = 0 \quad \text{for every} \quad f \in K\}\,. \tag{7.8}$$

L is the symplectic complement of K and it is nothing else but $iK^\perp$ if $\perp$ is understood with the real inner product $(f,g) = Re\langle f,g\rangle$ $(f,g \in \mathcal{K})$. Since $[W(h), W(f)] = 0$ for $h \in K$ and $f \in L$ we have $\mathcal{M}(L) \subset \mathcal{M}(K)'$. Assume that $K \cap iK = \{0\}$. Then $L+iL$ is dense in $\mathcal{K}$. (Indeed, $(L+iL)^\perp = K \cap iK$.) Thanks to Proposition 7.7 Φ is cyclic for $\mathcal{M}(L)$ and so is for $\mathcal{M}(K)'$. Hence under the hypothesis $K+iK$ is dense and $K \cap iK = \{0\}$ the vector Φ is cyclic and separating for the von Neumann algebra $\mathcal{M}(K)$.

The following result was obtained in [**Ar1**] and the presented proof comes from [**E-O**].

Theorem 7.8 *Let $K \subset \mathcal{K}$ be a closed real subspace such that $K+iK$ is dense in $\mathcal{K}$ and $K \cap iK = \{0\}$. Then the commutant of $\mathcal{M}(K)$ is $\mathcal{M}(L)$ where L is given by (7.8).*

The theorem will be proven in the important special case when the subspaces K and L are in generic position. By this we mean that any two of the subspaces $K, K^\perp, L, L^\perp$ have trivial intersection. Then $\mathcal{K}$ may be identified with a direct sum $\mathcal{K}_* \oplus \mathcal{K}_*$ and there exists a positive contraction T on $\mathcal{K}_*$ such that

$$ker\, T = ker(I-T) = \{0\}\,,$$

$$K = \{h \oplus Th \,:\, h \in \mathcal{K}_*\} \quad , \quad K^\perp = \{-Th \oplus h \,:\, h \in \mathcal{K}_*$$
$$L = \{Th \oplus h \,:\, h \in \mathcal{K}_*\} \quad , \quad L^\perp = \{h \oplus -Th \,:\, h \in \mathcal{K}_*\}\,.$$

Concerning the existence of $\mathcal{K}_*$ and T we refer to [**Ha**].

The mapping

$$f \mapsto B^+(f)\Phi \qquad (f \in \mathcal{K})$$

is an inner product preserving embedding of $\mathcal{K}$ into $\mathcal{H}$. Hence we may identify $\mathcal{K}$ with a subspace of $\mathcal{H}$. Our first goal is to show that the closure S of the operator

$$a\Phi \mapsto a^*\Phi \qquad (a \in \mathcal{M}(K))\,,$$

which is central in Tomita's theorem, leaves $\mathcal{K}$ invariant.

Since

$$t^{-1}(W(\pm tf) - I)\Phi \to \pm iB(f)\Phi$$

as $t \to 0$, we obtain that $B(f)\Phi \in \mathcal{D}(S)$ and $SB(f)\Phi = B(f)\Phi$ for every $f \in K$. It follows that

$$SB^+(f + ig)\Phi = B^+(f - ig)\Phi\,. \qquad (f, g \in K) \tag{7.9}$$

Using the assumption that K is closed it is easy to see that the operator

$$S_1 \,:\, B^+(f + ig)\phi \mapsto B^+(f - ig)\phi \qquad (f, g \in K)$$

is closed. Hence S_1 is the restriction of S to $\mathcal{K}$. One finds that

$$S_1 = \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix}$$

with respect to the decomposition $\mathcal{K} = \mathcal{K}_* \oplus \mathcal{K}_*$. (Note that $K = L^\perp$.)

Let $S_1 = J_1\Delta_1^{1/2}$ and $S = J\Delta^{1/2}$ be the polar decompositions. In order to conclude that

$$J_1 = J|\mathcal{K} \quad \text{and} \quad \Delta_1 = \Delta|\mathcal{K} \tag{7.10}$$

we have to establish that S^* leaves $\mathcal{K}$ invariant, too. Denote by $\mathcal{L}$ the orthogonal complement of $\mathcal{K}$ in $\mathcal{H}$, i.e. $\mathcal{L} = \{\xi \in \mathcal{H} \,:\, \langle \xi, \eta \rangle = 0$ for every $\eta \in \mathcal{K}\}$. Let $\eta \in \mathcal{D}(S) \cap \mathcal{K}$ and $\xi \in \mathcal{D}(S) \cap \mathcal{L}$. Then

$$\langle S^*\eta, \xi \rangle = \langle \eta, S\xi \rangle$$

and we need to know that $\mathcal{D}(S) \cap \mathcal{L}$ is dense in $\mathcal{L}$ and S leaves $\mathcal{L}$ invariant. This requires a bit more analysis and we sketch how to do it. First by differentiation we get

$$SB(f_1)B(f_2)\dots B(f_n)\Phi = B(f_n)B(f_{n-1})\dots B(f_1)\Phi$$

for every $f_1, f_2, \dots, f_n \in K$ and $n \in \mathbb{N}$. Then we deduce

$$SB^+(f_1)B^+(f_2)\dots B^+(f_n)\Phi = B^+(f_1)B^+(f_2)\dots B^+(f_n)\Phi \qquad (7.11)$$

(The detailed proof of (7.11) may be similar to that of Lemma 7.6.) The linear subspace spanned by the vectors

$$B^+(g_1)B^+(g_2)\dots B^+(g_n)\Phi$$

(where $n = 0$ or $n \geq 2$ and $g_j \in K + iK$) is in the domain of S, contained $\mathcal{L}$ and stable under S.

It is straightforward to see that

$$J_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{and} \quad \Delta_1 = \begin{pmatrix} T^2 & 0 \\ 0 & T^{-2} \end{pmatrix} \qquad (7.12)$$

So J_1 is a bijection of K onto L. We prove that

$$JB(f)J = B(J_1 f)\,. \qquad (f \in K) \qquad (7.13)$$

For $f \in K$ the selfadjoint operator $B(f)$ is affiliated with the von Neumann algebra $\mathcal{M}(K)$. Tomita's theorem tells us that $JB(f)J$ is affiliated with the commutant $\mathcal{M}(K)'$. For $f, g \in K$ we have

$$\begin{aligned} JB(f)JW(g)\Phi &= W(g)JB(f)J\Phi = \\ W(g)JB(f)\Phi &= W(g)JB^+(f)\Phi = \\ W(g)B^+(J_1 f)\Phi &= W(g)B(J_1 f)\Phi\,. \end{aligned}$$

Since $B(J_1 f)$ is affiliated with $\mathcal{M}(L) \subset \mathcal{M}(K)'$ we obtain

$$JB(f)JW(g)\Phi = B(J_1 f)W(g)\Phi \qquad (7.14)$$

whenever $f, g \in K$. Here we interrupt the proof of the theorem to state a lemma which is interesting also in its own right.

Lemma 7.9 *Assume that $K \subset \mathcal{K}$ is a real linear subspace such that $K + iK$ is dense in $\mathcal{K}$. Then the linear hull D of the set $\{W(h)\Phi : h \in K\}$ is a core for every field operator $B(f)$ $(f \in \mathcal{K})$.*

This statement improves Proposition 7.7. We know that $\mathcal{D}_B$ is a core for $B(f)$. So it suffices to show that the closure of D with respect to the graph norm associated with $B(f)$ contains $\mathcal{D}_B$. We may follow the proof of Proposition 7.7 and get that

$$B^+(f_1)B^+(f_2)\dots B^+(f_n)\Phi \tag{7.15}$$

is in the closure of D for $f_1, f_2, \dots, f_n \in K + iK$. Then by means of Lemma 7.5 we conclude that the vector (7.15) is in the closure of D also for every $f_1, f_2, \dots, f_n \in K$. (It is rather straightforward that the continuity in Lemma 7.5 holds if on $\mathcal{H}$ the graph norm is considered.)

Now we resume the proof of Theorem 7.8. Let D_0 be the (complex) linear subspace spanned by $\{W(g)\phi : g \in K\}$. It follows from (7.14) that

$$JB(f)J \supset B(J_1 f)|D_0 \,.$$

Since the latter operator is $B(J_1 f)$ according to Lemma 7.9 and $JB(f)J$ is selfadjoint we arrive at (7.13). Finally we infer

$$\begin{aligned} JW(f)J = J \quad & \sum_{n=0}^{\infty} \frac{i^n}{n!} B(f)^n J = B \\ & \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} (JB(f)J)^n = \\ & \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} B(J_1 f)^n = W(-J_1 f) \end{aligned}$$

for every $f \in K$. So the mapping $A \mapsto JAJ$ sends $CCR(K)$ into $CCR(L)$ and the proof is completed.

Chapter 8

Completely positive maps

Let $\mathcal{A}$ be a C^*-algebra. We denote by $M_n(\mathcal{A})$ the set of all $n \times n$-matrices $a = (a_{ij})$ with entries a_{ij} in $\mathcal{A}$. With the obvious matrix multiplication and involution $M_n(\mathcal{A})$ is a $*$-algebra and may be identified with $\mathcal{A} \otimes M_n$.

Let $\mathcal{A}$ and $\mathcal{B}$ C^*-algebras. For each linear map $\alpha : \mathcal{A} \to \mathcal{B}$ we define a linear map $\alpha_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ by

$$\alpha_n(a_{ij}) = (\alpha(a_{ij}))\,. \tag{8.1}$$

If α_n is positive for all n, then α is said to be completely positive. Completely positivity is the "right" generalization of a positive functional both from mathematical and physical point of view. Assume that the C^*-algebra acts on a Hilbert space $\mathcal{H}$, that is, $\mathcal{B} \subset B(\mathcal{H})$. Then $\alpha : \mathcal{A} \to \mathcal{B}$ is completely positive if and only if for every $a_1, a_2, \dots, a_k \in \mathcal{A}$, $\xi_1, \xi_2, \dots, \xi_k \in \mathcal{H}$ and for all $k \in \mathbb{N}$

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \langle \alpha(a_i^* a_j)\xi_j, \xi_i \rangle \geq 0 \tag{8.2}$$

holds.

Let (H, σ) and (H', σ') be symplectic spaces. A linear map $\beta : CCR(H, \sigma) \to CCR(H', \sigma')$ is called quasifree if there are a linear map $B : H \to H'$ and a function $F : H \to \mathbb{C}$ such that

$$\beta(W(h)) = F(h) W'(Bh) \qquad (h \in H)\,. \tag{8.3}$$

Clearly, the composition of quasifree maps is quasifree. In this chapter we shall deal with completely positive quasifree maps.

Theorem 8.1 *A quasifree map* $\beta : CCR(H,\sigma) \to CCR(H',\sigma')$ *given by (8.3) is completely positive if and only if the kernel*

$$(f,g) \mapsto F(g-f) \exp i(\sigma(g,f) - \sigma'(Bg,Bf)) \qquad (f,g \in H) \quad (8.4)$$

is positive-definite.

First we assume that (8.4) is positive-definite and show completely positivity in the form of (8.2). It suffices to choose

$$a_s = \sum_k \lambda_{sk} W(f_{sk}) \qquad (1 \le s \le k)\,.$$

Simple computation yields for $\xi_s \in \mathcal{H}$ $(1 \le s \le k)$ the following.

$$\sum_{s,t} \langle \beta(a_s^* a_t)\xi_t, \xi_s\rangle =$$

$$\sum_{s,t} \sum_{l,k} \lambda_{tl}\overline{\lambda}_{sk} F(f_{tl} - f_{sk}) \exp i(\sigma(f_{tl}, f_{sk}) - \sigma'(Bf_{tl}, Bf_{sk}))\times$$
$$\langle W'(Bf_{tl})\xi_t, W'(Bf_{sk})\xi_s\rangle$$

Here on the right hand side one recognizes the product of two kernels. The first one is positive-definite due to our hypothesis and the second one,

$$\langle (s,k),(t,l)\rangle \mapsto \langle W'(Bf_{tl})\xi_t, W'(Bf_{sk})\xi_s\rangle\,,$$

is positive-definite by a straightforward checking. Since the product of positive-definite kernels is also positive-definite (see Lemma 3.3) we have obtained that β is completely positive.

Let us recall that the C^*-algebra $CCR(H',\sigma')$ possesses a tracial state τ' (defined by (2.5)). Stand $(\pi, \mathcal{K}, \Omega)$ for the corresponding GNS-triplet. Consider the Hilbert space $\mathcal{K} \otimes \mathbb{C}^n$ and set a vector

$$\xi = \sum_{k=1}^{n} \pi(W'(-Bf_k))\Omega \otimes e_k$$

where $(e_1, e_2, \dots, e_n)$ is the canonical basis in $\mathbb{C}^n$. If β is completely positive then for every $x \in CCR(H,\sigma) \otimes M_n$ the inequality

$$\langle (\pi \otimes id)\beta_n(x^*x)\xi, \xi\rangle \ge 0 \qquad (8.5)$$

must hold. Choose

$$x = \sum_{l=1}^{n} \mu_l \, W(f_l) \otimes E_{1l}$$

(where E_{kl}'s form a system of matrix units in M_n), and compute

$$\beta_n(x^*x) = \sum_{s,t} \overline{\mu}_s \mu_t F(-f_s + f_t) \exp\, i\sigma(-f_s, f_t) W(-Bf_s + Bf_t) \otimes E_{st}$$

and

$$\langle (\pi \otimes id)\beta_n(x^*x)\xi, \xi\rangle =$$
$$\sum_{s,t} \overline{\mu}_s \mu_t F(f_t - f_s) \exp\; i(\sigma(f_t, f_s) - \sigma'(Bf_t, Bf_s))$$

which is nonnegative in consequence of (8.5). Hence we have verified that the kernel (8.4) is positive-definite.

Let $\mathcal{H}$ and $\mathcal{H}'$ be complex Hilbert spaces. Write

$$\sigma(f, g) = Im\langle f, g\rangle \qquad (f, g \in \mathcal{H})$$

and define σ' similarly on $\mathcal{H}'$. We are going to show that for every linear contraction $A : \mathcal{H} \to \mathcal{H}'$ there exists a completely positive quasifree map $\alpha_A : CCR(\mathcal{H}, \sigma) \to CCR(\mathcal{H}', \sigma')$ such that

$$\alpha_A(W(f)) = W'(Af) \exp\left(\frac{1}{2}\|Af\|^2 - \frac{1}{2}\|f\|^2\right) \tag{8.6}$$

for every $f \in \mathcal{H}$. In the light of Theorem 8.1 we need to show that the kernel

$$(f, g) \mapsto \exp\left(\frac{1}{2}\|A(g - f)\|^2 - \frac{1}{2}\|g - f\|^2 + i\sigma(g, f) - i\sigma'(Ag, Af)\right)$$

is positive-definite. It is so if and only if the kernel

$$(f, g) \mapsto \exp\left(Re\langle (I - A^*A)g, f\rangle + i\, Im\langle (I - A^*A)g, f\rangle\right)$$

is positive-definite. Lemma 3.2 tells us that

$$(f, g) \mapsto Re\langle (I - A^*A)g, f\rangle + i\, Im\langle (I - A^*A)g, f\rangle$$

is positive-definite and the pointwise exponentiation preserves this property.

On the language of categories the above example hides a functor. Let us consider the category whose objects are complex Hilbert spaces and whose morphisms are contractions. The correspondence $A \mapsto \alpha_A$ is functorial in the sense that

$$\alpha_{AB} = \alpha_A \alpha_B \tag{8.7}$$

whenever $B : \mathcal{H}_1 \to \mathcal{H}_2$ and $A : \mathcal{H}_2 \to \mathcal{H}_3$ are contractions. To the zero operator α associates the Fock state

$$\varphi_{\mathcal{H}}(W(f)) = \exp\left(-\frac{1}{2}\|f\|^2\right) \qquad (f \in \mathcal{H})\,.$$

Since $0A = 0$ we have

$$\varphi_{\mathcal{H}_2} \circ \alpha_B = \varphi_{\mathcal{H}_1}\,. \tag{8.8}$$

$\Gamma(\mathcal{H})$ is a frequently used notation for the GNS Hilbert space corresponding to $CCR(\mathcal{H}, \sigma)$ and $\varphi_{\mathcal{H}}$. It follows from (8.9) and the Schwarz inequality that the linear application

$$W(h_1)\Phi_1 \mapsto \alpha_B(W(h_1))\Phi_2 \qquad (h_1 \in \mathcal{H}_1)$$

extends to a contraction $\Gamma(B) : \Gamma(\mathcal{H}_1) \to \Gamma(\mathcal{H}_2)$. (Here Φ_1 and Φ_2 are the cyclic vectors.) $B \mapsto \Gamma(B)$ is also a functorial correspondence and it is called Fock functor. It is convenient to know the action of $\Gamma(B)$ on exponential vectors (see (4.10)). We have

$$\Gamma(B)e(f) = e(Bf) \qquad (f \in \mathcal{H}_1)\,. \tag{8.9}$$

Let $\mathcal{A}_i$ be a C^*-algebra and φ_i a faithful state on it ($i = 1, 2$). If the unital linear map $T : \mathcal{A}_1 \to \mathcal{A}_2$ satisfies the inequality

$$T(a^*a) \geq T(a)^*T(a) \qquad (a \in \mathcal{A}_1) \tag{8.10}$$

then it is called a Schwarz map. It is well known that a completely positive map is of Schwarz type. For $a_i, b_i \in \mathcal{A}_i$ and $i = 1, 2$ we set

$$\begin{aligned} \alpha_i(a_i, b_i) &= Re\, \varphi_i(b_i^* a_i) \\ \sigma_i(a_i, b_i) &= Im\, \varphi_i(b_i^* a_i) \end{aligned}$$

and denote by $CCR(\mathcal{A}_i, \varphi_i)$ the CCR-algebra associated with the symplectic space $(\mathcal{A}_i, \sigma_i)$ $(i = 1, 2)$. On the algebra $CCR(\mathcal{A}_i, \varphi_i)$ the Fock state

$$\rho_i(W(a_i)) = \exp\left(-\frac{1}{2}\alpha_i(a_i, a_i)\right) \qquad (a_i \in \mathcal{A}_i\,,\ i = 1, 2)$$

is at our disposal.

Assume that T is a Schwarz map and $\varphi_2 \circ T = \varphi_1$. Then T becomes a contraction with respect to the GNS-norms and there exists a quasifree completely positive map $\alpha : CCR(\mathcal{A}_1, \varphi_1) \to CCR(\mathcal{A}_2, \varphi_2)$ such that

$$\alpha(W(a_1)) = W(Ta_1) \exp\frac{1}{2}\beta(a_1, a_1) \qquad (a_1 \in \mathcal{A}_1) \tag{8.11}$$

where

$$\beta(a_1, b_1) = \varphi_2(T(b_1)^* T(a_1)) - \varphi_1(b_1^* a_1)\,.$$

We are going to prove that the quasifree mapping α is in some sense a central limit.

Consider the n-bold tensorproducts $\mathcal{A}_1^n = \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_1$ and $\mathcal{A}_2^n = \mathcal{A}_2 \otimes \ldots \otimes \mathcal{A}_2$. Let $T_n : \mathcal{A}_1^n \to \mathcal{A}_2^n$ be $T \otimes T \otimes \ldots \otimes T$ and $\omega_n = \varphi_2 \otimes \varphi_2 \otimes \ldots \otimes \varphi_2$. We recall that for $a \in \mathcal{A}_1$ its nth fluctuation is defined as

$$F_n(a_1) = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} I^{(1)} \otimes \ldots \otimes I^{(l-1)} \otimes (a - \varphi_1(a)) \otimes I^{(l+1)} \otimes \ldots \otimes I^{(n)}$$

Theorem 8.2 *Let* $a_1, a_2, \ldots, a_k \in \mathcal{A}_1^{sa}$ *and assume that* $\varphi_1(a_l) = 0$ $(1 \le l \le k)$. *Then*

$$\begin{aligned} \lim_{n\to\infty} & \omega_n(T_n(\exp\, i\, F_n(a_1)) \ldots T_n(\exp\, i\, F_n(a_k)) \\ & = \rho_2(\alpha(a_1)\alpha(a_2) \ldots \alpha(a_k)) \end{aligned} \tag{8.12}$$

In order to prove the theorem, we compute both sides of (8.13). First

$$\begin{aligned} & \omega_n\left(T_n(\exp\, i\, F_n(a_1)) \ldots T_n(\exp\, i\, F_n(a_k))\right) \\ & = \varphi_2\left(T\left(\exp\frac{2a_1}{\sqrt{n}}\right) \ldots T\left(\exp\frac{ia_k}{\sqrt{n}}\right)\right)^n \end{aligned}$$

and

$$T\left(\exp\frac{ia_1}{\sqrt{n}}\right)\dots T\left(\exp\frac{ia_k}{\sqrt{n}}\right) = I + \sum_{s=1}^{k}\frac{iT(a_s)}{n}$$
$$-\sum_{s=1}^{k}\frac{T(a_s^2)}{2n} - \sum_{s<t}\frac{T(a_s)T(a_t)}{n} + O(n^{3/2})\,.$$

Since $\varphi_2 T(a_s) = \varphi_1(a_s) = 0$ by assumption we obtain that the left hand side of (8.13) equals to

$$\exp\left(-\frac{1}{2}\sum_{s=1}^{k}\varphi_1(a_s^2) - \sum_{s<t}\varphi_2(T(a_s)T(a_t)\right)\,. \tag{8.13}$$

Applying (8.12) and using the CCR-relation we see that the right hand side of (8.13) is exactly

$$\exp\left(-\frac{1}{2}\alpha_2\left(\sum_{l=1}^{k}T_{al}, \sum_{l=1}^{k}T_{al}\right) - \frac{1}{2}\sum_{l=1}^{k}\beta(a_l, a_l) + i\sum_{s<t}\sigma_2(Ta_s, Ta_t)\right)\,.$$

It is straightforward to check that the latter expression is identical with (8.14).

More complicated central limit theorems for completely positive maps have been obtained in [**Qu**]. Here we presented a simpler generalization of Theorem 5.5.

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and Q a skewadjoint operator on H with $\|Q\| < 1$ and $Ker\ Q = \{0\}$. Setting $\sigma(f,g) = (Qf,g)$ we get a nondegenerate symplectic form and there is a quasifree state φ on $CCR(H,\sigma)$ such that

$$\varphi(W(f)) = \exp\left(-\frac{1}{2}(f,f)\right) \qquad (f \in H) \tag{8.14}$$

For a subspace $(H_0,\sigma_0) \subset (H,\sigma)$ we have

$$CCR(H_0,\sigma_0) \subset CCR(H,\sigma)$$

and we shall construct a completely positive quasifree map

$$E_\varphi : CCR(H,\sigma) \to CCR(H_0,\sigma_0)$$

which may be viewed a conditional expectation with respect to the quasifree state (8.15).

It is possible to express Q as

$$Q = J\, tanh\, L \tag{8.15}$$

where J is a complex structure and L is a selfadjoint operator on H satisfying

$$[L,J] = 0 \quad \text{and} \quad L \geq \epsilon\,. \tag{8.16}$$

It was shown in Chapter 6 that the automorphism group σ_t given by

$$\sigma_t(W(f)) = W(T_t f)\,,\; T_t = \exp(-2tJL) \qquad (t \in \mathbb{R},\; f \in H)$$

satisfies the KMS-condition with the state φ. Therefore, for every $a, b \in CCR(H,\sigma)$ the function

$$t \mapsto \varphi(a\sigma_t(b))$$

admits an analytic extension to the strip $\{z \in \mathbb{C} : 0 \leq Im\, z \leq 1\}$. We set

$$\ll a, b \gg = \varphi(a^*\sigma_{i/2}(b))$$

(i.e., the value of the extension at $i/2$). The three lines theorem tells us that

$$|\ll a, b \gg| \leq \|a\|\,\|b\| \qquad (a, b \in CCR(H,\sigma))$$

and for every fixed $a \in CCR(H,\sigma)$

$$\gamma_a : b \mapsto \ll a, b \gg$$

is a linear form on $CCR(H,\sigma)$. Let us compute $\ll W(f), W(g) \gg$. To do this we continue analytically the function

$$t \mapsto \varphi(W(-f)W(T_t g)) = \exp\left(-i(Qf, T_t g) - \frac{1}{2}\|f\|^2 - \frac{1}{2}\|g\|^2 + \langle f, T_t g\rangle\right)\,.$$

Lemma 6.2 yields

$$\begin{aligned}(Qf, T_{i/2}g) &= (Qf, \cosh\, Lg) - i(Qf, J \sinh\, Lg)\\ (f, T_{i/2}g) &= (f, \cosh\, Lg) - i(f, J \sinh\, Lg)\,.\end{aligned}$$

Using

$$\cosh\, L = (I - Q^*Q)^{-1/2} \quad , \quad J \sinh\, L = Q(I - Q^*Q)^{-1/2}$$

we obtain

$$\ll W(f), W(g) \gg = \exp\left(-\frac{1}{2}(f,f) - \frac{1}{2}(g,g) + (f, (I - Q^*Q)^{1/2}g)\right)\,. \tag{8.17}$$

Denote by P the orthogonal projection of H onto H_0.

Theorem 8.3 *There exists a completely positive map* $E_\varphi : CCR(H,\sigma) \to CCR(H_0, \sigma_0)$ *such that*

(i) $\ll a_0, b \gg = \ll a_0, E_\varphi(b) \gg_0$ *for every* $a_0 \in CCR(H_0, \sigma_0)$ *and* $b \in CCR(H, \sigma)\,.$

(ii) $E_\varphi(W(f)) = W(Tf) \exp\left(\frac{1}{2}\|Tf\|^2 - \frac{1}{2}\|f\|^2\right)$ *where* $T = (I - PQ^*PQP)^{-1/2}P(I - Q^*Q)^{1/2}\,.$

Since

$$(I - PQ^*PQP)^{1/2}TT^*(I - PQ^*PQP) = P - PQ^*QP \le P - PQ^*PQP$$

we see that T is a contraction. Theorem 4.9 tells us that there are complex Hilbert spaces $\mathcal{H} \supset H$ and $\mathcal{H}_0 \supset H_0$ such that

$$\begin{aligned}\sigma(f,g) &= Im\langle f, g\rangle && (f, g \in H)\\ \sigma_0(f,g) &= (PQPf, g) = Im\langle f, g\rangle_0 && (f, g \in H_0)\end{aligned}$$

and (8.6) yields a quasifree mapping exactly in the form (ii).

Having the concrete formula (8.18) at our disposal it is a simple verification that E_φ defined by (ii) satisfies also (i).

E_φ is the dual of the embedding $CCR(H_0, \sigma_0) \to CCR(H, \sigma)$ with respect to the sesquilinear form $\ll \cdot, \cdot \gg$. The peculiarity of the latter form is in the fact

$$\ll a, b \gg \ge 0$$

whenever $a, b \geq 0$. E_φ is called φ-conditional expectation in [**A-C**] and it was computed in the CCR setting in [**Fr**]. The paper [**Pe**] is a review on the subject.

Note that in the case $[P, Q] = 0$ we have $T = P$ and

$$E_\varphi(W(f_0)W(g)) = W(f_0)E_\varphi(W(g)) \tag{8.18}$$

holds for every $f_0 \in H_0$, $g \in H$.

We used [**E-L**] and [**D-V-V**] to prepare the general part of this chapter.

Chapter 9

Equivalence of states

Let $\mathcal{A}$ be a C^*-algebra and π_i its representation on a Hilbert space $\mathcal{K}_i$ $(i = 1, 2)$. It is said that π_1 and π_2 are unitary equivalent if there is a unitary $V : \mathcal{K}_1 \to \mathcal{K}_2$ such that

$$V\,\pi_1(a)V^* = \pi_2(a) \qquad (a \in \mathcal{A})\,.$$

Two states are unitary equivalent if the corresponding GNS-representations are unitary equivalent.

Let $\mathcal{H}$ be a complex Hilbert space and consider $CCR(\mathcal{H}, \sigma)$ where the symplectic form σ is the imaginary part of the inner product. For $\lambda \geq 1$ there is a quasifree state φ_λ on $CCR(\mathcal{H}, \sigma)$ such that

$$\varphi_\lambda(W(h)) = \exp\left(-\frac{\lambda}{2}\|h\|^2\right) \qquad (h \in \mathcal{H})\,. \tag{9.1}$$

If $\lambda = 1$ we obtain a pure state due to Theorem 4.7. The other states are not pure as it follows from Proposition 3.9. Therefore, φ_1 can not be unitary equivalent with φ_λ if $\lambda > 1$.

Let (H, σ) be a $2n$-dimensional symplectic space and choosing a symplectic basis in H we have an isomorphism $\theta : H \to \mathbb{C}^n$ such that

$$\sigma(f, g) = Im\langle \theta f, \theta g\rangle \qquad (f, g \in H)\,.$$

Since $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, the Lebesgue measure on $\mathbb{R}^{2n}$ induces a measure dg on H. This measure seems to depend on θ but it is not so. In order to see that it suffices to show that every symplectic transformation T on H preserves dg. Let

$$\sigma(f,g) = Re\langle Df, g\rangle \qquad (f, g \in H)\,.$$

Since T is supposed to be symplectic we have

$$Re\langle Df, g\rangle = Re\langle DTf, Tg\rangle$$

and it follows that

$$D = T^* DT\,.$$

$\det D \neq 0$ and we arrive at $|\det T| = 1$. As a consequence of the above argument we may speak of the Lebesgue measure on a finite dimensional symplectic space.

Let φ be a Fock state on $CCR(H,\sigma)$. The integral

$$\pi^{-n} \int_H \varphi(W(g))W(g)dg = P \tag{9.2}$$

is norm convergent and defines an element P of $CCR(H,\sigma)$. For a $2m$-dimensional subspace H_0 of H we set

$$P_0 = \pi^{-m} \int_{H_0} \varphi(W(g))W(g)dg \tag{9.3}$$

The following two lemmas are obtained by direct manipulations with Gaussian integrals in a symplectic basis.

Lemma 9.1 *For every $h \in H$ we have*

$$PW(h)P = \varphi(W(h))P$$

Lemma 9.2 $PP_0 = P$

The following result is a reformulation of von Neumann's uniqueness theorem.

Theorem 9.3 *For a finite dimensional nondegenerate symplectic space (H,σ) any two Fock states of $CCR(H,\sigma)$ are unitary equivalent.*

We follow the proof of Theorem 1.2. If φ is a Fock state then $\pi_\varphi(P)$ is a projection of rank one because π_φ is an irreducible representation. (Here the projection P is given by (9.2).) One can construct the intertwining unitary as in Chapter 1.

We shall see that over an infinite dimensional symplectic space there are nonequivalent Fock states on the CCR-algebra.

The representations π_1 and π_2 are called quasi-equivalent ($\pi_1 \approx \pi_2$ in notation) if there exists a von Neumann algebra isomorphism α between $\pi_1(\mathcal{A})''$ and $\pi_2(\mathcal{A})''$ such that

$$\alpha(\pi_1(a)) = \pi_2(a) \qquad (a \in \mathcal{A}).$$

Quasi-equivalence is weaker than unitary equivalence but they coincide on the class of irreducible representations.

Let $\mathcal{H} = L^2(0,1)$. We show that the states φ_λ given by (9.1) are not quasi-equivalence for different λ's. Let $\lambda > \mu \geq 1$ and argue by contradiction. Suppose that α is a von Neumann algebra isomorphism of $\pi_\lambda(CCR(\mathcal{H}))''$ onto $\pi_\mu(CCR(\mathcal{H}))''$. Since the field operators are affiliated with $\pi_\lambda(CCR(\mathcal{H}))''$, α acts on them. Clearly,

$$\alpha(B_\lambda(f)^2) = B_\mu(f)^2 \qquad (f \in \mathcal{H}).$$

Denote by h_n^k the characteristic function of the interval $[(k-1)/n\,,\, k/n]$ and set

$$A_\lambda^n = -\frac{1}{2}\left(B_\lambda(h_n^1)^2 + \ldots + B_\lambda(h_n^n)^2\right)$$

$\exp A_\lambda^n$ is a positive contraction in $\pi_\lambda(CCR(\mathcal{H}))''$ and A_μ^n is defined similarly. We have

$$\alpha(\exp A_\lambda^n) = \exp A_\mu^n\,.$$

Compute

$$\begin{aligned}&\langle \exp A_\lambda^n W(g)\Phi_\lambda, W(g)\Phi_\lambda\rangle = \\ &(2\pi)^{-n/2}\int_{\mathbb{R}^n} \langle \exp\left(-i\sum_{k=1}^n x_k B(h_n^k)\right) W(g)\Phi_\lambda, W(g)\Phi_\lambda\rangle \\ &\times \exp\left(-\frac{1}{2}\sum_{k=1}^n x_k^2\right) dx_1 dx_2 \ldots dx_n\,.\end{aligned}$$

By evaluation of the Gaussian integrals we obtain

$$\langle \exp A_\lambda^n W(g)\Phi_\lambda, W(g)\phi_\lambda\rangle = \left(1+\frac{\lambda}{n}\right)^{-n/2} \exp\left(-\frac{1}{2+2\lambda/n}\sum_{k=0}^{n}\sigma(g,h_n^k)\right).$$

Since

$$\sum_{k=0}^{n}\sigma(g,h_n^k)=0$$

for every real function g we have

$$\langle \exp A_\lambda^n W(g)\Phi_\lambda, W(g)\Phi_\lambda\rangle \to \exp\left(-\frac{\lambda}{2}\right).$$

One gets in this way that

$$\exp A_\lambda^n \to e^{-\lambda/2} \quad \text{and} \quad \exp A_\mu^n \to e^{-\mu/2}$$

in the weak operator topology. Since α is continuous in this topology (on bounded sets) we arrive at a contradiction.

Let H be a real Hilbert space with scalar product $(\cdot,\cdot)$. On the space $\mathcal{H}=H\oplus H$

$$\sigma(f_1\oplus f_2, g_1\oplus g_2) = (f_2,g_1)-(f_1,g_2)$$

defines a nondegenerate symplectic form. The standard Fock state φ on $CCR(\mathcal{H},\sigma)$ is given by

$$\varphi(W(f_1\oplus f_2)) = \exp\left(-\frac{1}{2}(f_1,f_1)-\frac{1}{2}(f_2,f_2)\right). \tag{9.4}$$

Let T be a positive bounded operator on H and suppose that T has a bounded inverse. The map $T_2 : (f_1\oplus f_2)\mapsto Tf_1\oplus T^{-1}f_2$ is symplectic and

$$\varphi_T(W(f)) = \exp\left(-\frac{1}{2}(T_2f,f)\right) \qquad (f\in\mathcal{H}) \tag{9.5}$$

defines another Fock state. Our next aim will be to discuss the equivalence of φ and φ_T. Below we shall assume that H and $\mathcal{H}$ are infinite dimensional separable Hilbert spaces.

Theorem 9.4 *Let* (H_0, σ_0) *be a finite dimensional symplectic space and* $\pi : CCR(H_0, \sigma_0) \to B(\mathcal{K})$ *a continuous representation. Then* π *is the direct sum of (irreducible) Fock representations.*

Let P_0 be the projection in $CCR(H_0, \sigma_0)$ given by (9.3) and (ξ_i) an orthonormal basis in the subspace $\mathcal{K}_0 = P_0\mathcal{K}$. $\mathcal{K}_i = [\pi(CCR(H_0, \sigma_0))\xi_i]$ and $\pi_i = \pi|\mathcal{K}_i$. Since

$$\langle W(f)\xi_i, W(g)\xi_j\rangle = e^{i\sigma_0(g,f)}\varphi_0(W(f-g))\langle \xi_i, \xi_j\rangle$$

we get $\mathcal{K}_i \perp \mathcal{K}_j$ if $i \neq j$. From the fact that $\pi_i(P_0)$ is a projection of rank one it follows that π_i must be irreducible.

Let $\mathcal{L}$ be the orthogonal complement of the sum $\sum_i \oplus \mathcal{K}_i$. This is the place to use the continuity hypothesis. It yields that $\mathcal{L} = \{0\}$ (cf. the proof of Theorem 1.2.).

One can guess from the above proof that every representation of $CCR(H_0, \sigma_0)$ is the direct sum of a continuous and a "singular" one.

Lemma 9.5 *Let* $\mathcal{H}_0$ *be a finite dimensional subspace of* $\mathcal{H}$ *and* π *a representation of* $CCR(\mathcal{H})$ *such that* $\pi|CCR(\mathcal{H}_0)$ *is continuous. Then*

$$\pi(CCR(\mathcal{H}_0))' \cap \pi(CCR(\mathcal{H}))'' = \pi(CCR(\mathcal{H}_0^\perp))''$$

The point is the containment $\subset$. It follows from the previous theorem that the von Neumann algebra $\mathcal{M}_0 = \pi(CCR(\mathcal{H}_0))''$ is the direct sum of type I factors. Therefore, the identity is a sum of pairwise orthogonal minimal projections of $\mathcal{M}_0$. It suffices to show that for every minimal projection E of $\mathcal{M}_0$. We have

$$\pi(CCR(\mathcal{H}))' \cap \pi(CCR(\mathcal{H}))''E \subset \pi(CCR(\mathcal{H}_0^\perp))''E\,. \qquad (9.6)$$

Evidently,

$$\pi(CCR(\mathcal{H}_0))' \cap \pi(CCR(\mathcal{H}))''E \subset E\pi(CCR(\mathcal{H}))''E\,. \qquad (9.7)$$

The mapping $\beta : T \mapsto ETE$ is strongly continuous. The von Neumann algebra $\pi(CCR(\mathcal{H}))''$ is generated by $\pi(CCR(\mathcal{H}_0))$ and $\pi(CCR(\mathcal{H}_0^\perp))$. For $A \in \pi(CCR(\mathcal{H}_0))$ and $B \in \pi(CCR(\mathcal{H}_0^\perp))$ we have

$$\beta(AB) = EAEB \in \pi(CCR(\mathcal{H}_0^\perp))''E\,.$$

(because EAE is a multiple of E.) Hence

$$E\pi(CCR(\mathcal{H}))''E \subset \pi(CCR(\mathcal{H}_0^\perp))''E\,.$$

This combined with (9.7) yields (9.6).

We say that $(\mathcal{H}_\alpha)$ is an absorbing net of subspaces if

(i) For every α $\mathcal{H}_\alpha$ is a finite dimensional complex linear subspace of $\mathcal{H}$.

(ii) For every α and β there is a γ such that $\mathcal{H}_\alpha, \mathcal{H}_\beta \subset \mathcal{H}_\gamma$.

(iii) $\bigcup_\alpha \mathcal{H}_\alpha$ is dense in $\mathcal{H}$.

Proposition 9.6 *Let $(\mathcal{H}_\alpha)$ be an absorbing net of subspaces and ψ_1, ψ_2 factor states of $CCR(\mathcal{H})$. Assume that π_{ψ_1} and π_{ψ_2} are continuous. Then ψ_1 and ψ_2 are quasi-equivalent if and only if for every $\varepsilon > 0$ there is an α such that*

$$\|(\psi_1 - \psi_2)|CCR(\mathcal{H}_\alpha^\perp)\| \le \varepsilon\,. \tag{9.8}$$

Consider $\pi = \pi_{\psi_1} \oplus \pi_{\psi_2}$ acting on the Hilbert space $\mathcal{K}_1 \oplus \mathcal{K}_2$. Since ψ_1 and ψ_2 are factor states the centre of the von Neumann algebra $\mathcal{M} = \pi(CCR(\mathcal{H}))''$ is contained in $\mathbb{C}\oplus\mathbb{C}$. It is well-known that $\psi_1 \approx \psi_2$ if and only if $\mathcal{M}$ is a factor (see [**Di**], 5.3).

Set $\overline{\psi}_1$ and $\overline{\psi}_2$ for the vector states on $\mathcal{M}$ generated by $\Psi_1 \oplus 0$ and $0 \oplus \Psi_2$. Assume that (9.8) is not true. Then

$$L_\alpha = \left\{x \in \mathcal{M} \cap \pi(CCR(\mathcal{H}_\alpha^\perp))'' \,:\, \|x\| \le 1\,,\ |(\overline{\psi}_1 - \overline{\psi}_2)(x)| \ge \varepsilon\right\}$$

is a weakly compact nonempty subset of $\mathcal{M}$. There exists

$$Z \in \bigcap_\alpha L_\alpha\,.$$

Since

$$[Z, \pi(CCR(\mathcal{H}_\alpha))] = 0$$

for every α, Z must be in the centre of $\mathcal{M}$. This yields that the centre is actually $\mathbb{C} \oplus \mathbb{C}$ and $\mathcal{M}$ is not a factor.

The centre of $\mathcal{M}$ is

$$\bigcap_\alpha \pi(CCR(\mathcal{H}_\alpha))' \cap \mathcal{M}\,.$$

Due to Lemma 9.5 this equals to

$$\bigcap_\alpha \pi(CCR(\mathcal{H}_\alpha^\perp))''\,.$$

To prove the converse we assume that $\mathcal{M}$ is not a factor. Then $Z_1 = 1 \oplus (-1)$ belongs to the von Neumann algebra

$$\pi(CCR(\mathcal{H}_\alpha^\perp))''$$

for every α. Then

$$\begin{aligned} \|(\psi_1 - \psi_2)|CCR(\mathcal{H}_\alpha^\perp)\| &= \|(\overline{\psi}_1 - \overline{\psi}_2)|\pi(CCR(\mathcal{H}_\alpha^\perp))''\| \geq |\psi_1(I) - \psi_2(I)| \\ &= 2 \end{aligned}$$

and (9.8) does not hold.

We return now to the Fock states φ and φ_T given in (9.4) and (9.5).

Lemma 9.7 *If φ and φ_T are quasi-equivalent then $I - T$ is compact.*

For a given $\varepsilon > 0$ Proposition 9.6 provides a finite dimensional subspace $\mathcal{H}_0$ such that

$$\left| \exp\left(-\frac{1}{2}(h,h)\right) - \exp\left(-\frac{1}{2}(T_2 h, h)\right)\right| \leq \varepsilon \tag{9.9}$$

whenever $h \in \mathcal{H}_0^\perp$. (Note that π_{φ_T} is continuous thanks to the boundedness of T and T^{-1}.) We deduce from (9.9) that

$$\left| 1 - \exp\left(-\frac{1}{2}((I_2 - T_2)h, h)\right)\right| \leq C\varepsilon$$

for $h \in \mathcal{H}_0^\perp$ with $\|h\| \leq 1$. Using the continuity of the exponential function one gets

$$|((I_2 - T_2)h, h)| \leq \delta(\varepsilon) \tag{9.10}$$

if $h \in \mathcal{H}_0^\perp$ and $\|h\| \le 1$. In the identity

$$\Delta - E_0^\perp \Delta E_0 - E_0 \Delta E_0^\perp - E_0 \Delta E_0 = E_0^\perp \Delta E_0^\perp$$

we choose $\Delta = I_2 - T_2$ and E_0 is the orthogonal projection onto $\mathcal{H}_0$. We infer from (9.10) that $I_2 - T_2$ may be approximated in operator norm arbitrary close by finite rank operators. This yields that $I_2 - T_2$ is compact as well as $I - T$ and $I - T^{-1}$.

Let $\{e_1, e_2, \dots\}$ be a basis of the real Hilbert space H such that every e_k is an eigenvector of T. Set $\mathcal{H}_n^m$ for the complex linear subspace of $\mathcal{H}$ spanned by the vectors $\{e_n, e_{n+1}, \dots, e_m\}$ $(n < m)$. Lemma 9.1 tells us that

$$P_n^m = \frac{1}{\pi^k} \int_{\mathcal{H}_n^m} \varphi(W(g)) W(g) dg \qquad (k = m - n + 1)$$

is a projection in $CCR(\mathcal{H}_n^m)$. Lemma 9.2 gives that

$$P_n^m \le P_{n'}^{m'}$$

if $[m, n] \supset [m', n']$.

$\varphi(P_n^m) = 1$ and we compute $\varphi_T(P_n^m)$.

$$\begin{aligned} \varphi_T(P_n^m) &= \pi^{-k} \int \exp\left(-\frac{1}{2}\langle (I_2 + T_2)h, h\rangle\right) dh \\ &= \prod_{i=n}^m \left(\frac{1+\lambda_i}{2}\right)^{-1/2} \left(\frac{1+\lambda_1^{-1}}{2}\right)^{-1/2} \\ &= \prod_{i=n}^m \left(1 + \frac{\mu_i}{2}\right)^{-1/2} \end{aligned} \tag{9.11}$$

where

$$\mu_i = \frac{\lambda_i + \lambda_i^{-1} - 2}{2}.$$

Theorem 9.8 *The Fock states φ and φ_T of $CCR(\mathcal{H}, \sigma)$ (defined by (9.4) and (9.5)) are quasi-equivalent if and only if the operator $I - T$ is in the Hilbert-Schmidt class.*

Assume that $\varphi \approx \varphi_T$. Then Lemma 9.7 tells us that $I - T$ is compact. $(\mathcal{H}_1^m)_m$ is an absorbing net of finite dimensional subspaces. Due to Proposition 9.6

$$\|(\varphi - \varphi_T)|CCR(\mathcal{H}_n^m)\|$$

can be arbitrary small if n big enough. Since

$$\|(\varphi - \varphi_T)|CCR(\mathcal{H}_n^m)\| \geq |(\varphi - \varphi_T)(P_n^m)| = 1 - \varphi_T(P_n^m)$$

we have

$$\varphi_T(P_n^m) \geq \frac{1}{1+\varepsilon} \tag{9.12}$$

for big n and $m > n$. The next elementary inequality may be varified easily.

$$1 + \frac{1}{2}\Sigma\mu_i \leq \Pi\left(1 + \frac{\mu_i}{2}\right) \leq \exp\left(\frac{1}{2}\Sigma\mu_i\right) \tag{9.13}$$

Combination of (9.11), (9.12) and (9.13) yields that

$$1 + \frac{1}{2}\sum_{i=n}^{m}\mu_i \leq \varphi_T(P_n^m)^{-2} \leq (1+\varepsilon)^2$$

if n is big enough. This shows that $\sum_{i=1}^{\infty}\mu_i$ is convergent. Since

$$2m\mu_i \leq (1-\lambda_i)^2 \leq 2M\mu_i \tag{9.14}$$

for $M \geq \lambda_i \geq m$ we conclude that $\sum_{i=1}^{\infty}(1-\lambda_i)^2$ is convergent, or in other words, $I - T$ is a Hilbert-Schmidt operator.

Before proving the converse we state a lemma.

Lemma 9.9 *If ω_1 and ω_2 are states of $B(\mathcal{K})$ given by vectors ξ_1 and ξ_2 then*

$$\|\omega_1 - \omega_2\| \leq 2(1 - |\langle\xi_1, \xi_2\rangle|^2)^{1/2}.$$

Let P_i be the projection onto $\mathbb{C}\xi_i$ $(i = 1, 2)$. It is easy to compute that $P_1 - P_2$ has spectral decomposition

$$\lambda Q_1 - \lambda Q_2 \quad \text{with} \quad \lambda = (1 - |\langle \xi_1, \xi_2 \rangle|^2)^{1/2}.$$

Hence

$$|\omega_1(A) - \omega_2(A)| = |\lambda\, Tr\, AQ_1 - \lambda\, Tr\, AQ_2| \le 2\lambda \|A\|.$$

Now we go back to the proof of Theorem 9.8. Let $(\pi, \Omega, \mathcal{K})$ be the GNS-triplet associated with $CCR(\mathcal{H}_n^m)$ and $\varphi_T|CCR(\mathcal{H}_n^m)$. It follows from Lemma 9.1 that

$$\varphi(W(h)) = \varphi_T(P_n^m)^{-1} \langle \pi(W(h))\pi(P_n^m)\Omega, \pi(P_n^m)\Omega \rangle.$$

Lemma 9.9 is applicable and yields

$$\|(\varphi - \varphi_T)|CCR(\mathcal{H}_m^n)\| \le 2(1 - \varphi_T(P_n^m))^{1/2}. \tag{9.15}$$

Suppose that $I - T$ is Hilbert-Schmidt. Then $\Sigma \mu_i < +\infty$ thanks to (9.14). We are going to benefit from Proposition 9.6 and to verify condition (9.8). On the one hand

$$\begin{aligned} \|(\varphi - \varphi_T)|CCR(\mathcal{H}_n^\perp)\|^2 &\le \limsup_{m\to\infty} \|(\varphi - \varphi_T)|CCR(\mathcal{H}_n^m)\|^2 \\ &\le \limsup_{m\to\infty} 4(1 - \varphi_T(P_n^m)) \\ &= \limsup_{m\to\infty} 4\left(1 - \sum_{i=n}^{m}\left(1 + \frac{\mu_i}{2}\right)^{-1/2}\right) \end{aligned}$$

and on the other hand from (9.13)

$$1 - \sum_{i=n}^{m}\left(1 + \frac{\mu_i}{2}\right)^{-1/2} \le 1 - \exp\left(-\frac{1}{4}\sum_{i=n}^{m}\mu_i\right).$$

Here the right hand side is arbitrary small when n is large enough. The proof of Theorem 9.8 is complete.

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space. Let A be a real linear bounded operator on $\mathcal{H}$ such that A^{-1} is bounded and

$$Im\langle f, g\rangle = Im\langle Af, Ag\rangle.$$

So A is symplectic with respect to the standard symplectic structure of $\mathcal{H}$ and there is a (Bogoliubov) automorphism α_A of $CCR(\mathcal{H})$ such that

$$\alpha_A(W(f)) = W(Af) \qquad (f \in \mathcal{H}) .$$

We denote by φ the standard Fock state on $CCR(\mathcal{H})$ and set

$$\varphi_A(W(h)) = \exp\left(-\frac{1}{2}\|Ah\|^2\right) = \varphi(\alpha_A(W(h)) .$$

So φ_A is another Fock state (sometimes called squeezed Fock state). Denoting by π the Fock representation associated with φ and acting on $\mathcal{K}$ we are ready to state and to prove Shale's theorem [**Sh**].

Theorem 9.10 *The following conditions are equivalent.*

(i) $A^*A - I$ *is in the Hilbert-Schmidt class.*

(ii) *The states* φ *and* φ_A *on* $CCR(\mathcal{H})$ *are quasi-equivalent.*

(iii) *There exists a unitary operator* U *on* $\mathcal{K}$ *such that* $\pi(\alpha_A(W(h)) = U^*\pi(W(h))U$ *for every* $h \in \mathcal{H}$.

We note that A^* is the adjoint of A with respect to the real inner product on $\mathcal{H}$. We may assume that $\mathcal{H} = L^2(0,1)$. Let H be the real part of $L^2(0,1)$, then $\mathcal{H} = H \oplus H$. Every linear operator on $\mathcal{H}$ may be given by a 2×2 matrix with entries in $B(H)$. If

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{9.16}$$

then A is symplectic if and only if

$$A^*JA = J \tag{9.17}$$

and a real linear operator on $H \oplus H$ is complex linear on $\mathcal{H}$ if and only if it commutes with J. Let

$$A = V_1 H_1$$

be the polar decomposition of A. From (9.17) by the uniqueness of the polar decomposition we obtain

$$JV_1 = V_1 J \quad \text{and} \quad H_1 = J^{-1} H_1^{-1} J\,.$$

Hence V_1 is complex linear and analysis of the second condition gives a unitary V_2 such that

$$JV_2 = V_2 J \quad \text{and} \quad H_1 = V_2 L V_2^*$$

where L is diagonal in the decomposition $H \oplus H$ and has the form

$$L = \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}$$

where $T \geq 0$ has a bounded inverse. It follows from the decomposition

$$A = V_1 V_2 L V_2^*$$

that $\varphi_A \approx \varphi_L$. Therefore, $\varphi \approx \varphi_A$ if and only if $\varphi \approx \varphi_L$. On the other hand, $A^*A - I = V_2(L^2 - I)V_2^*$ and $A^*A - I$ is Hilbert-Schmidt if and only if $L^2 - I$ is so. It is not difficult to see that $L^2 - I$ is in the Hilbert-Schmidt class if and only if $T - I$ is Hilbert-Schmidt. This way we have succeeded in reducing the equivalence of (i) and (ii) to Theorem 9.8.

The equivalence of (ii) and (iii) is based on the fact that φ and φ_A are pure states. Set $(\pi_1, \mathcal{H}_1, \Phi_1)$ for the GNS-triplet of φ_A. Assume that there exists a unitary $V : \mathcal{K} \to \mathcal{K}_1$ such that

$$\pi(W(h)) = V^* \pi_1(W(h)) V \qquad (h \in \mathcal{H})\,.$$

Then we have

$$\varphi_A(W(h)) = \langle \pi(W(h))\Omega, \Omega \rangle \qquad (h \in \mathcal{H})$$

for $\Omega = V^*\Phi_1$. Define a unitary by the formula

$$U : \pi(W(h))\Omega \mapsto \pi(W(A^{-1}h))\Phi$$

and simple computation shows the U implements the automorphism φ_A. The inverse implication (iii) $\to$ (ii) is trivial.

The study of quasi-equivalence of non-Fock quasifree states may go through the purification as it is done in the papers [**VD**] and [**Ho1**].

We do not enter this subject but give another proof of the fact that on an infinite dimensional Hilbert space $\mathcal{H}$ the states φ_λ in (9.1) are not quasi-equivalent.

More generally, let A and B be bounded operators on $\mathcal{H}$ such that $A, B \geq (1+\varepsilon)I$ for some $\varepsilon > 0$. Consider the quasifree states

$$\varphi_A(W(h)) = \exp\left(-\frac{1}{2}\langle Ah, h\rangle\right), \ \varphi_B(W(h)) = \exp\left(-\frac{1}{2}\langle Bh, h\rangle\right)$$

on $CCR(\mathcal{H})$. Using Theorem 6.1 one can get that φ_A and φ_B are factor states. (A nontrivial central element gives rise to another KMS-state for the appropriate group.) We are in a position to apply Proposition 9.6 and can use the argument in Lemma 9.7. Instead of (9.9) we have

$$\left|\exp\left(-\frac{1}{2}\langle Ah, h\rangle\right) - \exp\left(-\frac{1}{2}\langle Bh, h\rangle\right)\right| \leq \varepsilon \tag{9.18}$$

whenever $h \in \mathcal{H}_0^\perp$. The scalar product

$$\langle h_1, h_2\rangle_B = \langle Bh_1, h_2\rangle$$

is equivalent with the original one and from (9.18) we obtain

$$\langle (I - B^{-1/2}AB^{-1/2})h, h\rangle \leq \delta(\varepsilon)$$

whenever $h \in \mathcal{H}_0^\perp$. Hence the assumption $\varphi_A \approx \varphi_B$ yields that $I - B^{-1/2}AB^{-1/2}$ and so $B - A$ must be compact. This is enough to see that φ_λ and φ_μ are not quasi-equivalent if $\lambda > \mu > 1$.

Chapter 10

The selfdual approach

In this section we present the selfdual approach to the algebra of the canonical commutation relation. This formalism was developed mainly by Araki ([**Ar2**],[**A-S**]) and lays the emphasis on creation and annihilation operators rather than on Weyl unitaries.

A triplet (K, γ, Γ) is called phase space if K is a complex linear space, γ is a symmetric sesquilinear form and Γ is a conjugate linear involution (i.e., $\Gamma^2 = id$) such that

$$\gamma(\Gamma f, \Gamma g) = -\gamma(g, f) \qquad (f, g \in K)\,. \tag{10.1}$$

If (K, γ, Γ) is a phase space then on the real linear subspace $\{f \in K : \Gamma f = f\}$ γ is a symplectic form (it may be degenerate). On the other hand, if (H, σ) is a symplectic space then $H \oplus H$ may be endowed with a complex linear structure by letting

$$i(f, g) = (-g, f)\,.$$

The operator $\Gamma(f, g) = (f, -g)$ is an involution and σ extends to a symmetric form γ on $H \oplus H$. Hence phase space and symplectic space are somewhat close notions.

The CCR-algebra $\mathcal{A}(K, \gamma, \Gamma)$ over the phase space (K, γ, Γ) is the quotient of the free *-algebra generated by $a(f)$, its conjugate $a^*(f)$ and the identity over the two-sided $*$-ideal generated by the following relations :

(i) $a(f)$ is complex linear in f,

(ii) $a(f)a^*(g) - a^*(g)a(f) = -\gamma(f,g)$,

(iii) $a(\Gamma f) = a^*(f)$.

A linear mapping U of K into itself satisfying $\gamma(Uf, Ug) = \gamma(f,g)$ and $\Gamma U = U\Gamma$ preserves the relations (i)-(iii) and there exists a $*$-endomorphism $\tau(U)$ of $\mathcal{A}(K,\gamma,\Gamma)$ such that $\tau(U)a(f) = a(Uf)$ holds.

An operator P on K satisfying

(i) $P^2 = P$

(ii) $\gamma(f, Pf) > 0$ if $Pf \neq 0$

(iii) $\gamma(Pf, g) = \gamma(f, Pg)$

(iv) $\Gamma P \Gamma = 1 - P$

is called a basis projection. Let P be a basis projection and consider the complex linear subspace $\mathcal{H} = \mathrm{rng}\, P$. It follows from property (ii) that γ is a separating inner product on $\mathcal{H}$. So we may consider the C^*-algebra $CCR(\mathcal{H}, Im\,\gamma)$ over the symplectic space $(\mathcal{H}, Im\,\gamma)$ and we let it act in the standard Fock representation where creation and annihilation operators appear. Denote by $\Gamma(\mathcal{H})$ the Fock space. For $f \in K$ we define an application α by the formula

$$\alpha(a(f)) = B^+(Pf) + B^-(P\Gamma f)\,. \tag{10.2}$$

Since

$$\begin{aligned}
&\big[B^+(Pf) + B^-(P\Gamma f), B^+(P\Gamma g) + B^-(Pg)\big]\\
&\quad = -\langle Pf, Pg\rangle + \langle P\Gamma g, P\Gamma f\rangle\\
&\quad = -\langle Pf, g\rangle - \langle (1-P)f, g\rangle\\
&\quad = -\langle f, g\rangle
\end{aligned}$$

and

$$\alpha(a^*(f)) = \alpha(a(\Gamma f)) = \alpha(a(f))^*$$

α extends to a *-algebra homomorphism of $\mathcal{A}(K,\gamma,\Gamma)$.

When $\mathcal{A}$ is a *-algebra then by a state φ of $\mathcal{A}$ we mean a linear functional such that $\varphi(I) = 1$ and $\varphi(x^*x) \geq 0$ for every $x \in \mathcal{A}$. A state φ on $\mathcal{A}(K,\gamma,\Gamma)$ is called quasifree if

(i) $\varphi(a(f_1)a(f_2)\dots a(f_{2k+1}))=0 \qquad (k\in\mathbb{Z}^+, f_i\in K)$

(ii) $\varphi(a(f_1)a(f_2)\dots a(f_{2k}))=\sum\prod_{m=1}^{k}\varphi(a(f_{j_m})a(f_{l_m}))$
$(k\in\mathbb{N},\ f_i\in K)$

where the summation is over all partitions $\{H_1,H_2,\dots,H_k\}$ of $\{1,2,\dots,2k\}$ such that $H_m=\{j_m,l_m\}$ with $j_m<l_m$.

Proposition 10.1 *If P is a basis projection over the phase space (K,γ,Γ) then there exists a unique quasifree state φ on $\mathcal{A}(K,\gamma,\Gamma)$ such that*

$$\varphi(a^*(g)a(f))=\gamma(Pf,g)\,. \tag{10.3}$$

The uniqueness is obvious. To establish the existence we use the $*$-algebra homomorphism α defined above. Setting

$$\varphi(x)=\langle\alpha(x)\Phi,\Phi\rangle \qquad (x\in\mathcal{A}(K,\gamma,\Gamma)) \tag{10.4}$$

we obtain a state on $\mathcal{A}(K,\gamma,\Gamma)$. ($\phi$ is the vacuum vector here.) It follows from Proposition 3.8 and simple properties of the Fock states that φ is a quasifree state in the above sense and its two-point function is really given by (10.3).

It is worthwile to point out that the basis projection P determines that $a(f)$ becomes creation or annihilation operator in the corresponding representation.

Let s be a positive hermitian form on K such that

$$s(f,g)-s(\Gamma g,\Gamma f)=\gamma(f,g)\,. \qquad (f,g\in K) \tag{10.5}$$

Then s is called a polarization of γ. Observe that if P is a basis projection then

$$s_P(f,g)=\gamma(Pf,g)$$

is a polarization.

Lemma 10.2 *If s is a polarization of γ then*

$$\langle g,f\rangle_s=s(g,f)+s(\Gamma f,\Gamma g)$$

is a positive form such that

$$|\gamma(g,f)|^2 \le \langle f,f\rangle_s \langle g,g\rangle_s \tag{10.6}$$

holds.

We estimate simply as follows

$$\begin{aligned} |\gamma(g,f)| &\le |\langle g,f\rangle|_s + |\langle \Gamma f,\Gamma g\rangle|_s \\ &\le \langle g,g\rangle_s^{1/2}\langle f,f\rangle_s^{1/2} + \langle \Gamma f,\Gamma f\rangle_s^{1/2}\langle \Gamma g,\Gamma g\rangle_s^{1/2} \\ &\le (\langle g,g\rangle_s + \langle \Gamma g,\Gamma g\rangle_s)^{1/2}\,(\langle f,f\rangle_s + \langle \Gamma f,\Gamma f\rangle_s)^{1/2} \\ &= \langle g,g\rangle_s^{1/2}\langle f,f\rangle_s^{1/2}. \end{aligned}$$

Proposition 10.3 *Let (K,γ,Γ) be a nondegenerate phase space and let $s(\cdot,\cdot)$ be its polarization. Then there exists a quasifree state φ on $\mathcal{A}(K,\gamma,\Gamma)$ such that*

$$\varphi(a^*(g)a(f)) = s(f,g) \qquad (f,g\in K).$$

We perform a doubling procedure and reduce the problem to Proposition 10.1.

Since γ is assumed to be nondegenerate, Lemma 10.2 gives that $\langle\cdot,\cdot\rangle_s$ is an inner product. By completion with respect to $\langle\cdot,\cdot\rangle_s$ we obtain (K',γ',Γ'). There is a contraction S on K' such that

$$\gamma(f,g) = \langle Sf,g\rangle_s \qquad (f,g\in K').$$

We have

$$S^* = S \quad \text{and} \quad \Gamma' S \Gamma' = -S. \tag{10.7}$$

Set

$$K'' = K'\oplus K' \ , \ \Gamma'' = \Gamma'\oplus\Gamma'$$

and

$$\gamma''(f_1\oplus f_2, g_1\oplus g_2) = \langle Sf_1,g_1\rangle_s - \langle Sf_2,g_2\rangle_s.$$

In this way we arrive at the doubled phase space (K'',γ'',Γ''). Since

$$s(f,f) = \langle f,f\rangle_s - s(\Gamma f,\Gamma f) \le \langle f,f\rangle_s$$

there is a positive contraction $\overline{S}$ on K' such that

$$s(f,g) = \langle \overline{S} f, g\rangle \qquad (f,g \in K')\,.$$

It satisfies also the conditions

$$\Gamma' S \Gamma' = I - \overline{S} \quad \text{and} \quad \overline{S} - \Gamma'\overline{S}\Gamma' = S\,. \tag{10.8}$$

Set

$$\begin{aligned}\langle f_1 \oplus f_2, g_1 \oplus g_2\rangle &= \langle f_1, g_1\rangle_s + \langle f_2, g_2\rangle_s \\ &\quad +2\langle \overline{S}^{1/2}(I-\overline{S})^{1/2} f_1, g_2\rangle_s \\ &\quad +2\langle \overline{S}^{1/2}(I-\overline{S})^{1/2} f_2, g_1\rangle_s\,.\end{aligned}$$

This form is Γ''-invariant because $\langle\cdot,\cdot\rangle_s$ is Γ'-invariant and

$$\Gamma'\overline{S}^{1/2}\Gamma' = (I-\overline{S})^{1/2}\,.$$

The identity

$$\langle f_1 \oplus f_2, f_1 \oplus f_2\rangle = \|\overline{S}^{1/2} f_1 + (I-\overline{S})^{1/2} f_2\|_s^2 + \|\overline{S}^{1/2} f_2 + (I-\overline{S})^{1/2} f_1\|_s^2$$

shows that $\langle\cdot,\cdot\rangle$ is nonnegative. Assume that

$$\langle f_1 \oplus f_2, f_1 \oplus f_2\rangle = 0\,.$$

Short computation yields that this implies

$$(2\overline{S} - I) f_i = 0 \qquad (i = 1,2)\,.$$

We get from (10.8) that $f_i \in \mathcal{K}er\, S = \{0\}$ and $\langle\cdot,\cdot\rangle$ is an inner product.

We deduce from the relation

$$\begin{aligned}\gamma''(f_1 \oplus f_2, g_1 \oplus g_2) &= \langle \overline{S}^{1/2} f_1 + (I-\overline{S})^{1/2} f_2, \overline{S}^{1/2} g_1 + (I-\overline{S})^{1/2} g_2\rangle_s \\ &\quad -\langle (I-\overline{S})^{1/2} f_1 + \overline{S}^{1/2} f_2, (I-\overline{S})^{1/2} g_1 + \overline{S}^{1/2} g_2\rangle_s\end{aligned}$$

that

$$|\gamma''(f_1 \oplus f_2, g_1 \oplus g_2)| \le \|f_1 \oplus f_2\| \cdot \|g_1 \oplus g_2\|\,.$$

We may assume that $(K'', \langle\cdot,\cdot\rangle)$ is a complete inner product space. Let H_1 and H_2 be the closed subspaces spanned by the vectors

$$\left\{\overline{S}^{1/2} f \oplus -(I-\overline{S})^{1/2} f \,:\, f\in K'\right\}, \left\{(I-\overline{S})^{1/2} f \oplus -\overline{S}^{1/2} f \,:\, f\in K'\right\},$$

respectively. One can check that $K'' = H_1 \oplus H_2$, moreover H_1 and H_2 are orthogonal with respect to γ''. The operator T given by

$$\gamma''(f,g) = \langle Tf, g\rangle \qquad (f,g\in K'')$$

is diagonal in the decomposition $H_1 \oplus H_2$. In fact,

$$Tf_1 = f_1 \quad (f_1\in H_1) \quad \text{and} \quad Tf_2 = -f_2 \quad (f_2 \in H_2)$$

and $P_s = \frac{1}{2}(T+I)$ is a projection. It is straightforward to verify that P_s is a basis projection. Proposition 10.1 tells us that there is a quasifree state φ'' on $(K'', \gamma'', \Gamma'')$ such that

$$\varphi''(a^*(g)a(f)) = \gamma''(P_s f, g)\,.$$

The embedding $f_1 \mapsto f_1 \oplus 0$ of (K,γ,Γ) into (K'',γ'',Γ'') gives rise to an embedding α of $\mathcal{A}(K,\gamma,\Gamma)$ into $\mathcal{A}(K'',\gamma'',\Gamma'')$. $\varphi = \varphi''\circ\alpha$ is a quasifree state on $\mathcal{A}(K,\gamma,\Gamma)$. The only thing remained to show is

$$\gamma''(P_s(f\oplus 0),(g\oplus 0)) = s(f,g) \qquad (f,g\in K)\,.$$

However,

$$\begin{aligned}
\gamma''(P_s(f\oplus 0),(g\oplus 0)) &= \langle \frac{1}{2}T(I+T)(f\oplus 0),(g\oplus 0)\rangle \\
&= \frac{1}{2}\gamma''(f\oplus 0, g\oplus 0) + \frac{1}{2}\langle f\oplus 0, g\oplus 0\rangle \\
&= \frac{1}{2}\gamma(f,g) + \frac{1}{2}\langle f,g\rangle_s \\
&= \frac{1}{2}\gamma(f,g) + \frac{1}{2}s(f,g) + \frac{1}{2}s(\gamma g, \Gamma f)
\end{aligned}$$

which is $s(f,g)$ due to (10.5).

For degenerate γ the doubling is slightly more complicated but also this case is treated in the original paper [**A-S**]. The quasi-equivalence of quasifree states in a very general frame is contained in [**A-Y**].

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