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Abstract: This paper studies the Bondi–Metzner–Sachs group in homogeneous projective coordinates because it is then possible to write all transformations of such a group in a manifestly linear way. The 2-sphere metric, the Bondi–Metzner–Sachs metric, asymptotic Killing vectors, generators of supertranslations as well as boosts and rotations of Minkowski spacetime are all re-expressed in homogeneous projective coordinates. Lastly, the integral curves of vector fields which generate supertranslations are evaluated in detail. This work paves the way for more advanced applications of the geometry of asymptotically flat spacetime in projective coordinates by virtue of the tools provided from complex analysis in several variables and projective geometry.

Keywords: asymptotic symmetries; Bondi-Metzner-Sachs group; projective geometry

1. Introduction

The Bondi–Metzner–Sachs [1–3] asymptotic symmetry group of asymptotically flat spacetime has received much attention over the last decade by virtue of its relevance to black-hole physics [4–6], the group-theoretical structure of general relativity [7–20] and the infrared structure of fundamental interactions [21–25]. Moreover, since asymptotic symmetries can provide key constraints on the celestial dual to quantum gravity in flat spacetime, much work has been devoted to the celestial holography program and related issues [26–32].

The appropriate geometric framework can be summarized as follows. In spacetime models for which null infinity can be defined, the cuts of null infinity are spacelike two-dimensional surfaces orthogonal to the generators of null infinity [33]. Using the familiar stereographic coordinate

$$\zeta = e^{i\varphi} \cot \frac{\theta}{2}, \tag{1}$$

the first half of Bondi–Metzner–Sachs transformations read as

$$\zeta' = f(\zeta) = \frac{(a\zeta + b)}{(c\zeta + d)} = f_\Lambda(\zeta), \tag{2}$$

where the matrix $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has unit determinant $(ad - bc) = 1$ and belongs therefore to the group $SL(2, \mathbb{C})$. The resulting projective version of the special linear group can be defined as the space of pairs

$$PSL(2, \mathbb{C}) = \{(f, \Lambda) \mid f : \zeta \in \mathbb{C} \to f_\Lambda(\zeta), \Lambda \in SL(2, \mathbb{C})\}, \tag{3}$$

i.e., the group of fractional linear maps f_Λ according to Equation (2) with the associated matrix Λ. Since

$$f_\Lambda(\zeta) = \frac{(a\zeta + b)}{(c\zeta + d)} = \frac{(-a\zeta - b)}{(-c\zeta - d)} = f_{-\Lambda}(\zeta), \tag{4}$$

one can write that $\mathrm{PSL}(2, \mathbb{C})$ is the quotient space $\mathrm{SL}(2, \mathbb{C})/\delta$, where δ is the homeomorphism defined by

$$\delta(a, b, c, d) = (-a, -b, -c, -d). \tag{5}$$

The fractional linear maps (2) can be defined for all values of ζ upon requiring that

$$f_\Lambda(\infty) = \frac{a}{c}, \ f_\Lambda\left(-\frac{d}{c}\right) = \infty. \tag{6}$$

Moreover, using fractional linear maps, lengths along the generators of null infinity scale according to

$$du' = K_\Lambda(\zeta) du, \tag{7}$$

where the conformal factor is given by [19,33]

$$K_\Lambda(\zeta) = \frac{1 + |\zeta|^2}{|a\zeta + b|^2 + |c\zeta + d|^2}. \tag{8}$$

By integration, Equation (7) yields the second half of Bondi–Metzner–Sachs transformations:

$$u' = K_\Lambda(\zeta)\left[u + \alpha(\zeta, \overline{\zeta})\right]. \tag{9}$$

As was pointed out in Ref. [19], the complex homogeneous coordinates associated with the Bondi–Metzner–Sachs transformation (2) have modulus ≤ 1, which is the equation of a unit circle, and are

$$z_0 = e^{\frac{i\varphi}{2}} \cos\frac{\theta}{2}, \ z_1 = e^{-i\frac{\varphi}{2}} \sin\frac{\theta}{2}. \tag{10}$$

In other words, given that

$$\zeta = \frac{z_0}{z_1}, \tag{11}$$

Equation (2) is equivalent to the linear transformation law

$$\begin{pmatrix} z_0' \\ z_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}. \tag{12}$$

The next step of the program initiated in Ref. [19] consists of realizing that, much in the same way as the affine transformations in the Euclidean plane

$$x' = x + a, \ y' = y + b, \tag{13}$$

can be re-expressed with the help of a 3×3 matrix in the form

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + a \\ y + b \\ 1 \end{pmatrix}, \tag{14}$$

one can further re-express Equation (12) with the help of a 3×3 matrix in the form

$$\begin{pmatrix} w_0' \\ w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix}, \tag{15}$$

with the understanding that Equation (12) is the restriction to the unit circle Γ of the map (15), upon defining

$$w_0|_\Gamma = 1, \ w_1|_\Gamma = z_0, \ w_2|_\Gamma = z_1. \tag{16}$$

Within this extended framework, one can consider two complex projective planes [19]. Let P be a point of the first plane with coordinates (w_0, w_1, w_2), and let P' be a point of the second plane, with coordinates (u_0, u_1, u_2). One can now consider the nine products between a complex coordinate of P and a complex coordinate of P', i.e.,

$$Z_{hk} = w_h u_k, \ h, k = 0, 1, 2. \tag{17}$$

This equation provides the coordinate description of the *Segre manifold* [34,35], which is the projective image of the product of projective spaces. It is a natural tool for accommodating the transformations that reduce to the BMS transformations upon restriction to the unit circle Γ. It contains a complex double infinity of planes, two arrays of planes and a complex fourfold infinity of quadrics [19,34], but its differential geometry is still largely unexplored, as far as we know.

Unlike Ref. [19], we have a more concrete task: since the Bondi–Metzner–Sachs transformation (2) becomes linear when expressed in terms of z_0 and z_1, we are aiming to develop the Bondi–Metzner–Sachs formalism with the associated Killing vector fields by using the pair of variables (z_0, z_1) instead of $(\zeta, \overline{\zeta})$. For this purpose, the homogeneous projective coordinates for the 2-sphere are studied in Section 2, while the Bondi–Sachs metric in homogeneous coordinates is considered in Section 3. Asymptotic Killing fields for supertranslations are evaluated in Section 4, while their flow is investigated in Section 5. Concluding remarks and open problems are presented in Section 6, while technical details are provided in the Appendices A and B.

2. Homogeneous Coordinates on the 2-Sphere

It is useful to developing the BMS formalism as an instrument in homogeneous coordinates to re-write the 2-sphere metric in the desired coordinates. By using the definition (10), we obtain

$$z_0 z_1 = \sin(\theta/2)\cos(\theta/2) = \frac{\sin(\theta)}{2} \Rightarrow \theta = \sin^{-1}(2z_0 z_1), \tag{18}$$

while for φ, we obtain

$$\frac{z_0}{z_1} = e^{i\varphi}\cot(\theta/2) \Rightarrow \varphi = -i\,\log\left(\tan(\theta/2)\frac{z_0}{z_1}\right). \tag{19}$$

By virtue of the identity

$$\tan(\theta/2) = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = \frac{\sin(\theta)}{1 + \sqrt{1 - \sin^2(\theta)}}, \tag{20}$$

we obtain for φ the more convenient expression

$$\varphi = -i\,\log\left(\frac{2z_0^2}{1 + \sqrt{1 - 4z_0^2 z_1^2}}\right).$$

In order to re-express the 2-sphere metric, let us evaluate

$$
\begin{aligned}
d\theta^2 &= \left(\frac{\partial\theta}{\partial z_0}\right)^2 dz_0^2 + \left(\frac{\partial\theta}{\partial z_1}\right)^2 dz_1^2 + 2\frac{\partial\theta}{\partial z_0}\frac{\partial\theta}{\partial z_1}dz_0 dz_1 \\
&= \frac{4z_1^2}{1 - 4z_0^2 z_1^2}dz_0^2 + \frac{4z_0^2}{1 - 4z_0^2 z_1^2}dz_1^2 + \frac{8z_0 z_1}{1 - 4z_0^2 z_1^2}dz_0 dz_1,
\end{aligned} \tag{21}
$$

while

$$\sin^2(\theta)\,d\varphi^2 = 4z_0^2 z_1^2\,d\varphi^2 = 4z_0^2 z_1^2 \left\{ \left(\frac{\partial\varphi}{\partial z_0}\right)^2 dz_0^2 + \left(\frac{\partial\varphi}{\partial z_1}\right)^2 dz_1^2 + 2\frac{\partial\varphi}{\partial z_0}\frac{\partial\varphi}{\partial z_1}dz_0 dz_1 \right\}$$

$$= \frac{-16z_1^2\left(1 - 2z_0^2 z_1^2 + \sqrt{1 - 4z_0^2 z_1^2}\right)^2}{\left(1 - 4z_0^2 z_1^2 + \sqrt{1 - 4z_0^2 z_1^2}\right)^2}\,dz_0^2 - \frac{64z_0^6 z_1^4}{\left(1 - 4z_0^2 z_1^2 + \sqrt{1 - 4z_0^2 z_1^2}\right)^2}\,dz_1^2$$

$$- \frac{32z_0^3 z_1^3}{1 - 4z_0^2 z_1^2}\,dz_0 dz_1. \tag{22}$$

Eventually, we obtain the metric for the 2-sphere in homogeneous coordinates

$$\Omega_2 = d\theta^2 + \sin^2(\theta)d\varphi^2 = \sum_{\mu,\nu=0}^{1} g_{\mu\nu}dz^\mu dz^\nu$$

$$= -4z_1^2\left(\frac{1 - 4z_0^2 z_1^2 + 2\sqrt{1 - 4z_0^2 z_1^2}}{1 - 4z_0^2 z_1^2}\right)dz_0^2 + 8z_0 z_1 dz_0 dz_1$$

$$- 4z_0^2\left(\frac{1 - 4z_0^2 z_1^2 - 2\sqrt{1 - 4z_0^2 z_1^2}}{1 - 4z_0^2 z_1^2}\right)dz_1^2. \tag{23}$$

At this stage, upon defining the real-valued function

$$\gamma(z_0, z_1) = \frac{2}{\sqrt{1 - 4z_0^2 z_1^2}} = \frac{2}{\cos\theta}, \tag{24}$$

we can write the matrix of metric components in the form

$$\gamma_{AB} = \begin{pmatrix} -4z_1^2(1+\gamma) & 4z_0 z_1 \\ 4z_0 z_1 & -4z_0^2(1-\gamma) \end{pmatrix}, \tag{25}$$

with non-vanishing determinant $-16z_0^2 z_1^2 \gamma^2$ and inverse matrix

$$\gamma^{AB} = \begin{pmatrix} \dfrac{1-\gamma}{4z_1^2\gamma^2} & \dfrac{1}{4z_0 z_1\gamma^2} \\ \dfrac{1}{4z_0 z_1\gamma^2} & \dfrac{1+\gamma}{4z_0^2\gamma^2} \end{pmatrix}. \tag{26}$$

We can see from (18) that the terms

$$2z_0 z_1 = \sin(\theta) \rightarrow 4z_0^2 z_1^2 = \sin^2(\theta) \rightarrow 1 - 4z_0^2 z_1^2 = \cos^2(\theta) \rightarrow 4z_0 z_1 = 2\sin(\theta),$$

are real-valued, whereas

$$z_0^2 = e^{i\varphi}\cos^2(\theta/2),\ z_1^2 = e^{-i\varphi}\sin^2(\theta/2)$$

are complex.

3. Bondi–Sachs Metric in Homogeneous Coordinates

We can now write the Bondi–Sachs (hereafter BS) metric in homogeneous coordinates with the help of the previous formulae. For this purpose, let us first write the general BS metric in the form

$$ds^2 = -U du^2 - 2e^{2\beta} du\, dr + h_{AB}\left(dx^A + \frac{1}{2}U^A du\right)\left(dx^B + \frac{1}{2}U^B du\right). \tag{27}$$

On passing from (θ, φ) to (z_0, z_1) coordinates, we find the metric components of (27) expressed as follows (the material from our Equation (28) to our Equation (43) can be obtained from Equations (4.33), (4.35) and (4.37) in Ref. [36], which relies in turn upon the work in Ref. [37]):

$$g_{uu} = -U + \frac{1}{4}h_{z_0 z_0}(U^{z_0})^2 + \frac{1}{4}h_{z_1 z_1}(U^{z_1})^2 + \frac{1}{2}h_{z_0 z_1}U^{z_0}U^{z_1}, \tag{28}$$

$$g_{ur} = -e^{2\beta}, \tag{29}$$

$$g_{uz_0} = \frac{1}{2}(h_{z_0 z_0}U^{z_0} + h_{z_0 z_1}U^{z_1}), \tag{30}$$

$$g_{uz_1} = \frac{1}{2}(h_{z_0 z_1}U^{z_0} + h_{z_1 z_1}U^{z_1}), \tag{31}$$

$$g_{z_0 z_0} = h_{z_0 z_0},\ g_{z_0 z_1} = h_{z_0 z_1},\ g_{z_1 z_1} = h_{z_1 z_1}. \tag{32}$$

The Bondi gauge $\partial_r \det(r^{-2}g_{AB}) = 0$ implies that $\gamma^{AB}C_{AB} = 0$, where γ^{AB} is given in Equation (26). With our coordinates, this relation reads as

$$\gamma^{AB}C_{AB} = 0 \Leftrightarrow g^{z_0 z_0}C_{z_0 z_0} + g^{z_1 z_1}C_{z_1 z_1} + 2g^{z_0 z_1}C_{z_0 z_1} = 0.$$

We no longer have the simple result $C_{z\bar{z}} = 0$ for the mixed component as in the stereographic coordinates because in homogeneous coordinates we obtain

$$\frac{1-\gamma}{4z_1^2\gamma^2}C_{z_0 z_0} + \frac{1+\gamma}{4z_0^2\gamma^2}C_{z_1 z_1} + \frac{1}{2z_0 z_1\gamma^2}C_{z_0 z_1} = 0, \tag{33}$$

which implies that

$$C_{z_0 z_1} = -\frac{1}{2}(1-\gamma)\frac{z_0}{z_1}C_{z_0 z_0} - \frac{1}{2}(1+\gamma)\frac{z_1}{z_0}C_{z_1 z_1}. \tag{34}$$

The angular components of the metric are

$$g_{z_0 z_0} = r^2\gamma_{z_0 z_0} + rC_{z_0 z_0} + \mathcal{O}(r),\ g_{z_1 z_1} = r^2\gamma_{z_1 z_1} + rC_{z_1 z_1} + \mathcal{O}(r),$$

$$g_{z_0 z_1} = r^2\gamma_{z_0 z_1} + rC_{z_0 z_1} + \mathcal{O}(r)$$
$$= r^2\gamma_{z_0 z_1} - r\left(\frac{z_0}{z_1}\frac{(1-\gamma)}{2}C_{z_0 z_0} + \frac{z_1}{z_0}\frac{(1+\gamma)}{2}C_{z_1 z_1}\right) + \mathcal{O}(r),$$

where, of course, γ_{AB} is given in Equation (25). These formulae, jointly with the falloff conditions

$$U(u, r, x^A) = 1 - \frac{2m(u, r, x^A)}{r} + \frac{U_2(u, x^A)}{r^2} + \mathcal{O}(r^{-3})$$

$$\beta(u, r, x^A) = \frac{\beta_1(u, x^A)}{r} + \frac{\beta_2(u, x^A)}{r^2} + \mathcal{O}(r^{-3})$$

$$U^A(u, r, x^B) = \frac{U_2^A(u, x^B)}{r^2} + \frac{U_3^A(u, x^B)}{r^3} + \mathcal{O}(r^{-4})$$

$$g_{AB}(u, r, x^A) = r^2 \gamma_{AB}(x^A) + r C_{AB}(u, x^A) + D_{AB}(u, x^A) + \mathcal{O}(r^{-1}), \tag{35}$$

help to rewrite

$$g_{uu} = -\left(1 - \frac{2m}{r}\right) + \mathcal{O}(r^{-2}). \tag{36}$$

Upon assuming that $\beta_1/r \ll 1$, we obtain

$$g_{ur} = -\exp\left(\frac{2\beta_1}{r} + \mathcal{O}(r^{-2})\right) = -1 - \frac{2\beta_1}{r} + \mathcal{O}(r^{-2}), \tag{37}$$

while for g_{uz_0} and g_{uz_1}, we find

$$
\begin{aligned}
g_{uz_0} &= \frac{1}{2}\left(r^2 \gamma_{z_0 z_0} + r C_{z_0 z_0}\right)\left(\frac{U_2^{z_0}}{r^2} + \frac{U_3^{z_0}}{r^3}\right) + \frac{1}{2}\left\{r^2 \gamma_{z_0 z_1} - r\left[\frac{z_0}{z_1}\frac{(1-\gamma)}{2}C_{z_0 z_0}\right.\right. \\
&\quad \left.\left. + \frac{z_1}{z_0}\frac{(1+\gamma)}{2}C_{z_1 z_1}\right]\right\}\left(\frac{U_2^{z_1}}{r^2} + \frac{U_3^{z_1}}{r^3}\right) \\
&= \frac{\gamma_{z_0 z_0}}{2}U_2^{z_0} + \frac{\gamma_{z_0 z_1}}{2}U_2^{z_1} + \frac{1}{r}\left[\frac{\gamma_{z_0 z_0}}{2}U_3^{z_0} + \frac{C_{z_0 z_0}}{2}U_2^{z_0} + \frac{\gamma_{z_0 z_1}}{2}U_3^{z_1}\right. \\
&\quad \left. - \frac{z_0}{z_1}\frac{(1-\gamma)}{4}C_{z_0 z_0}U_2^{z_1} - \frac{z_1}{z_0}\frac{(1+\gamma)}{4}C_{z_1 z_1}U_2^{z_1}\right] + \mathcal{O}(r^{-2})
\end{aligned} \tag{38}
$$

and

$$
\begin{aligned}
g_{uz_1} &= \frac{1}{2}\left(r^2 \gamma_{z_1 z_1} + r C_{z_1 z_1}\right)\left(\frac{U_2^{z_1}}{r^2} + \frac{U_3^{z_1}}{r^3}\right) + \frac{1}{2}\left\{r^2 \gamma_{z_0 z_1} - r\left[\frac{z_0}{z_1}\frac{(1-\gamma)}{2}C_{z_0 z_0}\right.\right. \\
&\quad \left.\left. + \frac{z_1}{z_0}\frac{(1+\gamma)}{2}C_{z_1 z_1}\right]\right\}\left(\frac{U_2^{z_0}}{r^2} + \frac{U_3^{z_0}}{r^3}\right) \\
&= \frac{\gamma_{z_1 z_1}}{2}U_2^{z_1} + \frac{\gamma_{z_0 z_1}}{2}U_2^{z_0} + \frac{1}{r}\left[\frac{\gamma_{z_1 z_1}}{2}U_3^{z_1} + \frac{C_{z_1 z_1}}{2}U_2^{z_1} + \frac{\gamma_{z_0 z_1}}{2}U_3^{z_0}\right. \\
&\quad \left. - \frac{z_0}{z_1}\frac{(1-\gamma)}{4}C_{z_0 z_0}U_2^{z_0} - \frac{z_1}{z_0}\frac{(1+\gamma)}{4}C_{z_1 z_1}U_2^{z_0}\right] + \mathcal{O}(r^{-2}),
\end{aligned} \tag{39}
$$

where use has been made of (34). Eventually, we obtain the matrix of Bondi metric components

$$
g_{\mu\nu} = \begin{pmatrix}
-\left(1 - \dfrac{2m}{r}\right) & -1 - \dfrac{2\beta_1}{r} & g_{uz_0} & g_{uz_1} \\[2ex]
-1 - \dfrac{2\beta_1}{r} & 0 & 0 & 0 \\[2ex]
g_{uz_0} & 0 & r^2\gamma_{z_0 z_0} + r C_{z_0 z_0} & r^2\gamma_{z_0 z_1} + r C_{z_0 z_1} \\[2ex]
g_{uz_1} & 0 & r^2\gamma_{z_0 z_1} + r C_{z_0 z_1} & r^2\gamma_{z_1 z_1} + r C_{z_1 z_1}
\end{pmatrix} + \mathcal{O}(r^{-2}). \tag{40}
$$

The gauge condition $\det(g_{AB}/r^2) = 0$, instead of giving a solution for D_{AB} such as in stereographic coordinates, gives us a condition for C_{AB}

$$\det(g_{AB}) = \det \begin{pmatrix} r^2 \gamma_{z_0 z_0} + r C_{z_0 z_0} + D_{z_0 z_0} & r^2 \gamma_{z_0 z_1} + r C_{z_0 z_1} + D_{z_0 z_1} \\ r^2 \gamma_{z_0 z_1} + r C_{z_0 z_1} + D_{z_0 z_1} & r^2 \gamma_{z_1 z_1} + r C_{z_1 z_1} + D_{z_1 z_1} \end{pmatrix}$$

$$= r^4 \left(\gamma_{z_0 z_0} \gamma_{z_1 z_1} - \gamma_{z_0 z_1}^2 \right) + r^3 \left(\gamma_{z_0 z_0} C_{z_1 z_1} + \gamma_{z_1 z_1} C_{z_0 z_0} - 2 r^3 \gamma_{z_0 z_1} C_{z_0 z_1} \right)$$
$$+ r^2 \left(\gamma_{z_0 z_0} D_{z_1 z_1} + C_{z_0 z_0} C_{z_1 z_1} + \gamma_{z_1 z_1} D_{z_0 z_0} - C_{z_0 z_1}^2 - 2 \gamma_{z_0 z_1} D_{z_0 z_1} \right) + \mathcal{O}(r),$$

$$\det \left(\frac{g_{AB}}{r^2} \right) \Rightarrow \gamma_{z_0 z_0} D_{z_1 z_1} + C_{z_0 z_0} C_{z_1 z_1} + \gamma_{z_1 z_1} D_{z_0 z_0} - C_{z_0 z_1}^2 - 2 \gamma_{z_0 z_1} D_{z_0 z_1} \overset{!}{=} 0$$

$$\implies D_{z_0 z_1} = \frac{\gamma_{z_0 z_0} D_{z_1 z_1} + C_{z_0 z_0} C_{z_1 z_1} + \gamma_{z_1 z_1} D_{z_0 z_0} - C_{z_0 z_1}^2}{2 \gamma_{z_0 z_1}}$$

$$C_{z_0 z_1}^2 = C_{z_0 z_0} C_{z_1 z_1}.$$

In order to determine the various coefficients in the falloff conditions, we require that the Bondi metric should satisfy the Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

Upon restricting to the vacuum case $T = 0$, in the limit as r approaches ∞ in the Einstein tensor, first looking at G_{rr}, and neglecting the terms of order $\mathcal{O}(r^{-4})$, we obtain

$$G_{rr} = -\frac{4\beta_1}{r^3} + \mathcal{O}(r^{-4}) \overset{!}{=} 0 \Rightarrow \beta_1 \equiv 0.$$

Upon looking at $G_{r z_0}$ and $G_{r z_1}$, respectively, we obtain lengthy relations for $U_2^{z_1}$ and $U_2^{z_0}$, compared to the stereographic coordinates case, which depend on other coefficients. However, we still manage to solve directly for $U_2^{z_0}$ and $U_2^{z_1}$. On studying $G_{rA} = 0$, we find

$$U_2^{z_0} = \frac{2 z_0 z_1 \left(C_{z_1 z_1} U_2^{z_1} + \gamma_{z_0 z_1} U_3^{z_0} + \gamma_{z_1 z_1} U_3^{z_1} \right)}{z_1^2 (1 + \gamma) C_{z_1 z_1} + z_0^2 (1 - \gamma) C_{z_0 z_0}} = -\frac{C_{z_1 z_1} U_2^{z_1} + 2 \gamma_{z_0 z_1} U_3^{z_0}}{C_{z_0 z_1}}, \tag{41}$$

and

$$U_2^{z_1} = \frac{2 z_0 z_1 \left(C_{z_0 z_0} U_2^{z_0} + \gamma_{z_0 z_0} U_3^{z_0} + \gamma_{z_0 z_1} U_3^{z_1} \right)}{z_1^2 (1 + \gamma) C_{z_1 z_1} + z_0^2 (1 - \gamma) C_{z_0 z_0}} = -\frac{C_{z_0 z_0} U_2^{z_0} + 2 \gamma_{z_0 z_1} U_3^{z_1}}{C_{z_0 z_1}}, \tag{42}$$

where we recall that $C_{z_0 z_1}$ is given in Equation (34). By virtue of Equations (38) and (39) we eventually find the metric in the form

$$ds^2 = -du^2 - 2dudr + 2 \left(r^2 \gamma_{z_0 z_1} + r C_{z_0 z_1} \right) dz_0 dz_1 + \frac{2m}{r} du^2 + \left(r^2 \gamma_{z_0 z_0} + r C_{z_0 z_0} \right) dz_0^2$$

$$+ \left(r^2 \gamma_{z_1 z_1} + r C_{z_1 z_1} \right) dz_1^2$$

$$+ \left[\frac{\gamma_{z_0 z_0}}{2} U_2^{z_0} + \frac{\gamma_{z_0 z_1}}{2} U_2^{z_1} + \frac{1}{r} \left(\frac{C_{z_0 z_0}}{2} U_2^{z_0} + \frac{C_{z_0 z_1}}{2} U_2^{z_1} + \frac{\gamma_{z_0 z_0}}{2} U_3^{z_0} + \frac{\gamma_{z_0 z_1}}{2} U_3^{z_1} \right) \right] dudz_0$$

$$+ \left[\frac{\gamma_{z_0 z_1}}{2} U_2^{z_0} + \frac{\gamma_{z_1 z_1}}{2} U_2^{z_1} + \frac{1}{r} \left(\frac{C_{z_0 z_1}}{2} U_2^{z_0} + \frac{C_{z_1 z_1}}{2} U_2^{z_1} + \frac{\gamma_{z_0 z_1}}{2} U_3^{z_0} + \frac{\gamma_{z_1 z_1}}{2} U_3^{z_1} \right) \right] dudz_1$$

$$+ \mathcal{O}(r^{-2}). \tag{43}$$

For the discussion of Bondi's news tensor, mass and angular momentum aspects we refer again to the work in Refs. [36,37]. Now, we are ready to evaluate the BMS generators in homogeneous coordinates in order to determine the supertranslations.

4. Asymptotic Killing Fields

After finding the most general Bondi metric in homogeneous coordinates satisfying the asymptotically flat spacetime falloffs, our aim is to find the most general vector fields ξ satisfying the Bondi gauge condition and the asymptotically flat spacetime falloffs. As is well known, the Killing vectors solve by definition the equations

$$\left(\mathcal{L}_\xi g\right)_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu \xi^\rho + g_{\nu\rho}\partial_\mu \xi^\rho = 0.$$

Moreover, the preservation of the Bondi gauge condition yields

$$\left(\mathcal{L}_\xi g\right)_{rr} = 0, \ \left(\mathcal{L}_\xi g\right)_{rA} = 0 \ \text{and} \ g^{AB}\left(\mathcal{L}_\xi g\right)_{AB} = 0. \tag{44}$$

From these relations, one can calculate the four components of ξ^μ. At this stage, we can compute the asymptotic Killing fields in homogeneous coordinates by using the familiar transformation law of vector fields. In other words, the work in Ref. [22] has defined the stereographic variable (we write ψ rather than z used in Ref. [22], in order to avoid confusion with our ζ in Equation (1))

$$\psi = e^{i\varphi}\tan\frac{\theta}{2} = \frac{1}{\bar{\zeta}}, \tag{45}$$

and has found, in Bondi coordinates u, r, θ, φ, the asymptotic Killing fields ξ_T^+ where the components depend on a function f and on the Bondi coordinates. On denoting as usual by Y_l^m the spherical harmonics on the 2-sphere, one finds

$$\left.\xi_T^+\right|_{f=Y_0^0} = \frac{\partial}{\partial u}, \tag{46}$$

$$\left.\xi_T^+\right|_{f=Y_1^0} = \frac{(1-\psi\bar{\psi})}{(1+\psi\bar{\psi})}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) + \frac{\psi}{r}\frac{\partial}{\partial \psi} + \frac{\bar{\psi}}{r}\frac{\partial}{\partial\bar{\psi}}, \tag{47}$$

$$\left.\xi_T^+\right|_{f=Y_1^1} = \frac{\psi}{(1+\psi\bar{\psi})}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) + \frac{\psi^2}{2r}\frac{\partial}{\partial\psi} - \frac{1}{2r}\frac{\partial}{\partial\bar{\psi}}, \tag{48}$$

$$\left.\xi_T^+\right|_{f=Y_1^{-1}} = \frac{\bar{\psi}}{(1+\psi\bar{\psi})}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) - \frac{1}{2r}\frac{\partial}{\partial\psi} + \frac{\bar{\psi}^2}{2r}\frac{\partial}{\partial\bar{\psi}}. \tag{49}$$

Now, by virtue of the basic identities

$$\frac{\partial}{\partial\psi} = \frac{\partial z_0}{\partial\psi}\frac{\partial}{\partial z_0} + \frac{\partial z_1}{\partial\psi}\frac{\partial}{\partial z_1}, \tag{50}$$

$$\frac{\partial}{\partial\bar{\psi}} = \frac{\partial z_0}{\partial\bar{\psi}}\frac{\partial}{\partial z_0} + \frac{\partial z_1}{\partial\bar{\psi}}\frac{\partial}{\partial z_1}, \tag{51}$$

and upon exploiting the formulae (A7)–(A10) in the Appendix A, we find

$$\left.\xi_T^+\right|_{f=Y_1^0} = \frac{2}{\gamma}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) + \frac{z_0}{2r}\frac{(2-\gamma)}{\gamma}\frac{\partial}{\partial z_0} + \frac{z_1}{2r}\frac{(2+\gamma)}{\gamma}\frac{\partial}{\partial z_1}, \tag{52}$$

$$
\begin{aligned}
\left.\xi_T^+\right|_{f=Y_1^1} &= \frac{z_0}{2z_1}\frac{(\gamma-2)}{\gamma}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{(z_0)^2}{z_1}\left(\frac{1}{4} - \frac{1}{\gamma(\gamma+2)}\right)\frac{\partial}{\partial z_0} \\
&\quad + \frac{z_0}{2r}\left(\frac{1}{(\gamma+2)} - \frac{(\gamma+2)}{2\gamma}\right)\frac{\partial}{\partial z_1},
\end{aligned} \tag{53}
$$

$$\xi_T^+ \Big|_{f=Y_1^{-1}} = \frac{z_1}{2z_0} \frac{(\gamma+2)}{\gamma} \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r} \right) - \frac{z_1}{2r} \left(\frac{1}{(\gamma-2)} + \frac{(\gamma-2)}{2\gamma} \right) \frac{\partial}{\partial z_0}$$

$$+ \frac{1}{r} \frac{(z_1)^2}{z_0} \left(\frac{1}{4} - \frac{1}{\gamma(\gamma-2)} \right) \frac{\partial}{\partial z_1}. \tag{54}$$

Now, we denote by $\xi_0, \xi_1, \xi_2, \xi_3$ the vector fields (46), (52), (53) and (54), respectively. Nontrivial Lie brackets among them involve ξ_1, ξ_2, ξ_3 only. With our notation, we can re-write Equations (52)–(54) in the form

$$\xi_1 = A_{11} \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r} \right) + A_{12} \frac{\partial}{\partial z_0} + A_{13} \frac{\partial}{\partial z_1}, \tag{55}$$

$$\xi_2 = A_{21} \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r} \right) + A_{22} \frac{\partial}{\partial z_0} + A_{23} \frac{\partial}{\partial z_1}, \tag{56}$$

$$\xi_3 = A_{31} \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r} \right) + A_{32} \frac{\partial}{\partial z_0} + A_{33} \frac{\partial}{\partial z_1}, \tag{57}$$

where the values taken by the A_{ij} functions can be read off from (52)–(54). At this stage, a patient evaluation proves that such vector fields have vanishing Lie brackets:

$$[\xi_1, \xi_2] = [\xi_2, \xi_3] = [\xi_3, \xi_1] = 0. \tag{58}$$

The result is simple, but the actual proof requires several details, for which we refer the reader to Appendix B.

5. Flow of Supertranslation Vector Fields

The analysis in this section does not have a direct impact on unsolved problems but (as far as we can see) can help the general reader. More precisely, in order to appreciate that the familiar geometric constructions are also feasible in projective coordinates, we now consider the flow of supertranslation vector fields (52)–(54). For example, by virtue of (24), and defining $p = (u, r, z_0, z_1)$, the task of finding the flow of the supertranslation vector fields (52), (53) and (54) consists of solving a system of nonlinear and coupled differential equations. For this purpose, we denote by σ, Σ, χ, respectively, the appropriate flow, and define

$$\delta(W; \tau, p) = \sqrt{1 - 4 \left(W^3(\tau, p) W^4(\tau, p) \right)^2}, \tag{59}$$

where $W = \sigma, \Sigma, \chi$, respectively, with components W^1, W^2, W^3, W^4. Hence, we study the following coupled systems of nonlinear differential equations:

$$\frac{d\sigma^1}{d\tau} = \delta(\sigma; \tau, p), \tag{60}$$

$$\frac{d\sigma^2}{d\tau} = -\delta(\sigma; \tau, p), \tag{61}$$

$$\frac{d\sigma^3}{d\tau} = \frac{\sigma^3(\tau, p)}{2\sigma^2(\tau, p)} (\delta(\sigma; \tau, p) - 1), \tag{62}$$

$$\frac{d\sigma^4}{d\tau} = \frac{\sigma^4(\tau, p)}{2\sigma^2(\tau, p)} (\delta(\sigma; \tau, p) + 1), \tag{63}$$

$$\frac{d\Sigma^1}{d\tau} = \frac{\Sigma^3(\tau, p)}{2\Sigma^4(\tau, p)} (1 - \delta(\Sigma; \tau, p)), \tag{64}$$

$$\frac{d\Sigma^2}{d\tau} = -\frac{\Sigma^3(\tau, p)}{2\Sigma^4(\tau, p)} (1 - \delta(\Sigma; \tau, p)), \tag{65}$$

$$\frac{d\Sigma^3}{d\tau} = \frac{(\Sigma^3(\tau,p))^2}{4\Sigma^2(\tau,p)\Sigma^4(\tau,p)}\left[1 - \delta(\Sigma;\tau,p) + \frac{\delta(\Sigma;\tau,p)}{(1+\delta(\Sigma;\tau,p))}\right],\tag{66}$$

$$\frac{d\Sigma^4}{d\tau} = \frac{\Sigma^3(\tau,p)}{4\Sigma^2(\tau,p)}\left[\frac{\delta(\Sigma;\tau,p)}{(1+\delta(\Sigma;\tau,p))} - 1 - \delta(\Sigma;\tau,p)\right],\tag{67}$$

$$\frac{d\chi^1}{d\tau} = \frac{\chi^4(\tau,p)}{2\chi^3(\tau,p)}(1+\delta(\chi;\tau,p)),\tag{68}$$

$$\frac{d\chi^2}{d\tau} = -\frac{\chi^4(\tau,p)}{2\chi^3(\tau,p)}(1+\delta(\chi;\tau,p)),\tag{69}$$

$$\frac{d\chi^3}{d\tau} = -\frac{\chi^4(\tau,p)}{4\chi^2(\tau,p)}\left[\frac{\delta(\chi;\tau,p)}{(1-\delta(\chi;\tau,p))} + 1 - \delta(\chi;\tau,p)\right],\tag{70}$$

$$\frac{d\chi^4}{d\tau} = \frac{(\chi^4(\tau,p))^2}{4\chi^2(\tau,p)\chi^3(\tau,p)}\left[1 + \delta(\chi;\tau,p) - \frac{\delta(\chi;\tau,p)}{(1-\delta(\chi;\tau,p))}\right],\tag{71}$$

with the initial conditions

$$W^1(0,p) = u, \ W^2(0,p) = r, \ W^3(0,p) = z_0, \ W^4(0,p) = z_1.\tag{72}$$

The resulting equations can only be solved numerically to the best of our knowledge, and such solutions are displayed in Figures 1–9. Since the desired solutions are complex-valued, we display both real and imaginary parts, with three choices of initial conditions.
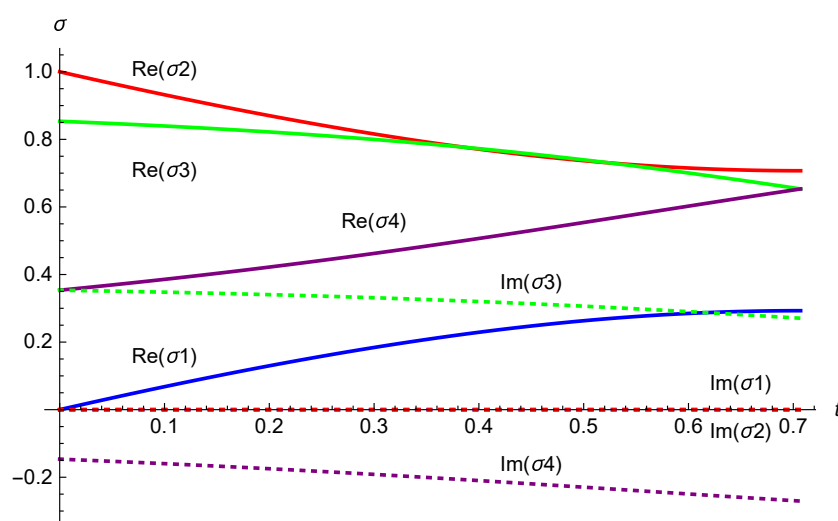
Figure 1. Numerical evaluation of the integral curve for the supertranslation vector field (52). The initial conditions (72) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{8}}\cos\frac{\pi}{8}, z_1 = e^{-i\frac{\pi}{8}}\sin\frac{\pi}{8}$.
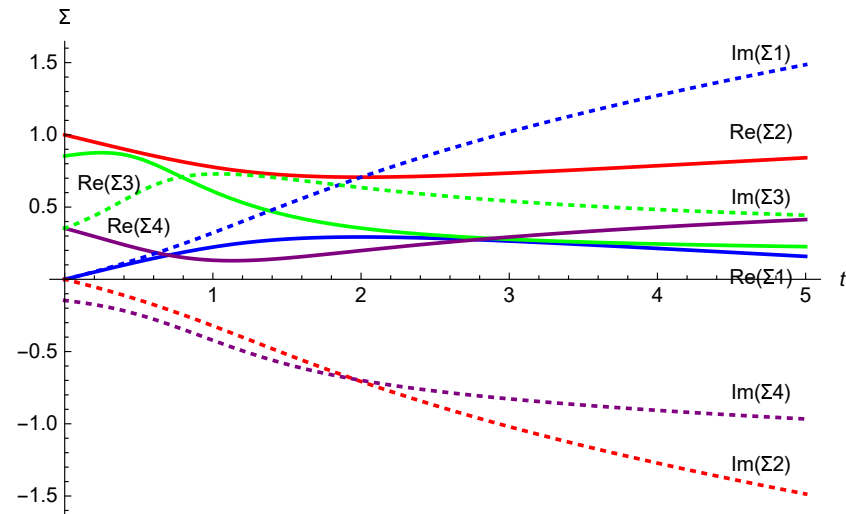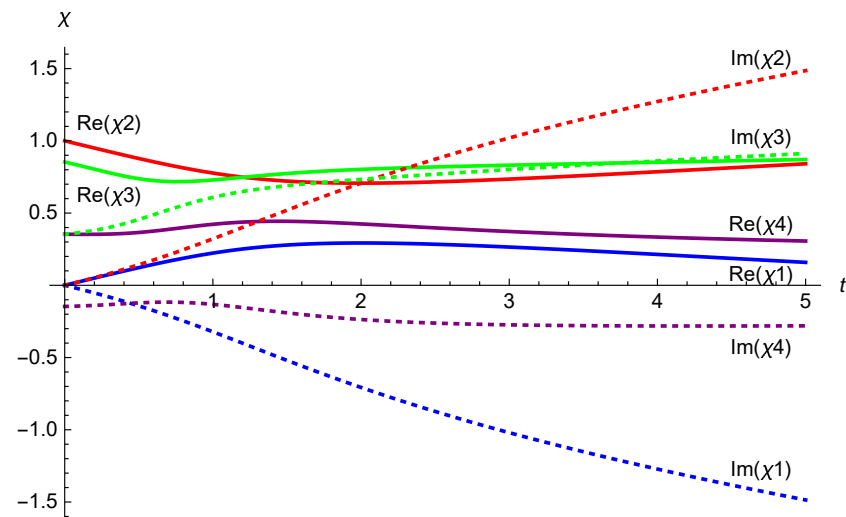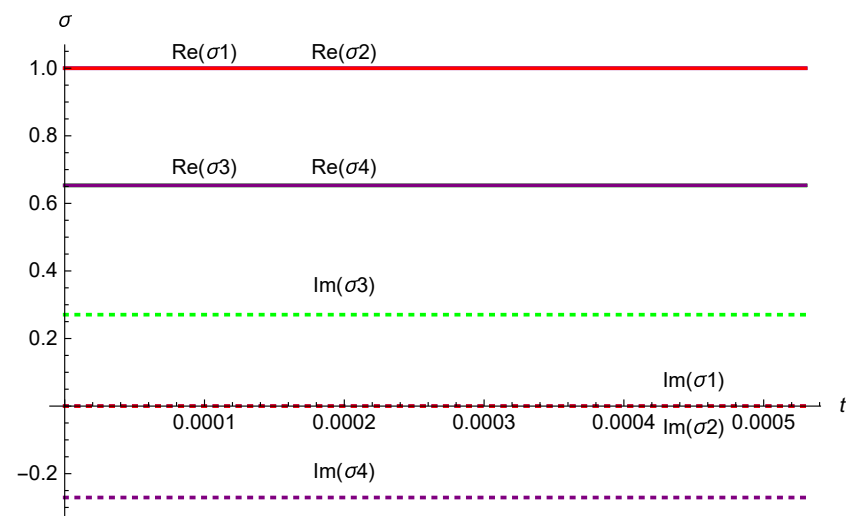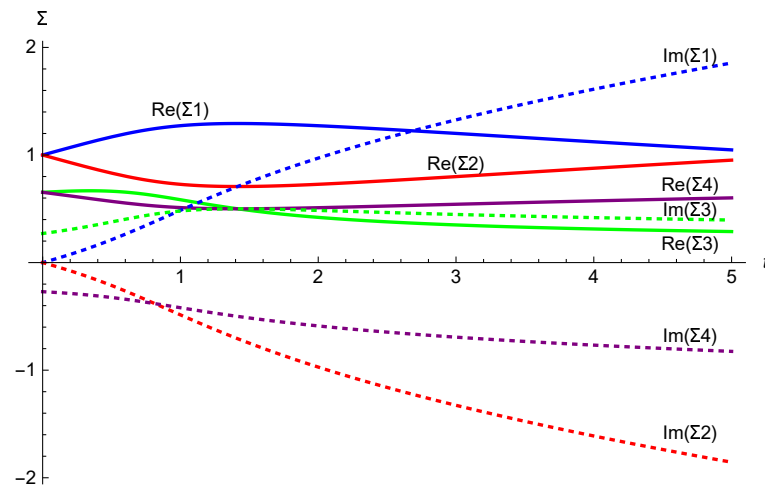
Figure 2. Numerical evaluation of the integral curve for the supertranslation vector field (53). The initial conditions (72) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{8}} \cos\frac{\pi}{8}, z_1 = e^{-i\frac{\pi}{8}} \sin\frac{\pi}{8}$.
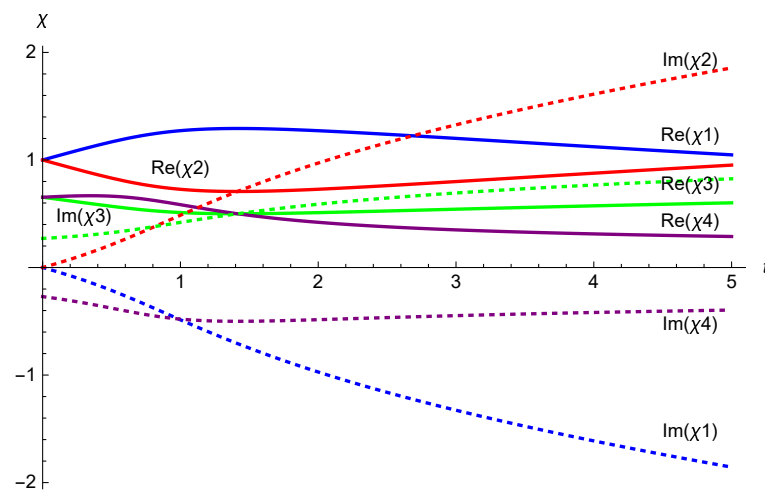

Figure 3. Numerical evaluation of the integral curve for the supertranslation vector field (54). The initial conditions (72) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{8}} \cos\frac{\pi}{8}, z_1 = e^{-i\frac{\pi}{8}} \sin\frac{\pi}{8}$.

Figure 4. Numerical evaluation of the integral curve for the supertranslation vector field (52). The initial conditions (72) are taken to be $u = 1, r = 1, z_0 = e^{i\frac{\pi}{8}} \cos\frac{\pi}{4}, z_1 = e^{-i\frac{\pi}{8}} \sin\frac{\pi}{4}$.
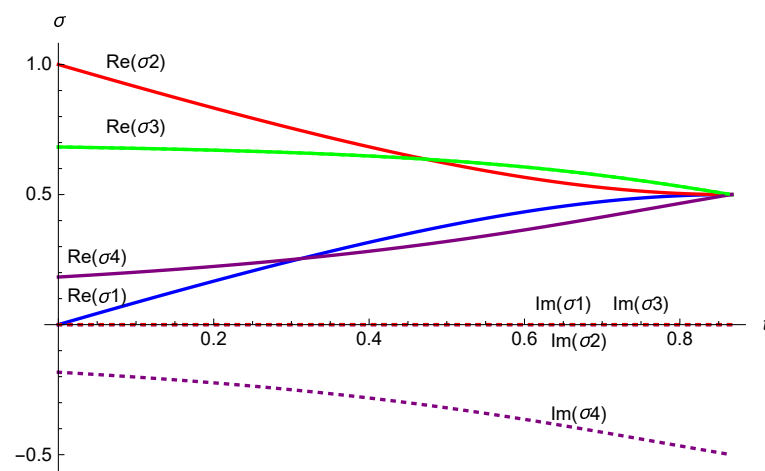
Figure 5. Numerical evaluation of the integral curve for the supertranslation vector field (53). The initial conditions (72) are taken to be $u = 1, r = 1, z_0 = e^{i\frac{\pi}{8}} \cos \frac{\pi}{4}, z_1 = e^{-i\frac{\pi}{8}} \sin \frac{\pi}{4}$.

Figure 6. Numerical evaluation of the integral curve for the supertranslation vector field (54). The initial conditions (72) are taken to be $u = 1, r = 1, z_0 = e^{i\frac{\pi}{8}} \cos \frac{\pi}{4}, z_1 = e^{-i\frac{\pi}{8}} \sin \frac{\pi}{4}$.

Figure 7. Numerical evaluation of the integral curve for the supertranslation vector field (52). The initial conditions (72) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{4}} \cos \frac{\pi}{12}, z_1 = e^{-i\frac{\pi}{4}} \sin \frac{\pi}{12}$. In this particular case, the real parts meet at a single point.
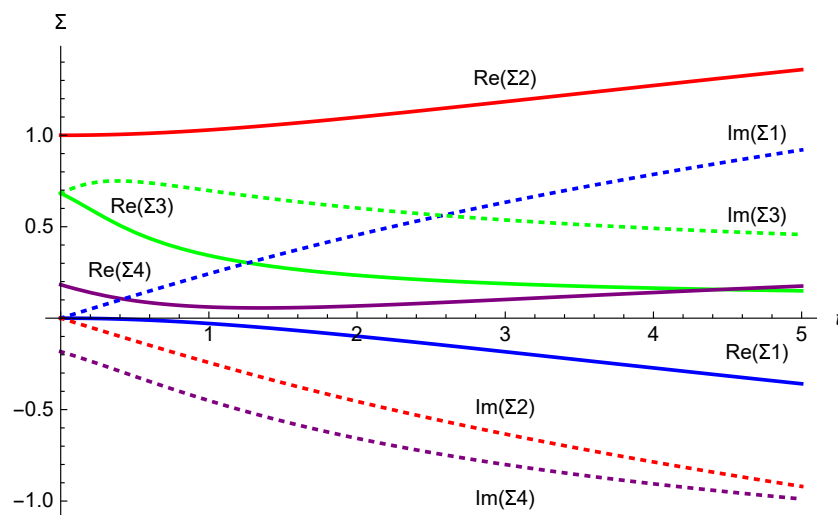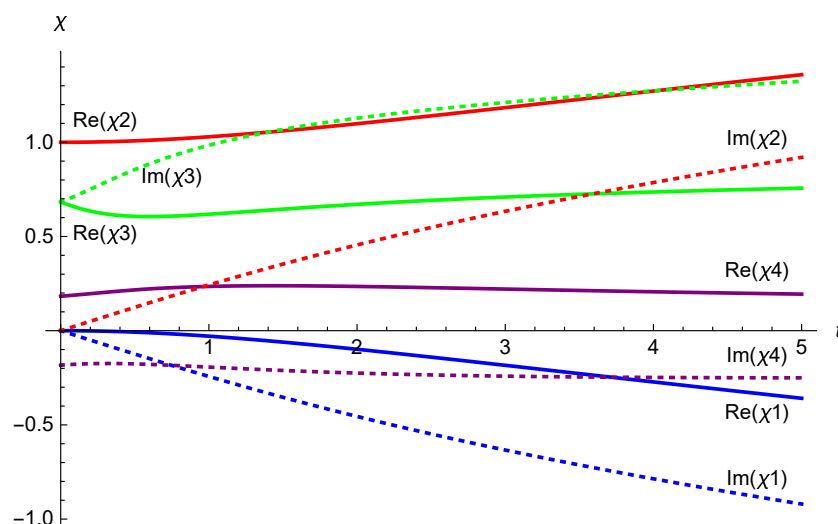
Figure 8. Numerical evaluation of the integral curve for the supertranslation vector field (53). The initial conditions (72) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{4}} \cos \frac{\pi}{12}, z_1 = e^{-i\frac{\pi}{4}} \sin \frac{\pi}{12}$.

Figure 9. Numerical evaluation of the integral curve for the supertranslation vector field (54). The initial conditions (72) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{4}} \cos \frac{\pi}{12}, z_1 = e^{-i\frac{\pi}{4}} \sin \frac{\pi}{12}$.

6. Concluding Remarks and Open Problems

As far as we can see, the interest of our investigation lies in having shown that homogeneous projective coordinates lead to a fully computational scheme for all applications of the BMS group. This might yield success when more advanced properties are studied. In particular, we have in mind the concept of super-rotations [22,23] on the one hand and the physical applications of the Segre manifold advocated in our Introduction and in Ref. [19]. In other words, since our Equation (15) contains Equation (12), which in turn is just a re-expression of the BMS transformation (2), one might aim at embedding the study of BMS symmetries into the richer mathematical framework of complex analysis in several variables [38] and algebraic geometry. The exploitation of the complex analysis approach to algebraic geometry appears promising because the singular points of functions of several complex variables form a continuum (see definitions and theorems in Refs. [34,38]). The potentialities of this framework for studying, e.g., super-rotations were unforeseen and deserve careful consideration in our opinion. Our paper has tried to pave the way for such a synthesis through manageable calculations.

Moreover, we would like to mention that the research in Refs. [36,39,40] has exploited the fact that one can actually work with a completely arbitrary metric on the asymptotic

2-sphere. By doing so, one can write the on-shell expression of U, U^A and β in our Section 3 in terms of this arbitrary 2-sphere metric. This might therefore provide a way to recover our results when taking the particular case in which the 2-sphere metric is expressed in homogeneous projective coordinates. We are grateful to M. Geiller for this remark and also for having brought to our attention the work in Ref. [41], where the authors have written the solution space and the asymptotic Killing vectors and their action in the case of an even more general gauge than Bondi–Sachs.

Appendix A. The Use of Homogeneous Coordinates

By virtue of Equations (10) and (24), we find

$$(z_0)^2 = e^{i\varphi} \frac{(1 + \cos\theta)}{2} \implies e^{i\varphi} = \frac{2(z_0)^2}{(1 + \cos\theta)} = \frac{2(z_0)^2 \gamma}{(\gamma + 2)}, \tag{A1}$$

and hence the variable ψ in Equation (45) can be re-expressed in the form

$$\psi = \frac{2(z_0)^2 \gamma}{(\gamma + 2)} \frac{(1 - \cos\theta)}{\sin\theta} = \frac{2(z_0)^2 \gamma}{(\gamma + 2)} \frac{\left(1 - \frac{2}{\gamma}\right)}{2z_0 z_1} = \frac{z_0}{z_1} \frac{(\gamma - 2)}{(\gamma + 2)}, \tag{A2}$$

while

$$\bar{\psi} = \frac{1}{\zeta} = \frac{z_1}{z_0}. \tag{A3}$$

Moreover, we need the identities

$$\psi\bar{\psi} = \frac{\left(1 - \cos^2\frac{\theta}{2}\right)}{\cos^2\frac{\theta}{2}} \implies \cos\frac{\theta}{2} = \frac{1}{\sqrt{(1 + \psi\bar{\psi})}}, \ \sin\frac{\theta}{2} = \sqrt{\frac{\psi\bar{\psi}}{(1 + \psi\bar{\psi})}}, \tag{A4}$$

$$e^{i\frac{\varphi}{2}} = \left(\frac{\psi}{\bar{\psi}}\right)^{\frac{1}{4}}, \ e^{-i\frac{\varphi}{2}} = \left(\frac{\bar{\psi}}{\psi}\right)^{\frac{1}{4}}, \tag{A5}$$

which, jointly with the definitions (10), lead to

$$z_0 = \left(\frac{\psi}{\bar{\psi}}\right)^{\frac{1}{4}} \frac{1}{\sqrt{1 + \psi\bar{\psi}}}, \ z_1 = \left(\frac{\bar{\psi}}{\psi}\right)^{\frac{1}{4}} \sqrt{\frac{\psi\bar{\psi}}{(1 + \psi\bar{\psi})}}. \tag{A6}$$

At this stage, we can evaluate the partial derivatives occurring in Equations (50) and (51) by patient application of Equations (A2), (A3) and (A6), i.e.,

$$\frac{\partial z_0}{\partial \psi} = \frac{z_1}{2} \frac{(\gamma + 2)}{\gamma(\gamma - 2)}, \tag{A7}$$

$$\frac{\partial z_1}{\partial \psi} = \frac{1}{2} \frac{(z_1)^2}{z_0} \frac{(\gamma + 2)}{\gamma(\gamma - 2)}, \tag{A8}$$

$$\frac{\partial z_0}{\partial \bar{\psi}} = -\frac{(z_0)^2}{2z_1} \frac{(\gamma - 1)}{\gamma}, \tag{A9}$$

$$\frac{\partial z_1}{\partial \bar{\psi}} = \frac{z_0}{2} \frac{(\gamma + 1)}{\gamma}, \tag{A10}$$

and we eventually find the asymptotic Killing fields in the form (52)–(54). Our homogeneous projective coordinates z_0 and z_1 have also been considered in Ref. [42], but in that case, upon writing

$$\zeta = \frac{(x + iy)}{(1 - z)}, \tag{A11}$$

one finds that the x, y, z coordinates for the embedding of the 2-sphere in three-dimensional Euclidean space are given by

$$x = \frac{2\text{Re}(\zeta)}{(1 + |\zeta|^2)} = \frac{(z_0 \bar{z}_1 + \bar{z}_0 z_1)}{(|z_0|^2 + |z_1|^2)}, \tag{A12}$$

$$y = \frac{2\text{Im}(\zeta)}{(1 + |\zeta|^2)} = \frac{(z_0 \bar{z}_1 - \bar{z}_0 z_1)}{i(|z_0|^2 + |z_1|^2)}, \tag{A13}$$

$$z = \frac{(|\zeta|^2 - 1)}{(|\zeta|^2 + 1)} = \frac{(|z_0|^2 - |z_1|^2)}{(|z_0|^2 + |z_1|^2)}. \tag{A14}$$

The global spacetime translations of Minkowski spacetime can be first re-expressed in $u, r, \xi, \bar{\xi}$ coordinates, and read eventually, in terms of the asymptotic Killing fields (52)–(54),

$$X_0 = -\xi_T^+ \Big|_{f=Y_0^0}, \quad X_1 = -\xi_T^+ \Big|_{f=Y_1^1} - \xi_T^+ \Big|_{f=Y_1^{-1}},$$

$$iX_2 = \xi_T^+ \Big|_{f=Y_1^{-1}} - \xi_T^+ \Big|_{f=Y_1^1}, \quad X_3 = -\xi_T^+ \Big|_{f=Y_1^0}. \tag{A15}$$

Explicitly, we find

$$
\begin{aligned}
X_1 &= \left(\frac{z_0}{2z_1} \frac{(\gamma - 2)}{\gamma} + \frac{z_1}{2z_0} \frac{(\gamma + 2)}{\gamma} \right) \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial r} \right) \\
&+ \frac{1}{2r} \left[\frac{(z_0)^2}{z_1} \left(-\frac{1}{2} - \frac{2}{\gamma(\gamma + 2)} \right) + z_1 \left(\frac{(\gamma - 2)}{2\gamma} + \frac{1}{(\gamma - 2)} \right) \right] \frac{\partial}{\partial z_0} \\
&+ \frac{1}{2r} \left[\frac{(z_1)^2}{z_0} \left(-\frac{1}{2} + \frac{2}{\gamma(\gamma - 2)} \right) + z_0 \left(\frac{(\gamma + 2)}{2\gamma} - \frac{1}{(\gamma + 2)} \right) \right] \frac{\partial}{\partial z_1}, \tag{A16}
\end{aligned}
$$

$$
\begin{aligned}
X_2 &= i\left(\frac{z_0}{2z_1} \frac{(\gamma - 2)}{\gamma} - \frac{z_1}{2z_0} \frac{(\gamma + 2)}{\gamma} \right) \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r} \right) \\
&+ \frac{i}{2r} \left[\frac{(z_0)^2}{z_1} \left(\frac{1}{2} - \frac{2}{\gamma(\gamma + 2)} \right) + z_1 \left(\frac{(\gamma - 2)}{2\gamma} + \frac{1}{(\gamma - 2)} \right) \right] \frac{\partial}{\partial z_0} \\
&- \frac{i}{2r} \left[\frac{(z_1)^2}{z_0} \left(\frac{1}{2} - \frac{2}{\gamma(\gamma - 2)} \right) + z_0 \left(\frac{(\gamma + 2)}{2\gamma} - \frac{1}{(\gamma + 2)} \right) \right] \frac{\partial}{\partial z_1}, \tag{A17}
\end{aligned}
$$

$$X_3 = \frac{2}{\gamma} \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial r} \right) + \frac{1}{2r} \left(z_0 \frac{\partial}{\partial z_0} - z_1 \frac{(\gamma + 2)}{\gamma} \frac{\partial}{\partial z_1} \right). \tag{A18}$$

The boost (K_i) and rotation (J_{ij}) vector fields for Lorentz transformations in Minkowski spacetime can be written in $u, r, \xi, \bar{\xi}$ coordinates as is shown, for example, in Ref. [22]. At that stage, by using again Equations (50), (51) and (A7)–(A10) we find

$$
\begin{aligned}
K_1 &= \frac{1}{2}\left(\frac{z_0}{z_1}\frac{(\gamma-2)}{\gamma} + \frac{z_1}{z_0}\frac{(\gamma+2)}{\gamma}\right)\left(-u\frac{\partial}{\partial u} + (u+r)\frac{\partial}{\partial r}\right) \\
&\quad - \frac{(u+r)}{2r}\left[\frac{(z_0)^2}{z_1}\left(\frac{1}{2} - \frac{2}{\gamma(\gamma+2)}\right) - z_1\left(\frac{(\gamma-2)}{2\gamma} + \frac{1}{(\gamma-2)}\right)\right]\frac{\partial}{\partial z_0} \\
&\quad + \left[\frac{(z_1)^2}{z_0}\left(\frac{1}{2} - \frac{2}{\gamma(\gamma-2)}\right) + z_0\left(-\frac{(\gamma+2)}{2\gamma} + \frac{1}{(\gamma+2)}\right)\right]\frac{\partial}{\partial z_1},
\end{aligned} \tag{A19}
$$

$$
\begin{aligned}
K_2 &= \frac{i}{2}\left(\frac{z_0}{z_1}\frac{(\gamma-2)}{\gamma} - \frac{z_1}{z_0}\frac{(\gamma+2)}{\gamma}\right)\left(u\frac{\partial}{\partial u} - (u+r)\frac{\partial}{\partial r}\right) \\
&\quad + i\frac{(u+r)}{2r}\left\{\left[\frac{(z_0)^2}{z_1}\left(\frac{1}{2} - \frac{2}{\gamma(\gamma+2)}\right) + z_1\left(\frac{(\gamma-2)}{2\gamma} + \frac{1}{(\gamma-2)}\right)\right]\frac{\partial}{\partial z_0}\right. \\
&\quad + \left.\left[\frac{(z_1)^2}{z_0}\left(-\frac{1}{2} + \frac{2}{\gamma(\gamma-2)}\right) + z_0\left(-\frac{(\gamma+2)}{2\gamma} + \frac{1}{(\gamma+2)}\right)\right]\frac{\partial}{\partial z_1}\right\},
\end{aligned} \tag{A20}
$$

$$
K_3 = \frac{2}{\gamma}\left(-u\frac{\partial}{\partial u} + (u+r)\frac{\partial}{\partial r}\right) + \frac{1}{2r}(u+r)\left(z_0\frac{\partial}{\partial z_0} - z_1\frac{(\gamma+2)}{\gamma}\frac{\partial}{\partial z_1}\right), \tag{A21}
$$

$$
J_{12} = \frac{i}{2}\left(z_0\frac{\partial}{\partial z_0} - z_1\frac{\partial}{\partial z_1}\right), \tag{A22}
$$

$$
\begin{aligned}
J_{23} &= \frac{i}{2}\left[\left(-\frac{(z_0)^2}{2z_1}\frac{\gamma}{(\gamma+2)} + z_1\left(\frac{1}{2} - \frac{1}{(\gamma-2)}\right)\right)\right]\frac{\partial}{\partial z_0} \\
&\quad + \frac{i}{2}\left[\left(-\frac{(z_1)^2}{2z_0}\frac{\gamma}{(\gamma-2)} + z_0\left(\frac{1}{2} + \frac{1}{(\gamma+2)}\right)\right)\right]\frac{\partial}{\partial z_1},
\end{aligned} \tag{A23}
$$

$$
\begin{aligned}
J_{31} &= -\frac{1}{2}\left[\left(\frac{(z_0)^2}{2z_1}\frac{\gamma}{(\gamma+2)} + z_1\left(\frac{1}{2} - \frac{1}{(\gamma-2)}\right)\right)\right]\frac{\partial}{\partial z_0} \\
&\quad + \frac{1}{2}\left[\left(\frac{(z_1)^2}{2z_0}\frac{\gamma}{(\gamma-2)} + z_0\left(\frac{1}{2} + \frac{1}{(\gamma+2)}\right)\right)\right]\frac{\partial}{\partial z_1}.
\end{aligned} \tag{A24}
$$

Appendix B. Lie Brackets of Asymptotic Killing Fields

Given the vector fields (55) and (56), the evaluation of their Lie bracket shows that

$$
[\xi_1, \xi_2] = \rho_1\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) + \rho_2\frac{\partial}{\partial z_0} + \rho_3\frac{\partial}{\partial z_1}, \tag{A25}
$$

where, upon defining the functions

$$
\alpha_1 = \frac{2(z_0)^3 z_1}{r}\gamma\left(\frac{1}{4} - \frac{1}{\gamma(\gamma+2)}\right), \tag{A26}
$$

$$
\alpha_2 = \frac{2}{\gamma}\frac{1}{r^2}\frac{(z_0)^2}{z_1}\left(\frac{1}{4} - \frac{1}{\gamma(\gamma+2)}\right), \tag{A27}
$$

$$
\alpha_3 = \frac{(z_0)^3 z_1}{r}\gamma\left[\frac{1}{(\gamma+2)} - \frac{(\gamma+2)}{2\gamma}\right], \tag{A28}
$$

$$
\alpha_4 = \frac{z_0}{\gamma}\frac{1}{r^2}\left[\frac{1}{(\gamma+2)} - \frac{(\gamma+2)}{2\gamma}\right], \tag{A29}
$$

$$\alpha_5 = \frac{z_0}{2r}\left(\frac{2}{\gamma} - 1\right)\left[\frac{1}{2z_1}\left(1 - \frac{2}{\gamma}\right) + (z_0)^2 z_1 \gamma\right], \tag{A30}$$

$$\alpha_6 = \frac{(z_0)^2}{4z_1 r^2}\left(\frac{2}{\gamma} - 1\right)^2, \tag{A31}$$

$$\alpha_7 = \frac{(z_0)^4 z_1 (8 + (\gamma - 2)\gamma^2)}{4r^2(\gamma + 2)^2}, \tag{A32}$$

$$\alpha_8 = \frac{(z_0)^4 z_1 \gamma}{2r^2}\left[\frac{1}{(\gamma + 2)} - \frac{(\gamma + 2)}{2\gamma}\right], \tag{A33}$$

$$\alpha_9 = \frac{z_0}{4r^2}\left(\frac{2}{\gamma} - 1\right)\left[\frac{1}{(\gamma + 2)} - \frac{(\gamma + 2)}{2\gamma} + \frac{4(z_0)^2(z_1)^2\gamma(\gamma + 1)}{(\gamma + 2)^2}\right], \tag{A34}$$

$$\alpha_{10} = \frac{z_0 z_1}{4r}\left(\frac{2}{\gamma} + 1\right)\left[-\frac{1}{(z_1)^2}\left(1 - \frac{2}{\gamma}\right) + 2(z_0)^2\gamma\right], \tag{A35}$$

$$\alpha_{11} = \frac{z_0}{4r^2}\left(\frac{4}{\gamma^2} - 1\right), \tag{A36}$$

$$\alpha_{12} = \frac{(z_0)^2 z_1}{2r^2}\left(\frac{2}{\gamma} + 1\right)\left[-\frac{1}{(z_1)^2}\left(\frac{1}{4} - \frac{1}{\gamma(\gamma + 2)}\right) + \frac{2(z_0)^2\gamma(\gamma + 1)}{(\gamma + 2)^2}\right], \tag{A37}$$

$$\alpha_{13} = \frac{(z_0)^3(z_1)^2\gamma}{r^2}\left(\frac{1}{4} - \frac{1}{\gamma(\gamma + 2)}\right), \tag{A38}$$

$$\alpha_{14} = \frac{z_0(8 - (\gamma - 6)\gamma^2)}{16r^2\gamma^2}, \tag{A39}$$

we find that

$$\rho_1 = \alpha_1 + \alpha_3 + \alpha_5 + \alpha_{10} = 0, \tag{A40}$$

$$\rho_2 = \alpha_2 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_{12} = 0, \tag{A41}$$

$$\rho_3 = \alpha_4 + \alpha_9 + \alpha_{11} + \alpha_{13} + \alpha_{14} = 0. \tag{A42}$$

In the course of performing the calculation, the definition (24) leads to the useful identity

$$\frac{1}{\gamma^2} = \frac{4(z_0)^2(z_1)^2}{(\gamma^2 - 4)}. \tag{A43}$$

An analogous procedure shows that

$$[\xi_2, \xi_3] = [\xi_3, \xi_1] = 0, \tag{A44}$$

with the help of two additional sets of 14 nonvanishing functions, one set for each Lie bracket in (A44). For example, in the Lie bracket among ξ_2 and ξ_3, the coefficient of $\frac{\partial}{\partial z_0}$ is the function

$$\begin{aligned}
\rho &= -\frac{z_0}{4r^2} - \frac{(z_0)^2}{16r^2}\left(1 - \frac{4}{\gamma(\gamma + 2)}\right)2z_0(z_1)^2\gamma\left(1 - \frac{\gamma^2}{(\gamma - 2)^2}\right)\\
&+ \frac{1}{16r^2}\left(1 + \frac{4}{\gamma(\gamma - 2)}\right)\left[2z_0\left(1 - \frac{4}{\gamma(\gamma + 2)}\right) - 2(z_0)^3(z_1)^2\gamma\left(\frac{\gamma^2}{(\gamma + 2)^2} - 1\right)\right]\\
&+ \frac{z_0}{8r^2}\left(1 - \frac{4}{\gamma(\gamma - 2)}\right)\left[\left(1 - \frac{4}{\gamma(\gamma + 2)}\right) + (z_0)^2(z_1)^2\gamma\left(\frac{\gamma^2}{(\gamma + 2)^2} - 1\right)\right]\\
&+ \frac{z_0}{8r^2}\left(1 + \frac{4}{\gamma(\gamma + 2)}\right)\left[\frac{1}{2}\left(1 + \frac{4}{\gamma(\gamma - 2)}\right) + (z_0)^2(z_1)^2\gamma\left(1 - \frac{\gamma^2}{(\gamma - 2)^2}\right)\right]\\
&= 0. \tag{A45}
\end{aligned}$$

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