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PAPER

Unextendible and uncompletable product bases in every bipartition

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Abstract

Unextendible product basis is an important object in quantum information theory and features a broad spectrum of applications, ranging from quantum nonlocality to quantum cryptography. A generalized concept called uncompletable product basis also attracts much attention. In this paper, we find some unextendible product bases that are uncompletable product bases in every bipartition, which answers a 19 year-old open question proposed by DiVincenzo *et al* (2003 *Commun. Math. Phys.* **238** 379–410). As a consequence, we connect such unextendible product bases to local hiding of information, positive-partial-transpose entangled states and genuinely entangled states. Furthermore, we give a sufficient condition for the existence of an unextendible product basis that is still unextendible in every bipartition, and the existence of such a UPB is another open question proposed by Demianowic *et al* (2018 *Phys. Rev.* A **98** 012313). Our results advance the understanding of the geometry of unextendible product bases.

1. Introduction

An unextendible product basis (UPB) in a multipartite quantum system is an incomplete orthogonal product basis (OPB) whose complementary subspace contains no product state [1]. UPBs have a lot of applications in quantum information. They can be used to construct bound entangled states [1]. UPBs cannot be perfectly distinguished under local positive operator-valued measures (POVMs) and classical communication [1], which shows the phenomenon of quantum nonlocality without entanglement [2]. Moreover, UPBs were also connected to strong quantum nonlocality [3–5], more nonlocality with less purity [6], Bell inequalities with no quantum violation [7, 8] and quantum secret sharing [9].

In 2003, DiVincenzo *et al* generalized the concept of UPBs [10]. An uncompletable product basis (UCPB) in a multipartite quantum system is an incomplete OPB, which cannot be extended to a complete OPB [10]. An incomplete OPB is a strongly uncompletable product basis (SUCPB), if it is a UCPB in any locally extended Hilbert space [10]. Actually, the set of all UPBs is a proper subset of the set of all SUCPBs, and the set of all SUCPBs is a proper subset of the set of all UCPBs. See also figure 2 for the inclusion relation of these three sets. It is known that UPBs and SUCPBs cannot be perfectly distinguished under local POVMs and classical communication, and UCPBs cannot be perfectly distinguished under local projective measurements and classical communication [1, 10]. In [10], DiVincenzo *et al* proposed an open question: whether there exists a UPB, which is a UCPB in every bipartition? This open question exists for 19 years because there are few constructions of UPBs in multipartite systems, and it is difficult to show UCPBs in bipartite systems. Such UPBs can be used to understand the geometry of UPBs. There exists another famous open question for UPBs [11]: can we find a UPB, which is still a UPB in every bipartition? Such UPBs cannot be perfectly distinguished under local POVMs and classical communication in every bipartition [1], and can be used to construct genuinely entangled subspaces [11]. Recently, Demianowice showed that such UPBs with the minimum size do not exist [12]. However, the existence of such UPBs is still unknown.

In this work, we address the 19 year-old open question in [10], by presenting a UPB with a stronger property, which is an SUCPB in every bipartition. We also show that such UPBs can be connected to local hiding of information, positive-partial-transpose entangled states and genuinely entangled states. Tile structures in bipartite systems provide an efficient method for constructing bipartite UPBs [10, 13]. We generalize the tile structures to multipartite systems, and give a sufficient condition for the existence of a UPB that is still a UPB in every bipartition. This sufficient condition is intuitive, and one can search such UPBs through computer under this condition. Thus this sufficient condition is hopeful for solving the open question in [11].

The rest of this paper is organized as follows. In section 2, we introduce the concepts of UPBs, UCPBs, and SUCPBs. Next, in section 3, we find a UPB that is an SUCPB in every bipartition for arbitrary three-, and four-partite system. In section 4, we give a sufficient condition for the existence of a UPB that is still a UPB in every bipartition. Finally, we conclude in section 5.

2. Preliminaries

In this paper, we do not normalize product states for simplicity. We denote $\mathbb{Z}_n := \{0, 1, \ldots, n-1\}$ and $w_n := \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$. For a matrix M, let sum(M) be the sum of all elements. Assume $\{|i\rangle_A\}_{i\in\mathbb{Z}_m}$ and $\{|j\rangle_B\}_{j\in\mathbb{Z}_n}$ are the computational bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. For any bipartite state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, it can be expressed by

$$|\psi\rangle = \sum_{i\in\mathbb{Z}_m, j\in\mathbb{Z}_n} a_{i,j}|i\rangle_A|j\rangle_B. \tag{1}$$

Then $|\psi\rangle$ corresponds to an $m \times n$ matrix

$$M = (a_{i,j})_{i\in\mathbb{Z}_m, j\in\mathbb{Z}_n}. \tag{2}$$

If rank$(M) = 1$, then $|\psi\rangle$ is a product state; if rank$(M) \geqslant 2$, then $|\psi\rangle$ is an entangled state. Assume $|\psi_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ corresponds to an $m \times n$ matrix M_i for $i = 1, 2$, then $\langle\psi_1|\psi_2\rangle = \mathrm{Tr}(M_1^\dagger M_2)$. Let $\mathcal{H} = \otimes_{i=1}^n \mathcal{H}_i$ be an n-partite Hilbert space. An orthogonal product set (OPS) in $\mathcal{H}$ is a set of orthogonal product states, and an OPB in $\mathcal{H}$ is an OPS which spans $\mathcal{H}$. Given $\mathcal{H} = \otimes_{i=1}^n \mathcal{H}_i$, let $\mathcal{H}_{\text{ext}} = \otimes_{i=1}^n (\mathcal{H}_i \oplus \mathcal{H}_i')$ be a locally extended Hilbert space of $\mathcal{H}$, where H_i' is a local extension. Now, we review some definitions.

Definition 1. Let $\mathcal{S}$ be an OPS in $\mathcal{H} = \otimes_{i=1}^n \mathcal{H}_i$. The set $\mathcal{S}$ spans a subspace $\mathcal{H}_\mathcal{S}$ in $\mathcal{H}$, and Dim$(\mathcal{H}_\mathcal{S}) <$ Dim$(\mathcal{H})$. If the complementary subspace $\mathcal{H}_\mathcal{S}^\perp$ contains no product state, then $\mathcal{S}$ is called a UPB. If $\mathcal{S}$ cannot be extended to an OPB in $\mathcal{H}$, then $\mathcal{S}$ is called a *UCPB*. Moreover, if $\mathcal{S}$ is a UCPB in any locally extended Hilbert space $\mathcal{H}_{\text{ext}} = \otimes_{i=1}^n (\mathcal{H}_i \oplus \mathcal{H}_i')$, then $\mathcal{S}$ is called an SUCPB.

From definition 1, a UPB or an SUCPB must be a UCPB. We can always obtain a UPB from a UCPB $\mathcal{S}$, by adding some orthogonal product states to $\mathcal{S}$ from $\mathcal{H}_\mathcal{S}^\perp$ till the new OPS is a UPB. Moreover, UPBs and SUCPBs cannot be perfectly distinguished under local POVMs and classical communication, and UCPBs cannot be perfectly distinguished under local projective measurements and classical communication [1].

For an OPS $\mathcal{S} = \{|\psi_i\rangle\}_{i=1}^s$ in $\mathcal{H} = \otimes_{i=1}^n \mathcal{H}_i$ with Dim$(\mathcal{H}) = D$ (where $s < D$), we can define a mixed state that is proportional to the projector on $\mathcal{H}_\mathcal{S}^\perp$,

$$\overline{\rho}_\mathcal{S} = \frac{1}{D-s}\left(\mathbb{I} - \sum_{i=1}^s |\psi_i\rangle\langle\psi_i|\right). \tag{3}$$

Applying partial transposition map to $\overline{\rho}_\mathcal{S}$ in any bipartition, then we can find that $(\mathbb{I} \otimes T)\overline{\rho}_\mathcal{S} \geqslant 0$. It means that $\overline{\rho}_\mathcal{S}$ has the positive partial transpose (PPT) property in any bipartition. If $\mathcal{S}$ is a UPB, then $\overline{\rho}_\mathcal{S}$ must be entangled from the definition. Thus $\overline{\rho}_\mathcal{S}$ is a PPT entangled state, which is also a bound entangled state (no pure entanglement can be distilled) [1, 10]. However, if $\mathcal{S}$ is a UCPB, $\overline{\rho}_\mathcal{S}$ is either separable or entangled [1, 10].

It is difficult to show that an OPS is an SUCPB from the definition. There exists a sufficient condition.

Lemma 1. *Let $\mathcal{S}$ be an OPS in $\mathcal{H} = \otimes_{i=1}^n \mathcal{H}_i$. If all the product states in $\mathcal{H}_\mathcal{S}^\perp$ cannot span $\mathcal{H}_\mathcal{S}^\perp$, then $\mathcal{S}$ is an SUCPB.*

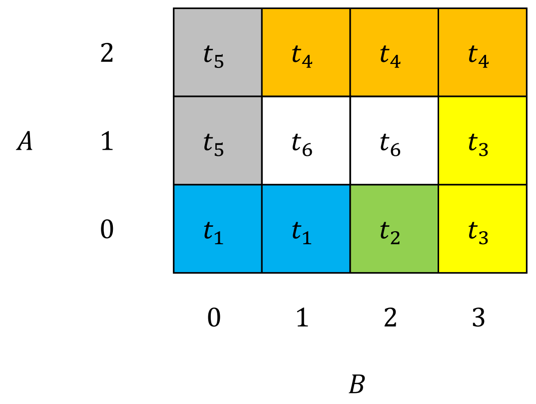

Figure 1. Tile structure with six tiles in $\mathbb{C}^3 \otimes \mathbb{C}^4$.

Proof. If all the product states in $\mathcal{H}_{\mathcal{S}}^{\perp}$ cannot span $\mathcal{H}_{\mathcal{S}}^{\perp}$, then $\overline{\rho}_{\mathcal{S}}$ must be entangled by theorem 2(b) in [14]. Further, according to proposition 1 in [10], $\mathcal{S}$ is an SUCPB. □

By lemma 1, a UPB must be an SUCPB. However, the converse is not true. We will give an example of SUCPB, which is not a UPB.

Tile structures can be used to construct UPBs [10, 13, 15]. Next, we show that tile structures can also be used to construct SUCPBs. A tile structure $\mathcal{T}$ in $\mathbb{C}^m \otimes \mathbb{C}^n$ is an $m \times n$ rectangle, which can be partitioned into s disjoint tiles $\{t_i\}_{i=1}^s$. Each tile t_i is a rectangle. We denote $\mathcal{T} := \cup_{i=1}^s t_i$. For example, figure 1 gives a tile structure $\mathcal{T} = \cup_{i=1}^6 t_i$ with six tiles in $\mathbb{C}^3 \otimes \mathbb{C}^4$. Any tile t_i of $\mathcal{T} = \cup_{i=1}^s t_i$ has row coordinates $\{p_0, p_1, \ldots, p_{k-1}\}_A$ and column coordinates $\{q_0, q_1, \ldots, q_{\ell-1}\}_B$, and we denote it as $t_i = \{p_0, p_1, \ldots, p_{k-1}\}_A \times \{q_0, q_1, \ldots, q_{\ell-1}\}_B$, where $\{p_0, p_1, \ldots, p_{k-1}\}$ and $\{q_0, q_1, \ldots, q_{\ell-1}\}$ are subsets of $\mathbb{Z}_m$ and $\mathbb{Z}_n$, respectively. For tile t_i, we can construct an OPS of size $k\ell$ in $\mathbb{C}^m \otimes \mathbb{C}^n$,

$$\mathcal{A}_i = \left\{ |\psi_i(a,b)\rangle := \left(\sum_{e \in \mathbb{Z}_k} m_{a,e} |p_e\rangle \right)_A \left(\sum_{e \in \mathbb{Z}_\ell} n_{b,e} |q_e\rangle \right)_B |(a,b) \in \mathbb{Z}_k \times \mathbb{Z}_\ell \right\}. \tag{4}$$

Here the coefficient matrix $M = (m_{a,e})_{a,e \in \mathbb{Z}_k}$ is a $k \times k$ row orthogonal matrix (row vectors are mutually orthogonal), and $m_{0,e} = 1$ for $e \in \mathbb{Z}_k$, and the coefficient matrix $N = (n_{b,e})_{b,e \in \mathbb{Z}_\ell}$ is an $\ell \times \ell$ row orthogonal matrix, and $n_{0,e} = 1$ for $e \in \mathbb{Z}_\ell$. For example, we can choose $M = (w_k^{ae})_{a,e \in \mathbb{Z}_k}$, and $N = (w_\ell^{be})_{b,e \in \mathbb{Z}_\ell}$. Since those tiles in $\mathcal{T} = \cup_{i=1}^s t_i$ are disjoint, we can obtain an OPB

$$\mathcal{B} := \cup_{i=1}^s \mathcal{A}_i \tag{5}$$

in $\mathbb{C}^m \otimes \mathbb{C}^n$. Further, we define the 'stopper' state as

$$|S\rangle = \left(\sum_{i \in \mathbb{Z}_m} |i\rangle \right)_A \left(\sum_{j \in \mathbb{Z}_n} |j\rangle \right)_B. \tag{6}$$

We mainly consider the following OPS,

$$\mathcal{S} := \cup_{i=1}^s (\mathcal{A}_i \backslash \{|\psi_i(0,0)\rangle\}) \cup \{|S\rangle\}. \tag{7}$$

For example, using the tile structure $\mathcal{T} = \cup_{i=1}^6 t_i$ in figure 1, we obtain an OPB $\mathcal{B} = \cup_{i=1}^6 \mathcal{A}_i$ in $\mathbb{C}^3 \otimes \mathbb{C}^4$, where

$$\begin{aligned}
\mathcal{A}_1 &= \{|\psi_1(0,b)\rangle = |0\rangle_A(|0\rangle + (-1)^b|1\rangle)_B | b \in \mathbb{Z}_2\}, \\
\mathcal{A}_2 &= \{|\psi_2(0,0)\rangle = |0\rangle_A |2\rangle_B\}, \\
\mathcal{A}_3 &= \{|\psi_3(a,0)\rangle = (|0\rangle + (-1)^a|1\rangle)_A |3\rangle_B | a \in \mathbb{Z}_2\}, \\
\mathcal{A}_4 &= \{|\psi_4(0,b)\rangle = |2\rangle_A(|1\rangle + w_3^b|2\rangle + w_3^{2b}|3\rangle)_B | b \in \mathbb{Z}_3\}, \\
\mathcal{A}_5 &= \{|\psi_5(a,0)\rangle = (|1\rangle + (-1)^a|2\rangle)_A |0\rangle_B | a \in \mathbb{Z}_2\}, \\
\mathcal{A}_6 &= \{|\psi_6(0,b)\rangle = |1\rangle_A(|1\rangle + (-1)^b|2\rangle)_B | b \in \mathbb{Z}_2\}.
\end{aligned} \tag{8}$$

The 'stopper' state is

$$|S\rangle = (|0\rangle + |1\rangle + |2\rangle)_A(|0\rangle + |1\rangle + |2\rangle + |3\rangle)_B. \tag{9}$$

Next, we show that

$$\mathcal{S} := \cup_{i=1}^{6}(\mathcal{A}_i \backslash \{|\psi_i(0,0)\rangle\}) \cup \{|S\rangle\} \tag{10}$$

is an SUCPB in $\mathbb{C}^3 \otimes \mathbb{C}^4$.

Example 1. In $\mathbb{C}^3 \otimes \mathbb{C}^4$, the OPS $\mathcal{S}$ given by equation (10) is an SUCPB.

Proof. Let $\mathcal{S}_1 := \cup_{i=1}^{6}(\mathcal{A}_i \backslash \{|\psi_i(0,0)\rangle\})$ and $\mathcal{S}_2 := \cup_{i=1}^{6}|\psi_i(0,0)\rangle$. We know that $\mathcal{S}_1 \cup \mathcal{S}_2$ is an OPB in $\mathbb{C}^3 \otimes \mathbb{C}^4$. Since $\mathcal{H}_{\mathcal{S}_1} \subset \mathcal{H}_{\mathcal{S}}$, it implies $\mathcal{H}_{\mathcal{S}}^{\perp} \subset \mathcal{H}_{\mathcal{S}_1}^{\perp} = \mathcal{H}_{\mathcal{S}_2}$. Then for any product state $|\psi\rangle \in \mathcal{H}_{\mathcal{S}}^{\perp}$, there exists $a_i \in \mathbb{C}$ for $1 \leqslant i \leqslant 6$, such that

$$|\psi\rangle = \sum_{i=1}^{6} a_i |\psi_i(0,0)\rangle.$$

Next, $|\psi\rangle$ corresponds to a 3×4 matrix,

$$M = \begin{pmatrix} a_5 & a_4 & a_4 & a_4 \\ a_5 & a_6 & a_6 & a_3 \\ a_1 & a_1 & a_2 & a_3 \end{pmatrix}.$$

Note that M has a similar structure to the tile structure in figure 1. The 'stopper' state $|S\rangle$ corresponds to a all-ones matrix J, where every element is equal to one. Since $|\psi\rangle$ is a product state and $\langle S|\psi\rangle = 0$, we have $\text{rank}(M) = 1$ and $\text{sum}(M) = 0$. This is only possible for

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_1 & a_2 & 0 \end{pmatrix}, \quad 2a_1 + a_2 = 0.$$

It means that $\mathcal{H}_{\mathcal{S}}^{\perp}$ contains only one product state $|0\rangle(|0\rangle + |1\rangle - 2|2\rangle)$. Since $\text{Dim}(\mathcal{H}_{\mathcal{S}}^{\perp}) = 5$, the OPS $\mathcal{S}$ is an SUCPB by lemma 1. □

Since there exists a product state $|\psi\rangle = |0\rangle(|0\rangle + |1\rangle - 2|2\rangle) \in \mathcal{H}_{\mathcal{S}}^{\perp}$, $\mathcal{S}$ is not a UPB. However, if we add $|\psi\rangle$ to $\mathcal{S}$, then $\mathcal{S}' = \mathcal{S} \cup \{|\psi\rangle\}$ must be a UPB in $\mathbb{C}^3 \otimes \mathbb{C}^4$. In fact, for any product state $|\phi\rangle \in \mathcal{H}_{\mathcal{S}'}^{\perp}$, $|\phi\rangle$ corresponds to a 3×4 matrix,

$$M = \begin{pmatrix} a_5 & a_4 & a_4 & a_4 \\ a_5 & a_6 & a_6 & a_3 \\ a_1 & a_1 & a_2 & a_3 \end{pmatrix},$$

where $a_1 = a_2$, $\text{rank}(M) = 1$ and $\text{sum}(M) = 0$. Such a matrix M does not exist. From the above discussion, we can obtain a sufficient condition for the construction of UPBs by tile structures.

Lemma 2. *For a tile structure $\mathcal{T} = \cup_{i=1}^{s} t_i$ $(s \geqslant 5)$ in $\mathbb{C}^m \otimes \mathbb{C}^n$, if any r $(2 \leqslant r \leqslant s-1)$ tiles cannot form a rectangle, then the OPS $\mathcal{S}$ given by equation (7) is a UPB in $\mathbb{C}^m \otimes \mathbb{C}^n$.*

The tile structure in lemma 2 is the U-tile structure proposed in reference [13]. In reference [1], the authors gave a UCPB, which is not an SUCPB. Let $\mathcal{D}(\text{UPB})$ be the set of all UPBs; $\mathcal{D}(\text{SUCPB})$ be the set of all SUCPBs; and $\mathcal{D}(\text{UCPB})$ be the set of all UCPBs. Then following set inclusion relation is obtained,

$$\mathcal{D}(\text{UPB}) \subsetneqq \mathcal{D}(\text{SUCPB}) \subsetneqq \mathcal{D}(\text{UCPB}).$$

See also figure 2 for the inclusion relation of these three sets. In reference [10], the authors proposed an open question: can we find a UPB which is a UCPB in every bipartition? We will give a positive answer, by showing a stronger UPB, which is an SUCPB in every bipartition.

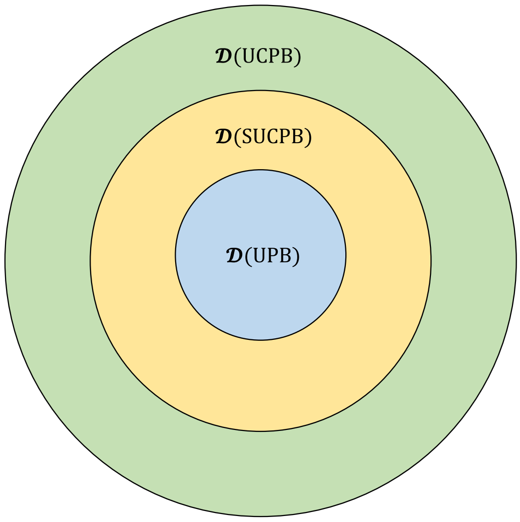

Figure 2. A set inclusion relation among the set of all UPBs $\mathcal{D}$(UPB), the set of all SUCPBs $\mathcal{D}$(SUCPB), and the set of all UCPBs $\mathcal{D}$(UCPB). The set $\mathcal{D}$(UPB) is a proper subset of $\mathcal{D}$(SUCPB), and $\mathcal{D}$(SUCPB) is a proper subset of $\mathcal{D}$(UCPB).

3. The existence of a UPB that is an SUCPB in every bipartition

In this section, we show that there exists a UPB which is an SUCPB in every bipartition in any three, and four-partite system. Since any OPS in $\mathbb{C}^2 \otimes \mathbb{C}^n$ can be extended to an OPB [1, 10], the minimum system for the existence of such UPBs is $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. Consider an OPB $\cup_{i=1}^{9} \mathcal{A}_i$ in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$,

$$
\begin{aligned}
\mathcal{A}_1 &:= \{|\psi_1(i,j)\rangle = |\xi_i\rangle_A |0\rangle_B |\eta_j\rangle_C | (i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_2 &:= \{|\psi_2(i,j)\rangle = |\xi_i\rangle_A |\eta_j\rangle_B |2\rangle_C | (i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_3 &:= \{|\psi_3(i,j)\rangle = |2\rangle_A |\xi_i\rangle_B |\eta_j\rangle_C | (i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_4 &:= \{|\psi_4(i,j)\rangle = |\eta_i\rangle_A |2\rangle_B |\xi_j\rangle_C | (i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_5 &:= \{|\psi_5(i,j)\rangle = |\eta_i\rangle_A |\xi_j\rangle_B |0\rangle_C | (i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_6 &:= \{|\psi_6(i,j)\rangle = |0\rangle_A |\eta_i\rangle_B |\xi_j\rangle_C | (i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_7 &:= \{|\psi_7(0,0)\rangle = |0\rangle_A |0\rangle_B |0\rangle_C\},\\
\mathcal{A}_8 &:= \{|\psi_8(0,0)\rangle = |1\rangle_A |1\rangle_B |1\rangle_C\},\\
\mathcal{A}_9 &:= \{|\psi_9(0,0)\rangle = |2\rangle_A |2\rangle_B |2\rangle_C\},
\end{aligned} \tag{11}
$$

where $|\eta_s\rangle_X = |0\rangle_X + (-1)^s|1\rangle_X$, $|\xi_s\rangle_X = |1\rangle_X + (-1)^s|2\rangle_X$ for $s \in \mathbb{Z}_2$, and $X \in \{A, B, C\}$. Actually, the OPB $\cup_{i=1}^{9} \mathcal{A}_i$ is obtained from the tile structure in tripartite system (we will introduce tile structures in multipartite systems in section 4). Define the 'stopper' state,

$$
|S\rangle = (|0\rangle + |1\rangle + |2\rangle)_A (|0\rangle + |1\rangle + |2\rangle)_B (|0\rangle + |1\rangle + |2\rangle)_C. \tag{12}
$$

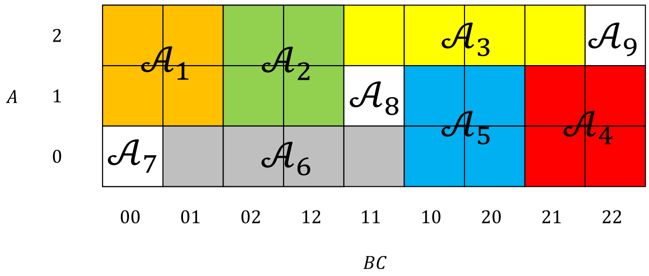

Figure 3. The corresponding tile structure in $\mathbb{C}^3 \otimes \mathbb{C}^9$ of the OPB $\cup_{i=1}^{9}\mathcal{A}_i$ (equation (11)) in $A|BC$ bipartition.

Then

$$\mathcal{U} := \cup_{i=1}^{6}(\mathcal{A}_i \backslash \{|\psi_i(0,0)\rangle\}) \cup \{|S\rangle\} \tag{13}$$

is a UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ [16]. Now, we have the following lemma.

Lemma 3. *In $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, the UPB $\mathcal{U}$ given by equation (13) is an SUCPB in every bipartition.*

Proof. First, we consider the bipartition $A|BC$. The OPB $\cup_{i=1}^{9}\mathcal{A}_i$ given by equation (11) in $A|BC$ bipartition corresponds the tile structure in figure 3. Next, we show that the OPS $\mathcal{U}_{A|BC}$ is an SUCPB in $\mathbb{C}^3 \otimes \mathbb{C}^9$. For the same discussion as example 1, we can assume that $|\psi\rangle \in \mathcal{H}^{\perp}_{\mathcal{U}_{A|BC}}$ is a product state.

Let $\mathcal{S}_1 := \cup_{i=1}^{6}(\mathcal{A}_i \backslash \{|\psi_i(0,0)\rangle\})_{A|BC}$ and $\mathcal{S}_2 := \cup_{i=1}^{9}|\psi_i(0,0)\rangle_{A|BC}$. We know that $\mathcal{S}_1 \cup \mathcal{S}_2$ is an OPB in $\mathbb{C}^3 \otimes \mathbb{C}^9$. Since $\mathcal{H}_{\mathcal{S}_1} \subset \mathcal{H}_{\mathcal{U}_{A|BC}}$, it implies $\mathcal{H}^{\perp}_{\mathcal{U}_{A|BC}} \subset \mathcal{H}^{\perp}_{\mathcal{S}_1} = \mathcal{H}_{\mathcal{S}_2}$. Then for any product state $|\psi\rangle \in \mathcal{H}^{\perp}_{\mathcal{U}_{A|BC}}$, there exists $a_i \in \mathbb{C}$ for $1 \leqslant i \leqslant 9$, such that

$$|\psi\rangle = \sum_{i=1}^{9} a_i |\psi_i(0,0)\rangle_{A|BC}. \tag{14}$$

Then $|\psi\rangle$ corresponds to a 3×9 matrix

$$M = \begin{pmatrix} a_1 & a_1 & a_2 & a_2 & a_3 & a_3 & a_3 & a_3 & a_9 \\ a_1 & a_1 & a_2 & a_2 & a_8 & a_5 & a_5 & a_4 & a_4 \\ a_7 & a_6 & a_6 & a_6 & a_6 & a_5 & a_5 & a_4 & a_4 \end{pmatrix},$$

where $a_i \in \mathbb{C}$ for $1 \leqslant i \leqslant 9$. Note that M has a similar structure to the tile structure in figure 3. Since $|\psi\rangle$ is a product state, then $\mathrm{rank}(M) = 1$. Further, since $|S\rangle_{A|BC}$ is orthogonal to $|\psi\rangle$, we have $\mathrm{sum}(M) = 0$. There are only four cases,

(i) $a_1 + a_2 = 0$, $a_i = 0$ for $1 \leqslant i \leqslant 9$ and $i \neq 1, 2$; (ii) $4a_3 + a_9 = 0$, $a_i = 0$ for $1 \leqslant i \leqslant 9$ and $i \neq 3, 9$; (iii) $a_4 + a_5 = 0$, $a_i = 0$ for $1 \leqslant i \leqslant 9$ and $i \neq 4, 5$; (iv) $4a_6 + a_7 = 0$, $a_i = 0$ for $1 \leqslant i \leqslant 9$ and $i \neq 6, 7$.

It means that there are only four product states in $\mathcal{H}^{\perp}_{\mathcal{U}_{A|BC}}$: $(|1\rangle + |2\rangle)_A(|00\rangle + |01\rangle - |02\rangle - |12\rangle)_{BC}$, $|2\rangle_A(|11\rangle + |10\rangle + |20\rangle + |21\rangle - 4|22\rangle)_{BC}$, $(|0\rangle + |1\rangle)_A(|10\rangle + |20\rangle - |21\rangle - |22\rangle)_{BC}$, and $|0\rangle_A(|01\rangle + |02\rangle + |12\rangle + |11\rangle - 4|00\rangle)_{BC}$. Since $\mathrm{Dim}(\mathcal{H}^{\perp}_{\mathcal{U}_{A|BC}}) = 8$, the OPS $\mathcal{U}_{A|BC}$ is an SUCPB by lemma 1.

Further, since the OPB $\cup_{i=1}^{9}\mathcal{A}_i$ given by equation (11) in any bipartition of $\{A|BC, B|AC, C|AB\}$ corresponds to a tile structure similar to figure 3, we obtain that $\mathcal{U}_{A|BC}$, $\mathcal{U}_{B|AC}$, and $\mathcal{U}_{C|AB}$ are all SUCPBs. Thus $\mathcal{U}$ is an SUCPB in every bipartition. □

Now, we can give a more general theorem.

Theorem 1. *In $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_C}$, $d_A, d_B, d_C \geqslant 3$, there exists a UPB which is an SUCPB in every bipartition.*

The proof of theorem 1 is given in appendix A. Next, we show that there also exists a four-partite UPB which is an SUCPB in every bipartition. Consider an OPB in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$,

$$\begin{aligned}
\mathcal{A}_1 &:= \{|\psi_1(i,j,k)\rangle = |\xi_i\rangle_A|\eta_j\rangle_B|0\rangle_C|\xi_k\rangle_D|(i,j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_2 &:= \{|\psi_2(i,j,k)\rangle = |\xi_i\rangle_A|2\rangle_B|\eta_j\rangle_C|\eta_k\rangle_D|(i,j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_3 &:= \{|\psi_3(i,j,k)\rangle = |\xi_i\rangle_A|\xi_j\rangle_B|\xi_k\rangle_C|2\rangle_D|(i,j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_4 &:= \{|\psi_4(i,0,0)\rangle = |\xi_i\rangle_A|2\rangle_B|0\rangle_C|2\rangle_D|i \in \mathbb{Z}_2\},\\
\mathcal{A}_5 &:= \{|\psi_5(i,j,k)\rangle = |2\rangle_A|\eta_i\rangle_B|\xi_j\rangle_C|\eta_k\rangle_D|(i,j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_6 &:= \{|\psi_6(i,0,0)\rangle = |2\rangle_A|\eta_i\rangle_B|0\rangle_C|0\rangle_D|i \in \mathbb{Z}_2\},\\
\mathcal{A}_7 &:= \{|\psi_7(i,0,0)\rangle = |2\rangle_A|0\rangle_B|\xi_i\rangle_C|2\rangle_D|i \in \mathbb{Z}_2\},\\
\mathcal{A}_8 &:= \{|\psi_8(i,0,0)\rangle = |2\rangle_A|2\rangle_B|2\rangle_C|\eta_i\rangle_D|i \in \mathbb{Z}_2\},\\
\mathcal{A}_9 &:= \{|\psi_9(i,j,k)\rangle = |\eta_i\rangle_A|\xi_j\rangle_B|2\rangle_C|\eta_k\rangle_D|(i,j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_{10} &:= \{|\psi_{10}(i,j,k)\rangle = |\eta_i\rangle_A|0\rangle_B|\xi_j\rangle_C|\xi_k\rangle_D|(i,j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_{11} &:= \{|\psi_{11}(i,j,k)\rangle = |\eta_i\rangle_A|\eta_j\rangle_B|\eta_k\rangle_C|0\rangle_D|(i,j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_{12} &:= \{|\psi_{12}(i,0,0)\rangle = |\eta_i\rangle_A|0\rangle_B|2\rangle_C|0\rangle_D|i \in \mathbb{Z}_2\},\\
\mathcal{A}_{13} &:= \{|\psi_{13}(i,j,k)\rangle = |0\rangle_A|\xi_i\rangle_B|\eta_j\rangle_C|\xi_k\rangle_D|(i,j,k) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\},\\
\mathcal{A}_{14} &:= \{|\psi_{14}(i,0,0)\rangle = |0\rangle_A|\xi_i\rangle_B|2\rangle_C|2\rangle_D|i \in \mathbb{Z}_2\},\\
\mathcal{A}_{15} &:= \{|\psi_{15}(i,0,0)\rangle = |0\rangle_A|2\rangle_B|\eta_i\rangle_C|0\rangle_D|i \in \mathbb{Z}_2\},\\
\mathcal{A}_{16} &:= \{|\psi_{16}(i,0,0)\rangle = |0\rangle_A|0\rangle_B|0\rangle_C|\xi_i\rangle_D|i \in \mathbb{Z}_2\},\\
\mathcal{A}_{17} &:= \{|\psi_{17}(0,0,0)\rangle = |1\rangle_A|1\rangle_B|1\rangle_C|1\rangle_D\},
\end{aligned} \tag{15}$$

where $|\eta_s\rangle_X = |0\rangle_X + (-1)^s|1\rangle_X$, $|\xi_s\rangle_X = |1\rangle_X + (-1)^s|2\rangle_X$ for $s \in \mathbb{Z}_2$, and $X \in \{A, B, C, D\}$. The 'stopper' state is,

$$|S\rangle = (|0\rangle + |1\rangle + |2\rangle)_A(|0\rangle + |1\rangle + |2\rangle)_B(|0\rangle + |1\rangle + |2\rangle)_C(|0\rangle + |1\rangle + |2\rangle)_D. \tag{16}$$

Then

$$\mathcal{V} := \cup_{i=1}^{16}(\mathcal{A}_i \backslash \{|\psi_i(0,0,0)\rangle\}) \cup \{|S\rangle\} \tag{17}$$

is a UPB in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ [4]. We can show that this UPB is an SUCPB in every bipartition.

Lemma 4. *In $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, the UPB $\mathcal{V}$ given by equation (17) is an SUCPB in every bipartition.*

Proof. We need to consider the bipartition set $\{A|BCD, B|ACD, C|ABD, D|ABC, AB|CD, AC|BD, AD|BC\}$. Since the OPB $\cup_{i=1}^{17}\mathcal{A}_i$ given by equation (15) in any bipartition of $\{A|BCD, B|ACD, C|ABD, D|ABC\}$ corresponds to a tile structure similar to figure 5, and $\cup_{i=1}^{17}\mathcal{A}_i$ in any bipartition of $\{AB|CD, AC|BD, AD|BC\}$ also corresponds to a tile structure similar to figure 6 (see figures 10 and 11 in appendix B). We only need to consider $\mathcal{V}_{A|BCD}$ and $\mathcal{V}_{AB|CD}$.

For $\mathcal{V}_{A|BCD}$, we assume that $|\psi\rangle \in \mathcal{H}^{\perp}_{\mathcal{V}_{A|BCD}}$ is a product state. Let $\mathcal{S}_1 := \cup_{i=1}^{16}(\mathcal{A}_i \backslash \{|\psi_i(0,0,0)\rangle\})_{A|BCD}$ and $\mathcal{S}_2 := \cup_{i=1}^{17}|\psi_i(0,0,0)\rangle_{A|BCD}$. We know that $\mathcal{S}_1 \cup \mathcal{S}_2$ is an OPB in $\mathbb{C}^3 \otimes \mathbb{C}^{27}$. Since $\mathcal{H}_{\mathcal{S}_1} \subset \mathcal{H}_{\mathcal{V}_{A|BCD}}$, it implies $\mathcal{H}^{\perp}_{\mathcal{V}_{A|BCD}} \subset \mathcal{H}^{\perp}_{\mathcal{S}_1} = \mathcal{H}_{\mathcal{S}_2}$. Then for any product state $|\psi\rangle \in \mathcal{H}^{\perp}_{\mathcal{V}_{A|BCD}}$, there exists $a_i \in \mathbb{C}$ for $1 \leqslant i \leqslant 17$, such that

$$|\psi\rangle = \sum_{i=1}^{17} a_i|\psi_i(0,0,0)\rangle_{A|BCD}. \tag{18}$$

Then $|\psi\rangle$ corresponds to a 3×27 matrix M (M is given in figure 4), where $a_i \in \mathbb{C}$ for $1 \leqslant i \leqslant 17$. Note that M has a similar structure to the tile structure in figure 5. Since $|\psi\rangle$ is a product state, and $|S\rangle_{A|BCD}$ is orthogonal to $|\psi\rangle$, we have $\text{rank}(M) = 1$, and $\text{sum}(M) = 0$. There are only four cases,

A \ BCD	001	002	101	102	200	210	201	211	122	222	112	212	202	111	020	010	110	000	100	011	021	012	022	120	121	220	221
2	a_1	a_1	a_1	a_1	a_2	a_2	a_2	a_2	a_3	a_3	a_3	a_3	a_4	a_5	a_5	a_5	a_5	a_6	a_6	a_5	a_5	a_7	a_7	a_5	a_5	a_8	a_8
1	a_1	a_1	a_1	a_1	a_2	a_2	a_2	a_2	a_3	a_3	a_3	a_3	a_4	a_{17}	a_{12}	a_{11}	a_{11}	a_{11}	a_{11}	a_{10}	a_{10}	a_{10}	a_{10}	a_9	a_9	a_9	a_9
0	a_{16}	a_{16}	a_{13}	a_{13}	a_{15}	a_{15}	a_{13}	a_{13}	a_{14}	a_{14}	a_{13}	a_{13}	a_{13}	a_{13}	a_{12}	a_{11}	a_{11}	a_{11}	a_{11}	a_{10}	a_{10}	a_{10}	a_{10}	a_9	a_9	a_9	a_9

$= M$,

Figure 4. The corresponding 3×27 matrix M of equation (18).

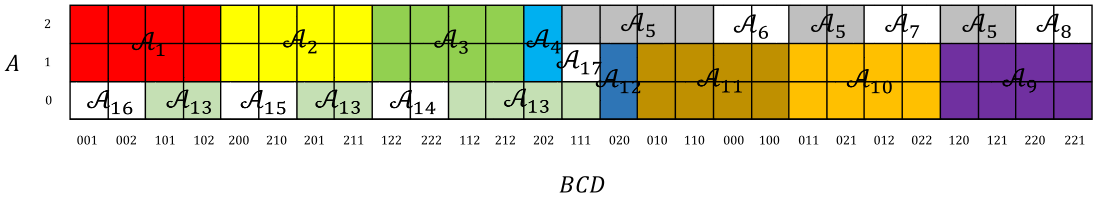

Figure 5. The corresponding tile structure in $\mathbb{C}^3 \otimes \mathbb{C}^{27}$ of the OPB $\cup_{i=1}^{17}\mathcal{A}_i$ (equation (15)) in $A|BCD$ bipartition.

(i) $4a_1 + 4a_2 + 4a_3 + a_4 = 0$, and $a_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 1, 2, 3, 4$; (ii) $4a_5 + a_6 + a_7 + a_8 = 0$, and $a_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 5, 6, 7, 8$; (iii) $4a_9 + 4a_{10} + 4a_{11} + a_{12} = 0$, and $a_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 9, 10, 11, 12$; (iv) $4a_{13} + a_{14} + a_{15} + a_{16} = 0$, and $a_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 13, 14, 15, 16$.

Then $|\psi\rangle$ must belong to one of the four subspaces,

(a) $O_1 = \{(|1\rangle + |2\rangle)_A(a_1(|001\rangle + |002\rangle + |101\rangle + |102\rangle) + a_2(|200\rangle + |210\rangle + |201\rangle + |211\rangle) + a_3(|122\rangle + |222\rangle + |112\rangle + |212\rangle) + a_4|202\rangle))_{BCD}|4a_1 + 4a_2 + 4a_3 + a_4 = 0\}$;
(b) $O_2 = \{(|2\rangle)_A(a_5(|111\rangle + |020\rangle + |010\rangle + |110\rangle + |011\rangle + |021\rangle + |120\rangle + |121\rangle) + a_6(|000\rangle + |100\rangle) + a_7(|012\rangle + |022\rangle) + a_8(|220\rangle + |221\rangle))_{\mathrm{BCD}}|4a_5 + a_6 + a_7 + a_8 = 0\}$;
(c) $O_3 = \{(|0\rangle + |1\rangle)_A(a_9(|221\rangle + |220\rangle + |121\rangle + |120\rangle) + a_{10}(|022\rangle + |012\rangle + |021\rangle + |011\rangle) + a_{11}(|100\rangle + |000\rangle + |110\rangle + |010\rangle) + a_{12}|020\rangle))_{\mathrm{BCD}}|4a_9 + 4a_{10} + 4a_{11} + a_{12} = 0\}$;
(d) $O_4 = \{(|0\rangle)_A(a_{13}(|111\rangle + |202\rangle + |212\rangle + |112\rangle + |211\rangle + |201\rangle + |102\rangle + |101\rangle) + a_{14}(|222\rangle + |122\rangle) + a_{15}(|210\rangle + |200\rangle) + a_{16}(|002\rangle + |001\rangle))_{\mathrm{BCD}}|4a_{13} + a_{14} + a_{15} + a_{16} = 0\}$, where $\mathrm{Dim}(O_i) = 3$ for $1 \leqslant i \leqslant 4$, and $O_i \perp O_j$ for $1 \leqslant i \neq j \leqslant 4$. Then $\mathrm{Dim}(O_1 + O_2 + O_3 + O_4) = 12$. Since $\mathrm{Dim}(\mathcal{H}^{\perp}_{\mathcal{V}_{A|BCD}}) = 16$, the OPS $\mathcal{V}_{A|BCD}$ is an SUCPB by lemma 1.

For $\mathcal{V}_{AB|CD}$, we assume that $|\phi\rangle \in \mathcal{H}^{\perp}_{\mathcal{V}_{AB|CD}}$ is a product state. Let $\mathcal{R}_1 := \cup_{i=1}^{16}(\mathcal{A}_i \backslash \{|\psi_i(0,0,0)\rangle\})_{AB|CD}$ and $\mathcal{R}_2 := \cup_{i=1}^{17}|\psi_i(0,0,0)\rangle_{AB|CD}$. We know that $\mathcal{R}_1 \cup \mathcal{R}_2$ is an OPB in $\mathbb{C}^9 \otimes \mathbb{C}^9$. Since $\mathcal{H}_{\mathcal{R}_1} \subset \mathcal{H}_{\mathcal{V}_{AB|CD}}$, it implies $\mathcal{H}^{\perp}_{\mathcal{V}_{AB|CD}} \subset \mathcal{H}^{\perp}_{\mathcal{R}_1} = \mathcal{H}_{\mathcal{R}_2}$. Then for any product state $|\phi\rangle \in \mathcal{H}^{\perp}_{\mathcal{V}_{AB|CD}}$, there exists $b_i \in \mathbb{C}$ for $1 \leqslant i \leqslant 17$, such that

$$|\phi\rangle = \sum_{i=1}^{17} b_i |\psi_i(0,0,0)\rangle_{AB|CD}. \tag{19}$$

Then $|\phi\rangle$ corresponds to a 9×9 matrix N (N is given in figure 7), where $b_i \in \mathbb{C}$ for $1 \leqslant i \leqslant 17$. Note that, N has a similar structure to the tile structure in figure 6. Since $|\phi\rangle$ is a product state, and $|S\rangle_{AB|CD}$ is orthogonal to $|\phi\rangle$, we have $\mathrm{rank}(N) = 1$, and $\mathrm{sum}(N) = 0$. There are only eight cases,

(a) $4b_1 + b_{16} = 0$, and $b_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 1, 16$;
(b) $4b_2 + b_4 = 0$, and $b_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 2, 4$;
(c) $4b_3 + b_7 = 0$, and $b_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 3, 7$;
(d) $4b_5 + b_6 = 0$, and $b_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 5, 6$;
(e) $4b_9 + b_8 = 0$, and $b_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 8, 9$;
(f) $4b_{10} + b_{12} = 0$, and $b_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 10, 12$;
(g) $4b_{11} + b_{15} = 0$, and $b_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 11, 15$;
(h) $4b_{13} + b_{14} = 0$, and $b_i = 0$ for $1 \leqslant i \leqslant 17$ and $i \neq 13, 14$.

It means that there are only eight product states in $\mathcal{H}^{\perp}_{\mathcal{V}_{AB|CD}}$:
$(|10\rangle+|11\rangle+|20\rangle+|21\rangle-4|00\rangle)_{AB}(|01\rangle+|02\rangle)_{CD}$, $(|12\rangle+|22\rangle)_{AB}(|00\rangle+|01\rangle+|10\rangle+|11\rangle-4|02\rangle)_{CD}$,
$(|11\rangle+|12\rangle+|21\rangle+|22\rangle-4|20\rangle)_{AB}(|12\rangle+|22\rangle)_{CD}$, $(|20\rangle+|21\rangle)_{AB}(|10\rangle+|11\rangle+|20\rangle+|21\rangle-4|00\rangle)_{CD}$,
$(|01\rangle+|02\rangle+|11\rangle+|12\rangle-4|22\rangle)_{AB}(|20\rangle+|21\rangle)_{CD}$, $(|00\rangle+|10\rangle)_{AB}(|11\rangle+|12\rangle+|21\rangle+|22\rangle-4|20\rangle)_{CD}$,
$(|00\rangle+|01\rangle+|10\rangle+|11\rangle-4|02\rangle)_{AB}(|00\rangle+|10\rangle)_{CD}$, $(|01\rangle+|02\rangle)_{AB}(|01\rangle+|02\rangle+|11\rangle+|12\rangle-4|22\rangle)_{CD}$.
Since $\mathrm{Dim}(\mathcal{H}^{\perp}_{\mathcal{V}_{AB|CD}})=16$, the OPS $\mathcal{V}_{AB|CD}$ is an SUCPB by lemma 1.

Above all, $\mathcal{V}$ is an SUCPB in every bipartition. □

The construction of UPB in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ was generalized to any four-partite system $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}\otimes\mathbb{C}^{d_C}\otimes\mathbb{C}^{d_D}$ for $d_A,d_B,d_C,d_D\geqslant 3$ [4]. Obviously, we can also show that these UPBs are all SUCPBs in every bipartition.

Theorem 2. *In $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}\otimes\mathbb{C}^{d_C}\otimes\mathbb{C}^{d_D}$, $d_A,d_B,d_C,d_D\geqslant 3$, there exists a UPB which is an SUCPB in every bipartition.*

The proof of theorem 2 is given in appendix C. However, not all UPBs have this property. For example, let

$$|\psi_1\rangle=|0\rangle_A(|0\rangle-|1\rangle)_B,$$
$$|\psi_2\rangle=(|0\rangle-|1\rangle)_A|2\rangle_B,$$
$$|\psi_3\rangle=|2\rangle_A(|1\rangle-|2\rangle)_B,$$
$$|\psi_4\rangle=(|1\rangle-|2\rangle)_A|0\rangle_B,$$
$$|\psi_5\rangle=(|0\rangle+|1\rangle+|2\rangle)_A(|0\rangle+|1\rangle+|2\rangle)_B,$$

then $\cup_{i=1}^5|\psi_i\rangle$ is a UPB in $\mathbb{C}^3\otimes\mathbb{C}^3$ [10]. We can construct a UPB in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ from $\cup_{i=1}^5|\psi_i\rangle$ as follows,

$$\mathcal{A}_1:=\{|\psi_i\rangle|0\rangle_C|1\leqslant i\leqslant 5\},$$
$$\mathcal{A}_2:=\{|i\rangle_A|j\rangle_B|1\rangle_C|i,j\in\mathbb{Z}_3\},$$
$$\mathcal{A}_3:=\{|i\rangle_A|j\rangle_B|2\rangle_C|i,j\in\mathbb{Z}_3\}.$$

Let $\mathcal{W}:=\cup_{i=1}^3\mathcal{A}_i$. For any product state $|\varphi\rangle=|\varphi_1\rangle_A|\varphi_2\rangle_B|\varphi_3\rangle_C\in\mathcal{W}^{\perp}$, since $|\varphi\rangle$ is orthogonal to any state in $\mathcal{A}_2\cup\mathcal{A}_3$, $|\varphi_3\rangle_C$ must be $|0\rangle_C$. Further, since $\cup_{i=1}^5|\psi_i\rangle$ is a UPB, it means that $|\varphi_1\rangle_A|\varphi_2\rangle_B$ cannot be a product state. Thus $\mathcal{W}$ is a UPB in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$. Nevertheless, $\mathcal{W}$ is not a UCPB in every bipartition. There must exist four orthogonal states $\{|\psi_6\rangle,|\psi_7\rangle,|\psi_8\rangle,|\psi_9\rangle\}$ such that $\cup_{i=1}^9|\psi_i\rangle$ is an orthogonal basis in $\mathbb{C}^3\otimes\mathbb{C}^3$. Then $\{\cup_{i=6}^9|\psi_i\rangle|0\rangle_C\}\cup\mathcal{W}_{AB|C}$ is an OPB in $AB|C$ bipartition. Thus, $\mathcal{W}$ is not a UCPB in $AB|C$ bipartition.

Next, we consider the application of these SUCPBs constructed in our paper. First of all, SUCPBs cannot be perfectly distinguished under local POVMs and classical communication [10]. Since all UPBs constructed in theorems 1 and 2 are SUCPBs in every bipartition, they cannot be perfectly distinguished under local POVMs and classical communication in any bipartition. Therefore, all UPBs in theorems 1 and 2 can be used for local hiding of information [17]. For example, assume the information is encoded in the UPB $\mathcal{U}$ given by equation (13), and the boss send it to his three subordinates: A, B and C. These three subordinates are from different offices. They can only perform local POVMs, and communicate classic information by telephones. In this case, the three subordinates cannot obtain the full information, even if any two of them join together. A and B join together means that A and B can perform joint measurements. See also figure 8. Further, reference [4] showed a stronger property: any UPB in theorems 1 and 2 is locally irreducible[6] in every bipartition, which shows the phenomenon of strong quantum nonlocality without entanglement [3].

For any UPB $\mathcal{U}$ in theorems 1 and 2, and any bipartition $A|A'$, in order to show that $\mathcal{U}_{A|A'}$ is an SUCPB, we have proved that all the product states in $\mathcal{H}^{\perp}_{\mathcal{U}_{A|A'}}$ cannot span $\mathcal{H}^{\perp}_{\mathcal{U}_{A|A'}}$ in the proofs of theorems 1 and 2. Then the mixed state $\overline{\rho}_{\mathcal{U}_{A|A'}}$ that is proportional to the projector on $\mathcal{H}^{\perp}_{\mathcal{U}_{A|A'}}$ must be entangled by theorem 2(b) in [14]. Thus $\overline{\rho}_{\mathcal{U}_{A|A'}}$ is a PPT entangled state. It means that the mixed state $\overline{\rho}_{\mathcal{U}}$ that is proportional to the projector on $\mathcal{H}^{\perp}_{\mathcal{U}}$ is a PPT entangled state in every bipartition. Finally, we consider the connection between those UPBs and genuinely entangled states. A genuinely entangled state is a multipartite pure state that is entangled in every bipartition [18]. Since $\mathcal{H}^{\perp}_{\mathcal{U}_{A|A'}}$ must contain entangled

[6] A set of multipartite orthogonal states is locally irreducible if it is not possible to eliminate one or more states from the set by orthogonality-preserving local POVMs [3].

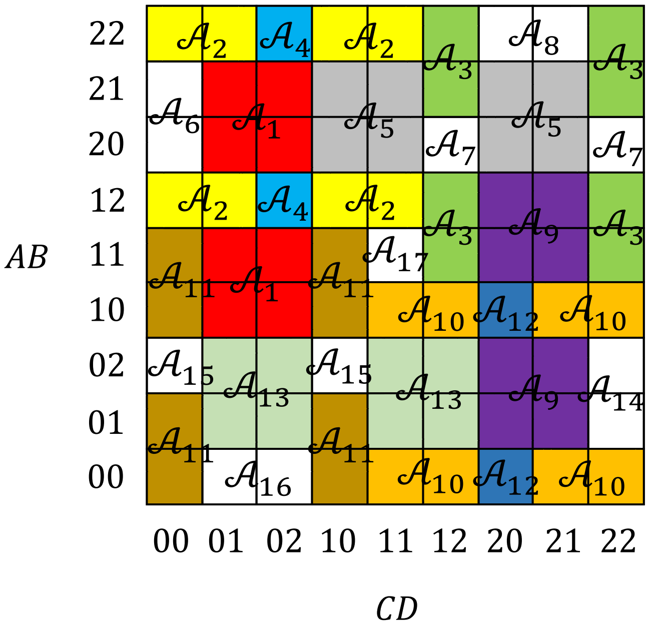

Figure 6. The corresponding tile structure in $\mathbb{C}^9 \otimes \mathbb{C}^9$ of the OPB $\cup_{i=1}^{17}\mathcal{A}_i$ (equation (15)) in $AB|CD$ bipartition.

states. It is possible to find genuinely entangled states in $\mathcal{H}_{\mathcal{U}}^{\perp}$. For example, we can find a genuinely entangled state $|000\rangle - |222\rangle$ in $\mathcal{H}_{\mathcal{U}}^{\perp}$, where $\mathcal{U}$ is the UPB in equation (13).

4. UPBs in every bipartition

There exists another open question for UPBs [11]: can we find a UPB, which is still a UPB in every bipartition? Such a UPB can be used to construct genuinely entangled subspace, and it cannot be perfectly distinguished under local POVMs and classical communication in any bipartition. Unfortunately, any UPB in section 3 is not a UPB in every bipartition. We will give a sufficient condition for the existence of such a UPB.

We can also generalize the tile structures to multipartite systems. A tile structure $\mathcal{T} = \cup_{i=1}^{s} t_i$ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \ldots \otimes \mathbb{C}^{d_n}$ is a $d_1 \times d_2 \times \cdots \times d_n$ hypercube, which can be partitioned into s disjoint tiles $\{t_i\}_{i=1}^{s}$. Each tile t_i is a hypercube, and it can be expressed by

$$t_i = \{x_0^{(1)}, x_1^{(1)}, \ldots, x_{k_1-1}^{(1)}\}_{A_1} \times \{x_0^{(2)}, x_1^{(2)}, \ldots, x_{k_2-1}^{(2)}\}_{A_2} \times \cdots \times \{x_0^{(n)}, x_1^{(n)}, \ldots, x_{k_n-1}^{(n)}\}_{A_n}, \tag{20}$$

where $\{x_0^{(j)}, x_1^{(j)}, \ldots, x_{k_j-1}^{(j)}\}$ is a subset of $\mathbb{Z}_{d_j}$ for $1 \leqslant j \leqslant n$. For tile t_i, we can construct an OPS of size $k_1 k_2 \ldots k_n$ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \ldots \otimes \mathbb{C}^{d_n}$,

$$\mathcal{A}_i = \left\{ \otimes_{j=1}^{n} \left(\sum_{e_j \in \mathbb{Z}_{k_j}} m_{a_j, e_j}^{(j)} |x_{e_j}^{(j)}\rangle \right)_{A_j} \middle| a_j \in \mathbb{Z}_{k_j}, 1 \leqslant j \leqslant n \right\}, \tag{21}$$

where each coefficient matrix $M^{(j)} = (m_{a_j,e_j}^{(j)})_{a_j, e_j \in \mathbb{Z}_{k_j}}$ is a $k_j \times k_j$ row orthogonal matrix, and $m_{0,e_j}^{(j)} = 1$ for $e_j \in \mathbb{Z}_{k_j}$. Then we obtain an OPB $\mathcal{B} := \cup_{i=1}^{s} \mathcal{A}_i$ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \ldots \otimes \mathbb{C}^{d_n}$. Note that

$$|\psi_i\rangle = \otimes_{j=1}^{n} \left(\sum_{e_j \in \mathbb{Z}_{k_j}} |x_{e_j}^{(j)}\rangle \right)_{A_j} \in \mathcal{A}_i. \tag{22}$$

The ‘stopper’ state is

$$|S\rangle = \otimes_{j=1}^{n}\left(\sum_{r\in\mathbb{Z}_{d_j}}|r\rangle\right)_{A_j}. \tag{23}$$

We may wonder whether the OPS

$$\mathcal{X} := \cup_{i=1}(\mathcal{A}_i\backslash\{|\psi_i\rangle\})\cup\{|S\rangle\} \tag{24}$$

is a UPB that is still a UPB in every bipartition.

For any bipartition $C|D$, where $C, D\subset\{1,2,\ldots,n\}$, and $C\cup D=\{1,2,\ldots,n\}$, the OPB $\mathcal{B}$ must correspond to a tile structure $\mathcal{T}_{C|D}$ in $\mathbb{C}^{h_1}\otimes\mathbb{C}^{h_2}$, where $h_1=\prod_{g\in C}d_g$ and $h_2=\prod_{g\in D}d_g$ (for example, see figure 3). By using lemma 2, if any r $(2\leqslant r\leqslant s-1)$ tiles in $\mathcal{T}_{C|D}$ cannot form a rectangle, then $\mathcal{X}_{C|D}$ is a UPB in $\mathbb{C}^{h_1}\otimes\mathbb{C}^{h_2}$. Note that in this case, $\mathcal{X}$ is also a UPB in $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\ldots\otimes\mathbb{C}^{d_n}$. This is because if $\mathcal{X}$ is not a UPB, then there exists a product state $|\psi\rangle$ in $\mathcal{H}_{\mathcal{X}}^{\perp}$, and the product state $|\psi\rangle_{C|D}$ in bipartition $C|D$ belongs to $\mathcal{H}_{\mathcal{X}_{C|D}}^{\perp}$, which contradicts $\mathcal{X}_{C|D}$ being a UPB in $\mathbb{C}^{h_1}\otimes\mathbb{C}^{h_2}$. Now, we have the following theorem.

Theorem 3. *Consider a tile structure $\mathcal{T}=\cup_{i=1}^{s}t_i$ $(s\geqslant 5)$ in $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\ldots\otimes\mathbb{C}^{d_n}$. For any bipartition $C|D$, if any r $(2\leqslant r\leqslant s-1)$ tiles in $\mathcal{T}_{C|D}$ cannot form a rectangle, then the OPS $\mathcal{X}$ given by equation (24) is a UPB in $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\ldots\otimes\mathbb{C}^{d_n}$, which is still a UPB in every bipartition.*

One can use computer to search the tile structure in theorem 3. By exhaustive search, we show that such a tile structure does not exist in $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$. However, we conjecture that such a tile structure may exist in a higher multipartite system. Note that the condition in theorem 3 is not necessary. This is because not all UPBs can be constructed from tile structures. For example, the following UPB in $\mathbb{C}^4\otimes\mathbb{C}^4$ is from reference [19]:

$$\begin{aligned}
&|0\rangle_A|0\rangle_B; && |1\rangle_A(|0\rangle-|2\rangle+|3\rangle)_B;\\
&|2\rangle_A(|0\rangle+|1\rangle-|3\rangle)_B; && |3\rangle_A|3\rangle_B;\\
&(|1\rangle+|2\rangle+|3\rangle)_A(|0\rangle-|1\rangle+|2\rangle)_B; && (|0\rangle-|2\rangle+|3\rangle)_A|2\rangle_B;\\
&(|0\rangle+|1\rangle-|3\rangle)_A|1\rangle_B; && (|0\rangle-|1\rangle+|2\rangle)_A(|1\rangle+|2\rangle+|3\rangle)_B.
\end{aligned}$$

This UPB cannot be constructed from the tile structure, since it does not contain the stopper state.

5. Conclusion and discussion

In this paper, we showed that there exist some UPBs that are UCPBs in every bipartition in $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}\otimes\mathbb{C}^{d_C}$ and $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}\otimes\mathbb{C}^{d_C}\otimes\mathbb{C}^{d_D}$ for $d_A, d_B, d_C, d_D\geqslant 3$, and $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ achieved the minimum system for the existence of such UPBs. This result answers an open question proposed in [10]. We also showed that such UPBs can be connected to local hiding of information, positive-partial-transpose entangled states and genuinely entangled states. Finding a UPB that is still a UPB in every bipartition can be challenging, and we gave a sufficient condition for the existence of such a UPB.

There are some interesting open questions left. How to find a UPB that is still a UPB in every bipartition by using theorem 3? What is the minimum size of a UPB that is a UCPB in every bipartition?

Acknowledgments

We thank Lin Chen for discussing this problem. The research of XZ and FS were supported by the NSFC under Grant Nos. 12171452 and 11771419, the Anhui Initiative in Quantum Information Technologies under Grant No. AHY150200, the Innovation Program for Quantum Science and Technology (2021ZD0302904), and the National Key Research and Development Program of China (2020YFA0713100). M-SL was supported by the NSFC under Grant No. 12005092, the China Postdoctoral Science Foundation (2020M681996). QZ acknowledges HKU Seed Fund for New Staff.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. The proof of theorem 1

Proof. Consider an OPB $\cup_{i=1}^{9}\mathcal{A}_i$ in $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}\otimes\mathbb{C}^{d_C}$, $d_A, d_B, d_C \geqslant 3$,

$$\begin{aligned}
\mathcal{A}_1 &:= \{|\psi_1(i,j)\rangle = |\xi_i\rangle_A|0\rangle_B|\eta_j\rangle_C|(i,j)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_C-1}\},\\
\mathcal{A}_2 &:= \{|\psi_2(i,j)\rangle = |\xi_i\rangle_A|\eta_j\rangle_B|d_C-1\rangle_C|(i,j)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_B-1}\},\\
\mathcal{A}_3 &:= \{|\psi_3(i,j)\rangle = |d_A-1\rangle_A|\xi_i\rangle_B|\eta_j\rangle_C|(i,j)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\},\\
\mathcal{A}_4 &:= \{|\psi_4(i,j)\rangle = |\eta_i\rangle_A|d_B-1\rangle_B|\xi_j\rangle_C|(i,j)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_C-1}\},\\
\mathcal{A}_5 &:= \{|\psi_5(i,j)\rangle = |\eta_i\rangle_A|\xi_j\rangle_B|0\rangle_C|(i,j)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_B-1}\},\\
\mathcal{A}_6 &:= \{|\psi_6(i,j)\rangle = |0\rangle_A|\eta_i\rangle_B|\xi_j\rangle_C|(i,j)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\},\\
\mathcal{A}_7 &:= \{|\psi_7(0,0)\rangle = |0\rangle_A|0\rangle_B|0\rangle_C\},\\
\mathcal{A}_8 &:= \{|\psi_8(i,j,k)\rangle = |\beta_i\rangle_A|\beta_j\rangle_B|\beta_k\rangle_C|(i,j,k)\in\mathbb{Z}_{d_A-2}\times\mathbb{Z}_{d_B-2}\times\mathbb{Z}_{d_C-2}\},\\
\mathcal{A}_9 &:= \{|\psi_9(0,0)\rangle = |d_A-1\rangle_A|d_B-1\rangle_B|d_C-1\rangle_C\},
\end{aligned} \tag{A1}$$

where $|\eta_s\rangle_X = \sum_{t=0}^{d_X-2} w_{d_X-1}^{st}|t\rangle_X$, $|\xi_s\rangle_X = \sum_{t=0}^{d_X-2} w_{d_X-1}^{st}|t+1\rangle_X$ for $s\in\mathbb{Z}_{d_X-1}$ and $X\in\{A,B,C\}$, and $|\beta_s\rangle_X = \sum_{t=0}^{d_X-3} w_{d_X-2}^{st}|t+1\rangle_X$ for $s\in\mathbb{Z}_{d_X-2}$ and $X\in\{A,B,C\}$. The 'stopper' state is,

$$|S\rangle = \left(\sum_{i=0}^{d_A-1}|i\rangle\right)_A\left(\sum_{j=0}^{d_B-1}|j\rangle\right)_B\left(\sum_{k=0}^{d_C-1}|k\rangle\right)_C. \tag{A2}$$

Then

$$\mathcal{U} := \cup_{i=1}^{6}(\mathcal{A}_i\backslash\{|\psi_i(0,0)\rangle\})\cup(\mathcal{A}_8\backslash\{|\psi_8(0,0,0)\rangle\})\cup\{|S\rangle\} \tag{A3}$$

is a UPB in $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}\otimes\mathbb{C}^{d_C}$ [4]. Next, we show that $\mathcal{U}$ given by equation (A3) is an SUCPB in every bipartition.

Since the OPB $\cup_{i=1}^{9}\mathcal{A}_i$ given by equation (A1) in any bipartition of $\{A|BC, B|AC, C|AB\}$ corresponds to a tile structure similar to figure 9, we only need to show that $\mathcal{U}_{A|BC}$ is an SUCPB. We can assume that $|\psi\rangle\in\mathcal{H}^{\perp}_{\mathcal{U}_{A|BC}}$ is a product state. Then $|\psi\rangle$ corresponds to a $d_A\times d_Bd_C$ matrix M. From the proof of lemma 3, we know that M has a similar structure to the tile structure in figure 9. That is, $\mathcal{A}_i$ is filled with a_i's for $1\leqslant i\leqslant 9$, where $a_i\in\mathbb{C}$ for $1\leqslant i\leqslant 9$. Further, it must have $\mathrm{rank}(M)=1$, and $\mathrm{sum}(M)=0$. There are only four cases,

(i) $(d_C-1)a_1+(d_B-1)a_2=0$, $a_i=0$ for $1\leqslant i\leqslant 9$ and $i\neq 1,2$; (ii) $(d_B-1)(d_C-1)a_3+a_9=0$, $a_i=0$ for $1\leqslant i\leqslant 9$ and $i\neq 3,9$; (iii) $(d_C-1)a_4+(d_B-1)a_5=0$, $a_i=0$ for $1\leqslant i\leqslant 9$ and $i\neq 4,5$; (iv) $(d_B-1)(d_C-1)a_6+a_7=0$, $a_i=0$ for $1\leqslant i\leqslant 9$ and $i\neq 6,7$.

It means that there are only four product states in $\mathcal{H}^{\perp}_{\mathcal{U}_{A|BC}}$:

(a) $\left(\sum_{i=0}^{d_A-2}|i+1\rangle\right)_A\left((d_B-1)\sum_{j=0}^{d_C-2}|0\rangle|j\rangle-(d_C-1)\sum_{j=0}^{d_B-2}|j\rangle|d_C-1\rangle\right)_{BC}$;

(b) $|d_A-1\rangle_A\left(\sum_{i=0}^{d_B-2}\sum_{j=0}^{d_C-2}|i+1\rangle|j\rangle-(d_B-1)(d_C-1)|d_B-1\rangle|d_C-1\rangle\right)_{BC}$;

(c) $\left(\sum_{i=0}^{d_A-2}|i\rangle\right)_A\left((d_B-1)\sum_{j=0}^{d_C-2}|d_B-1\rangle|j+1\rangle-(d_C-1)\sum_{j=0}^{d_B-2}|j+1\rangle|0\rangle\right)_{BC}$;

(d) $|0\rangle_A\left(\sum_{i=0}^{d_B-2}\sum_{j=0}^{d_C-2}|i\rangle|j+1\rangle-(d_B-1)(d_C-1)|0\rangle|0\rangle\right)_{BC}$.

Since $\mathrm{Dim}(\mathcal{H}^{\perp}_{\mathcal{U}_{A|BC}})=8$, the OPS $\mathcal{U}_{A|BC}$ is an SUCPB by lemma 1. This completes the proof.

Appendix B. The corresponding tile structures of the OPB $\cup_{i=1}^{17}\mathcal{A}_i$ (equation (15)) in $\{A|BCD, B|ACD, C|ABD, D|ABC, AB|CD, AC|BD, AD|BC\}$ bipartition

See figures 8 and 9.

$$
AB\begin{array}{c}22\\21\\20\\12\\11\\10\\02\\01\\00\end{array}\left[\begin{array}{ccccccccc}
b_2 & b_2 & b_4 & b_2 & b_2 & b_3 & b_8 & b_8 & b_3\\
b_6 & b_1 & b_1 & b_5 & b_5 & b_3 & b_5 & b_5 & b_3\\
b_6 & b_1 & b_1 & b_5 & b_5 & b_7 & b_5 & b_5 & b_7\\
b_2 & b_2 & b_4 & b_2 & b_2 & b_3 & b_9 & b_9 & b_3\\
b_{11} & b_1 & b_1 & b_{11} & b_{17} & b_3 & b_9 & b_9 & b_3\\
b_{11} & b_1 & b_1 & b_{11} & b_{10} & b_{10} & b_{12} & b_{10} & b_{10}\\
b_{15} & b_{13} & b_{13} & b_{15} & b_{13} & b_{13} & b_9 & b_9 & b_{14}\\
b_{11} & b_{13} & b_{13} & b_{11} & b_{13} & b_{13} & b_9 & b_9 & b_{14}\\
b_{11} & b_{16} & b_{16} & b_{11} & b_{10} & b_{10} & b_{12} & b_{10} & b_{10}
\end{array}\right] = N,
$$

$$\begin{array}{ccccccccc}00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22\end{array}$$

$$CD$$

Figure 7. The corresponding 9×9 matrix N of equation (19).

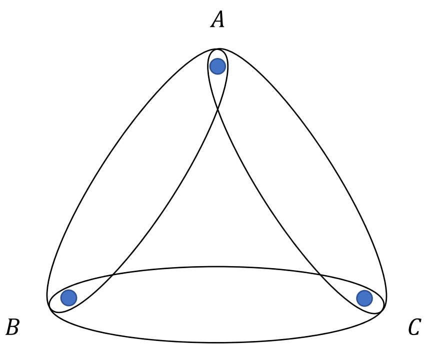

Figure 8. The information is encoded to a tripartite UPB which is an SUCPB is every bipartition, and the boss send it to his three subordinates: A, B, and C. Even if any two of them join together, the three subordinates cannot obtain the full information under local POVMs and classical communication.

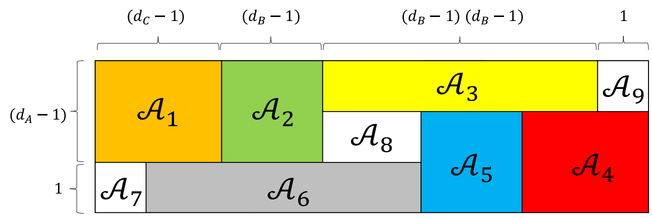

Figure 9. The corresponding tile structure in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B d_C}$ of the OPB $\cup_{i=1}^{9}\mathcal{A}_i$ (equation (A1)) in $A|BC$ bipartition.

Appendix C. The proof of theorem 2

Proof. Consider an OPB $\cup_{i=1}^{17}\mathcal{A}_i$ in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_C} \otimes \mathbb{C}^{d_D}$, $d_A, d_B, d_C, d_D \geqslant 3$,

$$\mathcal{A}_1 := \{|\psi_1(i,j,k)\rangle\} = |\xi_i\rangle_A|\eta_j\rangle_B|0\rangle_C|\xi_k\rangle_D|(i,j,k) \in \mathbb{Z}_{d_A-1} \times \mathbb{Z}_{d_B-1} \times \mathbb{Z}_{d_D-1}\},$$

$$\mathcal{A}_2 := \{|\psi_2(i,j,k)\rangle\} = |\xi_i\rangle_A|d_B-1\rangle_B|\eta_j\rangle_C|\eta_k\rangle_D|(i,j,k) \in \mathbb{Z}_{d_A-1} \times \mathbb{Z}_{d_C-1} \times \mathbb{Z}_{d_D-1}\},$$

$$\mathcal{A}_3 := \{|\psi_3(i,j,k)\rangle\} = |\xi_i\rangle_A|\xi_j\rangle_B|\xi_k\rangle_C|d_D-1\rangle_D|(i,j,k) \in \mathbb{Z}_{d_A-1} \times \mathbb{Z}_{d_B-1} \times \mathbb{Z}_{d_C-1}\},$$

$$\begin{aligned}
&\mathcal{A}_4 := \{|\psi_4(i,0,0)\rangle\} = |\xi_i\rangle_A|d_B-1\rangle_B|0\rangle_C|d_D-1\rangle_D|i\in\mathbb{Z}_{d_A-1}\},\\
&\mathcal{A}_5 := \{|\psi_5(i,j,k)\rangle\} = |d_A-1\rangle_A|\eta_i\rangle_B|\xi_j\rangle_C|\eta_k\rangle_D|(i,j,k)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\},\\
&\mathcal{A}_6 := \{|\psi_6(i,0,0)\rangle\} = |d_A-1\rangle_A|\eta_i\rangle_B|0\rangle_C|0\rangle_D|i\in\mathbb{Z}_{d_B-1}\},\\
&\mathcal{A}_7 := \{|\psi_7(i,0,0)\rangle\} = |d_A-1\rangle_A|0\rangle_B|\xi_i\rangle_C|d_D-1\rangle_D|i\in\mathbb{Z}_{d_C-1}\},\\
&\mathcal{A}_8 := \{|\psi_8(i,0,0)\rangle\} = |d_A-1\rangle_A|d_B-1\rangle_B|d_C-1\rangle_C|\eta_i\rangle_D|i\in\mathbb{Z}_{d_D-1}\},\\
&\mathcal{A}_9 := \{|\psi_9(i,j,k)\rangle\} = |\eta_i\rangle_A|\xi_j\rangle_B|d_C-1\rangle_C|\eta_k\rangle_D|(i,j,k)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_D-1}\},\\
&\mathcal{A}_{10} := \{|\psi_{10}(i,j,k)\rangle\} = |\eta_i\rangle_A|0\rangle_B|\xi_j\rangle_C|\xi_k\rangle_D|(i,j,k)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\},\\
&\mathcal{A}_{11} := \{|\psi_{11}(i,j,k)\rangle\} = |\eta_i\rangle_A|\eta_j\rangle_B|\eta_k\rangle_C|0\rangle_D|(i,j,k)\in\mathbb{Z}_{d_A-1}\times\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\},\\
&\mathcal{A}_{12} := \{|\psi_{12}(i,0,0)\rangle\} = |\eta_i\rangle_A|0\rangle_B|d_C-1\rangle_C|0\rangle_D|i\neq 0\in\mathbb{Z}_{d_A-1}\},\\
&\mathcal{A}_{13} := \{|\psi_{13}(i,j,k)\rangle\} = |0\rangle_A|\xi_i\rangle_B|\eta_j\rangle_C|\xi_k\rangle_D|(i,j,k)\in\mathbb{Z}_{d_B-1}\times\mathbb{Z}_{d_C-1}\times\mathbb{Z}_{d_D-1}\},\\
&\mathcal{A}_{14} := \{|\psi_{14}(i,0,0)\rangle\} = |0\rangle_A|\xi_i\rangle_B|d_C-1\rangle_C|d_D-1\rangle_D|i\in\mathbb{Z}_{d_B-1}\},\\
&\mathcal{A}_{15} := \{|\psi_{15}(i,0,0)\rangle\} = |0\rangle_A|d_B-1\rangle_B|\eta_i\rangle_C|0\rangle_D|i\in\mathbb{Z}_{d_C-1}\},\\
&\mathcal{A}_{16} := \{|\psi_{16}(i,0,0)\rangle\} = |0\rangle_A|0\rangle_B|0\rangle_C|\xi_i\rangle_D|i\in\mathbb{Z}_{d_D-1}\},\\
&\mathcal{A}_{17} := \{|\psi_{17}(i,j,k,\ell)\rangle\} = |\beta_i\rangle_A|\beta_j\rangle_B|\beta_k\rangle_C|\beta_\ell\rangle_D|(i,j,k,\ell)\in\mathbb{Z}_{d_A-2}\times\mathbb{Z}_{d_B-2}\times\mathbb{Z}_{d_C-2}\times\mathbb{Z}_{d_D-2}\},
\end{aligned} \tag{C1}$$

where $|\eta_s\rangle_X = \sum_{t=0}^{d_X-2} w_{d_X-1}^{st}|t\rangle_X$, $|\xi_s\rangle_X = \sum_{t=0}^{d_X-2} w_{d_X-1}^{st}|t+1\rangle_X$ for $s\in\mathbb{Z}_{d_X-1}$ and $X\in\{A,B,C,D\}$, and $|\beta_s\rangle_X = \sum_{t=0}^{d_X-3} w_{d_X-2}^{st}|t+1\rangle_X$ for $s\in\mathbb{Z}_{d_X-2}$ and $X\in\{A,B,C,D\}$. The 'stopper' state is,

$$|S\rangle := \left(\sum_{i=0}^{d_A-1}|i\rangle\right)_A\left(\sum_{j=0}^{d_B-1}|j\rangle\right)_B\left(\sum_{k=0}^{d_C-1}|k\rangle\right)_C\left(\sum_{\ell=0}^{d_D-1}|\ell\rangle\right)_D. \tag{C2}$$

Then

$$\mathcal{V} := \cup_{i=1}^{16}(\mathcal{A}_i\backslash\{|\psi_i(0,0,0)\rangle\})\cup(\mathcal{A}_{17}\backslash\{|\psi_{17}(0,0,0,0)\rangle\})\cup\{|S\rangle\} \tag{C3}$$

is a UPB in $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}\otimes\mathbb{C}^{d_C}\otimes\mathbb{C}^{d_D}$ [4]. Next, we show that $\mathcal{V}$ given by equation (C3) is an SUCPB in every bipartition.

Since the OPB $\cup_{i=1}^{17}\mathcal{A}_i$ given by equation (C1) in any bipartition of $\{A|BCD, B|ACD, C|ABD, D|ABC\}$ corresponds to a tile structure similar to figure 12, and $\cup_{i=1}^{17}\mathcal{A}_i$ in any bipartition of $\{AB|CD, AC|BD, AD|BC\}$ corresponds to a tile structure similar to figure 13, we only need to consider $\mathcal{V}_{A|BCD}$ and $\mathcal{V}_{AB|CD}$.

For $\mathcal{V}_{A|BCD}$, we assume that $|\psi\rangle\in\mathcal{H}^{\perp}_{\mathcal{V}_{A|BCD}}$ is a product state. Then $|\psi\rangle$ corresponds to a $d_A\times d_Bd_Cd_D$ matrix M, which has a similar structure to the tile structure in figure 12. That is, $\mathcal{A}_i$ is filled with a_i's for $1\leqslant i\leqslant 17$, where $a_i\in\mathbb{C}$ for $1\leqslant i\leqslant 17$. It must have $\mathrm{rank}(M)=1$, and $\mathrm{sum}(M)=0$. There are only four cases,

(i) $(d_B-1)(d_D-1)a_1+(d_C-1)(d_D-1)a_2+(d_B-1)(d_C-1)a_3+a_4=0$, and $a_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 1,2,3,4$; (ii) $(d_B-1)(d_C-1)(d_D-1)a_5+(d_B-1)a_6+(d_C-1)a_7+(d_D-1)a_8=0$, and $a_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 5,6,7,8$; (iii) $(d_B-1)(d_D-1)a_9+(d_C-1)(d_D-1)a_{10}+(d_B-1)(d_C-1)a_{11}+a_{12}=0$, and $a_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 9,10,11,12$; (iv) $(d_B-1)(d_C-1)(d_D-1)a_{13}+(d_B-1)a_{14}+(d_C-1)a_{15}+(d_D-1)a_{16}=0$, and $a_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 13,14,15,16$.

Then $|\psi\rangle$ must belong to one of the four subspaces,

$$\begin{aligned}
O_1 = \Bigg\{&\left(\sum_{i=0}^{d_A-2}|i+1\rangle\right)_A\Bigg(a_1\sum_{j=0}^{d_B-2}\sum_{k=0}^{d_D-2}|j\rangle|0\rangle|k+1\rangle+a_2\sum_{j=0}^{d_C-2}\sum_{k=0}^{d_D-2}|d_B-1\rangle|j\rangle|k\rangle\\
&+a_3\sum_{j=0}^{d_B-2}\sum_{k=0}^{d_C-2}|j+1\rangle|k+1\rangle|d_D-1\rangle+a_4|d_B-1\rangle|0\rangle|d_D-1\rangle\Bigg)_{BCD}\\
&\times\,|(d_B-1)(d_D-1)a_1+(d_C-1)(d_D-1)a_2+(d_B-1)(d_C-1)a_3+a_4=0\Bigg\};
\end{aligned} \tag{a}$$

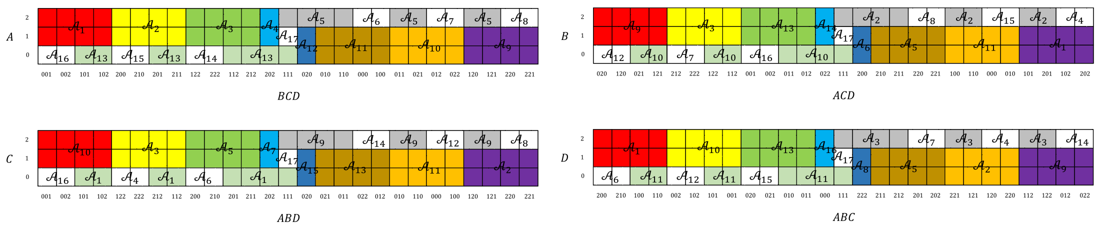

Figure 10. The corresponding tile structures of the OPB $\cup_{i=1}^{17}\mathcal{A}_i$ (equation (15)) in $\{A|BCD, B|ACD, C|ABD, D|ABC\}$ bipartition.

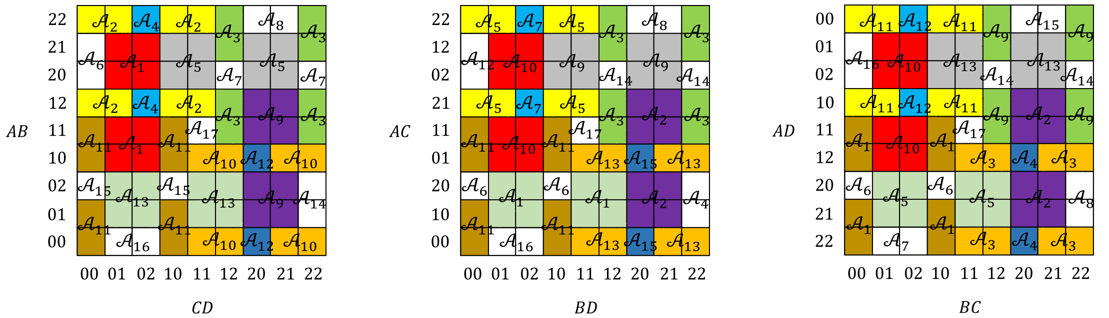

Figure 11. The corresponding tile structures of the OPB $\cup_{i=1}^{17}\mathcal{A}_i$ (equation (15)) in $\{AB|CD, AC|BD, AD|BC\}$ bipartition.

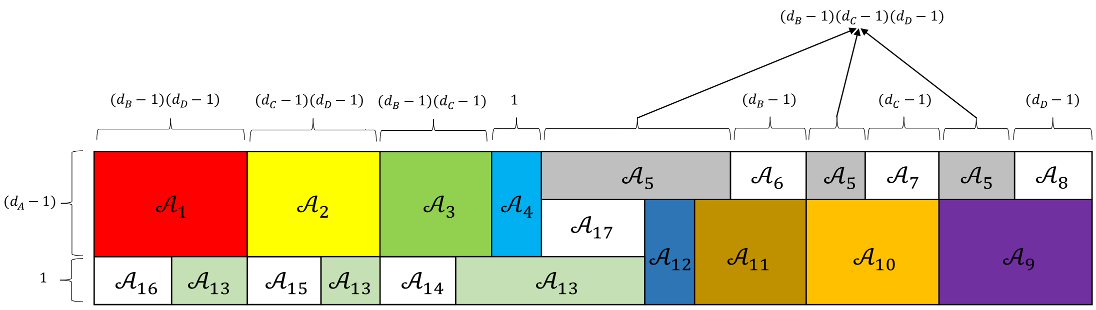

Figure 12. The corresponding tile structure in $\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_Bd_Cd_D}$ of the OPB $\cup_{i=1}^{17}\mathcal{A}_i$ (equation (C1)) in $A|BCD$ bipartition.

$$O_2=\Bigg\{|d_A-1\rangle_A\Bigg(a_5\sum_{i=0}^{d_B-2}\sum_{j=0}^{d_C-2}\sum_{k=0}^{d_D-2}|i\rangle|j+1\rangle|k\rangle+a_6\sum_{i=0}^{d_B-2}|i\rangle|0\rangle|0\rangle$$

$$\text{(b)}\qquad +a_7\sum_{i=0}^{d_C-2}|0\rangle|i+1\rangle|d_D-1\rangle+a_8\sum_{i=0}^{d_D-2}|d_B-1\rangle|d_C-1\rangle|i\rangle\Bigg)_{BCD}$$

$$\times\,|(d_B-1)(d_C-1)(d_D-1)a_5+(d_B-1)a_6+(d_C-1)a_7+(d_D-1)a_8=0\Bigg\};$$

$$O_3=\Bigg\{\left(\sum_{i=0}^{d_A-2}|i\rangle\right)_A\Bigg(a_9\sum_{j=0}^{d_B-2}\sum_{k=0}^{d_D-2}|j+1\rangle|d_C-1\rangle|k\rangle+a_{10}\sum_{j=0}^{d_C-2}\sum_{k=0}^{d_D-2}|0\rangle|j+1\rangle|k+1\rangle$$

$$\text{(c)}\qquad +a_{11}\sum_{j=0}^{d_B-2}\sum_{k=0}^{d_C-2}|j\rangle|k\rangle|0\rangle+a_{12}|0\rangle|d_C-1\rangle|0\rangle\Bigg)_{BCD}$$

$$\times\,|(d_B-1)(d_D-1)a_9+(d_C-1)(d_D-1)a_{10}+(d_B-1)(d_C-1)a_{11}+a_{12}=0\Bigg\};$$

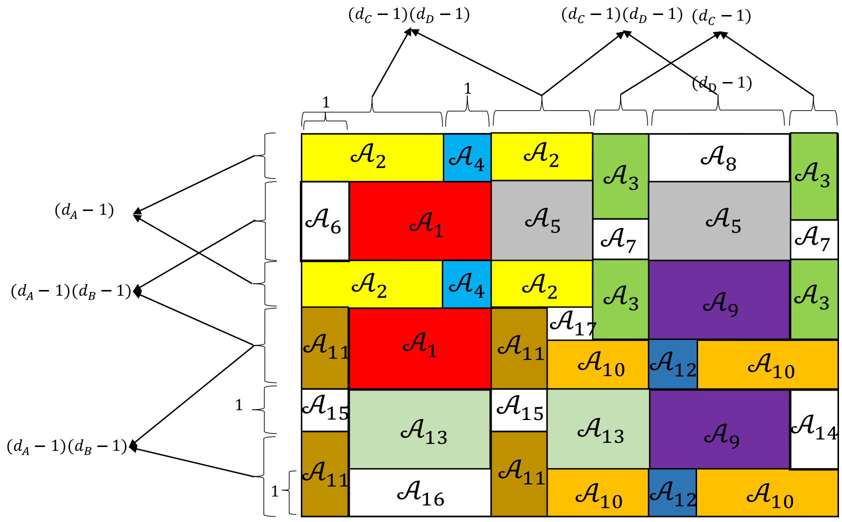

Figure 13. The corresponding tile structure in $\mathbb{C}^{d_Ad_B}\otimes\mathbb{C}^{d_Cd_D}$ of the OPB $\cup_{i=1}^{17}\mathcal{A}_i$ (equation (C1)) in $AB|CD$ bipartition.

(d)
$$O_4=\Big\{|0\rangle_A\Big(a_{13}\sum_{i=0}^{d_B-2}\sum_{j=0}^{d_C-2}\sum_{k=0}^{d_D-2}|i+1\rangle|j\rangle|k+1\rangle+a_{14}\sum_{i=0}^{d_B-2}|i+1\rangle|d_C-1\rangle|d_D-1\rangle$$
$$+a_{15}\sum_{i=0}^{d_C-2}|d_B-1\rangle|i\rangle|0\rangle+a_{16}\sum_{i=0}^{d_D-2}|0\rangle|0\rangle|i+1\rangle\Big)_{BCD}$$
$$\times\,|(d_B-1)(d_C-1)(d_D-1)a_{13}+(d_B-1)a_{14}+(d_C-1)a_{15}+(d_D-1)a_{16}=0\Big\},$$

where $\mathrm{Dim}(O_i)=3$ for $1\leqslant i\leqslant 4$, and $O_i\perp O_j$ for $1\leqslant i\neq j\leqslant 4$. Then $\mathrm{Dim}(O_1+O_2+O_3+O_4)=12$. Since $\mathrm{Dim}(\mathcal{H}^{\perp}_{\mathcal{V}_{A|BCD}})=16$, the OPS $\mathcal{V}_{A|BCD}$ is an SUCPB by lemma 1.

For $\mathcal{V}_{AB|CD}$, we assume that $|\phi\rangle\in\mathcal{H}^{\perp}_{\mathcal{V}_{AB|CD}}$ is a product state. Then $|\phi\rangle$ corresponds to a $d_Ad_B\times d_Cd_D$ matrix N, which has a similar structure to the tile structure in figure 13. That is, $\mathcal{A}_i$ is filled with b_i's for $1\leqslant i\leqslant 17$, where $b_i\in\mathbb{C}$ for $1\leqslant i\leqslant 17$. It must have $\mathrm{rank}(N)=1$, and $\mathrm{sum}(N)=0$. There are only eight cases,

(a) $(d_A-1)(d_B-1)b_1+b_{16}=0$, and $b_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 1,16$;
(b) $(d_C-1)(d_D-1)b_2+b_4=0$, and $b_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 2,4$;
(c) $(d_A-1)(d_B-1)b_3+b_7=0$, and $b_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 3,7$;
(d) $(d_C-1)(d_D-1)b_5+b_6=0$, and $b_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 5,6$;
(e) $(d_A-1)(d_B-1)b_9+b_8=0$, and $b_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 8,9$;
(f) $(d_C-1)(d_D-1)b_{10}+b_{12}=0$, and $b_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 10,12$;
(g) $(d_A-1)(d_B-1)b_{11}+b_{15}=0$, and $b_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 11,15$;
(h) $(d_C-1)(d_D-1)b_{13}+b_{14}=0$, and $b_i=0$ for $1\leqslant i\leqslant 17$ and $i\neq 13,14$.

It means that there are only eight product states in $\mathcal{H}^{\perp}_{\mathcal{V}_{AB|CD}}$:

(a) $\Big(\sum_{i=0}^{d_A-2}\sum_{j=0}^{d_B-2}|i+1\rangle|j\rangle-(d_A-1)(d_B-1)|0\rangle|0\rangle\Big)_{AB}\Big(\sum_{k=0}^{d_D-2}|0\rangle|k+1\rangle\Big)_{CD}$,
(b) $\Big(\sum_{i=0}^{d_A-2}|i+1\rangle|d_B-1\rangle\Big)_{AB}\Big(\sum_{j=0}^{d_C-2}\sum_{k=0}^{d_D-2}|j\rangle|k\rangle-(d_C-1)(d_D-1)|0\rangle|d_D-1\rangle\Big)_{CD}$,
(c) $\Big(\sum_{i=0}^{d_A-2}\sum_{j=0}^{d_B-2}|i+1\rangle|j+1\rangle-(d_A-1)(d_B-1)|d_A-1\rangle|0\rangle\Big)_{AB}\Big(\sum_{k=0}^{d_D-2}|k+1\rangle|d_D-1\rangle\Big)_{CD}$,
(d) $\Big(\sum_{i=0}^{d_A-2}|d_A-1\rangle|i\rangle\Big)_{AB}\Big(\sum_{j=0}^{d_C-2}\sum_{k=0}^{d_D-2}|j+1\rangle|k\rangle-(d_C-1)(d_D-1)|0\rangle|0\rangle\Big)_{CD}$,
(e) $\Big(\sum_{i=0}^{d_A-2}\sum_{j=0}^{d_B-2}|i\rangle|j+1\rangle-(d_A-1)(d_B-1)|d_A-1\rangle|d_B-1\rangle\Big)_{AB}\Big(\sum_{k=0}^{d_D-2}|d_C-1\rangle|k\rangle\Big)_{CD}$,
(f) $\Big(\sum_{i=0}^{d_A-2}|i\rangle|0\rangle\Big)_{AB}\Big(\sum_{j=0}^{d_C-2}\sum_{k=0}^{d_D-2}|j+1\rangle|k+1\rangle-(d_C-1)(d_D-1)|d_C-1\rangle|0\rangle\Big)_{CD}$,
(g) $\Big(\sum_{i=0}^{d_A-2}\sum_{j=0}^{d_B-2}|i\rangle|j\rangle-(d_A-1)(d_B-1)|0\rangle|d_B-1\rangle\Big)_{AB}\Big(\sum_{k=0}^{d_D-2}|k\rangle|0\rangle\Big)_{CD}$,

(h) $\left(\sum_{i=0}^{d_A-2}|0\rangle|i+1\rangle\right)_{AB}\left(\sum_{j=0}^{d_C-2}\sum_{k=0}^{d_D-2}|j\rangle|k+1\rangle-(d_C-1)(d_D-1)|d_C-1\rangle|d_D-1\rangle\right)_{CD}$,

Since $\mathrm{Dim}(\mathcal{H}^{\perp}_{\mathcal{V}_{AB|CD}})=16$, the OPS $\mathcal{V}_{AB|CD}$ is an SUCPB by lemma 1. This completes the proof. □

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