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On partial abelianization of framed local systems

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Abstract

Gaiotto, Moore and Neitzke introduced spectral networks to understand the framed G-local systems over punctured surfaces for G a split Lie group via a procedure called abelianization. We generalize this construction to groups G of the form $\mathrm{GL}_2(A)$, where A is a unital associative ring, and to some of its subgroups. This relies on a precise analysis of the two-fold ramified coverings associated with spectral networks and triangulations and on a matrix reinterpretation of their path lifting rules; along the way we provide another proof of the Laurent phenomenon brought to light by Berenstein and Retakh. The partial abelianization enables us to gives parametrizations of the moduli spaces of decorated G-local systems and of framed G-local systems over punctured surfaces. For (A, σ) a Hermitian involutive $\mathbb{R}$-algebra the group $G = \mathrm{Sp}_2(A, \sigma)$ is a classical Hermitian Lie group of tube type, and we are able to identify and parametrize the moduli space of maximal framed G-local systems.

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Contents

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1 Introduction

The theory of spectral networks was developed by Gaiotto, Moore and Neitzke [1–4] during their research on supersymmetric quantum field theory. However, the mathematical objects arising from this work proved to have an independent mathematical interest. The abelianization using spectral networks can be applied to the study of the geometry of the character varieties of surface groups into complex Lie groups and split real Lie groups [5].

Spectral networks can be seen as graphs on a n-fold ramified covering of a given surface. For $n > 2$ a generic spectral network is an infinite graph that is dense on the surface, however finite spectral networks exist for every $n \geq 2$. The case $n = 2$ is the simplest one, but the abelianization procedure in this case can be only applied to a very restricted class of split Lie groups of rank 1 (e.g. $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{C})$).

The main purpose of this paper is to generalize the abelianization procedure described by Gaiotto, Moore and Neitzke to Lie groups G that can be seen as $\mathrm{GL}_2(A)$ or some subgroups of $\mathrm{GL}_2(A)$ for some unital associative not necessarily commutative $\mathbb{R}$-algebra A. Although, such groups are not always split of rank 1, the abelianization procedure can be partially applied for these groups. In this way, we can understand the structure of the moduli space of decorated and framed G-local systems over punctured surfaces.

We now describe our results in more detail.

Let S be a surface without boundary of negative Euler characteristic $\chi(S)$ with punctures (we refer to Sect. 2 for the wider generality that can be allowed for S, for example disks with marked points on the boundary). A *decorated surface* is a surface as above together with a choice of a simple smooth loop (called a decorating loop) in a neighborhood of every puncture.

Let G be a subgroup of $\mathrm{GL}_2(A)$ for some unital associative not necessarily commutative $\mathbb{R}$-algebra A. A *twisted G-local system* on S is a local system on the unit tangent bundle $T'S$ of S with the holonomy around the fiber of $T'S \to S$ equal to -1.

Further, we consider A^2 (seen as the space of column vectors) as a right A-module. The group G acts on A^2 by the left multiplication. A *framing* of a twisted G-local system is a choice of a parallel line A-subbundle in a neighborhood of every puncture. A *decoration* of a twisted G-local system is a choice of a parallel regular section along every decorating loop. For a precise definition of those notions see Sect. 3. A twisted G-local system together with a framing (or decoration) is called a *framed (resp. decorated) twisted G-local system.*

Notice that a parallel regular section along a decorating loop always induces a parallel line A-subbundle in a neighborhood of the corresponding puncture. Hence, a decorated twisted G-local system always admits a natural framing.

Fixing an ideal triangulation $\mathcal{T}$ of S, we consider a subspace of the space of twisted (framed or decorated) G-local systems, that are *transverse* with respect to $\mathcal{T}$ (or just $\mathcal{T}$-transverse). Following [1, 3], we introduce the ramified covering $\Sigma \to S$ adapted to the triangulation $\mathcal{T}$ and the spectral network on Σ as a graph that satisfies some axioms (for more detail, we refer to Sect. 2.3).

For $\mathcal{T}$-transverse framed twisted G-local system, we describe the twisted abelianization procedure using spectral networks adapted to the triangulation $\mathcal{T}$. The result of this procedure is a twisted $A^\times$-local system on Σ. We also show the converse, i.e. that for every twisted $A^\times$-local system on Σ there exist a unique twisted $\mathcal{T}$-transverse framed G-local system for $G = \mathrm{GL}_2(A)$ (non-abelianization). We describe the abelianization and non-abelianization procedures by defining a path-lifting map from the *twisted path algebra* (see Sect. 2.5) of S to the twisted path algebra of Σ, and we show this map is homotopy-invariant. The partial abelianization and the partial non-abelianization defined in this article are identical to those constructed in [6], albeit described here in a slightly more general framework (working in linear group over any algebra rather than over a division algebra). The non-abelianization procedure is described here using spectral networks, whereas in [6] it is described using a reconstruction functor.

Using this construction, we define non-commutative $\mathcal{A}$-coordinates on the space of decorated twisted G-local systems, and using the path-lifting map we show that these coordinates provide a geometric realization of the non-commutative algebra introduced in [7]. This allows us to give a geometrical proof of the non-commutative Laurent phenomenon, first shown in [7].

Further, we use this abelianization procedure to understand the topology of $\mathcal{T}$-transverse (framed and decorated) twisted G-local systems:

Theorem 1.1 *Let S be a punctured orientable surface of negative Euler characteristic $\chi(S)$ without boundary. Then the moduli space of framed (twisted)* $\mathrm{GL}_2(A)$*-local systems on S that are transverse to a fixed triangulation $\mathcal{T}$ is homeomorphic to the moduli space of (twisted) $A^\times$-local systems on Σ which is homeomorphic to $(A^\times)^{1-4\chi(S)}/A^\times$ where $A^\times$ acts diagonally by conjugation on $(A^\times)^{1-4\chi(S)}$.*

The moduli space of decorated twisted unipotent $\mathrm{GL}_2(A)$*-local systems on S that are transverse to a fixed triangulation $\mathcal{T}$ is homeomorphic to the product of the moduli space of twisted $A^\times$-local systems on $\overline{\Sigma}$ and $(A^\times)^{2p}$ where p is the number of punctures of S.*

Finally, we introduce involutive algebras (A, σ), i.e. unital, associative $\mathbb{R}$-algebras with an $\mathbb{R}$-linear map $\sigma : A \to A$ such that $\sigma(ab) = \sigma(b)\sigma(a)$ for all $a, b \in A$ and $\sigma^2 = \mathrm{Id}$. Over involutive algebras, the symplectic group can be defined as follows: $\mathrm{Sp}_2(A, \sigma) := \{g \in \mathrm{GL}_2(A) \mid \sigma(g)^t \omega g = \omega\}$ where $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. These groups were studied in [8], and they are of particular interest for higher rank Teichmüller theory: For a special class of involutive algebras (A, σ) called Hermitian algebras, the groups $\mathrm{Sp}_2(A, \sigma)$ are Hermitian of tube type. This gives rise to so-called maximal $\mathrm{Sp}_2(A, \sigma)$-local systems on S and maximal representations of the fundamental group of S into $\mathrm{Sp}_2(A, \sigma)$. Maximal local systems and maximal representations were introduced and studied in [9–11]. They provide examples of so-called Higher Teichmüller spaces, i.e. subspaces of the character variety $\mathrm{Rep}(\pi_1(S), \mathrm{Sp}_2(A, \sigma)) = \mathrm{Hom}(\pi_1(S), \mathrm{Sp}_2(A, \sigma)) / \mathrm{Sp}_2(A, \sigma)$ that consist entirely of discrete and faithful representations. The topology of spaces of maximal representations for closed surfaces was studied in [12–15], partly using the theory of Higgs bundles. In [16], the spaces of framed and decorated maximal representations into the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ are parametrized using a non-commutative analog of the Fock–Goncharov parametrization [17] and the topology of them is studied.

We introduce (framed and decorated) twisted $\mathrm{Sp}_2(A, \sigma)$-local system and describe the topology of the moduli space of $\mathcal{T}$-transverse framed twisted $\mathrm{Sp}_2(A, \sigma)$-local systems:

Theorem 1.2 *Using the same notations as in the previous theorem, the moduli space of framed (twisted)* $\mathrm{Sp}_2(A, \sigma)$*-local systems on S that are transverse to a fixed triangulation* $\mathcal{T}$ *is homeomorphic to:*

$$\left(((A^\sigma)^\times)^{-2\chi(S)} \times (A^\times)^{1-\chi(S)}\right) / A^\times$$

where $A^\sigma = \mathrm{Fix}_A(\sigma)$*,* $A^\times$ *acts componentwisely by conjugation on* $(A^\times)^{1-\chi(S)}$ *and by congruence on* $((A^\sigma)^\times)^{-2\chi(S)}$.

For Hermitian A, we also introduce maximal (framed and decorated) twisted $\mathrm{Sp}_2(A, \sigma)$-local systems and describe the topology of the moduli space of maximal framed twisted symplectic local systems:

Theorem 1.3 *If A is Hermitian, then the moduli space of framed (twisted) maximal* $\mathrm{Sp}_2(A, \sigma)$*-local systems on S is homeomorphic to:*

$$\left((A_+^\sigma)^{-2\chi(S)} \times (A^\times)^{1-\chi(S)}\right) / A^\times$$

where $A_+^\sigma = \{a^2 \mid a \in (A^\sigma)^\times\}$*,* $A^\times$ *acts componentwisely by conjugation on* $(A^\times)^{1-\chi(S)}$ *and by congruence on* $(A_+^\sigma)^{-2\chi(S)}$.

This provides a new proof of the result of [16, 18, 19].

Structure of the paper

In Sect. 2 we introduce the topological and combinatorial data needed for the abelianization process, such as ramified coverings and a special class of graphs on them called spectral networks. We define the path-lifting map related to the spectral network. In Sect. 3 we describe the partial abelianization and non-abelianization processes for framed twisted $\mathrm{GL}_2(A)$-local systems. In Sect. 4 we apply this construction to decorated twisted $\mathrm{GL}_2(A)$-local systems, and relate it to the non-commutative algebra introduced in [7]. We also describe the topology of the moduli space of both framed and decorated twisted $\mathrm{GL}_2(A)$-local systems that are transverse with respect to a fixed ideal triangulation. In Sect. 5 we specify this construction for $\mathrm{Sp}_2(A,\sigma)$-local systems, and describe the topology of the moduli space of maximal framed twisted symplectic local systems.

2 Topological and combinatorial data

2.1 Punctured surface

Let $\overline{S}$ be a compact orientable smooth surface of finite type with or without boundary. Let P be a nonempty finite subset of $\overline{S}$ such that on every boundary component of $\overline{S}$ there is at least one element of P. We define $S := \overline{S} \backslash P$. Elements of P are called *punctures* of S. Sometimes we will distinguish between elements of P that lie in the interior of $\overline{S}$ – *internal punctures* and that lie on the boundary – *external punctures*. Surfaces that can be obtained in this way are called *punctured surfaces*, with the exception of the (closed) disk with one or two punctures on the boundary and the sphere with one or two punctures. Every punctured surface can be equipped with a complete hyperbolic structure of finite volume with totally geodesic boundary. For every such hyperbolic structure, all the internal punctures are cusps and all boundary curves are (infinite) geodesics. Once equipped with a hyperbolic structure as above, the universal covering S' of S can be seen as a closed convex subset of the hyperbolic plane $\mathbb{H}^2$ with totally geodesic boundary, which is invariant under the natural action of $\pi_1(S)$ on $\mathbb{H}^2$ by the holonomy representation. Punctures of S are lifted to points of the ideal boundary of $\mathbb{H}^2$ which we call punctures of S' and denote their set by $P' \subseteq \partial_\infty S' \subseteq \partial_\infty \mathbb{H}^2$. Notice, if $\overline{S}$ does not have boundary, then S' is the entire $\mathbb{H}^2$.

An *ideal triangulation* of S is a triangulation with oriented edges of $\overline{S}$ whose set of vertices agrees with P, such that if γ is an edge of the triangulation, then the opposite edge $\overline{\gamma}$ is also an edge of this triangulation. We always consider edges of an ideal triangulation as homotopy classes of oriented paths (relative to their endpoints) connecting points in P. Connected components of the compliment on S to all edges of an ideal triangulation $\mathcal{T}$ are called *faces* or *triangles* of $\mathcal{T}$. Every edge belongs to the boundary of one or two triangles. In the first case, an edge is called *external*, in the second – *internal*. Any ideal triangulation of S can be represented by an ideal geodesic triangulation when a hyperbolic structure as above on S is chosen.

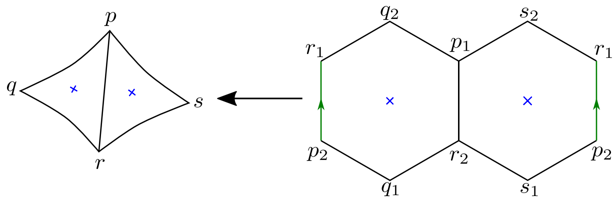


Fig. 1 The ramified two-fold covering of two glued triangles. The preimages of p are p_1 and p_2, same for q, r, s. The branched points are the blue crosses. The two outer edges with an arrow are glued according to arrow orientation

2.2 Ramified covering

Let $\mathcal{T}$ be an ideal triangulation of S. We can endow $\overline{S}$ with a Euclidean structure with conical points by choosing for each triangle T of $\mathcal{T}$ an orientation preserving diffeomorphism $\varphi_T : \mathsf{T} \to T$ where T is the Euclidean triangle in $\mathbb{R}^2 = \mathbb{C}$ with vertices $1, j = e^{\frac{2i\pi}{3}}$ and j^2. Then for each adjacent pair of triangles (not necessarily distinct) in S, glue the corresponding Euclidean triangles with the composition of a rotation and a translation. The conical points of this structure are exactly the points in P, meaning that this structure once restricted to S is smooth. Let $B = \{\varphi_T(0) \mid T \text{ triangle of } \mathcal{T}\} \subset S$. There is exactly one point of B in the interior of each triangle of $\mathcal{T}$. With this data, we can construct a two-fold branched covering $\pi : \overline{\Sigma} \to \overline{S}$ such that the branched points are precisely elements of B and $\overline{\Sigma}$ has a Euclidean structure. Let H be the Euclidean hexagon with vertices the sixth roots of unity in $\mathbb{C}$. Then the map $z \mapsto z^2$ is a ramified covering from H to T that has exactly one ramification of order 2 at the point 0. Then take as many copies of H as there are triangles in $\mathcal{T}$ and for each pair of adjacent triangles (not necessarily distinct) in S, glue the corresponding Euclidean hexagons on both edges that are mapped to the glued edge in S with rotation and a translation (Fig. 1).

This defines a two-fold ramified covering $\pi : \overline{\Sigma} \to \overline{S}$ with ramification points at B, and the conical points of $\overline{\Sigma}$ are a subset of $\pi^{-1}(P)$. This means the map π restricted to $\Sigma = \overline{\Sigma} \backslash \pi^{-1}(P)$ is a smooth two-fold branched covering from Σ to S, with simple ramifications at each point of B. The lift $\mathcal{T}^* := \pi^{-1}(\mathcal{T})$ of $\mathcal{T}$ to Σ induces a hexagonal tiling of Σ such that in every hexagon there is exactly one element of $\pi^{-1}(B)$.

Remark 2.1 By construction, triangles of $\mathcal{T}$ are in 1:1-correspondence with elements of B, and hexagons of $\pi^{-1}(\mathcal{T})$ are also in 1:1-correspondence with elements of B.

Definition 2.2 For a smooth manifold X, denote $T'X$ the spherical quotient of TX, i.e. $T'X = T^pX/\mathbb{R}_+^*$ where T^pX is the punctured tangent bundle of X and the group $\mathbb{R}_+^*$ acts fiberwise by multiplication. The space $T'X$ is then a sphere bundle over X, and we will write an element of $T'X$ as an ordered pair (x, v) with $x \in X$ and v a non-zero vector in T_xX, identified with the half-line it spans. With a slight abuse of terminology, we will call this sphere bundle the *unit tangent bundle* of X.

Remark 2.3 Since the map π is a local diffeomorphism on $\Sigma \backslash \pi^{-1}(B)$, it induces the tangent (differential) map $d\pi : T(\Sigma \backslash \pi^{-1}(B)) \to T(S \backslash B)$ that factorizes to unit tangent bundles $T'(\Sigma \backslash \pi^{-1}(B)) \to T(S\backslash B)'$. In order to simplify the notation, we will sometime write $\pi : T\Sigma \to TS$ and $\pi : T'\Sigma \to T'S$ instead of $d\pi$.

Remark 2.4 The unit tangent bundle of H is canonically identified to $\mathsf{H} \times \mathbb{S}^1$ as H is a subset of $\mathbb{R}^2$. With this identification, the preimages by $d\pi$ of $(x, v) \in T'S$ are of the form (x_1, v') and $(x_2, -v')$ where x_1 and x_2 are the preimages of x by π.

The following proposition describe the topology of the ramified covering Σ:

Proposition 2.5 *Let $\overline{S}$ be a compact orientable surface with $k \geq 0$ boundary components $C_1, \ldots, C_k$ and let P be a finite set of points of $\overline{S}$ such that for all $i \in \{1, \ldots, k\}$, $n_i = \#(C_i \cap P) > 0$. Let k_e (resp. k_o) be the number of components of $\partial\overline{S}$ with an even (resp. odd) number of punctures, such that $k = k_e + k_o$. Let $p = \#(P\backslash\partial\overline{S})$, let g be the genus of $\overline{S}$ and let $S = \overline{S}\backslash P$. Then the two-fold ramified covering $\Sigma = \overline{\Sigma}\backslash\pi^{-1}(P)$ of S is a surface such that:*

- *$\overline{\Sigma}$ is a compact orientable surface of genus*

$$g' = \frac{1}{2}\left(2p + 2k_e + 3k_o + 8g - 6 + \sum_{i=1}^{k} n_i\right),$$

- *for each of the k_e boundary components C of $\overline{S}$ with even number n of punctures, $\pi^{-1}(C)$ is the union of two distinct boundary components in $\overline{\Sigma}$, each with n punctures,*
- *for each of the k_o boundary components C of $\overline{S}$ with odd number n of punctures, $\pi^{-1}(C)$ is one boundary component in $\overline{\Sigma}$ with $2n$ punctures,*
- *Σ has $2p$ internal punctures.*

Proof First, note that the genus g' of Σ is an integer because $3k_o + \sum n_i$ is always even. It is clear from the construction that $\overline{\Sigma}$ is compact and orientable, and that Σ has $2p$ internal punctures. To compute the number of boundary components of $\overline{\Sigma}$, we will glue to each boundary of $\overline{S}$ a disk with the corresponding number of punctures on the boundary to get a surface $\hat{S}$ with no boundary, only internal punctures. Since a disk with one (resp. two) puncture on the boundary does not admit an ideal triangulation, we glue a disk with one (resp. two) puncture on the boundary and one internal puncture instead. In the corresponding ramified covering $\hat{\Sigma}$ of $\hat{S}$, we then remove the lifts of the interior of the glued disks to obtain Σ. The result follows from the following lemma:

Lemma 2.6 *If S is a closed disk with $n \geq 3$ punctures on the boundary, $\overline{\Sigma}$ has either one boundary component with $2n$ punctures if n is odd or two boundary components with n punctures each if n is even. If S is a disk with one internal puncture and one puncture on the boundary, $\overline{\Sigma}$ has one boundary component with two punctures. If S is a disk with one internal puncture and two punctures on the boundary, $\overline{\Sigma}$ has two boundary components with two punctures each.*

Proof The two cases with an internal puncture can be computed individually. Let S be a disk with $n \geq 3$ punctures on the boundary. Let $\mathcal{T}$ be a triangulation of S and Σ

the corresponding ramified covering. Let γ be a loop homotopic to the boundary of the disk going around all the $n-2$ branched points in S. Let x be the base point of γ, and x_1, x_2 the lifts of x to Σ. Let $\tilde{\gamma}$ the lift of γ starting at x_1. If $\tilde{\gamma}$ is a loop then there are two lifts of the boundary of $\overline{S}$ to $\overline{\Sigma}$, and if $\tilde{\gamma}$ is a path from x_1 to x_2 then the lift of the boundary of $\overline{S}$ is connected in $\overline{\Sigma}$. The loop γ is homotopic to the concatenations of loops $\gamma_1, \dots, \gamma_{\lfloor \frac{n-2}{2} \rfloor}, \gamma'$ based at x such that each γ_i goes around two branched points in S and γ' is either trivial if $n-2$ is even or goes around one branched point if $n-2$ is odd. Then $\tilde{\gamma}$ is the concatenation of the lifts $\tilde{\gamma}_1, \dots, \tilde{\gamma}_{\lfloor \frac{n-2}{2} \rfloor}, \tilde{\gamma}'$. Since the $\tilde{\gamma}_i$ are loops based at p_1 and $\tilde{\gamma}'$ is either trivial or a path from p_1 to p_2 (depending on the parity of n), we get the result. □

The Euler characteristic of $\overline{S}$ is

$$\chi(\overline{S}) = 2 - 2g - k = 2 - 2g - k_o - k_e$$

and the Euler characteristic of $\overline{\Sigma}$ is

$$\chi(\overline{\Sigma}) = 2 - 2g' - k_o - 2k_e.$$

The number of branched points is the same as the number of triangles in $\mathcal{T}$, which is $-2\chi(\overline{S}) + 2p + \sum n_i$. Riemann-Hurwitz formula gives us:

$$\begin{aligned}\chi(\overline{\Sigma}) = 2 - 2g' - k_o - 2k_e &= 2\chi(\overline{S}) - \left(-2\chi(\overline{S}) + 2p + \sum_{i=1}^{k} n_i\right) \\ &= 4\chi(\overline{S}) - 2p - \sum_{i=1}^{k} n_i \\ &= 8 - 8g - 2p - 4k_o - 4k_e - \sum_{i=1}^{k} n_i\end{aligned}$$

We can then solve for g' to get the result. □

Remark 2.7 In particular, the topology of Σ does not depend on the triangulation $\mathcal{T}$.

We denote by $\theta \colon \Sigma \to \Sigma$ the covering involution. The following result is a direct consequence of the above proposition.

Corollary 2.8 *The fundamental group $\pi_1(\Sigma)$ is a free group of rank*

$$1 - \chi(\overline{\Sigma}) + 2p = 1 - 4\chi(\overline{S}) + 4p + \sum n_i.$$

Let $b \in \Sigma$ be a ramification point of the covering $\pi : \Sigma \to S$. Let $\alpha_1, \dots, \alpha_s : [0, 1] \to S$ be free generators of the fundamental group $\pi_1(S, \pi(b))$ that do not pass through other ramification points. The fundamental group $\pi_1(\Sigma, b)$ is the free group freely generated by the following collection of loops on Σ:

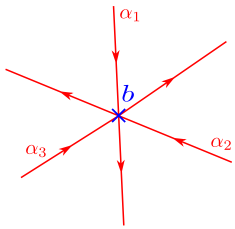


Fig. 2 Local description of a spectral network around a branch point

1. *For every generator α_i, there are two closed lifts γ_i^1 and $\gamma_i^2 = \theta \circ \gamma_i^1$ on Σ based at b (in total $2 - 2\chi(\overline{S}) + 2p$ curves);*
2. *For every ramification point $b' \neq b$ in Σ, we fix a simple segment on S connecting $\pi(b)$ and $\pi(b')$ and take the lift of this segment on Σ. It is a closed loop ξ based at b (in total $-2\chi(\overline{S}) + 2p - 1 + \sum n_i$ curves).*

The fundamental group $\pi_1(T'\Sigma, \tilde{b})$ where $\tilde{b} \in T'\Sigma$ is a lift of b to $T'\Sigma$ is generated by lifts of curves described above and the curve going once around the fiber of $T'\Sigma \to \Sigma$ at $\tilde{b}$.

2.3 Spectral network

Let $\mathcal{T}$ be an ideal triangulation of S and $\pi : \Sigma \to S$ be the corresponding two-fold ramified covering constructed in Sect. 2.2. A (small) *spectral network* associated with this ramified covering is a set $\mathcal{W}$ of paths $[-1, 1] \to \overline{\Sigma}$ (called *rays*) satisfying:

- for all $\alpha \in \mathcal{W}$, $\alpha(-1), \alpha(1) \in \pi^{-1}(P)$, $\alpha(0) \in \pi^{-1}(B)$ and if $t \notin \{-1, 0, 1\}$, then $\alpha(t) \notin \pi^{-1}(P \cup B)$
- for all $\alpha \in \mathcal{W}$ and for all $t \in [-1, 1]$, $\pi(\alpha(t)) = \pi(\alpha(-t))$
- for all $b \in \pi^{-1}(B)$, there are exactly 3 rays $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{W}$ passing through b, and locally around b the rays look like Fig. 2.
- for all $\alpha \neq \alpha'$, $\alpha(] - 1, 0[)$ – which we call the *past* of α – does not intersect $\alpha'(]0, 1[)$ – which we call the *future* of α'.

Remark 2.9 We can omit the last condition in rank 2 spectral networks (i.e. associated to a two-fold covering) as we will construct spectral networks without intersections in Σ.

We can construct a spectral network associated with a triangulation $\mathcal{T}$ in the following way: call the points of $\pi^{-1}(P)$ that are on third roots of unity (for the Euclidean structure) *sinks* and all other points of $\pi^{-1}(P)$ *sources*. This way, each point of P have two preimages, one source and one sink. For each hexagon H of $\pi^{-1}(\mathcal{T})$, the three rays going through the branch point in H are the three Euclidean segments going from the source to the sink for each of the three puncture in T (Fig. 3).

We fix a spectral network $\mathcal{W}$ on Σ adapted to the covering $\Sigma \to S$ and to the ideal triangulation $\mathcal{T}$. The complement of all lines of $\mathcal{W}$ on Σ is a collection of simply connected regions called *cells*. These are either quadrilaterals bounded by four lines of $\mathcal{W}$ or triangles bounded by two lines of $\mathcal{W}$ and one boundary component of S. The closure of every cell in $\overline{S}$ contains exactly two punctures.

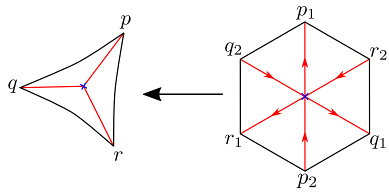


Fig. 3 Picture of the spectral network on each triangle of the triangulation. Here the sources are p_2, q_2, r_2 and the sinks are p_1, q_1, r_1

Let $p_0 \colon S' \to S$ be the universal covering of S. There exist a branched two-fold covering $\pi' \colon \Sigma' \to S'$ and an (infinite but in general not universal) covering $p_1 \colon \Sigma' \to \Sigma$ such that the following diagram commutes:

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\pi'} & S' \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_0} \\ \Sigma & \xrightarrow{\pi} & S \end{array}$$

The ideal hexagonal tiling of Σ lifts to an ideal hexagonal tiling of Σ'. We call the set of ends of edges of this tiling *the ideal boundary of* Σ'. In fact, the ideal boundary of Σ' does not depend on the choice of the tiling.

The map π' can be continuously extended to the ideal boundary of Σ'. Therefore, we can talk about images and preimages of punctures under π'.

We lift the triangulation $\mathcal{T}$ on S, the corresponding hexagonal tiling on Σ and the spectral network $\mathcal{W}$ to these coverings.

Now we are working on the universal covering S'. For every puncture p of S', we consider the union of all cells that have this puncture in their (ideal) boundaries, take the closure of this union in S' and then take the interior of this set. This is an open contractible set of S', we denote it by U_p and call the *standard neighborhood* of the puncture $p \in S'$. Let p_1 and p_2 be two lifts of the puncture p of S' under π'. The lift of U_p to Σ' consists of two connected components U_{p_1} and U_{p_2} every of which projects homeomorphically to U_p, i.e. U_p is evenly covered by U_{p_1} and U_{p_2}. We will call U_{p_1} (resp. U_{p_2}) the *standard neighborhood* of p_1 (resp. p_2).

Notice that two standard neighborhoods either do not intersect or intersect in a cell of the (lifted) spectral network $\mathcal{W}$. More precisely, two standard neighborhoods U_p and U_q intersect if and only if the punctures p and q of S' (resp. of Σ') are connected by an edge of $p_0^{-1}(\mathcal{T})$ in S' (resp. by an edge of $p_1^{-1}(\mathcal{T}^*)$ in Σ').

2.4 Peripheral decoration

To consider framed and decorated local systems, we will need the following additional data on the surface S: for every internal puncture $p \in P$ we fix a neighborhood $S_p \subset S$ of p that is diffeomorphic to a punctured disk. For every external puncture $p \in P$, we choose a neighborhood $S_p \subset S$ of p that is diffeomorphic to a punctured half-disk. We also assume that all S_p are so small that they are pairwise disjoint, and their union does not contain points of B. In this case, every S_p is evenly covered by Σ_{p_1} and Σ_{p_2}

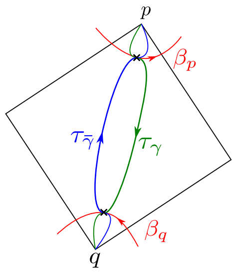


Fig. 4 The bending of an edge of the triangulation. In red are the peripheral decoration, in blue and green are the oriented edges of the triangulation, the crosses are the points in $I_{\mathcal{T}}$ and the thicker part of the edges in between the peripheral curves are the paths τ_γ and $\tau_{\overline{\gamma}}$ (Color figure online)

where $\{p_1, p_2\} = \pi^{-1}(p)$ and Σ_{p_1} and Σ_{p_2} are the two connected components of $\pi^{-1}(S_p)$.

Furthermore, for every internal puncture $p \in P$ we fix a simple smooth loop $\beta_p : [0,1] \to S_p$ around p such that $\dot{\beta}_p(0) = \dot{\beta}_p(1)$, oriented so that p is on the right of β_p according to the orientation of S. For every external puncture $p \in P$, we chose a simple smooth path $\beta_p : ([0,1], \{0,1\}) \to (S_p, S_p \cap \partial S)$ connecting two boundary connected components separated by p, once again with orientation given by the one on S.

In both cases, up to isotopy there is only one such β_p. Since all β_p are smooth, we can lift them to the $T'S$ namely to the curve $[\beta_p(t), \dot{\beta}_p(t)] \in T'S$, $t \in [0,1]$. We denote this lift by $T'\beta_p : [0,1] \to T'S$. Notice that for every internal puncture p, the lift $T'\beta_p$ is always a loop.

If for every $p \in P$ a curve β_p as above is chosen, then we say that the surface S is *decorated*, and the collection $\mathcal{D} = \{\beta_p \mid p \in P\}$ is called a *decoration* of S.

If a hyperbolic structure as above on S is chosen, then every β_p can be represented by projections of small enough horocycles around some $p' \in p_0^{-1}(p)$ under the universal covering map $p_0 : S' \to S$.

Let $\mathcal{T}$ be an ideal triangulation of S. We can assume the arcs of the triangulation are smooth, and if the surface is decorated, we will further assume that any arc of the triangulation intersects only once the peripheral curves associated to its endpoints and do so with matching derivatives, i.e. an arc $\gamma \in \mathcal{T}$ from $p \in P$ to $q \in P$ satisfies $\dot{\gamma}(t_0) = \dot{\beta}_p(t_0')$ and $\dot{\gamma}(t_1) = \dot{\beta}_q(t_1')$ where t_0, t_1, t_0', t_1' are such that $\gamma(t_0) = \beta_p(t_0')$ and $\gamma(t_1) = \beta_q(t_1')$. This is the same as assuming that every arc from p to q of the lift $\mathcal{T}'$ of $\mathcal{T}$ to $T'S$ intersects the lifts $T'\beta_p$ and $T'\beta_q$. This can be done by bending the arcs of $\mathcal{T}$ in a neighborhood of their intersections with the peripheral curves. Let $I_{\mathcal{T}}(S) \subset T'S$ be the set of intersection points between $\mathcal{T}'$ and the lifted decoration curves $T'\beta_p$. This means that now each edge of the triangulation $\mathcal{T}$ is endowed with two special points (one for each extremity) lying on the peripheral curves associated to its endpoints. For every edge $\gamma \in \mathcal{T}$ of the triangulation, let τ_γ be the path in $T'S$ with extremities in $I_{\mathcal{T}}$ obtained by restricting γ to the part in between the two special points on it. Note that since this is applied to all the oriented arcs of the triangulation, the chosen representative for τ_γ and $\tau_{\overline{\gamma}}$ are such that $T'(\tau_\gamma . \tau_{\overline{\gamma}})$ is homotopic to a lace that loop once around the fiber $T'S \to S$ (see Fig. 4).

We also apply the same construction in Σ to equip each edge of the hexagonal tiling $\mathcal{T}^*$ with two special points, and denote $I_{\mathcal{T}^*}(\Sigma)$ the set of all special points in $T'\Sigma$.

Note that both $I_{\mathcal{T}^*}(\Sigma)$ and $I_{\mathcal{T}}(S)$ are finite sets, and that $\pi : T'\Sigma \to T'S$ is 2:1 from $I_{\mathcal{T}^*}(\Sigma)$ to $I_{\mathcal{T}}(S)$.

Since the points in $I_{\mathcal{T}^*}(\Sigma)$ lie on peripheral curves associated with punctures, they inherit the source/sink naming from the puncture.

2.5 The spectral network map

Lifting paths to a ramified covering is not homotopy invariant: a contractible loop around a branch point $b \in B$ is lifted as two paths on Σ that are not loops, thus not homotopic to the lift of the trivial loop. The goal of this section is to construct a path-lifting map SN, which depends on the spectral network $\mathcal{W}$, from paths on $T'S$ to paths on $T'\Sigma$ such that SN is well-defined on homotopy classes.

We will use the symbol $\approx$ to represent homotopy (relative to extremities) of paths.

Let S be a punctured surface, $\mathcal{T}$ an ideal triangulation of S and $\pi : \Sigma \to S$ the two-fold branched covering constructed in Sect. 2.2. Let $\mathcal{W}$ be the spectral network adapted to this covering constructed in Sect. 2.3. Every path $\alpha :]-1,1[\to \Sigma$ of $\mathcal{W}$ is smooth since it is a straight line for the Euclidean structure. We can thus lift the paths of $\mathcal{W}$ to

$$T'\alpha : \begin{array}{rl}]-1,1[& \to T'\Sigma \\ t & \mapsto (\alpha(t), \dot{\alpha}(t)) \end{array}.$$

We will also call this set of paths in $T'\Sigma$ a spectral network and denote it $T'\mathcal{W}$.

Let H be a hexagonal tile of Σ. Note that the Euclidean structure on Σ allows us to identify $T'H$ with $H \times \mathbb{S}^1$. In the following, a path γ on $T'H \simeq H \times \mathbb{S}^1$ will be written as a couple (x, v) where x is the projection of γ on $H \subset \Sigma$ and v is the projection of γ on $\mathbb{S}^1$. Note that $\mathbb{S}^1$ has a natural orientation given by the one on Σ. For all $\theta \in \mathbb{S}^1$, define s_θ^+ to be the (homotopy class of the) path in $\mathbb{S}^1$ going from θ to $-\theta$ following the orientation of $\mathbb{S}^1$, and s_θ^- going from θ to $-\theta$ in the opposite direction. For a path v on $\mathbb{S}^1$, we will denote $-v$ the image of v under the involution $\theta \mapsto -\theta$. The path $-v$ goes from $-v(0)$ to $-v(1)$. In particular, we have $s_\theta^- = -\overline{s_\theta^+}$ and $(-s_\theta^\pm).s_\theta^\pm = \delta_\theta^\pm$ where $\delta_\theta^\pm : t \mapsto \theta \pm 2i\pi t$. When the context is clear, we will omit the subscript describing the starting point of the paths $s^\pm$ and $\delta^\pm$. The paths $\delta^\pm$ satisfy $\delta^- = \overline{\delta^+}$ and if v is a path on $\mathbb{S}^1$ from θ_1 to θ_2, we have $\delta_{\theta_2}^\pm.v \approx v.\delta_{\theta_1}^\pm$.

The Euclidean structure on Σ also defines a flat connection ∇ on $T\Sigma$ given by the restriction of the standard flat connection on $\mathbb{R}^2$. Since it is a bilinear map on the sections of $T\Sigma$ (denoted $\Gamma(T\Sigma)$), this connection induces a flat connection (which we also call ∇) on the unit tangent bundle

$$\nabla : \Gamma(T'\Sigma) \times \Gamma(T'\Sigma) \to \Gamma(T'\Sigma).$$

Definition 2.10 Let X be a topological space. The path algebra of X (denoted $\mathbb{Z}[\mathrm{Path}(X)]$) is the free $\mathbb{Z}$-algebra generated by homotopy classes (relative to extremities) of paths $[0, 1] \to X$, with the product given by concatenation of paths: if $\gamma_1(0) \neq \gamma_2(1)$ then $\gamma_1.\gamma_2 = 0$ and if $\gamma_1(0) = \gamma_2(1)$ then $\gamma_1.\gamma_2$ is the path obtained by following γ_2 then γ_1.

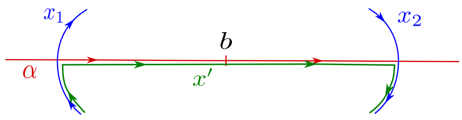


Fig. 5 The path x' added by the intersection with α

Now let X be a smooth surface. Define the *twisted path algebra* of X as

$$\mathrm{TPA}(X) = \mathbb{Z}[\mathrm{Path}(T'X)]/\mathcal{I}$$

where $\mathcal{I}$ is the two-sided ideal generated by the elements $e_{x,\theta} + \delta_{x,\theta}$ for $(x, \theta) \in T'X$, with

$$e_{x,\theta} : \begin{array}{rl} [0,1] \to & T'X \\ t \mapsto & (x,\theta) \end{array} \text{ and } \delta_{x,\theta} : \begin{array}{rl} [0,1] \to & T'X \\ t \mapsto & (x,\theta+2\pi t) \end{array}.$$

Remark 2.11 Given any non-empty subset $E \subset T'X$, the subset

$$\{\gamma_1 + \cdots + \gamma_r + \mathcal{I} \mid \text{for all } 1 \le i \le r,\ \text{endpoints of } \gamma_i \text{ are in } E\} \subset \mathrm{TPA}(X)$$

is a subring of $\mathrm{TPA}(X)$ because composition of paths preserves the set of endpoints. We will denote $\mathrm{TPA}_E(X)$ this subring.

Let x be a path on S intersecting only once (and not at its endpoints) the spectral network $\mathcal{W}$ and not going through a branch point. Let $\alpha \in \mathcal{W}$ be the path such that $\pi(\alpha)$ intersects x. The two standard lifts x_1 and x_2 of x to Σ each intersect once α, one of them intersecting the past of α and the other intersecting the future of α. Suppose x_1 is the one intersecting the past of α. We can then define a new path x' on Σ as the concatenation of 5 paths $x'_1, \ldots, x'_5$ defined as follows (Fig. 5):

- x'_1 is the part of x_1 from its starting point to the intersection point with α
- x'_2 is the part of α from the intersection with x_1 to the branch point
- x'_3 is a constant path at the branch point (it will be useful in the next paragraph when we will consider the lifted spectral network $T'\mathcal{W}$)
- x'_4 is the part of α from the branch point to the intersection with x_2
- x'_5 is the part of x_2 from the intersection with α to its endpoint.

Now let $\gamma : t \mapsto (x(t), v(t))$ be a path on $T'S$ such that the path x on S intersects only once the spectral network on a ray $\alpha \in \mathcal{W}$ at a time $t_0 \in\ ]0, 1[$. Let $\gamma_1 = (x_1, v_1)$ and $\gamma_2 = (x_2, v_2)$ be the standard lifts of γ to $T'\Sigma$, with the same numbering as above. Note that γ_1 and γ_2 do not intersect $T'\mathcal{W}$ in general, but x_1 and x_2 intersect $\alpha \in \mathcal{W}$. Let x' be the path on Σ obtained with the construction described in the previous paragraph. We now want a continuous map $v' : [0, 1] \to \mathbb{S}^1$ which coincide with the standard lifts v_1 and v_2 when x' coincides with either x_1 or x_2. Without loss of generality, suppose x is smooth at the intersection point with $\mathcal{W}$ and that the intersection is transverse. Then x_1 and x_2 are also smooth at their intersection points with α. We say the intersection of x_1 with α is *positively oriented* if $(\dot{x_1}(t_0), \dot{\alpha}(t_0))$ agrees with the orientation on Σ, *negatively oriented* if not.

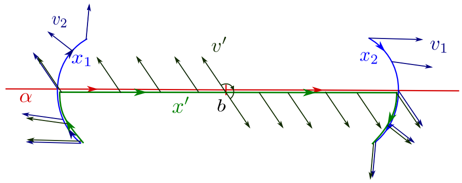


Fig. 6 The path (x', v') added by the intersection with α. The intersection of x_1 with α is positively oriented if Σ is oriented clockwise

Remark 2.12 The positivity of the intersection of a path (x, v) in $T'\Sigma$ with a ray of the spectral network is determined using the derivative of the underlying path x, and does not depend on the vector field v on x.

Let v' be the concatenation of 5 paths $v'_1, \dots, v'_5$ defined as follows (Fig. 6):

- v'_1 is the part of v_1 from its starting point to the intersection point with α
- v'_5 is the part of v_2 from the intersection with α to its endpoint
- v'_2 is obtained by parallel transport with respect to the flat connection ∇ on Σ from the vector $v_1(t_0)$ along the path x'_2
- v'_4 is obtained by parallel transport with respect to ∇ from the vector $v_2(t_0) = -v_1(t_0)$ along the path $\overline{x}'_4$
- v'_3 is the path $s^+_{v'_2(0)}$ in $T'_b\Sigma \simeq \mathbb{S}^1$ if the intersection of x_1 with α is positively oriented, and $s^-_{v'_2(0)}$ if the intersection is negatively oriented.

Remark 2.13 The resulting path v' on $\mathbb{S}^1$ is homotopic to $(-v_1^2).s^{\pm}_{v_1(t_0)}.v_1^1$ where $v_1^1 = v_1 \mid_{[0,t_0]}$ and $v_1^2 = v_1 \mid_{[t_0,1]}$. Note that for any path w on $\mathbb{S}^1$ from θ_0 to θ_1, we have

$$s^{\pm}_{\theta_1}.w \approx (-w).s^{\pm}_{\theta_0}$$

so the path v' is homotopic to $s^{\pm}_{v_1(1)}.v_1$.

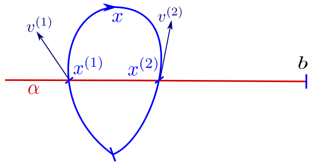
Let $\gamma' = (x', v')$ and $SN(\gamma)$ be the element $\gamma_1 + \gamma_2 + \gamma' \in \mathrm{TPA}(\Sigma)$. Let γ be a path in $T'S$. We can write γ as a concatenation of smaller paths $\gamma^{(1)}, \dots, \gamma^{(r)}$, each intersecting at most once the spectral network and for each of these small paths, apply the construction above to obtain $SN(\gamma^{(1)}), \dots, SN(\gamma^{(r)})$ (if $\gamma^{(i)}$ does not intersect the spectral network, define $SN(\gamma^{(i)})$ to be the sum of the two standard lifts of $\gamma^{(i)}$). Define the lift of γ with respect to the spectral network $\mathcal{W}$ to be the product $SN(\gamma) = SN(\gamma^{(1)}) \dots SN(\gamma^{(r)}) \in \mathrm{TPA}(\Sigma)$.

Theorem 2.1 *Let γ_1 and γ_2 be two homotopic paths in $T'S$. Then $SN(\gamma_1) = SN(\gamma_2)$. In particular, the map*

$$SN\colon \begin{array}{rcl} \mathrm{TPA}(S) & \to & \mathrm{TPA}(\Sigma) \\ \gamma & \mapsto & SN(\gamma) \end{array}$$

is well-defined.

Fig. 7 A loop intersecting twice the same ray of $\mathcal{W}$



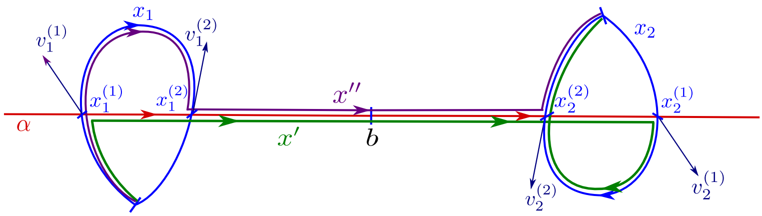


Fig. 8 Spectral network lift of γ

Remark 2.14 The map SN is not defined on the whole twisted path algebra of S as paths with endpoints on a ray of the spectral network cannot be lifted consistently, but we will never need to lift such paths. The subset of $\mathrm{TPA}(S)$ (resp. $\mathrm{TPA}(\Sigma)$) of elements where no term has an endpoint on $\mathcal{W}$ is a subring (see Remark 2.11), and with a slight abuse of notation we will still denote it $\mathrm{TPA}(S)$ (resp. $\mathrm{TPA}(\Sigma)$).

The theorem is a consequence of the two following lemmas:

Lemma 2.15 *Let $\gamma = (x, v)$ be a path in $T'S$ that intersects exactly twice the same ray α of the spectral network and no other ray of $\mathcal{W}$, as in* Fig. 7. *Then $SN(\gamma) = \gamma_1 + \gamma_2$ where γ_1 and γ_2 are the two standard lifts of γ.*

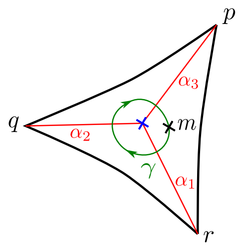
Proof Let $t_1 < t_2$ be the two elements of the interval $[0, 1]$ such that $x(t_1)$ and $x(t_2)$ are on α. Let $(x^{(1)}, v^{(1)}) = \gamma(t_1)$ and $(x^{(2)}, v^{(2)}) = \gamma(t_2)$, and let $\gamma_1 = (x_1, v_1)$ and $\gamma_2 = (x_2, v_2)$ the two standard lifts of γ, γ_1 being the lift intersecting the past of α. Then $SN(\gamma) = \gamma_1 + \gamma_2 + \gamma' + \gamma''$ where $\gamma' = (x', v')$ is such that x' follow α from $x_1^{(1)}$ to $x_2^{(1)}$ and $\gamma'' = (x'', v'')$ is such that x'' follows α from $x_1^{(2)}$ to $x_2^{(2)}$ (Fig. 8).

In order to prove the lemma, we need to show that the two paths γ' and γ'' added by the intersections with the spectral network cancel each other in $\mathrm{TPA}(\Sigma)$, i.e. that $\gamma' + \gamma'' = 0$. For this, we need to show that $\overline{\gamma}''.\gamma'$ is homotopic to an odd power of $\delta_{x_1(0),v_1(0)}$. The paths x' and x'' are homotopic on Σ so the concatenation $\overline{x}''.x'$ is trivial. What is left is to show that $\overline{v}''.v'$ is homotopic to an odd power of δ^+.

Suppose the intersection of x_1 with α at $x_1^{(1)}$ is positive, the other case being symmetric. Then the intersection of x_1 with α at $x_1^{(2)}$ is negative. Then by Remark 2.13, $v' \approx s^+.v_1$ and $v'' \approx s^-.v_1$, so we have

$$\begin{aligned}\overline{v}''.v' &\approx \overline{v}_1.\overline{s^-}.s^+.v_1\\ &\approx \overline{v}_1.\delta^+.v_1\\ &\approx \delta^+.\end{aligned}$$

Fig. 9 A small loop around a branch point



□

Lemma 2.16 *Let m be a point in S in a small neighborhood of a branch point b but not on a ray of $\mathcal{W}$ and $\theta \in T'_m S$. Let γ be a path homotopic to $e_{m,\theta}$ in $T'S$ that loops around a branch point b in S, intersecting exactly once each of the three rays of $\mathcal{W}$ going out of b, as in* Fig. 9. *Then $SN(\gamma) = e_{m_1,\theta_1} + e_{m_2,\theta_2}$ where (m_1, θ_1) and (m_2, θ_2) are the two lifts of (m, θ) in $T'\Sigma$.*

Proof Suppose the path γ is looping around b in the direction given by the orientation of Σ, the other case being symmetric. Then all the intersections of the standard lifts of γ with the spectral network in Σ are positive. By applying the spectral network lifting rule to γ, we get 8 paths: the two standard lifts $\gamma_1 = (x_1, v_1)$ and $\gamma_2 = (x_2, v_2)$, and 6 additional paths $\gamma'_1, \ldots, \gamma'_6$ shown in Fig. 10.

Let α_1, α_2 and α_3 be the three rays of $\mathcal{W}$ intersected by γ, in that order. Let γ_1 be the standard lift of γ intersecting the past of α_1, and let (m_1, θ_1) be its starting point and (m_2, θ_2) be its endpoint. We will label the spectral network lifts $\gamma'_i = (x'_i, v'_i)$ of γ as follows:

- γ'_1 follows γ_1 until the intersection with α_3, then α_3, then γ_2 until its end
- γ'_2 follows γ_2 until the intersection with α_2, then α_2, then γ_1 until its end
- γ'_3 follows γ_2 until the intersection with α_2, then α_2, then γ_1 until the intersection with α_3, then α_3, then γ_2 until its end
- γ'_4 follows γ_1 until the intersection with α_1, then α_1, then γ_2 until its end
- γ'_5 follows γ_1 until the intersection with α_1, then α_1, then γ_2 until the intersection with α_2, then α_2, then γ_1 until the intersection with α_3, then α_3, then γ_1 until its end
- γ'_6 follows γ_1 until the intersection with α_1, then α_1, then γ_2 until the intersection with α_2, then α_2, then γ_1 until its end.

The paths x'_1, x'_4 and x'_5 are homotopic to the trivial path e_{m_1}, x'_6 is homotopic to x_1, x'_2 is homotopic to e_{m_2} and x'_3 is homotopic to x_2. Since γ is homotopic to $e_{m,\theta}$ and x is looping around b in the direction given by the orientation of Σ, we have $v_1 \approx s^-_{\theta_1}$ and $v_2 \approx s^-_{\theta_2}$. Using the same reasoning as above, we get the following:

$$v'_1 \approx s^+.v_1 \approx e_{\theta_1}$$
$$v'_2 \approx s^+.v_2 \approx e_{\theta_2}$$
$$v'_3 \approx s^+.s^+.v_2 \approx \delta^+.v_2$$

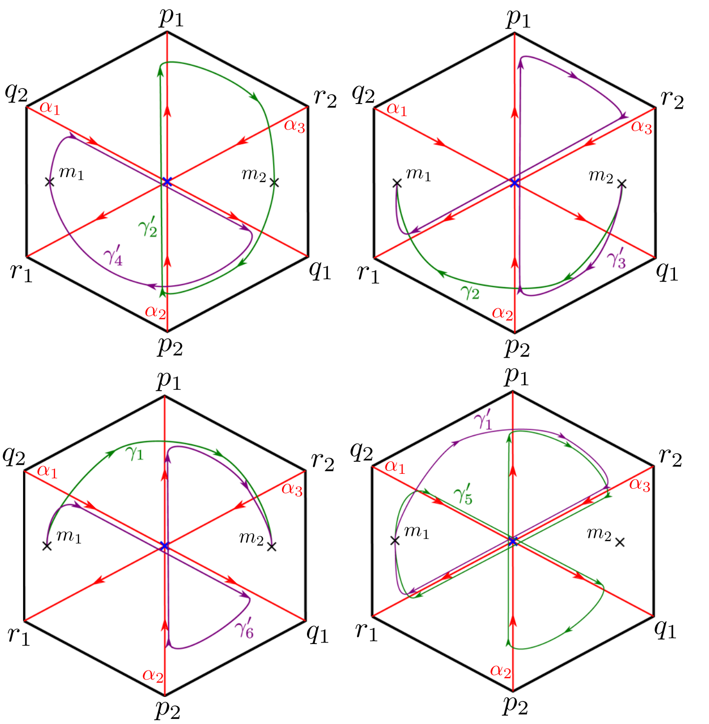


Fig. 10 All 6 paths added by intersections with the spectral network, together with the standard lifts. On the upper left picture are the paths homotopic to trivial paths, and on the other are the remaining lifts, grouped as pairs of paths cancelling each other out in TPA(Σ). Only the paths x_i' on Σ are drawn

$$
\begin{aligned}
v_4' &\approx s^+.v_1 \approx e_{\theta_1} \\
v_5' &\approx s^+.s^+.s^+.v_1 \approx \delta_{\theta_1}^+ \\
v_3' &\approx s^+.s^+.v_1 \approx \delta^+.v_1
\end{aligned}
$$

So in TPA(Σ), we have:

$$
\begin{aligned}
\gamma_2 + \gamma_3' &= 0 \\
\gamma_1 + \gamma_6' &= 0 \\
\gamma_1' + \gamma_5' &= 0 \\
\gamma_4' &= e_{m_1,\theta_1} \\
\gamma_2' &= e_{m_2,\theta_2}
\end{aligned}
$$

so

$$
SN(\gamma) = \gamma_1 + \gamma_2 + \gamma_1' + \gamma_2' + \gamma_3' + \gamma_4' + \gamma_5' + \gamma_6' = e_{m_2,\theta_2} + e_{m_2,\theta_2}. \tag{2.1}
$$

□

3 Partial abelianization of framed twisted local systems

In this section, we define framed and decorated twisted $\mathrm{GL}_2(A)$-local systems over a punctured surface S and describe a partial non-abelianization procedure for them. Using this, we describe the topology of the moduli space of framed and decorated twisted $\mathrm{GL}_2(A)$-local system that are transverse with respect to a fixed ideal triangulation of S.

In this section, A is finite dimensional unital $\mathbb{R}$-algebra, and algebra homomorphisms are required to preserve unity elements.

Let A^n be the set of columns ($n \times 1$ matrices) endowed with the structure of a right A-module.

Definition 3.1 We make the following definitions:

1. An n-tuple $(x_1, \dots, x_n)$ for $x_1, \dots, x_n \in A^n$ is called *basis* of A^n if the map
$$\begin{aligned} A^n &\to A^n \\ (a_1, \dots, a_n) &\mapsto \sum_{i=1}^{n} x_i a_i \end{aligned}$$
is an isomorphism of A-modules.
2. The element $x \in A^n$ is called *regular* if there exist $x_2, \dots, x_n \in A^n$ such that $(x, x_2, \dots, x_n)$ is a basis of A^n.
3. $\ell \subseteq A^n$ is called an A-*line* if $\ell = xA$ for a regular $x \in A^n$. We denote the space of A-lines of A^n by $\mathbb{P}(A^n)$.
4. Regular elements $x_1, \dots, x_k \in A^n$ for $k \le n$ are called *linearly independent* if there exist $x_{k+1}, \dots, x_n \in A^n$ such that $(x_1, \dots, x_n)$ is a basis of A^n.
5. Two A-lines ℓ, m are called *transverse* if $\ell = xA, m = yA$ for linearly independent $x, y \in A^n$.

Let $M_n(A)$ be the ring of all $n \times n$-matrices with entries in A, and $\mathrm{GL}_n(A)$ be the group of all invertible matrices of $M_n(A)$. Then $\mathrm{GL}_n(A)$ acts on A^n by left multiplication.

Definition 3.2 A $\mathrm{GL}_n(A)$-*local system* over a smooth manifold X is a A^n-bundle over X equipped with a flat connection.

Definition 3.3 Let U be an open subset of X. A *regular A-subbundle* L of a $\mathrm{GL}_n(A)$-local system $\mathcal{L}$ over U is a subbundle of $\mathcal{L}$ such that for every $p \in U$ there exists a neighborhood U_p containing p and a local trivialization

$$\Phi_p : \mathcal{L}\,|_{U_p} \to U_p \times A^n$$

such that $\Phi_p(L\,|_{U_p}) = U_p \times \ell$ where ℓ is an A-line in A^n.

A section $v : U \to \mathcal{L}$ is *regular* if vA is a regular A-subbundle of $\mathcal{L}\,|_U$.

3.1 Twisted local systems

In this section, X denotes either S or Σ.

Definition 3.4 A *twisted* $\mathrm{GL}_n(A)$*-local system* on X is a flat A^n-bundle over $T'X$ with monodromy around the fibers of the natural projection $T'X \to X$ equal to $-\mathrm{Id}$.

Let $x_0 \in X$ and $v \in T'_{x_0}X$. The natural projection $T'X \to X$ has fiber homeomorphic to $\mathbb{S}^1$, and we have the short exact sequence

$$1 \to \pi_1(T'_{x_0}X, (x_0, v)) \to \pi_1(T'X, (x_0, v)) \to \pi_1(X, x_0) \to 1$$

with $\pi_1(T'_{x_0}X, (x_0, v))$ being isomorphic to $\mathbb{Z}$, generated by the loop $\delta^+_{x_0,v}$ going around the fiber over x_0 once in the direction given by the orientation of X. This extension is central because X is oriented.

Since X is not closed, the group $\pi_1(X)$ is free, so the sequence above splits. The choice of a splitting corresponds to the choice of a non-vanishing vector field on X. Let $\pi_1^s(X)$ denote the quotient of $\pi_1(T'X, (x_0, v))$ by the normal subgroup $2\mathbb{Z} \subset \mathbb{Z} \simeq \pi_1(T'_{x_0}X, (x_0, v))$, so we have the short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1^s(X) \to \pi_1(X, x_0) \to 1 \tag{3.1}$$

that once again splits. Note that this second sequence also splits when X is closed (for instance for $X = \overline{\Sigma}$ and $\overline{\Sigma}$ has no boundary) since a closed surface of negative Euler characteristic always admits a vector field with zeroes of even indices only.

Proposition 3.5 *The set of twisted* $\mathrm{GL}_n(A)$*-local systems on X up to isomorphism is in 1:1-correspondence with the set of representations* $\rho : \pi_1^s(X) \to \mathrm{GL}_n(A)$ *such that* $\rho(\delta^+_{x_0,v}) = -\mathrm{Id}$*, up to the action of* $\mathrm{GL}_n(A)$ *by conjugation.*

Remark 3.6 Any splitting of the short exact sequence (3.1) induces a 1:1-correspondence between twisted $\mathrm{GL}_n(A)$-local systems on X and $\mathrm{GL}_n(A)$-local systems on X.

If $\mathcal{L}$ is a twisted local system on X and γ is a path on $T'X$, the flat connection defines a holonomy map m_γ from $\mathcal{L}_{\gamma(0)}$ to $\mathcal{L}_{\gamma(1)}$. Moreover, the path $\delta_{x,\theta}$ induces the linear map $-\mathrm{Id}$ on $\mathcal{L}_{x,\theta}$ by definition of a twisted local system. Thus, if $\gamma = \gamma_1 + \cdots + \gamma_r + \mathcal{I} \in \mathrm{TPA}(X)$ where all the $\gamma_i \in \mathrm{Path}(T'X)$ have the same extremities, the holonomy map $m_\gamma = m_{\gamma_1} + \cdots + m_{\gamma_r} : \mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(1)}$ is well-defined (if there is more than one term in γ the holonomy map m_γ may not be an isomorphism). However, if γ_1 and γ_2 do not have the same extremities, it is not possible to associate an element of $M_n(A)$ to $\gamma_1 + \gamma_2$, which is a problem we need to solve in order to consider representations of $\mathrm{TPA}(X)$. To make a link between twisted local systems and representations of $\mathrm{TPA}(S)$, we first need to modify the ring $M_n(A)$ to solve this issue of endpoints. Since multiplication in $\mathrm{TPA}(X)$ is zero for paths whose extremities do not match, we need a ring with the same behavior.

Definition 3.7 Let A be a unital ring and $E \subset X$ any non-empty subset. Let A_E be the ring $A^{(E\times E)}$ of finite formal sums of elements of the form $a_{(p,q)}$, $a \in A$, $p, q \in E$, endowed with the multiplication defined as follows:

- $\forall a, b \in A, \forall x, y, z \in E, a_{(x,y)}.b_{(y,z)} = (a.b)_{(x,z)}$
- $\forall a, b \in A, \forall x, y, z, t \in E, y \neq z, a_{(x,y)}.b_{(z,t)} = 0$

The elements of this ring are copies of elements of A indexed by pairs of points in E, thought as "endpoints" of these elements. The sum of two elements is a formal sum except when the indices match, it then agrees with the sum in A. The multiplication of two elements is made so it agrees with the composition of paths: multiplication of two elements with "non-composable" indices is zero and multiplication with "composable" indices agrees with the one on A, and the index of the result is the composition of the indices.

The ring A_E contains many isomorphic copies of A as subrings: for all $x \in E$,

$$A_x := \left\{a_{(x,x)} \mid a \in A\right\}$$

is a subring of A_E isomorphic to A. Note however that the ring A_E is not unital if E is infinite, but contains many idempotent elements.

Let $\mathrm{TPA}_{\mathcal{T}}(S) = \mathrm{TPA}_{I_{\mathcal{T}}(S)}(S)$ be the subring of $\mathrm{TPA}(S)$ of paths with endpoints in $I_{\mathcal{T}}(S)$ described in Remark 2.11. Similarly, let $\mathrm{TPA}_{\mathcal{T}^*}(\Sigma) = \mathrm{TPA}_{I_{\mathcal{T}^*}(\Sigma)}(\Sigma)$. For any unital ring, let $A_{\mathcal{T}} = A_{I_{\mathcal{T}}(S)}$ and $A_{\mathcal{T}^*} = A_{I_{\mathcal{T}^*}(\Sigma)}$. In the following, $\mathrm{TPA}_{\mathcal{T}^{(*)}}(X)$ denotes either $\mathrm{TPA}_{\mathcal{T}}(S)$ or $\mathrm{TPA}_{\mathcal{T}^*}(\Sigma)$ and similarly $A_{\mathcal{T}^{(*)}}$ denotes either $A_{\mathcal{T}}$ or $A_{\mathcal{T}^*}$. Since $I_{\mathcal{T}}(S)$ and $I_{\mathcal{T}^*}(\Sigma)$ are finite, $\mathrm{TPA}_{\mathcal{T}^{(*)}}(X)$ and $A_{\mathcal{T}^{(*)}}$ are unital, the units elements being respectively $\sum_{x \in I_{\mathcal{T}^{(*)}}(X)} e_x$ and $\sum_{x \in I_{\mathcal{T}^{(*)}}(X)} 1_{(x,x)}$. There is then a diagonal embedding

$$\begin{array}{rcl} A^{\# I_{\mathcal{T}^{(*)}}(X)} & \to & A_{\mathcal{T}^{(*)}} \\ (a_x)_{x \in I_{\mathcal{T}^{(*)}}(X)} & \mapsto & \sum_{x \in I_{\mathcal{T}^{(*)}}(X)} (a_x)_{(x,x)} \end{array}.$$

For every $x \in I_{\mathcal{T}^{(*)}}(X)$, there is an injective group homomorphism

$$\pi_1^s(X, x) \to \mathrm{TPA}_x(X)^\times \subset \mathrm{TPA}_{\mathcal{T}^{(*)}}(X).$$

Two elements $a, b \in A_{\mathcal{T}^{(*)}}$ are said *conjugated* if there exists an invertible element u in $A^{\# I_{\mathcal{T}^{(*)}}}$ such that $b = u.a.u^{-1}$. This is an equivalence relation.

Proposition 3.8 *There is a 1:1 correspondence between the set of twisted* $\mathrm{GL}_n(A)$-*local systems on* X *up to isomorphism and the set of ring homomorphisms* $\mathrm{TPA}_{\mathcal{T}^{(*)}}(X) \to M_n(A)_{\mathcal{T}^{(*)}}$ *up to the action of* $\mathrm{GL}_n(A)^{\# I_{\mathcal{T}^{(*)}}(X)}$ *by conjugation.*

Proof Given a twisted $\mathrm{GL}_n(A)$-local system $\mathcal{L}$ on X, for all $x \in I_{\mathcal{T}^{(*)}}(X)$ choose a basis of the fiber of $\mathcal{L}$ over x. The map

$$\varphi : \begin{array}{rcl} \mathrm{TPA}_{\mathcal{T}^{(*)}}(X) & \to & M_n(A)_{\mathcal{T}^{(*)}} \\ \sum \gamma & \mapsto & \sum (Hol_{\mathcal{L}}(\gamma))_{(t(\gamma), s(\gamma))} \end{array}$$

is a ring homomorphism, where $s(\gamma)$ (resp. $t(\gamma)$) is the source (resp. the sink) of γ (which are in $I_{\mathcal{T}^{(*)}}(X)$), and $Hol_{\mathcal{L}}(\gamma)$ is the holonomy of γ in $\mathcal{L}$ in the corresponding bases. The conjugacy class of φ does not depend on the choices of the bases.

Conversely, let φ be the conjugacy class of a representation $\mathrm{TPA}_{\mathcal{T}^{(*)}}(X) \to M_n(A)_{\mathcal{T}^{(*)}}$, and let $x \in I_{\mathcal{T}^{(*)}}(X)$. Then $\mathrm{TPA}_x(X)$ contains an isomorphic copy of $\pi_1^s(X,x)$ and the restriction of φ to $\pi_1^s(X,x)$ yield a representation $\pi_1^s(X,x) \to \mathrm{GL}_n(A)$ mapping $\delta_x^{\pm}$ to $-\mathrm{Id}$, which define a unique isomorphism class of twisted $\mathrm{GL}_n(A)$-local system by Proposition 3.5, having holonomies described by φ. □

3.2 Framing and decoration

Let $\mathcal{L}$ be a twisted $\mathrm{GL}_2(A)$-local system on S. We say that $\mathcal{L}$ is *peripherally parabolic* if for every puncture $p \in P$ there exist a parallel regular A-subbundle of $\mathcal{L}$ over $T'\beta_p$. A choice of such a parallel regular subbundle $L_p \subset \mathcal{L}_p \to T'\beta_p$, where $\mathcal{L}_p := \mathcal{L}\mid_{T'\beta_p}$ for every $p \in P$ is called a *framing* of $\mathcal{L}$. Since L_p is parallel, on the quotient bundle $\mathcal{L}_p/L_p$ (which is an A-bundle) over $T'\beta_p$ the flat connection is also well-defined. A *framed twisted* $\mathrm{GL}_2(A)$-*local system* is a pair $(\mathcal{L}, (L_p)_{p\in P})$ where $(L_p)_{p\in P}$ is a framing of $\mathcal{L}$.

Let $\mathcal{T}$ an ideal triangulation of S. We say a framed twisted local system $(\mathcal{L}, (L_p)_{p\in P})$ is $\mathcal{T}$-*transverse* if for every edge of the triangulation two subbundles corresponding to two ends of the edge are transverse.

A twisted $\mathrm{GL}_2(A)$-local system $\mathcal{L}$ over a decorated surface $(S, \mathcal{D})$ is called *peripherally unipotent* if for every $\beta_p \in \mathcal{D}$, $p \in P$ there exist a parallel regular section v_p of $\mathcal{L}$ along $T'\beta_p$ and a parallel regular section w_p of the bundle $\mathcal{L}_p/L_p$ along $T'\beta_p$, where L_p is the A-subbundle of $\mathcal{L}_p$ spanned by v_p. If for every β_p such parallel regular sections v_p of $\mathcal{L}_p$ and w_p of $\mathcal{L}_p/L_p$ along $T'\beta_p$ are chosen, then $(\mathcal{L}, (v_p)_{p\in P}, (w_p)_{p\in P})$ is called a *decorated* twisted $\mathrm{GL}_2(A)$-local system.

3.3 Non-abelianization of twisted local systems

In Sect. 2.5, we constructed an algebra homomorphism $SN : \mathrm{TPA}(S) \to \mathrm{TPA}(\Sigma)$. This homomorphism restricts to a ring homomorphism

$$SN : \mathrm{TPA}_{\mathcal{T}}(S) \to \mathrm{TPA}_{\mathcal{T}^*}(\Sigma)$$

as mentioned in Remark 2.11. Let $\gamma \in \mathrm{TPA}_{\mathcal{T}}(S)$ be a path from p to q, $p, q \in I_{\mathcal{T}}(S)$, and let p_1, p_2 be the two lifts of p to Σ, and q_1, q_2 the two lifts of q, with p_1, q_1 being the sinks and p_2, q_2 being the sources. Then

$$SN(\gamma) = \gamma_{1,1} + \gamma_{1,2} + \gamma_{2,1} + \gamma_{2,2}$$

where $\gamma_{j,i}$ is the sum of all terms of $SN(\gamma)$ from p_i to q_j ($\gamma_{i,j}$ may be 0). Instead of a formal sum, it will be more convenient to see $SN(\gamma)$ as a 2 by 2 matrix with coefficients in $\mathrm{TPA}_{\mathcal{T}^*}(\Sigma)$. The definition of the multiplication on $\mathrm{TPA}_{\mathcal{T}^*}(\Sigma)$ makes it so the map:

$$SN : \begin{array}{ccc} \mathrm{TPA}_{\mathcal{T}}(S) & \to & M_2(\mathrm{TPA}_{\mathcal{T}^*}(\Sigma)) \\ \gamma & \mapsto & \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix} \end{array}$$

is a ring homomorphism. We also have a ring homomorphism

$$\pi_A : \begin{array}{ccc} M_2(A_{\mathcal{T}^*}) & \to & M_2(A)_{\mathcal{T}} \\ \begin{pmatrix} a_{q_1,p_1} & b_{q_1,p_2} \\ c_{q_2,p_1} & d_{q_2,p_2} \end{pmatrix} & \mapsto & \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{q,p} \end{array}.$$

Note that we can always write an element of $M_2(A_{\mathcal{T}^*})$ as the sum of elements of the form

$$\begin{pmatrix} a_{q_1,p_1} & b_{q_1,p_2} \\ c_{q_2,p_1} & d_{q_2,p_2} \end{pmatrix}$$

(possibly with some coefficients equal to 0).

Proposition 3.9 *Let $\mathcal{E}$ be a twisted $A^\times$-local system over Σ and let $\varphi : \mathrm{TPA}_{\mathcal{T}^*}(\Sigma) \to A_{\mathcal{T}^*}$ the corresponding ring homomorphism given by* Proposition 3.8. *Then the ring homomorphism*

$$\psi = \pi_A \circ M_2(\varphi) \circ SN : \mathrm{TPA}_{\mathcal{T}}(S) \to M_2(A)_{\mathcal{T}}$$

corresponds to a peripherally parabolic twisted $\mathrm{GL}_2(A)$*-local system on S, together with a $\mathcal{T}$-transverse framing.*

Proof Let $\mathcal{E}$ be a twisted $A^\times$-local system on Σ, and let $\mathcal{L}$ be the $\mathrm{GL}_2(A)$-local system obtained on S. We need to show that $\mathcal{L}$ admits a flat section on any peripheral curve β_p on S, i.e. that the monodromy along β_p is upper triangular in some basis. Let $p \in P$ and p_1, p_2 the lifts of p to Σ, p_1 being the sink and p_2 the source. Let $q \in I_{\mathcal{T}}(S) \cap \beta_p$ and q_1, q_2 the lifts of q to Σ, $q_i \in \beta_{p_i}$. We will assume β_p is a loop based on q. The fiber $\mathcal{L}_q$ of $\mathcal{L}$ over q can be identified with the direct sum $\mathcal{E}_{q_1} \oplus \mathcal{E}_{q_2}$ of the fibers of $\mathcal{E}$ over q_1 and q_2. Every ray of the spectral network $\mathcal{W}$ crossed by β_p on S lifts to a ray from p_2 to p_1 on Σ. This means that the lifts added by the spectral network all go from q_1 to q_2, so the image of β_p via $SN : \mathrm{TPA}_{\mathcal{T}}(S) \to M_2(\mathrm{TPA}_{\mathcal{T}^*}(\Sigma))$ is upper triangular. Then $\psi(\beta_p) \in P_{\mathcal{T}}(M_2(A))$, the monodromy of β_p, is also upper triangular. The line $\mathcal{E}_{q_1} \subset \mathcal{L}_q$ is preserved by the peripheral monodromy which means that the parallel transport of $\mathcal{E}_{q_1}$ along β_p defines a framing $L_p \subset \mathcal{L}_{\beta_p}$ around p. This framing is $\mathcal{T}$-transverse because for every edge γ of $\mathcal{T}$ from p to q, $p, q \in P$, the map $L_p \to \mathcal{L}_q/L_q$ is the holonomy of $\mathcal{E}$ along one of the lifts of τ_γ so it is an isomorphism. □

The twisted $\mathrm{GL}_2(A)$-local system $\mathcal{L}$ on S obtained from a twisted $A^\times$-local system $\mathcal{E}$ on Σ via this construction is called the *non-abelianization* of $\mathcal{E}$. In the next part, we introduce an inverse construction.

3.4 Partial abelianization of transverse framed local systems

Let $(\mathcal{L}, (L_p)_{p\in P})$ be a $\mathcal{T}$-transverse framed twisted $\mathrm{GL}_2(A)$-local system on S. Our goal is to construct a twisted $A^\times$-local system $\mathcal{E} \to T'\Sigma$

that provides under the non-abelianization procedure the initial local system $\mathcal{L}$.

Let $\pi'\colon \Sigma' \to S'$ be the covering as in Sect. 2.3. A transverse framed twisted $\mathrm{GL}_2(A)$-local system $\mathcal{L}$ over S gives rise to a transverse framed twisted $\pi_1(S)$-equivariant $\mathrm{GL}_2(A)$-local system $\mathcal{L}'$ over S'. We obtain a parallel A-subbundle L'_p over the preimage $T'U_p$ under $T'S' \to S'$ of every standard neighborhood U_p of every puncture p of S'. And similarly to the discussion above, on the quotient A-bundle $\mathcal{L}'_p/L'_p$ over $T'U_p$ the flat connection is well-defined.

The spectral network $\mathcal{W}$ on Σ (resp. $\mathcal{W}'$ on Σ') divides the set of punctures of Σ (resp. Σ') in two classes: sinks of $\mathcal{W}$ (resp. $\mathcal{W}'$) and sources of $\mathcal{W}$ (resp. $\mathcal{W}'$). For every sink p of Σ' we define a flat A-bundle over $T'U_p$ as the pull-back of the A-subbundle $L'_{\pi'(T'U_p)}$. For every source p of Σ' we define a flat A-bundle over $T'U_p$ as the pull-back of the A-bundle $\mathcal{L}'/L'_{\pi'(T'U_p)}$.

To construct a twisted flat A-bundle over Σ' we need to "glue" the standard neighborhoods along cells of the (lifted) spectral network. For this, we notice that two standard neighborhoods that share a cell always correspond to punctures of different classes. Along the interior of such a cell c two A-bundles are defined: $L'_{\pi'(T'U_p)}$ corresponding to a sink p of Σ' and $\mathcal{L}'/L'_{\pi'(T'U_q)}$ corresponding to a source q of Σ'.

By transversality, for every point $z \in T'c \subset T'\Sigma'$ the A-submodules $L'_{\pi'(T'U_p)}(\pi'(z))$ and $L'_{\pi'(T'U_q)}(\pi'(z))$ of $\mathcal{L}$ are transverse. That means that for all $z \in T'c$ the natural projection map

$$a_{qp}(\pi'(z))\colon L'_{\pi'(T'U_p)}(\pi'(z)) \to \mathcal{L}'/L'_{\pi'(T'U_q)}(\pi'(z))$$

is an isomorphism. So we can identify $v \in L_{\pi'(T'U_p)}(\pi'(z))$ with $a_{qp}(\pi'(z))(v) \in \mathcal{L}/L_{\pi'(T'U_q)}(\pi'(z))$ for all $z \in T'c$. Since $a_{qp}(\pi'(z))$ is constant in any parallel frames of $L'_{\pi'(T'U_p)}$ and $\mathcal{L}'/L'_{\pi'(T'U_q)}$ along $T'c$, this provides a constant change of local trivialization over $T'c$. In other words, we obtain a flat A-bundle over $\Sigma'\backslash B'$ that we denote by $\mathcal{E}' \to \Sigma'\backslash B'$. The construction of $\mathcal{E}$ is obviously $\pi_1(S)$ equivariant and, therefore, defines a flat A-bundle $\mathcal{E} \to T'(\Sigma\backslash B)$.

Lemma 4.1 below provides that the holonomies around branch points are trivial. This implies that the bundle $\mathcal{E}$ can be extended also over B, i.e. we obtain a bundle $\mathcal{E} \to T'\Sigma$. The twisted $A^\times$-local system obtained on Σ is called the *abelianization* of $\mathcal{L}$.

Those processes are inverse to each other by construction.

We have thus shown:

Theorem 3.1 *The abelianization and non-abelianization processes define a bijection between the set of twisted $A^\times$-local systems on Σ up to isomorphism and the set of framed $\mathcal{T}$-transverse twisted local systems on S up to isomorphism.*

Remark 3.10 The construction of Σ depends on $\mathcal{T}$, which is implicit in the theorem.

Remark 3.11 The abelianized local system constructed on Σ is identical to the one constructed in [6]. In [20], the first author describe the partial abelianization and non-abelianization processes for n-fold ramified coverings, using spectral networks.

4 Partial abelianization of decorated twisted local systems

In this section, we apply the above construction to decorated twisted $\mathrm{GL}_2(A)$-local systems. This construction gives a geometrical representation of the non-commutative algebra introduced in [7], as well as geometrical proof of the non-commutative Laurent phenomenon. We also use this construction to describe the topology of the space of framed or decorated twisted $\mathrm{GL}_2(A)$-local systems.

4.1 Kashiwara-Maslov map

Let ℓ_1, ℓ_2, ℓ_3 be pairwise transverse A-lines in A^2. We denote $a_{ji}\colon \ell_i \to A^2/\ell_j$ the projection maps for all $i, j \in \{1, 2, 3\}$, $i \neq j$. By transversality, a_{ij} are A-linear isomorphisms.

The following lemma is immediate:

Lemma 4.1 *The map $-a_{31}^{-1}a_{32}a_{12}^{-1}a_{13}a_{23}^{-1}a_{21}\colon \ell_1 \to \ell_1$ is the identity map.*

The map $\mu_1^{23} := a_{13}a_{23}^{-1}a_{21}\colon \ell_1 \to A^2/\ell_1$ is called the *Kashiwara-Maslov map* of the triple of A-lines (ℓ_1, ℓ_2, ℓ_3). Notice that $\mu_1^{23} = -\mu_1^{32}$.

Let $\pi\colon \Sigma \to S$ be a ramified covering associated to an ideal triangulation $\mathcal{T}$ as before. Let $\mathcal{L} \to T'S$ be a framed $\mathcal{T}$-transverse twisted local system over S and $\mathcal{E} \to T'\Sigma$ be the twisted $A^\times$-local system obtained from $\mathcal{L}$ by the partial abelianization described in Sect. 3.4. Let $\tau \subset S$ be a triangle of $\mathcal{T}$ that is incident to punctures p_1, p_2, p_3, and the orientation of the triangle agrees with the cyclic order of the triple (p_1, p_2, p_3). Let $H = \pi^{-1}(\tau)$ be the hexagon of Σ that covers τ. The hexagon H is divided in six (open) triangles by lines of the spectral network, as on Fig. 3. As before, let L_i be a parallel A-subbundle of $\mathcal{L} \to T'\tau$ corresponding to the puncture p_i, $i \in \{1, 2, 3\}$. By construction of the local system on Σ, over every triangle of H two A-bundles L_i and $\mathcal{L}/L_j$, $i, j \in \{1, 2, 3\}$ are defined. These two bundles are glued along this triangle by the projection map $a_{ji}\colon L_i \to A^2/L_j$. We denote this triangle by t_{ji}. Let now $p \in T't_{21}$, then $\theta(p) \in T't_{12}$, where $\theta\colon T'\Sigma \to T'\Sigma$ is the involution associated with the covering $\pi\colon \Sigma \to S$.

We take a path $\gamma\colon [0, 1] \to T'H$ such that $\gamma(0) = p$, $\gamma(1) = \theta(p)$ and such that $(\theta \circ \gamma).\gamma$ is a loop in $T'H$ homotopic to the loop going once in the positive direction around the fiber of $T'\Sigma \to \Sigma$.

The following proposition follows directly from the construction of the $A^\times$-local system over $T'\Sigma$:

Proposition 4.2 *The parallel transport along the path γ agrees with the Kashiwara-Maslov map $\mu_1^{23}\colon L_1(\pi(p)) \to \mathcal{L}/L_1(\pi(p))$.*

4.2 Partial abelianization of transverse decorated local systems

Applying the construction of Sect. 3.4 to a peripherally unipotent local system, we get an $A^\times$-local system $\mathcal{E} \to T'\Sigma$. Moreover, the existence of sections v_p and w_p over $T'\beta_p$ for all $p \in P$ induces that the holonomies of $\mathcal{E}$ around punctures of Σ are all equal to -1.

A decoration of $\mathcal{L}$ provides additionally a parallel section of $\mathcal{E}_p \to T'\Sigma_p$. We call the set of all those parallel sections a *decoration* of the twisted $A^\times$-local system $\mathcal{E}$.

Theorem 4.1 *The abelianization and non-abelianization processes define a bijection between the set of decorated twisted $A^\times$-local systems on Σ with trivial monodromy around punctures up to isomorphism and the set of decorated $\mathcal{T}$-transverse twisted $\mathrm{GL}_2(A)$-local systems on S up to isomorphism.*

4.3 Non-commutative $\mathcal{A}$-coordinates and partial abelianization

Let $\mathcal{D}$ be a decoration of S and let $\mathcal{T}$ be a triangulation of S. Let $(\mathcal{L}, (v_p)_{p\in P}, (w_p)_{p\in P})$ be a decorated twisted $\mathrm{GL}_2(A)$-local system on the surface S, and assume $\mathcal{L}$ is $\mathcal{T}$-transverse. Then for every arc γ of $\mathcal{T}$ from $p \in P$ to $q \in P$, we can trivialize the $\mathrm{GL}_2(A)$-local system $\mathcal{L}$ over $T'\gamma$ and the A-subbundles spanned by the flat sections v_p and v_q are transverse. The natural projection

$$a_\gamma : L_p = \mathrm{Span}(v_p) \to \mathcal{L}/L_q = \mathrm{Span}(w_q)$$

is an isomorphism, and we can identify it with its (1 by 1) matrix in the bases v_p and w_q. We thus obtain a family $(a_\gamma)_{\gamma\in\mathcal{T}}$ of elements of $A^\times$ which we call *non-commutative $\mathcal{A}$-coordinates* of $\mathcal{L}$. The name "coordinates" is a slight abuse, since they are not independent.

Proposition 4.3 *For every oriented triangle $(\gamma_1, \gamma_2, \gamma_3)$ of $\mathcal{T}$, we have*

$$a_{\gamma_3} a_{\overline{\gamma}_2}^{-1} a_{\gamma_1} = a_{\overline{\gamma}_1} a_{\gamma_2}^{-1} a_{\overline{\gamma}_3} \tag{4.1}$$

The coordinates of a decorated twisted $\mathrm{GL}_2(A)$-local system are the holonomies of its abelianized system $\mathcal{E}$ along the lifts of the arcs τ_γ, $\gamma \in \mathcal{T}$: for each puncture $p \in P$, the two lifts of p to Σ are a sink p_1 and a source p_2. In the neighborhood of p_1, the bundle $\mathcal{E}$ is the pullback of L_p and in the neighborhood of p_2 the bundle is the pullback of $\mathcal{L}/L_p$. Now for an arc $\gamma \in \mathcal{T}$ from $p \in P$ to $q \in P$, the lifts of τ_γ to Σ join a sink and a source. Denote γ_1 the lift from p_1 to q_2 and γ_2 the lift from p_2 to q_1. Then a_γ is the holonomy of $\mathcal{E}$ along $T'\tau_{\gamma_1}$ and $a_{\overline{\gamma}}$ is the holonomy of $\mathcal{E}$ along $T'\tau_{\overline{\gamma_2}}$.

Let Γ be the graph embedded in $T'\Sigma$ with vertices the lifts of peripheral curves and edges the arcs τ_γ, $\gamma \in \mathcal{T}^*$ oriented from sink to source. To each oriented edge of this graph a coordinate is associated, and we assign to the edges with reversed orientation the inverse of this coordinate. The vertices of this graph are curves around which the monodromy of $\mathcal{E}$ is trivial because $\mathcal{L}$ is decorated, so given a path on Γ, its holonomy in $\mathcal{E}$ is well-defined. Then the triangle relation (4.1) imply that the monodromy of the abelianized system $\mathcal{E}$ restricted to the graph Γ is trivial around every hexagonal tile.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two triangulations differing only by one flip. Let p_1, p_2, p_3, p_4 be the four (not necessarily distinct) punctures at the vertices of the quadrilateral supporting the flip, in the cyclic order such that $\mathcal{T}_1 \backslash \mathcal{T}_2 = \{\gamma_{1,3}, \gamma_{3,1}\}$ and $\mathcal{T}_2 \backslash \mathcal{T}_1 = \{\gamma_{2,4}, \gamma_{4,2}\}$ where $\gamma_{i,j}$ is the arc of the quadrilateral going from p_j to p_i. Using the path-lifting map, we can compute the relations between the $\mathcal{A}$-coordinates associated to $\mathcal{T}_1$ and the $\mathcal{A}$-coordinates associated to $\mathcal{T}_2$.

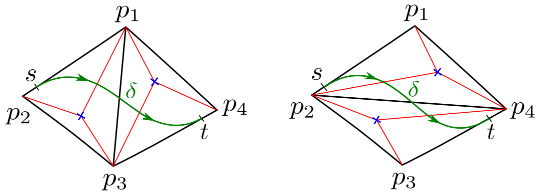


Fig. 11 The path δ on S with triangulation $\mathcal{T}_1$ on the left and with triangulation $\mathcal{T}_2$ on the right

Proposition 4.4 *Let $\mathcal{L}$ a decorated twisted $\mathrm{GL}_2(A)$-local system that is both $\mathcal{T}_1$-transverse and $\mathcal{T}_2$-transverse. Then its $\mathcal{A}$-coordinates with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy the following exchange relations:*

$$a_{\gamma_{2,4}} = a_{\gamma_{2,1}} a_{\gamma_{3,1}}^{-1} a_{\gamma_{3,4}} + a_{\gamma_{2,3}} a_{\gamma_{1,3}}^{-1} a_{\gamma_{1,4}}$$
$$a_{\gamma_{4,2}} = a_{\gamma_{4,1}} a_{\gamma_{3,1}}^{-1} a_{\gamma_{3,2}} + a_{\gamma_{4,3}} a_{\gamma_{1,3}}^{-1} a_{\gamma_{1,2}}$$

Proof For $i \in \{1, 2, 3, 4\}$, let p_i', p_i'' be the two lifts of p_i to Σ where p_i' is the sink and p_i'' is the source. Let $s \in I_{\mathcal{T}_1}(S) \cap I_{\mathcal{T}_2}(S)$ be the intersection of $T'\beta_{p_2}$ and $\gamma_{2,1}$ and let $t \in I_{\mathcal{T}_1}(S) \cap I_{\mathcal{T}_2}(S)$ be the intersection of $T'\beta_{p_4}$ and $\gamma_{3,4}$. Let δ be a path in $T'S$ from s to t as in Fig. 11.

The holonomy of $\mathcal{L}$ along δ does not depend on the triangulation.

Let Σ_1 and $\mathcal{W}_1$ be the ramified covering and the spectral network associated to the triangulation $\mathcal{T}_1$ and Σ_2, $\mathcal{W}_2$ the ones associated to $\mathcal{T}_2$. The corresponding path-lifting maps will be denoted SN_1 and SN_2, and the corresponding abelianizations of S will be denoted $\mathcal{E}_1$ and $\mathcal{E}_2$.

First, let's lift δ to Σ_2 using SN_2. Let s_1, s_2 the lifts of s to Σ_2, s_1 being the sink and s_2 the source. Similarly, let t_1, t_2 the lifts of t, t_1 being the sink and t_2 the source. We get

$$SN_2(\delta) = \delta_1 + \delta_2 + \delta_1' + \delta_2' + \delta_3'$$

where δ_1 is a standard lift from s_1 to t_2, δ_2 is a standard lift from s_2 to t_1, δ_1' is a spectral lift from s_2 to t_2, δ_2' is a spectral lift from s_2 to t_1 and δ_3' is a spectral lift from s_1 to t_1 (see Fig. 12). The path δ_1 is the only lift going from s_1 to t_2, and its holonomy in $\mathcal{E}_2$ in the corresponding bases is $a_{\gamma_{4,2}}$ since it is homotopic to $\tau_{\gamma_{4,2}}$ precomposed with a piece of β_{p_2} and postcomposed with a piece of β_{p_4}, both of which have trivial holonomies. Since $\mathcal{L}$ is the non-abelianization of $\mathcal{E}_2$, this means that the map $L_{p_2} \to \mathcal{L}_{p_4}/L_{p_4}$ obtained by trivializing $\mathcal{L}$ along δ is exactly $a_{\gamma_{4,2}}$.

Now we will lift δ to Σ_1 using SN_1. We will keep the same notations as in the previous paragraph. We get

$$SN_1(\delta) = \delta_1 + \delta_2 + \delta_1' + \delta_2' + \delta_3'$$

where δ_1 is a standard lift from s_1 to t_2, δ_2 is a standard lift from s_2 to t_1, δ_1' is a spectral lift from s_1 to t_1, δ_2' is a spectral lift from s_1 to t_2 and δ_3' is a spectral lift from s_2 to t_2 (see Fig. 13). The paths going from s_1 to t_2 are δ_1 and δ_2', and their holonomies in $\mathcal{E}_1$ in the corresponding bases are respectively $a_{\gamma_{4,3}} a_{\gamma_{1,3}}^{-1} a_{\gamma_{1,2}}$ and $a_{\gamma_{4,1}} a_{\gamma_{3,1}}^{-1} a_{\gamma_{3,2}}$. These

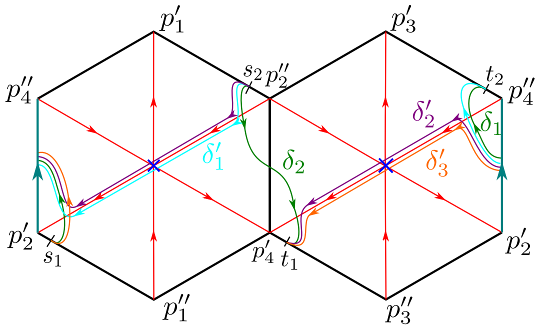


Fig. 12 All the lifts of δ to Σ_2 using SN_2

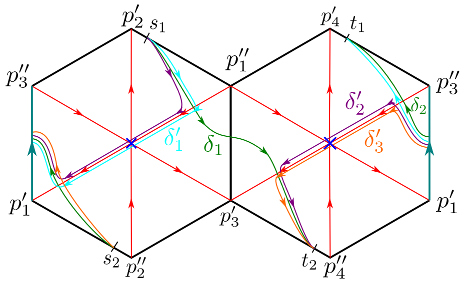


Fig. 13 All the lifts of δ to Σ_1 using SN_1

are obtained by retracting the paths on the graph Γ, as the oriented edges of Γ have holonomies given by the $\mathcal{A}$-coordinates. Since $\mathcal{L}$ is also the non-abelianization of $\mathcal{E}_1$, this means that the map $L_{p_2} \to \mathcal{L}_{p_4}/L_{p_4}$ obtained by trivializing $\mathcal{L}$ along δ must be equal to the holonomy of $\delta_1 + \delta'_2$, which give the formula:

$$a_{\gamma_{4,2}} = a_{\gamma_{4,1}} a_{\gamma_{3,1}}^{-1} a_{\gamma_{3,2}} + a_{\gamma_{4,3}} a_{\gamma_{1,3}}^{-1} a_{\gamma_{1,2}}$$

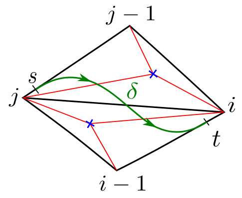
The formula for $a_{\gamma_{2,4}}$ is obtained similarly. □

Remark 4.5 The non-commutative $\mathcal{A}$-coordinates constructed here are identical to the one constructed in [6] in the case of a two-fold ramified covering. In [20], the first author describe non-commutative $\mathcal{A}$-coordinates for $\mathrm{GL}_n(A)$-local systems, $n \geq 2$, using the abelianization procedure. The coordinates constructed there only coincide with the coordinates of [6] when $n = 2$.

This gives a geometric realization of the non-commutative algebra $\mathcal{A}_S$ introduced in [7]. Using the same type of arguments as above, we can give a topological/geometrical proof of the Laurent phenomenon for the cluster algebra of a polygon:

Theorem 4.2 *Let $n \geq 3$ and let S_n the closed disk with n punctures on the boundary. Let $i, j \in \{1, \ldots, n\}$, $i \neq j$. Then for every triangulation $\mathcal{T}$ of S_n and every decorated twisted* $\mathrm{GL}_2(A)$*-local system $\mathcal{L}$ that is both $\mathcal{T}$-transverse and (i, j)-transverse, the $\mathcal{A}$-coordinate $a_{\gamma_{i,j}}$ is a non-commutative Laurent polynomial in the $\mathcal{A}$-coordinates $(a_\gamma)_{\gamma \in \mathcal{T}}$ associated to the triangulation $\mathcal{T}$.*

Fig. 14 The path δ in the triangulation $\mathcal{T}_0$. Only the quadrilateral $(i-1, i, j-1, j)$ is drawn



Proof All the edges of the form $\gamma_{i,i+1}$, with $i \in P$ ordered cyclically, belong to every triangulation of S_n so the result is immediate. Now let $i, j \in \{1, \ldots, n\}$, $i \neq j \pm 1$. Let $\mathcal{T}_0$ be a triangulation of S_n containing the edges $\gamma_{i,j}$, $\gamma_{i,j-1}$ and $\gamma_{i-1,j}$. Such a triangulation always exists when $i \neq j \pm 1$. Let $s \in I_{\mathcal{T}_0}(S_n) \cap I_{\mathcal{T}}(S_n)$ be the intersection of β_j and $\gamma_{j-1,j}$ and let $t \in I_{\mathcal{T}_0}(S_n) \cap I_{\mathcal{T}}(S_n)$ be the intersection of β_i and $\gamma_{i-1,i}$. Let δ be the path from s to t drawn in Fig. 14.

As we have seen in the proof of the flip relation (and keeping the same notations), in the spectral network lift of δ with respect to the triangulation $\mathcal{T}_0$ the only term from s_1 to t_2 has the holonomy $a_{\gamma_{i,j}}$ in the abelianization of $\mathcal{L}$ with respect to $\mathcal{T}_0$. This means that the map $L_j \to \mathcal{L}_i/L_i$ obtained by trivializing $\mathcal{L}$ on δ is $a_{\gamma_{i,j}}$.

Let $\mathcal{E}$ be the abelianization of $\mathcal{L}$ with respect to $\mathcal{T}$. In spectral network lift of δ with respect to the triangulation $\mathcal{T}$, let $\delta' = \delta'_1 + \cdots + \delta'_r$ be the sum of all paths from s_1 to t_2. Each δ'_k has a holonomy in $\mathcal{E}$ that is a monomial in the coordinates $(a_\gamma^{\pm 1})_{\gamma \in \mathcal{T}}$ as it retracts on the graph Γ. Since $\mathcal{L}$ is the non-abelianization of $\mathcal{E}$, the map $L_j \to \mathcal{L}_i/L_i$ obtained by trivializing $\mathcal{L}$ on δ is equal to the sum of the holonomies of the δ'_k in $\mathcal{E}$, so it is a Laurent polynomial in the $\mathcal{A}$-coordinates $(a_\gamma)_{\gamma \in \mathcal{T}}$. □

Remark 4.6 The above proposition implies a similar statement about $\mathcal{A}$-coordinates on a surface, as shown in [7]. The proof given above however relies on the fact the external edges of a polygon belongs to every triangulation, thus can not be extended to surfaces directly.

Using these $\mathcal{A}$-coordinates, we can describe precisely the changes of the $A^\times$-local system on Σ induced by a flip in the triangulation. We use the same notations as in Proposition 4.4. Let $\mathcal{L}$ be a framed twisted $\mathrm{GL}_2(A)$-local system on S that is transverse with respect to both $\mathcal{T}_1$ and $\mathcal{T}_2$. Let $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) be the $A^\times$-local system on Σ obtained by abelianizing $\mathcal{L}$ with respect to $\mathcal{T}_1$ (resp. $\mathcal{T}_2$). These changes on the abelianized local system are supported in the lift C_Q of the quadrilateral Q surrounding the flip, which is homeomorphic to a cylinder with four punctures on each boundary component in Σ. Let γ be a loop on $T'\Sigma$. If γ only crosses one of the two boundary components of $\overline{C}_Q$, then the monodromies of γ in $\mathcal{E}_1$ and $\mathcal{E}_2$ are equal. Suppose γ crosses exactly once each of the two boundary components of $\overline{C}_Q$. Let γ_Q be the loop going around C_Q with the same orientation as the boundary of C_Q containing the sinks lifts of p_2 and p_4 (we refer to this boundary as the *positive* one, and the other one as *negative*).

Remark 4.7 We think of the holonomy of γ_Q in $\mathcal{E}$ as a generalization in the non-commutative setting of Fock-Goncharov's $\mathcal{X}$-coordinate of the quadrilateral Q. If $A = \mathbb{R}$, the holonomy of γ_Q is the cross-ratio of the four lines in $\mathbb{R}^2$ given by the framing of $\mathcal{L}$.

Up to homotopy, we can assume γ is going through at least one point $x_0 \in I_{\mathcal{T}_1^*}(\Sigma) \cap I_{\mathcal{T}_2^*}(\Sigma)$ on one of the eight external edges of the hexagon tiling of Q. We also choose a representative of γ_Q based at x_0. Let b be a basis of the fiber of $\mathcal{L}_1$ over x_0. Since x_0 is not in the interior of the cylinder supporting the flip in Σ, the fibers of $\mathcal{E}_1$ and $\mathcal{E}_2$ over x_0 are the same. The holonomy X of γ_Q is the same in $\mathcal{E}_1$ and in $\mathcal{E}_2$. Let $Y_1 \in A^\times$ (resp. Y_2) be the holonomy of γ in $\mathcal{E}_1$ (resp. $\mathcal{E}_2$).

Proposition 4.8 *If the part of γ inside C_Q goes from the positive boundary to the negative boundary, then*

$$Y_2 = Y_2(1 + X).$$

If the part of γ inside C_Q goes from the negative boundary to the positive boundary, then

$$Y_2 = Y_1(1 + X^{-1})^{-1}$$

Remark 4.9 The element $1 + X^{-1} \in A$ is invertible because of the transversality of $\mathcal{L}$ with respect to $\mathcal{T}_2$.

4.4 Topology of the moduli space of framed twisted local systems

In this section, we describe the topology of the moduli space of framed twisted $\mathrm{GL}_2(A)$-local systems on S that are transverse to a fixed triangulation $\mathcal{T}$.

As we have seen, framed twisted $\mathrm{GL}_2(A)$-local systems on S that are transverse with respect to a fixed triangulation $\mathcal{T}$ are in 1:1-correspondence with twisted $A^\times$-local systems on Σ. Since Σ has punctures, the space of twisted and non-twisted $A^\times$-local systems are homeomorphic (see Remark 3.6). So we obtain the following theorem, using the same notations as in Proposition 2.5:

Theorem 4.3 *The moduli space of framed (twisted) $\mathrm{GL}_2(A)$-local systems on S that are transverse with respect to a fixed triangulation $\mathcal{T}$ is homeomorphic to the moduli space of (twisted) $A^\times$-local systems on Σ which is homeomorphic to $(A^\times)^{1-4\chi(\overline{S})+2p+\sum n_i}/A^\times$ where $A^\times$ acts diagonally by conjugation on $(A^\times)^{1-4\chi(\overline{S})+2p+\sum n_i}$.*

Remark 4.10 In [18] the authors prove the same result using different techniques. They define local systems on some appropriate graphs over S and parametrize them using coordinates that are similar to Fock-Goncharov's GL_n-cluster $\mathcal{X}$-coordinates [17].

Since any twisted peripherally unipotent $\mathrm{GL}_2(A)$-local system on S has exactly one framing, we obtain:

Corollary 4.11 *The moduli space of twisted peripherally unipotent $\mathrm{GL}_2(A)$-local systems on S whose unique framing is transverse with respect to a fixed triangulation $\mathcal{T}$ is homeomorphic to the moduli space of twisted $A^\times$-local systems on $\overline{\Sigma}$.*

Corollary 4.12 *The moduli space of decorated twisted peripherally unipotent $\mathrm{GL}_2(A)$-local systems on S that are transverse with respect to a fixed triangulation $\mathcal{T}$ is homeomorphic to the product of the moduli space of twisted $A^\times$-local systems on $\overline{\Sigma}$ and $(A^\times)^{2p}$.*

5 Symplectic groups over involutive algebras and symplectic local systems

Involutive algebras are an important class of non-commutative algebras. Over involutive algebras, generalizations of many classical groups can be constructed (e.g. orthogonal groups, symplectic groups). In this chapter, we define algebras with anti-involutions and symplectic groups over such algebras that were introduced and studied in [8]. Further, we introduce framed twisted symplectic local system and characterize them in terms of partial abelianization introduced before.

5.1 Involutive algebras

Let A be a unital associative, possibly non-commutative $\mathbb{R}$-algebra.

Definition 5.1 An *anti-involution* on A is a $\mathbb{R}$-linear map $\sigma : A \to A$ such that

- $\sigma(ab) = \sigma(b)\sigma(a)$;
- $\sigma^2 = \mathrm{Id}$.

An *involutive* $\mathbb{R}$-*algebra* is a pair (A, σ), where A is a $\mathbb{R}$-algebra and σ is an anti-involution on A.

Definition 5.2 Two elements $a, a' \in A$ are called *congruent*, if there exists $b \in A^\times$ such that $a' = \sigma(b)ab$.

Definition 5.3 An element $a \in A$ is called σ-*symmetric* if $\sigma(a) = a$. An element $a \in A$ is called σ-*anti-symmetric* if $\sigma(a) = -a$. We denote

$$A^\sigma := \mathrm{Fix}_A(\sigma) = \{a \in A \mid \sigma(a) = a\},$$

$$A^{-\sigma} := \mathrm{Fix}_A(-\sigma) = \{a \in A \mid \sigma(a) = -a\}.$$

Definition 5.4 The closed subgroup

$$U_{(A,\sigma)} = \{a \in A^\times \mid \sigma(a)a = 1\}$$

of $A^\times$ is called the *unitary group* of A. The Lie algebra of $U_{(A,\sigma)}$ agrees with $A^{-\sigma}$.

Definition 5.5 Let (A, σ) be an $\mathbb{R}$-algebra with an anti-involution. We define two set of squares:

$$A^\sigma_+ := \left\{a^2 \mid a \in (A^\sigma)^\times\right\},\ A^\sigma_{\geq 0} := \left\{a^2 \mid a \in A^\sigma\right\}.$$

Remark 5.6 Since the algebra A is unital, we always have the canonical copy of $\mathbb{R}$ in A, namely $\mathbb{R} \cdot 1$ where 1 is the unit of A. We will always identify $\mathbb{R} \cdot 1$ with $\mathbb{R}$. Moreover, since σ is linear, for all $k \in \mathbb{R}$, $\sigma(k \cdot 1) = k\sigma(1) = k \cdot 1$, i.e. $\mathbb{R} \cdot 1 \subseteq A^\sigma$ and $\mathbb{R}_{>0} \cdot 1 \subseteq A^\sigma_+$.

Definition 5.7 A unital associative finite dimensional $\mathbb{R}$-algebra with an anti-involution (A, σ) is called *Hermitian* if for all $x, y \in A^\sigma$, $x^2 + y^2 = 0$ implies $x = y = 0$.

Remark 5.8 In [8], the property to be Hermitian is defined in the same way for algebras with an anti-involution over any real closed field. In this paper, we are discussing only Hermitian algebras over $\mathbb{R}$.

Remark 5.9 In [8] is shown that (A, σ) is a Hermitian algebra if and only if A^σ_+ is an open proper convex cone in A^σ, where proper means that the set does not contain (affine) lines.

If (A, σ) is Hermitian, for an element $a \in A^\sigma$ the *signature* can be defined, which is a bounded function $\operatorname{sgn}\colon A^\sigma \to \mathbb{Z}$ that is invariant under congruence by elements of $A^\times$. The elements of maximal signature are precisely the elements of A^σ_+. For more details about the signature see [8].

5.2 Symplectic groups over non-commutative algebras

Let A be a unital associative finite dimensional $\mathbb{R}$-algebra with an anti-involution σ. We consider A^2 as a right A-module over A.

Definition 5.10 Let $\omega(x, y) := \sigma(x)^t \Omega y$ with $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The group

$$\mathrm{Sp}_2(A, \sigma) := \mathrm{Aut}(\omega) = \{g \in M_2(A) \mid \sigma(g)^t \omega g = \omega\}$$

is the *symplectic group* Sp_2 over (A, σ). The form ω is called the *standard symplectic form* on A^2.

We have

$$\mathrm{Sp}_2(A, \sigma) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \sigma(a)c,\ \sigma(b)d \in A^\sigma,\ \sigma(a)d - \sigma(c)b = 1 \right\} \subseteq \mathrm{GL}_2(A)$$

We can also determine the Lie algebra $\mathfrak{sp}_2(A, \sigma)$ of $\mathrm{Sp}_2(A, \sigma)$:

$$\mathfrak{sp}_2(A, \sigma) = \left\{ \begin{pmatrix} x & z \\ y & -\sigma(x) \end{pmatrix} \mid x \in A,\ \ y, z \in A^\sigma \right\} \subseteq M_2(A).$$

Remark 5.11 In [8] is shown that, if A is a Hermitian algebra, then $\mathrm{Sp}_2(A, \sigma)$ is a Hermitian Lie group of tube type.

Let (x, y) be a basis of A^2. We say that this basis is *isotropic* if $\omega(x, x) = \omega(y, y) = 0$. We say that this basis is *symplectic* if furthermore $\omega(x, y) = 1$.

Let $x \in A^2$ be a regular isotropic element, i.e. $\omega(x, x) = 0$. We call the set $xA := \{xa \mid a \in A\}$ an *isotropic A-line*. The space of all isotropic A-lines is denoted by $\mathrm{Is}(\omega)$.

5.3 Symplectic local systems

We consider a twisted $\mathrm{GL}_2(A)$-local system $\mathcal{L} \to T'S$ over S. We say that $\mathcal{L}$ is a *twisted* $\mathrm{Sp}_2(A, \sigma)$*-local system* (or just *twisted symplectic local system*) if there exists a parallel field of the standard symplectic 2-form $\omega\colon \mathcal{L} \times \mathcal{L} \to A$ on $T'S$. We say that $\mathcal{L}$ is *peripherally parabolic (or unipotent)* if it is parabolic (resp. unipotent) as a twisted $\mathrm{GL}_2(A)$-local system.

A framing of a parabolic twisted symplectic local system is called *isotropic* if the parallel subbundle defining the framing in a neighborhood of every puncture is isotropic with respect to the field of the form ω. A decoration $((v_p)_{p\in P}, (w_p)_{p\in P})$ of a unipotent twisted symplectic local system is called *symplectic* if $\omega(v_p, v_p) = 0$ and $\omega(v_p, w_p) = 1$.

Remark 5.12 Notice, that if $\omega(v_p, v_p) = 0$, then the expression $\omega(v_p, w_p)$ is well-defined. Indeed, let $\tilde{w}_p$ and $\tilde{w}'_p$ be two lifts of w_p to A^2. Then $\tilde{w}'_p = \tilde{w}_p + v_p a$ for some $a \in A$. Further,

$$\omega(v_p, \tilde{w}'_p) = \omega(v_p, \tilde{w}_p + v_p a) = \omega(v_p, \tilde{w}_p) =: \omega(v_p, w_p).$$

It is always enough to choose v_p for every $p \in P$. Then w_p becomes uniquely defined.

A *framed twisted symplectic local system* is a peripherally parabolic twisted symplectic local system with an isotropic framing. A *decorated twisted symplectic local system* is a peripherally unipotent twisted symplectic local system with a symplectic decoration.

Remark 5.13 Notice, that since ω is a parallel form of even degree, the parallel transport of ω around the fiber of $T'S$ is trivial.

Let $\pi : \Sigma \to S$ be the ramified two-fold covering as before. Let $\mathcal{E} \to T'\Sigma$ be an $A^\times$-local system over the spectral covering Σ of S that is obtained by the partial abelianization procedure.

Let $\theta\colon \Sigma \to \Sigma$ be the covering involution. Slightly abusing the notation, we also denote $\theta = \theta_*\colon T'\Sigma \to T'\Sigma$.

Remark 5.14 Notice that θ does not have fixed points in $T'\Sigma$.

We consider the pull-back of $\mathcal{E}$ with respect to θ and denote it by $\mathcal{E}' := \theta^*\mathcal{E}$. To simplify the notation, we will identify $\mathcal{E}'_p$ and $\mathcal{E}_{\theta(p)}$ for all $p \in \Sigma$.

We denote by $P_\gamma : \mathcal{E}_{\gamma(0)} \to \mathcal{E}_{\gamma(1)}$, $P'_\gamma = P_{\theta\circ\gamma} : \mathcal{E}_{\theta(\gamma(0))} \to \mathcal{E}_{\theta(\gamma(1))}$ the parallel transport along $\gamma\colon [0, 1] \to \Sigma$ in $\mathcal{E}$ and $\mathcal{E}'$. We denote by $P^S_\alpha\colon V_{\alpha(0)} \to V_{\alpha(1)}$ the parallel transport along $\alpha\colon [0, 1] \to S$ in $\mathcal{L}$.

Definition 5.15 Let V and V' be two right A-modules. A map $b\colon V \times V' \to A$ is called an A-*sesquilinear pairing* between V and V' if it is additive in every argument and if for all $v \in V$, $v' \in V'$, and for all $a, a' \in A$, $b(va, v'a') = \sigma(a)b(v, v')a$. An A-sesquilinear paring b is *non-degenerate* if for every regular $v \in V$ there exists $v' \in V'$ such that $b(v, v') \in A^\times$ and for every regular $v' \in V'$ there exists $v \in V$ such that $b(v, v') \in A^\times$.

We denote by $B(\mathcal{E}, \mathcal{E}') \to T'\Sigma$ the vector bundle of all A-sesquilinear parings between $\mathcal{E}$ and $\mathcal{E}'$. A section $\beta \in \Gamma(T'\Sigma, B(\mathcal{E}, \mathcal{E}'))$ is called *parallel* if

$$\beta_{\gamma(0)}(x, y) = \beta_{\gamma(1)}(P_\gamma(x), P'_\gamma(y)) = \beta_{\gamma(1)}(P_\gamma(x), P_{\theta\circ\gamma}(y))$$

for every $\gamma\colon [0, 1] \to T'\Sigma$ and for every $x \in \mathcal{E}_{\gamma(0)}$, $y \in \mathcal{E}'_{\gamma(0)} = \mathcal{E}_{\theta(\gamma(0))}$.

Remark 5.16 Notice that if $\beta \in \Gamma(T'\Sigma, B(\mathcal{E}, \mathcal{E}'))$ is parallel and β_p is non-degenerate for one $p \in T'\Sigma$, then β_p is non-degenerate for all $p \in T'\Sigma$.

Theorem 5.1 *The framed local system $\mathcal{L}$ is an $\mathrm{Sp}_2(A, \sigma)$-local system if and only if there exists a non-degenerate parallel section $\beta \in \Gamma(T'\Sigma, B(\mathcal{E}, \mathcal{E}'))$ such that $\beta_p(x, y) = -\sigma(\beta_{\theta(p)}(y, x))$ for every $p \in T'\Sigma$, for every $x \in \mathcal{E}_p$ and for every $y \in \mathcal{E}_{\theta(p)}$.*

Proof ($\Rightarrow$) Assume, $\mathcal{L}$ is an $\mathrm{Sp}_2(A, \sigma)$-local system. That means, there exists a field of standard symplectic forms ω on $\mathcal{L} \to T'S$ such that for every $\alpha\colon [0, 1] \to T'S$ and for every $v, w \in V_{\alpha(0)}$,

$$\omega_{\alpha(0)}(v, w) = \omega_{\alpha(1)}(P^S_\alpha(v), P^S_\alpha(w)).$$

Let $\gamma\colon [0, 1] \to T'\Sigma$ be a smooth path such that $\gamma(0)$, $\gamma(1)$ do not project to points on lines of the spectral network on Σ, and let $x \in \mathcal{E}_{\gamma(0)}$ and $y \in \mathcal{E}_{\theta(\gamma(0))}$ be regular elements. We consider $\gamma' = \theta \circ \gamma$ and $\alpha = \pi \circ \gamma = \pi \circ \gamma'$. Moreover, $(\pi_*(x), \pi_*(y))$ is an isotropic basis of $\mathcal{L}_{\alpha(0)}$. We can define

$$\beta_{\gamma(0)}(x, y) := \omega_{\alpha(0)}(\pi_*(x), \pi_*(y)).$$

Since ω is non-degenerate and skew-Hermitian, β is non-degenerate and sesquilinear pairing. Moreover, $\beta_{\gamma(0)}(x, y) = -\sigma(\beta_{\theta(\gamma(0))}(y, x))$ because $\omega_{\alpha(0)}(\pi_*(x), \pi_*(y)) = -\sigma(\omega_{\alpha(0)}(\pi_*(y), \pi_*(x)))$.

If γ does not intersect lines of the spectral network, then β along γ is parallel because in this case $P^S_\alpha = P_\gamma \oplus P_{\sigma\circ\gamma}$

If γ is a small segment intersecting a line of spectral network, then

$$\omega_{\alpha(1)}(P^S_\alpha(\pi_*(x)), P^S_\alpha(\pi_*(y))) = \omega_{\alpha(1)}(\pi_*(P_\gamma(x)) + \pi_*(P_{\tilde{\gamma}}(x)), \pi_*(P_{\theta\circ\gamma}(y)))$$

where $\tilde{\gamma}$ is a lift of α going along a line of spectral network from $\gamma(0)$ to $\theta(\gamma(1))$. But elements $P_{\tilde{\gamma}}(x), P_{\theta\circ\gamma}(y) \in \mathcal{E}_{\theta(\gamma(1))}$, therefore, $\omega(\pi_*(P_{\tilde{\gamma}}(x)), \pi_*(P_{\theta\circ\gamma}(y))) = 0$. So

$$\begin{aligned}
\beta_{\gamma(0)}(x, y) &= \omega_{\alpha(0)}(\pi_*(x), \pi_*(y)) \\
&= \omega_{\alpha(1)}(P^S_\alpha(\pi_*(x)), P^S_\alpha(\pi_*(y))) \\
&= \omega_{\alpha(1)}(\pi_*(P_\gamma(x)), \pi_*(P_{\theta\circ\gamma}(y))) \\
&= \beta_{\alpha(1)}(P_\gamma(x), P_{\theta\circ\gamma}(y)),
\end{aligned}$$

i.e. β is parallel and extends also along lines of the spectral network on Σ.

Finally, let $p \in T'\Sigma$. Let $x \in \mathcal{E}_p$ and $y \in \mathcal{E}_{\theta(p)}$ regular elements. Then $(\pi_*(x), \pi_*(y))$ is an isotropic basis of $\mathcal{L}_{\pi(p)}$, i.e. $\beta(x, y) = \omega(\pi_*(x), \pi_*(y)) \in A^\times$. So the pairing β is non-degenerate.

($\Leftarrow$) Assume, there exists a non-degenerate parallel sesquilinear pairing β. Let $p \in T'\Sigma$ that does not project to a point on a line of the spectral network on Σ. We define for every $x \in \mathcal{E}_p$, $y \in \mathcal{E}_{\theta(p)}$:

$$\omega_{\pi(p)}(\pi_*(x), \pi_*(y)) := \beta_p(x, y).$$

If x, y are regular, $(\pi_*(x), \pi_*(y))$ is a basis of $V_{\pi(p)}$. Further, ω extends by sesquilinearity on $V_{\pi(p)}$ if we assume

$$\omega_{\pi(p)}(\pi_*(x), \pi_*(x')) = \omega_{\pi(p)}(\pi_*(y), \pi_*(y')) = 0$$

for all $x, x' \in \mathcal{E}_p$ and $y, y' \in \mathcal{E}_{\theta(p)}$. Since β is non-degenerate, ω is non-degenerate as well.

Since $\beta_p(x, y) = -\sigma(\beta_{\theta(p)}(y, x))$, we get

$$\omega_{\pi(p)}(\pi_*(y), \pi_*(x)) = \beta_{\theta(p)}(y, x) = -\sigma(\beta_p(x, y)) = -\sigma(\omega_{\pi(p)}(\pi_*(x), \pi_*(y))).$$

Further, ω is parallel. Indeed, let $\alpha\colon [0, 1] \to T'S$ be a path such that the projections of $\alpha(0)$ and $\alpha(1)$ to S are not on the lines of the spectral network. Let $x, y \in \mathcal{L}_{\alpha(0)}$. Let $\alpha_1, \alpha_2 := \theta \circ \alpha_1$ are two standard lifts of α to $T'\Sigma$. Then $x = \pi_*(x_1) + \pi_*(x_2)$ and $y = \pi_*(y_1) + \pi_*(y_2)$ where $x_1, y_1 \in \mathcal{E}_{\alpha_1(0)}$ and $x_2, y_2 \in \mathcal{E}_{\alpha_2(0)}$. If the projection of α to Σ does not intersect the spectral network, then the projection $T'\Sigma \to T'S$ and the parallel transport along α and α_1, α_2 commute. So ω is parallel because β is parallel.

Assume now that the projection of α intersects the spectral network once. We denote by α_3 the additional lift of α along the spectral network. Without loss of generality, assume $\alpha_3(0) = \alpha_1(0)$ and $\alpha_3(1) = \alpha_2(1)$. Notice that the path $\theta \circ (\alpha_3.\overline{\alpha}_1).\alpha_3.\overline{\alpha}_1$ is homotopic to the fiber of $T'\Sigma \to \Sigma$. Therefore, $P_{\theta\circ\overline{\alpha}_1.\alpha_3} = -P_{\theta\circ\overline{\alpha}_3.\alpha_1}$. Therefore,

$$\begin{aligned}
\omega_{\alpha(1)}(P^S_\alpha(x), P^S_\alpha(y)) &= \omega_{\alpha(1)}(P^S_\alpha(x), P^S_\alpha(y))\\
&= \omega_{\alpha(1)}(P^S_\alpha(\pi_*(x_1)) + P^S_\alpha(\pi_*(x_2)), P^S_\alpha(\pi_*(y_1)) + P^S_\alpha(\pi_*(y_2)))\\
&= \omega_{\alpha(1)}(\pi_*(P_{\alpha_1}(x_1) + P_{\alpha_3}(x_1) + P_{\alpha_2}(x_2)),\\
&\quad \pi_*(P_{\alpha_1}(y_1) + P_{\alpha_3}(y_1) + P_{\alpha_2}(y_2)))\\
&= \omega_{\alpha(1)}(\pi_*(P_{\alpha_1}(x_1), \pi_*(P_{\alpha_3}(y_1) + P_{\alpha_2}(y_2))))\\
&\quad + \omega_{\alpha(1)}(\pi_*(P_{\alpha_3}(x_1) + P_{\alpha_2}(x_2)), \pi_*(P_{\alpha_1}(y_1)))\\
&= \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_3}(y_1) + P_{\alpha_2}(y_2)) + \beta_{\alpha_2(1)}(P_{\alpha_3}(x_1)\\
&\quad + P_{\alpha_2}(x_2), P_{\alpha_1}(y_1))\\
&= \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_3}(y_1)) + \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_2}(y_2))\\
&\quad + \beta_{\alpha_2(1)}(P_{\alpha_3}(x_1), P_{\alpha_1}(y_1)) + \beta_{\alpha_2(1)}(P_{\alpha_2}(x_2), P_{\alpha_1}(y_1))\\
&= \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_2}(y_2)) + \beta_{\alpha_2(1)}(P_{\alpha_2}(x_2), P_{\alpha_1}(y_1))\\
&\quad + \beta_{\alpha_1(0)}(x_1, P_{(\theta\circ\overline{\alpha}_3).\alpha_1}(y_1) + P_{(\theta\circ\overline{\alpha}_1).\alpha_3}(y_1))\\
&= \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_2}(y_2)) + \beta_{\alpha_2(1)}(P_{\alpha_2}(x_2), P_{\alpha_1}(y_1))
\end{aligned}$$

$$
\begin{aligned}
&+ \beta_{\alpha_1(0)}(x_1, P_{\theta\circ\overline{\alpha}_3.\alpha_1}(y_1) + P_{\theta\circ\overline{\alpha}_1.\alpha_3}(y_1))\\
&= \beta_{\alpha_1(1)}(P_{\alpha_1}(x_1), P_{\alpha_2}(y_2)) + \beta_{\alpha_2(1)}(P_{\alpha_2}(x_2), P_{\alpha_1}(y_1))\\
&= \beta_{\alpha_1(0)}(x_1, y_2) + \beta_{\alpha_2(0)}(x_2, y_1)\\
&= \omega_{\alpha(0)}(x, y).
\end{aligned}
$$

So ω is parallel and extends also along lines of the spectral network on S.

Finally, let $p \in \Sigma$ and $x \in \mathcal{E}_p$, $y \in \mathcal{E}_{\theta(p)}$ such that $\beta_p(x, y) = 1$, then $\omega(\pi_*(x), \pi_*(y)) = 1$. So ω is a field of standard symplectic forms. □

5.4 Topology of the moduli space of framed twisted symplectic local systems

We keep the same notations as in Proposition 2.5. Our goal in this section is to prove the following theorem:

Theorem 5.2 *The moduli space of framed (twisted)* $\mathrm{Sp}_2(A, \sigma)$*-local systems on S that are transverse with respect to a fixed triangulation* $\mathcal{T}$ *is homeomorphic to:*

$$\left(((A^\sigma)^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i} \times (A^\times)^{1-\chi(\overline{S})+p}\right)/A^\times$$

where the group $A^\times$ *acts componentwisely by conjugation on* $(A^\times)^{1-\chi(\overline{S})+p}$ *and by congruence on* $((A^\sigma)^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i}$.

Proof We use the 1:1-correspondence between framed twisted $\mathrm{Sp}_2(A, \sigma)$-local systems on S that are transverse to a fixed triangulation $\mathcal{T}$ and twisted $A^\times$-local systems on Σ equipped with a non-degenerate parallel pairing β as in Theorem 5.1.

Let $\tilde{b} \in T'\Sigma$ be such that it projects to a ramification point $b \in \Sigma$. Let $\alpha_1, \dots, \alpha_s \colon [0, 1] \to S$ are free generators of the fundamental group $\pi_1(S, \pi(b))$. Let γ_i^1, γ_i^2 are closed lifts of α_i to $T'\Sigma$ such that $\theta \circ \gamma_i^1 = \gamma_i^2$ and γ_i^1 is based at $\tilde{b}$. Notice, that then γ_i^2 is based at $\theta(\tilde{b})$.

Let $s^+_{\tilde{b}}$ be as before the path from $\tilde{b}$ to $\theta(\tilde{b})$ going along the fiber at b in the positive direction and $s^-_{\theta(\tilde{b})} := \overline{s^+_{\tilde{b}}}$ the path from $\theta(\tilde{b})$ to $\tilde{b}$ going along the fiber at b in the negative direction. If the context is clear, we just write s^+ or s^- to simplify the notation.

Let $x \in \mathcal{E}_{\tilde{b}}$. Then on one hand: $\beta_{\tilde{b}}(x, P_{s^+}(x)) = -\sigma(\beta_{\theta(\tilde{b})}(P_{s^+}(x), x))$. On the other hand, since β is parallel:

$$
\begin{aligned}
\beta_{\tilde{b}}(x, P_{s^+}(x)) &= \beta_{\theta(\tilde{b})}(P_{s^+}(x), P_{s^+}(P_{s^+}(x)))\\
&= \beta_{\theta(\tilde{b})}(P_{s^+}(x), -x)\\
&= -\beta_{\theta(\tilde{b})}(P_{s^+}(x), x).
\end{aligned}
$$

So we obtain:

$$\beta_{\tilde{b}}(x, P_{s^+}(x)) = -\beta_{\theta(\tilde{b})}(P_{s^+}(x), x) = -\sigma(\beta_{\theta(\tilde{b})}(P_{s^+}(x), x)) =: a_0 \in A^\sigma.$$

Let now γ be a loop based at $\tilde{b}$ and

$$a_0 = \beta_{\tilde{b}}(x, P_{s^+}(x)) = \beta_{\tilde{b}}(P_\gamma(x), P_{\theta\circ\gamma} P_{s^+}(x)).$$

For every $x \in \mathcal{E}_{\tilde{b}}$, $P_\gamma(x) = xa_\gamma$ where $a_\gamma \in A^\times$. Let $P_{\theta\circ\gamma} P_{s^+}(x) = P_{s^+}(x)a'_\gamma$ for $a'_\gamma \in A^\times$. Then

$$a_0 = \sigma(a_\gamma)\beta_{\tilde{b}}(x, P_{s^+}(x))a'_\gamma = \sigma(a_\gamma)a_0a'_\gamma,$$

$$a'_\gamma = a_0^{-1}\sigma(a_\gamma^{-1})a_0.$$

Let γ and $s^-.(\theta \circ \overline{\gamma}).s^+$ are different generators of $\pi_1(T'\Sigma, \tilde{b})$ (this corresponds to curves γ_i^1 and γ_i^2 of Lemma 2.8 case (1) lifted to $T'\Sigma$). In particular, they are not homotopic. Then a_γ and a_0 determine uniquely a'_γ.

Let $\gamma\colon [0, 1] \to T'\Sigma$ and $\theta\circ\gamma\colon [0, 1] \to T'\Sigma$ be two lifts to $T'\Sigma$ of a segment in S connecting $\pi(b)$ and $\pi(b')$ where b' is another ramification point on Σ. Let $\tilde{b} := \gamma(0)$ and $\tilde{b}' := \gamma(1)$. In this case, $\xi_{\tilde{b}'} := \xi := s^-_{\theta(\tilde{b})}.\theta(\overline{\gamma}).s^+_{\tilde{b}'}.\gamma$ and $s^-.(\theta \circ \overline{\xi}).s^+$ are homotopic in $T'\Sigma$. Therefore, $a_\xi = a_0^{-1}\sigma(a_\xi)a_0$, i.e. $a_0a_\xi \in A^\sigma$. Moreover, an easy calculation shows that $a_0a_\xi = \beta_{\tilde{b}'}(y, P_{s^+_{\tilde{b}'}}y)$ where $y = P_\gamma(x)$.

So the symplectic local system provides us with elements $a_i \in A^\times$ corresponding to $P_{\gamma_i^1}$, $a_0 \in A^\sigma$ and $a_0a_\xi \in A^\sigma$ for every ξ as in (2) of Lemma 2.8 (lifted to $T'\Sigma$). These elements are well-defined up to a common conjugation of all a_i and common congruence of all a_0 and a_0a_ξ by an element of $A^\times$.

Conversely, if elements a_i, a_0, a_ξ as above are given, then a twisted $A^\times$-local systems on Σ equipped with a non-degenerate parallel pairing β can be reconstructed uniquely. Equivalent local system correspond to a common conjugation of all a_i and common congruence of all a_0 and a_0a_ξ by an element of $A^\times$. □

5.5 Symplectic local system over Hermitian algebras

Let A be a Hermitian algebra. Let ℓ_1, ℓ_2, ℓ_3 be pairwise transverse isotropic A-lines. The Kashiwara-Maslov index of the triple (ℓ_1, ℓ_2, ℓ_3) is the signature of the element $\omega(x, \mu_1^{23}(x)) \in (A^\sigma)^\times$ for a regular $x \in \ell_1$ where μ_1^{23} is the Kashiwara-Maslov map defined in Sect. 4.1. In fact, this signature does not depend on $x \in \ell_1$, and it is invariant under cyclic permutations of the triple (ℓ_1, ℓ_2, ℓ_3) and it changes the sign by transposition of the elements of the triple.

Let $\tau \subset S$ be a triangle of the triangulation $\mathcal{T}$ that is incident to punctures p_1, p_2, p_3 and the orientation of the triangle agrees with the orientation of the triple (p_1, p_2, p_3). As in Sect. 4.1, let L_i be a parallel isotropic A-subbundle of $\mathcal{L} \to T'\tau$ corresponding to the puncture p_i, $i \in \{1, 2, 3\}$. Let $H = \pi^{-1}(\tau) \subset \Sigma$ be the hexagon that covers τ. Let b be the ramification point in H, let $\tilde{b}$ be a lift of b in $T'H$ and let s^+ be a path in $T'H$ going from $\tilde{b}$ to $\theta(\tilde{b})$ along the fiber in the positive direction.

The following proposition is immediate:

Proposition 5.17 *Let $z \in T'\tau$. The Kashiwara-Maslov index of $(L_1(z), L_2(z), L_3(z))$ agrees with the signature of the element $\beta_{\tilde{b}}(x, P_{s^+}(x)) \in A^\sigma$ for a regular $x \in \mathcal{E}_{\tilde{b}}$.*

Theorem 5.3 *If A is Hermitian, then the moduli space of framed (twisted) maximal $\mathrm{Sp}_2(A,\sigma)$-local systems on S is homeomorphic to:*

$$\left((A_+^\sigma)^{-2\chi(\overline{S})+p} \times (A^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i}\right)/A^\times$$

where $A^\times$ acts componentwisely by conjugation on $(A^\times)^{-2\chi(\overline{S})+2p-1+\sum n_i}$ and by congruence on $(A_+^\sigma)^{-2\chi(\overline{S})+p}$.

Proof Following the notation of the proof of Theorem 5.2, notice that the signature of $a_0 \in A^\sigma$ agrees with the Kashiwara-Maslov index of the oriented triangle where the ramification point $\pi(b) \in S$ lies, and the signature of $a_0 a_{\xi_{\tilde{b}'}} \in A^\sigma$ agrees with the Kashiwara-Maslov index of the oriented triangle where the ramification point $\pi(b') \in S$ lies. A twisted symplectic local system is maximal if and only if Kashiwara-Maslov indices of all oriented triangles are maximal. So we obtain the statement of the theorem. □

Remark 5.18 The results of this and previous sections agree with the results from [18] obtained using different techniques (see also Remark 4.10).

5.6 $\mathcal{A}$-coordinates for symplectic local systems

Since $\mathrm{Sp}_2(A,\sigma)$ is a subgroup of $\mathrm{GL}_2(A)$, the $\mathcal{A}$-coordinates defined in Sect. 4.3, a twisted symplectic local system have well-defined $\mathcal{A}$-coordinates, and because of the additional structure of symplectic local systems, they satisfy additional relations. The following proposition is immediate:

Proposition 5.19 *Let $\mathcal{L} \to S$ be a twisted decorated $\mathcal{T}$-transverse symplectic local system. Let γ be an arc of the triangulation $\mathcal{T}$ from $p \in P$ to $q \in P$. Then $a_\gamma = \omega(v_q, v_p)$. In particular, $a_{\overline{\gamma}} = -\sigma(a_\gamma)$.*

Proof By definition of non-commutative $\mathcal{A}$-coordinates, $v_p \in L_q$ projects to $w_q a_\gamma \in \mathcal{L}/L_q$, i.e. for some lift $\hat{w}_q \in A^2$ of w_q, $v_p = \hat{w}_q a_\gamma + v_q r$ for some $r \in A$. Therefore, $\omega(v_q, v_p) = \omega(v_q, \hat{w}_q a_\gamma + v_q r) = \omega(v_q, \hat{w}_q)a_\gamma = a_\gamma$. □

From Proposition 4.1 follows:

Corollary 5.20 *Let (A,σ) be an involutive algebra. A twisted decorated $\mathcal{T}$-transverse $\mathrm{GL}_2(A)$-local system $\mathcal{L} \to S$ is symplectic if and only if $a_{\overline{\gamma}} = -\sigma(a_\gamma)$ for all edges γ of the triangulation $\mathcal{T}$.*

For a twisted decorated $\mathcal{T}$-transverse symplectic local system $\mathcal{L} \to S$ and for each oriented triangle $T := (\gamma_1, \gamma_2, \gamma_3)$ of $\mathcal{T}$, we have $\beta_T := a_{\gamma_3} a_{\overline{\gamma}_2}^{-1} a_{\gamma_1} \in A^\sigma$.

If (A,σ) is Hermitian, the signature of β_T agrees with the Kashiwara-Maslov index of T.

The decorated local system is maximal if and only if $\beta_T \in A_+^\sigma$ for all oriented triangles T of $\mathcal{T}$.

Remark 5.21 A $\mathrm{GL}_2(A)$-equivalence class of twisted decorated $\mathcal{T}$-transverse $\mathrm{GL}_2(A)$-local system $\mathcal{L} \to S$ contains a representative which admits a reduction to $\mathrm{Sp}_2(A, \sigma)$ if and only for each oriented triangle $T := (\gamma_1, \gamma_2, \gamma_3)$ of $\mathcal{T}$, $\beta_T := a_{\gamma_3} a_{\overline{\gamma}_2}^{-1} a_{\gamma_1} \in A^\sigma$. However, this condition only guarantees that the framing of $\mathcal{L}$ is isotropic but does not guarantee the decoration is symplectic. To guarantee a symplectic decoration of $\mathcal{L}$, the stronger condition from Corollary 5.20 is necessary.

Remark 5.22 Since these additional relations on $\mathcal{A}$-coordinates involve the structure of (A, σ), it is not possible to define a corresponding non-commutative algebra for symplectic local systems as in [7] for $\mathrm{GL}_2(A)$-local systems.

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Declarations

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References

1. Gaiotto, D., Moore, G.W., Neitzke, A.: Spectral networks. Ann. Henri Poincaré **14**(7), 1643–1731 (2013). https://doi.org/10.1007/s00023-013-0239-7
2. Gaiotto, D., Moore, G.W., Neitzke, A.: Spectral networks and snakes. Ann. Henri Poincaré **15**(1), 61–141 (2014). https://doi.org/10.1007/s00023-013-0238-8
3. Hollands, L., Neitzke, A.: Spectral networks and Fenchel–Nielsen coordinates. Lett. Math. Phys. **106**(6), 811–877 (2016). https://doi.org/10.1007/s11005-016-0842-x
4. Moore, G.W.: Lecture notes for Felix Klein lectures (2012)
5. Alessandrini, D.: Introduction to Spectral Networks. Mini-course at University of Illinois at Urbana-Champaign (2014)
6. Goncharov, A., Kontsevich, M.: Spectral description of non-commutative local systems on surfaces and non-commutative cluster varieties (2022). arXiv:2108.04168 [math.AG]
7. Berenstein, A., Retakh, V.: Noncommutative marked surfaces. Adv. Math. **328**, 1010–1087 (2018). https://doi.org/10.1016/j.aim.2018.02.014

8. Alessandrini, D., Berenstein, A., Retakh, V., Rogozinnikov, E., Wienhard, A.: Symplectic groups over noncommutative algebras. Selecta Math. (N.S.) **28**(4), 82 (2022). https://doi.org/10.1007/s00029-022-00787-x
9. Burger, M., Iozzi, A., Wienhard, A.: Surface group representations with maximal Toledo invariant. Ann. Math. (2) **172**(1), 517–566 (2010). https://doi.org/10.4007/annals.2010.172.517
10. Burger, M., Iozzi, A., Labourie, F., Wienhard, A.: Maximal representations of surface groups: symplectic Anosov structures. Pure Appl. Math. Q. **1**(3, Special Issue: In memory of Armand Borel. Part 2), 543–590 (2005). https://doi.org/10.4310/PAMQ.2005.v1.n3.a5
11. Strubel, T.: Fenchel–Nielsen coordinates for maximal representations. Geom. Dedic. **176**, 45–86 (2015). https://doi.org/10.1007/s10711-014-9959-1
12. Gothen, P.B.: Components of spaces of representations and stable triples. Topology **40**(4), 823–850 (2001). https://doi.org/10.1016/S0040-9383(99)00086-5

13. Guichard, O., Wienhard, A.: Topological invariants of Anosov representations. J. Topol. **3**(3), 578–642 (2010). https://doi.org/10.1112/jtopol/jtq018
14. Bradlow, S.B., García-Prada, O., Gothen, P.B.: Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces. Geom. Dedic. **122**, 185–213 (2006). https://doi.org/10.1007/s10711-007-9127-y
15. Alessandrini, D., Collier, B.: The geometry of maximal components of the PSp $(4, \mathbb{R})$ character variety. Geom. Topol. **23**(3), 1251–1337 (2019). https://doi.org/10.2140/gt.2019.23.1251
16. Alessandrini, D., Guichard, O., Rogozinnikov, E., Wienhard, A.: Noncommutative coordinates for symplectic representations. Memoirs of the American Mathematical Society Mem. Amer. Math. Soc. **300**, 1504 (2024). https://doi.org/10.1090/memo/1504
17. Fock, V., Goncharov, A.: Moduli spaces of local systems and higher Teichmüller theory. Publ. Math. Inst. Hautes Étud. Sci. **103**, 1–211 (2006). https://doi.org/10.1007/s10240-006-0039-4
18. Guichard, O., Rogozinnikov, E., Wienhard, A.: Parametrizing positive representations (2022). arXiv:2210.11605
19. Rogozinnikov, E.: Symplectic groups over noncommutative rings and maximal representations. PhD thesis, Ruprecht-Karls-Universität Heidelberg (2020). https://doi.org/10.11588/heidok.00028737
20. Kineider, C.: Partial abelianization of GL_n-local systems and non-commutative $\mathcal{A}$-coordinates. PhD thesis, Université de Strasbourg (2023)