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Embedded Structure in Quantum Theory, Functional Operators and the Multiverse

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We explore a wider theoretical framework that has quantum field theory built in, taking the fact that quantum mechanics is reconstructed from quantum field theory as a hint. We formulate a quantum theory with an embedded structure by introducing functional operators, and we find that it could describe the level II multiverse. Topics related to the beginning of the universe such as inflation, the third quantization, and the landscape are discussed in our formulation.

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1. Introduction

Quantum field theory (QFT) or quantum theory of fields offers an excellent framework that explains a great variety of phenomena in our world and describes curious features such as the creation and annihilation of particles, the dichotomy between a particle and a wave, and so on [1,2]. In fact, QFT has been applied to various systems and has been massively successful. For example, particle physics at the electroweak scale is excellently explained by the standard model [3], and superconductivity becomes better understood by BCS theory [4] in condensed matter physics.

In spite of such prominent features and triumphs of QFT, it cannot be the ultimate framework of physics because it suffers from intrinsic problems. For instance, QFT has a divergence difficulty in which theoretical values of physical quantities diverge to infinity after radiative corrections are incorporated. In particular, this problem becomes serious after the gravitational interaction is introduced, because there appear infinities that cannot be removed by the renomalization procedure. Hence, QFT is currently understood as an effective theory of quantum fields [5,6,7].

Furthermore, we have several questions in mind.

(1) Why does QFT work so well as an effective theory of elementary particles? Why are particles or fields quantized in the first place?
(2) What is the origin of particles (fields) and spacetime? Which came first, particles or spacetime?
(3) Why is our universe described by the standard model at the electroweak scale?

For Q1, the relationship between QFT and quantum mechanics (QM) can be a key to solve the riddle by the following reasoning. QM describes a system with a definite number of particles very well in a simple fashion. In contrast, QFT is applied to a system in which the number of

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particles can vary as well. In other words, the range in application of QFT is wider than that of QM, and QM can actually be reconstructed from QFT by fixing a number of particles in a system. Hence, if there were a theory that rebuilds QFT, an answer to Q1 could be derived.

Q2 can be expanded and deepened as “what is the origin of physical laws and our universe?” and “which came first, physical laws or our universe?”, respectively. If there were a framework to deal with particles and spacetime (physical laws and our universe) as a unit, the which-came-first-particles-or-spacetime problem (which-came-first-physical-laws-or-our-universe problem) could be solved.

Q3 stems from the fact that QFT possesses no powerful principle to select realistic models theoretically and completely. If there were a huge variety of universes with different particle contents and physical parameters, called “the level II multiverse” [8], there is a possibility that the existence of our universe could be understood by the anthropic principle [9,10], and a profound riddle like the cosmological constant problem is neutralized [11]. Thus, Q3 is a substitute for the question of whether a framework to describe the level II multiverse can be constructed or not.

In this paper, we explore a wider theoretical framework that has QFT built in, taking the fact that QM is reconstructed from QFT as a hint. We formulate a quantum theory with an embedded structure by introducing functional operators, and we find that it could describe the level II multiverse. Topics related to the beginning of the universe such as inflation, the third quantization, and the landscape are discussed in our formulation.

The outline of this paper is as follows. In the next section, we review a framework of QFT and explain how QM is derived from QFT. In Section 3, we explore the underlying framework that embeds QFT, by the use of a toy model. In Section 4, we extend our framework in order to describe the level II multiverse and discuss the physical implications of the birth of the universe. In the last section, we give conclusions and discussions. Explicit forms of Hamiltonian operators are listed for several species of particle in Appendix A. More details on the wave functional are given in Appendix B.

2. Quantum field theory and quantum mechanics

2.1. *Framework of quantum field theory*

First, we review the framework of QFT, based on the Lagrangian density given by

$$\widehat{\mathscr{L}}_{\varphi} = \widehat{\varphi}^{\dagger}(\boldsymbol{x}, t) i\hbar \frac{\partial}{\partial t} \widehat{\varphi}(\boldsymbol{x}, t) - \widehat{\varphi}^{\dagger}(\boldsymbol{x}, t) \widehat{H} \widehat{\varphi}(\boldsymbol{x}, t), \tag{1}$$

where $\widehat{\varphi}(\boldsymbol{x}, t)$ is a quantum field, with $\boldsymbol{x} = (x^1, x^2, x^3)$ and t standing for coordinates of space and time, respectively, $\hbar$ is the reduced Planck constant, and $\widehat{H}$ is the Hamiltonian operator in QM containing $\boldsymbol{x}$ and its derivatives ∇, i.e. $\widehat{H} = \widehat{H}(\boldsymbol{x}, -i\hbar\nabla)$. Here, we choose the Lagrangian density with the first time-derivative term, because the compatibility (relationship) with QM is easily comprehensible, as will be seen.

For example, for a particle with a mass m in nonrelativistic QM, $\widehat{H}$ is given by

$$\widehat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x}), \tag{2}$$

where $V(\boldsymbol{x})$ is the potential energy. For a free Dirac fermion (a particle with spin 1/2 and a mass m) in relativistic QM, $\widehat{H}$ is given by

$$\widehat{H} = -i\hbar c\boldsymbol{\alpha} \cdot \nabla + \beta mc^2, \tag{3}$$

where c is the speed of light, and $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ and β are 4×4 Hermitian matrices satisfying $\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}I$ ($i, j = 1, 2, 3$, I: 4×4 unit matrix), $\alpha^i\beta + \beta\alpha^i = 0$, and $\beta^2 = I$.

For simplicity, we assume that our spacetime is the 4D Minkowski spacetime and the system is described by a free-field operator $\widehat{\varphi}(\boldsymbol{x}, t)$ in the Heisenberg picture.

The canonical conjugate of $\widehat{\varphi}(\boldsymbol{x}, t)$ is defined by

$$\widehat{\pi}(\boldsymbol{x}, t) \equiv \frac{\partial \widehat{\mathscr{L}}_\varphi}{\partial(\partial\widehat{\varphi}(\boldsymbol{x}, t)/\partial t)} = i\hbar\widehat{\varphi}^\dagger(\boldsymbol{x}, t), \tag{4}$$

and the Hamiltonian operator $\widehat{H}_\varphi$ in QFT is obtained as

$$\begin{aligned}\widehat{H}_\varphi &\equiv \int \left(\widehat{\pi}(\boldsymbol{x}, t)\frac{\partial\widehat{\varphi}(\boldsymbol{x}, t)}{\partial t} - \widehat{\mathscr{L}}_\varphi\right) d^3x \\ &= \int \widehat{\varphi}^\dagger(\boldsymbol{x}, t)\widehat{H}\widehat{\varphi}(\boldsymbol{x}, t)d^3x = \frac{1}{i\hbar}\int \widehat{\pi}(\boldsymbol{x}, t)\widehat{H}\widehat{\varphi}(\boldsymbol{x}, t)d^3x.\end{aligned} \tag{5}$$

We notice that the Hamiltonian operator in QFT is constructed by sandwiching the Hamiltonian operator in QM between two field operators. We refer to this construction as the "embedded structure", "nested construction", and so on. We list explicit forms of $\widehat{H}_\varphi$ for particles with spin 1/2 in Appendix A.

The following quantization conditions are imposed on field operators with spin 1/2:

$$\{\widehat{\varphi}(\boldsymbol{x}, t), \widehat{\pi}(\boldsymbol{y}, t)\} = i\hbar\delta^3(\boldsymbol{x} - \boldsymbol{y}), \quad \{\widehat{\varphi}(\boldsymbol{x}, t), \widehat{\varphi}(\boldsymbol{y}, t)\} = 0, \quad \{\widehat{\pi}(\boldsymbol{x}, t), \widehat{\pi}(\boldsymbol{y}, t)\} = 0, \tag{6}$$

where $\{\widehat{A}, \widehat{B}\} \equiv \widehat{A}\widehat{B} + \widehat{B}\widehat{A}$.

The field operators obey Heisenberg's equation of motion:

$$i\hbar\frac{\partial}{\partial t}\widehat{\varphi}(\boldsymbol{x}, t) = \left[\widehat{\varphi}(\boldsymbol{x}, t), \widehat{H}_\varphi\right], \quad i\hbar\frac{\partial}{\partial t}\widehat{\pi}(\boldsymbol{x}, t) = \left[\widehat{\pi}(\boldsymbol{x}, t), \widehat{H}_\varphi\right], \tag{7}$$

where $[\widehat{A}, \widehat{B}] \equiv \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$. Using Eqs. (5), (6), and (7), we derive the equations:

$$i\hbar\frac{\partial}{\partial t}\widehat{\varphi}(\boldsymbol{x}, t) = \widehat{H}\widehat{\varphi}(\boldsymbol{x}, t), \quad i\hbar\frac{\partial}{\partial t}\widehat{\pi}(\boldsymbol{x}, t) = -\widehat{\pi}(\boldsymbol{x}, t)\widehat{H}, \tag{8}$$

and these equations agree with the Euler–Lagrange equation:

$$\partial_\mu\left(\frac{\partial\widehat{\mathscr{L}}_\varphi}{\partial(\partial_\mu\widehat{\varphi}^\dagger)}\right) - \frac{\partial\widehat{\mathscr{L}}_\varphi}{\partial\widehat{\varphi}^\dagger} = 0, \quad \partial_\mu\left(\frac{\partial\widehat{\mathscr{L}}_\varphi}{\partial(\partial_\mu\widehat{\varphi})}\right) - \frac{\partial\widehat{\mathscr{L}}_\varphi}{\partial\widehat{\varphi}} = 0, \tag{9}$$

which is derived from the action integral $\widehat{S}_\varphi = \frac{1}{c}\int \widehat{\mathscr{L}}_\varphi d^4x$, based on the least-action principle.

In the Schrödinger picture, the quantum fields $\widehat{\varphi}(\boldsymbol{x})$ and $\widehat{\pi}(\boldsymbol{x})$ are independent of time, and they are related to those in the Heisenberg picture as

$$\widehat{\varphi}(\boldsymbol{x}, t) = e^{\frac{i}{\hbar}\widehat{H}_\varphi t}\widehat{\varphi}(\boldsymbol{x})e^{-\frac{i}{\hbar}\widehat{H}_\varphi t}, \quad \widehat{\pi}(\boldsymbol{x}, t) = e^{\frac{i}{\hbar}\widehat{H}_\varphi t}\widehat{\pi}(\boldsymbol{x})e^{-\frac{i}{\hbar}\widehat{H}_\varphi t}. \tag{10}$$

Using Eq. (10) and the conservation law of $\widehat{H}_\varphi$, i.e. $d\widehat{H}_\varphi/dt = 0$, $\widehat{H}_\varphi$ is rewritten in a time-independent form as

$$\widehat{H}_\varphi = \int \widehat{\varphi}^\dagger(\boldsymbol{x})\widehat{H}\widehat{\varphi}(\boldsymbol{x})d^3x = \frac{1}{i\hbar}\int \widehat{\pi}(\boldsymbol{x})\widehat{H}\widehat{\varphi}(\boldsymbol{x})d^3x. \tag{11}$$

The field operators $\widehat{\varphi}(\boldsymbol{x})$ and $\widehat{\pi}(\boldsymbol{x})$ obey the anticommutation relations:

$$\{\widehat{\varphi}(\boldsymbol{x}), \widehat{\pi}(\boldsymbol{y})\} = i\hbar\delta^3(\boldsymbol{x} - \boldsymbol{y}), \quad \{\widehat{\varphi}(\boldsymbol{x}), \widehat{\varphi}(\boldsymbol{y})\} = 0, \quad \{\widehat{\pi}(\boldsymbol{x}), \widehat{\pi}(\boldsymbol{y})\} = 0. \tag{12}$$

From the first condition in Eq. (12), $\widehat{\pi}(\boldsymbol{y})$ is given by $\widehat{\pi}(\boldsymbol{y}) = i\hbar\delta/\delta\varphi(\boldsymbol{y})$, i.e. $\widehat{\varphi}^\dagger(\boldsymbol{y}) = \delta/\delta\varphi(\boldsymbol{y})$, using the functional derivative, in the representative diagonalizing $\widehat{\varphi}(\boldsymbol{x})$ such as $\widehat{\varphi}(\boldsymbol{x})|\varphi\rangle = \varphi(\boldsymbol{x})|\varphi\rangle$ where $\varphi(\boldsymbol{x})$ is a Grassmann-valued field configuration.

Any state $|\Psi(t)\rangle$ is evolved by the Schrödinger equation:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \widehat{H}_{\varphi}|\Psi(t)\rangle, \tag{13}$$

and its formal solution is given by

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}\widehat{H}_{\varphi}t}|\Psi(0)\rangle. \tag{14}$$

Multiplying $\langle\varphi|$ by both sides of Eq. (13), we obtain the equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(\varphi, t) = \widehat{H}_{\varphi}\Psi(\varphi, t), \tag{15}$$

where $\Psi(\varphi, t) = \langle\varphi|\Psi(t)\rangle$ is a wave functional in QFT [12], and $\widehat{H}_{\varphi}$ is written by

$$\widehat{H}_{\varphi} = \int \frac{\delta}{\delta\varphi(\boldsymbol{x})}\widehat{H}\varphi(\boldsymbol{x})d^3x, \tag{16}$$

using $\widehat{\pi}(\boldsymbol{x}) = i\hbar\delta/\delta\varphi(\boldsymbol{x})$. The expectation value of $\widehat{H}_{\varphi}$ is given by

$$\langle\Psi(t)|\widehat{H}_{\varphi}|\Psi(t)\rangle = \int \mathscr{D}\varphi\ \Psi^{\dagger}(\varphi, t)\widehat{H}_{\varphi}\Psi(\varphi, t), \tag{17}$$

as seen in Eq. (B4). We explain more about the wave functional in Appendix B.

Here, for the sake of completeness, we comment on a boson with spin 0. For a free complex scalar particle ϕ with a mass m, the Lagrangian density in QFT is given by

$$\widehat{\mathscr{L}_{\phi}} = \hbar c\left\{\partial_{\mu}\widehat{\phi}^{\dagger}(\boldsymbol{x}, t)\partial^{\mu}\widehat{\phi}(\boldsymbol{x}, t) - \left(\frac{mc}{\hbar}\right)^2\widehat{\phi}^{\dagger}(\boldsymbol{x}, t)\widehat{\phi}(\boldsymbol{x}, t)\right\}, \tag{18}$$

where $\widehat{\phi}(\boldsymbol{x}, t)$ is the quantum field of ϕ. The canonical conjugates of $\widehat{\phi}(\boldsymbol{x}, t)$ and $\widehat{\phi}^{\dagger}(\boldsymbol{x}, t)$ are defined by

$$\widehat{\pi}_{\phi}(\boldsymbol{x}, t) \equiv \frac{\partial\widehat{\mathscr{L}_{\phi}}}{\partial(\partial\widehat{\phi}(\boldsymbol{x}, t)/\partial t)} = \frac{\hbar}{c}\frac{\partial}{\partial t}\widehat{\phi}^{\dagger}(\boldsymbol{x}, t),\ \ \widehat{\pi}_{\phi}^{\dagger}(\boldsymbol{x}, t) \equiv \frac{\partial\widehat{\mathscr{L}_{\phi}}}{\partial(\partial\widehat{\phi}^{\dagger}(\boldsymbol{x}, t)/\partial t)} = \frac{\hbar}{c}\frac{\partial}{\partial t}\widehat{\phi}(\boldsymbol{x}, t), \tag{19}$$

respectively, and the Hamiltonian operator $\widehat{H}_{\phi}$ is obtained as

$$\widehat{H}_{\phi} = \int \hbar c\left\{\frac{1}{\hbar^2}\widehat{\pi}_{\phi}^{\dagger}(\boldsymbol{x}, t)\widehat{\pi}_{\phi}(\boldsymbol{x}, t) + \nabla\widehat{\phi}^{\dagger}(\boldsymbol{x}, t)\cdot\nabla\widehat{\phi}(\boldsymbol{x}, t) + \left(\frac{mc}{\hbar}\right)^2\widehat{\phi}^{\dagger}(\boldsymbol{x}, t)\widehat{\phi}(\boldsymbol{x}, t)\right\}d^3x. \tag{20}$$

The following commutation relations are imposed on the field operators with spin 0:

$$\begin{aligned}
&\left[\widehat{\phi}(\boldsymbol{x}, t), \widehat{\pi}_{\phi}(\boldsymbol{y}, t)\right] = i\hbar\delta^3(\boldsymbol{x}-\boldsymbol{y}),\ \ \left[\widehat{\phi}(\boldsymbol{x}, t), \widehat{\phi}(\boldsymbol{y}, t)\right] = 0,\ \ \left[\widehat{\pi}_{\phi}(\boldsymbol{x}, t), \widehat{\pi}_{\phi}(\boldsymbol{y}, t)\right] = 0,\\
&\left[\widehat{\phi}^{\dagger}(\boldsymbol{x}, t), \widehat{\pi}_{\phi}^{\dagger}(\boldsymbol{y}, t)\right] = i\hbar\delta^3(\boldsymbol{x}-\boldsymbol{y}),\ \ \left[\widehat{\phi}^{\dagger}(\boldsymbol{x}, t), \widehat{\phi}^{\dagger}(\boldsymbol{y}, t)\right] = 0,\ \ \left[\widehat{\pi}_{\phi}^{\dagger}(\boldsymbol{x}, t), \widehat{\pi}_{\phi}^{\dagger}(\boldsymbol{y}, t)\right] = 0,\\
&\left[\widehat{\phi}(\boldsymbol{x}, t), \widehat{\pi}_{\phi}^{\dagger}(\boldsymbol{y}, t)\right] = 0,\ \ \left[\widehat{\phi}^{\dagger}(\boldsymbol{x}, t), \widehat{\pi}_{\phi}(\boldsymbol{y}, t)\right] = 0,\ \ \left[\widehat{\phi}(\boldsymbol{x}, t), \widehat{\phi}^{\dagger}(\boldsymbol{y}, t)\right] = 0,\\
&\left[\widehat{\pi}_{\phi}(\boldsymbol{x}, t), \widehat{\pi}_{\phi}^{\dagger}(\boldsymbol{y}, t)\right] = 0.
\end{aligned} \tag{21}$$

In the Schrödinger picture, quantum fields are independent of time, and $\widehat{\pi}_{\phi}(\boldsymbol{y})$ and $\widehat{\pi}_{\phi}^{\dagger}(\boldsymbol{y})$ are represented by functional derivatives such as $\widehat{\pi}_{\phi}(\boldsymbol{y}) = -i\hbar\delta/\delta\phi(\boldsymbol{y})$ and $\widehat{\pi}_{\phi}^{\dagger}(\boldsymbol{y}) = -i\hbar\delta/\delta\phi^{\dagger}(\boldsymbol{y})$, respectively. A wave functional $\Psi(\phi, t) = \langle\phi|\Psi(t)\rangle$ is evolved by the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(\phi, t) = \widehat{H}_{\phi}\Psi(\phi, t), \tag{22}$$

where the Hamiltonian operator is rewritten as

$$\widehat{H}_{\phi} = \int \hbar c\left\{-\frac{\delta^2}{\delta\phi^{\dagger}(\boldsymbol{x})\delta\phi(\boldsymbol{x})} + \nabla\phi^{\dagger}(\boldsymbol{x})\cdot\nabla\phi(\boldsymbol{x}) + \left(\frac{mc}{\hbar}\right)^2\phi^{\dagger}(\boldsymbol{x})\phi(\boldsymbol{x})\right\}d^3x. \tag{23}$$

In a similar way to a particle with spin 1/2 and a particle with spin 0, a particle with spin 1 (gauge boson) and a particle with a higher spin are also described by the use of QFT. Actually, particle physics at the electroweak scale is excellently controlled by the Lagrangian density $\widehat{\mathscr{L}}_{\rm SM}$ of the standard model, and the basic formula in the Schrödinger picture is given by

$$i\hbar\frac{\partial}{\partial t}\Psi(\varphi_{\rm SM},t)=\widehat{H}_{\rm SM}\Psi(\varphi_{\rm SM},t), \tag{24}$$

where $\widehat{H}_{\rm SM}$ is the Hamiltonian operator derived from $\widehat{\mathscr{L}}_{\rm SM}$, and $\varphi_{\rm SM}$ denotes a set of field variables of the standard model particles.

Let us come back to the scalar particle ϕ. When we deal with a relativistic scalar particle in QM, we encounter the problem that the probability interpretation breaks down in the absence of a positive definite expression of probability density. Because QM shows excellence in the description of a system with a definite number of particles, we do well to consider a nonrelativistic case. Using the redefinition of a quantum field:

$$\widehat{\phi}(\boldsymbol{x},t)=\sqrt{\frac{\hbar}{2mc}}e^{-\frac{i}{\hbar}mc^2t}\ \widehat{\psi}(\boldsymbol{x},t), \tag{25}$$

$\widehat{\mathscr{L}}_\phi$ is rewritten as

$$\begin{aligned}\widehat{\mathscr{L}}_\phi&=\frac{i\hbar}{2}\left(\widehat{\psi}^\dagger(\boldsymbol{x},t)\frac{\partial}{\partial t}\widehat{\psi}(\boldsymbol{x},t)-\frac{\partial}{\partial t}\widehat{\psi}^\dagger(\boldsymbol{x},t)\ \widehat{\psi}(\boldsymbol{x},t)\right)-\frac{\hbar^2}{2m}\nabla\widehat{\psi}^\dagger(\boldsymbol{x},t)\cdot\nabla\widehat{\psi}(\boldsymbol{x},t)+\cdots\\&=\widehat{\psi}^\dagger(\boldsymbol{x},t)i\hbar\frac{\partial}{\partial t}\widehat{\psi}(\boldsymbol{x},t)-\widehat{\psi}^\dagger(\boldsymbol{x},t)\left(-\frac{\hbar^2}{2m}\nabla^2\right)\widehat{\psi}(\boldsymbol{x},t)+(\text{total derivatives})+\cdots,\end{aligned} \tag{26}$$

where the ellipsis contains a second time-derivative term and this term is neglected in the nonrelativistic limit, i.e. $|mc^2\widehat{\psi}|\gg|i\hbar\partial\widehat{\psi}/\partial t|$. From Eq. (26), we find that the Hamiltonian operator has an embedded structure such as $\widehat{H}_\psi=\int\widehat{\psi}^\dagger(\boldsymbol{x},t)\widehat{H}\widehat{\psi}(\boldsymbol{x},t)d^3x$ with $\widehat{H}=-\frac{\hbar^2}{2m}\nabla^2$.

2.2. *Derivation of quantum mechanics*

Let us examine the relationship between QFT and QM, paying attention to physical states, operators of four-momenta, and the expectation values.

A physical state in QFT is represented in the $\boldsymbol{x}$-representation ($\boldsymbol{x}$-space) as a superposition of states with various numbers of particles as follows:[1]

$$|\Psi(t)\rangle=\psi^{(0)}(t)|0\rangle+\sum_{N=1}^{\infty}\int d^3x_1\cdots d^3x_N\ \psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t)|\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N\rangle, \tag{27}$$

where $|0\rangle$ is a vacuum state in QFT, $\psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t)$ is the wave function of an N-particle state in QM defined by

$$\psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t)\equiv\langle\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N|\Psi(t)\rangle, \tag{28}$$

and $|\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N\rangle$ is the ket vector satisfying $\widehat{\boldsymbol{x}}_l|\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N\rangle=\boldsymbol{x}_l|\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N\rangle$ $(l=1,\cdots,N)$, defined by

$$|\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N\rangle\equiv\frac{1}{\sqrt{N!}}\widehat{\varphi}^\dagger(\boldsymbol{x}_1)\cdots\widehat{\varphi}^\dagger(\boldsymbol{x}_N)|0\rangle, \tag{29}$$

using field operators $\widehat{\varphi}^\dagger(\boldsymbol{x})$. Note that $\widehat{\varphi}^\dagger(\boldsymbol{x})$ plays the role of a creation operator of a particle φ and an annihilation operator of its antiparticle $\overline{\varphi}$, and $\widehat{\varphi}(\boldsymbol{x})$ plays the role of a creation operator of $\overline{\varphi}$ and an annihilation operator of φ.

First, we consider a zero-particle state (a state of vacuum) such as

$$|\Psi(t)\rangle^{(0)}=\psi^{(0)}(t)|0\rangle, \tag{30}$$

[1]Strictly speaking, we need to include antiparticles, but we omit them for simplicity.

where $\psi^{(0)}(t)$ is the wave function of the zero-particle state in QM and is rewritten as $\psi^{(0)}(t) = \langle 0|\Psi(t)\rangle$. $\psi^{(0)}(t)$ satisfies the equation:

$$i\hbar\frac{d}{dt}\psi^{(0)}(t) = \mathscr{E}_0\psi^{(0)}(t), \tag{31}$$

and its solution is obtained as $\psi^{(0)}(t) = e^{-\frac{i}{\hbar}\mathscr{E}_0 t}\psi^{(0)}(0)$ with a vacuum energy $\mathscr{E}_0 = \langle 0|\widehat{H}_\varphi|0\rangle$.

Next we consider the one-particle state limited as

$$|\Psi(t)\rangle^{(1)} = |\psi(t)\rangle = \int d^3x\ \psi(\boldsymbol{x}, t)|\boldsymbol{x}\rangle, \tag{32}$$

where $\psi(\boldsymbol{x}, t)$ is a wave function of the one-particle state in QM and is rewritten by

$$\begin{aligned}\psi(\boldsymbol{x}, t) &= \langle \boldsymbol{x}|\Psi(t)\rangle = \langle 0|\widehat{\varphi}(\boldsymbol{x})|\Psi(t)\rangle = \langle 0|e^{\frac{i}{\hbar}\widehat{H}_\varphi t}\widehat{\varphi}(\boldsymbol{x})e^{-\frac{i}{\hbar}\widehat{H}_\varphi t}|\Psi(0)\rangle \\ &= \langle 0|\widehat{\varphi}(\boldsymbol{x}, t)|\Psi(0)\rangle,\end{aligned} \tag{33}$$

using Eq. (14), $\langle 0|e^{\frac{i}{\hbar}\widehat{H}_\varphi t} = \langle 0|$, i.e. $\langle 0|\widehat{H}_\varphi = 0$ and Eq. (10). Here, according to the ordinary procedure, we redefine $\widehat{H}_\varphi$ by using the normal ordering. From Eq. (33) and the first equation in Eq. (8), we find that $\psi(\boldsymbol{x}, t)$ satisfies the Schrödinger equation in QM:

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{x}, t) = \widehat{H}\psi(\boldsymbol{x}, t) \tag{34}$$

with the Hamiltonian operator $\widehat{H} = \widehat{H}(\boldsymbol{x}, -i\hbar\nabla)$ in QM. The Schrödinger equation is rewritten by

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \widehat{H}|\psi(t)\rangle, \tag{35}$$

and its formal solution is given by

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\widehat{H}t}|\psi(0)\rangle. \tag{36}$$

In a similar way, using Eq. (13), we find that $\psi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N, t)$ obeys the equation:

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N, t) = \widehat{H}^{(N)}\psi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N, t), \tag{37}$$

where $\widehat{H}^{(N)}$ is the total Hamiltonian operator of noninteracting N particles:

$$\widehat{H}^{(N)} = \sum_{l=1}^{N}\widehat{H}(\boldsymbol{x}_l, -i\hbar\nabla_l). \tag{38}$$

In a quantum theory, the four-momenta become generators of the spacetime translation such as $x^\mu \to x'^\mu = x^\mu - \varepsilon^\mu$ where $\mu = 0, 1, 2, 3$, $x^0 = ct$, $\boldsymbol{x} = (x^1, x^2, x^3)$ and ε^μ is an infinitesimal constant four-vector. Using the Noether procedure in analytical mechanics of fields, we derive the operators of four-momenta $\widehat{P}_\mu$ in QFT:

$$\widehat{P}_\mu = \frac{1}{c}\int\left(\frac{\partial\widehat{\mathscr{L}_\varphi}}{\partial(\partial_0\widehat{\varphi})}\partial_\mu\widehat{\varphi} - \delta^0{}_\mu\widehat{\mathscr{L}_\varphi}\right)d^3x = \frac{1}{c}\int\widehat{T}^0_\mu d^3x, \tag{39}$$

where $\partial_0\widehat{\varphi} = \partial\widehat{\varphi}/\partial x^0$, $\widehat{T}^0_\mu$ is a component of the energy–momentum tensor, and we use the fact that the field operator $\widehat{\varphi}(x)(= \widehat{\varphi}(x^0, \boldsymbol{x}))$ transforms as $\widehat{\varphi}'(x') = \widehat{\varphi}'(x - \varepsilon) = \widehat{\varphi}(x)$ under the translation $x'^\mu = x^\mu - \varepsilon^\mu$ and a change of $\widehat{\varphi}(x)$ is given by

$$\delta_\varepsilon\widehat{\varphi}(x) = \widehat{\varphi}'(x) - \widehat{\varphi}(x) = \widehat{\varphi}(x + \varepsilon) - \widehat{\varphi}(x) = \varepsilon^\mu\partial_\mu\widehat{\varphi}(x). \tag{40}$$

Because the Minkowski spacetime has a homogeneity, a physical system is invariant under the translation and then the four-momenta are conserved, i.e. $i\hbar d\widehat{P}_\mu/dt = [\widehat{P}_\mu, \widehat{H}] = 0$. Using Eqs. (1), (39), and the conservation law of $\widehat{P}_\mu$, we obtain the formula of $\widehat{P}_\mu$:

$$\widehat{P}_0 = \frac{1}{c}\int \widehat{\varphi}^\dagger(x)\widehat{H}(\boldsymbol{x}, -i\hbar\nabla)\widehat{\varphi}(x)d^3x = \frac{1}{c}\int \widehat{\varphi}^\dagger(\boldsymbol{x})\widehat{H}(\boldsymbol{x}, -i\hbar\nabla)\widehat{\varphi}(\boldsymbol{x})d^3x = \frac{\widehat{H}_\varphi}{c}, \tag{41}$$

$$\widehat{\boldsymbol{P}} = \int \widehat{\varphi}^\dagger(x)\,(-i\hbar\nabla)\,\widehat{\varphi}(x)d^3x = \int \widehat{\varphi}^\dagger(\boldsymbol{x})\,(-i\hbar\nabla)\,\widehat{\varphi}(\boldsymbol{x})d^3x. \tag{42}$$

In a similar way, we obtain the formula of the orbital angular momenta $\widehat{\boldsymbol{J}}$,

$$\widehat{\boldsymbol{J}} = \int \widehat{\varphi}^\dagger(\boldsymbol{x})\,\{\boldsymbol{x}\times(-i\hbar\nabla)\}\,\widehat{\varphi}(\boldsymbol{x})d^3x, \tag{43}$$

from the rotation. In this way, it is ascertained that a nested structure exists for $\widehat{\boldsymbol{P}}$ and $\widehat{\boldsymbol{J}}$ other than the Hamiltonian operator $c\widehat{P}_0$, as seen from Eqs. (42) and (43).

Alternatively, under the assumption that a nested structure is present, such as Eqs. (42) and (43), we reaffirm that the momenta are represented by the differential operator $\widehat{\boldsymbol{p}} = -i\hbar\nabla$ in the $\boldsymbol{x}$-representation and the following commutation relation in QM holds:

$$[\widehat{x}^i, \widehat{p}^j] = i\hbar\delta^{ij}, \quad [\widehat{x}^i, \widehat{x}^j] = 0, \quad [\widehat{p}^i, \widehat{p}^j] = 0. \tag{44}$$

Let us evaluate the expectation values of momenta $\boldsymbol{P}$ in the momentum space. There, a physical state in QFT is represented by

$$|\Psi(t)\rangle = \psi^{(0)}(t)|0\rangle + \sum_{N=1}^{\infty}\int d^3k_1\cdots d^3k_N\,\widetilde{\psi}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N,t)|\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N\rangle, \tag{45}$$

where $\widetilde{\psi}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N,t)$ is the wave function of an N-particle state in QM, $\boldsymbol{k}_l$ $(l = 1,\cdots,N)$ are wave-number vectors, and $|\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N\rangle$ is the ket vector defined by

$$|\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N\rangle \equiv \frac{1}{\sqrt{N!}}\widehat{b}^\dagger(\boldsymbol{k}_1)\cdots\widehat{b}^\dagger(\boldsymbol{k}_N)|0\rangle, \tag{46}$$

using creation operators $\widehat{b}^\dagger(\boldsymbol{k}_l)$ of particle φ with the momenta $\boldsymbol{p}_l = \hbar\boldsymbol{k}_l$.

The momentum operator in QFT is written as

$$\widehat{\boldsymbol{P}} = \int \hbar\boldsymbol{k}\left(\widehat{b}^\dagger(\boldsymbol{k})\widehat{b}(\boldsymbol{k}) + \widehat{d}^\dagger(\boldsymbol{k})\widehat{d}(\boldsymbol{k})\right)d^3k, \tag{47}$$

where $\widehat{b}(\boldsymbol{k})$ is the annihilation operator of φ with the momenta $\boldsymbol{p} = \hbar\boldsymbol{k}$, and $\widehat{d}^\dagger(\boldsymbol{k})$ and $\widehat{d}(\boldsymbol{k})$ are the creation and annihilation operators of the antiparticle $\overline{\varphi}$ with $\boldsymbol{p} = \hbar\boldsymbol{k}$, respectively.

For a one-particle state given by

$$|\Psi(t)\rangle^{(1)} = |\psi(t)\rangle = \int d^3k\,\widetilde{\psi}(\boldsymbol{k},t)\widehat{b}^\dagger(\boldsymbol{k})|0\rangle, \tag{48}$$

the expectation value of $\boldsymbol{P}$ is calculated as

$${}^{(1)}\langle\Psi(t)|\widehat{\boldsymbol{P}}|\Psi(t)\rangle^{(1)} = \langle\psi(t)|\widehat{\boldsymbol{P}}|\psi(t)\rangle = \int d^3k\,\widetilde{\psi}^\dagger(\boldsymbol{k},t)\,\hbar\boldsymbol{k}\,\widetilde{\psi}(\boldsymbol{k},t) = \langle\psi(t)|\widehat{\boldsymbol{p}}|\psi(t)\rangle, \tag{49}$$

where we use $\{\widehat{b}(\boldsymbol{k}), \widehat{b}^\dagger(\boldsymbol{k}')\} = \delta^3(\boldsymbol{k}-\boldsymbol{k}')$, $\widehat{b}(\boldsymbol{k})|0\rangle = 0$, $\widehat{d}(\boldsymbol{k})|0\rangle = 0$, $\langle 0|0\rangle = 1$, and so on. Together with the normalization condition

$${}^{(1)}\langle\Psi(t)|\Psi(t)\rangle^{(1)} = \langle\psi(t)|\psi(t)\rangle = \int d^3k\,\widetilde{\psi}^\dagger(\boldsymbol{k},t)\widetilde{\psi}(\boldsymbol{k},t) = 1, \tag{50}$$

we arrive at the probability interpretation that $\widetilde{\psi}^\dagger(\boldsymbol{k},t)\widetilde{\psi}(\boldsymbol{k},t)$ is the probability density and $\langle\psi(t)|\widehat{\boldsymbol{p}}|\psi(t)\rangle$ is the expectation value of momenta for the one-particle system in QM.

In a similar way, for a noninteracting N-particle system, the expectation value of the total momenta is given by

$$ {}^{(N)}\langle\Psi(t)|\widehat{\boldsymbol{P}}|\Psi(t)\rangle^{(N)} = \int d^3k_1\cdots d^3k_N\ \widetilde{\psi}^\dagger(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N,t)\sum_{l=1}^{N}\hbar\boldsymbol{k}_l\widetilde{\psi}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N,t), \tag{51} $$

where $|\Psi(t)\rangle^{(N)}$ is a state vector for the N-particle system given by

$$ |\Psi(t)\rangle^{(N)} = \int d^3k_1\cdots d^3k_N\ \widetilde{\psi}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N,t)|\boldsymbol{k}_1,\cdots,\boldsymbol{k}_N\rangle. \tag{52} $$

In the Schrödinger picture, observables in a system described by a Lagrangian density with the first time-derivative term are, in general, given in the form

$$ \widehat{\Omega}^a_\varphi = \int \widehat{\varphi}^\dagger(\boldsymbol{x})\widehat{\Omega}^a\widehat{\varphi}(\boldsymbol{x})d^3x, \tag{53} $$

where $\widehat{\Omega}^a_\varphi$ are operators including quantum fields in QFT, and $\widehat{\Omega}^a = \widehat{\Omega}^a(\boldsymbol{x}, -i\hbar\nabla)$ are operators operating a wave function in the position space of QM. It is shown that both $\widehat{\Omega}^a_\varphi$ and $\widehat{\Omega}^a$ satisfy the same type of algebraic relations:

$$ \left[\widehat{\Omega}^a_\varphi, \widehat{\Omega}^b_\varphi\right] = i\sum_c f^{abc}\widehat{\Omega}^c_\varphi, \quad \left[\widehat{\Omega}^a, \widehat{\Omega}^b\right] = i\sum_c f^{abc}\widehat{\Omega}^c. \tag{54} $$

Changing the Schrödinger picture into the Heisenberg picture, $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{p}}$ possess time dependence such as

$$ \widehat{\boldsymbol{x}}(t) = e^{\frac{i}{\hbar}\widehat{H}t}\widehat{\boldsymbol{x}}e^{-\frac{i}{\hbar}\widehat{H}t}, \quad \widehat{\boldsymbol{p}}(t) = e^{\frac{i}{\hbar}\widehat{H}t}\widehat{\boldsymbol{p}}e^{-\frac{i}{\hbar}\widehat{H}t}, \tag{55} $$

and $\widehat{\boldsymbol{x}}(t)$ and $\widehat{\boldsymbol{p}}(t)$ obey Heisenberg's equation of motion:

$$ i\hbar\frac{d\widehat{\boldsymbol{x}}(t)}{dt} = [\widehat{\boldsymbol{x}}(t), \widehat{H}], \quad i\hbar\frac{d\widehat{\boldsymbol{p}}(t)}{dt} = [\widehat{\boldsymbol{p}}(t), \widehat{H}]. \tag{56} $$

Here, for the sake of completeness, we point out that QM is also reconstructed from QFT in the case that particles interact with each other. In concrete terms, a Hamiltonian operator consists of two parts such as $\widehat{H}_\varphi = \widehat{H}^{(0)}_\varphi + \widehat{H}^{\rm int}_\varphi$ where $\widehat{H}^{(0)}_\varphi$ is the part relating to the kinetic energy and $\widehat{H}^{\rm int}_\varphi$ represents the interaction between particles. In the interaction picture, field operators $\widehat{\varphi}_{\rm I}(\boldsymbol{x}, t)$ and $\widehat{\pi}_{\rm I}(\boldsymbol{x}, t)$ obey Heisenberg's equation of motion:

$$ i\hbar\frac{\partial}{\partial t}\widehat{\varphi}_{\rm I}(\boldsymbol{x}, t) = \left[\widehat{\varphi}_{\rm I}(\boldsymbol{x}, t), \widehat{H}^{(0)}_\varphi\right], \quad i\hbar\frac{\partial}{\partial t}\widehat{\pi}_{\rm I}(\boldsymbol{x}, t) = \left[\widehat{\pi}_{\rm I}(\boldsymbol{x}, t), \widehat{H}^{(0)}_\varphi\right]; \tag{57} $$

they behave as free fields, and the Fock space is constructed using them. The physical state $|\Psi_{\rm I}(t)\rangle$ is evolved by the Schrödinger equation:

$$ i\hbar\frac{d}{dt}|\Psi_{\rm I}(t)\rangle = \widehat{H}^{\rm int}_\varphi(\widehat{\varphi}_{\rm I}, \widehat{\pi}_{\rm I})|\Psi_{\rm I}(t)\rangle, \tag{58} $$

and, for a one-particle state, we can effectively derive the Schrödinger equation in the interaction picture of QM:

$$ i\hbar\frac{d}{dt}|\psi_{\rm I}(t)\rangle = \widehat{V}_{\rm I}(t)|\psi_{\rm I}(t)\rangle \tag{59} $$

with a potential $\widehat{V}_{\rm I}(t)$, using the Born approximation in the nonrelativistic limit.

We list the central features of the relationship between QFT and QM, in a system described by a Lagrangian density with the first time-derivative term.

- QM is reconstructed from QFT under the condition that the number of particles is unchanged.

- From the transformation property of quantum fields under translation and rotation, it is understood that the momenta $\boldsymbol{p}$ are represented by the differential operator $\widehat{\boldsymbol{p}} = -i\hbar\nabla$ in the $\boldsymbol{x}$-representation of QM.
- Observables $\widehat{\Omega}^a_\varphi$ in QFT are, in general, constructed in the form of an embedded structure, e.g. $\widehat{\Omega}^a_\varphi = \int \widehat{\varphi}^\dagger(\boldsymbol{x})\widehat{\Omega}^a\widehat{\varphi}(\boldsymbol{x})d^3x$, where $\widehat{\Omega}^a = \widehat{\Omega}^a(\boldsymbol{x}, -i\hbar\nabla)$ are operators operating a wave function in $\boldsymbol{x}$-space of QM.

3. Quantum field functional theory

Let us come back to Q1, i.e. "Why does QFT work so well as an effective theory of elementary particles? Why are particles or fields quantized in the first place?" In concrete terms, why is a fermion (boson) described by a field operator such as $\widehat{\pi}(\boldsymbol{x}) = i\hbar\delta/\delta\varphi(\boldsymbol{x})$, i.e. $\widehat{\varphi}^\dagger(\boldsymbol{x}) = \delta/\delta\varphi(\boldsymbol{x})$ ($\widehat{\pi}(\boldsymbol{x}) = -i\hbar\delta/\delta\varphi(\boldsymbol{x})$), using the functional derivative?

If there were a fundamental framework that reached QFT in a similar way to the derivation of QM from QFT, we could answer the above questions. To explore such a framework, we will draw on the embedded structure in QFT, e.g.

$$\widehat{H}_\varphi = \int \widehat{\varphi}^\dagger(x)\widehat{H}\widehat{\varphi}(x)d^3x = \frac{1}{i\hbar}\int \widehat{\pi}(x)\widehat{H}\widehat{\varphi}(x)d^3x. \tag{60}$$

For simplicity, we consider a toy model containing a particle φ alone in this section.

3.1. *Framework of quantum field functional theory*

Taking a hint from the nested construction (60), we introduce basic operators denoted as $\widehat{\Phi}(\{\varphi\}, t)$ and the canonical conjugate $\widehat{\Pi}(\{\varphi\}, t)$, whose role will be discussed in the next subsection. We refer to $\widehat{\Phi}(\{\varphi\}, t)$ and $\widehat{\Pi}(\{\varphi\}, t)$ as "functional operators" or "field functionals", and call the functional operator theory the "quantum field functional theory", the "quantum theory of field functionals", or "QFFT" for short.

Let us start with a Lagrangian with a nested structure:

$$\widehat{L}_\Phi = \widehat{\Phi}^\dagger(\{\varphi\}, t)i\hbar\frac{\partial}{\partial t}\widehat{\Phi}(\{\varphi\}, t) - \widehat{\Phi}^\dagger(\{\varphi\}, t)\widehat{H}_\varphi\widehat{\Phi}(\{\varphi\}, t), \tag{61}$$

where $\widehat{\Phi}(\{\varphi\}, t)$ is a functional operator, $\{\varphi\}$ and t stand for a field of φ and time, respectively, and $\widehat{H}_\varphi$ is the Hamiltonian operator in QFT containing $\varphi(\boldsymbol{x})$ and its functional derivatives $\delta/\delta\varphi(\boldsymbol{x})$ (see Eq. (16)). For instance, $\widehat{H}_\varphi$ is given by

$$\widehat{H}_\varphi = \int \frac{\delta}{\delta\varphi(\boldsymbol{x})}\left(-i\hbar c\boldsymbol{\alpha}\cdot\nabla + \beta mc^2\right)\varphi(\boldsymbol{x})d^3x \tag{62}$$

for a free Dirac fermion.

The canonical conjugate of $\widehat{\Phi}(\{\varphi\}, t)$ is defined by

$$\widehat{\Pi}(\{\varphi\}, t) \equiv \frac{\partial\widehat{L}_\Phi}{\partial(\partial\widehat{\Phi}(\{\varphi\}, t)/\partial t)} = i\hbar\widehat{\Phi}^\dagger(\{\varphi\}, t), \tag{63}$$

and the Hamiltonian operator $\widehat{H}_\Phi$ in QFFT is obtained as

$$\begin{aligned}\widehat{H}_\Phi &\equiv \widehat{\Pi}(\{\varphi\}, t)\frac{\partial\widehat{\Phi}(\{\varphi\}, t)}{\partial t} - \widehat{L}_\Phi \\ &= \widehat{\Phi}^\dagger(\{\varphi\}, t)\widehat{H}_\varphi\widehat{\Phi}(\{\varphi\}, t) = \frac{1}{i\hbar}\widehat{\Pi}(\{\varphi\}, t)\widehat{H}_\varphi\widehat{\Phi}(\{\varphi\}, t).\end{aligned} \tag{64}$$

We notice that the Hamiltonian operator in QFFT is also constructed by sandwiching the Hamiltonian operator in QFT between two functional operators, and then it has a nested structure. We list explicit forms of $\widehat{H}_\Phi$ for particles with spin 1/2 in Appendix A.

We impose the following quantization conditions on the functional operators:[2]

$$\left\{\widehat{\Phi}(\{\varphi\},t),\widehat{\Pi}(\{\varphi\},t)\right\}=i\hbar,\ \ \left\{\widehat{\Phi}(\{\varphi\},t),\widehat{\Phi}(\{\varphi\},t)\right\}=0,\ \ \left\{\widehat{\Pi}(\{\varphi\},t),\widehat{\Pi}(\{\varphi\},t)\right\}=0 \quad (65)$$

or

$$\left[\widehat{\Phi}(\{\varphi\},t),\widehat{\Pi}(\{\varphi\},t)\right]=i\hbar,\ \ \left[\widehat{\Phi}(\{\varphi\},t),\widehat{\Phi}(\{\varphi\},t)\right]=0,\ \ \left[\widehat{\Pi}(\{\varphi\},t),\widehat{\Pi}(\{\varphi\},t)\right]=0. \quad (66)$$

The functional operators obey Heisenberg's equation of motion:

$$i\hbar\frac{\partial}{\partial t}\widehat{\Phi}(\{\varphi\},t)=\left[\widehat{\Phi}(\{\varphi\},t),\widehat{H}_\Phi\right],\ \ i\hbar\frac{\partial}{\partial t}\widehat{\Pi}(\{\varphi\},t)=\left[\widehat{\Pi}(\{\varphi\},t),\widehat{H}_\Phi\right]. \quad (67)$$

Using Eqs. (64), (65), and (67), we derive the equations:

$$i\hbar\frac{\partial}{\partial t}\widehat{\Phi}(\{\varphi\},t)=\widehat{H}_\varphi\widehat{\Phi}(\{\varphi\},t),\ \ i\hbar\frac{\partial}{\partial t}\widehat{\Pi}(\{\varphi\},t)=-\widehat{\Pi}(\{\varphi\},t)\widehat{H}_\varphi, \quad (68)$$

and these equations agree with the Euler–Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial\widehat{L}_\Phi}{\partial(\partial\widehat{\Phi}^\dagger/\partial t)}\right)-\frac{\partial\widehat{L}_\Phi}{\partial\widehat{\Phi}^\dagger}=0,\ \ \frac{d}{dt}\left(\frac{\partial\widehat{L}_\Phi}{\partial(\partial\widehat{\Phi}/\partial t)}\right)-\frac{\partial\widehat{L}_\Phi}{\partial\widehat{\Phi}}=0, \quad (69)$$

which is derived from the action integral $\widehat{S}_\Phi=\int\widehat{L}_\Phi dt$, based on the least-action principle.

In the Schrödinger picture, the functional operators $\widehat{\Phi}(\{\varphi\})$ and $\widehat{\Pi}(\{\varphi\})$ are independent of time, and they are related to those in the Heisenberg picture as

$$\widehat{\Phi}(\{\varphi\},t)=e^{\frac{i}{\hbar}\widehat{H}_\Phi t}\widehat{\Phi}(\{\varphi\})e^{-\frac{i}{\hbar}\widehat{H}_\Phi t},\ \ \widehat{\Pi}(\{\varphi\},t)=e^{\frac{i}{\hbar}\widehat{H}_\Phi t}\widehat{\Pi}(\{\varphi\})e^{-\frac{i}{\hbar}\widehat{H}_\Phi t}. \quad (70)$$

Using Eq. (70) and the conservation law of $\widehat{H}_\Phi$, i.e. $d\widehat{H}_\Phi/dt=0$, $\widehat{H}_\Phi$ is rewritten in a time-independent form as

$$\widehat{H}_\Phi=\widehat{\Phi}^\dagger(\{\varphi\})\widehat{H}_\varphi\widehat{\Phi}(\{\varphi\})=\frac{1}{i\hbar}\widehat{\Pi}(\{\varphi\})\widehat{H}_\varphi\widehat{\Phi}(\{\varphi\}). \quad (71)$$

The functional operators $\widehat{\Phi}(\{\varphi\})$ and $\widehat{\Pi}(\{\varphi\})$ obey the anticommutation relations:

$$\left\{\widehat{\Phi}(\{\varphi\}),\widehat{\Pi}(\{\varphi\})\right\}=i\hbar,\ \ \left\{\widehat{\Phi}(\{\varphi\}),\widehat{\Phi}(\{\varphi\})\right\}=0,\ \ \left\{\widehat{\Pi}(\{\varphi\}),\widehat{\Pi}(\{\varphi\})\right\}=0 \quad (72)$$

or commutation relations:

$$\left[\widehat{\Phi}(\{\varphi\}),\widehat{\Pi}(\{\varphi\})\right]=i\hbar,\ \ \left[\widehat{\Phi}(\{\varphi\}),\widehat{\Phi}(\{\varphi\})\right]=0,\ \ \left[\widehat{\Pi}(\{\varphi\}),\widehat{\Pi}(\{\varphi\})\right]=0. \quad (73)$$

From the first conditions in Eqs. (72) and (73), $\widehat{\Pi}(\{\varphi\})$ is given by $\widehat{\Pi}(\{\varphi\})=i\hbar\delta/\delta\Phi(\{\varphi\})$ or $\widehat{\Pi}(\{\varphi\})=-i\hbar\delta/\delta\Phi(\{\varphi\})$, respectively, in the representative diagonalizing $\widehat{\Phi}(\{\varphi\})$ such as $\widehat{\Phi}(\{\varphi\})|\Phi\rangle=\Phi(\{\varphi\})|\Phi\rangle$ with a configuration of field functional $\Phi(\{\varphi\})$.

Any state $|\Psi_{\rm M}(t)\rangle$ is evolved by the Schrödinger equation:

$$i\hbar\frac{d}{dt}|\Psi_{\rm M}(t)\rangle=\widehat{H}_\Phi|\Psi_{\rm M}(t)\rangle, \quad (74)$$

and its formal solution is given by

$$|\Psi_{\rm M}(t)\rangle=e^{-\frac{i}{\hbar}\widehat{H}_\Phi t}|\Psi_{\rm M}(0)\rangle. \quad (75)$$

[2]If the eigenvalues of $\widehat{H}_\varphi$ are bounded below, this system can also be quantized using commutation relations although the Lagrangian consists of the first time-derivative term and a (sign-flipping of a) Hamiltonian (see Eq. (61)). This feature is different from that of the quantization for particles with spin 1/2 in QFT. In fact, when we impose commutation relations on Dirac or Weyl fermions, the system becomes ill defined with the advent of negative norm states. This stems from the existence of negative-energy states in relativistic QM.

Multiplying $\langle \Phi |$ by both sides of Eq. (74), we obtain the equation:

$$i\hbar \frac{\partial}{\partial t} \Psi_{\mathrm{M}}(\Phi, t) = \widehat{H}_{\Phi} \Psi_{\mathrm{M}}(\Phi, t), \tag{76}$$

where $\Psi_{\mathrm{M}}(\Phi, t) = \langle \Phi | \Psi_{\mathrm{M}}(t) \rangle$ is a state functional in QFFT, and $\widehat{H}_{\Phi}$ is written as

$$\widehat{H}_{\Phi} = \frac{\delta}{\delta \Phi(\{\varphi\})} \widehat{H}_{\varphi} \Phi(\{\varphi\}) \quad \text{or} \quad \widehat{H}_{\Phi} = -\frac{\delta}{\delta \Phi(\{\varphi\})} \widehat{H}_{\varphi} \Phi(\{\varphi\}), \tag{77}$$

using $\widehat{\Pi}(\{\varphi\}) = i\hbar \delta / \delta \Phi(\{\varphi\})$ or $\widehat{\Pi}(\{\varphi\}) = -i\hbar \delta / \delta \Phi(\{\varphi\})$, respectively.

3.2. *Derivation of quantum field theory*

Let us investigate whether QFT can be rebuilt from QFFT or not.

Before proceeding to deal with this, we need to clarify the role of functional operators. To get a hint, we list empirical laws concerning particles.

- Elementary particles create and annihilate. Particles can appear in the vacuum, and a multiparticle state can return to the vacuum state. The vacuum state plays the role of a basis to construct any physical states, e.g. multiparticle states are obtained by multiplying creation operators.
- There is a hierarchical structure of matter, e.g. atoms consist of a nucleus and electrons, nuclei are composed of nucleons (protons and neutrons), and nucleons are made up of quarks.
- There is a hierarchical structure of physical laws, too. Equivalently, specific physical laws hold at each level of the structure of matter. Physical laws are universal in our universe.

From the above laws, we form a conjecture that *information on physical laws (particle contents and related parameters) is built in a vacuum state and the vacuum state is universal in our universe, in the sense that it obeys common physical laws everywhere, although it can be differently described depending on the situation of observers and the energy scale. Our vacuum state keeps a potential to create some definite elementary particles in accordance with physical systems. Additionally, if our universe has a beginning, our vacuum state must also be created by the operation of anything.*

According to the conjecture, let us assume that $\widehat{\Phi}^{\dagger}(\{\varphi\})$ is an operator to produce a vacuum state $|0\rangle_{\{\varphi\}}$ with the potential to create an elementary particle φ; φ obeys definite laws of QFT. This assumption is expressed by

$$\widehat{\Phi}^{\dagger}(\{\varphi\}) |\emptyset\rangle = |0\rangle_{\{\varphi\}}, \tag{78}$$

where $|\emptyset\rangle$ is a "nothingness state" or an "empty state", and it satisfies $\widehat{\Phi}(\{\varphi\}) |\emptyset\rangle = 0$ and $\langle \emptyset | \emptyset \rangle = 1$. Here, we list the relations:

$$\widehat{\Phi}^{\dagger}(\{\varphi\}) |\emptyset\rangle = |0\rangle_{\{\varphi\}}, \quad \widehat{\Phi}(\{\varphi\}) |\emptyset\rangle = 0, \quad \langle \emptyset | \widehat{\Phi}(\{\varphi\}) = {}_{\{\varphi\}}\langle 0 |, \quad \langle \emptyset | \widehat{\Phi}^{\dagger}(\{\varphi\}) = 0. \tag{79}$$

Note that $|\emptyset\rangle$ is not a vacuum state in QFT but a more fundamental one. If $\widehat{\Phi}^{\dagger}(\{\varphi\})$ produces a vacuum state, it must be accompanied by the emergence of spacetime where φ lives, and then the introduction of gravity is inevitable to formulate a complete theory. We will discuss a topic that is pertinent to gravity in Section 4.2.

If $\widehat{\Phi}^{\dagger}(\{\varphi\})$ and $\widehat{\Phi}(\{\varphi\})$ play the role of installing a vacuum state with a specific spacetime and removing it, it would be suitable to refer to $\widehat{\Phi}^{\dagger}(\{\varphi\})$ and $\widehat{\Phi}(\{\varphi\})$ as "installation operator" and "removal operator", respectively. Then, "Which came first, particles or spacetime?" in Q2 can be solved, because particles and a spacetime can be installed at the same time by $\widehat{\Phi}^{\dagger}(\{\varphi\})$ in QFFT.

Now let us derive QFT from QFFT, based on the $\boldsymbol{x}$-representation in QM.

For $\widehat{\Phi}^\dagger(\{\varphi\})$ satisfying the anticommutation relations (72), a state $|\Psi_{\rm M}(t)\rangle$ in QFFT is written as

$$|\Psi_{\rm M}(t)\rangle = \Psi^{(0)}(t)|\emptyset\rangle + \Psi(\{\varphi\}, t)\, \widehat{\Phi}^\dagger(\{\varphi\})|\emptyset\rangle, \tag{80}$$

where $\Psi(\{\varphi\}, t)$ is the state functional of one world made of φ and an expression for $\Psi(\{\varphi\}, t)$ will be given later (see Eq. (84)). In this case, one universe alone can appear and it has a literal meaning.

In contrast, for $\widehat{\Phi}^\dagger(\{\varphi\})$ satisfying the commutation relations (73), a state is written as a superposition of states constructed on various numbers of vacuum states,

$$|\Psi_{\rm M}(t)\rangle = \Psi^{(0)}(t)|\emptyset\rangle + \sum_{M=1}^{\infty} \Psi(\underbrace{\{\varphi\}, \cdots, \{\varphi\}}_{M}, t)\, \frac{1}{\sqrt{M!}} \left(\widehat{\Phi}^\dagger(\{\varphi\})\right)^M |\emptyset\rangle, \tag{81}$$

where $\Psi(\underbrace{\{\varphi\}, \cdots, \{\varphi\}}_{M}, t)$ is the state functional of M worlds and $\frac{1}{\sqrt{M!}} \left(\widehat{\Phi}^\dagger(\{\varphi\})\right)^M |\emptyset\rangle$ is a vacuum state described by

$$\frac{1}{\sqrt{M!}} \left(\widehat{\Phi}^\dagger(\{\varphi\})\right)^M |\emptyset\rangle = \underbrace{|0\rangle_{\{\varphi\}} \otimes \cdots \otimes |0\rangle_{\{\varphi\}}}_{M}. \tag{82}$$

In this case, M worlds can be interpreted as M identical universes developing according to the same physical laws.

Let us study a state on one world limited as

$$|\Psi_{\rm M}(t)\rangle^{(1)} = |\Psi(t)\rangle = \Psi(\{\varphi\}, t)\, \widehat{\Phi}^\dagger(\{\varphi\})|\emptyset\rangle = \Psi(\{\varphi\}, t)|0\rangle, \tag{83}$$

where $\Psi(\{\varphi\}, t)$ is a state functional depicted by

$$\Psi(\{\varphi\}, t) = \psi^{(0)}(t) + \sum_{N=1}^{\infty} \int d^3x_1 \cdots d^3x_N\, \psi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N, t) \frac{1}{\sqrt{N!}} \varphi^\dagger(\boldsymbol{x}_1) \cdots \varphi^\dagger(\boldsymbol{x}_N), \tag{84}$$

using Eq. (B19). Note that $\Psi(\{\varphi\}, t)$ should not be confused with the wave functional $\Psi(\varphi, t) \equiv \langle\varphi|\Psi(t)\rangle$ in QFT. As seen in Eq. (B20), they are related to each other as

$$\Psi(\varphi, t) = \Psi(\{\varphi\}, t)\mathscr{V}_0(\varphi), \tag{85}$$

with $\mathscr{V}_0(\varphi) = \langle\varphi|0\rangle$. Here and hereafter, we omit the subscript $\{\varphi\}$ attached to the vacuum state $|0\rangle$ and $\langle 0|$ in this subsection, for simplicity.

The vacuum wave function is defined by

$$\begin{aligned}\Psi_{\rm V}(t) &\equiv \langle 0|\Psi_{\rm M}(t)\rangle = \langle\emptyset|\widehat{\Phi}(\{\varphi\})|\Psi_{\rm M}(t)\rangle \\ &= \langle\emptyset|e^{\frac{i}{\hbar}\widehat{H}_\Phi t}\widehat{\Phi}(\{\varphi\})e^{-\frac{i}{\hbar}\widehat{H}_\Phi t}|\Psi_{\rm M}(0)\rangle = \langle\emptyset|\widehat{\Phi}(\{\varphi\}, t)|\Psi_{\rm M}(0)\rangle,\end{aligned} \tag{86}$$

using Eq. (75), $\langle\emptyset|e^{\frac{i}{\hbar}\widehat{H}_\Phi t} = \langle\emptyset|$, i.e. $\langle\emptyset|\widehat{H}_\Phi = 0$ and Eq. (70). From Eq. (86) and the first equation in Eq. (68), we find that $\Psi_{\rm V}(t)$ and $\widehat{\Phi}(\{\varphi\}, t)$ satisfy the same type of equation:

$$i\hbar\frac{\partial}{\partial t}\Psi_{\rm V}(t) = \mathscr{E}_0\Psi_{\rm V}(t), \quad i\hbar\frac{\partial}{\partial t}\widehat{\Phi}(\{\varphi\}, t) = \mathscr{E}_0\widehat{\Phi}(\{\varphi\}, t), \tag{87}$$

where $\mathscr{E}_0 = \langle 0|\widehat{H}_\varphi|0\rangle$ and $\widehat{H}_\varphi$ is the Hamiltonian operator in QFT. Then, the time evolution of $\widehat{\Phi}(\{\varphi\}, t)$ and $\widehat{\Phi}^\dagger(\{\varphi\}, t)$ is determined as

$$\widehat{\Phi}(\{\varphi\}, t) = e^{-\frac{i}{\hbar}\mathscr{E}_0 t}\widehat{\Phi}(\{\varphi\}), \quad \widehat{\Phi}^\dagger(\{\varphi\}, t) = \widehat{\Phi}^\dagger(\{\varphi\})e^{\frac{i}{\hbar}\mathscr{E}_0 t}. \tag{88}$$

Note that $\Psi_{\rm V}(t)$ agrees with $\langle 0|\Psi(t)\rangle = \psi^{(0)}(t)$ in QFT and it satisfies Eq. (B16).

From Eqs. (74) and (83), we can derive the Schrödinger equation in QFT:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \widehat{H}_\varphi|\Psi(t)\rangle, \tag{89}$$

under the assumption that $\widehat{H}_\varphi$ is the Hamiltonian operator in QFT, as follows:

$$\begin{aligned}i\hbar\frac{d}{dt}|\Psi(t)\rangle = i\hbar\frac{d}{dt}|\Psi_{\rm M}(t)\rangle^{(1)} = \widehat{H}_\Phi|\Psi_{\rm M}(t)\rangle^{(1)} = \widehat{\Phi}^\dagger(\{\varphi\})\widehat{H}_\varphi\widehat{\Phi}(\{\varphi\})|\Psi(t)\rangle \\ = \widehat{\Phi}^\dagger(\{\varphi\})\widehat{H}_\varphi\widehat{\Phi}(\{\varphi\})\,\Psi(\{\varphi\},t)\,\widehat{\Phi}^\dagger(\{\varphi\})|0\!\!\!/\rangle = \widehat{H}_\varphi|\Psi(t)\rangle.\end{aligned} \tag{90}$$

Multiplying $\langle\varphi|$ by both sides of Eq. (89), we obtain the equation on the wave functional in QFT:

$$i\hbar\frac{\partial}{\partial t}\Psi(\varphi,t) = \widehat{H}_\varphi\Psi(\varphi,t). \tag{91}$$

Next let us justify that a fermion (boson) field $\varphi(\boldsymbol{x})$ is quantized in order to satisfy the anticommutation relations (the commutation relations). Under an infinitesimal translation $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{\varepsilon}$, the field $\varphi(\boldsymbol{x})$ transforms as $\varphi'(\boldsymbol{x}') = \varphi'(\boldsymbol{x} - \boldsymbol{\varepsilon}) = \varphi(\boldsymbol{x})$, and then an infinitesimal change of $\varphi(\boldsymbol{x})$ is given by

$$\delta_\varepsilon\varphi(\boldsymbol{x}) = \varphi'(\boldsymbol{x}) - \varphi(\boldsymbol{x}) = \varphi(\boldsymbol{x}+\boldsymbol{\varepsilon}) - \varphi(\boldsymbol{x}) = \varepsilon^i\partial_i\varphi(\boldsymbol{x}). \tag{92}$$

If its field functional $\widehat{\Phi}(\{\varphi\},t)$ transforms as

$$\widehat{\Phi}'(\{\varphi'\},t) = \widehat{\Phi}'(\{\varphi+\delta_\varepsilon\varphi\},t) = \widehat{\Phi}(\{\varphi\},t) \tag{93}$$

under $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{\varepsilon}$, $\widehat{\Phi}(\{\varphi\},t)$ changes infinitesimally as

$$\begin{aligned}\delta_\varepsilon\widehat{\Phi}(\{\varphi\},t) &= \widehat{\Phi}'(\{\varphi\},t) - \widehat{\Phi}(\{\varphi\},t) = \widehat{\Phi}(\{\varphi-\delta_\varepsilon\varphi\},t) - \widehat{\Phi}(\{\varphi\},t) \\ &= -\left(\widehat{\Phi}(\{\varphi\},t) - \widehat{\Phi}(\{\varphi-\delta_\varepsilon\varphi\},t)\right) = -\int d^3x\,\delta_\varepsilon\varphi(\boldsymbol{x})\frac{\delta}{\delta\varphi(\boldsymbol{x})}\widehat{\Phi}(\{\varphi\},t) \\ &= -\int d^3x\,\varepsilon^i\partial_i\varphi(\boldsymbol{x})\frac{\delta}{\delta\varphi(\boldsymbol{x})}\widehat{\Phi}(\{\varphi\},t).\end{aligned} \tag{94}$$

Note that Eq. (93) implies the translational invariance of the vacuum state, which is one of the features in relativistic QFT.

Then, the action integral $\widehat{S}_\Phi = \int \widehat{L}_\Phi dt$ changes as

$$\begin{aligned}\delta_\varepsilon\widehat{S}_\Phi &= \int \delta_\varepsilon\widehat{L}_\Phi dt = \int \frac{\partial}{\partial t}\left(\widehat{\Phi}^\dagger(\{\varphi\},t)i\hbar\delta_\varepsilon\widehat{\Phi}(\{\varphi\},t)\right)dt \\ &= \int \frac{\partial}{\partial t}\left\{\widehat{\Phi}^\dagger(\{\varphi\},t)\int d^3x\,\varepsilon^i\partial_i\varphi(\boldsymbol{x})\left(-i\hbar\frac{\delta}{\delta\varphi(\boldsymbol{x})}\right)\widehat{\Phi}(\{\varphi\},t)\right\}dt,\end{aligned} \tag{95}$$

under the condition that $\widehat{\Phi}(\{\varphi\},t)$ obeys the equation of motion (68). From Eq. (95), the momentum operator $\widehat{P}_{\Phi\ i}$ can be read off as

$$\begin{aligned}\widehat{P}_{\Phi\ i} &\equiv \widehat{\Phi}^\dagger(\{\varphi\},t)\int d^3x\,\partial_i\varphi(\boldsymbol{x})\left(-i\hbar\frac{\delta}{\delta\varphi(\boldsymbol{x})}\right)\widehat{\Phi}(\{\varphi\},t) \\ &= \widehat{\Phi}^\dagger(\{\varphi\},t)\left(\int d^3x\,\widehat{\pi}(\boldsymbol{x})\partial_i\widehat{\varphi}(\boldsymbol{x})\right)\widehat{\Phi}(\{\varphi\},t) \\ &= \widehat{\Phi}^\dagger(\{\varphi\},t)\left(\frac{1}{c}\int d^3x\,\widehat{T}^0_i(\boldsymbol{x})\right)\widehat{\Phi}(\{\varphi\},t),\end{aligned} \tag{96}$$

where we use $\widehat{\varphi}(\boldsymbol{x}) = \varphi(\boldsymbol{x})$ and $\widehat{\pi}(\boldsymbol{x}) = i\hbar\delta/\delta\varphi(\boldsymbol{x})$ for the fermion ($\widehat{\varphi}(\boldsymbol{x}) = \varphi(\boldsymbol{x})$ and $\widehat{\pi}(\boldsymbol{x}) = -i\hbar\delta/\delta\varphi(\boldsymbol{x})$ for the boson), and $\widehat{T}^0_i(\boldsymbol{x})$ is a component of the energy–momentum tensor in QFT.

Note that $\varphi(\boldsymbol{x})$ is a Grassmann variable for the fermion and a constant term in the integration is subtracted.

In this way, we find that there also exists a nested structure for momenta and verify that fields become operators and satisfy the anticommutation relations for the fermion (the commutation relations for the boson). We notice that these features come from the transformation property of field functional and an answer to the question "Why are particles or fields quantized in the first place?" is obtained.

Let us evaluate the expectation value of momenta for the state given by Eq. (83). It is calculated as

$$\begin{aligned}
&{}^{(1)}\langle\Psi_{\rm M}(t)|\widehat{\boldsymbol{P}}_{\Phi}|\Psi_{\rm M}(t)\rangle^{(1)} = \langle\Psi(t)|\widehat{\boldsymbol{P}}_{\Phi}|\Psi(t)\rangle \\
&= \langle\emptyset|\widehat{\Phi}(\{\varphi\})\Psi^{\dagger}(\{\varphi\},t)\widehat{\Phi}^{\dagger}(\{\varphi\})\left(\int d^3x\,\widehat{\pi}(\boldsymbol{x})\nabla\widehat{\varphi}(\boldsymbol{x})\right)\widehat{\Phi}(\{\varphi\})\Psi(\{\varphi\},t)\widehat{\Phi}^{\dagger}(\{\varphi\})|\emptyset\rangle \\
&= \Psi^{\dagger}(\{\varphi\},t)\left(\int d^3x\,\widehat{\pi}(\boldsymbol{x})\nabla\widehat{\varphi}(\boldsymbol{x})\right)\Psi(\{\varphi\},t) = \int \mathscr{D}\varphi\ \Psi^{\dagger}(\varphi,t)\widehat{\boldsymbol{P}}\Psi(\varphi,t) \\
&= \langle\Psi(t)|\widehat{\boldsymbol{P}}|\Psi(t)\rangle,
\end{aligned} \tag{97}$$

using the conditions (72) or (73), $\langle\emptyset|\emptyset\rangle = 1$, and Eq. (B21).

In the same way, the expectation value of energy for the state given by Eq. (83) is calculated as

$$\begin{aligned}
{}^{(1)}\langle\Psi_{\rm M}(t)|\widehat{H}_{\Phi}|\Psi_{\rm M}(t)\rangle^{(1)} &= \langle\Psi(t)|\widehat{H}_{\Phi}|\Psi(t)\rangle = \Psi^{\dagger}(\{\varphi\},t)\widehat{H}_{\varphi}\Psi(\{\varphi\},t) \\
&= \int \mathscr{D}\varphi\ \Psi^{\dagger}(\varphi,t)\widehat{H}_{\varphi}\Psi(\varphi,t) = \langle\Psi(t)|\widehat{H}_{\varphi}|\Psi(t)\rangle.
\end{aligned} \tag{98}$$

In this way, we arrive at the expression for expectation values in QFT, and the expectation values in QM are also obtained by fixing the number of particles, as seen in Section 2.2.

4. Universes with different particle contents

Now it is time to tackle Q3, i.e. "Why is our universe described by the standard model at the electroweak scale?" Under the precondition that the existence of our universe can be understood by the combination of the level II multiverse (an ensemble of foreign universes) and the anthropic principle, and the above question is not really acknowledged as a problem, we face the question of whether a framework to describe the level II multiverse can be constructed or not. In the following, we investigate a theoretical framework to deal with a set of universes with different elementary particles and parameters based on the physical laws of QFT.

4.1. *Level II multiverse*

We extend our formulation to an assembly of universes with different elementary particles and parameters.

Let us first introduce installation operators $\widehat{\Phi}^{\dagger}_{(a)}(\{\varphi_{k^{(a)}}\})$ that produce a vacuum state $|0\rangle_{\{\varphi_{k^{(a)}}\}}$ where definite elementary particles $\varphi_{k^{(a)}}$ can be created and work obeying the laws of QFT. This is expressed by

$$\widehat{\Phi}^{\dagger}_{(a)}(\{\varphi_{k^{(a)}}\})\,|\emptyset\rangle = |0\rangle_{\{\varphi_{k^{(a)}}\}}, \tag{99}$$

where $|\emptyset\rangle$ is the nothingness state and it satisfies $\widehat{\Phi}_{(a)}(\{\varphi_{k^{(a)}}\})|\emptyset\rangle = 0$ and $\langle\emptyset|\emptyset\rangle = 1$. Here, $a(= 1,\cdots,\mathscr{N})$ is a label that specifies universes with a set of particles $\{\varphi_{k^{(a)}}\}$ and parameters, and

$k^{(a)}$ is a label that represents particles.[3] We have relations such as

$$\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right)|\emptyset\rangle = |0\rangle_{\{\varphi_{k^{(a)}}\}}, \quad \widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right)|\emptyset\rangle = 0,$$
$$\langle\emptyset|\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right) = {}_{\{\varphi_{k^{(a)}}\}}\langle 0|, \quad \langle\emptyset|\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right) = 0, \tag{100}$$

and impose the following conditions on $\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right)$ and $\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right)$:

$$\left\{\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right), \widehat{\Phi}^{\dagger}_{(b)}\left(\{\varphi_{k^{(b)}}\}\right)\right\} = \delta_{ab}, \quad \left\{\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right), \widehat{\Phi}_{(b)}\left(\{\varphi_{k^{(b)}}\}\right)\right\} = 0,$$
$$\left\{\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right), \widehat{\Phi}^{\dagger}_{(b)}\left(\{\varphi_{k^{(b)}}\}\right)\right\} = 0 \tag{101}$$

or

$$\left[\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right), \widehat{\Phi}^{\dagger}_{(b)}\left(\{\varphi_{k^{(b)}}\}\right)\right] = \delta_{ab}, \quad \left[\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right), \widehat{\Phi}_{(b)}\left(\{\varphi_{k^{(b)}}\}\right)\right] = 0,$$
$$\left[\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right), \widehat{\Phi}^{\dagger}_{(b)}\left(\{\varphi_{k^{(b)}}\}\right)\right] = 0. \tag{102}$$

Because production of the vacuum state $|0\rangle_{\{\varphi_{k^{(a)}}\}}$ by $\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right)$ must be associated with the emergence of spacetime, a graviton appears inevitably and each universe has invariance under the general transformation of coordinates. The consequence of this will be discussed in the next subsection.

When the field functionals satisfy Eq. (101), a state $|\Psi_{\mathrm{M_{II}}}(t)\rangle$ in the level II multiverse is written as a superposition of states constructed on various numbers of vacuum states as follows:

$$|\Psi_{\mathrm{M_{II}}}(t)\rangle = \Psi^{(0)}(t)|\emptyset\rangle + \sum_{M=1}^{\mathcal{N}}\sum_{a_1=1}^{\mathcal{N}}\cdots\sum_{a_M=1}^{\mathcal{N}}\Psi\left(\{\varphi_{k^{(a_1)}}\},\cdots,\{\varphi_{k^{(a_M)}}\},t\right)$$
$$\times\widehat{\Phi}^{\dagger}_{(a_1)}\left(\{\varphi_{k^{(a_1)}}\}\right)\cdots\widehat{\Phi}^{\dagger}_{(a_M)}\left(\{\varphi_{k^{(a_M)}}\}\right)|\emptyset\rangle, \tag{103}$$

where the summation is taken in the range $a_1 > \cdots > a_M$,[4] t is an auxiliary parameter that is identified as the time when the system is limited in some observable region of our universe, $\Psi\left(\{\varphi_{k^{(a_1)}}\},\cdots,\{\varphi_{k^{(a_M)}}\},t\right)$ is a state functional related to M kinds of universes, and $\widehat{\Phi}^{\dagger}_{(a_1)}\left(\{\varphi_{k^{(a_1)}}\}\right)\cdots\widehat{\Phi}^{\dagger}_{(a_M)}\left(\{\varphi_{k^{(a_M)}}\}\right)|\emptyset\rangle$ is a vacuum state described by

$$\widehat{\Phi}^{\dagger}_{(a_1)}\left(\{\varphi_{k^{(a_1)}}\}\right)\cdots\widehat{\Phi}^{\dagger}_{(a_M)}\left(\{\varphi_{k^{(a_M)}}\}\right)|\emptyset\rangle = |0\rangle_{\{\varphi_{k^{(a_1)}}\}}\otimes\cdots\otimes|0\rangle_{\{\varphi_{k^{(a_M)}}\}}. \tag{104}$$

When the field functionals satisfy Eq. (102), $|\Psi_{\mathrm{M_{II}}}(t)\rangle$ is written by

$$|\Psi_{\mathrm{M_{II}}}(t)\rangle = \Psi^{(0)}(t)|\emptyset\rangle$$
$$+\sum_{M_{(a_1)}=1}^{\infty}\cdots\sum_{M_{(a_{\mathcal{N}})}=1}^{\infty}\Psi(\underbrace{\{\varphi_{k^{(a_1)}}\},\cdots,\{\varphi_{k^{(a_1)}}\}}_{M_{(a_1)}},\cdots,\underbrace{\{\varphi_{k^{(a_{\mathcal{N}})}}\},\cdots,\{\varphi_{k^{(a_{\mathcal{N}})}}\}}_{M_{(a_{\mathcal{N}})}},t)$$
$$\times\frac{1}{\sqrt{M_{(a_1)}!}}\left(\widehat{\Phi}^{\dagger}_{(a_1)}\left(\{\varphi_{k^{(a_1)}}\}\right)\right)^{M_{(a_1)}}\cdots\frac{1}{\sqrt{M_{(a_{\mathcal{N}})}!}}\left(\widehat{\Phi}^{\dagger}_{(a_{\mathcal{N}})}\left(\{\varphi_{k^{(a_{\mathcal{N}})}}\}\right)\right)^{M_{(a_{\mathcal{N}})}}|\emptyset\rangle. \tag{105}$$

Any state $|\Psi_{\mathrm{M_{II}}}(t)\rangle$ is evolved by the Schrödinger equation:

$$i\hbar\frac{d}{dt}|\Psi_{\mathrm{M_{II}}}(t)\rangle = \widehat{H}_{\{\Phi\}}|\Psi_{\mathrm{M_{II}}}(t)\rangle, \tag{106}$$

[3] For the sake of completeness, installation operators and vacuum states should be denoted as $\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\},\{\alpha_{j^{(a)}}\}\right)$ and $|0\rangle_{\{\varphi_{k^{(a)}}\},\{\alpha_{j^{(a)}}\}}$, respectively, where $\{\alpha_{j^{(a)}}\}$ is a label representing a set of parameters, but here and hereafter $\{\alpha_{j^{(a)}}\}$ is omitted to avoid complications.

[4] When a label a_n takes continuous values, a summation should be replaced by an integration. The same applies to the following.

and its formal solution is given by

$$|\Psi_{\mathrm{M_{II}}}(t)\rangle = e^{-\frac{i}{\hbar}\widehat{H}_{\{\Phi\}}t}|\Psi_{\mathrm{M_{II}}}(0)\rangle, \tag{107}$$

where $\widehat{H}_{\{\Phi\}}$ is the Hamiltonian operator in QFFT given by

$$\widehat{H}_{\{\Phi\}} = \sum_{a=1}^{\mathcal{N}} \widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right) \widehat{H}_{\{\varphi_{k(a)}\}} \widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right). \tag{108}$$

In Eq. (108), $\widehat{H}_{\{\varphi_{k(a)}\}}$ are Hamiltonian operators in QFT containing $\varphi_{k^{(a)}}(x)$ and its functional derivatives $\delta/\delta\varphi_{k^{(a)}}(x)$, and we see that the nested construction is realized.

In the Heisenberg picture, installation and removal operators are given in a form with the time dependence by

$$\begin{aligned}\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) &= e^{\frac{i}{\hbar}\widehat{H}_{\{\Phi\}}t}\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right) e^{-\frac{i}{\hbar}\widehat{H}_{\{\Phi\}}t},\\ \widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) &= e^{\frac{i}{\hbar}\widehat{H}_{\{\Phi\}}t}\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right) e^{-\frac{i}{\hbar}\widehat{H}_{\{\Phi\}}t},\end{aligned} \tag{109}$$

and they are evolved by Heisenberg's equation of motion:

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) &= \left[\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right), \widehat{H}_{\{\Phi\}}\right],\\ i\hbar\frac{\partial}{\partial t}\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) &= \left[\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right), \widehat{H}_{\{\Phi\}}\right].\end{aligned} \tag{110}$$

The dynamics is summarized by the action integral:

$$\begin{aligned}\widehat{S}_{\{\Phi\}} &= \int \widehat{L}_{\{\Phi\}} dt\\ &= \int \sum_{a=1}^{\mathcal{N}} \Big(\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) i\hbar\frac{\partial}{\partial t}\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right)\\ &\qquad -\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) \widehat{H}_{\{\varphi_{k(a)}\}}\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right)\Big) dt.\end{aligned} \tag{111}$$

The canonical conjugate of $\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right)$ is defined by

$$\widehat{\Pi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) \equiv \frac{\partial \widehat{L}_{\{\Phi\}}}{\partial\left(\partial\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right)/\partial t\right)} = i\hbar\widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right), \tag{112}$$

and the Hamiltonian operator $\widehat{H}_{\{\Phi\}}$ in QFFT is obtained as

$$\begin{aligned}\widehat{H}_{\{\Phi\}} &\equiv \sum_{a=1}^{\mathcal{N}} \widehat{\Pi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right)\frac{\partial\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right)}{\partial t} - \widehat{L}_{\{\Phi\}}\\ &= \sum_{a=1}^{\mathcal{N}} \widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) \widehat{H}_{\{\varphi_{k(a)}\}}\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right)\\ &= \frac{1}{i\hbar}\sum_{a=1}^{\mathcal{N}} \widehat{\Pi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) \widehat{H}_{\{\varphi_{k(a)}\}}\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right).\end{aligned} \tag{113}$$

It is shown that this $\widehat{H}_{\{\Phi\}}$ agrees with that in Eq. (108), using the solutions of functional operators (see Eq. (116)).

The following quantization conditions are imposed on field functionals:

$$\begin{aligned}&\left\{\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right), \widehat{\Pi}_{(b)}\left(\{\varphi_{k^{(b)}}\}, t\right)\right\} = i\hbar\delta_{ab}, \quad \left\{\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right), \widehat{\Phi}_{(b)}\left(\{\varphi_{k^{(b)}}\}, t\right)\right\} = 0,\\ &\left\{\widehat{\Pi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right), \widehat{\Pi}_{(b)}\left(\{\varphi_{k^{(b)}}\}, t\right)\right\} = 0\end{aligned} \tag{114}$$

or

$$\left[\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right), \widehat{\Pi}_{(b)}\left(\{\varphi_{k^{(b)}}\}, t\right)\right] = i\hbar\delta_{ab}, \quad \left[\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right), \widehat{\Phi}_{(b)}\left(\{\varphi_{k^{(b)}}\}, t\right)\right] = 0,$$
$$\left[\widehat{\Pi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right), \widehat{\Pi}_{(b)}\left(\{\varphi_{k^{(b)}}\}, t\right)\right] = 0, \tag{115}$$

and then Heisenberg's equation of motion (110) agrees with the Euler–Lagrange equation derived from the action integral (111), based on the least-action principle.

4.2. *Physical implications*

Under the assumption that each universe contains specific elementary particle contents including a graviton and physical parameters, and it is evolved by the physical laws of QFT, we discuss the physical implications of the beginning of the universe based on QFFT.

4.2.1. *Inflation.* As a reference for the discussion from Eqs. (86)–(88), we obtain the following solutions:

$$\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) = e^{-\frac{i}{\hbar}\mathscr{E}_0^{(a)}t}\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right), \quad \widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) = \widehat{\Phi}^{\dagger}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right)e^{\frac{i}{\hbar}\mathscr{E}_0^{(a)}t}, \tag{116}$$

where $\mathscr{E}_0^{(a)} = \langle 0|\widehat{H}_{\{\varphi_{k^{(a)}}\}}|0\rangle$ is the vacuum energy of the universe labeled by a. The vacuum wave function $\Psi_{\mathrm{V}}^{(a)}(t)$ is evolved as

$$\Psi_{\mathrm{V}}^{(a)}(t) = e^{-\frac{i}{\hbar}\mathscr{E}_0^{(a)}t}\Psi_{\mathrm{V}}^{(a)}(0). \tag{117}$$

Here, we consider a universe labeled by a with uniformity and isotropy, whose geometry and dynamics are described by the Robertson–Walker metric. When the vacuum energy density $\rho_{\mathrm{V}}^{(a)} \equiv \mathscr{E}_0^{(a)}/V$ (V: the volume of the universe) dominates over other energy densities, the scale factor $a(t)$ varies based on the Friedmann equation:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G_{\mathrm{N}}}{3}\rho_{\mathrm{V}}^{(a)}, \tag{118}$$

where $\dot{a}(t) \equiv da(t)/dt$ and G_{N} is the gravitational constant. Then, in the case with $\rho_{\mathrm{V}}^{(a)} > 0$, inflation (an exponential expansion of the universe) (see Ref. [13] and references therein) occurs as

$$a(t) = a(0)e^{H_{\mathrm{V}}^{(a)}t}, \tag{119}$$

where $H_{\mathrm{V}}^{(a)} \equiv \sqrt{8\pi G_{\mathrm{N}}\rho_{\mathrm{V}}^{(a)}/3}$. In contrast, in the case with $\rho_{\mathrm{V}}^{(a)} < 0$, $a(t)$ oscillates.

In this way, we find that a universe can expand at a very early stage and a macroscopic world can emerge if a universe is dominated by positive vacuum energy shortly after its birth.

4.2.2. *Third quantization.* In a universe labeled by a whose spacetime varies, the spacetime itself is regarded as a dynamical object, and the theory must be invariant under the general transformation of coordinates and contain a gravitational field.

Let us study the invariance under the general transformation of coordinates in our formulation.

Under the infinitesimal transformation $x'^{\mu} = x^{\mu} - \varepsilon^{\mu}(x)$, the field $\varphi_{k^{(a)}}$ transforms as $\varphi'_{k^{(a)}}(\boldsymbol{x}') = \varphi'_{k^{(a)}}(\boldsymbol{x} - \boldsymbol{\varepsilon}) = \varphi_{k^{(a)}}(\boldsymbol{x})$, and then the change of $\varphi_{k^{(a)}}$ is given by

$$\delta_{\varepsilon}\varphi_{k^{(a)}}(\boldsymbol{x}) = \varphi'_{k^{(a)}}(\boldsymbol{x}) - \varphi_{k^{(a)}}(\boldsymbol{x}) = \varphi_{k^{(a)}}(\boldsymbol{x} + \boldsymbol{\varepsilon}) - \varphi_{k^{(a)}}(\boldsymbol{x}) = \varepsilon^{i}(x)\partial_{i}\varphi_{k^{(a)}}(\boldsymbol{x}). \tag{120}$$

If the field functional $\widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right)$ transforms as

$$\widehat{\Phi}'_{(a)}\left(\{\varphi'_{k^{(a)}}\}, t'\right) = \widehat{\Phi}'_{(a)}\left(\{\varphi_{k^{(a)}} + \delta_\varepsilon \varphi_{k^{(a)}}\}, t - \varepsilon_t\right) = \widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) \tag{121}$$

under $x'^\mu = x^\mu - \varepsilon^\mu(x)$ with $\varepsilon^0(x) = c\varepsilon_t(x)$, the change of $\widehat{\Phi}_{(a)}\left(\{\varphi\}_{k^{(a)}}, t\right)$ is induced such that

$$\begin{aligned}
\delta_\varepsilon \widehat{\Phi}_{(a)}\left(\{\varphi\}_{k^{(a)}}, t\right) &= \widehat{\Phi}'_{(a)}\left(\{\varphi\}_{k^{(a)}}, t\right) - \widehat{\Phi}_{(a)}\left(\{\varphi\}_{k^{(a)}}, t\right) \\
&= \widehat{\Phi}'_{(a)}\left(\{\varphi_{k^{(a)}} - \delta_\varepsilon \varphi_{k^{(a)}}\}, t + \varepsilon_t\right) - \widehat{\Phi}_{(a)}\left(\{\varphi\}_{k^{(a)}}, t\right) \\
&= -\int d^3x \sum_{k^{(a)}} \delta_\varepsilon \varphi_{k^{(a)}}(\boldsymbol{x}) \frac{\delta}{\delta \varphi_{k^{(a)}}(\boldsymbol{x})} \widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) + \varepsilon_t \frac{\partial}{\partial t} \widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) \\
&= -\int d^3x \sum_{k^{(a)}} \varepsilon^i(x) \partial_i \varphi_{k^{(a)}} \frac{\delta}{\delta \varphi_{k^{(a)}}} \widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) \\
&\quad + \varepsilon_t(x) \frac{\partial}{\partial t} \widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right),
\end{aligned} \tag{122}$$

and then the change of the action integral is given by

$$\begin{aligned}
\delta_\varepsilon \widehat{S}_{\{\Phi\}} &= \left[\widehat{\Phi}^\dagger_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right) \left(\frac{1}{c}\int d^3x\, \varepsilon^\mu(x) \widehat{T}^0_{\ \mu}(\boldsymbol{x})\right) \widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}, t\right)\right]_{t_i}^{t_f} \\
&= \left[\widehat{\Phi}^\dagger_{(a)}\left(\{\varphi_{k^{(a)}}\}\right) \left(\frac{1}{c}\int d^3x\, \varepsilon^\mu(x) \widehat{T}^0_{\ \mu}(x)\right) \widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right)\right]_{t_i}^{t_f},
\end{aligned} \tag{123}$$

using the Euler–Langrange equation.

When $\delta_\varepsilon \widehat{S}_{\{\Phi\}} = 0$ holds for arbitrary $\varepsilon^\mu(x)$, we have a physical state condition,

$$\widehat{\Phi}^\dagger_{(a)}\left(\{\varphi_{k^{(a)}}\}\right) \widehat{T}^0_{\ \mu}(x) \widehat{\Phi}_{(a)}\left(\{\varphi_{k^{(a)}}\}\right) |\Psi_{\mathrm{M_{II}}}(t)\rangle = 0, \tag{124}$$

and this leads to the condition and the equation

$$\widehat{T}^0_{\ \mu}(x) |\Psi(t)\rangle_{\{\varphi_{k^{(a)}}\}} = 0 \tag{125}$$

and

$$\widehat{T}^0_{\ \mu}(x) \Psi\left(\varphi_{k^{(a)}}, t\right) = 0, \tag{126}$$

respectively. The time component of Eq. (126) is equivalent to the Wheeler–deWitt equation [14], which represents the Hamiltonian constraint [15] at the quantum level, and we arrive at a fundamental formula on the third quantization [16].

4.2.3. *Landscape.* First, we list postulates about a vacuum state and multiverse.

- In each universe, a vacuum state can change by a phase transition (like the electroweak transition and QCD transition in our universe), and hence there can appear a variety of vacuum states, which are represented as $|0\rangle_{\{\varphi_{k^{(a)}}\}}$ together for a universe with a set of elementary particles $\{\varphi_{k^{(a)}}\}$.
- Even if universe A produces universe B [17] using a mechanism such as eternal inflation [18], as long as the mechanism works under physical laws in universe A, universe B must obey the same physical laws of universe A.

From the above postulates, we expect that a wider theoretical framework is necessary to study the creation of universes with different elementary particles and parameters and to discuss the relationship among them. We investigate such a framework by reference to the landscape of string theory vacua [19,20].

Here, we list features of string theories and those vacua [21,22,23,24].

- There are five kinds of superstring theories in 10D spacetime. Numerous 4D string models are constructed after compactifying an extra 6D space.
- A variety of models originate from a diversity of structures of the extra space and different configurations of strings. 4D string models are specified by (the vacuum expectation values of) scalar fields $\varphi_{\mathrm{L}m}$ including moduli that characterize the structure of the extra space. The space spanned by $\varphi_{\mathrm{L}m}$ is called the "landscape", and its altitude corresponds to a value of vacuum energy. The landscape can be regarded as a potential.
- Particle contents in each string model are stored in string fields symbolically denoted by $\widehat{\varphi}_{\mathrm{st}}(\boldsymbol{X}(\sigma), \boldsymbol{\psi}(\sigma))$.

Let us suppose that there is a Hamiltonian $\widehat{H}_{\{\varphi_{\mathrm{L}}\}}$ containing a potential $\widehat{V}_{\{\varphi_{\mathrm{L}}\}}$ whose local minima correspond to solutions describing universes with different particle contents and physical parameters, and the dynamics can be compactly expressed by the action integral:

$$\begin{aligned}\widehat{S}_{\Phi_{\mathrm{L}}} &= \int \widehat{L}_{\Phi_{\mathrm{L}}} dt \\ &= \int \left(\widehat{\Phi}^{\dagger}_{\mathrm{L}}(\{\varphi_{\mathrm{L}}\}, t)\, i\hbar \frac{\partial}{\partial t} \widehat{\Phi}_{\mathrm{L}}(\{\varphi_{\mathrm{L}}\}, t) - \widehat{\Phi}^{\dagger}_{\mathrm{L}}(\{\varphi_{\mathrm{L}}\}, t)\, \widehat{H}_{\{\varphi_{\mathrm{L}}\}} \widehat{\Phi}_{\mathrm{L}}(\{\varphi_{\mathrm{L}}\}, t) \right) dt, \qquad (127)\end{aligned}$$

where $\{\varphi_{\mathrm{L}}\}$ is a set of fields $\varphi_{\mathrm{L}m}$ living on the landscape, and $\widehat{\Phi}_{\mathrm{L}}(\{\varphi_{\mathrm{L}}\}, t)$ is the functional operator concerning $\{\varphi_{\mathrm{L}}\}$.

The functional operator satisfies the equation:

$$i\hbar \frac{\partial}{\partial t} \widehat{\Phi}_{\mathrm{L}}(\{\varphi_{\mathrm{L}}\}, t) = \widehat{H}_{\{\varphi_{\mathrm{L}}\}} \widehat{\Phi}_{\mathrm{L}}(\{\varphi_{\mathrm{L}}\}, t), \qquad (128)$$

and, from the stationary condition:

$$\frac{\delta}{\delta \varphi_{\mathrm{L}m}} \widehat{H}_{\{\varphi_{\mathrm{L}}\}} = 0, \qquad (129)$$

we obtain solutions $\{\varphi^{(a)}_{\mathrm{L}}\}$ that are identified with models owning a set of elementary particles $\{\varphi_{k^{(a)}}\}$. Then, $\widehat{\Phi}_{\mathrm{L}}\left(\{\varphi^{(a)}_{\mathrm{L}}\}, t\right)$ becomes $\widehat{\Phi}_{(a)}(\{\varphi_{k^{(a)}}\}, t)$, and, putting together them, we arrive at an effective formulation described by the action integral (111).

As a mechanism to produce the level II multiverse, eternal inflation using the potential $\widehat{V}_{\{\varphi_{\mathrm{L}}\}}$ in the landscape looks promising. Then, some of $\varphi_{\mathrm{L}m}$ can play the role of inflatons. For more detail, a vacuum state $|0\rangle_{\{\varphi_{\mathrm{L}}\}}$ and a spacetime can emerge by the operation of $\widehat{\Phi}^{\dagger}_{\mathrm{L}}(\{\varphi_{\mathrm{L}}\}, t)$ such as $|0\rangle_{\{\varphi_{\mathrm{L}}\}} = \widehat{\Phi}^{\dagger}_{\mathrm{L}}(\{\varphi_{\mathrm{L}}\}, t)\, |\emptyset\rangle$. Each region of the space expands while $\varphi_{\mathrm{L}m}$ are rolling down towards a different local minimum of the potential or they are transitioning from a local minimum to another one. When the potential has flat directions for some $\varphi_{\mathrm{L}m}$, those values can correspond to values of parameters. After $\varphi_{\mathrm{L}m}$ arrive at the local minimum labeled by $\{\varphi^{(a)}_{\mathrm{L}}\}$ including a set of parameters, the values of $\varphi_{\mathrm{L}m}$ are fixed, the vacuum $|0\rangle_{\{\varphi_{\mathrm{L}}\}}$ changes into $|0\rangle_{\{\varphi_{k^{(a)}}\}}$, and a universe with $|0\rangle_{\{\varphi_{k^{(a)}}\}}$ comes into existence. If the universe is dominated by a positive vacuum energy coming from some particle (called an inflaton) in $\{\varphi_{k^{(a)}}\}$, inflation can occur in the universe. After the inflation comes to an end, the inflaton decays into other particles in $\{\varphi_{k^{(a)}}\}$ and the universe becomes extremely hot and dense (like the big bang in our universe). In this way, various universes with different particle contents and parameters (different physical laws) can be generated.

A change between universes with different physical laws can also occur and its rate can be estimated in the setup based on the action integral (127), if there are no selection rules to forbid the change. According to the laws of quantum physics, we can calculate the transition amplitude $\widehat{S}_{\rm fi}$ from a vacuum state in a universe with $\{\varphi_{k^{(a)}}\}$ to a different one with $\{\varphi_{k^{(b)}}\}$ by using the formula

$$\widehat{S}_{\rm fi} = \langle \emptyset | \widehat{\Phi}_{(b)}(\{\varphi_{k^{(b)}}\}, t_{\rm f})\ e^{-\frac{i}{\hbar}\widehat{H}_{\Phi_{\rm L}}(t_{\rm f}-t_{\rm i})}\ \widehat{\Phi}^{\dagger}_{(a)}(\{\varphi_{k^{(a)}}\}, t_{\rm i})\,|\emptyset\rangle, \tag{130}$$

where $\widehat{H}_{\Phi_{\rm L}} = \widehat{\Phi}^{\dagger}_{\rm L}(\{\varphi_{\rm L}\}, t)\,\widehat{H}_{\{\varphi_{\rm L}\}}\widehat{\Phi}_{\rm L}(\{\varphi_{\rm L}\}, t)$.

5. Conclusions and discussions

Taking the embedded structure in QFT as a guiding principle, we have arrived at a wider theoretical framework called quantum field functional theory (QFFT) that has QFT built in. QFFT is constructed by using functional operators and has a nested structure.[5]

The rationale behind such a structure is that a physical state should be evolved by the same type of equation: $i\hbar d|\Psi_{\bigstar}(t)\rangle/dt = \widehat{H}_{\star}|\Psi_{\bigstar}(t)\rangle$ for any quantum system in a world where the laws of quantum theory hold. In concrete terms, for a system in the level II multiverse, it is given by $i\hbar d|\Psi_{\rm M_{II}}(t)\rangle/dt = \widehat{H}_{\{\Phi\}}|\Psi_{\rm M_{II}}(t)\rangle$. For a system in which the number of particles can vary, it becomes $i\hbar d|\Psi(t)\rangle/dt = \widehat{H}_{\varphi}|\Psi(t)\rangle$ based on QFT. For a system with a definite number of particles, it takes $i\hbar d|\psi(t)\rangle/dt = \widehat{H}|\psi(t)\rangle$ based on QM. Hence the Schrödinger equation for a state is worshiped as a master equation, and it makes us think that nature is not all that complicated at a fundamental level.

On the one hand, QFFT is not well defined mathematically in the same way as QFT, i.e. it is not yet known whether functional operators can be rigorously defined or not. On the other hand, QFFT has several advantages. First, it is relatively naturally understood that fermions (fields as Grassmann variables) and bosons are represented by field operators satisfying the anticommutation relations and the commutation relations, respectively. Second, the which-came-first-particles-or-spacetime problem (which-came-first-physical-laws-or-our-universe problem) can be solved, if particles and spacetime (physical laws and our universe) can be installed as a unit by a functional operator. Third, the level II multiverse could be described by QFFT. Last, topics related to the beginning of the universe such as inflation, the third quantization, and the landscape can be studied in our formulation. Our framework might also be applied to the level III multiverse where the level III multiverse means quantum mechanical many worlds [26].[6]

At present, QFFT is merely an effective framework to formulate the laws of particle physics or a tool in the understanding of quantum physics in the same way as QFT, and hence it might not possess such powerful predictability. It is intriguing to examine whether QFFT has both predictability and falsifiability or not, based on investigations of the level II multiverse.

There is a possibility that QFFT plays a supporting role as follows. For several decades, exploration of a fundamental theory has been actively carried out by using two approaches, a bottom-up one (which takes a route from the standard model to a theory beyond the standard model) and a top-down one (which takes a route from a theory of everything to the standard model). Superstring theory has been a hopeful candidate for an ultimate theory, and it offers vast numbers of string models that can be identified with members of the level II multiverse. Even if current superstring theory did not answer the question “How are universes created?”, it

[5]Another extension of QFT with a nested structure has been carried out by promoting a wave functional in QFT into an operator in reference to the second quantization [25]. In this formulation, the operator can create or annihilate a multiparticle state.

[6]It has been proposed that the eternally inflating multiverse and quantum mechanical many worlds are the same thing [27].

would be open to extend its framework. Thus, it is expected that QFFT serves as a good model. Furthermore, a combination of the two approaches mentioned above and an extension of the framework would be promising in unraveling the secrets of our universe and beyond.

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A. Hamiltonian operators in QM, QFT, and QFFT

We give explicit forms of Hamiltonian operators in QM, QFT, and QFFT for several cases.

A.1. Hamiltonian operators in nonrelativistic quantum theory

For a nonrelativistic electron with a mass $m_{\rm e}$, the Hamiltonian operator in QM is given by

$$\widehat{H}(\boldsymbol{x}, -i\hbar\nabla) = -\frac{\hbar^2}{2m_{\rm e}}\nabla^2 + V(\boldsymbol{x}) \tag{A1}$$

and then the Hamiltonian operators in QFT and QFFT are given by

$$\widehat{H}_\varphi = \int \widehat{\varphi}^\dagger(x)\widehat{H}\widehat{\varphi}(x)d^3x = \int \widehat{\varphi}^\dagger(x)\left(-\frac{\hbar^2}{2m_{\rm e}}\nabla^2 + V(\boldsymbol{x})\right)\widehat{\varphi}(x)d^3x \tag{A2}$$

and

$$\widehat{H}_\Phi = \widehat{\Phi}^\dagger(\{\varphi\}, t)\left\{\int \frac{\delta}{\delta\varphi(\boldsymbol{x})}\left(-\frac{\hbar^2}{2m_{\rm e}}\nabla^2 + V(\boldsymbol{x})\right)\varphi(\boldsymbol{x})d^3x\right\}\widehat{\Phi}(\{\varphi\}, t), \tag{A3}$$

respectively.

A.2. Hamiltonian operators in relativistic quantum theory

(1) Weyl fermion

For a free left-handed Weyl fermion (a massless particle with a helicity $-1/2$), the Hamiltonian operator in QM is given by

$$\widehat{H}(\boldsymbol{x}, i\hbar\nabla) = -i\hbar c\boldsymbol{\sigma}\cdot\nabla \tag{A4}$$

and then that in QFT is given by

$$\widehat{H}_\varphi = \int \widehat{\varphi}^\dagger(x)\widehat{H}\widehat{\varphi}(x)d^3x = \int \widehat{\varphi}^\dagger(x)\,(-i\hbar c\boldsymbol{\sigma}\cdot\nabla)\,\widehat{\varphi}(x)d^3x, \tag{A5}$$

where $\boldsymbol{\sigma}$ are Pauli matrices. For a free right-handed Weyl fermion (a massless particle with a helicity 1/2), $\widehat{H}(\boldsymbol{x}, i\hbar\nabla) = i\hbar c\boldsymbol{\sigma}\cdot\nabla$ and $\widehat{H}_\varphi = \int \widehat{\varphi}^\dagger(x)\,(i\hbar c\boldsymbol{\sigma}\cdot\nabla)\,\widehat{\varphi}(x)d^3x$ are obtained by replacing $\boldsymbol{\sigma}$ by $-\boldsymbol{\sigma}$ in Eqs. (A4) and (A5).

The Hamiltonian operator in QFFT is given by

$$\widehat{H}_\Phi = \widehat{\Phi}^\dagger(\{\varphi\})\left\{\int \frac{\delta}{\delta\varphi(\boldsymbol{x})}\,(-i\hbar c\boldsymbol{\sigma}\cdot\nabla)\,\varphi(\boldsymbol{x})d^3x\right\}\widehat{\Phi}(\{\varphi\}) \tag{A6}$$

for a left-handed Weyl fermion.

(2) Dirac fermion

For a free Dirac fermion (a particle with a mass m and spin 1/2), the Hamiltonian in QM is given by

$$\widehat{H}(\boldsymbol{x}, i\hbar\nabla) = -i\hbar c\boldsymbol{\alpha}\cdot\nabla + \beta mc^2, \tag{A7}$$

and then the Hamiltonian operators in QFT and QFFT are given by

$$\widehat{H}_\varphi = \int \widehat{\varphi}^\dagger(x)\widehat{H}\widehat{\varphi}(x)d^3x = \int \widehat{\varphi}^\dagger(x)\left(-i\hbar c\boldsymbol{\alpha}\cdot\nabla + \beta mc^2\right)\widehat{\varphi}(x)d^3x \tag{A8}$$

and

$$\widehat{H}_\Phi = \widehat{\Phi}^\dagger(\{\varphi\})\left\{\int \frac{\delta}{\delta\varphi(\boldsymbol{x})}\left(-i\hbar c\boldsymbol{\alpha}\cdot\nabla + \beta mc^2\right)\varphi(\boldsymbol{x})d^3x\right\}\widehat{\Phi}(\{\varphi\}), \tag{A9}$$

respectively.

B. Wave functional

A wave functional in QFT is defined by

$$\Psi(\varphi, t) \equiv \langle\varphi|\Psi(t)\rangle, \tag{B1}$$

where $\langle\varphi|$ is a bra vector, its ket vector $|\varphi\rangle$ is an eigenvector satisfying

$$\widehat{\varphi}(\boldsymbol{x})|\varphi\rangle = \varphi(\boldsymbol{x})|\varphi\rangle, \quad \langle\varphi|\widehat{\varphi}^\dagger(\boldsymbol{x}) = \langle\varphi|\varphi^\dagger(\boldsymbol{x}), \tag{B2}$$

and $\varphi(\boldsymbol{x})$ is the configuration of a classical field. For the fermion, $\varphi(\boldsymbol{x})$ takes Grassmann values. The eigenvectors $\langle\varphi|$ and $|\widetilde{\varphi}\rangle$ satisfy the relations:

$$\langle\varphi|\widetilde{\varphi}\rangle = \delta(\varphi - \widetilde{\varphi}) = \int \mathscr{D}\alpha\; e^{i\int \alpha(\boldsymbol{x})(\varphi(\boldsymbol{x})-\widetilde{\varphi}(\boldsymbol{x}))d^3x}, \quad \int \mathscr{D}\varphi\; |\varphi\rangle\langle\varphi| = 1. \tag{B3}$$

The expectation value of $\widehat{H}_\varphi$ is written by

$$\langle\Psi(t)|\widehat{H}_\varphi|\Psi(t)\rangle = \int \mathscr{D}\varphi \int \mathscr{D}\widetilde{\varphi}\; \langle\Psi(t)|\varphi\rangle\langle\varphi|\widehat{H}_\varphi|\widetilde{\varphi}\rangle\langle\widetilde{\varphi}|\Psi(t)\rangle$$
$$= \int \mathscr{D}\varphi \int \mathscr{D}\widetilde{\varphi}\; \Psi^\dagger(\varphi, t)\widehat{H}_\varphi\delta(\varphi - \widetilde{\varphi})\Psi(\widetilde{\varphi}, t) = \int \mathscr{D}\varphi\; \Psi^\dagger(\varphi, t)\widehat{H}_\varphi\Psi(\varphi, t), \tag{B4}$$

using Eq. (B3).

A general solution of the wave functional is given by

$$\Psi(\varphi, t) = \sum_{N=0}^{\infty} e^{-\frac{i}{\hbar}\mathscr{E}_N t}\mathcal{U}_N(\varphi), \tag{B5}$$

where $\mathscr{E}_N$ and $\mathscr{U}_N(\varphi)$ are eigenvalues and eigenvectors of the eigenvalue equation:

$$\widehat{H}_\varphi\mathcal{U}_N(\varphi) = \mathscr{E}_N\mathcal{U}_N(\varphi). \tag{B6}$$

Note that $\Psi(\varphi, t)$ satisfies Eq. (15), i.e. $i\hbar\partial\Psi(\varphi, t)/\partial t = \widehat{H}_\varphi\Psi(\varphi, t)$.

In the $\boldsymbol{x}$-representation, the state $|\Psi(t)\rangle$ is expressed by

$$|\Psi(t)\rangle = \psi^{(0)}(t)|0\rangle + \sum_{N=1}^{\infty}\int d^3x_1\cdots d^3x_N\psi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N, t)|\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N\rangle, \tag{B7}$$

where $|0\rangle$ is a vacuum state, $\psi(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N, t)$ is a wave function of N-particle states and $|\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N\rangle$ is a ket vector satisfying $\widehat{\boldsymbol{x}}_l|\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N\rangle = \boldsymbol{x}_l|\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N\rangle$ $(l = 1, \cdots, N)$ that is defined by

$$|\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N\rangle \equiv \frac{1}{\sqrt{N!}}\widehat{\varphi}^\dagger(\boldsymbol{x}_1)\cdots\widehat{\varphi}^\dagger(\boldsymbol{x}_N)|0\rangle, \tag{B8}$$

using $\widehat{\varphi}^\dagger(\boldsymbol{x})$, which plays the role of a creation operator of a particle φ or an annihilation operator of its antiparticle $\overline{\varphi}$.

Multiplying $\langle\varphi|$ by both sides of Eq. (B7) and using Eq. (B2), the following relation is obtained:

$$\begin{aligned}\Psi(\varphi,t) &= \langle\varphi|\Psi(t)\rangle \\ &= \psi^{(0)}(t)\langle\varphi|0\rangle + \sum_{N=1}^{\infty}\int d^3x_1\cdots d^3x_N\psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t)\langle\varphi|\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N\rangle \\ &= \psi^{(0)}(t)\langle\varphi|0\rangle \\ &\quad + \sum_{N=1}^{\infty}\int d^3x_1\cdots d^3x_N\psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t)\frac{1}{\sqrt{N!}}\varphi^\dagger(\boldsymbol{x}_1)\cdots\varphi^\dagger(\boldsymbol{x}_N)\langle\varphi|0\rangle. \end{aligned} \tag{B9}$$

From Eqs. (B5) and (B9), by taking $\mathcal{V}_0(\varphi) = \langle\varphi|0\rangle$ and $\mathcal{U}_N(\varphi) \equiv C_N(\varphi)\mathcal{V}_0(\varphi)$, we obtain the relations:

$$e^{-\frac{i}{\hbar}\mathscr{E}_0 t}C_0(\varphi) = \psi^{(0)}(t), \tag{B10}$$

$$e^{-\frac{i}{\hbar}\mathscr{E}_N t}C_N(\varphi) = \int d^3x_1\cdots d^3x_N\psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t)\frac{1}{\sqrt{N!}}\varphi^\dagger(\boldsymbol{x}_1)\cdots\varphi^\dagger(\boldsymbol{x}_N). \tag{B11}$$

Multiplying $|\varphi\rangle$ by $\langle\boldsymbol{x}|$, we obtain the relation:

$$\langle\boldsymbol{x}|\varphi\rangle = \langle 0|\widehat{\varphi}(\boldsymbol{x})|\varphi\rangle = \varphi(\boldsymbol{x})\langle 0|\varphi\rangle = \varphi(\boldsymbol{x})\mathcal{V}_0^\dagger(\varphi) \tag{B12}$$

with $\mathcal{V}_0^\dagger(\varphi) = \langle 0|\varphi\rangle$.

Here, we present formulae including $\mathcal{V}_0(\varphi)$ and $\mathcal{V}_0^\dagger(\varphi)$:

$$\textstyle\int \mathscr{D}\varphi\ \mathcal{V}_0^\dagger(\varphi)\mathcal{V}_0(\varphi) = \int \mathscr{D}\varphi\ \langle 0|\varphi\rangle\langle\varphi|0\rangle = \langle 0|0\rangle = 1, \tag{B13}$$

$$\begin{aligned}&\int \mathscr{D}\varphi\ \frac{1}{N!}\varphi(\boldsymbol{x}_1)\cdots\varphi(\boldsymbol{x}_N)\varphi^\dagger(\boldsymbol{x}'_1)\cdots\varphi^\dagger(\boldsymbol{x}'_N)\mathcal{V}_0^\dagger(\varphi)\mathcal{V}_0(\varphi) \\ &\qquad = \delta^3(\boldsymbol{x}_1-\boldsymbol{x}'_1)\cdots\delta^3(\boldsymbol{x}_N-\boldsymbol{x}'_N), \end{aligned} \tag{B14}$$

$$\begin{aligned}&\mathcal{V}_0(\varphi)\mathcal{V}_0^\dagger(\widetilde{\varphi}) + \sum_{N=1}^{\infty}\int d^3x_1\cdots d^3x_N\ \frac{1}{N!}\varphi^\dagger(\boldsymbol{x}_1)\cdots\varphi^\dagger(\boldsymbol{x}_N)\widetilde{\varphi}(\boldsymbol{x}_1)\cdots\widetilde{\varphi}(\boldsymbol{x}_N)\mathcal{V}_0(\varphi)\mathcal{V}_0^\dagger(\widetilde{\varphi}) \\ &\quad = \delta(\varphi-\widetilde{\varphi}). \end{aligned} \tag{B15}$$

As a reference, the vacuum wave function $\Psi_{\rm V}(t) = \langle 0|\Psi(t)\rangle = \psi^{(0)}(t)$ is written by

$$\Psi_{\rm V}(t) = \psi^{(0)}(t) = \langle 0|\Psi(t)\rangle = \int \mathscr{D}\varphi\ \langle 0|\varphi\rangle\langle\varphi|\Psi(t)\rangle = \int \mathscr{D}\varphi\ \mathcal{V}_0^\dagger(\varphi)\Psi(\varphi,t). \tag{B16}$$

In the same way, a component of the functional $\langle\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N|\Psi(t)\rangle = \psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t)$ is written by

$$\psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t) = \int \mathscr{D}\varphi\ \frac{1}{\sqrt{N!}}\varphi(\boldsymbol{x}_1)\cdots\varphi(\boldsymbol{x}_N)\ \mathcal{V}_0^\dagger(\varphi)\Psi(\varphi,t). \tag{B17}$$

The state $|\Psi(t)\rangle$ is rewritten as

$$|\Psi(t)\rangle = \psi^{(0)}(t)|0\rangle + \sum_{N=1}^{\infty}\int d^3x_1\cdots d^3x_N \psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t)\frac{1}{\sqrt{N!}}\widehat{\varphi}^{\dagger}(\boldsymbol{x}_1)\cdots\widehat{\varphi}^{\dagger}(\boldsymbol{x}_N)|0\rangle$$
$$= \psi^{(0)}(t)|0\rangle$$
$$+\sum_{N=1}^{\infty}\int d^3x_1\cdots d^3x_N \psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t)\frac{1}{\sqrt{N!}}\int \mathscr{D}\varphi|\varphi\rangle\langle\varphi|\widehat{\varphi}^{\dagger}(\boldsymbol{x}_1)\cdots\widehat{\varphi}^{\dagger}(\boldsymbol{x}_N)|0\rangle$$
$$= \psi^{(0)}(t)|0\rangle + \sum_{N=1}^{\infty}\int d^3x_1\cdots d^3x_N \psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t)\frac{1}{\sqrt{N!}}\varphi^{\dagger}(\boldsymbol{x}_1)\cdots\varphi^{\dagger}(\boldsymbol{x}_N)|0\rangle$$
$$= \Psi(\{\varphi\},t)|0\rangle, \tag{B18}$$

where $\Psi(\{\varphi\},t)$ is a wave functional defined by

$$\Psi(\{\varphi\},t) \equiv \psi^{(0)}(t) + \sum_{N=1}^{\infty}\int d^3x_1\cdots d^3x_N \psi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,t)\frac{1}{\sqrt{N!}}\varphi^{\dagger}(\boldsymbol{x}_1)\cdots\varphi^{\dagger}(\boldsymbol{x}_N). \tag{B19}$$

From Eqs. (B9) and (B19), we obtain the relation:

$$\Psi(\varphi,t) = \Psi(\{\varphi\},t)\mathscr{V}_0(\varphi). \tag{B20}$$

For an operator $\widehat{\Omega}_{\varphi}$, it is shown that the following formula holds:

$$\langle\Psi(t)|\widehat{\Omega}_{\varphi}|\Psi(t)\rangle = \int \mathscr{D}\varphi\, \Psi^{\dagger}(\varphi,t)\widehat{\Omega}_{\varphi}\Psi(\varphi,t)$$
$$= \Psi^{\dagger}(\{\varphi\},t)\widehat{\Omega}_{\varphi}\Psi(\{\varphi\},t), \tag{B21}$$

using Eq. (B13).

References

[1] P. A. M. Dirac, Proc. R. Soc. Lond. A **114**, 243 (1927).
[2] W. Heisenberg and W. Pauli, Z. Phys. **56**, 1 (1929).
[3] S. Navas et al. [Particle Data Group], Phys. Rev. D **110**, 030001 (2024).
[4] J. Bardeen, L. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[5] S. Weinberg, The Quantum Theory of Fields I (Cambridge University Press, Cambridge, UK, 1995).
[6] S. Weinberg, The Quantum Theory of Fields II, (Cambridge University Press, Cambridge, UK, 1996).
[7] S. Weinberg, The Quantum Theory of Fields III, (Cambridge University Press, Cambridge, UK, 2000).
[8] M. Tegmark, Our Mathematical Universe: My Quest for the Ultimate Nature of Reality (Knopf, New York, 2014).
[9] R. H. Dick, Nature **192**, 440 (1961).
[10] B. Carter, IAU Symp. 63: Confrontation of Cosmological Theories with Observational Data, p. 291 (1974). doi:10.1007/978-94-010-2220-0_25
[11] S. Weinberg, Phys. Rev. Lett. **59**, 2607 (1987).
[12] K. Symanzik, Nucl. Phys. B **190**, 1 (1981).
[13] D. H. Lyth and A. Riotto, Phys. Rept. **314**, 1 (1999).
[14] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).
[15] R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. **116**, 1322 (1959).
[16] S. B. Giddings and A. Strominger, Nucl. Phys. B **321**, 481 (1989).
[17] K. Sato, H. Kodama, M. Sasaki, and K. Maeda, Phys. Lett. **14**, 103 (1982).
[18] A. D. Linde, Nonsingular Regenerating Inflationary Universe (Cambridge University Press, Cambridge, UK, 1982).
[19] L. Susskind, arXiv:hep-th/0302219 [Search inSPIRE].
[20] M. Douglas, J. High Energy Phys. **0305**, 46 (2003).

[21] M. B. Green, J. H. Schwarz, and E. Witten, Superstring Theory I (Cambridge University Press, Cambridge, UK, 1987).
[22] M. B. Green, J. H. Schwarz, and E. Witten, Superstring Theory II, (Cambridge University Press, Cambridge, UK, 1987).
[23] J. Polchinski, String Theory I (Cambridge University Press, Cambridge, UK, 1998).
[24] J. Polchinski, String Theory II, (Cambridge University Press, Cambridge, UK, 1998).
[25] M. Pavšič, Mod. Phys. Lett. A **34**, 1950186 (2019).
[26] , H. Everett, III, Rev. Mod. Phys. **29**, 454 (1957).
[27] Y. Nomura, J. High Energy Phys. **1111**, 063 (2011).

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