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THEORETICAL ASPECTS OF SOME COLLECTIVE INSTABILITIES IN HIGH-ENERGY PARTICLE STORAGE RINGS

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ABSTRACT

After an introduction to single-particle dynamics, based on a unified Hamiltonian treatment of betatron and synchrotron oscillations, we consider two examples of collective instabilities which can limit the performances of high-energy storage rings: the transverse mode coupling instability, due to wake fields, and the incoherent beam − beam instability.

Special emphasis is placed on the localization of the interactions between particles and surrounding structures, such as the accelerating RF cavities. We derive an exact invariant for the linearized synchrotron motion and, starting from the Vlasov equation, we discuss the coherent synchro-betatron resonances caused by localized impedance. Under suitable assumptions, we show that the effect of the beam − beam kicks in electron − positron machines can be described by new diffusive terms in a "renormalized" Fokker − Planck equation and is therefore equivalent to an additional source of noise for the betatron oscillations.

CONTENTS

GENERAL INTRODUCTION

A storage ring is a machine which can control the behaviour of a large number (in the range $10^{10} - 10^{14}$) of high-energy charged particles, during a very long time (typically 10 hours), by means of electromagnetic fields. Owing to the form of the Lorentz force, energy exchanges are governed by electric fields, whereas magnetic fields are generally used to guide the particles along stable trajectories.

The importance of storage rings in high-energy physics depends on the possibility of colliding two counter-rotating beams of relativistic particles, thus attaining high values of the centre-of-mass energy: however the resulting luminosity, i.e. the counting rate of events for a process of unit cross-section, is low compared to that of fixed-target accelerators, since the density of particles in a fixed target is higher than the beam density. In order to obtain the maximum luminosity, one has to store the largest possible number of particles and to minimize the cross-section area of the beams.

Owing to collective effects, a system of many charged particles in interaction with their surroundings and with one another may become unstable when the number or the density of particles exceeds a threshold value. The study of these collective instabilities is therefore of great importance for the design of new accelerators and for the improvement of the already existing ones. Indeed, in order to optimize the performances of a given machine, one has to understand the dependence of the different instability thresholds upon physical parameters such as the particle energy, the ring radius, the beam size or the betatron and synchrotron tunes. Two main effects give rise to collective instabilities in a high-energy storage ring, namely:

i) The Beam − Environment Interaction.
It consists in the electromagnetic interaction of a beam with the structures of its environment, such as the accelerating RF cavities or any other cross-section variation of the vacuum chamber. As a consequence of the charges induced in these structures, particles generate wake fields which react on the beam, thus affecting its behaviour.

ii) The Beam − Beam Interaction.
It is the interaction between two counter-rotating beams. When a charged particle of a beam intersects a bunch of particles of the opposite beam, it experiences a collective electromagnetic force having a non-linear dependence on the particle transverse coordinates.

We shall consider two examples of collective instabilities, in electron − positron storage rings, which set an upper limit to the maximum achievable current and to the maximum attainable luminosity, respectively. The first one is a coherent single-bunch instability due to the beam − environment interaction. It is known as "transverse mode coupling instability" or "fast head − tail effect" and leads to the exponential growth of a transverse oscillation mode of the bunch, consisting in a tidy motion of the particles. The second one is an incoherent instability due to the beam − beam interaction. In this case the transverse motion of each particle may be greatly enhanced, but there is no phase correlation between the oscillations of different particles. It is worth mentioning, however, that the beam − beam interaction can give rise to a coherent instability as well.

Part 1 provides an introduction to the physics of high-energy storage rings. The starting point is a unified Hamiltonian treatment of the betatron and synchrotron oscillations in strong-focusing machines. After linearizing the equations of motion around a reference equilibrium orbit, we show that each particle can be considered as an anisotropic oscillator, characterized by a (relativistic) transverse mass $m_{\perp} = \gamma m$ and by a longitudinal mass $m_{\parallel} = \gamma^3 m$. A series of canonical

transformations allows then to decouple the three normal modes of the system. By a last transformation, leading to the corresponding action−angle variables, we obtain the Courant−Snyder betatron invariants as well as an exact invariant for the linearized synchrotron oscillations associated with a localized RF cavity. This synchrotron invariant can be useful when computing the emittance growth caused by intra-beam scattering in proton−antiproton storage rings. The stability of the betatron motion is then discussed in smooth approximation.

In the case of electron−positron machines, because of the small mass of the particles, there is considerable synchrotron radiation leading to damping of the normal modes. Moreover, quantum fluctuations of the emitted photons give rise to a random force, similar to white noise, which drives the particle oscillations: as a consequence, the phase-space distribution function satisfies the Fokker−Planck equation and, after a few damping times, it relaxes to a Gaussian steady state.

In Part 2 we consider the beam−environment interaction, focusing our attention on the transverse mode coupling instability caused by localized impedance. As we have seen, a relativistic charged particle passing through the surrounding structures of a storage ring, induces electromagnetic wake fields which react on the following particles. If the beam current is increased beyond a threshold value, this phenomenon leads to a coherent instability generally described in terms of transverse mode coupling.

Starting from the Vlasov equation for a simplified model of an electron−positron machine, we show the existence of instability stop bands at currents below threshold, which are due to the coupling between high-order and low-order dipole modes. Since the global effect of wake fields is represented by a transverse kick localized at a single point of the machine, the stop-band pattern repeats periodically (every half-integer) in the betatron tune ν_β. Denoting by $\Delta\nu_\beta = \nu_\beta \bmod 1/2$ the fractional betatron tune, the bunch may become unstable at very low currents near the resonant values $\Delta\nu_\beta = n\,\nu_s$ or $\Delta\nu_\beta = 1/2 - n\,\nu_s$, where ν_s is the synchrotron tune.

In Part 3, it is shown that the beam−beam interaction in electron−positron storage rings is equivalent to an additional source of noise for the betatron oscillations.

A white noise acting upon a non-linear oscillator causes a fast loss of coherence in its phase. This loss of coherence induces a broadening of the resonances, thus avoiding the problem of the divergent perturbative series which arises in the study of non-integrable Hamiltonian systems. A "renormalized" Fokker−Planck equation is established which contains new diffusive terms corresponding to the presence of resonances. The solution of this equation is exhibited explicitly in a simplified case. This allows an analytical approach to the problem of the incoherent beam−beam instability, which sets an upper limit to the maximum attainable luminosity in storage rings.

Part 1

UNIFIED HAMILTONIAN APPROACH TO SINGLE-PARTICLE DYNAMICS

We present an introduction to single-particle dynamics in high-energy storage rings. Starting from a unified Hamiltonian treatment of the betatron and synchrotron motion, we derive the Courant−Snyder invariants as well as an exact invariant for the linearized synchrotron oscillations associated with a localized RF cavity. The stability of the betatron motion and the effects related to synchrotron radiation in electron−positron machines are then discussed in smooth approximation. This leads to a few qualitative relations between the basic parameters of a storage ring, which will be used in Parts 2 and 3.

1.1 INTRODUCTION

The Hamiltonian formulation for synchrotron or for betatron oscillations in high-energy particle storage rings has been extensively used in the study of stability problems and resonances [1−5]. It allows a straightforward computation of invariant quantities and supplies the basis for the development of perturbation techniques. Also the effect of radiation damping in electron−positron storage rings can be studied by means of Lagrange invariants [6], which are closely related to the Hamiltonian formalism. On the other hand, synchrotron oscillations are generally discussed separately from betatron oscillations and the few existing unified approaches [7−10] tend either to be very complicated, owing to the effect of damping and non-linearities, or to oversimplify the resulting coupled equations, adopting the so-called "smooth approximation" [11].

As an introduction to single-particle dynamics in high-energy storage rings, we present a unified Hamiltonian treatment of the linearized motion around a reference equilibrium orbit. We shall consider only one RF cavity, giving rise to a longitudinal electric field localized at a single azimuth of the reference orbit. Under these assumptions, the search for the normal modes of the system, which can be carried out by means of subsequent canonical transformations, leads automatically to the betatron and synchrotron oscillations. This procedure is equivalent to the separation of variables in the Hamilton−Jacobi equation: the final transformation to action−angle variables is then a relatively simple task and, besides the well-known Courant−Snyder betatron invariants, it yields an exact invariant for the linearized synchrotron oscillations. The derivation of such an invariant has not only didactic purposes, but can be useful in connection with intra-beam scattering in strong-focusing machines [12, 13].

The content of the next three sections can be summarized in the following logical steps:

i) Using as independent variable the ratio s/c, where s is the curvilinear abscissa along the reference orbit $\mathbf{r}_0(s)$, the new Hamiltonian becomes $H_1 = -cP_s(x,y,t,p_x,p_y,-H;s/c)$. Here P_s is the covariant longitudinal component of the canonical momentum along the reference orbit, x, y, p_x and p_y are the transverse curvilinear coordinates and momenta, and H is the initial single-particle Hamiltonian.

ii) By a first canonical transformation, we can choose a reference frame moving with constant speed $v_0 \cong c$ along the equilibrium orbit, corresponding to the particle nominal energy E_0. The new Hamiltonian is then $H_2 = H_1 + H\, c/v_0 \cong H - cP_s$. It is the curvilinear analogue of the

Hamiltonian $H - \omega P_\theta$, which describes the motion in a rotating system with constant angular velocity ω. By expanding H_2 up to second-order terms, we obtain the Hamiltonian of an anisotropic oscillator, characterized by the (relativistic) transverse and longitudinal masses $m_\perp$ and $m_\parallel$, respectively.

iii) A particle with energy deviation ϵ from the nominal value E_0 will have a displaced closed orbit. By a second canonical transformation, we can decouple the radial betatron motion from the energy-dependent closed-orbit displacement, which is proportional to the so-called dispersion function D(s). The new Hamiltonian H_3 is the sum of three independent terms, corresponding to the normal modes of the system.

iv) By a last canonical transformation, we can go over to action−angle variables both for betatron and for synchrotron oscillations. The final Hamiltonian H_4 does not depend on the phases and therefore the action variables are invariants of motion. The stability of the synchrotron motion associated with a localized RF cavity is then discussed in terms of the dispersion function.

In Section 1.5, we derive the stability condition for betatron oscillations in smooth approximation, while in Section 1.6 we consider the effects associated with synchrotron radiation in electron−positron machines. This leads to a few qualitative relations between the basic parameters of a storage ring.

1.2 REFERENCE ORBIT AND SYNCHRONOUS FRAME

Expressing the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ in terms of the scalar potential Φ and of the vector potential $\mathbf{A}$

$$\mathbf{E} = -\nabla\Phi - (1/c)(\partial\mathbf{A}/\partial t), \qquad \mathbf{B} = \nabla\times\mathbf{A}, \tag{1.1}$$

the relativistic Hamiltonian of a particle with rest mass m and electric charge e, in interaction with the electromagnetic field, reads [14]

$$H(\mathbf{r},\mathbf{P},t) = \{m^2c^4 + c^2[\mathbf{P} - (e/c)\mathbf{A}(\mathbf{r},t)]^2\}^{1/2} + e\Phi(\mathbf{r},t), \tag{1.2}$$

where $\mathbf{P}$ is the canonical momentum of the particle.

The fields $\mathbf{E}$ and $\mathbf{B}$ are invariant under gauge transformations of the potentials

$$\begin{aligned} \Phi' &= \Phi - (1/c)\,(\partial\chi/\partial t), \\ \mathbf{A}' &= \mathbf{A} + \nabla\chi, \end{aligned} \tag{1.3}$$

while the Hamiltonian H and the canonical momentum $\mathbf{P}$ are not gauge invariant. Thus neither H nor $\mathbf{P}$ have a direct physical meaning: this is the main drawback of the Hamiltonian formulation. However we can choose the Coulomb gauge, in which $\Phi = 0$ and H can be identified with the mechanical energy of the particle. Then, as a consequence of Eqs. (1.1) and (1.3), the vector potential is given by

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}^{st}(\mathbf{r}) - c\int_0^t \mathbf{E}(\mathbf{r},t)\,dt. \tag{1.4}$$

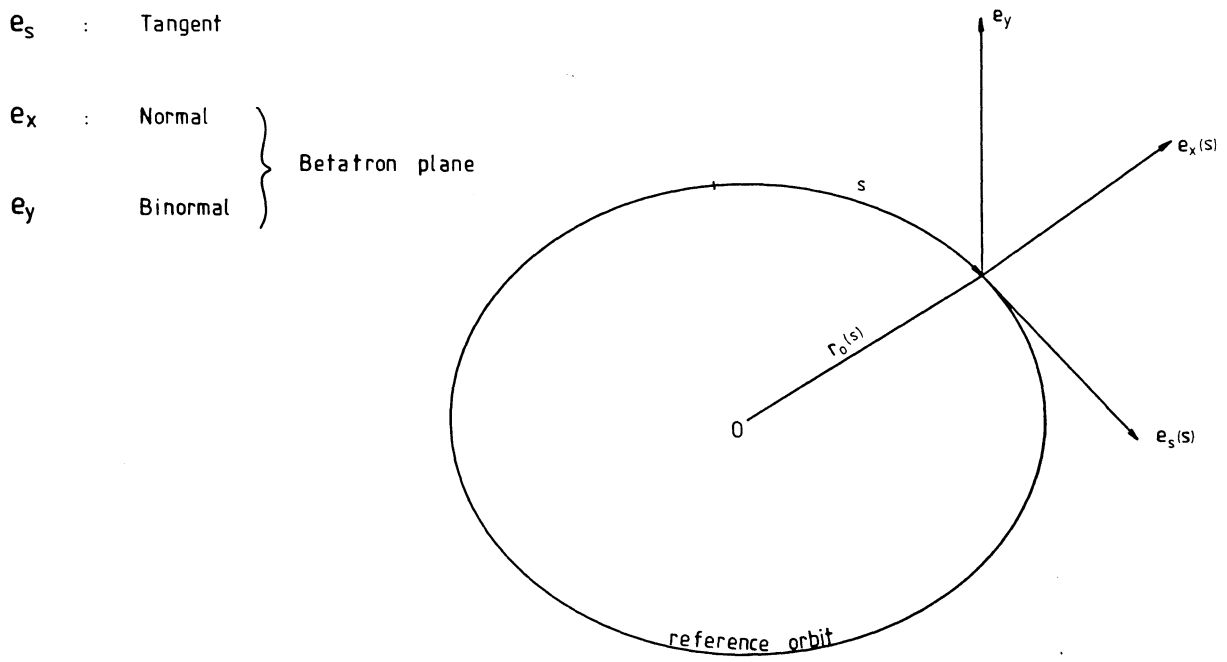


Fig. 1 Reference orbit and Frenet-Serret unit vectors.

We see that the vector potential is naturally split into a static part, corresponding to the static magnetic field, plus a time-dependent part, describing the effects of the accelerating electric field.

Instead of the particle Cartesian coordinates $\mathbf{r}$, we introduce curvilinear coordinates (x,y,s) with respect to a closed reference trajectory $\mathbf{r}_0(s)$ (see Fig. 1), lying in the plane $y = 0$. We can express $\mathbf{r}$ as a function of (x,y,s)

$$\mathbf{r}(x,y,s) = \mathbf{r}_0(s) + x\,\mathbf{e}_x(s) + y\,\mathbf{e}_y. \tag{1.5}$$

The Frenet–Serret unit vectors $\mathbf{e}_s(s)$ and $\mathbf{e}_x(s)$ satisfy the following differential relations:

$$\mathbf{r}_0'(s) = \mathbf{e}_s(s), \qquad \mathbf{e}_s'(s) = -G(s)\,\mathbf{e}_x(s), \qquad \mathbf{e}_x'(s) = G(s)\,\mathbf{e}_s(s), \tag{1.6}$$

where $G(s) = 1/\rho(s)$ is the curvature of the reference orbit and the prime denotes derivative with respect to the azimuth s.

If the vector potential has no transverse curvilinear components A_x and A_y, the corresponding canonical momenta P_x and P_y reduce to the particle mechanical momenta p_x and p_y and the Hamiltonian (1.2) reads [1, 2]

$$H(x,y,s,p_x,p_y,P_s,t) = \{m^2c^4 + c^2p_x^2 + c^2p_y^2 + [cP_s - eA_s(x,y,s,t)]^2/[1 + G(s)x]^2\}^{1/2}, \tag{1.7}$$

where A_s and P_s are the covariant longitudinal components of $\mathbf{A}$ and $\mathbf{P}$. Let us remark that, if A_s depends on the azimuth s, the assumption of vanishing A_x and A_y is not compatible with Maxwell's equations. However, since a storage-ring lattice consists of a series of long uniform magnets placed along the reference orbit, we shall consider the case in which A_s is a piecewise constant function of s, thus neglecting the end fields associated with the magnet edges.

The longitudinal motion in high-energy particle storage rings is ultrarelativistic and, instead of time t, it is convenient to choose the ratio s/c as the independent variable. Then the phase-space variables become $(x,y,t,p_x,p_y,-H)$ and the new Hamiltonian is $H_1 = -cP_s$, i.e.

$$H_1(x,y,t,p_x,p_y,-H;s/c) = -eA_s(x,y,s,t) - [1 + G(s)x] \{H^2 - c^2p_x^2 - c^2p_y^2 - m^2c^4\}^{1/2}. \quad (1.8)$$

Let us remark that, at this point, all the phase-space variables are gauge-invariant quantities.

We assume the reference trajectory to be an equilibrium orbit [15], corresponding to a nominal energy E_0. In order to discuss the linearized particle motion around this equilibrium orbit, we consider only dipole and quadrupole magnetic fields, so that B_x and B_y depend only linearly on x and y:

$$B_x = -(E_0v_0/ec)\, K(s)\, y, \qquad B_y = (E_0v_0/ec)\, [G(s) - K(s)\, x]. \quad (1.9)$$

The function $K(s)$ is known as the quadrupole strength [1, 11, 16] and, from Fig. 1, we see that a positive value of K corresponds to a transverse Lorentz force which is focusing in the vertical direction and defocusing in the radial one. The particle velocity v_0 along the equilibrium orbit is

$$v_0 = c\, \{E_0^2 - m^2c^4\}^{1/2} / E_0 \quad (1.10)$$

and in high-energy storage rings it is very close to c. Neglecting terms of third order in x and y, the static part of the vector potential A_s, corresponding to the magnetic field (1.9), is given by

$$A_s^{st}(x,y,s) = -(E_0v_0/ec)\, \{G(s)[x + G(s)x^2/2] + K(s)[y^2 - x^2]/2\}. \quad (1.11)$$

The RF cavities are generally placed along the straight sections of the machine, where the curvature $G(s)$ vanishes. Thus the covariant longitudinal component of the electric field can be identified with its projection $\mathbf{E}\cdot\mathbf{e}_s$ along the reference orbit. The longitudinal electric field E_s, generated by a single RF cavity localized at $s = 0$, will be approximated by

$$E_s(s,t) = \delta(s \bmod C)\, V(t), \quad (1.12)$$

where $\delta(s \bmod C)$ is a periodic Dirac delta-function, with period equal to the ring circumference C, and $V(t)$ is the instantaneous voltage across the cavity. Also $V(t)$ is a periodic function of time, with a period which is an integer submultiple of the particle revolution period C/v_0. This is an idealized model, which does not contain the complicated field pattern of a real accelerating cavity: in particular, we have neglected the transverse dependence of E_s.

We now perform a canonical transformation to the "synchronous frame", which is a frame moving with velocity v_0 along the reference orbit. The generating function [14], depending on the old coordinate t and the new momentum p_z, is

$$g(t,p_z;s) = (s - v_0t)\, (p_z + E_0/v_0), \quad (1.13)$$

corresponding to the transformation formulae

$$z = \partial g/\partial p_z = s - v_0t, \qquad H = -\partial g/\partial t = E_0 + v_0p_z. \quad (1.14)$$

The variable z measures the longitudinal displacement of the particle with respect to a "synchronous particle", travelling along the reference orbit at constant speed v_0, and $p_z = \varepsilon/v_0$ denotes the energy deviation from the nominal energy E_0 divided by v_0. The new Hamiltonian is

$$H_2 = H_1 + c\,\partial g/\partial s = H_1 + (E_0 + v_0 p_z)\, c/v_0. \tag{1.15}$$

It is the curvilinear analogue of the Hamiltonian $H - \omega P_\theta$, which describes the motion in a rotating system with constant angular velocity ω. From Eqs. (1.4), (1.8), (1.11) and (1.12), we obtain

$$H_2(x,y,z,p_x,p_y,p_z;s/c) = E_0\,\{Gx + [(G^2-K)x^2 + Ky^2]/2\}\, v_0/c + (E_0 + v_0 p_z)\, c/v_0$$

$$- (1 + Gx)\,[(E_0 + v_0 p_z)^2 - c^2 p_x^{\,2} - c^2 p_y^{\,2} - m^2c^4]^{1/2} + ec \int_0^{(s-z)/v_0} E_s(s,t)\, dt. \tag{1.16}$$

1.3 NORMAL MODES OF OSCILLATION AROUND THE REFERENCE ORBIT

In order to linearize the particle motion in the synchronous frame, we expand the Hamiltonian (1.16) up to second-order terms in p_x, p_y, p_z and z. In particular, using the periodicity of the oscillating RF voltage $V(t)$ appearing in Eq. (1.12), we obtain

$$\int_0^{(s-z)/v_0} E_s(s,t)\, dt = \delta(s \bmod C) \int_0^{-z/v_0} V(t)\, dt = \delta(s \bmod C)\,[-V_0\,(z/v_0) + \dot{V}_0\,(z/v_0)^2/2]. \tag{1.17}$$

Here $V_0 = V(0)$ is the voltage experienced by a synchronous particle and $\dot{V}_0 = \dot{V}(0)$ is the slope of the function $V(t)$ at $t = 0$. In this section we consider the case of a hadron storage ring, corresponding to $V_0 = 0$: then, since the radiation losses are negligible, the energy of a synchronous particle remains constant.

Expanding the square root appearing in Eq. (1.16) and using (1.10), we have

$$[(E_0 + v_0 p_z)^2 - c^2 p_x^{\,2} - c^2 p_y^{\,2} - m^2c^4]^{1/2} \cong E_0 v_0/c + c p_z - (1/2\gamma_0 m)\,[p_x^{\,2} + p_y^{\,2} + (p_z/\gamma_0)^2]\, c/v_0, \tag{1.18}$$

where $\gamma_0 = E_0/mc^2$ is the Lorentz factor corresponding to the nominal energy E_0. Then, replacing $v_0 \cong c$, the Hamiltonian H_2 becomes

$$H_2(x,y,z,p_x,p_y,p_z;s/c) = (1/2)\,\{[p_x^{\,2}/m_\perp + m_\perp c^2 (G^2-K)\, x^2] + [p_y^{\,2}/m_\perp + m_\perp c^2 K\, y^2]$$

$$+ [p_z^{\,2}/m_\parallel + (e\dot{V}_0/c)\,\delta(s \bmod C)\, z^2]\} - Gc\, x p_z. \tag{1.19}$$

Here $m_\perp = \gamma_0 m$ is the relativistic (transverse) mass of the particle, while $m_\parallel = \gamma_0^{\,3} m$ is the so-called longitudinal mass and expresses the inertia of the particle in the direction of motion. We see that H_2 is the sum of three quadratic terms, depending on the pairs of conjugate variables (x, p_x), (y, p_y) and (z, p_z), plus a curvature term $- Gc\, x p_z$ which couples radial and longitudinal oscillations. The origin of this coupling term is in the energy dependence of the equilibrium closed orbit.

In order to find the normal modes of the linearized particle motion, it is customary to decompose the radial displacement x into a betatron displacement x_β and a closed-orbit displacement, proportional to the energy deviation ϵ [16−18]. This can be accomplished in a more formal way, by means of a canonical transformation whose generating function, depending on the new coordinates x_β, z_s and the old momenta p_x, p_z, is given by

$$g(x_\beta,z_s,p_x,p_z;s) = -(x_\beta p_x + z_s p_z) + [D'(s)x_\beta - D(s)p_x/m_\perp c]\, p_z + D(s)D'(s)\, p_z^2/2m_\perp c. \quad (1.20)$$

The periodic function $D(s)$ is known as the dispersion. The transformation formulae corresponding to (1.20) are

$$\begin{aligned} x &= -\partial g/\partial p_x = x_\beta + D\, p_z/m_\perp c, & p_{x\beta} &= -\partial g/\partial x_\beta = p_x - D'\, p_z, \\ z &= -\partial g/\partial p_z = z_s - D'\, x_\beta + D\, p_{x\beta}/m_\perp c, & p_{zs} &= -\partial g/\partial z_s = p_z, \end{aligned} \quad (1.21)$$

and the new Hamiltonian is

$$H_3 = H_2 + c\,\partial g/\partial s = H_2 + (D''cx_\beta - D'p_{x\beta}/m_\perp)\, p_z + [(DD'/2)' - D'^2]\, p_z^2/m_\perp. \quad (1.22)$$

We start by assuming that the dispersion $D(s)$ and its derivative $D'(s)$ are both zero in correspondence with the azimuthal position $s = 0$ of the RF cavity

$$D(0) = 0, \qquad D'(0) = 0. \quad (1.23)$$

Then the term $\delta(s \bmod C)\, z^2$, appearing in Eq. (1.19), can be replaced by $\delta(s \bmod C)\, z_s^2$ and the new Hamiltonian reads

$$\begin{aligned} H_3 = (1/2)\, \{&[p_{x\beta}^2/m_\perp + m_\perp c^2(G^2-K)\, x_\beta^2] + [p_y^2/m_\perp + m_\perp c^2 K\, y^2] \\ &+ [1/\gamma_0^2 + (G^2-K)\, D^2 - 2GD + (DD')' - D'^2]\, p_{zs}^2/m_\perp + (e\dot{V}_0/c)\, \delta(s \bmod C)\, z_s^2\} \\ &+ [D'' + (G^2-K)\, D - G]c\, x_\beta p_{zs}. \end{aligned} \quad (1.24)$$

It follows that, in order to decouple x_β from p_{zs}, the dispersion must satisfy the well-known equation [16]

$$D''(s) + [G(s)^2 - K(s)]\, D(s) = G(s). \quad (1.25)$$

Since the curvature $G(s)$ and the quadrupole strength $K(s)$ are periodic functions of s, this is a non-homogeneous Hill's equation. The dispersion $D(s)$ is defined as the unique periodic solution of Eq. (1.25), so that the closed-orbit displacement is a single-valued function of the azimuth s. From Eq. (1.25), we also have

$$(G^2-K)\, D^2 - 2GD + (DD')' - D'^2 = -GD \quad (1.26)$$

and the Hamiltonian H_3 can be written as the sum of three independent terms

$$H_3(x_\beta,y,z_s,p_{x\beta},p_y,p_{zs};s/c) = H_x(x_\beta,p_{x\beta},s/c) + H_y(y,p_y,s/c) + H_z(z_s,p_{zs},s/c), \quad (1.27)$$

corresponding to radial and vertical betatron oscillations and to synchrotron oscillations, respectively:

$$H_x(x_\beta,p_{x\beta};s/c) = (1/2)\,\{p_{x\beta}^2/m_\perp + m_\perp c^2\,[G(s)^2 - K(s)]\,x_\beta^2\},$$

$$H_y(y,p_y;s/c) = (1/2)\,[p_y^2/m_\perp + m_\perp c^2\,K(s)\,y^2], \tag{1.28}$$

$$H_z(z_s,p_{zs},s/c) = (1/2)\,[p_{zs}^2/m_z(s) + (e\dot{V}_0/c)\,\delta(s \bmod C)\,z_s^2].$$

These are the three normal modes of oscillation associated with the linearized particle motion around the equilibrium orbit. The mass $m_z(s)$ appearing in the last equation (1.28) is given by

$$m_z(s) = \gamma_0^3 m / [1 - \gamma_0^2 G(s) D(s)]. \tag{1.29}$$

We want to emphasize that both the normal coordinates x_β and z_s, defined by Eqs. (1.21) and (1.25), differ from the physical coordinates x and z. However, since the synchrotron frequency is typically very low compared to the betatron frequency, one usually identifies the synchrotron coordinate z_s with the longitudinal displacement z. This allows a separate discussion of betatron and synchrotron oscillations. Although the average longitudinal motion of the particles (over many betatron periods) can be adequately described in this approximation, the exact transformation formulae (1.21) should be used to compute the local geometry of the beam, especially in those regions where the dispersion D or its derivative D′ are not negligible.

1.4 BETATRON AND SYNCHROTRON INVARIANTS

Although the Hamiltonian (1.27) is the sum of three decoupled terms, none of these terms is an invariant of motion, since they depend explicitly upon s. By a last canonical transformation, however, we can go over to action−angle variables for the three normal modes: the corresponding actions J_x, J_y and J_z are invariants of motion, and the equations for the phases Φ_x, Φ_y and Φ_z can be solved by quadratures.

The three Hamiltonians (1.28) are associated with pseudo-harmonic oscillators having frequencies which depend periodically on the azimuth s. Thus, according to Floquet's theory [19, 20], we consider the following type of canonical transformation:

$$\begin{aligned} x_\beta(\Phi_x,J_x,s) &= [2\,J_x\,\beta_x(s)/m_\perp c]^{1/2}\,\sin\Phi_x, \\ p_{x\beta}(\Phi_x,J_x,s) &= [2\,J_x\,m_\perp c/\beta_x(s)]^{1/2}\,\{\cos\Phi_x - \alpha_x(s)\,\sin\Phi_x\}, \end{aligned} \tag{1.30}$$

$$\begin{aligned} y(\Phi_y,J_y,s) &= [2\,J_y\,\beta_y(s)/m_\perp c]^{1/2}\,\sin\Phi_y, \\ p_y(\Phi_y,J_y,s) &= [2\,J_y\,m_\perp c/\beta_y(s)]^{1/2}\,\{\cos\Phi_y - \alpha_y(s)\,\sin\Phi_y\}, \end{aligned} \tag{1.31}$$

$$\begin{aligned} z_s(\Phi_z,J_z,s) &= [2\,J_z\,\beta_z(s)/|m_s|c]^{1/2}\,\sin\Phi_z, \\ p_{zs}(\Phi_z,J_z,s) &= [2\,J_z\,|m_s|c/\beta_z(s)]^{1/2}\,\{\cos\Phi_z - \alpha_z(s)\,\sin\Phi_z\}, \end{aligned} \tag{1.32}$$

where $\alpha_x(s)$, $\alpha_y(s)$, $\alpha_z(s)$ and $\beta_x(s)$, $\beta_y(s)$, $\beta_z(s)$ are unknown periodic functions of s. This set of equations can be considered as an "ansatz", defining the form of the canonical transformation. Then the free parameters α_x, α_y, α_z, β_x, β_y and β_z are determined by requiring that the action variables J_x, J_y and J_z be invariants of motion. It is worth noting that the beta functions β_x, β_y and β_z have the dimensions of a length, while α_x, α_y and α_z are dimensionless. The constant synchrotron mass m_s is defined so that the average value $\langle m_s/m_z(s)\rangle$ over the ring circumference is equal to one; therefore, from Eq. (1.29)

$$m_s = \gamma_0{}^3 m / [1 - (\gamma_0/\gamma_t)^2], \tag{1.33}$$

where the Lorentz factor γ_t, corresponding to the transition energy [11], depends on the so-called momentum compaction factor α_c:

$$1/\gamma_t{}^2 = \langle G(s)D(s)\rangle = \alpha_c. \tag{1.34}$$

Above transition, the synchrotron mass becomes negative: this means that the average longitudinal motion of a particle having a positive momentum p_z (and thus a positive energy deviation $\varepsilon = cp_z$ with respect to the nominal energy E_0) is in the negative z-direction. The reason is that the particle velocity cannot exceed c, while an energy increase leads to a lengthening of the equilibrium orbit. In Eqs. (1.32) we have used the absolute value of m_s, so that the synchrotron action variable J_z is positive definite.

The generating function of the transformations (1.30), (1.31) and (1.32) can be written

$$g(x_\beta,y,z_s,\Phi_x,\Phi_y,\Phi_z;s) = g_x(x_\beta,\Phi_x;s) + g_y(y,\Phi_y;s) + g_z(z_s,\Phi_z;s), \tag{1.35}$$

where

$$\begin{aligned} g_x(x_\beta,\Phi_x;s) &= [m_\perp c/2\beta_x(s)]\, x_\beta{}^2\, [\cot\Phi_x - \alpha_x(s)], \\ g_y(y,\Phi_y;s) &= [m_\perp c/2\beta_y(s)]\, y^2\, [\cot\Phi_y - \alpha_y(s)], \\ g_z(z_s,\Phi_z;s) &= [|m_s|c/2\beta_z(s)]\, z_s{}^2\, [\cot\Phi_z - \alpha_z(s)]. \end{aligned} \tag{1.36}$$

When the beta functions are constant and the α's are zero, each of these expressions corresponds to the generator of the transformation to action−angle variables for a standard harmonic oscillator [14]. Using Eqs. (1.27) and (1.35), the new Hamiltonian becomes

$$H_4 = H_3 + c\,\partial g/\partial s = K_x(\Phi_x,J_x;s/c) + K_y(\Phi_y,J_y;s/c) + K_z(\Phi_z,J_z;s/c), \tag{1.37}$$

with

$$\begin{aligned} K_x &= c\,(J_x/\beta_x)\,\{\cos^2\Phi_x - [2\alpha_x+\beta_x{}']\sin\Phi_x\cos\Phi_x + [\alpha_x\beta_x{}'-\beta_x\alpha_x{}'+\alpha_x{}^2+(G^2-K)\beta_x{}^2]\sin^2\Phi_x\}, \\ K_y &= c\,(J_y/\beta_y)\,\{\cos^2\Phi_y - [2\alpha_y+\beta_y{}']\sin\Phi_y\cos\Phi_y + [\alpha_y\beta_y{}'-\beta_y\alpha_y{}'+\alpha_y{}^2+K\beta_y{}^2]\sin^2\Phi_y\}, \\ K_z &= c\,(J_z/\beta_z)\,[|m_s|/m_z(s)]\,\{\cos^2\Phi_z - [2\alpha_z+(m_z(s)/|m_s|)\beta_z{}']\sin\Phi_z\cos\Phi_z \\ &\quad + (m_z(s)/|m_s|)\,[\alpha_z\beta_z{}'-\beta_z\alpha_z{}'+(|m_s|/m_z(s))\alpha_z{}^2 + (e\dot{V}_0/|m_s|c^3)\,\delta(s \bmod C)\,\beta_z{}^2]\sin^2\Phi_z\}. \end{aligned} \tag{1.38}$$

If the Hamiltonian (1.37) does not depend on the phases Φ_x, Φ_y and Φ_z, the action variables J_x, J_y and J_z are invariants of motion. This is true, provided the following conditions are satisfied

$$\begin{aligned}
&\alpha_x\beta_x' - \beta_x\alpha_x' + \alpha_x^2 + (G^2 - K)\,\beta_x^2 = 1, && 2\alpha_x + \beta_x' = 0,\\
&\alpha_y\beta_y' - \beta_y\alpha_y' + \alpha_y^2 + K\,\beta_y^2 = 1, && 2\alpha_y + \beta_y' = 0,\\
&\alpha_z\beta_z' - \beta_z\alpha_z' + [|m_s|/m_z(s)]\,\alpha_z^2 + (e\dot{V}_0/|m_s|c^3)\,\delta(s \bmod C)\,\beta_z^2 = |m_s|/m_z(s), && \\
& && 2\alpha_z + [m_z(s)/|m_s|]\,\beta_z' = 0.
\end{aligned} \tag{1.39}$$

In terms of the quantities

$$\gamma_x = (1 + \alpha_x^2)/\beta_x, \qquad \gamma_y = (1 + \alpha_y^2)/\beta_y, \qquad \gamma_z = (1 + \alpha_z^2)/\beta_z, \tag{1.40}$$

these conditions can be written

$$\alpha_x' = (G^2 - K)\,\beta_x - \gamma_x, \qquad \beta_x' = -2\alpha_x, \tag{1.41}$$

$$\alpha_y' = K\,\beta_y - \gamma_y, \qquad \beta_y' = -2\alpha_y, \tag{1.42}$$

$$\alpha_z' = (e\dot{V}_0/|m_s|c^3)\,\delta(s \bmod C)\,\beta_z - [|m_s|/m_z(s)]\,\gamma_z, \tag{1.43}$$

$$\beta_z' = -2\,[|m_s|/m_z(s)]\,\alpha_z. \tag{1.44}$$

By combining the first-order coupled equations (1.41) or (1.42), we obtain

$$\beta_x\beta_x''/2 - \beta_x'^2/4 + [G(s)^2 - K(s)]\,\beta_x^2 = 1, \tag{1.45}$$

$$\beta_y\beta_y''/2 - \beta_y'^2/4 + K(s)\,\beta_y^2 = 1, \tag{1.46}$$

These equations for the betatron functions $\beta_x(s)$ and $\beta_y(s)$ are well known [1, 11, 16] and will be discussed in the next section. Thanks to the presence of the delta-function, the remaining two coupled equations (1.43) and (1.44) can be solved explicitly in terms of the dispersion D(s), which appears in the mass $m_z(s)$. Then the final Hamiltonians (1.38) become

$$\begin{aligned}
&K_x(J_x;s/c) = c\,J_x/\beta_x(s), \qquad K_y(J_y;s/c) = c\,J_y/\beta_y(s),\\
&K_z(J_z;s/c) = c\,[|m_s|/m_z(s)]\,J_z/\beta_z(s)
\end{aligned} \tag{1.47}$$

and the azimuthal dependence of the phase variables is given by

$$\begin{aligned}
&\Phi_x(s) = \int ds/\beta_x(s), \qquad \Phi_y(s) = \int ds/\beta_y(s),\\
&\Phi_z(s) = \int [|m_s|/m_z(s)]\,ds/\beta_z(s).
\end{aligned} \tag{1.48}$$

To solve the coupled equations (1.43) and (1.44), we start by remarking that the periodic function $\gamma_z(s)$ defined by Eq. (1.40) satisfies the equation

$$\gamma_z'(s) = 2\,(e\dot{V}_0/|m_s|c^3)\,\alpha_z(s)\,\delta(s \bmod C) \tag{1.49}$$

and is therefore a constant, since $\gamma_z'(s) = 0$ for $s \neq 0$. Thus we have

$$\gamma_z(s) = \gamma_z, \qquad \alpha_z(0) = 0. \tag{1.50}$$

From Eq. (1.43), we see that the function $\alpha_z(s)$ has a discontinuity corresponding to the azimuthal position $s = 0$ of the RF cavity. Therefore the only possibility to have $\alpha_z(0) = 0$ is that $\alpha_z(0^-) = -\alpha_z(0^+)$, i.e. the left and the right limits of $\alpha_z(s)$, at $s = 0$, have opposite values. Thus, using Eq. (1.50) and solving Eq. (1.43) in the range $0 < s < C$, we find

$$\alpha_z(s) = \gamma_z\,\mathrm{sgn}(m_s)\,[C/2 - \int_0^s ds\, m_s/m_z(s)]. \tag{1.51}$$

From Eqs. (1.40) and (1.50), $\beta_z(s)$ can be written

$$\beta_z(s) = [1 + \alpha_z(s)^2]/\gamma_z \tag{1.52}$$

and, as a consequence of Eq. (1.51), it is a continuous function of s. At this point, we can determine the constant γ_z by inserting (1.51) back into Eq. (1.43):

$$\gamma_z\,C\,\mathrm{sgn}(m_s)\,\delta(s \bmod C) = (e\dot{V}_0/|m_s|c^3)\,\delta(s \bmod C)\,[1 + \alpha_z(s)^2]/\gamma_z \tag{1.53}$$

Since, from Eq. (1.51),

$$\alpha_z(0)^2 = \lim_{s \to 0^\pm} \alpha_z(s)^2 = (1/4)\,C^2\,\gamma_z^2, \tag{1.54}$$

we eventually obtain

$$\gamma_z = (1/C)\,\{\Delta\Phi_0^2/[1 - (\Delta\Phi_0/2)^2]\}^{1/2}, \tag{1.55}$$

where $\Delta\Phi_0$ is the synchrotron phase advance as computed in smooth approximation [see (1.113)]

$$\Delta\Phi_0^2 = (e\dot{V}_0/m_s c^3)\,C. \tag{1.56}$$

Expressions (1.51), (1.52) and (1.55) represent an explicit solution of Eqs. (1.43) and (1.44) in terms of the dispersion D(s), which enters in the definition (1.29) of the mass $m_z(s)$.

By inversion of the transformations (1.30) and (1.31), the betatron action variables J_x and J_y can be written as follows

$$\begin{aligned} J_x &= (m_\perp c/2)\,\{x_\beta^2/\beta_x(s) + \beta_x(s)\,[p_{x\beta}/m_\perp c + \alpha_x(s)\,x_\beta/\beta_x(s)]^2\}, \\ J_y &= (m_\perp c/2)\,\{y^2/\beta_y(s) + \beta_y(s)\,[p_y/m_\perp c + \alpha_y(s)\,y/\beta_y(s)]^2\}. \end{aligned} \tag{1.57}$$

They are proportional to the well-known Courant – Snyder invariants [1]. On the other hand, the synchrotron motion is usually discussed in smooth approximation (although the non-linearity of the sinusoidal RF voltage V(t) is generally taken into account [11, 16]). From Eq. (1.32), however, we obtain

$$J_z = (|m_s|c/2)\,\{z_s^2/\beta_z(s) + \beta_z(s)\,[p_{zs}/|m_s|c + \alpha_z(s)\,z_s/\beta_z(s)]^2\}, \tag{1.58}$$

where $\alpha_z(s)$ and $\beta_z(s)$ are given by Eqs. (1.51) and (1.52), respectively (see Fig. 2). Expression (1.58) provides an exact invariant for the linearized synchrotron oscillations associated with a localized RF cavity. This invariant should be properly used to build a steady-state distribution function: it can lead to more refined results when computing the emittance growth caused by intra-beam scattering in

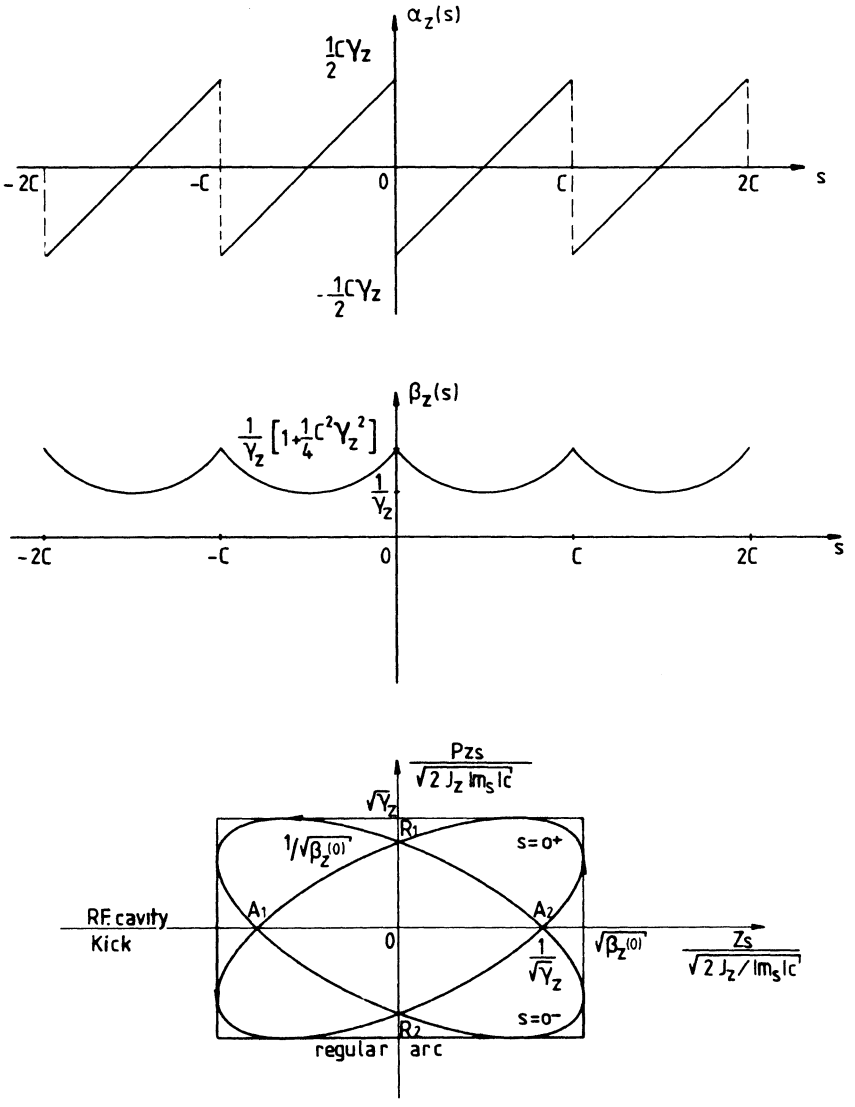


Fig. 2 Linearized synchrotron motion above transition energy in the case where G(s)D(s) = const. The first two pictures are the plots of the periodic functions $\alpha_z(s)$ and $\beta_z(s)$. The last picture represents the phase-space ellipses corresponding to the invariant $J_z(z_s,p_{zs},s)$ for $s = 0^+$ and $s = 0^-$, i.e. just after and just before the passage through the RF cavity. During the regular arc, the points A_1 and A_2 are fixed and the ellipse remains tangent to the upper and lower sides of the rectangle. During the RF kick, the points R_1 and R_2 are fixed and the ellipse remains tangent to the right and left sides of the rectangle.

hadron machines [12, 13] and is useful to discuss the thermodynamic properties of bunched beams [21, 22]. The transformation formulae (1.21) allow then to obtain the beam distribution as a function of the physical variables x, y, z, p_x, p_y and p_z.

Before concluding this section, let us discuss the stability of the linearized synchrotron oscillations. Using Eqs. (1.43) and (1.52) and denoting by $\alpha_z(C^-)$ the left limit of the discontinuous function $\alpha_z(s)$ at $s = C$, the actual synchrotron phase advance obtained by Eq. (1.48) is given by

$$\Delta\Phi_z = -\int_{0^+}^{C^-} ds\, \alpha_z'(s)/[1 + \alpha_z(s)^2] = \arctan[\alpha_z(0^+)] - \arctan[\alpha_z(C^-)]. \tag{1.59}$$

From Eqs. (1.51) and (1.55) this can be written as

$$\Delta\Phi_z = 2\arctan\{(1/2)\,\mathrm{sgn}(m_s)\,[\Delta\Phi_0{}^2/(1 - (\Delta\Phi_0/2)^2)]^{1/2}\} = 2\,\mathrm{sgn}(m_s)\arcsin(\Delta\Phi_0/2) \tag{1.60}$$

and, using Eqs. (1.33) and (1.56), we see that the stability condition for the linearized synchrotron oscillations associated with a localized RF cavity is the following [23, 24]:

$$0 < (e\dot{V}_0 C/\gamma_0 mc^3)\,(1/\gamma_0{}^2 - 1/\gamma_t{}^2) < 4. \tag{1.61}$$

Below transition energy [11], when $1/\gamma_0{}^2$ is larger than $1/\gamma_t{}^2$, this implies that the slope $\dot{V}_0$ of the RF voltage must be positive, while the opposite is true above transition. However, in both cases the absolute value of $\dot{V}_0$ cannot exceed a threshold value given by (1.61): this result is contrary to the simple-minded argument, valid in smooth approximation, according to which the slope of the RF voltage is a direct measure of the linear restoring force toward the synchronous particle. It is worth noting that, in the case where $G(s)D(s) = \text{const}$, we have $m_z(s) = m_s$ and the function $\beta_z(s)$ is parabolic in the range $0 < s < C$ (see Fig. 2). Indeed the effect of a localized RF cavity on the synchrotron motion is similar to that of a thin focusing quadrupole on the betatron motion. Therefore $\beta_z(s)$ corresponds to the betatron function for a "FOFO" lattice [11].

1.5 THE SMOOTH APPROXIMATION

In this section, we discuss the stability of the betatron motion in smooth approximation [1, 11]. The Hamilton equations associated with the betatron Hamiltonians H_x and H_y, given by Eq. (1.28), lead to the following Hill's equations for x_β and y

$$\begin{aligned} x_\beta'' + [G(s)^2 - K(s)]\, x_\beta &= 0, \\ y'' + K(s)\, y &= 0. \end{aligned} \tag{1.62}$$

If the curvature $G(s)$ and the quadrupole strength $K(s)$ are constant along the reference orbit, the stability condition for both radial and vertical betatron oscillations is

$$0 < K < G^2. \tag{1.63}$$

This corresponds to the so-called "weak-focusing" scheme: the number of betatron oscillations per ring revolution, i.e. the betatron tune, is less than one, and the quadrupole focusing in the vertical plane is accompanied by a reduction of the geometrical focusing (associated with curvature) in the horizontal plane.

However, if the absolute value of the quadrupole strength $|K(s)|$ is much larger than $G(s)^2$, it is possible to achieve stability in both betatron planes, provided $K(s)$ is a rapidly oscillating function of s. This focusing scheme is known as "alternating gradient" or "strong focusing" [1] and allows a very efficient transverse confinement of the particles around the equilibrium orbit in high-energy storage rings. We can rewrite both equations (1.62) in the form

$$y'' + g(s)\, y = 0, \tag{1.64}$$

where the focusing function $g(s)$ corresponds to $K(s)$ for the vertical oscillations and to $G(s)^2 - K(s) \cong -K(s)$ for the radial ones. As we will see, the smooth approximation consists in replacing the focusing function $g(s)$ by a constant "effective" focusing g_{eff} : this is possible if the betatron functions $\beta_x(s)$ and $\beta_y(s)$, which are the unique periodic solutions of Eqs. (1.45) and (1.46), can be replaced by their averages.

We discuss Eq. (1.64) in the case where $g(s)$ is piecewise constant and takes the two values g_1 and g_2 over subsequent intervals of length L (see Fig. 3a), namely

$$\begin{gathered} g(s) = \bar{g} + \Delta g(s), \\ \bar{g} = (g_1 + g_2)/2, \quad \Delta g(s) = \pm (g_1 - g_2)/2. \end{gathered} \tag{1.65}$$

Following an idea of Kapitza (see Ref. [25]), we look for an approximate solution of Eq. (1.64) by splitting $y(s)$ into the sum of a "slow" term $Y(s)$ and a small, oscillating term $\xi(s)$, having the same periodicity as the focusing function $g(s)$:

$$y(s) = Y(s) + \xi(s). \tag{1.66}$$

We assume that $Y(s)$ varies very little over one period $\Delta s = 2L$ of the focusing $g(s)$ and we choose $\xi(s)$ such that its average $\bar{\xi}$ over Δs be zero. Inserting Eq. (1.66) into Eq. (1.64) and using Eq. (1.65) yields

$$Y'' + \xi'' + [\bar{g} + \Delta g(s)]\,(Y + \xi) = 0. \tag{1.67}$$

Taking the average of the l.h.s. over one period Δs, we get

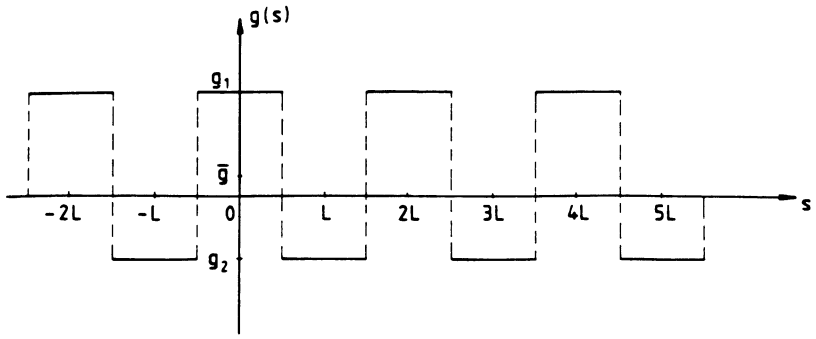
$$Y'' + \bar{g}\, Y + \overline{\Delta g\, \xi} = 0, \tag{1.68}$$

and, by subtracting Eq. (1.68) from Eq. (1.67), we have

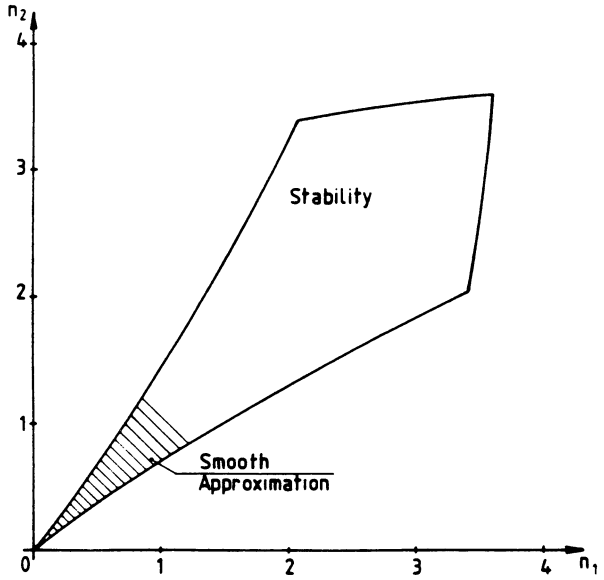
$$\xi'' + \Delta g(s)\, Y + [\, \bar{g}\, \xi + \Delta g(s)\, \xi - \overline{\Delta g\, \xi}\,] = 0. \tag{1.69}$$

Compared to $\xi'' \cong \xi/L^2$, the terms in square brackets can be neglected provided

$$|\bar{g}\, L^2| \ll 1, \quad |\Delta g\, L^2| \ll 1. \tag{1.70}$$



a) The periodic focusing function g(s).



b) The "necktie" stability region: the shaded part can be obtained in smooth approximation.

Fig. 3 Stability of the betatron motion in alternating-gradient machines.

The validity of these conditions, together with the assumption that Y(s) varies very little over one period $\Delta s = 2L$, will be checked at the end of the calculations.

Then Eq. (1.69) can be easily integrated, yielding

$$\xi(s) = \xi_0 - Y(s) \int_0^s ds' \int_0^{s'} ds'' \, \Delta g(s''). \tag{1.71}$$

The integration constant ξ_0 is chosen so that the average $\bar{\xi}$ be zero. However, since we are mainly interested in solving Eq. (1.68) for the slow term Y(s), we have to compute only $\overline{\Delta g \, \xi}$ which does not depend on ξ_0. Indeed

$$\overline{\Delta g \, \xi} = (1/12) \, (\Delta g \, L)^2 \, Y. \tag{1.72}$$

At this point, Eq. (1.68) can be written

$$Y'' + g_{eff} Y = 0, \tag{1.73}$$

where the effective focusing is given by

$$g_{eff} = \bar{g} + (\Delta g\, L)^2/12. \tag{1.74}$$

The stability condition is therefore $g_{eff} > 0$ and we see that the quadratic term, associated with the oscillating component $\Delta g(s)$ of the focusing function, contributes to stabilizing the motion even if $\bar{g}$ is negative. A similar effect occurs, for example, when the suspension point of a pendulum performs vertical oscillations at a frequency ω much higher than the pendulum frequency: for ω high enough, the unstable equilibrium position of the pendulum may become stable [25].

The smooth approximation consists in replacing the s-dependent focusing $g(s)$ of Eq. (1.64) by the constant effective focusing g_{eff} of Eq. (1.73). It is valid only if the frequency $\sqrt{g_{eff}}$, at which $Y(s)$ oscillates, is much lower than the frequency $1/2L$ of the focusing function. Thus, recalling that $g(s) = \pm K(s)$ for vertical or for radial betatron oscillations, respectively, we can combine the stability condition and the condition of validity of the smooth approximation as follows

$$0 < (\Delta g\, L)^2/12 \pm \bar{g} \ll 1/L^2. \tag{1.75}$$

Let us remark that also (1.70) follows from (1.75).

By introducing the dimensionless quantities

$$n_1 = g_1 L^2, \quad n_2 = -g_2 L^2, \tag{1.76}$$

the inequality (1.75) can be written, using (1.65),

$$0 < (n_1 + n_2)^2/48 \pm (n_1 - n_2)/2 \ll 1. \tag{1.77}$$

A graphic representation of (1.77) is given by the shaded region in Fig. 3b: the complete "necktie" stability region can be obtained by a more refined calculation [1], leading to the expressions

$$\begin{aligned} \cos \Delta\Phi_y &= \cos \sqrt{n_1} \cosh \sqrt{n_2} - [(n_1 - n_2)/2\sqrt{n_1 n_2}\,] \sin \sqrt{n_1} \sinh \sqrt{n_2}, \\ \cos \Delta\Phi_x &= \cos \sqrt{n_2} \cosh \sqrt{n_1} + [(n_1 - n_2)/2\sqrt{n_1 n_2}\,] \sin \sqrt{n_2} \sinh \sqrt{n_1}. \end{aligned} \tag{1.78}$$

Here $\Delta\Phi_y$ and $\Delta\Phi_x$ are the betatron phase advances over one "cell" of length $\Delta s = 2L$ and the stability condition follows from the condition that both the phase advances be real numbers.

In smooth approximation, from Eq. (1.73), we have

$$\Delta\Phi = 2L \sqrt{g_{eff}}. \tag{1.79}$$

If the number of cells in the ring circumference is N, the total betatron tune is

$$\nu = N\, \Delta\Phi/2\pi = (N/\pi)\, (g_{eff}\, L^2)^{1/2}. \tag{1.80}$$

Thus, from Eqs. (1.65), (1.74) and (1.76), the vertical and radial betatron tunes are given by

$$(\nu_{y,x})^2 = (N/\pi)^2 [(n_1 + n_2)^2/48 \pm (n_1 - n_2)/2]. \tag{1.81}$$

The same result can be obtained by expanding the exact formulae (1.78) up to second order in n_1 and n_2 [1]. Thus, from (1.77), the smooth approximation is valid when the betatron tunes are not too large

$$\nu \ll N/\pi. \tag{1.82}$$

Comparing Eq. (1.73) with Eqs. (1.30), (1.31) and (1.48), we see that in smooth approximation the betatron functions $\beta_x(s)$ and $\beta_y(s)$ are constant and can be written

$$\beta \cong (g_{eff})^{-1/2}. \tag{1.83}$$

This is equivalent to neglecting β' and β'' in Eqs. (1.45) and (1.46) and to replacing the alternating focusing $\pm K(s)$ by the constant effective focusing $g_{eff} = (\Delta g\, L)^2/12 \pm \bar{g}$. By applying the same procedure to Eq. (1.25), the dispersion D is given by

$$D \cong G/g_{eff} \cong 1/(R\, g_{eff}), \tag{1.84}$$

where R is the average machine radius. Since $R = N\, L/\pi$, from Eq. (1.80) we obtain the following approximate relations between the basic optical parameters of a storage ring:

$$\beta \cong R/\nu, \quad D \cong R/\nu^2, \quad \alpha_c = \langle GD \rangle \cong 1/\nu^2. \tag{1.85}$$

The stability condition associated with Eqs. (1.78) does not take into account the effect of non-linear magnetic elements such as sextupoles, which are indispensable to compensate for the natural chromaticity of the machine [3]. Owing to these non-linear elements, the transverse motion of a particle may become unstable if the oscillation amplitude exceeds a threshold value, known as the dynamic aperture of the machine [5]. Moreover, in a real storage ring there are unavoidable magnet imperfections, leading to field and gradient errors as well as to the coupling between radial and vertical oscillations. Because of field errors in the bending magnets, for example, the closed orbit is distorted and the betatron oscillations around the ideal equilibrium orbit are driven by a periodic force F(s), with period equal to the machine circumference C. Thus we have to consider the inhomogeneous Hill's equation associated with Eq. (1.64), namely

$$y'' + g(s)\, y = F(s). \tag{1.86}$$

If we introduce the normalized betatron variable q and the phase ϕ, defined as

$$q = y/\sqrt{\beta}, \qquad \phi = (1/\nu) \int ds/\beta(s), \tag{1.87}$$

using Eqs. (1.45) and (1.46) our equation (1.86) becomes

$$(d^2q/d\phi^2) + \nu^2\, q = \nu^2\, \beta^{3/2}\, F. \tag{1.88}$$

The forcing term on the r.h.s. can be regarded as a function of the new independent variable ϕ, periodic with period 2π in ϕ corresponding to the period C in s. Thus we have reduced an inhomogeneous Hill's equation to the equation of a driven harmonic oscillator. From Eq. (1.88), we

see that the motion becomes unbounded when the tune ν is integral, i.e. when the perturbing force is in resonance with the free betatron oscillations. Taking into account the other possible effects of magnet errors together with the non-linearities of the machine, it is possible to show that the betatron motion may become unstable if the following resonance condition is satisfied [26 – 28]:

$$n \nu_x + m \nu_y = p, \tag{1.89}$$

where n, m and p are integer numbers. The lowest-order resonances, corresponding to $n + m \leq 4$, are in practice the most dangerous ones [1] and have to be carefully avoided when choosing the working point of a storage ring.

1.6 DAMPING AND NOISE IN ELECTRON STORAGE RINGS

In electron – positron machines, because of the small mass of the particles, there is considerable synchrotron radiation leading to damping of the normal modes. Moreover, quantum fluctuations of the emitted photons give rise to a random force, similar to white noise, which drives the particle oscillations: as a consequence, the phase-space distribution function satisfies the Fokker – Planck equation [29 – 31] and, after a few damping times, it relaxes to a Gaussian steady state.

In the extreme relativistic limit ($\gamma \gg 1$), the radiation reaction force $\mathbf{R}$ acting on a particle moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$, orthogonal to $\mathbf{v}$, is given by

$$\mathbf{R} = -[W(t)/c^2]\, \mathbf{v}, \tag{1.90}$$

where $W(t)$ is the instantaneous radiated power which fluctuates owing to quantum effects. This expression corresponds to neglecting angular deviations of the emitted photons, of order $1/\gamma$, from the direction of $\mathbf{v}$ [32]. Equation (1.90) shows that $\mathbf{R}$ is a friction force and thus gives rise to an irreversible particle flux in momentum space. The non-Hamiltonian character of $\mathbf{R}$ remains true even at the classical level, when quantum fluctuations are neglected. This is related to the dependence of the microscopic radiation fields of each particle on its own trajectory.

Since the emission of an individual photon takes place within an azimuthal angle of order $1/\gamma$, the correlation time between two emissions is negligible with respect to the ring revolution period. Thus the stochastic variable $W(t)$ can be assumed to be Gaussian and to satisfy the following conditions [10]:

$$\begin{aligned} \overline{W(t)} &= \overline{W}, \\ \overline{W(t)W(t')} - \overline{W(t)}\,\overline{W(t')} &= \kappa\, \overline{W}\, \varepsilon_{ph}\, \delta(t-t'), \end{aligned} \tag{1.91}$$

where κ is a dimensionless constant of order unity; $\overline{W}$ is the mean instantaneous radiated power and ε_{ph} is the critical energy of the emitted photons. Their expressions, in terms of particle energy and magnetic field intensity, are [16]:

$$\begin{aligned} \overline{W} &= (2/3)\, (r_e\, \gamma\, B)^2\, c \\ \varepsilon_{ph} &= 3\, \gamma^2\, \mu\, B. \end{aligned} \tag{1.92}$$

Here $r_e = e^2/mc^2$ is the classical electron radius and $\mu = e\hbar/2mc$ the Bohr magneton.

The mean radiation reaction force leads to a progressive reduction of the six-dimensional phase-space volume occupied by the particles. By computing the rate of such volume reduction, it is possible to obtain a general relation between the damping constants of the three normal modes [33]. From Eqs. (1.9) and (1.14), the total particle energy and the magnetic field intensity are given by

$$\gamma mc^2 = E_0 + \varepsilon = E_0 + cp_z,$$
$$B^2 = B_x^2 + B_y^2 = (E_0/e)^2 \{[K(s)\, y]^2 + [G(s) - K(s)\, x]^2\}. \tag{1.93}$$

Using these expressions in Eqs. (1.92) and recalling Eq. (1.90), the mean radiation reaction force reads

$$\bar{\mathbf{R}} = -f(x,y,p_z,s)\,(\mathbf{v}/c), \tag{1.94}$$

where

$$f = \bar{W}/c = f_0(s)\,\{1 + 2\,(cp_z/E_0) - 2\,[K(s)/G(s)]\,x\} + O(x^2,y^2,p_z^2),$$
$$f_0(s) = (2/3)\, e^2\, \gamma_0^4\, G(s)^2. \tag{1.95}$$

Using the Hamiltonian H_2, given by Eq. (1.16), and denoting by $q = (x,y,z)$ and $p = (p_x,p_y,p_z)$ the curvilinear coordinates and momenta in the synchronous frame [see Eq. (1.14)], the equations of motion can be written in the following differential form:

$$dq = (dH_2/dp)\, ds/c, \qquad dp = -(dH_2/dq)\, ds/c + F\, ds/c. \tag{1.96}$$

The generalized force $F = (F_x,F_y,F_z)$ represents the non-Hamiltonian rate of change of the momenta p as a function of the curvilinear abscissa s along the reference orbit. Therefore, in the extreme relativistic limit, it is related to the radiation reaction force $\mathbf{R}$ through the geometric factor $c(dt/ds) \cong 1 + G(s)\,x$. Since we want to consider the average effect of the synchrotron radiation, we can replace $\mathbf{R}$ by $\bar{\mathbf{R}}$ and, using Eqs. (1.94) and (1.95), in linear approximation we have [6]

$$F_x = c\,(dt/ds)\,\bar{\mathbf{R}}\cdot\mathbf{e}_x = -f\,x' \cong -f_0\, cp_x/E_0,$$
$$F_y = c\,(dt/ds)\,\bar{\mathbf{R}}\cdot\mathbf{e}_y = -f\,y' \cong -f_0\, cp_y/E_0, \tag{1.97}$$
$$F_z = (dt/ds)\,\bar{\mathbf{R}}\cdot\mathbf{v} \cong -f\,[1 + G(s)\,x] \cong -f_0\,\{1 + 2\,(cp_z/E_0) + [(G^2 - 2K)/G]\,x\}.$$

The Jacobian determinant D of the infinitesimal phase-space transformation associated with the equations of motion (1.96) is given by

$$D = |\partial(q+dq,p+dp)/\partial(q,p)| = 1 + \mathrm{tr}(\partial F/\partial p)\, ds/c + O(ds^2), \tag{1.98}$$

where the first term is a consequence of Liouville's theorem and the second one represents the contribution of the non-Hamiltonian, diagonal terms of the Jacobian

$$\mathrm{tr}(\partial F/\partial p) = \partial F_x/\partial p_x + \partial F_y/\partial p_y + \partial F_z/\partial p_z. \tag{1.99}$$

To first order in ds, from Eq. (1.97), we obtain

$$D = 1 - 4\,[f_0(s)/E_0]\,ds. \tag{1.100}$$

If we assume that the average energy loss U_0 of a particle over one machine revolution is a small fraction of the nominal energy E_0, which is the case in most storage rings, the ratio of the final to the initial phase-space volume occupied by the particles, after one turn, is

$$\Omega/\Omega_0 \cong 1 - 4 \int_0^C ds\, f_0(s)/E_0 = 1 - 4\,(U_0/E_0). \tag{1.101}$$

In linear approximation, this volume contraction is the same for any region of phase space. In particular, we can consider the region delimited by a torus [34] corresponding to given values J_x, J_y and J_z of the action variables: its six-dimensional volume is proportional to the product $J_x J_y J_z$. Recalling that the action variables are quadratic functions of the particle coordinates and momenta, we can write

$$\begin{aligned} J_x &= J_{xo} \exp(-2\alpha_{x\beta}), \\ J_y &= J_{yo} \exp(-2\alpha_{y\beta}), \\ J_z &= J_{zo} \exp(-2\alpha_{zs}), \end{aligned} \tag{1.102}$$

where $\alpha_{x\beta}$, $\alpha_{y\beta}$ and α_{zs} are (small) dimensionless damping constants, describing the relative change in the amplitude of the normal modes after one machine revolution. Thus we have

$$\Omega/\Omega_0 = \exp[-2\,(\alpha_{x\beta} + \alpha_{y\beta} + \alpha_{zs})] \cong 1 - 2\,(\alpha_{x\beta} + \alpha_{y\beta} + \alpha_{zs}). \tag{1.103}$$

Comparing this expression with Eq. (1.101), we obtain the following general relation for the sum of the three damping constants:

$$\alpha_{x\beta} + \alpha_{y\beta} + \alpha_{zs} = 2\,(U_0/E_0). \tag{1.104}$$

In general, it is not possible to define normal modes of oscillation for a linear system under the effect of dissipative forces. Indeed the three matrices associated, in linear approximation, with the kinetic energy, with the potential energy and with the dissipation function, respectively, cannot be diagonalized simultaneously [14]. From Eq. (1.97), however, we see that the mean radiation reaction force couples only the radial with the longitudinal oscillations. Since the synchrotron frequency is typically very low compared to the betatron frequency, the mixing of the corresponding modes is weak and can be neglected in the adiabatic limit.

We shall now determine the damping constant of each normal mode. The calculation of the vertical damping constant $\alpha_{y\beta}$ is straightforward, while α_{zs} will be derived in smooth approximation. Then, from the general relation (1.104), we will obtain the radial betatron damping constant $\alpha_{x\beta}$. Using Eqs. (1.96) and (1.97) and expanding the Hamiltonian (1.16) as in Section 1.3, the linearized equations of motion including the effect of the mean radiation reaction force can be written

$$
\begin{aligned}
x' &= p_x/m_\perp c, & p_x' &= -(E_0/c)\,(G^2-K)\,x + G\,p_z - (f_0/E_0)\,p_x,\\
y' &= p_y/m_\perp c, & p_y' &= -(E_0/c)\,K\,y - (f_0/E_0)\,p_y,\\
z' &= cp_z/E_0\gamma_0^2 - G\,x, & p_z' &= (e/c)\,(V_0 - \dot{V}_0\,z/c)\,\delta(s \bmod C)\\
& & & - (f_0/c)\,\{1 + 2(cp_z/E_0) + [(G^2-2K)/G]\,x\}.
\end{aligned}
\tag{1.105}
$$

In the equations for the slopes x' and y', we have to retain the dependence of the relativistic (transverse) mass $m_\perp = \gamma m$ on the total particle energy and thus on p_z [see Eq. (1.93)]. The reason is that the radiation reaction force cannot affect the slope of the particle trajectory [16] since it has the same direction as the velocity v. This can be verified by combining the equations for x', p_x' and y', p_y': using the equation for p_z' and neglecting second-order terms, we obtain

$$
x'' + (eV_0/E_0)\,\delta(s \bmod C)\,x' + (G^2-K)\,x = G\,cp_z/E_0, \tag{1.106}
$$

$$
y'' + (eV_0/E_0)\,\delta(s \bmod C)\,y' + K\,y = 0. \tag{1.107}
$$

The terms proportional to the radiation reaction force f_0 cancel each other, but the longitudinal kicks experienced by the particle in the RF cavity reduce the slopes x' and y'. The effect of the RF cavity is included in the Hamiltonian (1.16) and, as a consequence of Liouville's theorem, it cannot change the phase-space volume occupied by the particles. This is confirmed by a phenomenon known as adiabatic damping [2, 11], which occurs during acceleration in proton machines. In order to have acceleration, the magnetic field is slowly increased so that the energy E_0 of the synchronous particle becomes higher [see Eq. (1.9)], while the orbit radius and the optical parameters of the machine remain constant. Then, since the action variables are adiabatic invariants, the oscillation amplitudes in x_β, y and z_s decrease, whereas those in $p_{x\beta}$, p_y and p_{zs} increase [see Eqs. (1.30)–(1.32)]. This example shows that, in general, we have to distinguish between "damping of the coordinates" and "damping of the momenta".

In electron storage rings, there is a balance between the synchronous RF energy gain eV_0 and the average radiation loss per turn U_0

$$
eV_0 = \int_0^C ds\, f_0(s) = U_0. \tag{1.108}
$$

(During acceleration this is not strictly true, but in this case the net energy gain of a particle after each revolution is generally small compared to U_0.) From the virial theorem, it follows that the average oscillation amplitudes in x_β, y and z_s are proportional, via constant factors, to the corresponding amplitudes in $p_{x\beta}$, p_y and p_{zs}. Therefore each normal mode is characterized by a unique damping constant, which is the same for the coordinate and the momentum. In particular, from Eq. (1.107) it is simple to show that the dimensionless damping constant $\alpha_{y\beta}$ of the vertical betatron motion is

$$
\alpha_{y\beta} = (1/2)\,(eV_0/E_0) = (1/2)\,(U_0/E_0). \tag{1.109}
$$

In smooth approximation, we can replace the longitudinal variables z and p_z by their average values $\langle z \rangle$ and $\langle p_z \rangle$ over one machine revolution. This amounts to neglecting the discrete nature of

the synchrotron motion, which is related to the localization of the RF cavity. By averaging the last two equations in (1.105) and using Eq. (1.108), we have

$$\langle z\rangle' = c\langle p_z\rangle/E_0\gamma_0^2 - \langle Gx\rangle,$$
$$\langle p_z\rangle' = -(e\dot{V}_0/c^2)\langle z\rangle/C - 2\langle f_0\rangle\langle p_z\rangle/E_0 - \langle f_0[(G^2-2K)/G]x\rangle/c. \tag{1.110}$$

From the usual decomposition (1.21) of the radial displacement x into a betatron displacement x_β plus a closed-orbit displacement $D\,cp_z/E_0$, we obtain the following approximate relations

$$\langle Gx\rangle \cong \langle GD\rangle c\langle p_z\rangle/E_0,$$
$$\langle f_0[(G^2-2K)/G]x\rangle \cong \langle f_0[(G^2-2K)/G]D\rangle c\langle p_z\rangle/E_0. \tag{1.111}$$

Thus, substituting these into Eq. (1.110) and combining the equations for $\langle z\rangle'$ and $\langle p_z\rangle'$, we finally get

$$\langle z\rangle'' + 2(\alpha_{zs}/C)\langle z\rangle' + (\Delta\Phi_0/C)^2\langle z\rangle = 0, \tag{1.112}$$

where the synchrotron phase advance $\Delta\Phi_0$ is given by

$$\Delta\Phi_0^2 = (e\dot{V}_0C/\gamma_0mc^3)(1/\gamma_0^2 - 1/\gamma_t^2) \tag{1.113}$$

and the dimensionless damping constant α_{zs} can be written

$$\alpha_{zs} = (1/2)(U_0/E_0)(2 + d),$$
$$d = \langle f_0[(G^2-2K)/G]D\rangle/\langle f_0\rangle = \langle DG(G^2-2K)\rangle/\langle G^2\rangle. \tag{1.114}$$

In the last equation for the parameter d, which is typically positive and small compared to unity, we have used the definition (1.95) of f_0. Having computed the vertical damping constant $\alpha_{y\beta}$ and the synchrotron one α_{zs}, from the general relation (1.104) we can deduce the radial betatron damping constant $\alpha_{x\beta}$. Indeed, using Eqs. (1.109) and (1.114), we obtain [35]

$$\alpha_{x\beta} = (1/2)(U_0/E_0)(1 - d). \tag{1.115}$$

In electron machines with combined-function alternating-gradient magnets [36], the parameter d is greater than unity and, as a consequence of (1.115), the radial betatron oscillations are antidamped. This problem can be overcome by choosing the defocusing magnets stronger than the focusing ones [33].

Let us now consider the effect produced by the quantum fluctuations of the synchrotron radiation [37]. Adopting the semi-classical approach leading to Eqs. (1.90) and (1.91), the radiation reaction force $\mathbf{R}$ contains a random component which behaves like white noise. The same is true for the generalized force F obtained by inserting $\mathbf{R}$, instead of $\bar{\mathbf{R}}$, in Eq. (1.97). Then the stochastic equations of motion (1.96) are equivalent to a Fokker–Planck equation for the phase-space particle density [10, 22, 29–31]. In linear approximation, the steady-state solution is Gaussian and depends on the action variables of the three normal modes (see Section 3.2). It is characterized by the corresponding standard deviations $\sigma_{x\beta}$, σ_y and σ_ε.

The energy spread σ_ε is related to the fluctuations of the number of photons N_{ph} emitted in one synchrotron damping time [16]. This number is roughly given by the ratio between the particle energy E_0 and the typical photon energy ε_{ph}

$$N_{ph} \cong E_0/\varepsilon_{ph}. \tag{1.116}$$

Since the number of photons emitted during a given time interval has a Poisson distribution, the r.m.s. deviation from the mean is $(N_{ph})^{1/2}$ and the corresponding energy spread can be approximated by

$$\sigma_\varepsilon \cong (N_{ph})^{1/2}\, \varepsilon_{ph} \cong (E_0\, \varepsilon_{ph})^{1/2}. \tag{1.117}$$

From Eqs. (1.92) and (1.93), we obtain

$$(\sigma_\varepsilon/E_0)^2 \cong \gamma_0^2\, \lambda\!\!\!^{-}_e/R, \tag{1.118}$$

where $\lambda\!\!\!^{-}_e = \hbar/mc$ is the Compton wavelength of the electron and R is the average machine radius.

The fluctuations of the particle energy are associated with a fluctuation of the closed-orbit radial displacement $D\, \varepsilon/E_0$. Moreover, during the emission of a single photon, the total radial displacement $x = x_\beta + D\, \varepsilon/E_0$ does not change and, as a consequence, the radial betatron motion is also affected by quantum excitation. Since the synchrotron and the betatron frequencies are very different, the resulting mean square radial spread σ_x^2 can be written as the sum of two statistically independent contributions

$$\sigma_x^2 = \sigma_{x\beta}^2 + D^2\, (\sigma_\varepsilon/E_0)^2. \tag{1.119}$$

In smooth approximation, these two contributions are of the same order [16] and, from Eqs. (1.85) and (1.118), we obtain

$$\sigma_x^2 \cong \gamma_0^2\, \lambda\!\!\!^{-}_e\, R/\nu^4. \tag{1.120}$$

The normalized radial betatron spread [see Eq. (1.87)] is therefore

$$\sigma = \sigma_{x\beta}/\sqrt{\beta_x} \cong \gamma_0\, (\lambda\!\!\!^{-}_e/\nu^3)^{1/2}. \tag{1.121}$$

The angular deviations of the emitted photons from the direction of the particle velocity lead to a very small vertical betatron spread σ_y, which is of order $1/\gamma_0$ compared to $\sigma_{x\beta}$. However, the vertical dimension of the beam in a real machine is mainly determined by the coupling between radial and vertical betatron oscillations [1, 38]. Owing to imperfections in the construction of the magnets, misalignments, solenoids or orbit offsets in the sextupoles, such a coupling is unavoidable. It can be (partially) compensated by the introduction of skew quadrupoles, which thus allow control of the vertical beam size [39].

Part 2

TRANSVERSE MODE COUPLING INSTABILITY DUE TO LOCALIZED STRUCTURES

A relativistic charged particle passing through localized structures of a storage ring, such as RF cavities, induces electromagnetic wake fields which react on the following particles. If the beam current is increased beyond a threshold value, this phenomenon leads to a fast single-bunch instability generally described in terms of transverse mode coupling.

Starting from the Vlasov equation for a simplified model of an electron−positron machine, we show the existence of instability stop bands at currents below threshold, which are due to the coupling between high-order and low-order dipole modes. Since the global effect of wake fields is represented by a transverse kick localized at a single point of the machine, the stop-band pattern repeats periodically (every half-integer) in the betatron tune ν_β. Denoting by $\Delta\nu_\beta = \nu_\beta \bmod 1/2$ the fractional betatron tune, the bunch may become unstable at very low currents near the resonant values $\Delta\nu_\beta = n\,\nu_s$ or $\Delta\nu_\beta = 1/2 - n\,\nu_s$, where ν_s is the synchrotron tune.

2.1 INTRODUCTION

Electromagnetic fields in a storage ring are the superposition of external fields, such as those due to magnets and RF cavities, and the fields generated by particles in interaction with their environment. The latter will be referred to as wake fields: they give rise to collective instabilities which set an upper limit to the maximum achievable current.

Among these collective phenomena, the so-called "fast head−tail effect" or "transverse mode coupling instability" is of great importance for large electron−positron machines such as PETRA [40−42], PEP [43] and TRISTAN [44], and it will play a crucial role for LEP at injection energy [45]. The physical mechanism driving the instability can be described as follows: the leading particles of a bunch create wake fields proportional to their transverse displacement, giving rise to a collective force on the trailing particles. Under the effect of such a force alone, the transverse oscillations of the bunch tail could grow linearly with time and this phenomenon has been discussed in relation to linacs [46]. In circular storage rings, however, synchrotron oscillations periodically exchange leading particles and trailing ones, thus stabilizing the system up to a threshold current (this is strictly true only for vanishing chromaticity [47]). Above threshold, the combined effect of wake fields and longitudinal motion leads to a coherent transverse blowup, characterized by a rise-time comparable to the synchrotron period. The stabilizing effect due to longitudinal oscillations increases with increasing frequency of head−tail exchange; thus the threshold current is proportional to the synchrotron tune. It is also proportional to particle energy, because high-energy particles are more rigid against perturbations: that is why the instability is important at injection.

Most of the analytical theories on beam stability [9, 46−54] have considered a "distributed impedance", corresponding to a collective force smeared out all along the ring. On the other hand, for particle tracking by computer simulation [55−58] the effect of wake fields is represented by localized kicks. These two models lead to different predictions when the betatron tune ν_β happens to be close to some resonant values, depending on the synchrotron tune ν_s. In this case, both the results of simulation and the conclusions drawn from a discrete two-particle model [45, 59] indicate that the

stabilizing effect due to synchrotron oscillations may fail at quite low currents, although it works again for higher currents up to nearly the same threshold as between the resonances.

The distributed impedance of a machine is associated with many different small objects in the vacuum chamber, such as bellows or pick-ups, placed at essentially random betatron phases along the ring. The localized impedance, on the contrary, depends on a few large structures whose longitudinal extent is still short compared to the betatron wavelength: in particular, the accelerating cavities of an RF station can often be considered as a single localized object. Let us remark that the concept of localized impedance also implies that the attenuation length of the travelling high-frequency wake fields is much shorter than a betatron wavelength, which is generally the case. The relative importance of distributed versus localized impedance is different for different machines and in some cases, e.g. for PETRA, the smooth approximation is a good one. However, the effect of the transverse localized impedance is enhanced by larger values of the betatron function at the RF cavities (corresponding to larger transverse displacements of the particles) and, for machines such as PEP, this represents the dominant contribution. In the design of LEP, the distributed impedance times the average beta function in the lattice is kept below the localized impedance times the average beta function in the cavities.

In order to obtain a better understanding of the coherent synchro-betatron resonances due to localized impedance, we present an analytical approach based on the Vlasov equation for a simplified model of a storage ring. We consider a single localized structure and confine our analysis to transverse dipole wake fields. In Section 2.2, we derive the Vlasov equation for our system, specifying the form of the collective force. In Section 2.3, we obtain a linear integral equation for the transverse dipole density which, in the case of a bunched beam (see Section 2.4), can be simplified by assuming linear synchrotron oscillations and by neglecting the exact azimuthal coordinate of the kicks on the scale of the bunch length. By Fourier analysis, the integral equation is then reduced to an equivalent eigenvalue problem, whose solution represents a dispersion relation for the frequencies of the dipole modes. In Section 2.5, we consider electron−positron machines: for a bunch with Gaussian distribution, the dipole motion is expanded in the so-called Hermite modes. In this basis, the infinite matrix to be diagonalized can be truncated to quite small dimensions and its eigenvalues can be computed easily.

Owing to the assumption of transverse kicks localized at a single point of the storage ring, the frequency of the dipole modes is only defined modulo the revolution frequency of the bunch. This means that high-order synchro-betatron satellites are "reflected back" toward low-order satellites. Thus, for a given range of currents, there can be coupling between modes apparently quite far apart and such a mechanism gives rise to instability stop bands at currents below threshold.

Section 2.6 presents a numerical study of the eigenvalue problem: we also introduce an empirical selection rule for mode coupling, which accounts for the stop-band pattern. Section 2.7 contains a few concluding remarks.

2.2 VLASOV EQUATION AND COLLECTIVE FORCE

In our simplified model of a storage ring, all particles perform betatron oscillations with the same frequency ω_β and are assumed to have an unperturbed synchrotron motion. At a fixed azimuth θ_0, particles experience a transverse kick at each turn which depends on their relative longitudinal positions.

The pseudo-harmonic betatron motion in the physical phase space (y, p_y) reduces to harmonic motion in the normalized phase space (q, p), with respect to a new periodic time coordinate known as quasi-time [42, 60]. Since we are only concerned with particle dynamics at the fixed azimuth θ_0, quasi-time can be replaced by time. Moreover the normalized variable q can be related to the transverse displacement y through the beta function β at θ_0 [see Eq. (1.87)]

$$q = y/\sqrt{\beta}. \tag{2.1}$$

Then, if τ and p_τ denote conjugate synchrotron variables, the single-particle equations of motion can be written as follows:

$$\dot{\tau} = \omega_s p_\tau, \qquad \dot{p}_\tau = -(\omega_s/\omega_{RF}) \sin(\omega_{RF}\tau), \tag{2.2a}$$

$$\dot{q} = \omega_\beta p, \qquad \dot{p} = -\omega_\beta q + \sqrt{\beta}\, F_c(\tau,t)/\gamma mc, \tag{2.2b}$$

where $F_c(\tau,t)$ is the transverse collective force, ω_{RF} the RF frequency, ω_s the synchrotron frequency corresponding to small-amplitude oscillations and γmc is the particle momentum in the extreme relativistic case.

Let us now introduce the distribution function $\psi(\tau,p_\tau,q,p,t)$, giving the phase-space particle density at position $\mathbf{r} = (\tau,p_\tau,q,p)$ and at time t. It satisfies the Vlasov equation [46, 47]

$$(\partial\psi/\partial t) + \dot{\tau}(\partial\psi/\partial\tau) + \dot{p}_\tau(\partial\psi/\partial p_\tau) + \dot{q}(\partial\psi/\partial q) + \dot{p}(\partial\psi/\partial p) = 0. \tag{2.3}$$

Since the collective force $F_c(\tau,t)$ is localized at a given azimuth θ_0 of the storage ring, we need a relation between time, synchrotron delay τ and azimuth θ. Denoting by ω_0 the angular revolution frequency of particles around the machine, this relation can be written

$$\theta = \theta_0 + \omega_0 (t + \tau). \tag{2.4}$$

Thus the distribution function ψ must be a periodic function of τ with period $T_0 = 2\pi/\omega_0$.

In order to specify the form of the collective force $F_c(\tau,t)$, we introduce the transverse dipole distribution $D(\tau,p_\tau,t)$, which is the first moment of ψ with respect to q:

$$D(\tau,p_\tau,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\, dp\; q\, \psi(\tau,p_\tau,q,p,t). \tag{2.5}$$

Also D must be periodic in τ with period T_0.

Using (2.4), the collective force $F_c(\theta,t)$ produced by wake fields can be expressed in the following general form:

$$F_c(\theta,t) = \int_0^{2\pi}\int_{-\infty}^{\infty} d\theta'\, dt'\; G(\theta,t,\theta',t')\, \Lambda(\theta',t'), \tag{2.6}$$

where $G(\theta,t,\theta',t')$ is a Green function specifying the properties of the storage ring, while $\Lambda(\theta,t)$ describes the collective motion of the charged particles. Since we want to confine our analysis to wake fields generated by electric dipole moments, we assume $\Lambda(\theta,t)$ to be given by the average transverse

displacement at azimuth θ and at time t, weighted by the longitudinal charge density at that point. Then, from (2.1), (2.4) and from the definition of the usual dipole density $\bar{D}(\tau,t)$

$$\bar{D}(\tau,t) = \int_{-\infty}^{\infty} dp_\tau \; D(\tau,p_\tau,t), \tag{2.7}$$

we put

$$\Lambda(\theta,t) = e\sqrt{\beta}\; \bar{D}[(\theta-\theta_0-\omega_0 t)/\omega_0,t]. \tag{2.8}$$

The Green function $G(\theta,t,\theta',t')$ is the transverse force on a particle at azimuth θ and at time t resulting from a point-like dipole excitation at azimuth θ' and at time t'. Our assumption of representing the global effect of wake fields by a kick localized at $\theta = \theta_0$ leads to the following expression for G:

$$G(\theta,t,\theta',t') = (e/R)\,\delta(\theta-\theta_0)\,\delta(\theta-\theta')\,w(t-t'), \tag{2.9}$$

where R is the average machine radius and $w(\Delta t)$ the transverse wake potential associated with the localized structure.

It is worth explaining the physical meaning of the factors appearing in Eq. (2.9). The delta-function $\delta(\theta-\theta')$ guarantees that wake fields which give rise to a kick at azimuth θ are excited at the same azimuth $\theta' = \theta$. However, since our storage-ring model is not invariant under "azimuthal translation", the Green function $G(\theta,t,\theta',t')$ cannot depend only on the difference $\theta - \theta'$ [61]: we need to include the factor $\delta(\theta-\theta_0)$ in order to describe the localization of the kicks.

The wake potential $w(\Delta t)$ can be considered as the "memory" of our localized structure and, owing to causality, it vanishes for negative values of the argument Δt. The appearance of the ring radius R is related to the usual definition of $w(\Delta t)$, having the dimensions of a force divided by the square of a charge or, in standard units, volt/(coulomb×metre). It is the integral over the ring circumference of the transverse force on a unit charge, following an exciting unit electric dipole at a fixed time delay Δt. We represent the global effect of wake fields by an azimuthally localized kick whose intensity is proportional to the azimuthally integrated force $w(\Delta t)/R$.

Inserting (2.8) and (2.9) into (2.6) and going back to the synchronous frame, which amounts to replacing θ by its expression (2.4) in terms of τ and t, we obtain our final form of the collective force $F_c(\tau,t)$

$$F_c(\tau,t) = \frac{e^2}{c}\sqrt{\beta}\,\delta[(t+\tau) \bmod T_0] \int_{-\infty}^{t} dt'\, w(t-t')\,\bar{D}(-t',t'). \tag{2.10}$$

The periodic delta-function $\delta[(t+\tau) \bmod T_0]$ comes from the factor $\delta(\theta-\theta_0)$ in (2.9) and specifies that particles undergo a localized kick at each machine revolution.

Since the collective force depends on the transverse dipole density $\bar{D}$, which in turn is defined through the distribution function ψ, the Vlasov equation (2.3) is a non-linear integro-differential equation for ψ. In the next section we will show that, without any further simplifying assumption, it is possible to derive from (2.3) a linear integral equation for the dipole density $\bar{D}$.

2.3 THE INTEGRAL EQUATION

The Vlasov equation can be used to obtain a second-order differential equation for the dipole distribution which is formally identical to that of a driven harmonic oscillator, the driving term being proportional to the collective force $F_c(\tau,t)$. Since the synchrotron motion is assumed to be known, it is possible to make use of the harmonic oscillator Green function to get a relation between D and the driving force (2.10), i.e. a linear integral equation for the transverse dipole distribution.

Starting from the definition (2.5) and using Eqs. (2.2) and (2.3), we obtain the following equation for the dipole distribution $D(\tau,p_\tau,t)$ [62]

$$\ddot{D} + \omega_\beta^2 D = \sqrt{\beta}\,\omega_\beta\,\rho(\tau,p_\tau,t)\,F_c(\tau,t)/\gamma mc, \tag{2.11}$$

where $\rho(\tau,p_\tau,t)$ is the particle distribution function in the synchrotron phase space

$$\rho(\tau,p_\tau,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq\,dp\;\psi(\tau,p_\tau,q,p,t). \tag{2.12}$$

It is a periodic function of τ with period T_0 and, from (2.3) and (2.12), it satisfies the "reduced" Vlasov equation

$$(\partial\rho/\partial t) + \dot{\tau}(\partial\rho/\partial\tau) + \dot{p}_\tau(\partial\rho/\partial p_\tau) = 0. \tag{2.13}$$

Equation (2.11) is formally identical to that of a driven harmonic oscillator, but the further dependence on the synchrotron variables τ and p_τ has to be taken into account.

Owing to (2.13), the longitudinal distribution $\rho(\tau,p_\tau,t)$ is invariant under synchrotron motion and thus can be considered a constant. Then, if we are able to express the τ-dependence of $F_c(\tau,t)$ through a suitable starting value τ^0, Eq. (2.11) can be solved by means of the harmonic oscillator Green function $(1/\omega_\beta)\sin(\omega_\beta t)$. This is indeed the case and we obtain

$$D(\tau,p_\tau,t) = (\sqrt{\beta}/\gamma mc)\,\rho(\tau,p_\tau,t)\int_{-\infty}^{t} dt'\,\sin[\omega_\beta(t-t')]\,F_c[\tau^0(\tau,p_\tau,t-t'),t'], \tag{2.14}$$

provided the invariant $\tau^0(\tau,p_\tau,t-t')$ reduces to τ when $t' = t$: therefore it is the initial longitudinal position at time t' of a particle having final synchrotron coordinates τ and p_τ at time t. The r.h.s. of Eq. (2.14) takes into account the effect of the collective force F_c at all previous times $t' \le t$ and at the corresponding longitudinal positions $\tau^0(\tau,p_\tau,t-t')$. An explicit expression for τ^0 will be given in the next section, where we consider the case of linear synchrotron oscillations. Inserting the collective force (2.10) into our solution (2.14) and integrating over p_τ, we obtain the following integral equation for $\bar{D}(\tau,t)$:

$$\bar{D}(\tau,t) = (e^2\beta/E)\int_{-\infty}^{\infty} dp_\tau\,\rho(\tau,p_\tau,t)\int_{-\infty}^{t} dt'\,\sin[\omega_\beta(t-t')]\,\delta[(t'+\tau^0)\bmod T_0]\int_0^{\infty} dx\,w(x)\,\bar{D}(x-t',t'-x), \tag{2.15}$$

where $E = \gamma mc^2$ is the relativistic particle energy and we have changed the integration variable t', in Eq. (2.10), to x.

According to our model, wake fields are excited by the dipole density $\bar{D}$ at the fixed azimuth θ_0 and, as a consequence, the r.h.s. of Eq. (2.15) depends only on $\bar{D}(x-t',t'-x)$ [see (2.4)]. Thus we can focus our attention on the transverse dipole density $\bar{D}_n(\tau)$ at $\theta = \theta_0$ after n machine revolutions: thanks to the periodicity of $\bar{D}(\tau,t)$ as a function of τ, it can be written

$$\bar{D}_n(\tau) = \bar{D}(\tau, nT_0 - \tau). \tag{2.16}$$

In order to avoid ambiguities, we assume $-T_0/2 \le \tau \le T_0/2$.

The "reduced" Vlasov equation (2.13) can be solved independently from (2.15) and we assume the existence of a steady-state longitudinal distribution $\rho(\tau,p_\tau)$. Furthermore, we neglect long-range terms in the wake potential w(x). Then, substituting (2.16) into (2.15) and carrying out the integration over t', we obtain an equation for $\bar{D}_n(\tau)$

$$\bar{D}_n(\tau) = (e^2\beta/E) \int_{-\infty}^{\infty} dp_\tau\, \rho(\tau,p_\tau) \sum_{k=0}^{\infty} \frac{\sin[\omega_\beta(kT_0+\tau_k^0-\tau)]}{|1-[\partial\tau^0(\tau,p_\tau,t)/\partial t]_{t=kT_0+\tau_k^0-\tau}|} \int_0^{\infty} dx\, w(x)\, \bar{D}_{n-k}(x+\tau_k^0), \tag{2.17}$$

where, because of the delta-function in Eq. (2.15), τ_k^0 is implicitly defined as follows:

$$\tau_k^0(\tau,p_\tau) = \tau^0(\tau,p_\tau,kT_0+\tau_k^0-\tau). \tag{2.18}$$

This happens because the arrival time of a particle at $\theta = \theta_0$ depends on its longitudinal position.

By a proper choice of the synchrotron distribution ρ and of the invariant τ^0, the integral equation (2.17) can also be used to discuss the transverse effects of a localized impedance on a coasting beam.

2.4 BUNCHED BEAMS AND FOURIER ANALYSIS

To simplify the analysis in the case of a bunched beam, we will assume that particles perform harmonic synchrotron oscillations. Then the invariant τ^0, appearing in Eqs. (2.14) and (2.18), reads

$$\tau^0(\tau,p_\tau,t-t') = \tau\cos[\omega_s(t-t')] - p_\tau \sin[\omega_s(t-t')]. \tag{2.19}$$

Up to now we have always considered periodic functions of τ with period T_0; nevertheless τ^0 is not periodic in τ. However, Eq. (2.17) contains the longitudinal distribution $\rho(\tau,p_\tau)$ which, for bunch lengths σ_t much shorter than T_0, is sharply peaked around $\tau = 0$ and $p_\tau = 0$. Thus the lack of periodicity in τ has no practical consequence.

From Eqs. (2.18) and (2.19), we see that τ, p_τ and τ_k^0 are of the same order of magnitude, given by the bunch length σ_t. Since, for a normal storage ring, both betatron and synchrotron phase advances are completely negligible over the length of a bunch, this allows considerable simplification of (2.17) and (2.18), yielding our final integral equation

$$\bar{D}_n(\tau) = (e^2\beta/E) \int_{-\infty}^{\infty} dp_\tau\, \rho(\tau,p_\tau) \sum_{k=0}^{\infty} \sin(k\omega_\beta T_0) \int_0^{\infty} dx\, w(x)\, \bar{D}_{n-k}(x+\tau_k), \tag{2.20}$$

where

$$\tau_k(\tau,p_\tau) = \tau \cos(k\omega_s T_0) - p_\tau \sin(k\omega_s T_0). \tag{2.21}$$

By Fourier analysis, Eq. (2.20) can be reduced to an equivalent eigenvalue problem. As we will see in the following, this procedure has two main advantages: it leads to a direct interpretation of the instability in terms of transverse mode coupling and allows a numerical solution based on standard computer routines for matrix diagonalization.

We start by defining the Fourier transform $\tilde{D}(\omega)$, with respect to time $t = nT_0 - \tau$ [see Eq. (2.16)], of the dipole density $\bar{D}_n(\tau)$ at the azimuth θ_0 after n machine revolutions

$$\tilde{D}(\omega) = \sum_{n=-\infty}^{\infty} \int_{-T_0/2}^{T_0/2} d\tau \, \exp[-i\omega(nT_0-\tau)] \, \bar{D}_n(\tau). \tag{2.22}$$

We also introduce the Fourier transform $Z_\perp(\omega)$ of the transverse wake potential w(t). This is generally known as the transverse impedance, having the dimensions of ohm/metre, and is defined as

$$Z_\perp(\omega) = -i \int_{-\infty}^{\infty} dt \, \exp(-i\omega t) \, w(t). \tag{2.23}$$

Let us now express the betatron and synchrotron frequencies in units of the revolution frequency ω_0: thus $\nu_\beta = \omega_\beta/\omega_0$ is the betatron tune and $\nu_s = \omega_s/\omega_0$ the synchrotron tune. We can also write the frequency ω (which is a complex number) as an integer multiple of ω_0 plus a fractional tune ν, whose real part is assumed to be in the range $[-1/2, 1/2]$

$$\omega = (\ell+\nu)\,\omega_0\,, \qquad |\,\mathrm{Re}(\nu)\,| \le 1/2, \tag{2.24}$$

where ℓ is an integer.

Since the particles undergo periodic kicks at time intervals T_0, the Fourier transform $\tilde{D}(\omega)$ of the dipole density contains only discrete spectral lines. As we will see, it can be obtained as a superposition of "dipole modes" characterized by a regular pattern of spectral lines with frequency intervals ω_0. Then it is convenient to use the decomposition (2.24) in order to define the following two functions:

$$\begin{aligned} D_\ell(\nu) &= \tilde{D}[(\ell+\nu)\,\omega_0], \\ Z_m(\nu) &= Z_\perp[(m+\nu)\,\omega_0]. \end{aligned} \tag{2.25}$$

Indeed a Fourier analysis of the integral equation (2.20) yields the following eigenvalue problem:

$$D_\ell(\nu) = \sum_{m=-\infty}^{\infty} A_{\ell m}(\nu)\, D_m(\nu), \tag{2.26}$$

where the matrix $A_{\ell m}(\nu)$ is given by [62]

$$A_{\ell m}(\nu) = (ie^2\beta\omega_0/E)\, Z_m(\nu) \sum_{k=0}^{\infty} \sin(2\pi k\nu_\beta) \exp(-i2\pi k\nu)$$

$$\int_0^\infty dJ\, \rho(J)\, J_0(\omega_0\{2J\,[(\ell+\nu)^2 + (m+\nu)^2 - 2(\ell+\nu)(m+\nu)\cos(2\pi k\nu_s)]\}^{1/2}). \qquad (2.27)$$

In this formula, $J_0(x)$ is the Bessel function of the first kind of order 0 and $\rho(J)$ is the steady-state longitudinal distribution expressed as a function of the synchrotron action variable J, which is defined by $J = (\tau^2 + p_\tau^2)/2$. Since we consider bunches whose length in time σ_t is much shorter than T_0, corresponding to a longitudinal distribution $\rho(J)$ strongly localized near $J = 0$, the limits of integration over J have been extended from 0 to ∞. Equations (2.26) and (2.27) are similar to Eqs. (13) and (18)–(20) in Ref. [52] and to Eqs. (77)–(79) in Ref. [9], where a distributed impedance was assumed. However, instead of an integral equation for $\tilde{D}(\omega)$, we obtain the eigenvalue problem (2.26) relating the discrete spectral lines $D_\ell(\nu)$. Moreover, our kernel $A_{\ell m}(\nu)$ contains a sum over k (contrasted with an integral over time) which leads to the resonant stop bands discussed in Section 2.1.

Equation (2.26) can be solved only for a discrete set of values of the fractional tune ν, fulfilling the condition

$$\det\{\delta_{\ell m} - A_{\ell m}(\nu)\} = 0. \qquad (2.28)$$

This represents a dispersion relation for the frequency spectrum of the transverse dipole modes $D_\ell(\nu)$, obtained as corresponding eigenvectors of Eq. (2.26).

If condition (2.28) is satisfied by a complex value of ν with negative imaginary part, the system is unstable. In the following, we shall study the fractional tune of each mode as a function of the bunch current I_b: when two of these modes get coupled, i.e. when they happen to have the same fractional tune ν, a further increase of current leads to instability.

2.5 GAUSSIAN BUNCHES

In order to compute the matrix elements $A_{\ell m}(\nu)$ given by Eq. (2.27), we have to specify the impedance $Z_m(\nu)$ and the longitudinal distribution $\rho(J)$. Since we want to consider the case of electron–positron storage rings, let us start by assuming a Gaussian distribution with standard deviation σ_t in the synchrotron phase space (τ, p_τ). This distribution will be normalized to the total number of particles per bunch N_b and, owing to the definition of J, it becomes

$$\rho(J) = (N_b/2\pi\sigma_t^2) \exp(-J/\sigma_t^2). \qquad (2.29)$$

Then the integration over the action variable J, appearing in Eq. (2.27), can be carried out explicitly and the matrix $A_{\ell m}(\nu)$ reduces to a very simple form, namely [62]:

$$A_{\ell m}(\nu) = K\, Z_m(\nu) \sum_{n=0}^{\infty} C_n(\nu)\, H_n[\sigma(\ell+\nu)]\, H_n[\sigma(m+\nu)], \qquad (2.30)$$

where $\sigma = \omega_0 \sigma_t$ is the angular bunch length and the parameter K, given by

$$K = i\,(N_b e^2 \beta \omega_0 / 2\pi E) = i\, e I_b \beta / E, \tag{2.31}$$

is proportional to the bunch current $I_b = N_b e / T_0$.

The functions $H_n[\sigma(\ell + \nu)]$ represent the so-called Hermite modes, defined by

$$H_n(x) = (n!)^{-1/2} \exp(-x^2/2)\, (x/\sqrt{2})^n. \tag{2.32}$$

Indeed their Fourier transform corresponds to a Gaussian times a Hermite polynomial of order n.

The coefficients $C_n(\nu)$ are given by

$$C_n(\nu) = \sum_{k=0}^{\infty} \sin(2\pi k \nu_\beta) \exp(-i 2\pi k \nu)\, [2 \cos(2\pi k \nu_s)]^n. \tag{2.33}$$

The infinite sum over k takes into account the effect of the localized kicks at all previous turns. As we will see, these coefficients contain "small denominators" corresponding to synchrotron satellites of the betatron tune $\nu = \nu_\beta + m \nu_s$ and their periodicity leads to the instability stop-band pattern described in Section 2.1.

Since the Hermite modes $H_n[\sigma(\ell + \nu)]$ form a complete set, we can expand the eigenvectors $D_\ell(\nu)$ of Eq. (2.26) in this basis

$$D_\ell(\nu) = \sum_{n=0}^{\infty} \alpha_n H_n[\sigma(\ell + \nu)]. \tag{2.34}$$

Then, inserting (2.30) and (2.34) into (2.26), we get our final eigenvalue problem for the expansion coefficients α_n:

$$\alpha_n = K\, C_n(\nu) \sum_{m=0}^{\infty} M_{nm}(\nu)\, \alpha_m, \tag{2.35}$$

where $M_{nm}(\nu)$ is given by

$$M_{nm}(\nu) = \sum_{\ell=-\infty}^{\infty} H_n[\sigma(\ell + \nu)]\, Z_\ell(\nu)\, H_m[\sigma(\ell + \nu)]. \tag{2.36}$$

This represents the "impedance matrix" in the basis of the Hermite modes. Equations (2.35) and (2.36) are again similar to those obtained for a distributed impedance (see in particular Ref. [9]), but our assumption of localized kicks leads to the appearance of the resonant coefficients $C_n(\nu)$.

The eigenvalue problem (2.35) is associated with the following compatibility condition:

$$\det \{\delta_{nm} - K\, C_n(\nu)\, M_{nm}(\nu)\} = 0. \tag{2.37}$$

This is a dispersion relation giving the fractional tune ν as a function of the parameter K, which is proportional to the bunch current I_b [see (2.31)].

Let us remark that our problem has two main "inputs", namely the longitudinal distribution $\rho(J)$ and the impedance $Z_\perp(\omega)$. The choice of a Gaussian distribution leads to a natural basis given by the Hermite modes [see (2.30)]. The impedance matrix $M_{nm}(\nu)$ in this basis, multiplied by the resonant coefficients $C_n(\nu)$, characterizes the final eigenvalue problem.

To investigate the effects of localized structures on a short bunch of electrons or positrons, we assume the following model impedance:

$$Z_\perp(\omega) = R_\perp (\omega_r/\omega) / [1 - i\, Q(\omega/\omega_r - \omega_r/\omega)]. \tag{2.38}$$

For a quality factor $Q \cong 1$, this represents the transverse impedance of a broad-band resonator with resonant frequency ω_r and peak value $R_\perp$. These parameters are chosen so as to obtain a best fit of the realistic impedance computed by numerical methods [63, 64]. The resonant frequency ω_r turns out to be of the order of the pipe cut-off frequency, i.e. the minimum frequency required for field propagation inside the pipe, and the ratio ω_r/ω_0 is generally very much larger than unity. Therefore the impedance matrix elements $M_{nm}(\nu)$, defined by (2.36), become independent of the fractional tune ν and can be computed analytically with very good accuracy [65]. They are proportional to $(n!\, m!\, 2^{n+m})^{-1/2}$ and decrease very quickly for large values of n and m. Thus we can truncate the infinite-dimensional eigenvalue problem (2.35) to a reasonable size (usually a matrix dimension of a few tens is sufficient), relying upon numerical methods for its solution.

Before concluding this section, let us come back to the resonant coefficients $C_n(\nu)$. Expressing the trigonometric functions $\sin(2\pi k\nu_\beta)$ and $\cos(2\pi k\nu_s)$ by means of complex exponentials and using the Newton binomial formula, Eq. (2.33) becomes

$$C_n(\nu) = (1/2i) \sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{\infty} \{\exp[-i2\pi k(\nu + (n-2m)\nu_s - \nu_\beta)] - \exp[-i2\pi k(\nu + (n-2m)\nu_s + \nu_\beta)]\}, \tag{2.39}$$

where $\binom{n}{m}$ are binomial coefficients. Thus the infinite sum over k reduces to a geometric series: in the convergence half-plane $\mathrm{Im}(\nu) < 0$, we obtain the following result

$$C_n(\nu) = (1/2) \sum_{m=0}^{n} \binom{n}{m} \frac{\sin\{2\pi[\nu_\beta - (n-2m)\nu_s]\}}{\cos(2\pi\nu) - \cos\{2\pi[\nu_\beta - (n-2m)\nu_s]\}}. \tag{2.40}$$

The form of this expression is a direct consequence of representing the global effect of wake fields by localized kicks and a careful discussion of its implications is given below.

We start by remarking that the coefficients $C_n(\nu)$ depend on the fractional coherent tune ν only through $\cos(2\pi\nu)$ and therefore $C_n(\nu)$ is equal to $C_n(-\nu)$. Since we have assumed a broad-band impedance, the dependence of the matrix elements M_{nm} on ν is negligible. Thus, at a given current I_b, if $\nu = \nu_0$ is a solution of the dispersion relation (2.37), the opposite value $\nu = -\nu_0$ is a solution as well. Since a negative imaginary part of the fractional tune ν corresponds to bunch instability, we see that a stable situation is only possible when all the solutions of the dispersion relation are real numbers. Moreover, from (2.24), we can consider $\mathrm{Re}(\nu)$ to be in the range [0, 1/2].

If the betatron tune ν_β is replaced by $\nu_\beta + 1/2$, the coefficients $C_n(\nu)$ remain the same provided ν is replaced by $1/2 - \nu$. Since bunch instabilities depend only on the imaginary part of ν, this means that the behaviour of our system at a given current repeats periodically, every half-integer in ν_β. Thus let us choose also ν_β in the range $[0, 1/2]$.

Expression (2.40) shows that, if ν approaches a synchro-betatron satellite $\nu_\beta + m\,\nu_s$, then all the coefficients $C_n(\nu)$ with n having the same parity as m and $n \geq |m|$ become unbounded. From the dispersion relation (2.37), we see that a resonant value for these coefficients is only compatible with a vanishing value of the parameter K, proportional to the bunch current I_b. Thus, in the limit of vanishing current, the orthogonal dipole modes are characterized by a fractional coherent tune ν approaching the synchrotron side bands $\nu_\beta + m\,\nu_s$: in the following, we will refer to each of these modes as "mode of order m" or simply "mode m".

Let us consider the case $\nu \to \nu_\beta$, i.e. the case of mode 0 for vanishing current. From (2.40), it follows that all the odd coefficients $C_{2n+1}(\nu_\beta)$ have a finite value, whereas the even coefficients $C_{2n}(\nu_\beta)$ diverge. We have

$$\lim_{\nu \to \nu_\beta} [C_{2n}(\nu)/C_0(\nu)] = \binom{2n}{n}. \tag{2.41}$$

If we factorize $C_0(\nu)$ in the product $C_n(\nu)\,M_{nm}$ appearing in the eigenvalue problem (2.35), in the limit $\nu \to \nu_\beta$ the odd rows of the impedance matrix will be multiplied by zero and the even rows by the binomial coefficients $\binom{2n}{n}$. Then, for vanishing current, the orthogonal mode 0 is a mixing of all the even Hermite modes H_{2n} starting from H_0. By similar considerations we conclude that, always for vanishing current, each orthogonal mode m is a mixing of all the Hermite modes with the same parity starting from $H_{|m|}$.

Since the denominators in expression (2.40) contain only cosines and we have chosen $\mathrm{Re}(\nu)$ in the range $[0, 1/2]$, any synchro-betatron satellite $\nu = \nu_\beta + m\,\nu_s$ falling out of this interval is "reflected back" at the edges 0 or 1/2. Thus, in the limit of vanishing current, the dipole modes have a fractional tune ν approaching the synchrotron side bands $\nu_\beta + m\,\nu_s$: if a side band falls out of the interval $[0, 1/2]$, it is reflected back into this range.

2.6 RESULTS OF THE NUMERICAL ANALYSIS

The conclusions presented in this section have been drawn from a numerical study of the linear system (2.35). These conclusions therefore have an empirical character, although their simplicity and regularity suggest a deeper connection with the structure of the resonant coefficients $C_n(\nu)$, given by (2.40), and thus a more general validity.

Let us recall that Eq. (2.35) represents our final eigenvalue problem for the expansion coefficients α_n, expressing the transverse dipole modes in the basis of the Hermite modes. The parameter K, defined by (2.31), is proportional to the bunch current I_b and the matrix M_{nm} can be considered independent of the fractional tune ν because we have assumed a broad-band impedance.

The first result of the numerical analysis can be stated as follows:

i) When the fractional betatron tune $\Delta\nu_\beta = \nu_\beta \bmod 1/2$ is close to the resonant values $\Delta\nu_\beta = n\,\nu_s$ or $\Delta\nu_\beta = 1/2 - n\,\nu_s$, an instability stop band appears at very low currents.

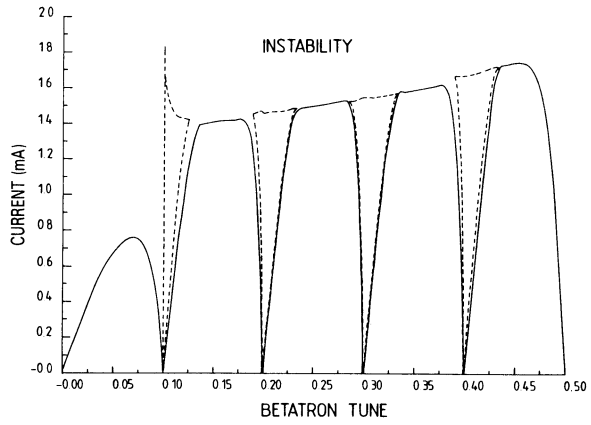


Fig. 4 Stop-band pattern for a synchrotron tune $\nu_s = 1/10$. The regions delimited by solid or dashed lines correspond to bunch stability.

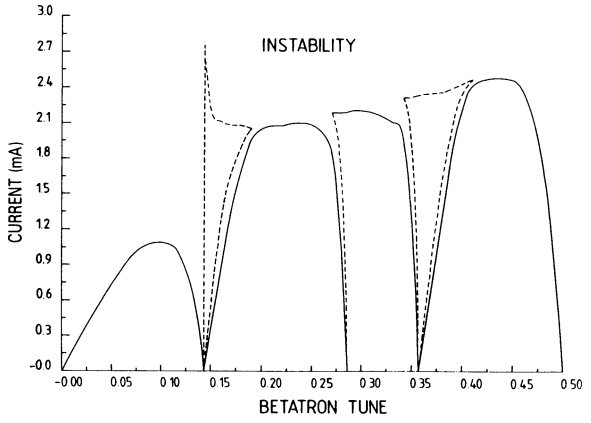


Fig. 5 Stop-band pattern for a synchrotron tune $\nu_s = 1/7$.

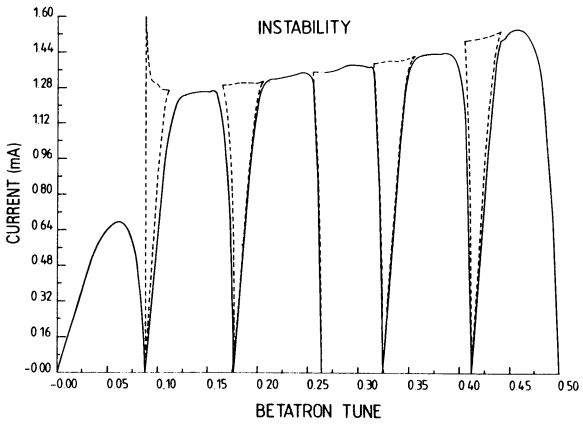


Fig. 6 Stop-band pattern for a synchrotron tune $\nu_s = 11/125$.

If the current is increased beyond this stop band, the system is stable again, but it becomes definitively unstable above a threshold value I_{th} (see Figs. 4, 5, 6 and Table 1).

The threshold current is weakly dependent on the betatron tune and can be approximated by the same formula obtained for a distributed impedance, namely [41, 42, 45]

$$I_{th} = 2\pi \frac{\nu_s (E/e)}{\beta (\omega_r/\omega_0) (R_\perp/Q)} F(\sigma). \qquad (2.42)$$

The form factor $F(\sigma)$ is minimum when the bunch length is of the order of the beam-pipe radius (which is true for LEP) and in this case it is near unity.

According to the discussion following (2.40), the behaviour of our system at a given current repeats periodically every half-integer in ν_β. Thus let us choose ν_β in the range [0, 1/2]. As already remarked in the last section, if a synchrotron side band $\nu_\beta + m\,\nu_s$ is out of the interval [0, 1/2], it is "reflected back" into this range. If we choose a sufficiently high mode number |m|, depending on the values of ν_β and ν_s, it will happen that $\nu_\beta + m\,\nu_s$ is reflected back and falls between modes 0 and -1 (i.e. between ν_β and $\nu_\beta - \nu_s$). This phenomenon accounts for the second result of the numerical analysis:

ii) The instability stop bands are due to the coupling of mode 0 or mode -1 with a higher-order mode.

Table 1: Parameter List

In all numerical calculations we have assumed the following set of parameter values (suitable for LEP):

Energy (at injection)	$E = 20$ GeV
Revolution angular frequency	$\omega_0 = 2\pi$ 11245.5 Hz
Bunch length ($\ell = \sigma_t c$)	$\ell = 2$ cm
Betatron function in the RF cavities	$\beta = 40$ m
Synchrotron damping time	$\tau_s = 0.202$ s
Betatron damping time	$\tau_\beta = 0.404$ s

Broad-band resonator model:

Localized transverse impedance	$R_\perp = 1920$ kΩ/m
Resonant frequency	$\omega_r = 10^5\ \omega_0$
Quality factor	$Q = 1$

The case of the coupling between mode 0 and mode 4 is illustrated in Fig. 7. Starting from current $I_b = 0$ and increasing I_b, mode 0 moves downward until it reaches mode 4 (whose fractional tune ν changes very little with current). After the coupling and the instability stop band, where ν becomes a complex number, mode 0 "emerges" and moves again downward: when it couples to mode -1, the ultimate threshold I_{th} is attained. We should point out that the role of modes 0 and -1 in the statement (ii) depends on our choice of a bunch length σ suitable for LEP (see Table 1). Indeed the impedance matrix M_{nm} is a function of the bunch length and thus, changing σ, modes 0 and -1 might be replaced by different modes, such as -1 and -2 or -2 and -3 [59].

The statement (ii) does not specify if it is mode 0 or mode -1 which couples to a higher-order mode. This is an important question because, as shown in Fig. 8, the appearance of a stop band for a given value of ν_β depends on the answer to this question. From the results of the numerical analysis, we have established the following "selection rule", accounting for the instability stop-band pattern:

iii) Only modes with the same parity can couple giving rise to an instability stop band.

It is worth explaining what is the parity of a mode and to clarify the meaning of this selection rule. In the last section we showed that, for vanishing current, the eigenvectors of the linear system (2.26) do not reduce to pure Hermite modes, but rather to a mixing of an infinite number of them. However, owing to the structure of the resonant coefficients $C_n(\nu)$, which is a direct consequence of our assumption of localized kicks, each eigenvector can only contain even Hermite modes or odd ones, thus allowing a definition of its parity. Our selection rule refers to this "vanishing-current parity",

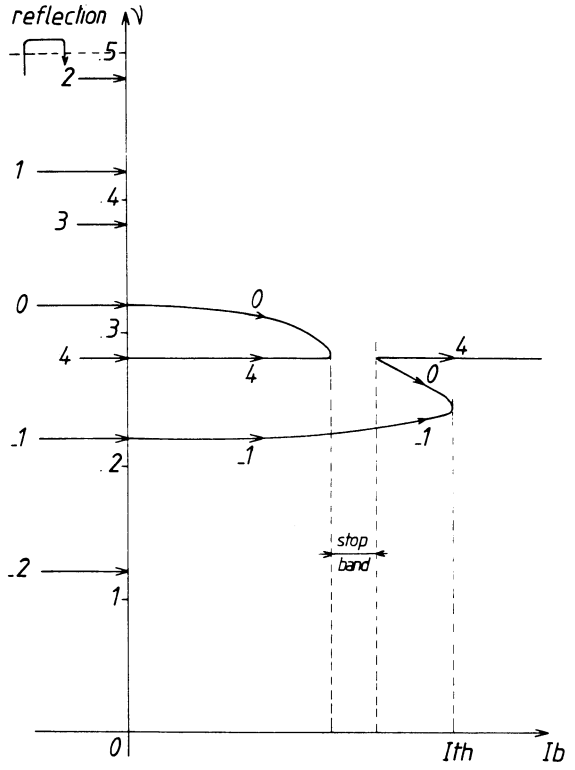


Fig. 7 Fractional coherent tune ν of modes 0, -1, and 4 as a function of the bunch current I_b, for $\nu_s = 0.1$ and $\nu_\beta = 0.32$.

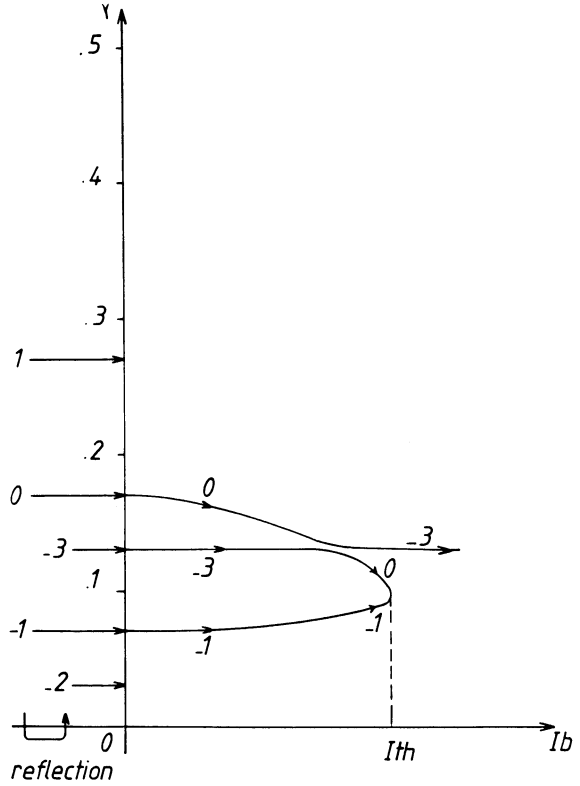


Fig. 8 Fractional coherent tune ν of modes 0, -1, and -3 as a function of the bunch current I_b, for $\nu_s = 0.1$ and $\nu_\beta = 0.17$.

although, at finite currents, the eigenvectors become a mixing of the whole set of Hermite modes. The numbers marked near each branch in Figs. 7 and 8 express only the dominant Hermite component at that current.

In order to illustrate the role of the selection rule (iii) in explaining the instability stop-band pattern, let us compare Figs. 6 and 9. In Fig. 9 we have indicated the coupling modes giving rise to each instability gap. When the betatron tune ν_β is in the ranges $[0, \nu_s/2]$ or $[\nu_s/2, \nu_s]$, we find only an ultimate threshold due to the coupling of modes 0 or -1, respectively, with "themselves". This

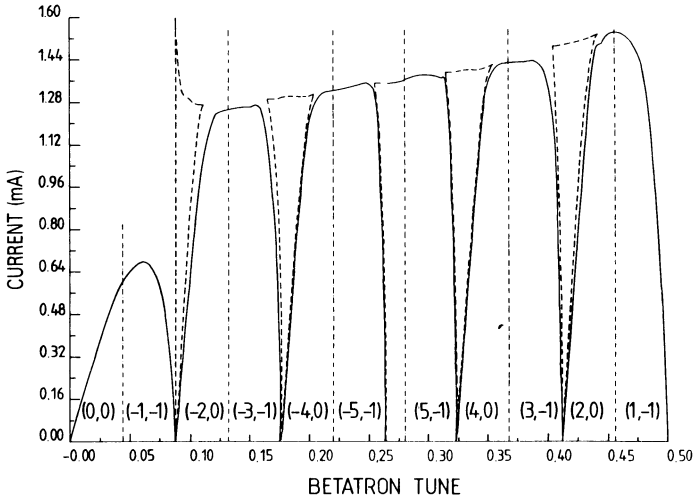


Fig. 9 Stop-band pattern for $\nu_s = 11/125$. The coupling modes giving rise to the instability stop bands are marked.

means that the fractional tune ν at threshold current is 0 and it is a consequence of the symmetry $C_n(\nu) = C_n(-\nu)$ of the resonant coefficients (2.40). For increasing values of ν_β, we see that modes -2, -3, -4 and -5, reflected back at $\nu = 0$, are alternatively coupled with modes 0 or -1, according to their parity. A similar pattern is obtained starting from $\nu_\beta = 1/2$ and considering descending values. When ν_β is in the range $[(1-\nu_s)/2, 1/2]$, we find only an ultimate threshold due to the coupling between modes -1 and 1. Then, for decreasing values of ν_β, modes 2, 3, 4 and 5, which are reflected back at $\nu = 1/2$, get coupled alternatively with modes 0 or -1, according to our selection rule (iii). The reason why we do not consider higher-order modes will be explained in the following.

We should remark that, in the instability stop bands due to the coupling between even modes and mode 0, the transverse dimensions of the bunch and the oscillation amplitude of its barycentre are comparable. On the contrary, when the instability is due to the coupling between odd modes and mode -1, the main blowup concerns the transverse dimensions of the bunch, without a correspondingly large barycentre displacement. This phenomenon can be understood by remembering that the orthogonal mode 0 has a dominant component given by the Hermite mode H_0. Since the Hermite modes H_n correspond to a dipole distribution described by a Gaussian times a Hermite polynomial of order n, only the excitation of H_0 leads to a transverse displacement of the bunch barycentre. From Fig. 9, we see that the stop bands associated with mode 0 always occur on the right of a resonance, whereas those associated with mode -1 occur always on the left of a resonance.

The following is another result of the numerical analysis.

iv) The instability growth rate in a stop band is proportional to the width of the stop band itself and this width decreases for increasing mode numbers.

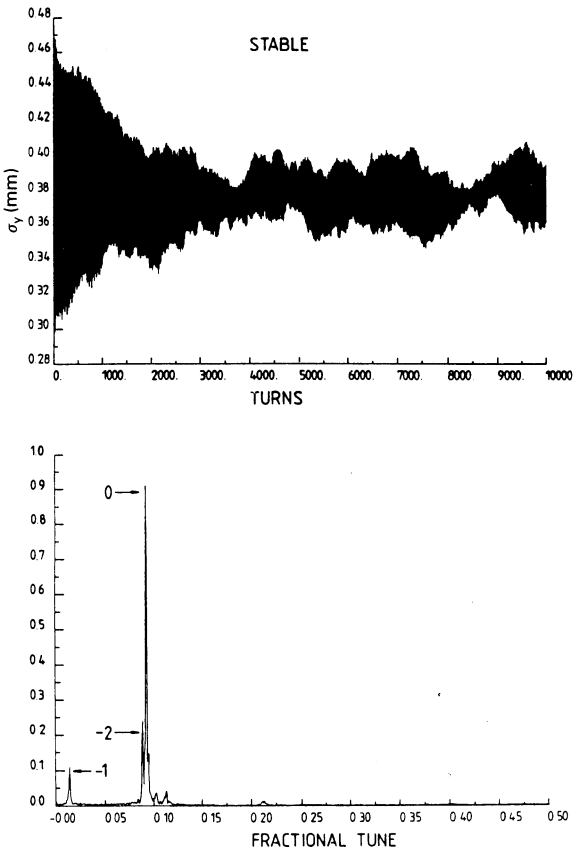
In an electron$-$positron storage ring, only those instabilities with a growth rate larger than the damping rate are effective. Then, from the statement (iv), we see that stop bands associated with very-high-order modes will disappear.

We have shown that the synchrotron side bands $\nu_\beta + m\,\nu_s$ are reflected back at the edges of the interval [0, 1/2] and, for |m| large enough, multiple reflections can take place. If ν_s is an irrational number, this mechanism gives rise to an infinite set of resonant values for ν_β, which is dense in the interval [0, 1/2]. Thanks to damping (or to other stabilizing effects such as Landau damping), statement (iv) leads to a high-order mode cut-off which limits the number of resonances. Considering typical damping times and synchrotron tunes in electron$-$positron machines, the only stop bands actually observable correspond to mode numbers up to ± 5. For LEP at injection energy ($E \cong 20$ GeV), the instability rise-time in these stop bands is of the order of a thousand machine revolutions, i.e. about one-fifth of the betatron damping time.

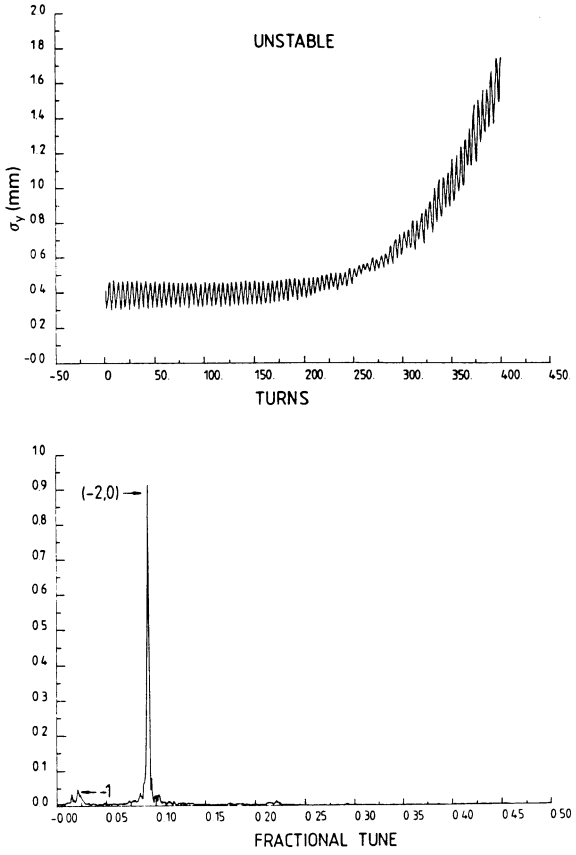
2.7 CONCLUSIONS

Starting from the Vlasov equation and confining our analysis to transverse wake fields generated in a single localized structure, we have derived a linear integral equation for the dipole density. By a numerical study of the associated eigenvalue problem, we have shown the existence of instability stop bands at currents below threshold, which are due to the coupling between high-order and low-order modes.

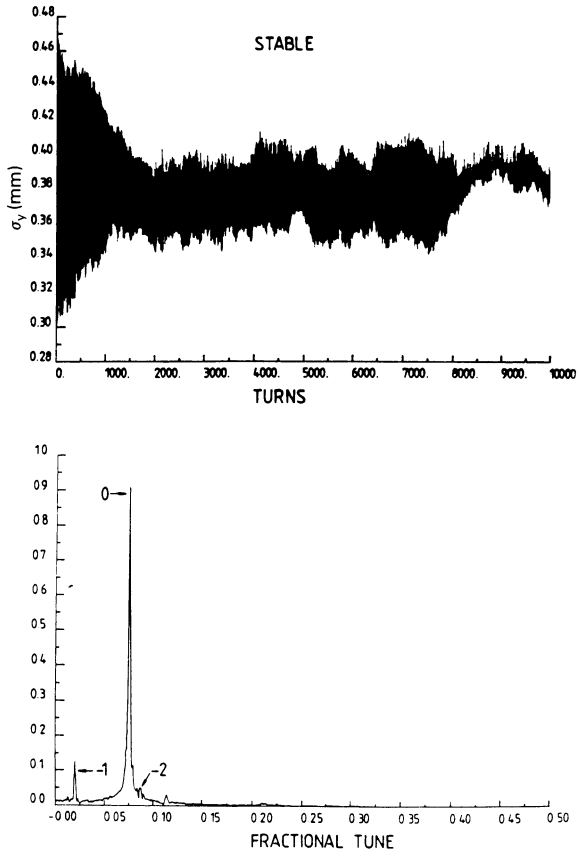
This prediction is in good agreement with the results of particle tracking, based on computer simulation, and with the conclusions drawn from a multi-particle model recently investigated [66]. For



a) I_b = 0.5 mA, below the instability gap.



b) I_b = 0.7 mA, within the instability gap.



c) I_b = 1.0 mA, above the instability gap.

Fig. 10
Parts of the output of the simulation program SIMTRAC, with longitudinal wake fields switched off, showing an instability stop band due to the coupling between mode 0 and mode −2. The upper pictures give the bunch r.m.s. transverse dimension as a function of the number of machine revolutions; the lower ones show the Fourier spectrum of the bunch barycentre motion (in arbitrary units) and contain peaks corresponding to the fractional tunes of modes 0, −1, and −2. These plots have been obtained by tracking 500 particles distributed in 100 "bins"; the tunes are ν_s = 0.1 and ν_β = 66.11.

example, Fig. 10 shows the transverse evolution of the bunch near a stop band, for three increasing values of current: it was obtained by means of the simulation program SIMTRAC [57], with longitudinal wake fields switched off. Up to now, the existence of such instability gaps has not been confirmed by direct observation; however, a machine experiment aimed at measuring the rise-time in these stop bands would give information about the ratio between the localized impedance (mainly due to RF cavities) and the distributed impedance [67].

A more realistic model of electron−positron storage rings should include effects not considered here. For example, in order to obtain better estimates of the instability growth rates and to prove the disappearance of stop bands associated with very-high-order modes, the Vlasov equation should be replaced by the Fokker−Planck equation [68, 69] taking into account damping and noise caused by synchrotron radiation. Other possible mechanisms which can affect the instability stop-band pattern are the particle tune spread, leading to Landau damping [70, 71], and the combined effect of two or more localized structures [72, 73].

Part 3

BEAM – BEAM INTERACTION AND RENORMALIZED FOKKER – PLANCK EQUATION

It is shown that the beam – beam interaction in electron – positron storage rings is equivalent to an additional source of noise for the betatron oscillations.

A white noise acting upon a non-linear oscillator causes a fast loss of coherence in its phase. This loss of coherence induces a broadening of the resonances, thus avoiding the problem of the divergent perturbative series which arises in the study of non-integrable Hamiltonian systems. A "renormalized" Fokker – Planck equation is established which contains new diffusive terms corresponding to the presence of resonances. The solution of this equation is exhibited explicitly in a simplified case. This allows an analytical approach to the problem of the incoherent beam – beam instability, which sets an upper limit to the maximum attainable luminosity in storage rings.

3.1 INTRODUCTION

The study of the beam – beam interaction in high-energy particle storage rings is of great importance for the design of new accelerators and for the improvement of the already existing ones. It is experimentally observed [16] that, when the number of stored particles exceeds a threshold value which depends on the physical parameters of the machine, the size of the beams blows up; this means that the luminosity ceases to increase or that one of the beams is lost. The main problem is to understand the dependence of this threshold value on the physical parameters, for example, on the particle energy, the ring radius, the number of bunches per beam and the betatron frequency. Such an understanding would allow the designer to choose the optimum values for these parameters and to suggest a way to prevent beams from blowing up.

In the case of electron – positron storage rings, as we have seen in Section 1.6, there is large betatron damping and quantum noise due to synchrotron radiation. This leads to a fast relaxation to the steady-state distribution and makes it possible to rely on computer simulation [74 – 77]. Analytical approaches to the problem of the beam – beam interaction in storage rings have usually been attempted by means of simplified models [78] based on the theory of non-linear resonances [79]. These models do not take into account the fundamental role of damping and noise. To our knowledge only two attempts have been made to tackle this problem [80, 81], but unfortunately none of them leads to a satisfactory explanation of the diffusion induced by the resonances in the presence of noise.

We present a statistical approach to the simplified problem of the "strong beam – weak beam" interaction which was suggested by the phenomenological analysis quoted in Ref. [82]. This approach allows us to calculate on an idealized model (see Section 3.5) the dependence of the maximum luminosity on the physical parameters of the storage ring. Our starting point is the Fokker – Planck equation for the particle distribution function in betatron phase space, with the effect of damping and noise included. The beam – beam interaction adds to this equation a term proportional to the first-order derivatives in the phase-space variables with a time-dependent coefficient. We will show that the main contribution of this term is equivalent to the effect of an additional source of noise. The new equation, obtained by a properly defined average process, will be referred to as the "renormalized" Fokker – Planck equation. It will be shown further that, under suitable assumptions, the problem can

be reduced to a discrete noisy map. Indeed our approach is similar to the one used to deduce the Chapman−Kolmogorov equation [83−85] and the path-integral solutions [86] which appear in the study of period-doubling bifurcations and transition to chaos [87]. Though our discussion is specifically aimed at the problem of the beam−beam interaction in electron storage rings, the method we shall present has a wider applicability.

We now specify the parameters which characterize our system. Let us consider a linear oscillator of given mass and frequency. Its phase-space trajectories are circles, when the momentum is normalized using the oscillator mass and frequency. According to the fluctuation−dissipation theorem [88], dissipation is always accompanied by a noise which drives the oscillations and causes a fluctuation of their amplitude. We shall assume this noise to be white, which is justified in the case of betatron oscillations of high-energy particles in storage rings (see Section 1.6). The dissipation is described by a damping time τ_d giving the time interval after which the phase-space distribution function relaxes to a Gaussian steady-state distribution, characterized by a standard deviation σ.

Let us now suppose that our system undergoes a series of instantaneous interactions (kicks) equally spaced in time at intervals Δt. The interaction potential is assumed to be a function of the phase-space variables and to vary strongly on a scale equal to σ. The natural unit for the potential is $\sigma^2/\Delta t$; in this unit, the strength of the interaction is expressed by a dimensionless parameter ε.

Owing to damping and noise, our system is not Hamiltonian and the strength of its non-Hamiltonian part is characterized by the dimensionless parameter $\rho = \Delta t/\tau_d$, which is the inverse of the number of kicks per damping time. We shall assume both the parameters ε and ρ to be small compared to unity.

For times much shorter than the damping time, the white noise gives rise to a stochastic change of the oscillation amplitude similar to that of a Brownian motion. (It also gives rise to a fluctuation in the phase of the oscillations, which is negligible since it is much smaller than 2π.) Owing to the non-linearity of the interactions, the perturbed frequency of the oscillations depends upon their amplitude. Thus the "diffusion in amplitude", caused by the white noise, gives rise to a "diffusion in phase", i.e. to a loss of correlation in the phase of the oscillations.

Our renormalization procedure for the Fokker−Planck equation is based on the assumption that the following inequality holds:

$$(\rho/\varepsilon)^{1/3} \ll 1. \tag{3.1}$$

As we will see in the next section, this condition is equivalent to assuming the diffusion in phase to be much faster that the diffusion in amplitude.

When applying these considerations to the betatron oscillations in electron−positron storage rings, it is found convenient to introduce new coordinates and momenta. These are related to the usual variables through the square root of the beta function [see Eq. (1.87)]. Then, apart from a factor of order unity depending on the geometry of the storage ring, the quantity σ for the radial betatron oscillations is given by [see Eq. (1.121)]

$$\sigma \cong \gamma\,(\lambda\!\!\!^{-}_e/\nu^3)^{1/2}, \tag{3.2}$$

where $\lambda\!\!\!^{-}_e$ is the Compton wavelength of the electron, γ is the particle relativistic dilation factor and ν is the betatron tune, i.e. the betatron frequency measured in units of the ring revolution frequency. The corresponding value of σ for vertical oscillations is given by the same expression, but is multiplied by a factor depending on the coupling between radial and vertical betatron oscillations. The time interval Δt between two kicks is

$$\Delta t \cong \pi R/cn_b, \tag{3.3}$$

where R is the average machine radius and n_b is the number of bunches per beam.

The beam−beam interaction is the interaction of a single particle of one beam with a whole bunch of particles of the other beam. Thus, if N_b is the number of particles per bunch, related to the beam current and to the number of bunches n_b, the dimensionless parameter ε expressing the strength of the interaction is given by

$$\varepsilon \cong N_b \, (r_e/\lambda\!\!\!^{-}_e) \, (\nu/\gamma)^3, \tag{3.4}$$

where r_e is the classical radius of the electron. The value of ε divided by 2π is the so-called "linear tune shift", provided the following condition is satisfied [89]:

$$\varepsilon \cot(2\pi\nu) \ll 1. \tag{3.5}$$

From Eqs. (1.95), (1.108) and (1.109), the dimensionless parameter ρ, i.e. the inverse of the number of kicks per damping time, can be written

$$\rho \cong (\pi/n_b) \, (r_e/R) \, \gamma^3. \tag{3.6}$$

A typical value of ρ for the existing storage rings [75, 78] is $\rho \cong 5\times 10^{-4}$. The value of ε is chosen so as to obtain the maximum increase of the beam luminosity without causing the beam to blow up. For the existing accelerators, the maximum values of ε are of order $\varepsilon \cong 2\times 10^{-1}$ (corresponding to a linear tune shift of about 0.04). These values satisfy our assumption (3.1).

We will, however, restrict our study of the beam−beam interaction to values of ε somewhat smaller than the maximum value. Indeed, computer simulation [77] shows that the first basic modifications of the particle distribution function, such as "exponential tails", already occur at these smaller values of ε. For example, in the case of LEP [90] at a particle energy of 50 GeV and with a beam current of about 1 mA, we obtain $\varepsilon \cong 5\times 10^{-2}$ and $\rho \cong 5\times 10^{-4}$, which give $(\rho/\varepsilon)^{1/3} \cong 0.2$. Thus, besides assumption (3.1), which is still quite well satisfied, we will in the following assume

$$\varepsilon \ll (\rho/\varepsilon)^{1/3}. \tag{3.7}$$

Section 3.2 contains a discussion of the assumptions used and gives a qualitative interpretation of the role of noise as a "detuning" mechanism. In Section 3.3, we shall derive the renormalized Fokker−Planck equation for a rather general class of systems. Section 3.4 contains a mathematical description of the detuning mechanism based on considerations of operator algebra, stochastic differential equations and symplectic maps [29−31, 91, 92]. In Section 3.5, we shall apply our results to an idealized model of the strong beam−weak beam interaction in electron−positron storage rings. There we derive the dependence of the maximum luminosity on the physical parameters of the storage ring. This dependence is in qualitative agreement with the experimental data. Section 3.6 contains a summary of the results.

3.2 THE BEAM − BEAM INTERACTION IN ELECTRON STORAGE RINGS

3.2.1 The Fokker−Planck Equation for Betatron Oscillations

For simplicity we shall only consider one-dimensional betatron oscillations; the extension to the realistic two-dimensional case is in principle straightforward. The equations of motion are customarily

written for the normalized variables q and p, which are related to the particle displacement y and its conjugate momentum p_y through the formulae [see Eqs. (1.57) and (1.87)]

$$q = y/\sqrt{\beta}, \quad p = \sqrt{\beta}\,[(c/E)\,p_y - y\,\beta'/2\beta], \tag{3.8}$$

where β is the betatron function and E the particle energy.

According to the discussion of Section 1.6, in the new variables q and p the equations of motion including the effect of the synchrotron radiation are equivalent to those of a linear, damped oscillator, driven by white noise

$$\dot{q} = \omega_0\,p, \quad \dot{p} = -\omega_0\,q - \alpha\,p + (2K)^{1/2}\,\xi(t). \tag{3.9}$$

Here ω_0 is the betatron frequency, α is the damping constant and K is the intensity of the quantum noise, associated with the stochastic variable $\xi(t)$ which has zero mean value and is delta-correlated, i.e.

$$\langle\xi(t)\rangle = 0, \quad \langle\xi(t)\,\xi(t+\tau)\rangle = \delta(\tau). \tag{3.10}$$

Let us consider the distribution function $\psi(q,p;t)$ in the phase space with coordinates q and p at time t. The Fokker–Planck equation [29–31] associated with the system (3.9) of stochastic equations can be written

$$\partial\psi/\partial t = L\,\psi, \tag{3.11}$$

where L is an elliptic differential operator given by

$$L = \omega_0\,(q\,\partial/\partial p - p\,\partial/\partial q) + (\partial/\partial p)\,(\alpha\,p + K\,\partial/\partial p). \tag{3.12}$$

If $\psi(t_0)$ is the initial distribution function, the formal solution of Eq. (3.11) is

$$\psi(t) = \exp[(t-t_0)\,L]\,\psi(t_0). \tag{3.13}$$

The diffusive term in L, i.e. the one containing the second-order derivative with respect to p, gives rise to a "spreading" of $\psi(t)$ while the streaming terms, i.e. those containing at most first-order derivatives, determine the motion of the "centre of mass" of the distribution. The deterministic part of Eq. (3.9) causes this centre of mass to move with frequency ω_0 along a spiral toward the origin.

With the damping constant α and the intensity of the quantum noise K, we can form a quantity having the same dimensions of q and p,

$$\sigma = (K/\alpha)^{1/2}. \tag{3.14}$$

It can be used as a natural unit for q and p; from Eqs. (3.11) and (3.12), it follows that after a time of the order of the damping time $\tau_d \cong 1/\alpha$ the system reaches a steady state characterized by a distribution function $\psi_s(q,p)$ which is a Gaussian with standard deviation σ,

$$\psi_s(q,p) = (1/2\pi\sigma^2)\,\exp[-(q^2+p^2)/2\sigma^2]. \tag{3.15}$$

The quantity σ^2, related to the beam emittance [60], is proportional to the average energy of the betatron oscillations and its thermodynamical interpretation is therefore that of a temperature.

3.2.2 The Effect of the Beam − Beam Interaction

When a charged particle of a beam intersects a bunch of particles of the other beam, it experiences an electric and a magnetic force which at ultrarelativistic velocities are essentially equal. Their components in the betatron plane arise from a potential as in the electrostatic case.

Since the bunches are equally spaced and since their longitudinal extension is much shorter than the betatron wavelength, we can describe the effect of the beam − beam interaction on a charged particle as a sequence of instantaneous kicks with a constant time interval Δt.

To simplify the analysis, we shall assume that one of the two beams (the "strong" beam) contains many more particles than the other one (the "weak" beam). Therefore the former is not affected in practice by the latter and the problem is reduced to the study of the influence of the strong beam on the weak one. In the one-dimensional case, the distribution function of the strong beam relaxes to the steady state $\psi_s(q,p)$ of Eq. (3.15). Such a distribution generates an interaction potential which varies on the natural "length" scale given by σ and can be written as εU, where the dimensionless parameter ε measures the strength of the interaction and U is of order $\sigma^2/\Delta t$. We shall further assume that the beam − beam interaction does not affect the focusing properties of the storage ring. This is true provided condition (3.5) is satisfied [16, 89].

Since the interactions are assumed to be instantaneous, instead of the distribution function $\psi(t)$ of the weak beam we shall use its "snapshots" immediately after and immediately before the k^{th} kick, namely

$$\begin{aligned} P^+(k) &= \lim_{\tau \to 0^+} \psi(k\,\Delta t + \tau), \\ P^-(k) &= \lim_{\tau \to 0^+} \psi(k\,\Delta t - \tau). \end{aligned} \tag{3.16}$$

As we will see in Section 3.3, the effect of a single kick on the distribution function can be described by the evolution operator $\exp(\Delta t\, M)$. Here M is the so-called Lie operator associated with the interaction potential εU and defined by

$$M f = \varepsilon\,[U,f\,], \tag{3.17}$$

where f is an arbitrary (regular) function on the phase space and $[U,f\,]$ is the Poisson bracket [14] between U and f. The factor Δt, which multiplies M in the evolution operator, has been introduced only for dimensional reasons and depends on the normalization chosen for the interaction potential εU. Taking into account Eqs. (3.13) and (3.16), we have

$$\begin{aligned} P^+(k) &= \exp(\Delta t\, M)\, P^-(k), \\ P^-(k+1) &= \exp(\Delta t\, L)\, P^+(k). \end{aligned} \tag{3.18}$$

Putting together these two equations one obtains

$$\begin{aligned} P^+(k+1) &= \exp(\Delta t\, M) \exp(\Delta t\, L)\, P^+(k), \\ P^-(k+1) &= \exp(\Delta t\, L) \exp(\Delta t\, M)\, P^-(k). \end{aligned} \tag{3.19}$$

Denoting the limits for $k \to \infty$ of $P^+(k)$ and $P^-(k)$ by P^+ and P^-, respectively, the steady-state distributions P^+ and P^-, if they exist, satisfy the equations

$$P^+ = \exp(\Delta t\, M) \exp(\Delta t\, L)\, P^+ = \exp(R^+)\, P^+,$$
$$P^- = \exp(\Delta t\, L) \exp(\Delta t\, M)\, P^- = \exp(R^-)\, P^-, \tag{3.20}$$

where the operators R^+ and R^- are related to M and L through the so-called Campbell – Baker – Hausdorff formula [91]. A class of solutions of Eqs. (3.20) can be obtained by solving the differential equations

$$R^+ P^+ = 0, \quad R^- P^- = 0. \tag{3.21}$$

From Eq. (3.12) we see that for a Hamiltonian system, containing neither damping nor noise terms, the evolution operator $\exp(\Delta t\, L)$ reduces to a rotation in phase space of an angle $\omega_0\, \Delta t$. In this case the series defining R^+ and R^- [through Eq. (3.20) and the Campbell – Baker – Hausdorff formula] would not converge [92] owing to the effect of resonances, which leads to the so-called problem of "small denominators".

3.2.3 Qualitative Interpretation of the Role of the Noise

Before discussing the problem of the convergence of the perturbative series, we want to give a qualitative interpretation of the effect of the white noise on the resonances. Instead of the Cartesian coordinates q and p in phase space, we will use the polar coordinates A and ϕ, where A is the amplitude and ϕ is the phase of the betatron oscillations. As we will see, the main effect of the white noise is a loss of coherence in the phase ϕ. This loss of coherence gives rise to a "broadening" of the resonances and introduces an effective cut-off in the series defining R^+ and R^-.

From our previous assumption on the interaction potential εU, it follows that the maximum change in ϕ due to a single kick is of order ε. In the absence of noise, and to first order in ε, the average effect of the beam – beam interaction on ϕ is to change the betatron frequency from its unperturbed value ω_0 to $\omega_0 + \varepsilon\, \Delta\omega(A)$. Here $\Delta\omega(A)$, which depends on the interaction potential, is a non-linear function of the amplitude A with natural scale σ. The equation of motion for the phase ϕ is then

$$\dot{\phi} = \omega_0 + \varepsilon\, \Delta\omega[A(t)] + O(\varepsilon^2). \tag{3.22}$$

For times t much shorter than the damping time $\tau_d \cong 1/\alpha$, the effect of the white noise is to introduce a stochastic change in the oscillation amplitude A similar to that of a Brownian motion. Indeed, from Eq. (3.9),

$$A(t) = A_0 + \Delta A(t), \quad \langle \Delta A(t) \rangle = 0,$$
$$\langle \Delta A(t)\, \Delta A(t+\tau) \rangle = K\, t \quad (\tau \geq 0), \tag{3.23}$$

follows. Using natural units σ for the amplitude and Δt for time, from Eqs. (3.14) and (3.23)

$$\langle [\Delta A(t)/\sigma]^2 \rangle = (\alpha\, \Delta t)\, (t/\Delta t) = \rho\, (t/\Delta t), \tag{3.24}$$

follows. The dimensionless quantity $\rho = \alpha \Delta t$ gives a measure of the intensity of the white noise. Expanding $\Delta\omega[A(t)]$ in powers of $\Delta A(t)$, we obtain

$$\Delta\omega[A(t)] = \Delta\omega(A_0) + \Delta\omega'(A_0)\,\Delta A(t) + \mathcal{O}([\Delta A(t)/\sigma]^2), \tag{3.25}$$

where $\Delta\omega'$ is the derivative of $\Delta\omega$ with respect to A.

To solve Eq. (3.22), we begin by splitting the variable $\phi(t)$ into the sum of a deterministic part and a stochastic one $\Delta\phi(t)$. Using (3.22) and (3.25), the equation of motion for the stochastic part is

$$\Delta\dot{\phi}(t) \equiv \dot{\phi}(t) - \langle\dot{\phi}(t)\rangle = \varepsilon\,\Delta\omega'\,\Delta A(t) + \mathcal{O}(\varepsilon^2). \tag{3.26}$$

Integrating this equation, squaring, and averaging the result, from (3.23) and (3.24) one obtains

$$\langle[\Delta\phi(t)]^2\rangle \cong \varepsilon^2\,(\Delta\omega')^2\,\langle[\Delta A(t)]^2\rangle\,t^2 \cong (\Delta\omega'\,\sigma\,\Delta t)^2\,\varepsilon^2\,\rho\,(t/\Delta t)^3. \tag{3.27}$$

In the natural units σ and Δt, the dimensionless quantity $\Delta\omega'\,\sigma\,\Delta t$ is of order one, so that

$$\langle[\Delta\phi(t)]^2\rangle \cong \varepsilon^2\,\rho\,(t/\Delta t)^3. \tag{3.28}$$

We can now define a correlation time τ_{corr} by

$$\tau_{corr} \cong \Delta t\,(\varepsilon^2\,\rho)^{-1/3}. \tag{3.29}$$

After a time $t \cong \tau_{corr}$ the uncertainty $\Delta\phi$ in the phase of the betatron oscillations becomes of order one. This is equivalent to saying that the kicks become uncorrelated. The series defining R^+ and R^- can thus be limited to a finite number of terms, which corresponds to the number of correlated kicks N_{corr} given by

$$N_{corr} = \tau_{corr}/\Delta t \cong (\varepsilon^2\,\rho)^{-1/3}. \tag{3.30}$$

In order that expansion (3.25), in which only the first order term in $[\Delta A(t)/\sigma]$ has been retained, be valid after a time $t \cong \tau_{corr}$, the amplitude variation $\Delta A(\tau_{corr})$ must be much smaller than σ. Since, from Eqs. (3.24) and (3.29), we have

$$\Delta A(\tau_{corr})/\sigma \cong \{\langle[\Delta A(\tau_{corr})/\sigma]^2\rangle\}^{1/2} \cong [\rho\,(\tau_{corr}/\Delta t)]^{1/2} \cong (\rho/\varepsilon)^{1/3}, \tag{3.31}$$

we must require the validity of (3.1). This assumption implies that the process of "phase diffusion" is much faster than the process of "amplitude diffusion". In order that the term $\varepsilon\,\Delta\omega'\,\Delta A(\tau_{corr})$ in Eq. (3.26) be larger than the term of order ε^2, which has been neglected, a further assumption is needed, namely, condition (3.7).

In the next section, we shall formulate the whole problem in a more general way, deriving an equation which is equivalent to (3.21) and which will be referred to as the renormalized Fokker – Planck equation.

3.3 THE RENORMALIZED FOKKER – PLANCK EQUATION

In this section we shall derive the renormalized Fokker – Planck equation for a general class of systems. By Fourier analysis of the distribution function with respect to the phase variable ϕ, it is possible to consider separately the problem of the interaction between the resonances and the effect of the white noise on each resonance. Though our method applies to the general case, we shall assume for simplicity that each resonance is isolated, i.e. that there is no overlap between resonances in phase space [79].

Instead of Eq. (3.9), let us consider a more general system of stochastic differential equations of the type

$$\dot{q} = \partial H(q,p,t)/\partial p, \quad \dot{p} = -\partial H(q,p,t)/\partial q + F[q,p,\xi(t)]. \tag{3.32}$$

The Hamiltonian H is given by an unperturbed term $H_0(q,p)$ plus a perturbation $\varepsilon\, U(q,p)\, \eta(t)$ representing a series of instantaneous interactions (kicks) at time intervals Δt

$$\begin{aligned} H(q,p,t) &= H_0(q,p) + \varepsilon\, U(q,p)\, \eta(t), \\ \eta(t) &= \Delta t \sum_{n=-\infty}^{\infty} \delta(t - n\,\Delta t). \end{aligned} \tag{3.33}$$

The function F in Eq. (3.32) is the sum of two terms, one representing the effect of the damping $\rho\, f(q,p)$ and the other that of the white noise $\sqrt{\rho}\, g(q,p)\, \xi(t)$, related to the Gaussian, delta-correlated stochastic variable $\xi(t)$

$$\begin{aligned} F[q,p,\xi(t)] &= \rho\, f(q,p) + \sqrt{\rho}\, g(q,p)\, \xi(t), \\ \langle \xi(t) \rangle &= 0, \quad \langle \xi(t)\, \xi(t+\tau) \rangle = \delta(\tau). \end{aligned} \tag{3.34}$$

As in Section 3.2, we shall assume all the dimensionless quantities constructed from U, f and g to be of order one; thus the parameter ε represents the intensity of the perturbation, while the parameter ρ gives the strength of the non-Hamiltonian part of the system, i.e. the intensity of noise and damping. According to the qualitative discussion of Section 3.2, we will make the assumption

$$\varepsilon \ll (\rho/\varepsilon)^{1/3} \ll 1 \tag{3.35}$$

and any expansion in powers of ε and ρ will be limited to second-order in ε and to first order in ρ, respectively.

Let us introduce, instead of the coordinates q and p, the action – angle variables J and ϕ of the unperturbed Hamiltonian H_0. Thus Eqs. (3.32) become

$$\begin{aligned} \dot{\phi} &= \omega_0 + \varepsilon\, (\partial U/\partial J)\, \eta(t) + \rho\, f_1 + \sqrt{\rho}\, g_1\, \xi(t), \\ \dot{J} &= -\varepsilon\, (\partial U/\partial \phi)\, \eta(t) + \rho\, f_2 + \sqrt{\rho}\, g_2\, \xi(t), \end{aligned} \tag{3.36}$$

where ω_0 is the unperturbed frequency of the system (which in general depends upon the action variable J)

$$\omega_0(J) = \partial H_0(J)/\partial J, \tag{3.37}$$

while the functions f_1, g_1, f_2 and g_2 are related to f and g by the transformations

$$f_i = f\,\partial x_i/\partial p, \quad g_i = g\,\partial x_i/\partial p, \quad i=1,2,$$
$$x_1 \equiv \phi, \qquad x_2 \equiv J. \tag{3.38}$$

The Fokker–Planck equation for the distribution function $\psi(\phi,J;t)$ associated with the system (3.36) of stochastic equations is [30]

$$\partial\psi/\partial t = [L + M\,\eta(t)]\,\psi, \tag{3.39}$$

where the differential operators L and M are given by

$$L = -\omega_0\,(\partial/\partial\phi) - \rho\,[\sum_i (\partial/\partial x_i)\,f_i - (1/2)\sum_{i,j}(\partial/\partial x_i)\,g_i\,(\partial/\partial x_j)\,g_j],$$
$$M = \varepsilon\,[(\partial U/\partial\phi)\,\partial/\partial J - (\partial U/\partial J)\,\partial/\partial\phi]. \tag{3.40}$$

Integrating Eq. (3.39) over t and introducing the distributions $P^+(k)$ and $P^-(k)$ given by (3.16), we obtain the relations (3.18)–(3.20).

To separate the effect of the noise on each resonance from that of the interaction between resonances, we perform a Fourier analysis with respect to the phase ϕ. Thus we write

$$P_n^+ = (1/2\pi)\int_0^{2\pi} e^{in\phi}\,P^+\,d\phi, \qquad U_n = (1/2\pi)\int_0^{2\pi} e^{in\phi}\,U\,d\phi,$$
$$L_{nm} = (1/2\pi)\int_0^{2\pi} e^{in\phi}\,L\,e^{-im\phi}\,d\phi, \qquad M_{nm} = (1/2\pi)\int_0^{2\pi} e^{in\phi}\,M\,e^{-im\phi}\,d\phi. \tag{3.41}$$

Performing the Fourier analysis of the first of Eqs. (3.20) written in the form

$$\exp(-\Delta t\,M)\,P^+ = \exp(\Delta t\,L)\,P^+ \tag{3.42}$$

and isolating diagonal terms, one obtains

$$[(e^{-\varepsilon\mathbf{A}})_{nn} - (e^{\mathbf{B}})_{nn}]\,P_n^+ = \sum_{m\neq n}[(e^{\mathbf{B}})_{nm} - (e^{-\varepsilon\mathbf{A}})_{nm}]\,P_m^+. \tag{3.43}$$

The dimensionless matrix operator **A** represents the effect of the perturbation, whereas **B** is the sum of two terms, $\mathbf{B} = \mathbf{C} + \rho\,\mathbf{D}$, one representing the unperturbed evolution corresponding to the Hamiltonian H_0 and the other the combined effect of damping and noise. The explicit expressions for **A** and **B** are

$$\varepsilon\,A_{nm} = \Delta t\,M_{nm} = \varepsilon\,i\,\Delta t\,[m\,(dU_{n-m}/dJ) - (n-m)\,U_{n-m}\,d/dJ],$$
$$B_{nm} = \Delta t\,L_{nm} = i\,(n\omega_0\Delta t)\,\delta_{nm} + \rho\,D_{nm}. \tag{3.44}$$

The operator D_{nm} has a complicated form, which is given in Appendix A. It is only important to notice that, owing to the effect of the noise, D_{nm} contains at most second-order derivatives with respect to J. For any realistic choice of the interaction potential U, its Fourier components U_n decrease rapidly with increasing $|n|$. Thus the off-diagonal terms A_{nm}, which contain U_{n-m}, decrease rapidly with increasing $|n-m|$.

In Appendix B it is shown that, under the assumption (3.35), Eq. (3.43) becomes

$$\{[1 - \exp(\varepsilon A_{nn}) \exp(B_{nn})] + (\varepsilon^2/2) \sum_{m\neq n} A_{nm} A_{mn}\} P_n^+ = \varepsilon \sum_{m\neq n} A_{nm} P_m^+. \tag{3.45}$$

Before attempting a solution of the system (3.45), we remark that the equation for $n = 0$ has a radically different structure from those corresponding to $n \neq 0$. The main difference comes from B_{nn}, which is of order ρ only for $n = 0$. Indeed in Section 3.2, we have seen that the noise gives rise to a stochastic change of the oscillation amplitude similar to that of a Brownian motion. Moreover, owing to the non-linearity of the interactions, the noise causes a loss of correlation in phase. Now P_o^+ represents the amplitude distribution function averaged over the phase. On the other hand, the P_n^+ with $n \neq 0$ vanish unless $P^+(J,\phi)$ depends explicitly on ϕ. Therefore the P_n^+ with $n \neq 0$ give a measure of the correlation in phase. Thus we expect that the effect of the noise will be represented by a second-order differential operator acting on P_o^+, i.e. by a diffusion, and by a multiplicative factor (smaller than one) on P_n^+ with $n \neq 0$.

We first consider Eq. (3.45) for $n = 0$. From (3.44) it follows that $A_{oo} = 0$ and $B_{oo} = \rho D_{oo}$, thus to order ε^2 and ρ we have

$$\rho D_{oo} P_o^+ - (\varepsilon^2/2) \left(\sum_{m\neq 0} A_{om} A_{mo} \right) P_o^+ + \varepsilon \sum_{m\neq 0} A_{om} P_m^+ = 0. \tag{3.46}$$

The effect of the noise is represented by the diffusive operator ρD_{oo}, which contains second-order derivatives with respect to J. The second term in Eq. (3.46) takes into account the direct effect of the interaction on the average distribution P_o^+, whereas the last term, of order ε, represents the coupling to the higher "modes" P_n^+ due to the interaction.

For the sake of simplicity, let us suppose that the unperturbed frequency ω_0 does not depend on the action J. This approximation is compatible with the condition prevailing in storage rings, i.e. the unperturbed system is a linear oscillator. To study Eq. (3.45) in the case $n \neq 0$, we begin by remarking that the term in square brackets can be written as

$$1 - \exp(\varepsilon A_{nn}) \exp(B_{nn}) = 1 - \exp[in(\omega_0 + \varepsilon \, dU_o/dJ)\Delta t] \exp(\rho D_{nn}). \tag{3.47}$$

Far from resonances, i.e. when the condition

$$n (\omega_0 + \varepsilon \, dU_o/dJ) \Delta t = 2\pi m \tag{3.48}$$

is not satisfied, we can neglect the factor $\exp(\rho D_{nn})$ in Eq. (3.47). Therefore the operator C_n defined by

$$C_n = [1 - \exp(\varepsilon A_{nn}) \exp(B_{nn})]^{-1} \tag{3.49}$$

reduces to the multiplicative factor $\{1 - \exp[in(\omega_0 + \varepsilon \, dU_o/dJ)\Delta t]\}^{-1}$. The operator C_n contains the effect of resonances and of the detuning caused by the white noise. In the next section we shall

show that even when resonances are present, C_n can still be approximated by a multiplicative coefficient, having the following upper bound

$$|C_n| < (\varepsilon^2 \rho)^{-1/3}. \tag{3.50}$$

This upper bound is consistent with the qualitative discussion preceding Eq. (3.30). Making use of (3.49), Eq. (3.45) for $n \neq 0$ becomes

$$[1 + \mathcal{O}(\varepsilon^2 C_n)]\, P_n^+ = \varepsilon\, C_n \sum_{m \neq n} A_{nm}\, P_m^+, \qquad n \neq 0. \tag{3.51}$$

Thanks to the inequalities (3.35) and (3.50), the term of order $\varepsilon^2 C_n$ is small compared to unity,

$$\varepsilon^2 C_n < \varepsilon\, (\varepsilon/\rho)^{1/3} \ll 1, \tag{3.52}$$

and therefore can be neglected.

The derivation contained in Appendix B and the discussion following Eq. (3.45), which leads to the definition (3.49) of the resonant coefficients C_n, are different from those appearing in Ref. [93]. This is a consequence of the remarks contained in Ref. [94] and is related to the difficulty of dealing, at the same time, with conventional power expansions and with power expansions in the exponents. However, it is worth mentioning that a similar difficulty occurs in canonical perturbation theory too [14], as soon as one considers "small denominators" depending on the perturbed frequencies. In the case of the beam−beam interaction, the betatron frequency ω_0 can be treated as a constant and so it is indispensable to take into account the amplitude-dependent perturbation $\varepsilon\, dU_0(J)/dJ$ [see Eqs. (3.22) and (3.48)].

Our aim is now to solve Eq. (3.51) expressing P_n^+, with $n \neq 0$, in terms of P_0^+. Substituting the result into (3.46), we obtain the renormalized Fokker−Planck equation for the steady-state distribution function $P_0^+(J)$. Equation (3.51), which is an infinite set of coupled first-order differential equations, contains the mutual interactions between the "modes" P_n^+, and some simplifying assumption is needed in order to get an approximate solution. The coefficients C_n are functions of the action variable J and take their maximum value when J passes through a resonance. In the next section we will prove that the resonant values of C_n tend to unity as n increases, so that only a small number of these coefficients may become much greater than one. Moreover, A_{nm} decreases with increasing $|n-m|$, which reflects the fact that the coupling between distant modes tends to zero. In view of these considerations we can assume the following approximate solution of Eq. (3.51):

$$P_n^+ \cong \varepsilon\, C_n\, A_{no}\, P_o^+. \tag{3.53}$$

This formula is valid, to first order in ε, provided one of these conditions holds (for a fixed value of J):

i) There is no resonance corresponding to a small value of n; then all the C_n are of order (or smaller than) unity.

ii) There is only one, isolated resonance corresponding to a small value of n, with the relative C_n much greater than unity.

In the latter case, the solution (3.53) is correct only for the resonant value of n. However, the corresponding P_n^+ is the largest one and gives rise to the dominant contribution when substituted into Eq. (3.46). As the perturbation parameter ε increases, it becomes more and more likely that, for some value of J, neither condition (i) nor (ii) hold. This means that two, or more than two, simultaneous low-order resonances may overlap and in this case a more refined, but not significantly different calculation is needed. For simplicity, we will assume the "single resonance" solution (3.53).

Substituting (3.53) in Eq. (3.46) we obtain

$$\rho\, D_{oo}\, P_o^+ - (\varepsilon^2/2) \Big(\sum_{m\neq 0} A_{om}\, A_{mo} \Big) P_o^+ + \varepsilon^2 \Big(\sum_{m\neq 0} A_{om}\, C_m\, A_{mo} \Big) P_o^+ = 0. \tag{3.54}$$

Making use of (3.44), Eq. (3.54) can be written as

$$\{\rho\, D_{oo} + \varepsilon^2 \sum_m A_{om}\, [\mathrm{Re}(C_m) - 1/2]\, A_{mo}\}\, P_o^+ = 0, \tag{3.55}$$

where $\mathrm{Re}(C_m)$ is the real part of C_m. Equation (3.55) is our renormalized Fokker–Planck equation. The differential operator in curly brackets is a diffusive operator: the first term $\rho\, D_{oo}$ takes into account the direct effect of noise and damping; the second one represents the diffusion induced by the non-linear interactions. From Eqs. (3.47) and (3.48) we see that, far from resonances, the real part of C_m reduces to 1/2. Thus the diffusive effect induced by the interactions [see Eq. (3.55)] is localized near the values of J corresponding to resonances.

3.4 THE EFFECT OF THE WHITE NOISE ON THE RESONANCES

In this section we shall compute the operators C_n defined by Eq. (3.49) and we shall show that, for $n \neq 0$, they reduce to multiplicative coefficients. The maximum values of these coefficients occur in the presence of resonances and they remain finite due to the loss of phase correlation induced by the white noise.

Let us expand the r.h.s. of Eq. (3.49) in a power series of the product $\exp(\varepsilon A_{nn})\, \exp(B_{nn})$

$$C_n = \sum_{k=0}^{\infty} [\exp(\varepsilon A_{nn})\, \exp(B_{nn})]^k. \tag{3.56}$$

We shall now make use of the following algebraic result: let a and b be two arbitrary operators, then [91, 92, 95]

$$\exp(b)\, \exp(a) = \exp(a)\, \exp(\{\exp(-a), b\}), \tag{3.57}$$

where

$$\{e^{-a}, b\} \equiv b - [a,b] + (1/2!)\, [a,[a,b]] - (1/3!)\, [a,[a,[a,b]]] + \ldots . \tag{3.58}$$

From Eq. (3.57) it follows, by induction on k, that

$$[\exp(a)\exp(b)]^k = \exp(ka) \prod_{m=k-1}^{0} \exp(\{\exp(-ma),b\}). \tag{3.59}$$

In our case, setting $a = \varepsilon A_{nn}$ and $b = B_{nn}$, from (3.44) we have

$$\{\exp(-m\varepsilon A_{nn}),B_{nn}\} = i(n\omega_0\Delta t) + \rho(D_{nn} - m\varepsilon\,[A_{nn},D_{nn}] + (m^2\varepsilon^2/2)\,[A_{nn},[A_{nn},D_{nn}]]), \tag{3.60}$$

where only the first- and second-order commutators appear. This is due to the fact that A_{nn} is a function of J, while D_{nn}, which is a diffusive operator, contains at most second-order derivatives with respect to J; thus all commutators between A_{nn} and D_{nn} of order higher than two vanish.

Since we have assumed that the unperturbed frequency ω_0 does not depend on the action J, from Eqs. (3.59) and (3.60) it follows that

$$[\exp(\varepsilon A_{nn})\exp(B_{nn})]^k = \exp[k(in\omega_0\Delta t + \varepsilon A_{nn})] \prod_{m=k-1}^{0} \exp\{\rho(D_{nn}$$

$$- m\varepsilon\,[A_{nn},D_{nn}] + (m^2\varepsilon^2/2)\,[A_{nn},[A_{nn},D_{nn}]])\}. \tag{3.61}$$

The product in the r.h.s. of this equation, which we denote by S_k, can be further simplified by considering the following two cases:

a) $k \le 1/\varepsilon$. In this case, $m\varepsilon$ is smaller than one and we have

$$S_k = \prod_{m=k-1}^{0} \exp\{\rho[D_{nn} + O(m\varepsilon)]\} = \exp\{k\rho[D_{nn} + O(k\varepsilon)]\} = 1 + O(\rho/\varepsilon), \tag{3.62}$$

where, as a consequence of (3.35), ρ/ε is much smaller than one.

b) $k \gg 1/\varepsilon$. Then S_k can be written as the product of the terms with m smaller than $1/\varepsilon$ and of those larger than $1/\varepsilon$

$$S_k = S_{1/\varepsilon} \prod_{m=k-1}^{1/\varepsilon} \exp\{\rho(D_{nn} - m\varepsilon\,[A_{nn},D_{nn}] + (m^2\varepsilon^2/2)\,[A_{nn},[A_{nn},D_{nn}]])\}. \tag{3.63}$$

In the second product, $m\varepsilon$ is larger than one and, using (3.62), we have

$$S_k = [1 + O(\rho/\varepsilon)] \prod_{m=k-1}^{1/\varepsilon} \exp\{\rho((m^2\varepsilon^2/2)\,[A_{nn},[A_{nn},D_{nn}]] + O(m\varepsilon))\}$$

$$= [1 + O(\rho/\varepsilon)]\exp\{(1/2)\rho(k^3/3)\varepsilon^2\,[A_{nn},[A_{nn},D_{nn}]] + O(\rho k^2\varepsilon)\}. \tag{3.64}$$

It is worth noticing that, if k is smaller than or comparable to $1/\varepsilon$ as in case (a), formula (3.64) becomes

$$S_k = [1 + O(\rho/\varepsilon)] \exp\{O(\rho/\varepsilon)\} = 1 + O(\rho/\varepsilon), \tag{3.65}$$

which is the same as (3.62). Thus, except for a critical range of values of k which will be considered later, Eq. (3.64) is valid in general. Inserting (3.64) into (3.61), we obtain

$$[\exp(\varepsilon A_{nn}) \exp(B_{nn})]^k \cong \exp[k(in\omega_0\Delta t + \varepsilon A_{nn})] \exp\{(1/6)\rho k^3\varepsilon^2[A_{nn},[A_{nn},D_{nn}]]\}. \tag{3.66}$$

We have to check that, when k takes the critical value N_{corr}, such that [see Eq. (3.30)]

$$\rho N_{corr}^3 \varepsilon^2 \cong 1, \tag{3.67}$$

the quantity $\rho k^2\varepsilon$, that we have neglected in Eq. (3.64), is much smaller than one and thus actually negligible. Indeed from (3.67)

$$\rho N_{corr}^2 \varepsilon \cong (\rho/\varepsilon)^{1/3}, \tag{3.68}$$

which is much smaller than one, thanks to the assumption (3.35). Inserting (3.66) into Eq. (3.56) we obtain

$$C_n = \sum_{k=0}^{\infty} \exp[k(in\omega_0\Delta t + \varepsilon A_{nn})] \exp\{(1/6)\rho k^3\varepsilon^2[A_{nn},[A_{nn},D_{nn}]]\}. \tag{3.69}$$

Making use of expression (3.44) for A_{nn} we can write C_n in the form

$$C_n = \sum_{k=0}^{\infty} \exp[ikn(\omega_0 + \varepsilon\, dU_0/dJ)\Delta t] \exp\{-k^3(1/6)\rho(\varepsilon n\Delta t)^2[dU_0/dJ,[dU_0/dJ,D_{nn}]]\}. \tag{3.70}$$

This expression shows that, even at resonance, the operator C_n does not contain derivatives with respect to J and thus it reduces to a simple multiplicative function of J. When J takes a value corresponding to a resonance, i.e. when an integer m exists such that condition (3.48) is satisfied, C_n becomes of order

$$(C_n)^{max} \cong (\rho\varepsilon^2 n^2)^{-1/3}. \tag{3.71}$$

Equation (3.71) justifies the condition (3.50), used in the last section. From Eq. (3.70) we see that, at resonance, C_n tends to unity for very large values of n, i.e. that only the k = 0 term in the summation gives its contribution to C_n.

3.5 SOLUTION OF THE RENORMALIZED FOKKER – PLANCK EQUATION IN A SIMPLE CASE

In this section we shall apply our results to the case of the beam – beam interaction in electron – positron storage rings. We consider the renormalized Fokker – Planck equation written for a "quasi-logarithmic" interaction potential. By requiring that the size of the weak beam does not blow

up, we obtain a relation between the maximum attainable interaction strength ε and the noise intensity ρ. This relation can be used to express the dependence of the maximum attainable luminosity on the physical parameters of the storage ring, such as particle energy, machine tune and number of bunches per beam.

The renormalized Fokker−Planck equation (3.55), for the system described in Section 3.2, reads (see Appendix A)

$$(d/dJ)\{JP_o^+ + [\sigma^2 J + (2\varepsilon^2/\rho) \sum_{m>0} (\mathrm{Re}(C_m) - 1/2)\, |mU_m \Delta t|^2]\, dP_o^+/dJ\} = 0, \tag{3.72}$$

where, from Section 3.4, we have

$$\mathrm{Re}[C_m(J)] = \sum_{k=0}^{\infty} \cos[km\alpha(J)] \exp[-(\Delta_m(J)k)^3], \tag{3.73}$$

with

$$\alpha(J) = \omega(J)\Delta t, \quad \omega(J) = \omega_0 + \varepsilon\, dU_0(J)/dJ,$$
$$\Delta_m(J) = [(1/3)\rho(m\omega'(J)\Delta t)^2 \sigma^2 J]^{1/3}. \tag{3.74}$$

We recall that Eq. (3.72) applies to an idealized storage ring, with one-dimensional betatron oscillations, where the beam−beam interaction is treated in the strong beam−weak beam approximation.

The solution of Eq. (3.72), which satisfies the condition that its integral over J is finite, is

$$P_o^+(J) = Z \exp\{-(1/\sigma^2) \int_0^J dJ/[1 + (2\varepsilon^2/\rho\sigma^2 J) \sum_{m>0} (\mathrm{Re}(C_m) - 1/2)\, |mU_m \Delta t|^2]\}. \tag{3.75}$$

where Z is a normalization constant.

When the denominator of the integrand in Eq. (3.75) approaches one, the diffusive effect of the interaction is negligible. Indeed, in this case we obtain a steady-state distribution function $P_o^+ = Z \exp(-J/\sigma^2)$, which goes over to the unperturbed Gaussian distribution (3.15) when the action J is expressed as a function of q and p. If instead, in a given interval ΔJ, the denominator of the integrand in Eq. (3.75) is much greater than one, which can only be the case if there is a low-order resonance, the distribution $P_o^+(J)$ is essentially constant over ΔJ. Thus the diffusive effect of the interaction causes a considerable flattening of the distribution $P_o^+(J)$ when the following condition is satisfied:

$$(2\varepsilon^2/\rho\sigma^2 J) \sum_{m>0} (\mathrm{Re}(C_m) - 1/2)\, |mU_m \Delta t|^2 \geq 1. \tag{3.76}$$

When the interaction strength ε increases, the "flattening condition" (3.76) holds for a large number of resonances, corresponding to intervals ΔJ which tend to overlap. Thus the width of the distribution P_o^+ becomes much larger than σ and, when ε approaches a threshold value ε_{th}, the size of the weak beam blows up, causing the incoherent beam−beam instability described in Section 3.1.

The flattening condition (3.76) has the following functional form: $G(U;\varepsilon,\rho,J) \geq 1$. Thus, once we have specified the interaction potential U, this condition becomes a relation between ε, ρ and J. In order to obtain ε_{th} as a function of ρ, we have to fix J. This reflects the fact that the beam−beam instability starts at different values of ε, depending on the region of phase space. For any realistic choice of the interaction potential U, the smallest ε_{th} corresponds to intermediate values of J, of the order of a few times σ^2.

To evaluate ε_{th}, we need to estimate the width ΔJ and the spacing DJ between resonances. Instead of the variable J, it is more convenient to consider the "phase" $\alpha(J)$ defined by Eq. (3.74). We begin by remarking that the strong-beam Gaussian distribution function (3.15) gives rise to an interaction potential $\varepsilon U(q)$, which is an even function of the betatron displacement $q = (2J)^{1/2} \sin\phi$. As a consequence the odd Fourier components $U_{2n+1}(J)$ vanish. From the resonance condition (3.48) and from Eq. (3.74), it appears that in an interval $\Delta\alpha$ there are $(\Delta\alpha/2\pi)2n$ resonances of order 2n. The total number of even resonances of order smaller than or equal to 2n is therefore given by $(\Delta\alpha/2\pi)$ times the sum of the first n even numbers, i.e. times $n(n+1)$. For $n \gg 1$ the "average spacing in phase" $(D\alpha)_{2n}$ between any two even resonances of order smaller than or equal to 2n is

$$(D\alpha)_{2n} \cong 2\pi/n^2. \tag{3.77}$$

According to the qualitative discussion leading to Eq. (3.30), we expect the width of the resonances of order 2n to be inversely proportional to $N_{corr} \cong 1/\Delta_{2n}$. Indeed, from Eq. (3.73), it follows that the "width in phase" $(\Delta\alpha)_{2n}$ of the resonances of order 2n is given by

$$(\Delta\alpha)_{2n} \cong 2\pi\, \Delta_{2n}/2n. \tag{3.78}$$

From Eqs. (3.74) and (3.78), $(\Delta\alpha)_{2n}$ decreases as $n^{-1/3}$. Thus a sufficient condition for a complete overlap between even resonances of order smaller than or equal to 2n is that the width $(\Delta\alpha)_{2n}$ is equal to the spacing $(D\alpha)_{2n}$. However, this condition is too restrictive, because the width of the resonances of order lower than 2n is larger than $(\Delta\alpha)_{2n}$. Furthermore, one can expect that when a complete overlap is reached, the beam has already blown up. Thus, to obtain an estimate of the threshold value ε_{th}, we shall assume the width $(\Delta\alpha)_{2n}$ to be only a fraction, let us say about 1/4, of the spacing $(D\alpha)_{2n}$. From (3.77) and (3.78), this is equivalent to the condition

$$\Delta_{2n}(J) \cong 1/2n. \tag{3.79}$$

Combining (3.79) with the expression of $\Delta_{2n}(J)$ given by Eq. (3.74), we obtain n as a function of ε, ρ and J:

$$n \cong \{3/[32\,\rho\varepsilon^2(U_o''(J)\sigma^2\Delta t)^2 J/\sigma^2]\}^{1/5}. \tag{3.80}$$

Making use of Eqs. (3.73) and (3.79), we can approximate the resonant value of $\mathrm{Re}(C_{2n})$ by $N_{corr} \cong 1/\Delta_{2n} \cong 2n$. The flattening condition (3.76) is then satisfied if

$$16(\varepsilon^2/\rho)n^3|U_{2n}(J)\Delta t|^2/(\sigma^2 J) \geq 1. \tag{3.81}$$

At fixed J, from Eqs. (3.80) and (3.81) we obtain the threshold value ε_{th} as a function of ρ. This relation, combined with Eqs. (3.4) and (3.6), gives the dependence of the maximum beam current on the physical parameters of the storage ring.

We now consider the case of a "quasi-logarithmic" interaction potential

$$U(q) = (\sigma^2/\Delta t)\log(1 + q^2/\sigma^2), \tag{3.82}$$

which is a good approximation to the potential generated by a beam with circular cross-section and with bunches whose longitudinal extension is much larger than their transverse size σ. Then the components $U_0(J)$ and $U_{2n}(J)$, in Eqs. (3.80) and (3.81), have the expressions

$$\begin{aligned} U_0(J) &= (\sigma^2/\Delta t)\log[R(J)/2], \\ U_{2n}(J) &= -(\sigma^2/n\Delta t)\,[J/\sigma^2 R(J)]^n, \end{aligned} \tag{3.83}$$

where

$$R(J) = 1 + (J/\sigma^2) + [1 + 2(J/\sigma^2)]^{1/2}. \tag{3.84}$$

Taking for J the typical value $J \cong 4\sigma^2$, Eq. (3.80) becomes

$$n \cong (20/\rho\varepsilon^2)^{1/5}, \tag{3.85}$$

which, inserted into Eq. (3.81), gives

$$(\rho\varepsilon^2)^{1/5}\log(150\,\varepsilon^4/\rho^3) \geq 6. \tag{3.86}$$

When this inequality is satisfied, the weak beam blows up. In Fig. 11 we have plotted the threshold value ε_{th} against ρ obtained from (3.86). This plot is only valid for those values of ε and ρ satisfying our previous assumptions. In Appendix C it is shown that this is the case, provided ρ is in the range $0.02 < \rho < 0.08$. Let us remark that, in the range of validity of the theory, the maximum attainable interaction strength ε_{th} is an increasing function of ρ (see Fig. 11). This behaviour has actually been observed experimentally [89], though for different values of ε and ρ. We should have expected such a discrepancy since we have used only a one-dimensional idealized model, but it is encouraging that our results are at least in qualitative agreement with experience.

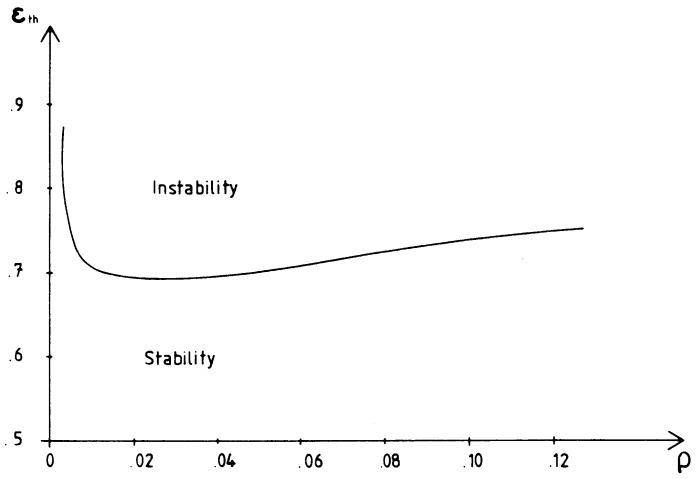


Fig. 11 Maximum attainable interaction strength ε_{th} as a function of the inverse of the number of kicks per damping time ρ.

3.6 CONCLUSIONS

We have considered a damped, noisy oscillator subject to the effect of a non-linear perturbation, consisting of a regular sequence of kicks. Our main result is the derivation of an equation satisfied by the steady-state distribution function averaged over the phase variable ϕ. This equation, referred to as the renormalized Fokker – Planck equation, takes into account the effect of the noise, of the damping and of the non-linear perturbation. The results presented here apply to the general theory of discrete maps with noise, establishing a formal procedure to deal with the classical problem of the "small denominators".

In the case of the strong beam – weak beam interaction in an electron – positron storage ring, the solution of the renormalized Fokker – Planck equation, subject to the condition that the size of the weak beam does not blow up, gives the dependence of the maximum attainable luminosity on the physical parametrs of the storage ring. Since we have considered only an idealized model of one-dimensional betatron oscillations, it is not possible to make a detailed comparison between our results and the experimental data. However, these results are at least in qualitative agreement with experience.

APPENDIX A

In this appendix we will give the explicit expression of the operator D_{nm}, introduced in Eq. (3.44), in terms of the functions f_1, f_2, g_1 and g_2 defined by (3.34) and (3.38). Then we shall apply this formula to the case of the betatron oscillations described by Eq. (3.9).

We begin by recalling that the Fourier transform of the product of two functions is the convolution integral between the Fourier transforms of those two functions. When the functions under consideration are periodic, the convolution integral becomes a "convolution series". Thus, if we denote by a subscript the Fourier components of a function with respect to the phase variable ϕ, from Eqs. (3.40), (3.41) and (3.44) we obtain

$$\begin{aligned} D_{nm} = \Delta t \{ & in(f_1)_{n-m} - (d/dJ)(f_2)_{n-m} \\ & + (1/2) \sum_k [(d/dJ)(g_2)_{n-m-k}(d/dJ)(g_2)_k - i(k+m)(d/dJ)(g_2)_{n-m-k}(g_1)_k \\ & - in(g_1)_{n-m-k}(d/dJ)(g_2)_k - n(k+m)(g_1)_{n-m-k}(g_1)_k] \}. \end{aligned} \tag{A.1}$$

From Eq. (3.70) we see that the coefficients C_n depend only on the diffusive part of D_{nn}, i.e. on the term with second-order derivatives. Equation (A.1) for D_{nn} can thus be written as

$$D_{nn} = (\text{non-diffusive terms}) + (1/2)\, \Delta t\, [(g_2)^2]_0\, d^2/dJ^2, \tag{A.2}$$

where, according to our previous notation, $[(g_2)^2]_0$ is the average over ϕ of the function $(g_2)^2$.

In the case of the betatron oscillations discussed in Section 3.2, by a comparison between Eqs. (3.9), (3.14), (3.32) and (3.34), we see that the functions f and g are

$$f(q,p) = -p/\Delta t, \qquad g(q,p) = \sigma\, (2/\Delta t)^{1/2}. \tag{A.3}$$

Thus from Eq. (3.38) we have

$$f_1 = (1/2\Delta t)\sin 2\phi, \qquad f_2 = -(2J/\Delta t)\cos^2\phi,$$
$$g_1 = -\sigma\,(J\Delta t)^{-1/2}\sin\phi, \qquad g_2 = 2\sigma\,(J/\Delta t)^{1/2}\cos\phi. \tag{A.4}$$

From (A.1) and (A.2) it then follows that

$$D_{oo} = (d/dJ)\{J + \sigma^2 J\, d/dJ\},$$
$$D_{nn} = (\text{non-diffusive terms}) + \sigma^2 J\, d^2/dJ^2. \tag{A.5}$$

These results are used in Section 3.5.

APPENDIX B

In this appendix we shall show that, under the assumption (3.35), Eq. (3.43) becomes Eq. (3.45). We begin by multiplying both sides of Eq. (3.43) by the factor $\exp(\varepsilon A_{nn})$, thus obtaining

$$[\exp(\varepsilon A_{nn})\,(e^{-\varepsilon \mathbf{A}})_{nn} - \exp(\varepsilon A_{nn})\,(e^{\mathbf{B}})_{nn}]\,P_n^+ = \exp(\varepsilon A_{nn}) \sum_{m\neq n} [(e^{\mathbf{B}})_{nm} - (e^{-\varepsilon \mathbf{A}})_{nm}]\,P_m^+. \tag{B.1}$$

From (3.44) it follows

$$(e^{\mathbf{B}})_{nn} = \exp[B_{nn} + O(\rho^2)],$$
$$\exp(\varepsilon A_{nn})\,(e^{-\varepsilon \mathbf{A}})_{nn} = 1 + (\varepsilon^2/2) \sum_{m\neq n} A_{nm} A_{mn} + O(\varepsilon^3), \tag{B.2}$$

and, for $n \neq m$,

$$(e^{\mathbf{B}})_{nm} = O(\rho),$$
$$(e^{-\varepsilon \mathbf{A}})_{nm} = -\varepsilon A_{nm} + O(\varepsilon^2). \qquad n \neq m, \tag{B.3}$$

Thus Eq. (B.1) becomes

$$\{1 + (\varepsilon^2/2) \sum_{m\neq n} A_{nm} A_{mn} + O(\varepsilon^3) - \exp(\varepsilon A_{nn}) \exp[B_{nn} + O(\rho^2)]\}\,P_n^+$$
$$= [1 + O(\varepsilon)] \sum_{m\neq n} [O(\rho) + O(\varepsilon^2) + \varepsilon A_{nm}]\,P_m^+. \tag{B.4}$$

From (3.35) we see that ρ is much smaller than ε and so, in the r.h.s. of Eq. (B.4), we can neglect $O(\rho)$ and $O(\varepsilon^2)$ with respect to the term of order ε. In the l.h.s. of the same equation, we can neglect $O(\rho^2)$ with respect to B_{nn} and $O(\varepsilon^3)$ with respect to the term of order ε^2, thus obtaining Eq. (3.45).

APPENDIX C

In this appendix we check if there is some range of values of ρ, such that our previous assumption (3.35) is satisfied by the corresponding threshold value of ε, given by (3.86) with the equal sign assumed.

We remark that the first inequality in (3.35) is needed in order to satisfy condition (3.52). Since in the derivation of Eq. (3.86) we have only used an estimate of C_{2n}, with n given by (3.85), from (3.71) we see that condition (3.52) can be replaced by

$$\varepsilon^2 C_{2n} < \varepsilon\,[\varepsilon/\rho(2n)^2]^{1/3} = [\varepsilon/(2n)^{2/3}]\,(\varepsilon/\rho)^{1/3} \ll 1. \tag{C.1}$$

Thus our previous assumption (3.35) becomes

$$\varepsilon/(2n)^{2/3} < (\rho/\varepsilon)^{1/3} \ll 1, \tag{C.2}$$

where n is given by Eq. (3.85).

When ρ is in the range $0.02 < \rho < 0.08$, the maximum attainable ε given by (3.86) is about $\varepsilon_{th} \cong 0.7$. Since in this range $(\rho/\varepsilon)^{1/3} \cong 0.3$, $n \cong 5$ and thus $\varepsilon/(2n)^{2/3} \cong 0.15$, conditions (C.2) are quite well satisfied. Therefore the range of validity of our results is $0.02 < \rho < 0.08$.

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