CLOSED AND OPEN STRING THEORIES IN NON-CRITICAL BACKGROUNDS

Sameer Murthy

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Abstract

This thesis is a study of closed and open string theories in low dimensional space-times, and the various relations between these theories. In particular, we focus on the theory of the two-dimensional black hole.

We first study closed strings in the background of the Euclidean two-dimensional black hole ($SL_2(\mathbb{R})/U(1)$) tensored with flat space, using the duality relating these theories to non-critical superstrings described by the supersymmetric sine-Liouville interaction on the worldsheet. We point out a subtlety in their geometric interpretation, and clarify the symmetry structure of the theories based on the understanding of these theories as near horizon limits of wrapped NS5-branes. In one such example (cigar $\times \mathbb{R}^6$), we use the brane description to understand the enhancement of the global symmetry in the coset theory from $U(1)$ to $SO(3)$ under which the sine-Liouville and cigar interactions are related. In the same example, a worldsheet description of the moduli space $\mathbb{R}^4/\mathbb{Z}_2$ is presented.

We then study open strings in the topologically twisted Euclidean two-dimensional black hole which is equivalent to noncritical $c = 1$ bosonic string theory compactified on a circle at self-dual radius. These strings live on D-branes that are extended along the Liouville direction. We present explicit expressions for the disc two- and three-point functions of boundary operators in this theory, as well as the bulk-boundary two-point function. The expressions obtained are divergent because of resonant behavior at self-dual radius. However, these can be regularised and renormalized in a precise way to get finite results. The boundary correlators are found to depend only on the differences of boundary cosmological constants, suggesting a fermionic behaviour. We initiate a study of the open-string field theory localized to the physical states, which leads to an interesting matrix model.

Finally, we present evidence that the worldvolume theory of N unstable D-particles in type IIB superstring theory in two-dimensions is represented by the supersymmetric matrix model of Marinari and Parisi. This identification suggests that the matrix model gives a holographic descriptions of superstrings in a two-dimensional black hole geometry.

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Contents

Chapter 1

Introduction, Summary and Discussion

1. What is this dissertation about?

This dissertation is an investigation of a few related topics in the quantum theory of one-dimensional objects - *strings* - living in certain spacetimes having the feature that the strength of the string interactions - the *string coupling* - varies as a function of one of the spatial directions. These theories are also known as *non-critical* string theories, and there are many known examples with a small number of dimensions (two or less). In particular, this thesis shall focus on a solution of these theories which, from the spacetime point of view, looks like a black hole in two dimensions. Since string theory is a consistent theory of quantum gravity, this is exciting because it might throw some light on issues in the physics of black holes, like the problem of information loss, *i.e.* understanding unitarity of the quantum theory, and the origin of entropy.

Following this introductory chapter, this thesis consists of three chapters based on the works [1–3]. The second chapter discusses the gravitational theory of closed superstrings in various spacetime backgrounds in less than ten dimensions which contain the two dimensional black hole geometry as the most important part. The main focus is on the geometry of these backgrounds, and where these theories fit into the space of solutions of superstring theory. The third and the fourth chapter, motivated by the concept of *holography* as manifested by *open-closed string duality*, study *open* strings in the black hole with the intention of getting information about the *closed* string gravitational theories. The third chapter sets forth the quantum field theory of open strings on *D-branes* in the topological Euclidean black hole. The fourth chapter studies the *tachyonic* open string theory in the physical black hole background and based on this, proposes a dual unitary quantum theory to describe the black hole.

The current chapter has two goals. Firstly, it is meant to put this thesis in perspective for the reader with a general interest in particle physics, gravitation and string theory by introducing the main ideas in the thesis along with some relevant background. Secondly, for the reader with more specific interest in the topics discussed – superstrings in low dimensions, the two dimensional black hole, open-closed string duality and matrix models – a brief summary of results and some discussion relating the various threads is provided towards the end of the chapter. More detailed discussions are present in the corresponding chapters and each chapter can be read as a logically independent unit with its own conclusions. In view of this, there is no separate concluding chapter.

We shall begin with some general motivation to study the non-critical theories, and the rest of the chapter is devoted to introducing the background material for the three chapters to follow, and to explain some of the *terms, ideas and concepts* appearing in the above paragraphs in a heuristic manner. A general reference for the background material on string theory is [4]. All other references to topics at hand are indicated in the chapters where they arise.

The motivation to study non-critical string theories is manifold, among the primary ones are:

1. These theories are quantum theories of gravity, which are tractable, and in some cases exactly solvable. They do not exactly model the world we live in; nevertheless, they contain many lessons about a theory of quantum gravity in four large spacetime dimensions, which has been difficult to understand so far, despite a lot of progress along many directions.
2. The theory of strong interactions of particle physics – Quantum Chromo Dynamics (QCD) – which governs the force that holds a nucleus inside an atom together has string-like excitations of the flux of "glue", the carrier of the strong force. These flux tubes bind quarks to one another. The theory of these strings of glue is very similar to the non-critical string theories that we will study. One hopes to learn lessons about QCD by studying the non-critical string theories.
3. In recent years, it has been realized through many examples, that the theory of strong interactions, and gauge theories in general, have deep connections with the theory of gravity in certain spacetimes. This idea is part of a more general set of ideas which together go by the name of holography, which relates the quantum theory of gravity of a particular spacetime to non-gravitational

theories like QCD, in a related spacetime of one dimension less. Non-critical string theories are another fertile field to explore and extend this connection further.

4. Non critical string theories admit solutions which resemble black holes in four dimensions. The possibility of studying black holes via a holographic unitary quantum theory is obviously very exciting towards understanding the issue of unitarity, or information loss in a black hole.

We shall begin below with a brief introduction to string theory as a quantum theory of gravity. Next we shall briefly describe the tool used to compute amplitudes for processes in this theory which is the two-dimensional quantum field theory living on the surface that a propagating string sweeps out in spacetime. We shall then turn to general conditions that this theory must satisfy in order to make the theory consistent and discuss the fact that if the spacetime has less than a certain critical number of dimensions, then it cannot be flat. In the process of trying to formulate a theory of strings in a few flat dimensions[1], the string worldsheet develops another dynamical mode which can be regarded as an extra dimension along which the curvature and the string coupling varies. This extra dimension, known as the Liouville field is a hallmark of non-critical string theories.

Then follows a brief description of supersymmetry in string theory, a symmetry which is a natural extension of the Poincare symmetries which we observe at low energies, and at the technical level, is an ingredient which makes the theory stable.

As a quantum theory of gravity, string theory must be able to describe the physics of black holes. We turn next to a brief discussion of black holes in string theory, and the relation among black holes in two dimensions and the supersymmetric non-critical string.

We next discuss the idea of a manifestation of holography as a duality between open and closed string theories. This gives us a very precise set-up in which to explore the idea of holography further and in particular, we shall discuss how the setting of non-critical string theories is a natural environment in which this kind of relation arises, and also, how due to their tractability, they hold the promise of being a very important laboratory for testing these ideas.

[1] One exmaple being the QCD string, which one might at first try to formulate in four flat dimensions

In Chapters 2,3 and 4, we present detailed investigations along some of the lines sketched above. In the rest of this introductory chapter, we shall point out along the way where and how the three bulk chapters fit into the scheme of things.

2. String theory in general spacetimes

String theory is presently the best candidate for a quantum theory of gravity. It is a quantum theory of one-dimensional objects (strings) which may or not have a boundary (open/closed strings). The theory is perturbatively defined by a sum (functional integral) over Riemann surfaces, analogous to the Feynman diagram expansion in field theory. Different modes of vibration of the string are interpreted as different particles with different mass, charge, spin and other quantum numbers. A Riemann surface then corresponds to a dynamical process in spacetime where some strings (particles) come in, interact with each other and some other particles go out. The amplitude associated to such a diagram has a factor of the string coupling constant g_s for each vertex where three strings meet.

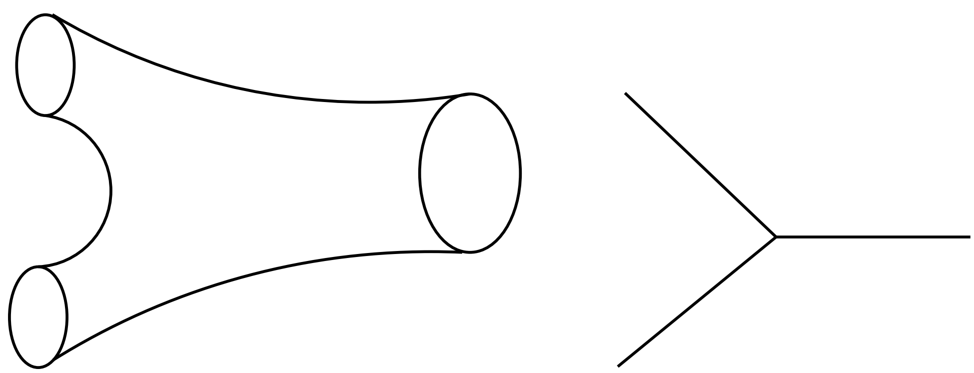

Fig. 1: The string worldsheet (left) naturally spreads the interaction over a finite region of spacetime; point particle theory (right) has interactions at one spacetime point.

One of the main reasons this theory is so attractive as a theory of quanutm gravity is that the spectrum of vibrations of the closed string always contains a spin two massless particle, *i.e.* a graviton, a quantum of the gravitational field. The fact that the interaction between the strings is not localized to one point in spacetime renders the theory perturbatively finite and consistent.

To formulate a perturbative quantum theory of gravity, one thinks of fluctuations around a given background for the metric and the other matter fields in the theory. The theory must specify the interaction between these fluctuations. For the sake of illustration, let us suppose that the background spacetime parametrized by variables X^μ is completely specified by the metric $G_{\mu\nu}(X)$, that is to say that there are no macroscopic sources which produce other background fields in the spacetime. The string propagating in this spacetime describes a two dimensional surface $X^\mu(\sigma)$, known as the *worldsheet* of the string. Perturbation theory is then defined in terms of a two dimensional quantum field theory living on the worldsheet parameterized by the variables (σ_1, σ_2):

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{h_{\alpha\beta}} G_{\mu\nu}(X) \partial_\alpha X^\mu \partial^\alpha X^\nu. \tag{2.1}$$

$h_{\alpha\beta}$ is the fluctuating metric on the two dimensional surface which has to be integrated over, α' is the one dimensionful parameter in the theory (Planck scale).

Extremizing this two-dimensional action describes the classical shape of the worldsheet. The full quantum theory is defined by a path integral over the fields describing the embedding of the worldsheet in spacetime, as well as the metric on the worldsheet. For example, the partition function is

$$Z = \int [DX^\mu][Dh_{\alpha\beta}] e^{-S[X^\mu, h_{\alpha\beta}]}. \tag{2.2}$$

The perturbative expansion of this path integral gives a Feynman diagram-like expansion:

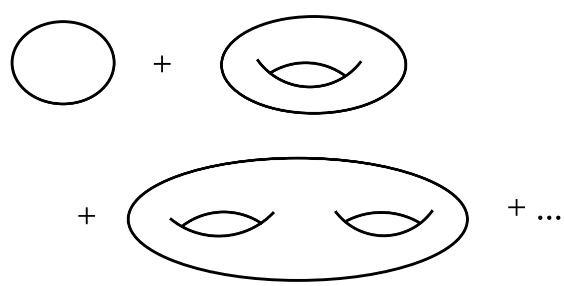

Fig. 2: The perturbative expansion over different genera

Fluctuations of the worldsheet about this stationary point are the dynamical fields of the theory. As mentioned above, one of the fluctuations is the perturbations of the metric itself, or the graviton. Along with the graviton, the string spectrum contains other fields, which could be thought of as arising from matter sources. For example, there could be sources corresponding to $p-$forms which are the generalizations of the electromagnetic field strength. At low energies compared to the string scale, the light modes should be observable and there is an effective field theory governing their interactions.

Among the fields in the theory, there is always a scalar field Φ called the dilaton which is the trace of the metric tensor. The zero mode of this field couples to the integral of the scalar curvature of the worldsheet, which is simply its Euler number. In the perturbation theory of (2.2) in Fig[2], the dilaton is related to the string coupling as $g_s = e^{\Phi}$. The interaction between the strings is, in this sense naturally encoded in the topology of the Riemann surface.

In order to reproduce known physics at low energies, the two dimensional quantum theory on the worldsheet is required to be conformal, *i.e.* invariant under local changes of scale on the worldsheet. Conformal invariance of the theory plays an important role in making the theory consistent. At the classical level (in the two dimensional theory), any flat background is conformal. At the quantum level, the background is required to satisfy certain conditions. The departure from conformal invariance of the theory can be measured in terms of a number known as the conformal anamoly, or central charge c. The theory on the worldsheet includes reparameterization ghosts which contribute $c = -26$. Thus the condition for conformal invariance is that the matter theory which represents the spacetime has $c = 26$.

Each spacetime dimension contributes to the central charge by one unit. The conformal invariance condition is satisfied by 26 flat dimensions (for the bosonic string – the supersymmetric string has more fields and satisfies the condition with 10 flat dimensions). More generally, any conformal field theory with central charge $c = 26$ serves as a consistent background for the bosonic string. For more general backgrounds with curvature and sources, the condition for conformal invariance involves a relation between these sources and the curvature of the metric which, at lowest order in α' reduces to Einstein's equations. In this fashion, string theory is a well defined theory at high energies (distance scales of $\sqrt{\alpha'}$), and reduces to general relativity at larger length scales where we know its veracity.

3. Strings in low dimensions: Non-critical backgrounds

The spectrum of the bosonic string contains a scalar field called the tachyon, which in general, is an unstable mode in the perturbation theory. A very simple non-trivial background which is conformally invariant is one where the spacetime in d dimensions has a flat metric and the only field which has an expectation value is the scalar tachyon:

$$
\begin{aligned}
&G_{\mu\nu}(X) = \eta_{\mu\nu}, \quad \Phi(X) = \left(\frac{26-d}{6\alpha'}\right)^{\frac{1}{2}} X^1 \\
&T(X) = \exp(qX^1), \quad q = \left(\frac{26-d}{6\alpha'}\right)^{\frac{1}{2}} - \left(\frac{2-d}{6\alpha'}\right)^{\frac{1}{2}}
\end{aligned}
\tag{3.1}
$$

Translational invariance in this spacetime is broken by the background tachyon and dilaton. Notice that the string coupling which is determined by the dilaton grows indefinitely along the X^1 direction. If the number of dimensions $d > 2$, the theory is unstable and there is really a tachyon in the spectrum. For $d \leq 2$, the tachyon field is stable, (and the word tachyon is a misnomer).

Since the tachyon field has a background, it should be added to the worldsheet action (2.1):

$$
S = S_{free} + \int d^2\sigma \, T(X) \tag{3.2}
$$

For $d \leq 2$, the tachyon field is real and the exponential potential suppresses the path integral (2.2) in the strong coupling region. Thus it effectively acts as a barrier and prevents strings from penetrating too deep into the strong coupling region. Perturbation theory is thus valid in this theory in spite of the fact that the string coupling diverges in a certain region of spacetime.

If the free field direction is compactified on a circle, the spacetime manifold is a flat cylinder. Asymptotically, there is no potential, and the only non-trivial field is the dilaton which varies linearly along the non-compact direction.

In fact, if one does not demand that the worldsheet theory be conformally invariant, and instead thinks of the worldsheet theory as matter coupled to 2-dimensional gravity, the measure in the path integral over metrics makes the scale factor in the metric a dynamical field, known as the Liouville field which behaves like the direction X^1 above, and supplies the additional central charge. This was historically known as non-critical string theory.

3.1. Supersymmetry

In order to look for a realistic (stable) theory of the world in more than two dimensions, one must get around the problem of the tachyons described above. One way to do that is to introduce fermions and supersymmetry in the theory. In addition to the bosonic degrees of freedom corresponding to the embedding dimensions, one can consider theories with fermionic variables on the worldsheet. Instead of two dimensional gravity coupled to matter, one has two dimensional supergravity coupled to a supersymmetric matter theory.

If there are a sufficient number of fermionic degrees of freedom, (so that there is $\mathcal{N} = (2,2)$ supersymmetry[2] on the worldsheet), one gets consistent supersymmetric theories in spacetime which are stable and do not contain any tachyonic degrees of freedom.

The theory now also contains reparameterization ghosts for the fermionic degrees of freedom, and they contribute with central charge $c = 11$, so the critical central charge for the superstring is $c = 26 - 11 = 15$. Since each bosonic field with $c = 1$ has a fermionic partner with $c = 1/2$, this corresponds to 10 flat dimensions. In lower dimensions, one can have supersymmetric generalizations of the non-critical string (3.1). It turns out (as we shall study in detail in chapter 2), that the $\mathcal{N} = 2$ supersymmetric version that we are interested in has, in addition to the Liouville coordinate ρ, an angular bosonic coordinate θ, and two fermions ψ_ρ, ψ_θ which are coupled to each other via the worldsheet interaction:

$$S_{int}^{sine-Liouville} = \psi\tilde{\psi}\, e^{-\frac{1}{Q}(\rho+\tilde{\rho}+i(\theta+\tilde{\theta}))} + c.c \tag{3.3}$$

where $\psi = \psi_\rho + i\psi_\theta$ is the superpartner of $\rho + i\theta$ and $\tilde{\psi}$ is its rightmoving counterpart.

Supersymmetry ties the two fields together into a complex field which we shall consider to have Euclidean signature in spacetime. The bosonic part of the interaction is called the sine-Liouville interaction. The purely bosonic sine-Liouville theory is classically unstable, the supersymmetric counterpart of course is well-defined. Asymptotically, this theory also reduces to a linear dilaton theory on a cylinder like the compactified bosonic $d = 2$ theory (3.1) considered above. However, the symmetries of the theory are different. The bosonic $d = 2$ theory had a

[2] These numbers correspond to the number of fermionic supercharges in the theory, of positive and negative chirality.

global $U(1) \times U(1)$ symmetry, one of the $U(1)$'s corresponding to the momentum around the circle, the other to the winding of strings around the circle which is also conserved. The sine-Liouville potential breaks the $U(1)$ of momentum by giving an expectation value to a momentum mode. We shall see the same phenomenon in another, seemingly different context later.

4. Black holes in low dimensions

As a theory of quantum gravity, string theory should be able to describe processes involving strong gravitational fields like black holes. String theory, or its low energy limit supergravity, is known to admit solutions which are black, in the sense that there is an event horizon which causally disconnects one region of spacetime from another.

A particular example which is understood as an exact perturbative string background, and not just in the supergravity limit is that of a black hole in two spacetime dimensions. The spacetime manifold can actually be thought of as parameterizing a coset of the group $SL(2, \mathbb{R})$ by a $U(1)$ subgroup, and this symmetry can be exploited to understand the solution to all orders in α', beyond the gravity approximation. There is a discrete parameter in the solution which is a positive integer, and in fact, for small values of the integer, the solution is highly curved and is well beyond the regime of classical gravity.

If we perform a Wick rotation of the time coordinate into Euclidean space, the region behind the horizon is excised and we have a smooth manifold which has the shape of a cigar, labelled by the coordinates $(\rho \geq 0, \theta \sim \theta + \frac{4\pi}{Q})$. The non-trivial fields in spacetime are the metric and the dilaton:

$$\begin{aligned} ds^2 &= d\rho^2 + \tanh^2(\frac{Q\rho}{2})d\theta^2; \\ \Phi &= -\log\cosh(\frac{Q\rho}{2}). \end{aligned} \tag{4.1}$$

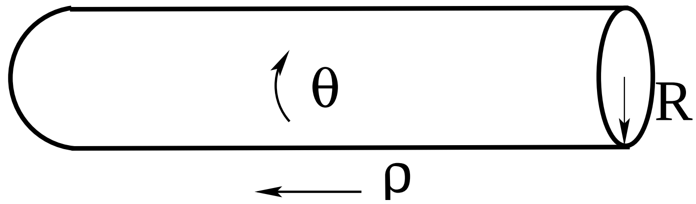

Fig. 3: The Euclidean black hole or the cigar

This solution, apart from being of intrinsic interest as a black hole, arises in many contexts in string theory - for example as the near-horizon limit of $NS5$-branes, and blow-ups of singular Calabi-Yau manifolds.

Asymptotically, the manifold looks like a cylinder with the dilaton varying linearly along its length. This resembles the compactified $d = 2$ bosonic theory and the sine-Liouville theory, both of which we encountered above. The cigar manifold also has only a single $U(1)$ symmetry of momentum around the cigar. Strings winding around the circle can slip off the tip of the cigar and can unwind. The cigar and the sine-Liouville solutions have the same behavior, that of a linear dilaton on a cylinder and one of the $U(1)$ symmetries of the cylinder being broken by the behavior of the solution in the strong coupling region. In fact, it is easy to see that asymptotically, the two solutions are related by a duality called T-duality which takes a circle of small radius into a circle of large radius, and interchanges the momemtum and winding of the strings.

It is a non-trivial fact that this duality is a true relation between the two exact CFT's. More precisely, the supersymmetric sine-Liouville theory and the supersymmetric version of the cigar theory are related by a duality called Mirror symmetry, which in the asymptotic region reduces to T-duality.

In chapter 2, we shall study the supersymmetric version of the 2-d black hole using this duality and discuss the geometry of these solutions in detail, as well as higher dimensional solutions which include the 2-d black hole as a part. We find that there is a subtlety in the geometric interpretation of the higher dimensional theories and that the spectrum is most naturally interpreted as couplings to currents in a putative holographic boundary theory. We also explain how the symmetries of the various theories can be understood as those of the near horizon limit of certain brane configurations in superstring theory. Based on this understanding, we discover a certain global symmetry relating the sine-Liouville and the cigar worldsheet theories. This throws some light on the duality between the two theories,

In chapter 4, we shall discuss a proposal for an exact non-perturbative description of the 2-d black hole using a duality between open and closed strings. In chapter 3, we shall study the open string theory in the example of the bosonic two-dimensional theory (3.1), (3.2). Apart from being a simpler example than the black hole, this bosonic theory actually turns out to capture the topological aspects of the 2d black hole. As a build-up to these studies, in the next part of the introduction, we shall discuss at an introductory level, the powerful open-closed string duality in string theory.

5. Open-closed string duality, branes, holography

As we mentioned earlier, strings can be either open (with two endpoints as boundaries), or closed loops. The closed string theory is a consistent quantum theory of gravitons and other closed strings. The open string theory does not contain gravitons in the perturbative spectrum and generally contains gauge bosons.

In closed string theory, there are solutions to the classical equations of motion analogous to instantons in gauge theory, in the sense that their action is proportional to $1/g_s^2$. There are also other solutions whose action is proportional to $1/g_s$. These are called D-branes. They are sources for gravitons – meaning they have mass, and curve space around them – and other closed strings (in the case of p-forms, this means that they are charged under those potentials). They can be thought of as a condensate of closed strings. It was realized by Polchinski in 1995 that D-branes have another description in terms of *open* string theory as hypersurfaces on which open strings can end. This realization led to a very important discovery of a duality between gravitational theories in the presence of these D-brane sources, and ordinary non-gravitational theories like gauge theories which live on these D-branes.

At the nuts-and-bolts level, this duality can be be understood by the following simple observation:

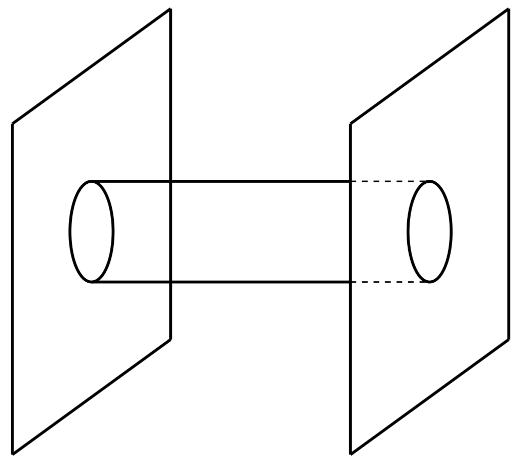

Fig. 4: Closed string exchange between two branes or equivalently, an open string running in a loop.

The picture [Fig.4] can be understood as a closed string tree-level exchange between the two D-branes, *or* as a open string one-loop diagram, where the open strings has its ends on the two branes.

More physically, the duality between closed string (gravitational) theories, and the corresponding open string (non-gravitational) ones can be understood as a realization of *holography* - which is a paradigm which states that the degrees of freedom of a gravitational theory in a certain region of spacetime live on the boundary of that region. This has been a powerful paradigm to understand the entropy of black holes as the well-understood entropy of the non-gravitational theories living on the boundary of spacetime. A much-studied example of this is the duality between type IIB superstring theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ Yang-Mills theory on $\mathbb{R}^4$. The radial direction in AdS_5 is understood as the energy scale in the Yang-Mills theory.

Another fertile field of examples of holography are the non-critical string theories we discussed earlier. The Liouville direction, which is the scale factor in the metric is naturally associated with energy scale of the worldsheet. If we choose a gauge in which the time on the worldsheet is related to the time in spacetime, we can understand that the Liouville direction is naturally associated with the energy scale in spacetime.

In the early nineties, a lot of progress was made in understanding the non-critical bosonic theories by discretizing the worldsheet, and performing a path-integral over random surfaces. This procedure led to a non-perturbative formulation of these theories as a quantum mechanics of square matrices of size N. These matrices have a self coupling κ, and we need to take a special limit where $N \to \infty$, and the coupling is tuned to a critical value $\kappa \to \kappa_0$ keeping a particular combination $(\kappa - \kappa_0)^x N$ fixed[3] which corresponds to the string coupling.

This description of two dimensional closed string theory as a one-dimensional (quantum mechanical) theory of matrices can also be thought of as holography. Recently, it was realized that this duality can also be understood as a open-closed string duality. The matrix model was realized as an effective open string theory living on D-branes in the non-critical background.

This duality has recently been under intense study, and one tool which has been added to the string theorist's arsenal to understand closed string theories in the background of D-branes is to study the corresponding open string theories living on the branes. The complete theory of open strings on the D-branes can be described by what is called *open string field theory* (OSFT). Sen has made a precise conjecture that the full OSFT on D-branes is dual to the theory of closed strings that the branes couple to.

In many cases, the cost one has to pay for completeness is complication - in most examples, the open string field theory is only understood using approximations like truncating the theory to a finite set of modes. In the case of bosonic $d = 2$,[4] one actually has a handle on the full OSFT on branes in the theory.

In chapter 3, we enunciate and explore the open string field theory of the compactified $c = 1$ theory in some detail. We find that the OSFT reduces to a model of matrices living on a two-sphere with coefficients depending purely on the open string parameters. The form of the action also suggests that the branes can be thought of as fermions in the theory.

In the cases where the full open string is not easily tractable, one technique which has been often used successfully is to consider limits in which most of the open string modes decouple and the open string theory reduces to something simple (like the quantum mechanics of free fermions for bosonic $d = 2$).

In chapter 4, we shall use this technique to study the two-dimensional superstring which, as we saw above is equivalently a theory of the supersymmetric

[3] The number x depends on the particular non-critical theory in question

[4] Because of the non-critical string interpretation, this is referred to, in the literature as the $c = 1$ theory.

two-dimensional black hole. We present evidence that the theory of the black hole is described by a certain supersymmetric quantum mechanics of large N matrices, known as the model of Marinari and Parisi.

6. Discussion

A full quantum mechanical description of a black hole is obviously very exciting, one would then hope to describe strong gravitational processes like formation of a black hole and Hawking radiation in a unitary fashion. We have now quite a few examples of black hole solutions in string theory, some of which even have brane descriptions whose entropy can be microscopically counted. However, most of them are at the level of supergravity, and the entropy counting can be really done for solutions which (unrealistically) preserve a lot of supersymmetry. The two-dimensional black hole, by virtue of its being an exact CFT overcomes these limitations and is a system which certainly deserves a lot of attention.

One of the fundamental questions which must be asked is: what is the nature of the true degrees of freedom of the black hole, or more generally of a gravitating system like a closed string theory? One answer which has arisen time and again in different guises[5] is simple: open strings. If one accepts this, the natural question to follow is: what is the theory which governs the dynamics of the open strings? Two kinds of answers have arisen in previous examples: the first kind of theory is when there is a special limit one must take which consistently focuses on certain open string modes, decoupling most of the infinite degrees of freedom of a typical string theory. The second kind is open string field theory - the quantum theory of all the open strings present in the background in question.

Examples of the first kind have produced in many cases quantum field theories which we understand well – analytically at weak coupling, and in principle on a computer in general. These have been extremely useful to describe gravitational theories in specific examples where there have been consistent limits which retain interesting dynamics. Adding to this list of examples is an important task which may help us understand some more general principles of this kind of duality.

The second kind of theories contain much more information in principle, but in practice has proven to be hard to understand, because of the nature of these theories

[5] Eg: AdS/CFT, Matrix theory, Old matrix models, Topological models.

– there are infinite modes, usually coupled to each other and with many derivatives in spacetime and it is not clear what a good approximation scheme is. Examples where open string field theories are tractable – even the first step of being able to write down the action for all the modes – are rare and therefore very important.

In the setting of non-critical string backgrounds, the rest of the chapters of this thesis take steps in the directions sketched above. We hope that it leads to a better understanding of the various issues alluded to in this introductory chapter.

Chapter 2

Non-Critical Superstrings in Various Dimensions

1. Introduction and summary

It has been known for a long time that consistent string theories can live in low number of dimensions. These theories typically develop a dynamical Liouville mode [5] on the worldsheet, and have been called non-critical string theories. These theories can be thought of as Weyl invariant string theories in a spacetime background which is one dimension higher, and contains a varying dilaton and a non-trivial 'tachyon' profile which corresponds to the Liouville interaction on the worldsheet. While the bosonic non-critical string is perturbatively consistent for one or less spacetime dimensions, the authors of [6] constructed non-critical theories with $\mathcal{N} = (2,2)$ supersymmetry on the worldsheet, which have the $\mathcal{N} = 2$ super-Liouville, also known as the sine-Liouville interaction, on the worldsheet giving rise to consistent string theories with spacetime supersymmetry in all even dimensions less than ten.

One aspect of this construction which has not been completely clear is the geometric interpretation of these theories. Fateev, Zamolodchikov and Zamolodchikov [7] conjectured that the sine-Liouville theory is equivalent to the conformal field theory describing the two dimensional Euclidean black hole or the cigar [8]. This conjecture was extended to the $\mathcal{N} = 2$ supersymmetric case by [9] and proved using the techniques of mirror symmetry by [10] (another derivation of the duality was recently given in [11]) . String theory in the black hole background has been previously studied by many authors [see e.g.[12]]. This duality provides a possible interpretation of the non-critical superstrings in d spacetime dimensions as an (infinite) set of fields propagating on the cigar tensored with $\mathbb{R}^{d-1,1}$, analogous to string theory in ten flat dimensions.

We shall study the non-critical theories in flat space for various values of d using the above two dual worldsheet conformal field theories:

1. The $\mathcal{N} = 2$ supersymmetric sine-Liouville theory tensored with flat spacetime[6] $\mathbb{R}^{d-1,1}$ as defined by [6].
2. The $\mathcal{N} = 2$ supersymmetric version of the cigar defined as the Kazama-Suzuki [13] supercoset $SL_2(\mathbb{R})/U(1)$, tensored with $\mathbb{R}^{d-1,1}$.

More recently, it has been understood how these theories fit into the moduli space of superstring theories. Motivated by the search for a holographic description of these theories, [9,14] conjectured that the non-critical theories arise as a certain double scaling limit of ten dimensional string theory. One approaches a point in moduli space of string theory on a Calabi-Yau manifold where it develops an isolated singularity, taking the string coupling to zero at the same time in such a way as to keep a combination of the two parameters fixed. To study the theory of the singular Calabi-Yau in the limit of $g_s \to 0$, one replaces the Calabi-Yau by its (non-compact) form near the singularity. To study the double scaling limit, one smoothes out the singularity by deforming the non-compact surface. The precise descriptions of the non-critical theories defined above is:

3. Superstring theory on $\mathbb{R}^{d-1,1}$ tensored with the non-compact manifold $\sum_{i=1}^{n} z_i^2 = \mu$, $n = (12-d)/2$, $z_i \in \mathbb{C}$.

A T-dual [15,16,17] of this description is given in terms of wrapped NS5-branes:

4. Superstring theory in the near-horizon background of NS5-branes with d flat spacetime directions and $6-d$ directions wrapped on $\sum_{i=3}^{n} z_i^2 = \mu$.

The appearance of NS5-branes is not surprising considering that the near horizon geometry of a stack of NS5-branes involves an infinite tube with the dilaton varying linearly along the length of the tube. It has also been noted [18] that singular geometries like (3) with $\mu = 0$[7] also involve an infinite tube for the winding modes.

[6] $d = 0, 2, 4, 6$; $d = 0$ is interpreted as the pure sine-Liouville theory. For $d = 8$, the extra two dimensions are flat, producing ten dimensional flat space string theory.

[7] Note that all the four descriptions presented above are singular at $\mu = 0$ - the first two descriptions in this limit have an infinite tube with a linearly growing dilaton, and the latter two have geometric singularities at the point $z_i = 0$. Turning on μ resolves the singularities - in (1), the sine-Liouville interaction is turned on, which provides a potential preventing strings from falling into the strong coupling region; in (2), the topology of the tube is changed to that of a cigar with the string coupling at the tip of the cigar determined by μ, thus eliminating the strong coupling singularity; in (3) and (4), the geometric singularities are smoothed out by the deformation.

In both the descriptions (3) and (4), some of the ten dimensions decouple in the limit - roughly speaking, (3) describes the string modes which are localised near the singularity and these only fluctuate in $d+2$ dimensions.

In this chapter, we use the above four descriptions in order to study and clarify some properties of the non-critical theories, in particular their geometric structure. The worlsheet description (1) gives us a handle on perturbative calculations like the spectrum. Using this, we find that there is a clear geometric interpretation in terms of a set of fields propagating on the cigar (2) *except* for the appearance of some discrete symmetries which are non-geometric from the cigar viewpoint. On the other hand, we show that these symmetries are natural from the point of view of descriptions (3) and (4). Indeed, they are simply the global symmetries of the above 5-brane (or Calabi-Yau) configurations which do not decouple after taking the limit described above. Our analysis also uncovers some new features of some of the sine-Liouville–cigar duality.

As just mentioned, in general, the physical spectrum cannot be interpreted as a set of fields propagating on a perhaps singular cigar-like manifold. From the point of view of the worldsheet CFT, this is due to the fact that the chiral $\mathbb{Z}_2$ symmetry used in the GSO projection is not the naive one of changing the sign of the chiral fermions, rather it acts by translation on the chiral part of the compact boson of the cigar in addition to the above action on the fermions. This is the only consistent choice for the GSO projection because the pure chiral rotation of the fermions on the worldsheet is anamolous due to the curvature of the cigar, and the conserved chiral $U(1)$ current acts on the compact boson of the cigar in addition to the fermions. This means that for a given field on the $d+2$ dimensional geometry, its spin in flat space is correlated with its momentum around the cigar. In the Green-Schwarz picture, this is due to the fact that the conserved current corresponding to the momentum around the cigar is a combination of the naive momentum and a piece acting on the worldsheet fermions.

The theories we consider asymptote to a linear dilaton geometry and are conjectured [19] to have holographic non-gravitational duals. In agreement with this statement is the structure of the target space supersymmetry in these theories; the supercharges anticommute to the flat spacetime[8] momentum generators. We

[8] In [6], the dilaton direction was interpreted as the time direction, and this was called space-supersymmetry.

demonstrate that the spectrum can be classified as current multiplets in the boundary theory, as consistent with the holographic interpretation. The conserved $U(1)$ momentum around the cigar is part of the R symmetry in the boundary theory. We shall exhibit this for the first few Kaluza-Klein modes for the $d = 4$ theory.

For $d = 6$, there is an explicit CFT interpretation as the near-horizon geometry of two parallel non-coincident NS5-branes [15]. The CFT describing the near-horizon geometry of $k \geq 2$ coincident five branes is $\mathbb{R}^{5,1} \times \mathbb{R} \times SU(2)_k$ [20]. There is an infinite throat along which the dilaton increases indefinitely as one approaches the location of the branes, and a trasverse 3-sphere. This theory on k coincident NS5-branes has the global rotation symmetry $SO(5,1) \times SO(4)$. Separating the 5-branes in the four transverse directions in a ring-like structure partially smoothes out the singularity, and the resulting theory has a symmetry $SO(5,1) \times U(1) \times Z_k$, and is conjectured to be string theory on $\mathbb{R}^{5,1} \times \frac{SL_2(\mathbb{R})_k}{U(1)} \times \frac{SU(2)_k}{U(1)}$.[9] [10] In the case $k = 2$, the bosonic sphere and the flux disappear and the deformed geometry is precisely the one we want to study for $d = 6$.

This geometric description of the $d = 6$ theory reveals some new features of the sine-Liouville–cigar duality. The moduli space of the theory is $\mathbb{R}^4/\mathbb{Z}_2$ corresponding to the separation of the two 5-branes, and the global symmetry of this configuration is $SO(4)$ broken to $O(3)$. We shall discuss these features in the conformal field theory. We find that the action of the CFT in the curved directions is not purely the sine-Liouville action, or the cigar action - only specific linear combinations of the two preserves the global symmetry. The $d = 6$ theory thus is an example where the duality between the sine-Liouville and the cigar is present already at the kinematic level, giving a better understanding of the duality which was conjectured based on dynamical reasons.

The plan of this chapter is the following: In section 2, we shall begin by reviewing the Euclidean black hole background and the non-critical superstring construction. Then we shall lay down the general features of the construction for all the dimensions. In section 3, we describe the special features of all the theories on a case-by-case basis. In particular, we describe many interesting features of the $d = 6$

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[9] The GSO projection relates the different factors and so the product is not direct.

[10] The WZW model corresponding to the sphere consists of bosonic $SU(2)$ currents at level $k - 2$ and fermionic currents at level 2.

theory. We also make a short note on the $\mathcal{N}=(4,4)$ algebra in this case. In section 4, we present the explanation of the global symmetries of the various theories using the embedding in ten dimensional flat space string theory. In Appendices A and B, we record the spectrum and one loop partition function of the various theories. Appendix C summarizes the details of the Green-Schwarz formalism, and Appendix D discusses the details of the conformal field theory at second order.

2. Superstring theories on the Cigar

The spacetime directions are $X^a = \rho, \theta, X^\mu$, $(\rho \geq 0, \mu = 0, 1..d-1)$. The geometry in the string frame[11] is that of a cigar tensored with flat spacetime:

$$\begin{aligned} ds^2 &= d\rho^2 + \tanh^2(\frac{Q\rho}{2})d\theta^2 + dX^\mu dX_\mu, \qquad \theta \sim \theta + \frac{4\pi}{Q}; \\ \Phi &= -\log\cosh(\frac{Q\rho}{2}), \qquad B_{ab} = 0. \end{aligned} \tag{2.1}$$

with the string coupling $g_s = e^\Phi$. This metric is a good one for string propagation because the dilaton obeys the equation $2D_aD_b\Phi + R_{ab} = 0$, where D_a is the spacetime covariant derivative, and R_{ab} is the spacetime curvature.

In the asymptotic region $\rho \to \infty$, the geometry reduces to $\mathbb{R}^{d-1,1}$ tensored with a cylinder of radius $R = \frac{2}{Q}$ with the dilaton varying linearly along its length. When $d+2$ fermions are added to this theory, it also has $\mathcal{N}=2$ supersymmetry. The currents of the cigar part of the theory in this region are

$$\begin{aligned} T_{\text{cig}} &= -\frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(\partial\theta)^2 - \frac{1}{2}(\psi_\rho\partial\psi_\rho + \psi_\theta\partial\psi_\theta) - \frac{1}{2}Q\partial^2\rho \\ G^\pm_{\text{cig}} &= \frac{i}{2}(\psi_\rho \pm i\psi_\theta)\partial(\rho \mp i\theta) + \frac{i}{2}Q\partial(\psi_\rho \pm i\psi_\theta) \\ J_{\text{cig}} &= -i\psi_\rho\psi_\theta + iQ\partial\theta \equiv i\partial H + iQ\partial\theta \equiv i\partial\phi \end{aligned} \tag{2.2}$$

In the exact cigar background the $\mathcal{N}=2$ supersymmetry is preserved but the exact expressions are more complicated. Away from the asymptotic region ∂H and $\partial\theta$ cannot be extended to conserved currents but their leftmoving linear combination $\partial\phi$ is exactly conserved. The current which asymptotically looks like $(j^\theta(z), \tilde{j}^\theta(\overline{z})) = \frac{iR}{2}(\partial\theta, \overline{\partial}\tilde{\theta})$ is conserved in the full theory and the corresponding charge P^θ is the

[11] We shall set $\alpha' = 2$ throughout this chapter

momentum around the cigar in units of the inverse radius. Since it is not purely leftmoving, the winding number is not conserved.

We conclude that the exact theory on the cigar has three conserved currents. $\partial\phi$ is leftmoving, $\overline{\partial}\tilde{\phi}$ is rightmoving and j^θ has both a leftmoving and a rightmoving component. In the asymptotic region where the theory looks like a linear dilaton theory one can describe the operators in terms of H, $\tilde{H}$, θ and $\tilde{\theta}$. But in the full theory it is better to use $\phi \approx H + Q\theta$ and $\tilde{\phi} \approx \tilde{H} + Q\tilde{\theta}$.

When $\mathbb{R}^{d-1,1}$ is tensored with the cigar, the currents $T_d, G_d^\pm, J_d$ of the d free bosons and fermions should be added to this CFT. All the conserved chiral currents now have additional pieces from the free CFT. The momentum around the cigar as defined above is still a symmetry, and there are also d conserved momenta in the flat directions. We can add the superconformal ghosts to this theory to form a consistent string background. The central charge of this theory, $\hat{c} = 2(1+Q^2)+d$ should be set to 10 which gives $Q = \sqrt{\frac{1}{2}(8-d)}$.

To construct theories with spacetime supersymmetry, we need to introduce supercharge operators. As seen above, the $U(1)$ R current of the worldsheet algebra in these theories involves the compact boson in addition to the worldsheet fermions. This means that the standard R-NS construction of the theory yields a spectrum which does not have a good spacetime interpretation as particles propagating in the $d+2$ dimensional curved spacetime; the spin of a particle and its momentum around the cigar are not independent. Rather, the theories have a natural holographic interpretation as a non-gravitational theory living in d dimensions. As we shall see, this feature is intimately related to the structure of supersymmetry in such curved spacetimes.

To emphasize the above point, we shall first construct the bosonic Type 0 theories, and then generate the supersymmetric Type II theories as a $\mathbb{Z}_2$ orbifold of the former. The bosonic Type 0 theories have no surprises and we shall quickly review the standard construction. We shall then see that the chiral GSO projection to get to the Type II theories has a subtlety involved, and the spectrum looks slightly unusual from the cigar perspective. We shall then present the classification of the particles of the Type II theories as off shell operators in the d dimensional holographic theory.

2.1. Type 0 theories on the cigar $\times \mathbb{R}^d$

We first construct the chiral (left or right moving) part of the vertex operators by demanding that they are physical operators in the theory. For example, in the pure cigar ($d = 0$) case, the the form of the vertex operators in the asymptotic region where all the worldsheet fields are free is (ignoring discrete states)

$$\begin{aligned} \mathcal{O}^{NS}_{nkp} &= e^{-\varphi+inH+ik\theta+(-1+p)\rho} \\ \mathcal{O}^{R}_{nkp} &= e^{-\frac{1}{2}\varphi+i(n+\frac{1}{2})H+ik\theta+(-1+p)\rho} \end{aligned} \tag{2.3}$$

Here $n \in Z$ and $p \geq 0$, and φ is the bosonized superconformal ghost [21]. In the higher dimensional theories, the operators are functions of the free fields as well. Physical operators also obey the condition of BRST invariance. To form the closed string theories, we put together the left and right movers with the following conditions:

1. Modular invariance demands a diagonal GSO projection, i.e. the same boundary conditions for the left and rightmoving fermions, together with

$$\begin{aligned} 0B &: (-)^{j_L} = (-)^{j_R}; \\ 0A &: (-)^{j_L} = (-)^{j_R} \text{ in } NS, \\ & \quad (-)^{j_L} = (-)^{j_R+1} \text{ in } R. \end{aligned} \tag{2.4}$$

   where $j_{L,R}$ are the leftmoving and rightmoving fermion number currents.

2. The parity even combination of the lowest NS-NS winding operators $\mathcal{O}^{NS}_{n=0,k=\pm\frac{1}{2}}$, $\tilde{\mathcal{O}}^{NS}_{n=0,k=\mp\frac{1}{2}}$ is the interaction term in the sine-Liouville theory written in the dual variables. The duality between the two theories shows that the spectrum must have the above two NS vertex operators. Imposing locality of the rest of the spectrum relative to these vertex operators, demanding that the Liouville momentum of ρ and $\tilde{\rho}$ must be the same, and the level matching condition determines the spectrum.

At this point, we should make a remark concerning the nature of these operators. In theories which asymptote to linear dilaton backgrounds, there is no state-operator correspondence. Non-normalizable modes correspond to local operators in the theory, and the normalizable modes are the states, or vacuum deformations in spacetime [22]. For the non-critical superstring theories with $d \leq 4$, the sine-Liouville interaction is a non-normalizable local operator and can be put in the action. The cigar metric is normalizable for all the theories. For $d > 4$, the sine-Liouville operator is normalizable as well. We shall not grapple with this issue in the following, and shall study all the theories, including $d = 6$.

The Type 0 theories as defined above for all d have a straightforward interpretation in terms of the cigar geometry. The spectrum can be classified as a set of particles with increasing masses. Asymptotically, all the propagating modes are determined by a free field propagating on the geometry, so that the particles have all integer momenta and winding (which is not a good quantum number) allowed by the equation of motion.

The lowest lying modes are a tachyonic scalar, and a graviton multiplet with d^2 degrees of freedom in the NS-NS sector, and a set of massless R-R fields appropriate to the particular dimension[12]. The winding modes in the R-R sector have a possible interpretation as Wilson lines of R-R potentials around the tube. This interpretation is only valid asymptotically, where the field strength vanishes. The type 0 spectra are presented in Appendix A with an example.

We can study the high energy behaviour of these theories and get an estimate for the asymptotic density of states by a saddle point approximation of the partition sum. The result for the mass density of states as a function of the spacetime dimension d is (Appendix B):

$$\rho(m) \sim m^{-(d+2)} \exp(\frac{m}{m_0}), \qquad m_0 = (\pi\sqrt{d\alpha'}) \tag{2.5}$$

As noted in [6], [23], this is unlike compactification to d dimensions wherein the string at high energies does see all the ten dimensions. These string theories are in this sense, truly d dimensional.

2.2. The chiral GSO projection and Type II theories

The symmetries of the 0A and 0B theories are the momentum around the cigar, and naively two (vector and axial) $U(1)$ R symmetries on the worldsheet. Only the first one under which all the R-R fields pick up a negative sign and is the one used in the type 0 projection, is a true symmetry of the theory. This is clear from the sine-Liouville interaction $\mathcal{L}^{SL}_{int} = \psi\tilde{\psi}\, e^{-\frac{1}{Q}(\rho+\tilde{\rho}+i(\theta-\tilde{\theta}))} + c.c$ where $\psi = \psi_\rho + i\psi_\theta$ is the

[12] For $d = 0$, there are no transverse oscillators and no graviton, there are only a few field theoretic states in the spectrum.

superpartner of $\rho + i\theta$ and $\tilde{\psi}$ is its rightmoving counterpart. The chiral rotation of the fermions is not a symmetry, the rotation of the left and right moving fermions in opposite directions is. It is also clear that there *is* a conserved chiral $U(1)$ current which rotates the left moving fermion by an angle α and simultaneously translates the left moving boson θ by $Q\alpha$.

From the cigar point of view, the non-conservation of the chiral rotation can be understood as due to the anomaly at one-loop in the $U(1)$ current j_L which rotates only the left moving fermions caused by the curvature of the cigar:

$$\partial_\alpha j_L^\alpha = R(\epsilon^{\alpha\beta}\,\partial_\alpha\rho\,\partial_\beta\theta), \tag{2.6}$$

where $R = -2D^a D_a \Phi = \frac{-Q^2}{2\cosh^2\frac{Q\rho}{2}}$ is the Ricci curvature of the cigar.

Due to the special form of the curvature in two dimensions, we can define a new current which *is* conserved. Changing to complex coordinates on the worldsheet, this current is the sum of the chiral rotation and another piece proportional to the left moving momentum:

$$\overline{\partial} j_G := \overline{\partial}(j_L + Q(\tanh\frac{Q\rho}{2})\partial\theta) = 0 \tag{2.7}$$

which reduces to the $U(1)$ R current of the $\mathcal{N} = 2$ SCFT (2.2) in the asymptotic region.

We conclude that to perform a chiral $\mathbb{Z}_2$ projection to get the type II theories, we *must* use the $\mathbb{Z}_2$ symmetry generated by the conserved current above, which acts in the asymptotic region as

$$G = (-)^{j_L + Qk_L}. \tag{2.8}$$

This GSO projection is implemented by introducing the target space supercharge in the twisted sector, as in [6]. For example, in the pure cigar case, we demand that the OPE of the $(1,0)$ operator

$$S = e^{-\frac{\varphi}{2} + i\frac{\phi}{2}} \tag{2.9}$$

with the physical operators is local. When $\mathbb{R}^d$ is tensored to this background, $S = e^{-\frac{\varphi}{2}+i\frac{\phi}{2}}$ and $\overline{S} = e^{-\frac{\varphi}{2}-i\frac{\phi}{2}}$ are each multiplied by spin fields which are spinors of $Spin(d)$ (for $d/2$ even these are conjugate spinors and for $d/2$ odd they are the same spinor). There is also a similar condition on the rightmoving side depending on whether the theory is IIA or IIB.

The algebra of the supercharges can be deduced by examining the currents in the asymptotic region. For $d/2$ odd or even, one has respectively

$$\{\mathcal{S}_\alpha, \overline{\mathcal{S}}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu, \qquad \text{or} \qquad \{\mathcal{S}_\alpha, \overline{\mathcal{S}}_{\dot\beta}\} = 2\gamma^\mu_{\alpha\dot\beta} P_\mu. \tag{2.10}$$

Note, in particular that it does not contain translation in θ. In fact, the symmetry generator P^θ corresponding to the translation around the cigar is an R symmetry.

$$[P^\theta, \mathcal{S}_\alpha] = \frac{1}{2}\mathcal{S}_\alpha, \qquad [P^\theta, \overline{\mathcal{S}}_{\dot\alpha}] = -\frac{1}{2}\overline{\mathcal{S}}_{\dot\alpha}. \tag{2.11}$$

This means that the fields in a given supersymmetry multiplet do not all have the same value of momentum. Thus, the allowed momenta around the cigar of a particle is correlated with its behaviour under Lorentz rotations in spacetime. For instance, in the zero winding sector, the graviton has only even momenta, and the tachyon[13] has only odd momenta. The fermionic states which arise in the twisted sector have half integer momenta. This means that they are antiperiodic around the tube. This is the expected behaviour for spinors which are single valued on the cigar (the spin structure which can be extended). We find also a discrete winding symmetry which is not natural from the cigar perspective. This discrete symmetry has its most natural interpretation from the Calabi-Yau or NS5-brane description of the theories, as described later in section 4. The type II spectrum is presented in Appendix A with an example.

In the Green-Schwarz formulation of the superstring, the states of the theory are in a representation of the zero modes of the spinors which are the superpartners of the bosonic fields. The quantization is treated in Appendix C. Here we point out two features of these theories. The first one is how the above non-smooth spacetime structure manifests itself in the Green-Schwarz formalism. In this description, the conserved current which corresponds to momentum around the cigar is really a combination of the naive momentum and a piece which acts on the worldsheet fermions. The spin of a particle and the momentum around the cigar are thus not independent of each other.

[13] The tachyonic zero mode is projected out and the field is no longer tachyonic. The type II theories are stable.

The second issue regards the comparision to the 10-dimensional Green-Schwarz string. In that case, in the light-cone gauge, the symmetry group $SO(8)$ has the property of triality relating the chiral spinor and vector representations which is crucial in showing that the spectrum is the same as that obtained by the R-NS formalism. In our theories in lower dimensions, there is no triality, but correspondingly, the NS-NS ground state is not a graviton built out of left and right moving vectors, rather it is the scalar (tachyon) field with one unit of momentum. There is no need for triality in these theories.

2.3. Holographic interpretation

The bosonic spectrum of the supersymmetric theories cannot be organized as multiplets of the $d+2$ dimensional bosonic Poincare group, but the full spectrum *can* be organized into multiplets of spacetime supersymmetry. This algebra (2.10) is effectively d dimensional and the modes of the $d+2$ dimensional fields are naturally classified as d dimensional off-shell currents (operators). This is consistent with the holographic interpretation of such theories [19]. We shall illustrate this using the $d=4$ example.

The spacetime supersymmetry algebra of the theory is a four dimensional $\mathcal{N}=2$ algebra with a $U(1)$ R charge which we identified as the momentum around the cigar $R=2P^\theta$. The six dimensional fields arrange themselves into multiplets of this supersymmetry. The Kaluza-Klein modes with a certain value of momentum around the cigar will be on-shell particles in five dimensions, and so they will fall into the current representations of the four dimensional algebra [*e.g.* Table 1,2 below]. These off shell four dimensional currents are conserved because of the gauge invariance in five dimensions.

Table 1: Tachyon multiplet

6d rep:	$(A_a)_{\rm RR}$	$(\psi_1,\psi_2),(\overline{\psi}_1,\overline{\psi}_2)$	$(T,T^*),(T_w,T_w^*)$	Tachyon
4d rep:	(V_μ,D)	$(\psi_1,\psi_2),(\overline{\psi}_1,\overline{\psi}_2)$	$(\phi_1,\phi_1^*),(\phi_2,\phi_2^*)$	$\mathcal{N}=2$ vector current
$n=\frac{1}{2}R$:	0	$(\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},-\frac{1}{2})$	$(1,-1,),(0,0)$	

Table 2: Graviton multiplet

6d rep:	$(G, B, \phi)_{\rm NSNS}$	$(\chi, \psi_\mu), (\overline{\chi}, \overline{\psi}_\mu)$	$(F_{\pm\mu}, F^{(\pm)}_{\mu\nu})_{\rm RR}$	Graviton
4d rep:	$(T_{\mu\nu}, B_{\mu\nu}, A_\mu, B_\mu)$	$(\chi, \psi_\mu), (\overline{\chi}, \overline{\psi}_\mu)$	$(\partial_\mu D \pm k^2_\nu V_\mu), (V^{(\pm)}_{\mu\nu})$	$\mathcal{N} = 2$ supercurrent
$n = \frac{1}{2}R$:	0	$(\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})$	$(1, -1), (0, 0)$	

Supersymmetry structure of the $d = 4$ spectrum: These are the first two Kaluza-Klein modes of the six dimensional fields in the type IIA theory. In six dimensions, these are the tachyon multiplet and the graviton multiplet. They are classified by their properties under the 6d Poincare algebra and the 4d SuperPoincare algebra. The momentum around the cigar is proportional to the U(1) R charge. The spinors are two component spinors in $\mathbf{2}$ and $\overline{\mathbf{2}}$ of $SO(4)$.

The details of the various type II theories differ slightly from each other. Some of the higher dimensional theories have been studied in [24,25,26]. These authors however, were interested in constructing modular invariant partition functions in particular cases, and the spacetime picture is not dwelt upon. In Appendix B, we present the modular invariant partition functions for the perturbative string spectrum for the various values of d. These are constructed by the standard technique [4] of counting the physical momentum and winding forced by the above GSO projection in the light cone gauge. In the next section, we shall visit the various theories and highlight the interesting features that each of them have.

3. Special features of the various theories

3.1. $d = 0$

In the theory of the pure cigar, the spectrum looks smooth in that it can be interpreted as a set of particles propagating on the cigar[14]. There are no transverse oscillators, and hence no infinite tower of states that we find in higher dimensional theories. We can explicitly write down the all the vertex operators easily in this

[14] This is loose usage of language - in two dimensions, there is no notion of a particle.

case. In the IIB theory, we have (all operators with $p = |k|$):

$$\begin{array}{ll}
e^{-\varphi-\tilde{\varphi}+i(N+\frac{1}{2})(\theta-\tilde{\theta})+(-1+|N+\frac{1}{2}|)(\rho+\tilde{\rho})} & N = 0, \pm 1, \pm 2, ... \\
e^{-\varphi-\frac{1}{2}\tilde{\varphi}-\frac{1}{2}i\tilde{H}-i(N+\frac{1}{2})(\theta+\tilde{\theta})+(-\frac{1}{2}+N)(\rho+\tilde{\rho})} & N = 0, 1, 2... \\
e^{-\frac{1}{2}\varphi-\tilde{\varphi}-\frac{1}{2}iH-i(N+\frac{1}{2})(\theta+\tilde{\theta})+(-\frac{1}{2}+N)(\rho+\tilde{\rho})} & N = 0, 1, 2... \\
e^{-\frac{1}{2}\varphi-\frac{1}{2}\tilde{\varphi}+\frac{1}{2}i(H+\tilde{H})+iN(\theta+\tilde{\theta})+(-1+N)(\rho+\tilde{\rho})} & N = 0, 1, 2...
\end{array} \tag{3.1}$$

In the NS-NS sector we have only winding modes of the scalar. In the NS-R and the R-NS sector, we find spacetime fermions. As can be seen from the H and $\tilde{H}$ behavior of the vertex operators they both have spin $-\frac{1}{2}$. As mentioned earlier, the half integer momentum is the expected behavior for spacetime spinors which are single valued on the cigar. This agrees with the interpretation of a black hole at finite temperature. From the R-R sector we find a spacetime boson which is periodic around the tube. We interpret this boson as the R-R scalar of the IIB theory. Note that the two fermions have only negative k and the R-R scalar has only positive k.

In the IIA theory we find

$$\begin{array}{ll}
e^{-\varphi-\tilde{\varphi}+i(N+\frac{1}{2})(\theta-\tilde{\theta})+(-1+|N+\frac{1}{2}|)(\rho+\tilde{\rho})} & N = 0, \pm 1, \pm 2, ... \\
e^{-\varphi-\frac{1}{2}\tilde{\varphi}+\frac{1}{2}i\tilde{H}+i(N+\frac{1}{2})(\theta+\tilde{\theta})+(-\frac{1}{2}+N)(\rho+\tilde{\rho})} & N = 0, 1, 2... \\
e^{-\frac{1}{2}\varphi-\tilde{\varphi}-\frac{1}{2}iH-i(N+\frac{1}{2})(\theta+\tilde{\theta})+(-\frac{1}{2}+N)(\rho+\tilde{\rho})} & N = 0, 1, 2... \\
e^{-\frac{1}{2}\varphi-\frac{1}{2}\tilde{\varphi}+\frac{1}{2}i(-H+\tilde{H})-i(N+\frac{1}{2})(\theta-\tilde{\theta})+(-\frac{1}{2}+N)(\rho+\tilde{\rho})} & N = 0, 1, 2... \\
e^{-\frac{1}{2}\varphi-\frac{1}{2}\tilde{\varphi}+\frac{1}{2}i(H-\tilde{H})+iN(\theta-\tilde{\theta})+(-1+N)(\rho+\tilde{\rho})} & N = 0, 1, 2...
\end{array} \tag{3.2}$$

We again find only winding modes of the scalar in the NS-NS sector. The NS-R and R-NS sectors lead to antiperiodic spacetime fermions. Their spins are $+\frac{1}{2}$ and $-\frac{1}{2}$, which is consistent with the spacetime parity of the IIA theory. The R-R sector leads to winding modes which have an asymptotic interpretation as Wilson lines of the R-R one form around the tube.

One aspect worth mentioning about this theory is that the physical spectrum is extremely constrained due to the lack of transverse oscillators. There are no gravitons, and the supercharges are not part of the spectrum. In fact, to obtain a modular invariant partition function[Appendix B], we cannot impose the Dirac equation, and we need to demand it as a further condition to read off the physical spectrum. Even when we do this, the bosons and fermions are not paired and so the partition function does not vanish.

3.2. $d = 2$

In the theories with $d = 2, 6$, there is a further subtlety in the chiral projection which arises because of the way the conserved fermion number current $j_G = j_L + Qk_L$ acts in these theories. The symmetry $G = e^{\pi i j_G}$ has a $\mathbb{Z}_4$ (and not $\mathbb{Z}_2$) action on the fields of the type 0 theories. The symmetry $G_1 = (-)^{2j_G}$ is a $\mathbb{Z}_2$ subgroup of the $\mathbb{Z}_4$ symmetry in the type 0 theories. Orbifolding by G_1 gives another bosonic theory which we call the type $0'$ theories. These theories has a $\mathbb{Z}_4/\mathbb{Z}_2 = \mathbb{Z}_2$ symmetry G_2 by which they can be further orbifolded to get the supersymmetric type II theory.

In the $d = 2$ dimensional case, $Q = \sqrt{3}$ which gives $R = \frac{2}{\sqrt{3}}$. The $\mathbb{Z}_2$ symmetry in the type 0 theories is $G_1 = (-)^{2j_L + 2\sqrt{3}k_L}$. The NS-NS operators have charge $(-)^{2\sqrt{3}k_L}$ and the R-R operators have charge $(-)^{2\sqrt{3}k_L+1}$ under this symmetry (This is because there are only two spin fields in four dimensions). The construction of the type $0'$ theories is implemented by demanding the presence of the $R\pm R\mp$ operators in the 0'B and $R \pm R\pm$ operators in the 0'A with winding $w = \pm\frac{1}{2}$. The chiral $\mathbb{Z}_2$ symmetry of these theories is $(-)^{j_L + \sqrt{3}k_L}$, orbifolding by which we get the type II theories. This is implemented as usual by demanding the supercharges to be present in the spectrum. The supercharges in this theory have $(n, w) = \pm(\frac{1}{2}, \pm\frac{3}{4})$.

3.3. $d = 4$

We have been presenting examples from the $d = 4$ theory above to illustrate the various general features, and shall only make one remark here. The asymptotic radius of the cigar is the self-dual radius. Asymptotically, the current $\partial\theta$ is a physical current, but the currents $e^{\pm i \frac{1}{\sqrt{2}}\theta}$ are not. The $SU(2)$ symmetry of the free boson is therefore not present in this model.

3.4. $d = 6$ or $k = 2$ NS5-branes

This example is particularly rich because of the explicit geometric description in terms of NS5-branes referred to in the introduction. We shall focus on three main points:

1. The equivalence between the cigar and sine-Liouville theories. We shall see that the two interactions are related by a symmetry.
2. The description of the moduli space of the theory, and
3. A discussion of the discrete $\mathbb{Z}_2$ symmetries present in the various 5-brane and cigar theories.

A quick review of the picture we shall use is the following: The conformal field theory in the asymptotic region is that of two parallel NS5-branes. This can be explicitly seen by fermionizing the angular coordinate θ on the tube to two free fermions $e^{\pm i\theta} \equiv \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2)$, which along with the fermion $\psi_\theta \equiv \psi_3$ generates a left moving $SU(2)_2$.[15] The theory has an $SU(2)_L \times SU(2)_R = SO(4)$ symmetry generated on the worldsheet by the rotation of the three leftmoving and three rightmoving fermions. The worldsheet interaction which resolves the singularity at the origin breaks the symmetry down to a global diagonal $SO(3)$ corresponding to the rotation of the three directions transverse to the two separated 5-branes.

The relationship between the cigar and sine-Liouville theories is a strong-weak coupling duality on the worldsheet. For large k, the cigar has small curvature and the classical description in terms of the metric is a good approximation to the full theory. The sine-Liouville term decays much faster than the cigar metric asymptotically, and the sine-Liouville lagrangian is strongly coupled. For small k on the other hand, the sine-Liouville term asymptotically dominates over the cigar metric which has large curvature, and it is the sine-Liouville which is a better description in terms of a weakly coupled worldsheet theory. For general k, the effective lagrangian in the full quantum theory has both these terms, and the dominance of one of these terms over the other is governed by the value of k. This has been confirmed by explicit

[15] In this case, $Q = 1 \Rightarrow R = 2$ which is the free fermion radius.

calculations of scattering amplitudes [*e.g.* [27] and refs. therein] where one sees poles corresponding to both the terms which can be used to compute an explicit relation between the two couplings.

In our theory with $k = 2$, both the terms decay at the same rate. As we shall see, they are related by a rotation in the $SO(3)$ symmetry group mentioned above. This gives an example where the kinematic structure determines explicitly that both the terms are present in the lagrangian and also determines the relation between the strength of the two couplings.

The three $\mathcal{N} = 2$ invariant currents $\psi_i e^{-\rho}$ (in the -1 picture) are in a triplet under $SU(2)_L$. The meaning of these currents is better understood when expressed in the 0 picture in the variables of the cigar: $(\psi_\rho \mp i\psi_\theta)e^{-\rho\pm i\theta}$, $(\psi_\rho\psi_\theta + \partial\theta)e^{-\rho}$ - the first two terms are nothing but the sine-Liouville interaction, and the third term is the first order correction from the cylinder towards the cigar metric. We can express these currents in a manifestly $SU(2)$ covariant manner as the fermion bilinears $A_i = (\psi_\rho\psi_i - \frac{1}{2}\epsilon_{ijk}\psi_j\psi_k)e^{-\rho}$. There are also the corresponding rightmovers. Noting the Clebsch-Gordon coefficients relating the **4** of the $SO(4)$ and the **3**'s of the left and the right $SU(2)$ (the 'tHooft symbols),

$$\eta^i_{\mu\nu}, \tilde{\eta}^i_{\mu\nu} = \delta^0_{[\mu}\delta^i_{\nu]} \pm \frac{1}{2}\epsilon^i_{jk}\delta^j_\mu\delta^k_\nu, \tag{3.3}$$

we can write down the proposed interaction for the theory of the non-coincident parallel NS5-branes. The matrix $\mathcal{L}_{\mu\nu} = \eta^{i\,\sigma}_\mu \tilde{\eta}^j_{\sigma\nu} A_i \tilde{A}_j$ is in the symmetric traceless (**9**) of the $SO(4)$, and the interaction

$$S_{int} = \int d^2z \, X^\mu X^\nu \mathcal{L}_{\mu\nu} \tag{3.4}$$

corresponds to separating the branes in the direction X^μ with the center of mass at the origin. A choice of X^μ breaks the $SO(4)$ symmetry to an $SO(3)$ rotation symmetry. Changing the position of the branes by a $SO(4)$ transformation leads to a different $SO(3)$ being preserved. This is reflected in the lagrangian by simultaneously conjugating the matrix $\mathcal{L}_{\mu\nu}$ by the same transformation. In keeping with the conventions used in the earlier sections, we shall relabel the four transverse directions as $\mu = 6, ..9$.

Since the two 5-branes are identical, the moduli space of the theory is $\mathbb{R}^4/\mathbb{Z}_2$, parameterized above by X^μ. In the $SL_2(\mathbb{R})/U(1)$ description of the theory, there are only four independent $(1,1)$ physical states. Two of the three chiral operators A_i (and correspondingly the rightmovers) are related by the spectral flow operation by one unit in the $SL_2/U(1)$ theory. This means that only two of them are independent states. The wavefunction of a state is in general a linear combination of the wavefunctions of the states reached by spectral flow from a given state. The coefficients of these wavefunctions depends on the given point in moduli space, because this determines the boundary conditions at the tip of the cigar [28,29]. The above four states are in the representation $\mathbf{1}+\mathbf{3}$ of the $SO(3)$, corresponding to the radial motion of the branes and the motion on the three-sphere of given radius.

In the subspace that preserves the momentum symmetry of the cigar and the parity of the angular coordinate, there is only one exactly marginal deformation. This was interpreted in [10] as the metric deformation. As mentioned above, this interpretation is only correct for large k where it dominates - as we see here, the spectral flow relation means that the deformation is a combination of operators as given by (3.4).

In the lagrangian (3.4) proposed above, the fact that there are only four independent operators is seen as a condition of second order conformal invariance of the theory. In the linear dilaton background, all the nine operators are conformal. In the background given by the interaction corresponding to say $X^6 \neq 0, X^{7,8,9} = 0$, only the operators $\mathcal{L}_{6\mu}$ remain conformal. To determine exactly marginal deformations, we add an arbitrary combination of the nine operators $C_{\mu\nu}\mathcal{L}^{\mu\nu}$ to the free action, and find that the theory remains conformal iff $C_{\mu\nu} = x_\mu x_\nu$ [Appendix D] which is exactly the four dimensional moduli space written above.

To summarize, there is no pure sine-Liouville or pure cigar theory, only particular linear combinations given by (3.4) are consistent with the $SO(4)$ being broken to an $SO(3)$ rotation symmetry. The theory has a four dimensional moduli space

$\mathbb{R}^4/\mathbb{Z}_2$.

We now make a few remarks:

1. The discrete winding symmetry in the theory is identified in the NS5-brane picture as the rotation by π in a plane containing the two branes, exchanging the two. This $\mathbb{Z}_2$ symmetry completes the $SO(3)$ into an $O(3)$.
2. The separation of the branes means that the origin is not singular, and this gives a unique spin structure in the transverse space. This forces the fermions to pick up a phase of $\pm i$ under the action of the exchange symmetry generator, precisely as seen in the spectrum. The fermions also transform in the $SU(2)$ cover of the global $SO(3)$.
3. We can make precise the relation of the cigar and 5-brane theories defined by the various GSO projections. The type $0'$ and II theories on the cigar correspond to the type 0 and II theories on the two 5-branes. The $\mathbb{Z}_2$ orbifold by the exchange symmetry mentioned above takes us from the type $0'$ to the type 0 theory on the cigar. The smooth (but tachyonic) type 0 theory on the cigar is identified with the theory of one 5-brane on $\mathbb{R}^4/\mathbb{Z}_2$ away from the origin. This theory does not have the $SO(3)$ symmetry.
4. To restate a point, the orbifold which gives the 5-brane theories from the Type 0 theory on the cigar does not have a geometric interpretation like changing the radius. The type $0A(B)$ and Type IIA(B) theory on the 5-branes can be obtained by quotienting by a chiral $\mathbb{Z}_2$ either the type $0A(B)$ theory with radius $R = 2$, or the type $0B(A)$ with radius $R = 1$ (which is produced from the former $0B(A)$ by a geometric quotient).

3.5. A note on the $\mathcal{N} = (4,4)$ algebra in the $d = 6$ theory.

This small subsection is slightly outside the main flow of the chapter. The main points in this subsection are the existence of an extended superconformal algebra in the $d = 6$ theory, and a comment on the related $D1/D5$ system.

Consider first the theory of $k = 2$ coincident NS5-branes. In a physical gauge where the lightcone directions along the NS5-branes are fixed, the degrees of freedom on the worldsheet are four free bosons (parallel to the branes) + four free fermions which have an $\mathcal{N}=(4,4)$ superconformal structure with $c=6$, and a linear dilaton (ρ) + four free fermions ($\psi_\mu=(\psi_\rho,\psi_i)$). There is also an $\mathcal{N}=(4,4)$ superconformal algebra involving the latter fields [20] of central charge $c=6$, which extends our algebra (2.2). There are four fermionic currents and three R currents forming an $SU(2)_1$. The currents of the algebra are:

$$
\begin{aligned}
T &= -\frac{1}{2}(\partial\rho)^2 - \frac{1}{2}\psi_\mu\partial\psi_\mu - \frac{1}{2}\partial^2\rho \\
G_\mu &= \frac{1}{\sqrt{2}}\psi_\mu\partial\rho + \frac{1}{6\sqrt{2}}\epsilon_{\mu\nu\sigma\tau}\psi^\nu\psi^\sigma\psi^\tau + \frac{1}{\sqrt{2}}\partial\psi_\mu \\
J_i &= \frac{1}{2}(\psi_\rho\psi_i + \frac{1}{2}\epsilon_{ijk}\psi_j\psi_k) = \frac{1}{2}\eta_i^{\mu\nu}\psi_\mu\psi_\nu .
\end{aligned}
\tag{3.5}
$$

As shown above, when the branes are noncoincident, the moduli are parameterized by the operators $\mathcal{L}_{\mu\nu}$ constructed using the chiral operators $A_i = (\psi_\rho\psi_i - \frac{1}{2}\epsilon_{ijk}\psi_j\psi_k)e^{-\rho} \equiv \overline{J}_i e^{-\rho}$ and their right moving counterparts. These operators commute with the above algebra and the $\mathcal{N}=(4,4)$ superconformal symmetry is preserved at all points in the moduli space. This is consistent with the fact that there are 16 supercharges in spacetime even when the two branes are noncoincident (but parallel). The $SO(4)=SU(2)\times SU(2)$ of the $J_i, \tilde{J}_i$ is identified with the rotation of the four directions parallel to the brane.

A fact worth pointing out is that for a system of k NS5-branes, in addition to the above algebra (3.5) with $c=6$, there is a second $\mathcal{N}=4$ algebra with $c=6(k-1)$. The R currents of this algebra contains $\overline{J}_i$ and another piece from the bosonic $SU(2)_{k-2}$, and the slope of the dilaton is different. Both these algebras have the same algebraic structure (called the 'small' $\mathcal{N}=4$ algebra), and they are both subalgebras of the 'large' $\mathcal{N}=4$ algebra. This 'large' algebra contains both the sets of $SU(2)$ R currents and its generators do not contain the improvement terms from the linear dilaton [20,30].

As explained in [30], these two conformal field theories are spacetime descriptions of the short and long string excitations respectively in the background of k NS5-branes and many fundamental strings.[16] Equivalently, they describe the

[16] In the gauge we have chosen, at distances where we are in the near horizon limit of the

Coulomb and Higgs branches respectively of the gauge theory on a D1-brane in the presence of D1-branes and D5-branes in the IIB theory. In general, the slope of the dilaton in the Higgs branch differs in sign and magnitude from the Coulomb branch, and the symmetries of the two branches are different.

For the case $k = 2$, there is a symmetry mapping one theory to the other. In this special case, both the chiral subalgebras have $c = 6$, and the string couplings are inverses of each other. There are two $SO(4) = SU(2)_L \times SU(2)_R$ - one corresponding to the rotation of the four directions parallel to the 5-brane and the other to the rotation of the four transverse directions. In the Coulomb branch, the parallel $SO(4)$ generated by the $J_i, \tilde{J}_i$ is preserved and the transverse $SO(4)$ is broken to a global diagonal $SU(2)$ by turning on the moduli parameterized by $\overline{J}^i, \overline{\tilde{J}^i}$ corresponding to the separation of the branes. In the Higgs branch, the roles of J^i's and $\overline{J}^i$'s are reversed (in this system, the moduli are the self dual NSNS B field and a linear combination of the RR zero form and four form). Given the identification of the long tube of the two branches as in [30], one can summarize the above by saying that there is a symmetry relating the short and long string systems given by: $(\rho, \psi_\rho) \to -(\rho, \psi_\rho)$. Starting from the weak coupling end of either theory, turning on the moduli to a non-zero value caps off the infinite tube to a semi-infinite cigar.

4. The global symmetries of the various theories

The global symmetries of the various theories are the following:

Table 3: *Global Symmetry structure*

Theory	$d = 6$	$d = 4$	$d = 2$	$d = 0$
Supersymmetry	$\mathcal{N} = (2,0)$ or $\mathcal{N} = (1,1)$	$\mathcal{N} = 2$	$\mathcal{N} = (4,0)$ or $\mathcal{N} = (2,2)$	$\mathcal{N} = 2$
Bosonic Symmetry	$SO(5,1) \times O(3)_R$	$SO(3,1) \times U(1)_R$	$SO(1,1) \times (U(1) \times \mathbb{Z}_2)_R$	$U(1)_R$

Only the Lorentz part of the full Poincare group in the flat directions is written above. The supersymmetry algebra could be chiral or

5-branes but not near horizon of the strings, the theory of short strings in this background is the same as the one we have been studying with 5-branes alone [14].

non-chiral in $d = 6, 2$ depending on if the theory is IIA or IIB. The fermions in the theories with $d = 6, 4, 2$ transform in the $Spin(d-1, 1)$ in the flat directions. They also transform in the spin cover of the R symmetry group: In $d = 6$, they are charged under $SU(2)$ of rotation, they have half integer charge under the $U(1)$'s in the $d = 4, 2, 0$ theories, and they pick up a phase of $\pm i$ under the $\mathbb{Z}_2$ generator in the $d = 6, 2$ theories.

From the CFT description of the non-crtitical string theories, we saw in section 2 the appearance of all these symmetries. From the spacetime point of view, the super-Poincare group in the flat directions is natural. As we saw earlier, the $U(1)$ part of the R symmetry group is interpreted as the conserved momentum around the cigar. In the previous section, we saw using the NS5-brane picture, exactly how all the symmetries of the $d = 6$ theory arise. The full R symmetry groups for all the theories including the discrete $\mathbb{Z}_2$ symmetries are naturally seen in the singular Calabi-Yau or NS5-brane description, which is what the rest of this section is devoted to.

As mentioned in the introduction, all the non-critical theories above are conjectured to arise as near horizon geometries of wrapped NS5-branes, which are dual to string theory near the singularity of certain singular Calabi-Yau manifolds tensored with flat space in a double scaling limit [9,14,15]. The global symmetries of our theories are nothing but the symmetries of these brane configurations in the near-horizon limit, or equivalently, those of the dual geometry whose action remains non-trivial in the double scaling limit. We will now describe the T-duality between the singular spaces and wrapped NS5-branes [[17], and refs. therein] and their respective deformations, and track which symmetries survive in the scaling limit. A linear sigma model analysis of this T-duality has been done in [31].

The non-critical theory in d dimensions is conjectured to be equivalent to string theory on $\mathbb{R}^{d-1,1} \times M^{10-d}$ where M^{10-d} is defined as the hypersurface $\sum_{i=1}^{n} z_i^2 = \mu$ in$\mathbb{C}^n$ where $n = (12-d)/2$.[17] By considering as above non-compact geometries, we

[17] For $d = 0$, we should consider the Euclidean theory, and in terms of the brane picture, it is the theory of an Euclidean NS5-brane completely wrapped on a non-compact Calabi-

have already taken the limit in which we zoom in on the singularity of the Calabi-Yau. In terms of the dual 5-brane description which we present below, the branes are stretched out to infinity even in the non-flat directions, describing deformed intersecting surfaces in these directions for $d \leq 4$. The only global symmetries left are the ones involving the directions fully transverse to the brane. Another way to see this is that in the holographic dual which is the decoupled theory on the brane world-volume, the degrees of freedom (D-strings in type IIB) are stuck near the intersection of the branes and only have kinetic terms in the flat directions. All the states in the theory are therefore singlets under internal rotations in the curved directions.

Let us fix coordinates so that the flat ones are $x^0 ... x^{d-1}$. For $n = 3$ and $\mu = 0$, the above Calabi-Yau is just the ALE space $\mathbb{R}^4/\mathbb{Z}_2$. This has an isometry which rotates (z_2, z_3) into each other. Performing a T-duality along this circle gives us two NS5-branes at the origin of $\mathbb{R}^4$. The deformation of the ALE space into a two-center Taub-Nut is dual to the separation of the location of the 5-branes in the $\mathbb{R}^4$. This is the $d = 6$ theory. We can chose coordinates so that x^6 is the circle of isometry. This circle shrinks to zero size at the location of the singularity, and thus in the brane picture, this grows to an infinite direction near the branes. The separation of the branes is in the direction (x^6, x^8, x^9) at $x^7 = 0$. The former three coordinates can be rotated so that the branes are at $x^{7,8,9} = 0$ giving rise to the $SO(3)$ as in the previous section. The exchange of the two 5-branes $x^6 \to -x^6$ is the symmetry which has a non-trivial $\mathbb{Z}_2$ action on the bosons.

For $n = 4, 5, 6$ which corresponds to the lower dimensional theories, we can write the Calabi-Yau near the deformed singularity as $z_1^2 + z_2^2 + z^2 = \mu$ where $z^2 = z_3^2 + .. z_n^2$. This should be thought of as an ALE fibration over the curve $z^2 = \mu$. We can perform the same T-duality as above, and we get an NS5-brane wrapped on $z^2 = \mu$. For $\mu = 0$, the curved part of the configuration is embedded in $\mathbb{R}^{2(n-2)}$ parameterized by $x^d ... x^5, x^8, x^9$, and the coordinates x^6, x^7 are transverse to this brane. When μ is turned on, there is a deformation in the internal space $\mathbb{R}^{2(n-2)}$

Yau 3-fold embedded in $\mathbb{R}^8$.

and in x^6 which causes the brane configuration to posses a minimum size S^{n-3}. Suppressing all the coordinates except the transverse directions (x^6, x^7), there are two point-like branes at the origin of the plane for $\mu = 0$, and the deformation separates them by μ in this plane.

Let us now look at the global symmetries. As mentioned above, the only global symmetries are those involving the transverse directions. First let us consider the singular case $\mu = 0$. For $d = 4, 2, 0$ ($n = 4, 5, 6$), the symmetry is $U(1) \times U(1)$ of momentum and winding around the circle that rotates the two transverse directions (x^6, x^7). We can identify the rotation of the transverse directions in complex coordinates as $z \to e^{i\phi} z$, which is produced by $z_i \to e^{i\phi} z_i$. The deformation preserves the $U(1)$ of winding and breaks the rotation to the discrete rotation $z \to -z$. It is clear that, upto an $SO(n-2)$ symmetry rotating the branes, this is equivalent to $z_3 \to -z_3$ for odd n and to a trivial rotation for even n. That the former is a rotation by π can be seen by looking at the projection of the curve to $(z_4, ..., z_n) = 0$ which is invariant under the above symmetry.

From the Calabi-Yau point of view, we can present the following argument. At a particular value of the radius in the geoemtry, the bosonic symmetry group is $O(n) = SO(n) \ltimes \mathbb{Z}_2$ for the theory specified by n. For odd n, the $SO(n)$ commutes with the $\mathbb{Z}_2$ and the product is really a direct product. In this case, the $\mathbb{Z}_2$ is just that of parity, and wavefunctions of particles propagating on this manifold are labelled by the quantum numbers of $SO(n)$, and a sign. For even n, the $SO(n)$ does not commute with the $\mathbb{Z}_2$, and the good quantum numbers are those of the $SO(n)$ alone. The wavefunctions with non-zero spin are peaked away from the origin and in the limit of going near the singularity, they decouple. Only the singlets under the $SO(n)$ remain, and only for odd n, there is also a $\mathbb{Z}_2$ symmetry. This $\mathbb{Z}_2$ symmetry acts on the top form of the Calabi-Yau as $\Omega \to -\Omega$, which is equivalent to a rotation of π. These facts can be checked in the cases that there are explicit metrics which have been written down. For instance, in the conifold theory [see e.g. [32] and refs. therein] , the angle ψ has period 4π in the full conifold. Near the tip though, a rotation by 2π along with a rotation in the sphere brings you back to the same point.

We end with two comments:

1. The difference between the $d = 6$ case and the lower dimensional ones is the following: for $n = 3$, turning on μ deforms the curve $z_3^2 = 0$ in the directions (x^8, x^9) and in x^6 just like the other cases. Because the curve in this case is a pair of points, the symmetry structure is $SO(4) \to O(3)$. In the other cases, because the two directions (x^8, x^9) are filled in by the brane, the symmetry is $U(1) \times U(1) \to U(1) \times \mathbb{Z}_2$. The deformation in the $d = 4, 2, 0$ theories correspond to turning on the sine-Liouville term in the action. As we saw earlier, the enhanced symmetry in the $d = 6$ theory could be seen from the conformal field theory as well where the sine-Liouville and cigar terms are related by the symmetry.
2. When n is even $(d = 4, 0)$, we saw that the there is only a rotation by 2π in the curved directions of the brane that remains as a symmetry. The fermions pick up a negative sign under this and this gives the discrete winding symmetry which is equivalent to $(-)^{F_S}$ for $d = 4$. For $d = 0$, the supercharges are charged under this rotation by 2π, but because of the lack of flat directions, the physical bosons and fermions are not paired by the supercharges. The physical fermionic operators in the lists (3.1) , (3.2) are thus not charged under the discrete winding symmetry.

Appendix A. Spectrum of the higher dimensional theories

The vertex operators become complicated as the mass in spacetime increases. To construct the states at a general level, we can use the oscillator algebra of the worldsheet fields which are free asymptotically. If we fix the reparameterization invariance on the worldsheet by going to the light cone gauge, we can write down the form of the general state in the theory asymptotically in terms of worldsheet oscillators in d transverse directions.

Table 1: *Type* 0 *spectrum*

Theory	**Sector**	$(\mathbf{n},\mathbf{w})$	**Example** $(d=4)$
$0B$ and $0A$	NS+NS+ NS-NS-	(N,M) (N,M)	Graviton multiplet (G,B,ϕ) Tachyonic scalar T
$0B$	R+R+ R-R-	(N,M) (N,M)	Scalar, two form (ϕ_1, B_{ab}^{sd}) Scalar, two form (ϕ_2, B_{ab}^{asd})
$0A$	R+R- R-R+	(N,M) (N,M)	Vector A_a^1 Vector A_a^2

The operators are labelled by the momentum n and winding w (N, M integers). The $\pm$ signs in the sectors are the naive chirality on the worldsheet defined asymptotically. These are not good quantum numbers, and they are only indicated to classify the spin of a particle in the asymptotic region. The superscripts in the R-R sector indicate the (anti)self duality condition that the field strenghts of the listed potential obeys. If there is no superscript, the field strength is unconstrained.

Table 2: *Type II spectrum* $(d=4)$

Theory	**Sector**	$(\mathbf{n},\mathbf{w})$	**Example** $(d=4)$
IIB and IIA	NS+NS+ NS-NS-	$(2N,2M),(2N+1,2M+1)$ $(2N+1,2M),(2N,2M+1)$	(G,B,ϕ) T (non-tachyonic)
IIB	R+R+ R-R-	$(2N,2M),(2N+1,2M+1)$ $(2N+1,2M),(2N,2M+1)$	(ϕ_1, B_{ab}^{sd}) (ϕ_2, B_{ab}^{asd})
IIA	R+R- R-R+	$(2N,2M),(2N+1,2M+1)$ $(2N+1,2M),(2N,2M+1)$	A_a^1 A_a^2
IIB	R+NS-, NS+R- R-NS+, NS-R+	$(2N+\frac{1}{2},2M-\frac{1}{2}),(2N-\frac{1}{2},2M+\frac{1}{2})$ $(2N+\frac{1}{2},2M+\frac{1}{2}),(2N-\frac{1}{2},2M-\frac{1}{2})$	(Supercharge, Photino, Gravitino)
IIA	R+NS-, NS+R+ R-NS+, NS-R-	$(2N+\frac{1}{2},2M-\frac{1}{2}),(2N-\frac{1}{2},2M+\frac{1}{2})$ $(2N+\frac{1}{2},2M+\frac{1}{2}),(2N-\frac{1}{2},2M-\frac{1}{2})$	(Supercharge, Photino, Gravitino)

The notation is as in Table 1. Notice that the particles cannot be described by an effective action, because not all the modes are present. The fermionic states are obtained by the action of the supercharges on the bosons. Their Lorentz representation is correllated with the momentum.

Appendix B. Partition functions of the various theories

On the sphere, the partition function vanishes in all even dimensions due to the presence of two extra fermionic zero modes on the cigar [6]. To compute the torus partition sum, we shall go to light cone gauge. In this gauge in d dimensional theories, the physical oscillators are $\alpha^I_n, \tilde{\alpha}^I_n, \psi^I_n, \tilde{\psi}^I_n$, $I = \rho, \theta, 1, 2, ..d-2$.

A general state is built out of the raising operators acting on the vacuum, and has momentum p, k_μ in the $d+1$ non-compact directions and momentum n and winding w around the circle. The left and right moving momentum around the circle are given by $k_L = \frac{Qn}{2} + \frac{w}{Q}$; $k_R = \frac{Qn}{2} - \frac{w}{Q}$. The physical state condition gives us the mass of a state in terms of the oscillators. In the left moving sector (the right moving mass shell equation is defined likewise):

$$\begin{aligned} L_0 &:= \sum_{n=1}^{\infty} n\alpha^I_{-n}\alpha^I_n + \sum_{r=\frac{1}{2}}^{\infty} r\psi^I_{-r}\psi^I_r + \frac{1}{2}(k_\mu^2 + k_L^2 - p^2) - \frac{d}{16} = 0 \qquad \text{(NS)} \\ L_0 &:= \sum_{n=1}^{\infty} n\alpha^I_{-n}\alpha^I_n + \sum_{m=1}^{\infty} m\psi^I_{-m}\psi^I_m + \frac{1}{2}(k_\mu^2 + k_L^2 - p^2) = 0 \qquad \text{(R)} \end{aligned} \tag{B.1}$$

The energy of the NS sector vacuum state (tachyon) can be understood as arising from the presence of the dilaton, or the curvature of spacetime, as in the previous sections.[18] It can also be seen by adding the zero point energies of d free bosons and d antiperiodic fermions on the worldsheet. The energy of the R sector vacuum is 0.

The one loop partition sum is split into the integration over the non-compact momenta, the sum over the allowed compact momentum and winding, and oscillator sums; taking into account the GSO projection:

$$\mathbf{Z_{T^2}} = V_{d+2} \int \frac{d^2\tau}{4\tau_2} \int \frac{d^dk^\mu}{(2\pi)^d} \int \frac{dp}{4\pi} \sum_{n,w} \frac{1}{R} \mathrm{Tr}(-1)^{F_S} q^{L_0} \overline{q}^{\overline{L}_0} \tag{B.2}$$

where $q = e^{2\pi i \tau}$ and the trace is over the physical Hilbert space of the theories.

[18] As a reference, in the 10 dimensional string theory, the energy of the NS vacuum in our units is $-\frac{1}{2}$.

The definition of the Θ and η functions are:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$$
$$\Theta_{ab}(\tau) = q^{\frac{a}{8}} e^{\pi i a b/2} \prod_{n=1}^{\infty} (1-q^n)(1+e^{\pi i b} q^{n-(1-a)/2})(1+e^{-\pi i b} q^{n-(1+a)/2}) \tag{B.3}$$

The modular invariance of the partition sum of the type 0 theories is straightforward in all cases, relying on the transformation properties of the Θ functions defined above.

The partition function for the type $0'$ and type II theories is computed by taking into account the GSO projection. We present below the modular invariant partition sums for the theories as a function of the dimension.

B.1. d=0

In this case, fixing to light-cone gauge leaves us with no oscillators. As we found in the text, there are only the few field-theoretic degrees of freedom in the pure black hole, and we can compute the partition function purely as a sum over winding and momenta, and integration over the dilaton direction. The type 0 partition function is:

$$\mathbf{Z}_{\mathbf{T^2}} \sim V_2 \int \frac{d^2\tau}{\tau_2^2} \sum_{m,w=-\infty}^{\infty} \exp\left(\frac{-\pi |m - w\tau|^2}{2\tau_2}\right) \tag{B.4}$$

which is equal to half the partition function of the two dimensional bosonic theory [33].

In the type II theory, the sum over the NS-NS sector cancels the sum over the spacetime fermions, and the RR field has all momentum and winding (we do not impose the Dirac equation at this level). The full partition function is therefore half of the type 0 partition function:

$$\mathbf{Z}_{\mathbf{T^2}} \sim V_2 \int \frac{d^2\tau}{\tau_2^2} \sum_{m,w=-\infty}^{\infty} \exp\left(\frac{-\pi |m - w\tau|^2}{2\tau_2}\right) \tag{B.5}$$

B.2. d=2

The torus partition function of the type 0 theory is:

$$\mathbf{Z_{T^2}} \sim V_4 \int \frac{d^2\tau}{\tau_2^2}\frac{1}{\tau_2} \sum_{m,w=-\infty}^{\infty} \exp\left(\frac{-2\pi|m-w\tau|^2}{3\tau_2}\right)\frac{1}{|\eta(\tau)|^4}\times \left[\left|\frac{\Theta_{00}(\tau)}{\eta(\tau)}\right|^2+\left|\frac{\Theta_{01}(\tau)}{\eta(\tau)}\right|^2+\left|\frac{\Theta_{10}(\tau)}{\eta(\tau)}\right|^2\mp\left|\frac{\Theta_{11}(\tau)}{\eta(\tau)}\right|^2\right] \tag{B.6}$$

The partition function of the type $0'$ theory is given by:

$$\mathbf{Z_{T^2}} \sim V_4 \int \frac{d^2\tau}{\tau_2^2}\frac{1}{\tau_2^{\frac{1}{2}}}\frac{1}{|\eta(\tau)|^2}\times \sum_{k=0}^{2}\left[\left|\frac{\Theta_{00}(\tau)\Theta_{\frac{2k}{3}0}(3\tau)}{\eta^2(\tau)}\right|^2-\left|\frac{\Theta_{01}(\tau)\Theta_{\frac{2k}{3}1}(3\tau)}{\eta^2(\tau)}\right|^2+\left|\frac{\Theta_{10}(\tau)\Theta_{\frac{2k+1}{3}0}(3\tau)}{\eta^2(\tau)}\right|^2\right] \tag{B.7}$$

The modular invariance of the above function can be seen by noticing the identity [34]

$$\Theta_{0,\frac{2k}{3}}(\frac{-1}{3\tau}) = \sum_{l=0}^{l=2}\Theta_{\frac{2l}{3},2k}(\frac{-3}{\tau}). \tag{B.8}$$

The partition function for the type II theory is given by:

$$\mathbf{Z_{T^2}} \sim V_4 \int \frac{d^2\tau}{\tau_2^2}\frac{1}{\tau_2^{\frac{1}{2}}}\frac{1}{|\eta(\tau)|^2}\times\left|\frac{\Theta_{00}(\tau)\Theta_{00}(\frac{\tau}{3})}{\eta(\tau)^2}-\frac{\Theta_{01}(\tau)\Theta_{01}(\frac{\tau}{3})}{\eta(\tau)^2}-\frac{\Theta_{10}(\tau)\Theta_{10}(\frac{\tau}{3})}{\eta(\tau)^2}\right|^2 \tag{B.9}$$

This function is modular invariant [34], and in fact, it vanishes, as it should for a supersymmetric theory.

B.3. d=4

The type 0 partition sum is

$$\mathbf{Z_{T^2}} \sim V_6 \int \frac{d^2\tau}{\tau_2^2}\frac{1}{\tau_2^2} \sum_{m,w=-\infty}^{\infty} \exp\left(\frac{-\pi|m-w\tau|^2}{\tau_2}\right)\frac{1}{|\eta(\tau)|^8}\times \left[\left|\frac{\Theta_{00}(\tau)}{\eta(\tau)}\right|^4+\left|\frac{\Theta_{01}(\tau)}{\eta(\tau)}\right|^4+\left|\frac{\Theta_{10}(\tau)}{\eta(\tau)}\right|^4\mp\left|\frac{\Theta_{11}(\tau)}{\eta(\tau)}\right|^4\right] \tag{B.10}$$

where the $\mp$ sign is for the 0A and 0B theories respectively. The tachyon is contained in the sum of the first two terms.

To fill a gap in the text, we will use this example to get an estimate of the high energy behaviour of our theories. This calculation can easily be generalized to the other cases. In the left moving R sector with no momentum, the partition sum is

$$\sum d_n q^n = \mathrm{Tr} q^{L_0} = 4 \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^4 \sim_{q \to 1} (1-q)^2 \exp\left(\frac{\pi^2}{1-q} \right) \tag{B.11}$$

The chiral density of states d_n for large n can be estimated by a saddle point approximation of the contour integral, and it is given by $d_n^{Op} \sim n^{-\frac{7}{4}} \exp(2\pi\sqrt{n})$. In the closed string, we have $d_n^{Cl} = (d_n^{Op})^2 \sim n^{-\frac{7}{2}} \exp(4\pi\sqrt{n})$. Considering that the mass of the string is related to the level as $\frac{\alpha'}{4} m^2 = n$ gives the mass density of states:

$$\rho(m) \sim m^{-6} \exp(\frac{m}{m_0}), \qquad m_0 = (\pi\sqrt{4\alpha'}) \tag{B.12}$$

The partition sum on the torus (for IIB and IIA) is

$$\mathbf{Z_{T^2}} \sim V_6 \int \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^{3/2}} \frac{1}{|\eta(\tau)|^6} \times \left[\left| \left(\frac{\Theta_{00}^2(\tau)}{\eta^2(\tau)} + \frac{\Theta_{01}^2(\tau)}{\eta^2(\tau)} \right) \frac{\alpha_{11}(\tau)}{\eta(\tau)} - \frac{\Theta_{10}^2(\tau)}{\eta^2(\tau)} \frac{\alpha_{01}(\tau)}{\eta(\tau)} \right|^2 + \left| \left(\frac{\Theta_{00}^2(\tau)}{\eta^2(\tau)} - \frac{\Theta_{01}^2(\tau)}{\eta^2(\tau)} \right) \frac{\alpha_{01}(\tau)}{\eta(\tau)} - \frac{\Theta_{10}^2(\tau)}{\eta^2(\tau)} \frac{\alpha_{11}(\tau)}{\eta(\tau)} \right|^2 \right] \tag{B.13}$$

where the functions α_{m1} are:

$$\alpha_{m1}(\tau) = \sum_{n \in \mathbf{Z}} q^{(n+m/2)^2} \quad (m = 0, 1). \tag{B.14}$$

The partition function defined above vanishes as expected for spacetime supersymmetric theories [26]. The first expression in the oscillator sum contains the tachyon (with odd momenta) and the tachyno, and the second contains the graviton multiplet (with even momenta) and its superpartner.

B.4. d=6

The torus partition function of the type 0 theory is:

$$\mathbf{Z_{T^2}} \sim V_8 \int \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^3} \sum_{m,w=-\infty}^{\infty} \exp\left(\frac{-2\pi|m-w\tau|^2}{\tau_2}\right) \frac{1}{|\eta(\tau)|^{12}} \times \left[\left|\frac{\Theta_{00}(\tau)}{\eta(\tau)}\right|^6 + \left|\frac{\Theta_{01}(\tau)}{\eta(\tau)}\right|^6 + \left|\frac{\Theta_{10}(\tau)}{\eta(\tau)}\right|^6 \mp \left|\frac{\Theta_{11}(\tau)}{\eta(\tau)}\right|^6\right] \tag{B.15}$$

The partition function of the type 0′ theory (which is the type 0 theory on two NS5-branes is given by:

$$\mathbf{Z_{T^2}} \sim V_8 \int \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^{\frac{5}{2}}} \frac{1}{|\eta(\tau)|^{10}} \times \left[\left|\frac{\Theta_{00}(\tau)}{\eta(\tau)}\right|^8 + \left|\frac{\Theta_{01}(\tau)}{\eta(\tau)}\right|^8 + \left|\frac{\Theta_{10}(\tau)}{\eta(\tau)}\right|^8\right] \tag{B.16}$$

The partition function for the type II theories is given by:

$$\mathbf{Z_{T^2}} \sim V_8 \int \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^{\frac{5}{2}}} \frac{1}{|\eta(\tau)|^{10}} \times \left|\frac{\Theta_{00}^4(\tau)}{\eta^4(\tau)} - \frac{\Theta_{01}^4(\tau)}{\eta^4(\tau)} - \frac{\Theta_{10}^4(\tau)}{\eta^4(\tau)}\right|^2 \tag{B.17}$$

Appendix C. Quantization in the Green-Schwarz formalism ($d = 4$)

The six bosonic coordinates are $X^a = X^a + \tilde{X}^a$ and their spacetime superpartners are θ_α, $\tilde{\theta}_\alpha$ which are six dimensional Weyl spinors in the $\mathbf{4}$ or $\overline{\mathbf{4}}$ of $SU(4)$. We shall work in light cone gauge where the coordinates are X^I $(I = \rho, \theta, 1, 2)$, and Weyl spinors $S_\alpha, \overline{S}_\alpha$ in the $\mathbf{2}$ and $\tilde{S}_\alpha, \overline{\tilde{S}}_\alpha$ in the $\mathbf{2}$ or $\mathbf{2}'$ (IIB or IIA) of $SO(4)$. In the T-dual variables, the interaction is the sine-Liouville interaction:

$$S = \int d^2\sigma \left(\frac{1}{2}\partial_\mu X^I \partial^\mu X^I - i\overline{S}^\alpha \partial_+ S^\alpha - i\overline{\tilde{S}^\alpha}\partial_- \tilde{S}^\alpha - e^{-\frac{1}{\sqrt{2}}(\rho+i\theta)} S^2 \tilde{S}^2 - e^{-\frac{1}{\sqrt{2}}(\rho-i\theta)} \overline{S}^2 \overline{\tilde{S}}^2 + e^{-\sqrt{2}\rho}\right). \tag{C.1}$$

In the above action, a specific choice of σ matrices has been made which breaks the $U(1)$ symmetry which rotates the spinors S^1, S^2 into each other.

The right and left moving Lorentz generators are given by:

$$\begin{aligned} J^{12} &= X^1 P^2 - P^2 X^1 + \frac{1}{\sqrt{2}}\dot{\theta} - \frac{1}{2}\overline{S}^2 S^2 + \frac{1}{2}\overline{S}^1 S^1, \\ \tilde{J}^{12} &= \tilde{X}^1 \tilde{P}^2 - \tilde{P}^2 \tilde{X}^1 + \frac{1}{\sqrt{2}}\dot{\tilde{\theta}} - \frac{1}{2}\overline{\tilde{S}^2}\tilde{S}^2 + \frac{1}{2}\overline{\tilde{S}^1}\tilde{S}^1, \end{aligned} \tag{C.2}$$

The rotation in the 1-2 direction is given the sum of the left and right moving parts which also we shall call J^{12}. The only other physical symmetry of the theory (consistent with the R-NS analysis) is:

$$w = \frac{1}{\sqrt{2}}(\dot{\theta} - \dot{\tilde{\theta}}) - \frac{1}{2}(\overline{S}^2 S^2 - \overline{\tilde{S}^2}\tilde{S}^2) - \frac{1}{2}(\overline{S}^1 S^1 - \overline{\tilde{S}^1}\tilde{S}^1) \tag{C.3}$$

In the cigar theory, this current is the conserved momentum which we saw earlier. Note that in this formalism, the above conserved current is a combination of naive momemtum around the cigar and a piece which acts on the worldsheet fermions.

The free action has eight leftmoving (and another eight rightmoving) supersymmetries - four shifts of the spinors and four linearly realised supersymmetries. The interaction preserves only half of the above symmetries - only shifts of $S^1, \overline{S}^1$ are preserved, and two of the linear supersymmetries (which are modified due to the superpotential so that they are no longer purely left or right moving) are preserved. These supercharges are charged under the currents J^{12}, w (here $\sigma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$):

$$\begin{aligned}
&[J^{12}, Q^\alpha] = \frac{1}{2}\sigma^3_{\alpha\beta} Q^\beta, \qquad [J^{12}, \tilde{Q}^\alpha] = \frac{1}{2}\sigma^3_{\alpha\beta}\tilde{Q}^\beta; \\
&[J^{12}, \overline{Q}^\alpha] = -\frac{1}{2}\sigma^3_{\alpha\beta}\overline{Q}^\beta, \qquad [J^{12}, \overline{\tilde{Q}^\alpha}] = -\frac{1}{2}\sigma^3_{\alpha\beta}\overline{\tilde{Q}^\beta}; \\
&[w, Q^\alpha] = \frac{1}{2}Q^\alpha, \qquad [w, \tilde{Q}^\alpha] = -\frac{1}{2}\tilde{Q}^\alpha; \\
&[w, \overline{Q}^\alpha] = -\frac{1}{2}\overline{Q}^\alpha, \qquad [w, \overline{\tilde{Q}^\alpha}] = \frac{1}{2}\overline{\tilde{Q}^\alpha}.
\end{aligned} \tag{C.4}$$

The ground state is annihilated by all bosonic and fermionic lowering operators, and is in a irrep of the susy algebra which (for the left moving part alone) consists of four states, all of vanishing energy. Below, we list the states and the charges under the two conserved currents winding and the spacetime rotation J^{12}. We start with a state which asymptotically obeys $\overline{S}^\alpha|0\rangle = \dot{\theta}|0\rangle = 0$, and build three other left moving states of the same energy (and repeat the construction on the rightmoving side):

Table 5: Ground states in the Green-Schwarz formalism

State	$\|0\rangle$	$\overline{S}^2\|0\rangle$	$\overline{S}^1\|0\rangle$	$\overline{S}^2\overline{S}^1\|0\rangle$	$\|0\rangle$	$\overline{\tilde{S}^2}\|0\rangle$	$\overline{\tilde{S}^1}\|0\rangle$	$\overline{\tilde{S}^2\tilde{S}^1}\|0\rangle$
$\mathbf{J^{12}}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0
$\mathbf{w}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	0	$\frac{1}{2}$	$\frac{1}{2}$	1

The closed string spectrum is built by tensoring these two together. The space-time spin properties are given by the charge under J^{12}. At this level, we get 16 states with zero energy which form a scalar(tachyon) multiplet. The four tachyon states have winding number $-1, 0, 0, 1$. The conserved charges are the same as in the NSR formalism.

The states with least non-zero energy $E = \frac{1}{2}$ is built on states which obeys $\overline{S}^\alpha| \pm 1\rangle = 0$, $\dot{\theta}| \pm 1\rangle = \pm\frac{1}{\sqrt{2}}| \pm 1\rangle$ tensored with the rightmovers. This gives 64 states with spins corresponding to the graviton multiplet. The gravitons have zero winding, and the conserved charges of all the states are the same as in the NSR formalism.

The general state in the spectrum is constructed by the action of the raising operators of the four bosons $\alpha^I_{-n}, \tilde{\alpha}^I_{-n}$ and the eight fermions $S^\alpha_{-n}, \overline{S}^\alpha_n, \tilde{S}^\alpha_{-n}, \overline{\tilde{S}}^\alpha_n$, on the states annihilated by all the lowering operators with arbitrary integer or half integer value of momemtum. The algebra of the conserved charges being the same as in the NSR formalism, and the ground state being the same guarantees that the full spectrum is identical as well.

Appendix D. Conformal invariance at second order in the $d = 6$ theory

The operators $\mathcal{L}^{\mu\nu}$ are all of dimension $(1, 1)$. The nine operators generate other operators when they come close to each other, and the coefficients of these operators should vanish for the theory to remain conformal.

1. As in the Liouville theory, we assume that in the OPE of $e^{p_1\rho}$ with $e^{p_2\rho}$, the operator that is generated is normalizable ($p < 0$) if $e^{(p_1+p_2)\rho}$ is.
2. Asymptotically, we can consistently choose to keep operators which decay slower than a certain rate (i.e., $p > p_0$ for some p_0). In the five brane the-

ory, we can think of this as an expansion in x^2/r^2 where x is the separation of the branes and r is the radial distance from the center of mass.

It can be checked to order $e^{-\rho}$, that the are the only operators that preserve $\mathcal{N} = (2, 2)$ superconformal symmetry on the worldsheet and are not total derivatives are the above nine. This means that the above set of operators are closed under the OPE (upto perhaps change in the zero modes of fields).

We have then, $A_i(z)A_j(0) \sim \frac{c_1}{2z}\epsilon_{ijk}A_k(0)$. The free field part of this OPE can be explicitly computed, and the dilaton part - $e^{-\rho}$ with itself - gives again $e^{-\rho}$ with a coefficient c_1 which we *assume* is non-zero. This relation is consistent with charge conservation after taking into account the background charge of the dilaton.

If we consider separating the branes in the directions (X^8, X^9) alone, we turn on only $\mathcal{L}^{int} = C_{66}\mathcal{L}^{66} + C_{77}\mathcal{L}^{77} + 2C_{67}\mathcal{L}^{67}$. At second order in the coefficients, there are new operators generated - the contribution to the beta function by these new operators is proportional to $((C_{66} + C_{77})\mathcal{L}^{int} + (C_{66}C_{77} - C_{67}^2)\mathcal{L}^{newint})e^{-\rho-\tilde{\rho}}$. Demanding that the second term vanishes gives us $C_{66}C_{77} - C_{67}^2 = 0$. We get five other such conditions for other separations.

The solution to these constraints is labelled by four parameters x^μ such that $C_{\mu\nu} = x_\mu x_\nu$. It can be checked that this is a good solution for the most general interaction allowed. The interaction $\mathcal{L}^{int} = x_\mu x_\nu \mathcal{L}^{\mu\nu}e^{-\rho-\tilde{\rho}}$ generates at second order a contribution to the beta function proprtional to $(x_\mu x^\mu)\mathcal{L}^{int}$ which is equivalent to a change in the zero mode of the dilaton. More precisely, the expectation value of $e^{-\rho_0}$ is identified with $(x_\mu x^\mu)$. The value of the dilaton at the tip corresponds to the distance of the branes from the origin, and the other three operators correspond to moving the branes on a sphere at constant radius.

Chapter 3

Stable D-branes and Open String Field Theory in Two Dimensions

1. Introduction

Non-critical string theories, describing strings propagating in two dimensions or less, were instrumental in shaping our understanding of the behaviour of string theory beyond the perturbative regime. The $\mathcal{O}(1/g_s)$ nonperturbative effect, so characteristic of D-branes, first emerged from the study of these systems[35]. Only recently, thanks to the advancement in our understanding of boundary Liouville dynamics[7–47] (following earlier work [48,49]), has a physical understanding of the nonperturbative effects begun to emerge[50–55]. (See the review [56] for an exhaustive list of references.)

In another development, the dynamics of tachyon condensation led Sen to propose a new duality between open and closed strings[57–59]. Noncritical string theories are likely to be ideally suited for understanding this duality and indeed they have already played an important role in the shaping of these ideas. Recently in an interesting work[55], Gaiotto and Rastelli applied this philosophy to Liouville theory coupled to $c = -2$ matter. This system has certain topological symmetries[60] constraining its dynamics. Using these symmetries the authors obtain the Kontsevich topological matrix model[61] describing the *closed-string* theory starting from the *open-string field theory*.

Among the non-critical string theories, the theory of a single scalar field coupled to worldsheet gravity has perhaps the richest structure. The matter theory has central charge $c = 1$, while the Liouville field with its central charge $c_L = 25$ provides an interpretation as a *critical* string theory with two-dimensional target space. Closed strings in this background have been studied in the past from quite a few different angles: matrix quantum mechanics (see Ref.[62] and references therein), worldsheet conformal field theory[63–65], topological field theory[66–68] and topological matrix models[69–72] related to the moduli space of Riemann surfaces (see [73] for a recent review). As shown in [67], this string theory actually captures the topological sector of string theory on the Euclidean black hole that we studied in the previous chapter.

The $c = 1$ closed-string theory has a marginal deformation, corresponding to changing the radius R of compactification of the scalar field. At a particular value of this radius, $R = 1$ in our conventions, the theory is self-dual under T-duality and an $SU(2) \times SU(2)$ symmetry gets restored as a result of which momentum and winding modes become degenerate with each other.

In this chapter, we will consider the open-string version of this two-dimensional string theory – more precisely, a scalar field compactified at the self-dual radius on a worldsheet with the topology of a disc or an upper half plane, coupled to the Liouville mode. Various types of branes are possible depending on the choice of boundary condition on the fields. We will choose to work with (generalized) Neumann boundary conditions on the Liouville field φ. On the matter field X we impose Dirichlet boundary conditions, as a result of which the brane is localised in X and there are no momentum modes in that direction, only winding modes. Because the radius is self-dual, one can equally well impose Neumann boundary conditions in X and then there are momentum but no winding modes. The physics is identical in the two cases.

The resulting branes are stable and are known as FZZT branes[7–37]. We will compute the two- and three-point disc correlation functions of the fields living on the FZZT branes, as well as the bulk-boundary two-point function of such fields with 'bulk' fields. The Liouville contributions to these correlators are non-trivial and general expressions are available in the literature[7,41,40,47]. Some of them are only known in the form of contour integrals. From the point of view of these theories, the $c_L = 25$ Liouville field coupled to $c = 1$ matter is at the 'boundary' of the theories studied in Refs.[7--49]. In the specific case of interest, we take a careful limit to obtain the desired correlators. In particular we are able to evaluate the relevant contour integrals in our case, leading to expressions that are much simpler and more explicit than those previously given in the literature for the more general $c \leq 1$ case.

The resulting expressions satisfy the expected consistency conditions and other recursion relations. When the Liouville theory is combined with matter, one gets a massless 'tachyon' field[19] labelled by integer winding numbers. The matter contribution to the tachyon correlators are just winding number conserving delta functions. In addition to the tachyons, there are discrete states at ghost numbers one and zero[74,75]. The former are the remnants of massless and massive states of critical strings and their correlators are determined by the SU(2) symmetry at the self-dual radius. (The latter class of operators are characteristic of non-critical theories and in particular, they form a ring on which the symmetry of the theory can be realized in a geometric way[76].) We have not attempted to study this ring in the FZZT brane background (for general results on the $c \leq 1$ boundary ground ring, see Ref.[77]). As mentioned above, the expressions for these correlators are divergent. As we will see, once we perform renormalizations of the bulk and boundary cosmological constants, the divergence is a common multiplicative factor for both the two- and three-point boundary correlators.

The simple and elegant form of the answers obtained is suggestive of a simple physical interpretation, perhaps in terms of fermions, as we will see. The answers share some of the properties of the (simpler) case of $c = -2$[55], notably that they are independent of the bulk cosmological constant. All this encourages us to try and understand the corresponding open-string field theory, following the ideas in Ref.[55]. Accordingly, in the last section of this chapter, we begin to study the open-string field theory of the FZZT branes. Motivated by the fact that the disc path integral describing classical processes of non-critical string theory localizes to

[19] No operator in this chaper is truly unstable on the worldsheet. The FZZT boundary conditions do not allow such modes. With this in mind, and since there is also no unstable operator in the bulk, we use the word tachyon as is conventional.

the BRS cohomology, we evaluate the action for the 'on-shell' states (tachyons and the discrete states). This results in a theory of infinitely many matrices. We hope to analyse this theory in more detail in the future.

The organisation of this chapter is as follows. Sec. 2 describes the background and sets up notation. The Liouville contributions to the two- and three-point functions are evaluated in Sections 3 and 4. In Sec. 5 we calculate the bulk-boundary two-point function. Sec. 6 is devoted to string field theory. We end with some comments in the final Sec. 7. Appendix A contains some properties of the special functions that appear in the Liouville correlators. Some details of the contour integral relevant for Sec. 4 are given in Appendix B.

2. Two-dimensional Open String Theory and the FZZT branes

The theory we are interested in is described by the worldsheet action[20]

$$\begin{aligned}&\frac{1}{4\pi}\int_{\mathcal{D}}\Big((\partial X)^2+(\partial\varphi)^2+Q\hat{R}\varphi+4\pi\mu_0\,e^{2b\varphi}\Big)\\&+\frac{1}{2\pi}\int_{\partial\mathcal{D}}\Big(Q\hat{K}\varphi+2\pi\mu_{B,0}\,e^{b\varphi}\Big)\,,\end{aligned}\tag{2.1}$$

where X,ϕ are the matter and Liouville fields and Q,b are numerical coefficients. In this action, $\mathcal{D}$ has the topology of a disc/UHP, $\hat{R}$ is the curvature of the (reference) metric, $\hat{K}$ the induced curvature of the boundary and μ_0 and $\mu_{B,0}$ are the (bare) bulk and boundary cosmological constants respectively.

With the action above, the matter sector has central charge $c=1$ while the Liouville sector has $c_L=1+6Q^2$. The coefficient b appearing in the exponents satisfies $Q=b+\frac{1}{b}$. Criticality requires the choice $c_L=25$, from which we determine $Q=2$ and $b=1$. Because of divergences that appear at $b=1$, we will need to carefully take the limit $b\to 1$ and regularise the divergences appropriately.

On the field φ, we will impose

$$i(\partial-\overline{\partial})\varphi=4\pi\mu_{B,0}\,e^{b\varphi},\tag{2.2}$$

[20] We work in $\alpha'=1$ units.

the generalized Neumann boundary condition.

The field X, which we take to be Euclidean is, in general, compactified on a circle of radius R. We can impose a suitable boundary condition on the field X; for instance, at a generic radius, we could impose Dirichlet or Neumann boundary conditions $(\partial \pm \overline{\partial})X = 0$ which, in conjunction with the boundary condition on the Liouville field above, describe the non-compact D-instanton and $D0$-brane respectively. As noted in the introduction, the two choices are physically equivalent at self-dual radius.

In the bulk, the observables of the theory are a massless scalar field in the two dimensional target space known as the 'tachyon' field, and an infinite set of quantum mechanical states which arise at special values of the momentum, known as the discrete states. The vertex operators[21] corresponding to the tachyon field take the following form at weak coupling:

$$\mathcal{T}_k = c\bar{c}\, \exp\left(ik(X \pm \overline{X}) + (2 - |k|)\varphi\right). \tag{2.3}$$

The tachyon vertex operator on the boundary on the other hand carries additional indices (σ_1, σ_2) corresponding to the boundary conditions on the two ends of the open string:

$$T_k \equiv c\, \left[e^{ikX}\, V_\beta\right]^{\sigma_1\sigma_2} = c\, \left[\exp\left(ikX + \beta\varphi\right)\right]^{\sigma_1\sigma_2}, \tag{2.4}$$

where the second expression is the asymptotic form. From this we see that β labels the Liouville momentum, and the conformal dimension of the Liouville vertex operator is $\Delta = \beta(Q - \beta)$ where $Q = b + \frac{1}{b} = 2$. Requiring that the full vertex operator has dimension one, one finds the on-shell condition $\beta = 1 - |k|$. The boundary label σ is related to the (bare) cosmological constants μ_0 and $\mu_{B,0}$ by:

$$\cos 2\pi b\left(\sigma - \frac{Q}{2}\right) = \frac{\mu_{B,0}}{\sqrt{\mu_0}}\sqrt{\sin \pi b^2}. \tag{2.5}$$

As we shall discuss later, the cosmological constants require renormalisation in the $c = 1$ string theory.

[21] We are only considering local operators, which correspond to non-normalizable modes.

We are specifically interested in the theory at self-dual radius $R = 1$, where the worldsheet theory is an $SU(2)_L \times SU(2)_R$ current algebra at level 1. The symmetry of the closed-string theory is generated by $(J^{\pm}, J^3) = (e^{\pm i2X}, i\partial X)$ and their right moving counterparts. The physical vertex operators at ghost number one are[76]:

$$Z_{k;m,\overline{m}} = c\overline{c}\, V^{mat}(k,m)\overline{V}^{mat}(k,\overline{m}) \exp\left((2-k)\varphi\right). \tag{2.6}$$

where k is a non-negative integer or half integer; $V^{mat}(k,k) \equiv e^{ikX}$, and the operators $V^{mat}(k, m < k)$ are defined by acting with the $SU(2)_L$ lowering operator. Hence $m = k, k-1, \cdots, -k$. The corresponding right movers are defined in a similar manner. The physical content of the theory can also be summarized as a massless field $\mathcal{T}(\theta, \phi, \psi; \varphi)$ living on an S^3 times the non-compact Liouville direction.

The open string imposes a boundary condition relating the left and right moving currents J^a and $\overline{J}^a$. The branes in the $SU(2)_n$ theory are labelled by a half-integer $J = 0, \cdots, \frac{n}{2}$ which labels the conjugacy class in the group, and continuous moduli which take values in $SO(3)$ which label the origin of the 3-sphere viewed as a group manifold[78]. The conjugacy classes are topologically 2-spheres in the group manifold.

For our case, level $n = 1$, there are only two possible discrete labels $J = 0, 1/2$ and the full moduli space is $SU(2)$ which is topologically S^3[79]. A brane is simply a point on this sphere, which can be thought of as a degenerate S^2. It breaks the $SO(4) = SU(2) \times SU(2)$ symmetry of the 3-sphere to a diagonal $SU(2)$ symmetry group of the degenerate 2-sphere. The open-string modes are classified as representations of this $SU(2)$.

For example, the boundary states which correspond to Neumann and Dirichlet for generic radii are labelled by the two poles on the S^3, and are given respectively by $J^a = \pm\overline{J}^a$. The generators of the diagonal $SU(2)$ subgroup which is preserved are $J^a \pm \overline{J}^a$. The allowed representations of the diagonal SU(2) are $k \pm \overline{k}$ where both k and $\overline{k}$ are integer or half integer, so that the allowed representations of the diagonal subgroup are integer. Note that half the representations of $SU(2)$ (the half-integer spins) do not correspond to physical operators.

Thus the physical vertex operators of the open string at ghost number one are:

$$Y^{\sigma_1\sigma_2}(k,m) \equiv c\,\left[V^{mat}(k,m)V_\beta\right]^{\sigma_1,\sigma_2} = c\,\left[V^{mat}(k,m)\exp((1-k)\varphi)\right]^{\sigma_1\sigma_2}, \quad (2.7)$$

where, (k,m) are the usual $SU(2)$ labels with spin k an integer and $m = k, k-1, \ldots, -k$.

3. Boundary Two-Point Function

In this section, we shall compute the two-point function of the Liouville vertex operators $V_\beta^{\sigma_1\sigma_2}$ which enter the physical open-string vertex operators (2.7). The two-point function of boundary operators in Liouville theory, of arbitrary central charge $c_L = 1+6Q^2$, is given by[7]:

$$\left\langle V_{\beta_1}^{\sigma_1\sigma_2}(x)V_{\beta_2}^{\sigma_2\sigma_1}(0)\right\rangle \equiv \frac{\delta(\beta_1+\beta_2-Q)+d(\beta|\sigma_1,\sigma_2)\delta(\beta_1-\beta_2)}{|x|^{2\Delta_{\beta_1}}}, \quad (3.1)$$

where $d(\beta|\sigma_1,\sigma_2)$ is the *reflection amplitude*, the expression for which is given below. The delta functions can be understood as arising due to the reflection from the Liouville potential, and is not present in the higher-point functions. Every non-normalizable operator in the theory is related to a normalizable operator by this reflection, $V_\beta^{\sigma_1\sigma_2} = d(\beta|\sigma_1,\sigma_2)V_{Q-\beta}^{\sigma_1\sigma_2}$.

The reflection amplitude $d(\beta|\sigma_1,\sigma_2)$ is given by[7]:

$$\begin{aligned} d(\beta|\sigma_1,\sigma_2) &= \mathcal{A}_1\,\mathcal{A}_2\,\mathcal{A}_3, \\ \mathcal{A}_1 &= \left(\pi\mu_0\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-2\beta}{2b}}, \\ \mathcal{A}_2 &= \frac{\Gamma_b(2\beta-Q)}{\Gamma_b(Q-2\beta)}, \\ \mathcal{A}_3 &= \frac{S_b(2Q-\sigma_1-\sigma_2-\beta)\,S_b(\sigma_1+\sigma_2-\beta)}{S_b(\beta+\sigma_1-\sigma_2)\,S_b(\beta+\sigma_2-\sigma_1)}. \end{aligned} \quad (3.2)$$

In the above, $\gamma(x) \equiv \Gamma(x)/\Gamma(1-x)$ and the special functions $\Gamma_b(x)$ and $S_b(x)$ are defined in [7,47]. We record the relevant details in Appendix A.

As mentioned above, to specialise to $c=1$ we must carefully take the limit $b \to 1, Q \to 2$. This limit is singular and requires us first of all to renormalize both the bulk and boundary cosmological constants. In the first line in Eq.(3.2), we set $b = 1 - \frac{\varepsilon}{2}$ and find that

$$\mathcal{A}_1 \rightarrow (\pi\mu_0\gamma(1-\varepsilon))^{1-\beta}\,. \tag{3.3}$$

Using $\gamma(1-\varepsilon) \rightarrow \varepsilon$, we see that the above expression becomes finite if we define the renormalised[22] bulk cosmological constant by:

$$\mu = 4\pi\mu_0\,\epsilon. \tag{3.4}$$

Using this and recalling from (2.7) that $\beta = 1-k$ with k a non-negative integer, it follows that the first factor in the two-point amplitude is:

$$\mathcal{A}_1 = \left(\frac{\mu}{4}\right)^{1-\beta} = \left(\frac{\sqrt{\mu}}{2}\right)^{2k}\,. \tag{3.5}$$

The renormalisation of the bare bulk cosmological constant μ_0 performed above is well-known, and leads to the result that the cosmological operator for $c = 1$ closed strings is not the naive one, $e^{2\varphi}$, but rather $\varphi\, e^{2\varphi}$.

Now coming back to Eq.(2.5) and taking $b = 1 - \frac{\varepsilon}{2}$ we have, for small ε:

$$\cos 2\pi\sigma = \sqrt{\pi\varepsilon}\,\frac{\mu_{B,0}}{\sqrt{\mu_0}} = 2\pi\varepsilon\,\frac{\mu_{B,0}}{\sqrt{\mu}}, \tag{3.6}$$

which means that we also need to define a renormalised[23] boundary cosmological constant $\mu_B = 2\pi\varepsilon\,\mu_{B,0}$. Hence finally the relation between the σ parameter and the renormalised (bulk and boundary) cosmological constants is:

$$\cos 2\pi\sigma = \frac{\mu_B}{\sqrt{\mu}}. \tag{3.7}$$

The parameter σ can be real or imaginary depending on whether $\mu_B < \sqrt{\mu}$ or $\mu_B > \sqrt{\mu}$. In what follows, we keep all the σ_i generic.

[22] This differs by a factor of 4 from the normalisation used in Refs.[42,51]. However, it is more natural as the area of a unit 2-sphere is 4π.

[23] Once again this differs (now by a factor of 2) from the normalisations of Refs.[51,42], and is consistent with the length of a unit circle being 2π.

The factor $\mathcal{A}_2$ depends only on β and not on σ_i. Using Eq.(3.2), we find:

$$\mathcal{A}_2 = \frac{\Gamma_1(-2k)}{\Gamma_1(2k)}. \tag{3.8}$$

This expression is actually divergent. However, we can regulate it by going slightly off-shell. We can do this by shifting β from the integer value by an amount ϵ: $k \to k + \epsilon$ and extract the leading divergence. We could use a different regulator and deform b away from 1 to $1 - \epsilon$ and we get the same answer. As detailed in Appendix A, $\mathcal{A}_2$ is determined to be:

$$\mathcal{A}_2 = \frac{(-1)^k}{(2\pi)^{2k}\,\Gamma(2k+1)\Gamma(2k)}\frac{1}{\epsilon^{2k+1}}. \tag{3.9}$$

Finally we turn to the third factor in Eq.(3.1):

$$\mathcal{A}_3 = \frac{S_1(2Q - \sigma_1 - \sigma_2 - \beta)\,S_1(\sigma_1 + \sigma_2 - \beta)}{S_1(\beta + \sigma_1 - \sigma_2)\,S_1(\beta + \sigma_2 - \sigma_1)}. \tag{3.10}$$

Now using the inversion relation $S_b(x)S_b(Q-x) = 1$ and substituting $\beta = 1 - k$, $\mathcal{A}_3$ can be rewritten as:

$$\begin{aligned}\mathcal{A}_3 &= \frac{S_1(\sigma_2 + \sigma_1 - \beta)}{S_1(-Q + \beta + \sigma_2 + \sigma_1)}\frac{S_1(Q - \beta + \sigma_1 - \sigma_2)}{S_1(\beta + \sigma_1 - \sigma_2)} \\ &= \frac{S_1(-1 + k + \sigma_2 + \sigma_1)}{S_1(-1 - k + \sigma_2 + \sigma_1)}\frac{S_1(1 + k + \sigma_1 - \sigma_2)}{S_1(1 - k + \sigma_1 - \sigma_2)}.\end{aligned} \tag{3.11}$$

Next we define the combinations $\sigma^{\pm} = \sigma_1 \pm \sigma_2$ and invoke the recursion relation (see Appendix A) $S_1(x+1) = 2\sin\pi x\, S_1(x)$ to write:

$$\begin{aligned}\mathcal{A}_3 &= \prod_{m=1}^{2k}\big(2\sin\pi(\sigma^+ + k - 1 - m)\big)\prod_{n=1}^{2k}\big(2\sin\pi(\sigma^- + k - 1 - n)\big) \\ &= \big(4\,\sin\,\pi\sigma^+\,\sin\,\pi\sigma^-\big)^{2k}.\end{aligned} \tag{3.12}$$

This can be rewritten in terms of the original boundary parameters σ_1 and σ_2:

$$\mathcal{A}_3 = \left(2\left(\cos 2\pi\sigma_1 - \cos 2\pi\sigma_2\right)\right)^{2k} = \left(2\,\frac{\mu_{1B} - \mu_{2B}}{\sqrt{\mu}}\right)^{2k}. \tag{3.13}$$

Putting everything together, we finally get:

$$d(1-k|\mu_{1B},\mu_{2B}) = \frac{(-1)^k}{\epsilon^{2k+1}}\frac{(\mu_{1B} - \mu_{2B})^{2k}}{(2\pi)^{2k}\,\Gamma(2k+1)\Gamma(2k)}. \tag{3.14}$$

We will find it convenient to renormalize the open-string operators (2.7) as

$$\tilde{Y}^{\sigma_1,\sigma_2}_{k,m} = (2\pi\epsilon)^k \Gamma(2k)\, Y^{\sigma_1\sigma_2}_{k,m}. \tag{3.15}$$

This redefinition is different from the standard one found in the literature for closed strings, as it has an additional factor of $(2\pi\epsilon)^k$. For the cosmological operator ($k = 0$) this extra factor is absent and the renormalisation is the standard one. The matter contribution to the two-point function being trivial, let us put the renormalization factor in the Liouville vertex operator alone and define

$$\tilde{V}^{\sigma_1,\sigma_2}_{1-k} = (2\pi\epsilon)^k \Gamma(2k)\, V^{\sigma_1,\sigma_2}_{1-k}. \tag{3.16}$$

Expressed in these variables, the reflection amplitude is:

$$\tilde{d}(1-k|\mu_{1B},\mu_{2B}) = \frac{(-1)^k}{\epsilon} \frac{(\mu_{2B}-\mu_{1B})^{2k}}{2k}. \tag{3.17}$$

Several features of this result are noteworthy. First, it is independent of the bulk cosmological constant μ. A similar feature was noticed[55] for the correlators of $c = 28$ Liouville theory (corresponding to strings propagating in a $c = -2$ matter background). Second, the result depends only on the difference of the two boundary cosmological constants μ_{1B}, μ_{2B}. We will see later that these features persist for the boundary three-point function. They are reminiscent of the identification of the extended B-type branes of topological field theories to fermions[80]. Finally, we see that after renormalization, the reflection amplitude has a simple pole singularity (as a function of ϵ). Again this turns out to be the case for the boundary three-point function as well. Later, when we use this in the string field theory action, we will need to absorb this singularity by a redefinition of the string coupling constant.

4. Boundary Three-Point Function

The three-point function in boundary Liouville theory is defined by:

$$\langle V^{\sigma_2\sigma_3}_{\beta_1}(x^1) V^{\sigma_3\sigma_1}_{\beta_2}(x^2) V^{\sigma_1\sigma_2}_{\beta_3}(x^3)\rangle = \frac{C^{\sigma_2\sigma_3\sigma_1}_{\beta_1\beta_2\beta_3}}{|x_{21}|^{\Delta_1+\Delta_2-\Delta_3}|x_{32}|^{\Delta_2+\Delta_3-\Delta_1}|x_{13}|^{\Delta_3+\Delta_1-\Delta_2}}. \tag{4.1}$$

An expression was found in Ref.[41] (see also Refs.[42,47] for subsequent discussions) for the coefficient C as a product of four factors:

$$
\begin{aligned}
C^{\sigma_2\sigma_3\sigma_1}_{\beta_1\beta_2\beta_3} &= \mathcal{B}_1\,\mathcal{B}_2\,\mathcal{B}_3\,\mathcal{B}_4,\,, \\
\mathcal{B}_1 &= \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{1}{2}(Q-\beta_1-\beta_2-\beta_3)}, \\
\mathcal{B}_2 &= \frac{\Gamma_b(\beta_2+\beta_3-\beta_1)\Gamma_b(2Q-\beta_1-\beta_2-\beta_3)\Gamma_b(Q-\beta_1-\beta_2+\beta_3)\Gamma_b(Q-\beta_1+\beta_2-\beta_3)}{\Gamma_b(Q)\,\Gamma_b(Q-2\beta_1)\,\Gamma_b(Q-2\beta_2)\,\Gamma_b(Q-2\beta_3)}, \\
\mathcal{B}_3 &= \frac{S_b(Q-\beta_3+\sigma_1-\sigma_3)\,S_b(2Q-\beta_3-\sigma_1-\sigma_3)}{S_b(\beta_2+\sigma_2-\sigma_3)\,S_b(Q+\beta_2-\sigma_2-\sigma_3)}, \\
\mathcal{B}_4 &= \frac{1}{i}\int_{-i\infty-0}^{+i\infty-0} ds\ \prod_{i=1}^{4}\frac{S_b(U_i+s)}{S_b(V_i+s)}.
\end{aligned}
\tag{4.2}
$$

In the factor $\mathcal{B}_4$, the quantities U_i, V_i, $i=1,\cdots,4$ are defined as follows:

$$
\begin{aligned}
U_1 &= \sigma_1+\sigma_2-\beta_1, & V_1 &= 2Q+\sigma_2-\sigma_3-\beta_1-\beta_3, \\
U_2 &= Q-\sigma_1+\sigma_2-\beta_1, & V_2 &= Q+\sigma_2-\sigma_3-\beta_1+\beta_3, \\
U_3 &= \beta_2+\sigma_2-\sigma_3, & V_3 &= 2\sigma_2, \\
U_4 &= Q-\beta_2+\sigma_2-\sigma_3, & V_4 &= Q.
\end{aligned}
\tag{4.3}
$$

We want to compute the above for the values $b=1$, $\beta_i = 1-k_i$ for our case of $c=1$. In this section, we choose the kinematic regime $k_3 > k_1, k_2 > 0$. We shall later need to take a careful limit as k_i approach integers. The first two factors are evaluated as before, and we get

$$
\begin{aligned}
\mathcal{B}_1 &= \left(\frac{\mu}{4}\right)^{\frac{1}{2}(k_1+k_2+k_3-1)} = \left(\frac{\sqrt{\mu}}{2}\right)^{\sum_i k_i-1}, \\
\mathcal{B}_2 &= \frac{\Gamma_1(1+\sum_i k_i)\Gamma_1(1+k_1+k_2-k_3)\Gamma_1(1+k_1-k_2+k_3)\Gamma_1(1+k_1-k_2-k_3)}{\Gamma_1(2)\,\Gamma_1(2k_1)\,\Gamma_1(2k_3)\,\Gamma_1(2k_2)} \\
&= \frac{(-1)^{\lfloor (k_2+k_3-k_1)/2\rfloor}}{(2\pi\epsilon)^{k_2+k_3-k_1}} \times \frac{\Gamma_1\left(1+\sum_i k_i\right)}{\Gamma_1(2)} \\
&\quad\times \frac{\Gamma_1(1+k_1+k_2-k_3)\,\Gamma_1(1+k_1-k_2+k_3)\,\Gamma_1(1-k_1+k_2+k_3)}{\Gamma_1(2k_1)\,\Gamma_1(2k_2)\,\Gamma_1(2k_3)} \\
&= \frac{(-1)^{\lfloor (k_2+k_3-k_1)/2\rfloor}}{(2\pi\epsilon)^{k_2+k_3-k_1}}\frac{\Gamma_1\left(1+\sum_i k_i\right)}{\Gamma_1(2)}\prod_{j=1}^{3}\frac{\Gamma_1\left(1+\sum_i k_i-2k_j\right)}{\Gamma_1(2k_j)},
\end{aligned}
\tag{4.4}
$$

where $\lfloor x \rfloor$ is the integer part of x. We have used the properties of the special function $\Gamma_1(x)$ at integer arguments given in Appendix A to rewrite the last factor in the numerator of $\mathcal{B}_2$.

For the factor $\mathcal{B}_3$, we insert the values of the parameters to write it as:

$$\mathcal{B}_3 = \frac{S_1(1+k_1+k_2+\sigma_1-\sigma_3)\, S_1(3+k_1+k_2-\sigma_1-\sigma_3)}{S_1(1-k_2+\sigma_2-\sigma_3)\, S_1(3-k_2-\sigma_2-\sigma_3)}. \tag{4.5}$$

It turns out that this simplifies when combined with a similar factor in $\mathcal{B}_4$.

Finally we must evaluate the contribution $\mathcal{B}_4$. This is carried out in Appendix B, where the contour integral in the last line of Eq.(4.2) is evaluated explicitly. That is then combined with $\mathcal{B}_3$ of Eq.(4.5) above to give the following amazingly simple result for the product:

$$\mathcal{B}_3\, \mathcal{B}_4 = \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \left(\frac{2\mu_{21}}{\sqrt{\mu}}\right)^{-1} \left\{ \left(\frac{2\mu_{23}}{\sqrt{\mu}}\right)^{\sum_i k_i} - \left(\frac{2\mu_{13}}{\sqrt{\mu}}\right)^{\sum_i k_i} \right\}. \tag{4.6}$$

Putting everything together, we arrive at the three-point function (with $\beta_i = 1 - k_i$):

$$\begin{aligned} C^{\mu_2\mu_3\mu_1}_{\beta_1,\beta_2,\beta_3} &= \mathcal{B}_1\, \mathcal{B}_2\, \mathcal{B}_3\, \mathcal{B}_4 \\ &= \frac{(-1)^{\lfloor \Sigma_i k_i/2 \rfloor}}{(2\pi\epsilon)^{1+\Sigma_i k_i}} \frac{\mu_{23}^{\Sigma_i k_i} - \mu_{13}^{\Sigma_i k_i}}{\mu_{21}} \frac{\Gamma_1\left(1+\sum_i k_i\right)}{\Gamma_1(2)} \prod_{j=1}^{3} \frac{\Gamma_1\left(1+\sum_i k_i - 2k_j\right)}{\Gamma_1(2k_j)}. \end{aligned} \tag{4.7}$$

In terms of the renormalised operators defined in Eq.(3.15) and (3.16), the three-point function becomes:

$$\tilde{C}^{\mu_2\mu_3\mu_1}_{\beta_1,\beta_2,\beta_3} = \frac{(-1)^{\lfloor \Sigma_i k_i/2 \rfloor}}{2\pi\epsilon} \frac{\mu_{23}^{\Sigma_i k_i} - \mu_{13}^{\Sigma_i k_i}}{\mu_{21}} \frac{\Gamma_1\left(1+\sum_i k_i\right)}{\Gamma_1(2)} \prod_{j=1}^{3} \frac{\Gamma_1\left(1+\sum_i k_i - 2k_j\right)\Gamma(2k_j)}{\Gamma_1(2k_j)}. \tag{4.8}$$

In the special case of tachyons, the momenta of the three operators obey $k_1+k_2 = k_3$, and the three point function takes the simpler form

$$\tilde{C}^{\mu_2\mu_3\mu_1}_{\beta_1,\beta_2,\beta_1+\beta_2-1} = \frac{(-1)^{\lfloor \Sigma_i k_i/2 \rfloor}}{\epsilon} \frac{\mu_{23}^{2k_1+2k_2} - \mu_{13}^{2k_1+2k_2}}{\mu_{21}}. \tag{4.9}$$

Like the boundary reflection amplitude Eq.(3.17), the boundary three-point function obtained here also turns out to be independent of the bulk cosmological constant, depends only on pairwise differences of boundary cosmological constants, and has a simple pole singularity in ϵ. We conjecture that these three properties also hold for all n-point functions of boundary operators in this theory.

As a check, we consider the three-point function with momenta $k_1 = k$, $k_2 = 1$ and $k_3 = k$ for the three operators. In this case the middle operator has $\beta = 0$ and hence, if we choose $\sigma_1 = \sigma_3$ (which implies $\mu_{1B} = \mu_{3B}$), it reduces to the identity. Now the above correlator should reduce to the two-point function. From Eq.(4.7) we find:

$$C^{\mu_2\mu_1\mu_1}_{1-k,1,1-k} = \frac{(-1)^k}{(2\pi\epsilon)^{2k+2}}\,\frac{2\pi\,\mu_{21}^{2k}}{\Gamma(2k+1)\Gamma(2k)}. \tag{4.10}$$

Comparing with Eq.(3.14), we see that this is related to the (bare) reflection amplitude by:

$$C^{\mu_2\mu_1\mu_1}_{1-k,1,1-k} = \frac{1}{2\pi\epsilon}\,d(1-k|\mu_1,\mu_2). \tag{4.11}$$

If we interpret $\frac{1}{2\pi\epsilon}$ as the $\delta(0)$ factor arising from $\delta(\beta_1-\beta_2)$ in Eq.(3.1), we may conclude that in the special case being considered, the three-point function indeed reduces to the two-point function as expected.

5. Bulk-Boundary Two-Point Function

The bulk-boundary two-point function on the disc involves a boundary operator $V_\beta^{\sigma\sigma}$ and a bulk operator $\mathcal{V}_\alpha$. This was computed in Ref.[40] (see also Refs.[42,47]) and the result is:

$$\left\langle \mathcal{V}_\alpha(z,\overline{z}) V_\beta^{\sigma\sigma}(x)\right\rangle = \frac{A^\sigma_{\alpha\beta}}{|z-\overline{z}|^{2\Delta_\alpha-\Delta_\beta}\,|z-x|^{2\Delta_\beta}}, \tag{5.1}$$

where,

$$\begin{aligned}
A^\sigma_{\alpha\beta} &= \mathcal{C}_1\,\mathcal{C}_2\,\mathcal{C}_3,\\
\mathcal{C}_1 &= 2\pi\left(\pi\mu_0\gamma(b^2)b^{2-2b^2}\right)^{(Q-2\alpha-\beta)/2},\\
\mathcal{C}_2 &= \frac{\Gamma_1^3(Q-\beta)\Gamma_1(2\alpha-\beta)\Gamma_1(2Q-2\alpha-\beta)}{\Gamma_1(Q)\Gamma_1(Q-2\beta)\Gamma_1(\beta)\Gamma_1(2\alpha)\Gamma_1(Q-2\alpha)},\\
\mathcal{C}_3 &= \frac{1}{i}\int_{-i\infty}^{i\infty} dt\; e^{2\pi i(2\sigma-Q)t}\,\frac{S_1\left(t+\frac12\beta+\alpha-\frac12 Q\right)S_1\left(t+\frac12\beta-\alpha+\frac12 Q\right)}{S_1\left(t-\frac12\beta-\alpha+\frac32 Q\right)S_1\left(t-\frac12\beta+\alpha+\frac12 Q\right)}.
\end{aligned} \tag{5.2}$$

We want to evaluate this in the $c = 1$ string theory where, as usual, we need to take the singular limit $b = 1$. Let us also recall that $\beta = 1 - k$ and $\alpha = 1 - \frac{1}{2}k$ — the bulk and boundary windings are related due to the winding number conservation condition from the matter sector. We will assume that $k > 0$. The first factor $\mathcal{C}_1$ can be rewritten using the the by-now familiar renormalized bulk cosmological constant:

$$\mathcal{C}_1 = 2\pi \left(\frac{\mu}{4}\right)^{k-\frac{1}{2}}, \tag{5.3}$$

while the second factor is easily evaluated to be:

$$\mathcal{C}_2 = \frac{(-1)^{1+k}}{2\pi\Gamma(2k)\left(\Gamma(k)\right)^2} (2\pi\epsilon)^{2k-1}. \tag{5.4}$$

Finally, we come to the third factor $\mathcal{C}_3$ which involves an integral similar to the one encountered in the evaluation of the boundary three-point function. Specifically, we have to evaluate

$$\mathcal{C}_3 = \frac{1}{i}\int_{-i\infty}^{i\infty} dt\ \exp\left(4\pi i(\sigma-1)t\right) \frac{S_1\left(t-k+\frac{1}{2}\right)S_1\left(t+\frac{1}{2}\right)}{S_1\left(t+k+\frac{3}{2}\right)S_1\left(t+\frac{3}{2}\right)}. \tag{5.5}$$

For large imaginary values of t, the integrand falls off exponentially. This makes the integral (5.5) convergent. In the kinematic region where k is negative, all the poles of the integrand arising from the numerator are in the left half-plane while those from the denominator are in the right half-plane. For other values of k the integral is defined by analytic continuation described in detail in Appendix B. Once again, the integral is dominated by its singular part, which comes from the collision of the poles from the two half-planes. Denoting $t + \frac{1}{2} = n$, the conditions for collision are met for integer values of n between $1 - k$ and k. Evaluating the (singular) residues at these poles we find

$$\mathcal{C}_3 = \frac{1}{(2\pi\epsilon)^{2k+1}} e^{-2\pi i\sigma} \sum_{n=1-k}^{k} e^{4\pi i\sigma n} = \frac{1}{(2\pi\epsilon)^{2k+1}} \frac{\sin(4\pi\sigma k)}{\sin 2\pi\sigma}. \tag{5.6}$$

Combining Eqs.(5.3), (5.4) and (5.6), the bulk-boundary two point function of tachyons is found to be:

$$\begin{aligned} A^\sigma_{\alpha\beta} &= \left(\frac{\sqrt{\mu}}{2}\right)^{2k-1} \frac{(-1)^{k-1}}{\epsilon^2\,\Gamma(2k)\,(\Gamma(k))^2}\,\frac{\sin(4\pi\sigma k)}{\sin(2\pi\sigma)} \\ &= \left(\frac{\sqrt{\mu}}{2}\right)^{2k-1} \frac{(-1)^{k-1}}{\epsilon^2\,\Gamma(2k)\,(\Gamma(k))^2}\,\sum_{\ell=1}^{k}(-1)^{\ell+1}\binom{2k-\ell}{\ell-1}\left(\frac{2\mu_B}{\sqrt{\mu}}\right)^{2k-2\ell+1}. \end{aligned} \tag{5.7}$$

Like the previous correlators, this too is conveniently expressed in terms of renormalised bulk and boundary operators, the latter being given by Eqn.(3.15) and for the former we choose:

$$\tilde{Z}(k;m,\overline{m}) = (2\pi\epsilon)^{-k}\frac{\Gamma(k)}{\Gamma(1-k)}\,Z(k;m,\overline{m}). \tag{5.8}$$

Once again, this redefinition differs from the standard one and is chosen so as simplify the form of the renormalized expression. Specialising to the tachyons (2.3),(2.4) for simplicity,

$$\left\langle \tilde{T}^{\sigma\sigma}(k)\tilde{\mathcal{T}}(k)\right\rangle = \left(\frac{\sqrt{\mu}}{2}\right)^{2k-1}\frac{(-1)^{k-1}}{\epsilon^2}\frac{\sin(\pi k)\sin(4\pi\sigma k)}{\pi\sin(2\pi\sigma)}. \tag{5.9}$$

Unlike the boundary two- and three-point functions, we see that the bulk-boundary correlator does depend explicitly on the bulk cosmological constant μ, through σ. It also lacks the translational symmetry in μ_B that we found in the boundary correlators. (This was to be expected, since there is only one boundary operator $V^{\sigma\sigma}_\beta$ and this is necessarily diagonal in the boundary cosmological constant. However, we suspect that with more boundary operators too, the bulk-boundary correlators will lack translational symmetry in the μ_B.) Finally, we see that this correlator has a double pole singularity in ϵ, unlike the simple pole found in the boundary correlators.

Specialising further to $k=0$ (the cosmological operators) we find:

$$\left\langle \tilde{T}^{\sigma\sigma}(0)\tilde{\mathcal{T}}(0)\right\rangle = \left\langle \tilde{T}^{\sigma\sigma}(k)\tilde{\mathcal{T}}(k)\right\rangle\Big|_{k\to 0} = -\frac{2}{\sqrt{\mu}}\,\frac{4\pi\sigma}{\sin 2\pi\sigma}. \tag{5.10}$$

Interestingly, in this case the correlator is non-singular.

As a consistency check, the bulk-boundary two-point function, if correctly normalised, should reduce to the bulk one point function when the boundary Liouville momentum vanishes, $\beta \to 0$. This corresponds to $k = 1$ in our case. The bulk one-point function of Liouville theory is given by[7]:

$$\langle \mathcal{V}_\alpha(z, \overline{z}) \rangle_\sigma = \frac{U_\sigma(\alpha)}{|z - \overline{z}|^{2\Delta_\alpha}}, \tag{5.11}$$

where,

$$\begin{aligned} U_\sigma(\alpha) = \frac{2}{b} \left(\pi\mu\gamma(b^2)\right)^{\frac{Q-2\alpha}{2b}} \Gamma(2b\alpha - b^2)\, \Gamma\left(\frac{2\alpha}{b} - \frac{1}{b^2} - 1\right) \\ \times \cos\left(\pi(2\alpha - Q)(2\sigma - Q)\right). \end{aligned} \tag{5.12}$$

Due to the momentum conservation condition from the matter sector, this should strictly be evaluated only at $k = 0$. Nevertheless, let us keep k arbitrary at this stage. Putting $b = 1$ in (5.12) and performing the familiar renormalization of cosmological constants, as well as renormalization of the bulk tachyon as in Eq.(5.8), the one-point function is:

$$\tilde{U}_\sigma(k) = -\left(\frac{\sqrt{\mu}}{2}\right)^k (2\pi\epsilon)^{-k} \frac{2\pi\cos(2\pi\sigma k)}{k\sin(\pi k)}. \tag{5.13}$$

The bulk-one point function above is seen to satisfy the expected functional equation $\tilde{U}_{\sigma+\frac{1}{2}}(k) + \tilde{U}_{\sigma-\frac{1}{2}}(k) = 2\cos(\pi k)\, \tilde{U}_\sigma(k)$ rather trivially[42]. Substituting $k = 1$ (hence $\alpha = 1 - \frac{k}{2} = \frac{1}{2}$) formally,

$$\tilde{U}_\sigma(k = 1) = -\frac{\sqrt{\mu}}{2} \frac{\cos 2\pi\sigma}{\pi\epsilon^2}, \tag{5.14}$$

On the other hand, Eq.(5.9) evaluated at $k = 1$ gives:

$$\left\langle \tilde{T}^{\sigma\sigma}(1,1)\tilde{\mathcal{T}}(1) \right\rangle = \frac{\sqrt{\mu}}{2} \frac{2\cos 2\pi\sigma}{\epsilon}. \tag{5.15}$$

Recalling from Eq.(3.15) that $\tilde{T}(1) = 2\pi\epsilon\, T(1)$ we see that

$$\left\langle T^{\sigma\sigma}(1)\tilde{\mathcal{T}}(1) \right\rangle = \frac{\sqrt{\mu}}{2} \frac{\cos 2\pi\sigma}{\pi\epsilon^2}. \tag{5.16}$$

This agrees with Eq.(5.14) (upto a sign).

6. Physical Correlators and Open String Field Theory

According to a recent proposal of Sen[57–59], open-string field theory on D-branes in a certain background is dual to a theory of closed strings to which the branes in that background couple. In most known examples, the complete string field theory is extremely complicated, and lacking the necessary analytic tools, is only accessible through approximation schemes such as level truncation. Having examples of D-branes on which the full open-string field theory can be analysed is clearly important. Non-critical string theories, with their relatively simple yet rich physical content and high degree of symmetry, are string backgrounds where we may further our understanding of this duality[58,59]. Indeed in Ref.[55], the topological Kontsevich matrix model of topological gravity (equivalent to $c = -2$ closed-string theory) is shown to arise from the open-string field theory of the branes of $c = -2$ matter coupled to Liouville theory.

In this section, we take the first step in this direction for the case of the N FZZT branes in the $c = 1$ theory compactified at $R = 1$. Open-string field theory has an infinite number of fields, but it also has infinite gauge redundancy. The closed-string sector of the $c = 1$ theory at the self-dual radius, and indeed of all non-critical string theories, possesses a topological symmetry due to which only degenerate worldsheets at the boundary of the moduli space of Riemann surfaces contribute to correlators. In other words, only physical ('on-shell') states in the cohomology of the BRS operator Q_B contribute to quantum string amplitudes, at all genus. This is the well-known topological localisation.

When D-branes, *i.e.* open strings, are included, we lack a direct proof that this property continues to hold. However, an important source of intuition comes from the relation of the bulk theory to the topological SL(2)/U(1) coset[67] and the deformed conifold[81]. There has been progress in understanding localisation in the open-string sector of the topological SU(2)/U(1) cosets[82], closely related to the first description. On the other hand, a classic result due to Witten[83] tells us that the open-string field theory on D3-branes wrapping a 3-cycle of the deformed conifold localises to pure Chern-Simons theory (this considerably preceded the discovery of D-branes!). With this motivation, for the moment we simply assume that the string field theory localises onto the physical states (as defined by the BRS cohomology) and arrive at an action for these modes, postponing a detailed analysis of localisation and the resulting model for future work[24].

Let the CFT Hilbert space of the states of the first-quantized string between the ith and the jth brane be $\mathcal{H}_{ij}$ $(i, j = 1, \cdots, N)$. The open-string field $|\Psi_{(ij)}\rangle$ is a ghost-number one state in this Hilbert space. The action defining the classical string field theory is

$$S[\Psi] = -\frac{1}{2g_{s0}} \sum_{ij} \left\langle \Psi_{(ij)}, Q_B \Psi_{(ji)} \right\rangle - \frac{1}{3g_{s0}} \sum_{ijk} \left\langle \Psi_{(ij)} \Psi_{(jk)} \Psi_{(ki)} \right\rangle , \tag{6.1}$$

where the quadratic and the cubic terms are given in terms of CFT correlators and Q_B is the BRST operator (see [84] for a review). The linearised equation of motion of the theory $Q_B|\Psi\rangle = 0$ is the statement that the worldsheet configuration is physical in the free theory — the cubic term then describes interactions.

Our task now is to compute the OSFT action (6.1) on the FZZT branes localized onto the physical states. This amounts to the evaluation of correlators in the CFT of $c = 1$ matter plus Liouville plus the (b, c) ghosts. In Liouville theory, there is no sense in which the string coupling is weak, therefore we cannot really regard the cubic term as a perturbation. This is reflected in the fact that there is an infrared divergence in the two-point correlators of the physical states. We shall use the same regulator that we did earlier and see that the kinetic and the cubic term, evaluated on the on-shell states, contribute to the same order.

We have seen earlier that the physical states of the background CFT are summarized as an $N \times N$ matrix field living on a 2-sphere. The expansion of the

[24] In [55], a powerful nilpotent symmetry of the gauge fixed quantum action of the $c = -2$ noncritical string theory is exploited for localisation. The existence of such a symmetry is stronger in that it takes into account the effect of worldsheet instantons. We note, however, that the absence of compact two-cycles in the deformed conifold geometry will forbid potential instanton correction.

open-string field in terms of these states is:

$$\left|\Psi_{(ij)}\right\rangle = \sum_{k,m} T_{ij}(k,m)\left|\tilde{Y}^{ij}(k,m)\right\rangle, \tag{6.2}$$

where $|\tilde{Y}^{ij}(k,m)\rangle$ is the ghost number one primary state in the boundary CFT corresponding to the open string with ends on branes (i,j) transforming as spherical harmonics $\mathcal{Y}_k^m(\theta,\phi)$ under rotations of S^2. Under the assumptions described above, the open-string field theory action reduces to an action for these matrices.

The coefficients of the string field in the OSFT action are determined by the CFT correlation functions for primary operators. The matter and ghost contribution to the two- and three-point correlators are very simple. As we saw in section 2, the physical operators behave exactly like the spherical harmonics. For the two-point function, the matter contribution is the condition $k_1 = k_2 \equiv k$ and the conservation condition $m_1 = -m_2 \equiv m$ of the J_3 component of angular momentum. The full two-point function is:

$$D_{ij}(k) \equiv \left\langle \tilde{Y}^{ij}(k,m), c_0\tilde{Y}^{ji}(k,-m)\right\rangle = \frac{(-1)^k}{\epsilon^2}\frac{(z_j - z_i)^{2k}}{2k}, \tag{6.3}$$

where, we have introduced the notation $\mu_B \equiv z$. Let us note that for the special case of the cosmological operators, both the terms in Eq. (3.1) contribute.

The full three-point function is determined from the SU(2) addition condition from the matter sector and, using Eq.(4.8), its expression is:

$$\begin{aligned} C_{jki}(k_1,k_2,k_3) &\equiv \left\langle \tilde{Y}^{jk}(k_1,m_1)\tilde{Y}^{ki}(k_2,m_2)\tilde{Y}^{ij}(k_3,m_3)\right\rangle \\ &= \frac{(-1)^{\Sigma_i k_i/2}}{(2\pi\epsilon)}\left(\frac{z_{jk}^{\Sigma_i k_i} - z_{ik}^{\Sigma_i k_i}}{z_{ji}}\right)\frac{\Gamma_1(1+\Sigma_i k_i)}{\Gamma_1(2)}\prod_{j=1}^{3}\frac{\Gamma_1(1+\Sigma_i k_i - 2k_j)\Gamma(2k_j)}{\Gamma_1(2k_j)} \\ &\quad \times \int d(\cos\theta)d\phi\, \mathcal{Y}_{k_1}^{m_1}(\theta,\phi)\mathcal{Y}_{k_2}^{m_2}(\theta,\phi)\mathcal{Y}_{k_3}^{m_3}(\theta,\phi), \end{aligned} \tag{6.4}$$

where $\mathcal{Y}_m^k(\theta,\phi)$ are the spherical harmonics of S^2. Let us recall that the three-point function is evaluated with the condition $k_3 > k_1, k_2$, a choice made in evaluating the Liouville correlators in Sec. 4. We would also like to point out that the extra divergence present in the two-point function (6.3) compared to the three-point function (6.4) is due to the delta function in (3.1), which can be understood as an infra-red divergence arising from the volume of the target space.

It is now straightforward to evaluate the action (6.1) localised on the physical states. The kinetic operator Q_B simplifies to $c_0 L_0$ in the Siegel gauge. This has a zero acting on the physical states $|\tilde{Y}^{ij}(k,m)\rangle$, (which are dimension zero primaries of the underlying CFT):

$$\begin{aligned}\left\langle \tilde{Y}^{ij}(k',m'), Q_B \tilde{Y}^{ji}(k,m)\right\rangle &= \left\langle \tilde{Y}^{ij}(k',m')\right| c_0 L_0 \left|\tilde{Y}^{ji}(k,m)\right\rangle \\ &= \left\langle \tilde{Y}^{ij}(k',m')\right| c_0 \left(k^2 + \beta_2(2-\beta_2) - 1\right) \left|\tilde{Y}^{ji}(k,m)\right\rangle \\ &= 2k\epsilon \left\langle \tilde{Y}_{ij}(k',m')\right| c_0 \left|\tilde{Y}^{ji}(k,m)\right\rangle \\ &= 2k\epsilon\, D_{ij}(k)\, \delta_{k,k'} \delta_{m+m',0}.\end{aligned} \tag{6.5}$$

This zero absorbs the volume divergence in the two point function. Using (6.3) in (6.5), we get the coefficient of the kinetic term

$$\left\langle \tilde{Y}^{ij}(k,m), Q_B \tilde{Y}^{ji}(k,-m)\right\rangle = \frac{(-1)^k}{\epsilon} \left(z_j - z_i\right)^{2k}, \tag{6.6}$$

which has a simple pole in ϵ. The coefficient of the cubic term is simply the three-point function (6.4) with the same singularity. Thus, the two terms in the action (6.1) have an identical singular coefficient. Then we can renormalise the string coupling as

$$g_s \equiv \epsilon g_{s0}, \tag{6.7}$$

to get a sensible matrix theory with a finite action. The novelty of this matrix model, compared to the existing ones in the literature, is that it has the $SU(2)$ symmetry of the theory manifest from the beginning.

Let us note that the (singular) renormalisation (6.7) of the string coupling was also necessary in Ref.[55] in order to get the Kontsevich model. It is actually implicit in [55], where the $1/\epsilon$ singularity of the three-point function as well as the delta-functions in the two-point functions have been suppressed[85].

The complete action involving all the modes in (6.2) is a little cumbersome. However, if we restrict to the tachyons (thereby giving up $SU(2)$ symmetry), we find the action $\mathcal{S} = \mathcal{S}_2 + \mathcal{S}_3$, where

$$\mathcal{S}_2 = -\frac{1}{2g_s} \sum_{k=0}^{\infty} (-1)^k \sum_{ij} \tilde{T}^{ij}(k) \left(z_j - z_i\right)^{2k} \tilde{T}^{ji}(-k)$$
$$\mathcal{S}_3 = -\frac{1}{3g_s} \sum_{k_1,k_2} (-1)^{k_1+k_2} \sum_{ijl} \frac{z_{jl}^{2k_2+2k_2} - z_{il}^{2k_2+2k_2}}{z_{ji}} \tilde{T}^{jl}(k_1)\tilde{T}^{li}(k_2)\tilde{T}^{ij}(-k_1-k_2). \tag{6.8}$$

In terms of a matrix $Z = \mathrm{diag}\,(iz_1, iz_2, \cdots)$, the kinetic term may be written as

$$\mathcal{S}_2 \sim \sum_k \mathrm{Tr}\, \tilde{T}(k) \left[Z, \cdots \left[Z, \tilde{T}(k)\right] \cdots\right]. \tag{6.9}$$

The fact that our Liouville correlators depend only on the difference of boundary cosmological constants shows up as a symmetry of the above term under a shift of the matrix $\tilde{T}$ by an arbitrary diagonal matrix. This symmetry is shared by the cubic term which can also be written down similarly.

7. Discussion

We have studied correlators of the boundary Liouville theory in the limit that the Liouville central charge c_L tends to 25, or equivalently $c \to 1$. The results are embodied in Eqs.(3.14)–(3.17), (4.8)–(4.9) and (5.9). The principal motivation to present these results is that they are far more explicit than the boundary correlators known for the $c < 1$ theory (as embodied in Eqs.(3.2),(4.2) and (5.2)). The latter are given in terms of special functions $S_b(x), \Gamma_b(x)$ and some of the correlators are known only as contour integrals over products of such functions. These contour integrals can be explicitly evaluated for $c = 1$ only, as far as we know, at the self-dual radius[25]. The boundary correlators we obtain in this way are all divergent, but as we have noted, the divergence factors out from the two- and three-point functions and can be absorbed in a rescaling of the string coupling leading to a well-defined open-string field theory action.

[25] Of course, rational multiples of this radius which correspond to orbifolds of the theory also have a similar behaviour.

The fact that the boundary correlators are independent of the bulk cosmological constant is reminiscent of a similar fact in Ref.[55]. There, the dependence of the two-point function on μ_B is crucial in recovering the Kontsevich model[61], where the different $\mu_{B,i}$ turn into the eigenvalues of the Kontsevich matrix. In similar vein, our matrix model depends only on $\mu_{B,i}$ which are the eigenvalues of a constant matrix Z.

We did not find a proof that the boundary correlators at $c = 1$ and selfdual radius are all independent of the bulk cosmological constant. However, if we assume this to be true, then we can see that the n-point tree-level boundary correlators must scale with the boundary cosmological constant μ_B as:

$$\langle V(k_1)V(k_2)\cdots V(k_n)\rangle \sim \mu_B^{\sum_{i=1}^n k_i - n+2} \tag{7.1}$$

where a factor of $(k_i - 1)$ comes from each Liouville vertex operator and an additional 2 comes from the linear dilaton factor in the path integral. This scaling is satisfied by the two- and three-point correlators that we computed. It is tempting to also conjecture that the n-point correlators will depend only on the pairwise differences μ_{ij} of boundary cosmological constants.

The natural matrix model that we might have expected to find from our computations, which is the analogue of the Kontsevich model for $c = 1$ at self-dual radius, is the model of Ref.[72]. But this is a one-matrix model, and here we find a model with infinitely many matrices. Moreover the model of [72] incorporates amplitudes for (closed-string) tachyon external states only, based as it is on the amplitudes computed in Ref.[71] from matrix quantum mechanics, in which the other discrete states have not yet been constructed. So there is in fact no candidate matrix model presently available that incorporates the full $SU(2)$ symmetry of the $c = 1$ string at self-dual radius. In contrast, the approach in the present chapter does lead to such a model, presented in embryonic form in Eqs.(6.4)–(6.6) More work is needed to understand this model and confirm whether open/closed duality works as expected.

Appendix A. Special Functions at $c = 1$

The correlators of Liouville theory are expressed in terms of some special functions[7,47]. In the case of $c_L = 25$, *i.e.*, $b = 1$, they are:

$$\begin{aligned}\ln \Gamma_1(x) &= \int_0^\infty \frac{dt}{t}\left(\frac{e^{-xt}-e^{-t}}{(1-e^{-t})^2} - \frac{(1-x)^2}{2e^t} - \frac{2(1-x)}{t}\right),\\ \ln S_1(x) &= \int_0^\infty \frac{dt}{t}\left(\frac{\sinh 2t(1-x)}{2\sinh^2 t} - \frac{1-x}{t}\right).\end{aligned} \tag{A.1}$$

Both are meromorphic functions and are related to each other via:

$$S_1(x) = \frac{1}{S_1(2-x)} = \frac{\Gamma_1(x)}{\Gamma_1(2-x)}, \tag{A.2}$$

where, we have also made use of the unitarity relation $S_1(x)S_1(2-x) = 1$. The function Γ_1 has poles at zero and negative integer arguments. Therefore, from Eq.(A.2), $S_1(x)$ has poles at these arguments and zeroes at integers larger than 1.

The functions $\Gamma_1(x)$ and $S_1(x)$ satisfy the recursion relations:

$$\begin{aligned}\Gamma_1(x+1) &= \frac{\sqrt{2\pi}}{\Gamma(x)}\,\Gamma_1(x),\\ S_1(x+1) &= 2\sin(\pi x)\,S_1(x);\end{aligned} \tag{A.3}$$

where, $\Gamma(x)$ is the usual Euler gamma function. The values of these special functions at (half)-integer arguments turn out to be of interest. In particular, we would need the ratio $\Gamma_1(-n)/\Gamma_1(n)$, which, as a matter of fact, is divergent. However, using the recursion relations above, one can show that the leading divergence, near an integer n is

$$\begin{aligned}\frac{\Gamma_1(-n)}{\Gamma_1(n)} &\equiv \lim_{\epsilon\to 0+}\frac{\Gamma_1(-n-\epsilon)}{\Gamma_1(n+\epsilon)}\\ &= \frac{(-1)^{n(n+1)/2}}{(2\pi)^n\Gamma(n)\Gamma(n+1)}\,\frac{1}{\epsilon^{n+1}}.\end{aligned} \tag{A.4}$$

However, for half-integer arguments:

$$\frac{\Gamma_1\left(-\frac{2m+1}{2}\right)}{\Gamma_1\left(\frac{2m+1}{2}\right)} = \frac{(-1)^{(m+1)(m+2)/2}\sqrt{2}}{\pi^{m+\frac{3}{2}}\,(2m-1)!!\,(2m+1)!!}, \qquad m \in \mathbf{Z}, \tag{A.5}$$

the corresponding ratio is finite.

Likewise, using the relations above, $S_1(1-x) = \frac{\Gamma(x)\Gamma_1(-x)}{\Gamma(-x)\Gamma_1(x)}$. Therefore, one finds that

$$\begin{aligned} S_1(1-n) &= \frac{(-1)^{n(n-1)/2}}{(2\pi\epsilon)^n}, \\ S_1\left(1-\frac{2n+1}{2}\right) &= \frac{(-1)^{-n(n+1)/2}}{2^{2n+\frac{1}{2}}\pi^{n+\frac{3}{2}}}, \end{aligned} \tag{A.6}$$

for an integer n. Once again, the first of the above is to be defined as a limit.

Appendix B. Evaluation of a Contour Integral for the Three-Point Function

Here we will evaluate the contour integral

$$\mathcal{B}_4 = \frac{1}{i}\int_{-i\infty-0}^{+i\infty-0} ds \ \prod_{i=1}^{4} \frac{S_b(U_i+s)}{S_b(V_i+s)} \tag{B.1}$$

where U_i, V_i are given in Eq.(4.3). The definition of the contour integral as an analytic function of the momenta is explained in Ref. [47]. We shall summarize and use that prescription for our case in which k_i approach positive integers, and $b \to 1$. We shall use an off-shell parameter ϵ here which is the deformation b away from $b=1$. As mentioned in section 3, an equivalent deformation is one where the Liouville momenta is shifted away from integers.

For large imaginary $|s|$, the integrand decays exponentially, so the integral is convergent in that region. Near the origin, the contour needs to be defined because the integrand has poles which lie on the origin. We do this by shifting the contour a little to the left of the imaginary axis.

Let us list the arguments of the functions S_1 for our case:

$$\begin{aligned} U_1 &= -1+\sigma_1+\sigma_2+k_1, \qquad & V_1 &= 2+\sigma_2-\sigma_3+k_1+k_3, \\ U_2 &= 1-\sigma_1+\sigma_2+k_1, \qquad & V_2 &= 2+\sigma_2-\sigma_3+k_1-k_3, \\ U_3 &= 1+\sigma_2-\sigma_3-k_2, \qquad & V_3 &= 2\sigma_2, \\ U_4 &= 1+\sigma_2-\sigma_3+k_2, \qquad & V_4 &= 2. \end{aligned} \tag{B.2}$$

The poles from the numerator and the denominator are at[26]

$$s + U_i = -n_i \quad \text{and} \quad s + V_i = 2 + m_i, \qquad (n_i, m_i = 0, 1, 2, \cdots) \tag{B.3}$$

respectively. For Re $U_i > 0$ and Re $V_i \leq 2$, the poles arising from the numerator are all in the left half-plane and those from the denominator are in the right half-plane[27]. The imaginary axis is therefore a well-defined contour and thanks to the asymptotic behaviour, the integral has a finite value.

For general values of k_i, the integral is defined by analytic continuation of the above prescription. Specifically, this means that as we vary k_i (or equivalently, U_i, V_i) smoothly, some of the poles from the LHP cross the imaginary axis and enter the RHP, and vice-versa. In such a case, one deforms the contour such that the poles from the numerator and the denominator are always separated by the contour. Alternatively, this could be done by an equivalent deformation as follows. Suppose, a pole of the numerator migrates to the LHP. The new (deformed) contour now consists of two parts, one is the old one and another a small circle around the 'migrating' pole. The latter will pick up the residue of the integrand around that pole. However, this also gives a finite contribution and will not be of our final interest.

The integral diverges if two poles, one originating in numerator and another in denominator, approach towards each other to coincide. In this case, the contour is 'pinched between' the two poles. Alternatively, the migrating pole hits another pole. This divergence dominates over the finite piece and it is this which is of interest to us. In order to extract the leading divergence in such cases, let us deform b away from the value $b = 1$ by an amount ϵ and make the circle around a migrating pole very small. As it hits a would-be singularity at $b = 1$, we determine the divergent residue as a power of ϵ.

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[26] Here we have already plugged in $b = 1$, the general formula has simple poles at $s + U_i = -nb - mb^{-1}$. At $b = 1$, these simple poles coalesce to a pole of high order.

[27] The V_4 factor has a pole at the origin, but we have shifted the contour a little to the left as indicated in (4.2). With this understanding, we shall continue to call it the imaginary axis.

The condition for collision between the poles (B.3) is $s = -U_i - n_i = 2 - V_j + m_j$, $(n_i, m_j = 0, 1, 2, \cdots)$, *i.e.*,

$$V_j - U_i = 2 + m, \; m = 0, 1, 2, \cdots. \tag{B.4}$$

For generic σ_i, this can only happen when V_1 collides with U_3 or U_4. Moreover, $V_1 - U_3 = 1 + k_1 + k_2 + k_3 = V_1 - U_4 + 2k_2$, so the divergence from the collision of V_1 and U_3 dominates and it is sufficient to consider only that. Let

$$s + (\sigma_2 - \sigma_3) = n \in \mathbb{Z}. \tag{B.5}$$

Then, a collision between the poles (B.3) happens when

$$1 + n - k_2 = -n_3, \qquad n + k_1 + k_3 = m_1, \qquad (n_3, m_1 = 0, 1, 2, \cdots). \tag{B.6}$$

This happens when $-k_1 - k_3 \leq n \leq k_2 - 1$. This set is non-empty for $(k_1, k_2) \neq (0, 0)$.

The divergence of the integrand for a particular value of n, as defined in Eq.(B.5) above, contributes an amount to the integral $\mathcal{B}_4$ that we denote $\mathcal{B}_4^{(n)}$. Hence,

$$\mathcal{B}_4 = \sum_n \mathcal{B}_4^{(n)} \tag{B.7}$$

The range of values of n over which the sum is to be performed will be determined below.

The net order of divergence of the integrand at a given value of n comes from counting the poles/zeroes in U_3, U_4, V_1 and V_2 (keeping σ_i are generic), and (using Eq.(B.5) and the formula for the divergence of the S_1-function given in Appendix A) is equal to:

$$-(n - k_2) - (n + k_2) + (1 + n + k_1 + k_3) + (1 + n + k_1 - k_3) = 2 + 2k_1. \tag{B.8}$$

One of these poles is the migrant one with a circular contour around it, so the divergent part of the residue[28] is $1/(2\pi\epsilon)^{2k_1+1}$.

[28] Let us see how the same result is obtained with the equivalent regulator in which $b = 1$ but k is shifted away from an integer. The contour integral is about a pole of higher order, say $M \equiv n + k_1 + k_3$, if the migrant pole is from V_1. The residue is then the $(M - 1)$th derivative of the other factor which has a pole of order $2k_1 + 2 - M$. The dominant singularity comes from differentiating this singular part, leading to the same final answer.

The finite piece of the residue is due to the other four S_1-functions. Once again, using (B.5), we can write the contribution $\mathcal{B}_4^{(n)}$ as:

$$\mathcal{B}_4^{(n)} = \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \frac{S_1(-1+n+k_1+\sigma_1+\sigma_3)\, S_1(1+n+k_1-\sigma_1+\sigma_3)}{S_1(n+\sigma_2+\sigma_3)\, S_1(2+n-\sigma_2+\sigma_3)}. \tag{B.9}$$

Combining $\mathcal{B}_3$ and $\mathcal{B}_4^{(n)}$ and using some inversions of S_1 along the way,

$$\begin{aligned}
\mathcal{B}_3\, \mathcal{B}_4^{(n)} =& \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \frac{S_1(-1+n+k_1+\sigma_1+\sigma_3)}{S_1(-1-k_3+\sigma_1+\sigma_3)} \frac{S_1(-1+k_2+\sigma_2+\sigma_3)}{S_1(n+\sigma_2+\sigma_3)} \\
&\times \frac{S_1(1+k_3+\sigma_1-\sigma_3)}{S_1(1-n-k_1+\sigma_1-\sigma_3)} \frac{S_1(-n+\sigma_2-\sigma_3)}{S_1(1-k_2+\sigma_2-\sigma_3)} \\
=& \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \left(2\sin\pi(\sigma_1+\sigma_3)\right)^{k_1+k_3+n} \left(2\sin\pi(\sigma_2+\sigma_3)\right)^{k_2-n-1} \\
&\times \left(2\sin\pi(\sigma_1-\sigma_3)\right)^{k_1+k_3+n} \left(2\sin\pi(\sigma_2-\sigma_3)\right)^{k_2-n-1} \\
=& \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \left(2\,\frac{\mu_{1B}-\mu_{3B}}{\sqrt{\mu}}\right)^{k_1+k_3+n} \left(2\,\frac{\mu_{2B}-\mu_{3B}}{\sqrt{\mu}}\right)^{k_2-n-1}.
\end{aligned} \tag{B.10}$$

Finally we have to sum over all these residues, since the contour is a disjoint sum of all these circles at various values of s labelled by an integer n, which ranges from $-k_1-k_3$ to k_2-1. This is a geometric series. Evaluating the sum, we get:

$$\begin{aligned}
\mathcal{B}_3\, \mathcal{B}_4 &= \mathcal{B}_3 \sum_{n=-k_1-k_3}^{k_2-1} \mathcal{B}_4^{(n)} \\
&= \frac{(-1)^{k_1}}{(2\pi\epsilon)^{2k_1+1}} \left(\frac{2\mu_{21}}{\sqrt{\mu}}\right)^{-1} \left\{ \left(\frac{2\mu_{23}}{\sqrt{\mu}}\right)^{\sum_i k_i} - \left(\frac{2\mu_{13}}{\sqrt{\mu}}\right)^{\sum_i k_i} \right\},
\end{aligned} \tag{B.11}$$

where we have defined:

$$\mu_{ij} \equiv \mu_{iB} - \mu_{jB}. \tag{B.12}$$

Chapter 4

Unstable D-branes in Two-dimensions and the Supersymmetric Matrix Model

1. Introduction

Matrix models provide an elegant and powerful formalism for describing low-dimensional string theories. Recently, it was proposed that the large N matrix variables can be viewed as the modes of N unstable D-particles in the corresponding string theory, in a decoupling limit [86,44]. This proposal reinterprets the matrix-model/string-theory correspondence as a holographic open-string/closed-string duality, and suggests a search algorithm for more examples. It has been clarified [87,88,43,89] and very recently extended to type 0 strings [90,91]. In this note, we apply this perspective to shed some new light on the physical identification of the supersymmetric matrix model of Marinari and Parisi [92].

This chapter is organized as follows. We begin by recollecting the basic features of the Marinari-Parisi model and its proposed continuum limit. In section 3, we review some of the target space properties of 2-d superstring theory. In section 4 we collect a list of correspondences between the two theories. Most notably, we find that the open string spectrum on unstable D-particles of the 2-d string theory is that of (a minor improvement of) the MP model, expanded around the maximum of its potential. We also make a direct comparison between the vacuum structure and instantons of both models. We end with some concluding remarks and open problems. Some technical discussions are sequestered to appendices A and B.

2. The Marinari-Parisi model

The Marinari-Parisi model is the quantum mechanics of an $N \times N$ hermitian matrix in a one-dimensional superspace

$$\Phi(\tau,\theta,\overline{\theta}) = M(\tau) + \overline{\theta}\Psi(\tau) + \overline{\Psi}(\tau)\theta + \theta\overline{\theta}F(\tau). \tag{2.1}$$

The action is

$$S = -N\int d\tau\, d\overline{\theta}\, d\theta\, \mathrm{Tr}\left\{\frac{1}{2}\overline{D}\Phi D\Phi + W_0(\Phi)\right\}, \tag{2.2}$$

where $D, \overline{D}$ are superspace derivatives. We can choose a cubic superpotential

$$W_0(\Phi) = \frac{1}{2}\Phi^2 - \frac{1}{3}\lambda^2\Phi^3. \tag{2.3}$$

The Feynman graph expansion for the model generates a discretization of random surfaces in superspace. Related work on supersymmetric matrix models includes [93,94,95,96,97,98,99,100,101,102,103].

The model we will discuss is actually a slight modification of the original MP model. We will take the derivatives appearing in (2.2) to be covariant with respect to gauge transformations which are local in superspace; their form is described in appendix A. In the case of the $c = 1$ matrix model, its identification with the worldline theory of D-particles made clear that the $U(N)$ conjugation symmetry of the matrix model should be gauged. As we will see, the same correspondence in our case suggests that we should introduce a superfield gauge symmetry in the model (2.2), which naturally effects the truncation to singlet states [95].

The model with superpotential (2.3) has two classical supersymmetric extrema $W_0'(\Phi) = 0$. These are minima of the bosonic potential $V(M) = M^2(1-\lambda^2 M)^2$. In addition, V has an unstable critical point at $M_c = \frac{1}{2\lambda^2}$. The quadratic form of the action, when expanded near this non-supersymmetric critical point is (defining $Y = M - M_c$)

$$S = -N\int d\tau \mathrm{Tr}\left\{\frac{1}{2}(D_\tau Y)^2 + \overline{\Psi}D_\tau\Psi + \frac{1}{2}\lambda^2 Y^2 - \frac{1}{16\lambda^2}\right\}. \tag{2.4}$$

In the following we will argue that this action can be viewed as that of N unstable D-particles, localized in the strong coupling/curvature region of the 2d string theory background.

In the MP model, the fermi level is not an independent parameter, in that it is determined by the form of the potential [95]. Critical behavior arises instead in this model through a singularity in the norm of the ground-state wavefunction [95]. We will discuss the ground states of the matrix model further in §4, but for now it suffices to study an exemplary one, $|f_0\rangle$, whose norm is given by

$$e^{\mathcal{F}} \equiv |f_0|^2 = \int \prod_i dz_i \prod_{i<j} (z_i - z_j)^2 \; e^{-2W_0(z)}; \tag{2.5}$$

this is a $c < 1$ matrix integral. For odd W_0, there is an irrelevant divergence at large $|z|$ which we simply cut off. A critical limit arises by tuning the potential W_0 to the $m = 2$ pure-gravity critical point of [104], near which the tree-level free energy is $\mathcal{F} \propto \kappa^{-2}$, with $\kappa^{-1} = (\lambda - \lambda_c)^{5/4} N$ providing the string coupling.

This limit naively gives a supersymmetric sigma model on 1d superspace $(\tau, \theta, \overline{\theta})$ coupled to 2d Liouville supergravity. In [105], however, the following argument was presented against such a description of the continuum limit: the matter part of the action is necessarily interacting, and has a one-loop beta function predicting that the coupling grows in the IR and that the matter fields become disordered. This would seem to indicate that the superspace coordinates $(\tau, \theta, \overline{\theta})$ become massive, and that spacetime does not survive in the critical theory. We consider this conclusion premature. The reasoning assumes that the matter theory and the worldsheet gravity are coupled only via the gauge constraints. This is not the case for the supersymmetric string in two dimensions [106].

3. Two-dimensional Superstrings

We saw in Chapter 2 how to construct 2d superstring theory. We review this briefly and move on to discuss some non-perturbative aspects of this theory.

To formulate 2d superstring theory, one starts from $\mathcal{N} = 2$ Liouville theory [107] and then performs a consistent GSO projection to obtain a string theory with target space supersymmetry [106,108]. Unlike bosonic and $\mathcal{N} = 1$ supersymmetric Liouville theory, the time direction τ is involved in the $\mathcal{N} = 2$ supersymmetry algebra, and it is involved in the $\mathcal{N}=2$ Liouville interaction as well:

$$\mathcal{L}_{int}^{SL} = \psi\widetilde{\psi}\, e^{-\frac{1}{2}(\rho+\widetilde{\rho}+i(\tau-\widetilde{\tau}))} + c.c. \tag{3.1}$$

with $\psi = \psi_\rho + i\psi_\tau$.

Interestingly, $\mathcal{N} = 2$ Liouville theory has been shown [109] to be dual to superstrings propagating inside the 2d black hole defined by the supercoset $SL(2,\mathbb{R})/U(1)$ [8]. The semiclassical background is

$$\begin{aligned} &ds^2 = d\rho^2 + \tanh^2\rho d\tau^2, \qquad \tau \equiv \tau + 2\pi; \\ &\Phi = \Phi_0 - \log\cosh\rho, \end{aligned} \tag{3.2}$$

with $g_s = e^\Phi$.[29] This explicitly shows that in the infrared region of small e^ρ, the τ direction degenerates. The dependence of the string background on ρ can be attributed to a gravitational dressing of the operators. This does however not preclude the existence of a two-dimensional continuum description (*c.f.* the above discussion of the scaling of the MP matrix model).

We now summarize a few properties of the target space theory. A good starting point is the worldsheet description of the Euclidean theory, the fermionic cigar at the free-fermion radius [108]. The string worldsheet theory on the cigar has three conserved currents: the left-moving and right-moving chiral currents J and $\widetilde{J}$, with

$$J = -\overline{\psi}\psi + i2\partial\tau \equiv i\partial(H + 2\tau), \tag{3.3}$$

and a non-chiral current whose integral charge P_τ is the quantized euclidean energy, *i.e.* the discrete momentum around the cigar. The chiral projection that defines the type II theories is the condition that physical operators should have a local OPE with the spectral flow operators, which in type IIB string theory takes the form

$$S = e^{-\frac{1}{2}\varphi+\frac{i}{2}(H+2\tau)} \qquad \widetilde{S} = e^{-\frac{1}{2}\widetilde{\varphi}+\frac{i}{2}(\widetilde{H}+2\widetilde{\tau})} \tag{3.4}$$

[29] Type IIB string theory based on this CFT is also equivalent, via a more trivial T-duality, to type IIA string theory on the circle of the inverse radius, with (3.1) replaced by the corresponding momentum condensate.

(here φ denotes the bosonized superghost current). Since the $U(1)$ R current of the $\mathcal{N}=2$ algebra involves the compact boson τ in addition to the worldsheet fermions, the symmetry generator P_τ is an R symmetry (here $\mathcal{S} = \int dz\ S$ is the supercharge)

$$[P_\tau, \mathcal{S}] = \frac{1}{2}\mathcal{S} \qquad\qquad [P_\tau, \widetilde{\mathcal{S}}] = \frac{1}{2}\widetilde{\mathcal{S}}\,. \tag{3.5}$$

States in a given supersymmetry multiplet therefore do not all have the same energy.

The perturbative closed string spectrum in the euclidean IIB string theory consists [108] of an NSNS (non-tachyonic) tachyon with odd winding modes, a left-moving periodic RR scalar (the self-dual axion), and a right moving complex fermion Υ with half-integer momenta. This is the expected behaviour for spinors which are single valued on the cigar. Therefore, in the compact theory, a rotation $\tau \to \tau + 2\pi$ acts on the spacetime fields as $e^{2\pi i P_\tau} = (-1)^{F_s}$, where F_s is the target-space fermion number. There is another $\mathbf{Z}_2$ symmetry $(-1)^{F_L}$ which acts by

$$\chi \mapsto -\chi, \quad \Upsilon \mapsto \overline{\Upsilon}. \tag{3.6}$$

The $\mathcal{N}=2$ Liouville which has primarily been considered in the literature has a euclidean time direction. On the other hand, matrix quantum mechanics is most easily described in a real-time Hamiltonian language. It will therefore be convenient for us to hypothesize a consistent analytic continuation of this theory. However, this analytic continuation needs to be understood better. Translating to a Minkowskian spectrum, we find a left-moving scalar χ and a complex right-moving fermion Υ. In addition to these propagating degrees of freedom, there are also discrete physical states at special energies.

D-Instantons and Flux sectors

The physics of the RR axion is closely linked to that of D-instantons. Two-dimensional IIB string theory, however, has some special features. First, the axion is a self-dual middle-rank form; it couples both electrically and magnetically to the D-instanton. One important implication of this is that the axion itself does not have a well-defined constant zeromode. Secondly, unlike the 10d case, where the BPS D-instanton breaks sixteen supercharges and thus carries an even number of fermion zeromodes, it seems that the 2d D-instanton only breaks *one* supersymmetry and therefore carries only one fermion zeromode. It thus interpolates between sectors with opposite fermion parity. A preliminary study of the D-instanton boundary state in appendix B bears this out.

At this point it is natural to introduce the right-moving scalar field U via bosonization $\Upsilon = e^{iU}$. In this bosonized language, the entire field-theoretic spectrum of 2d type IIB can thus be reassembled into a single non-chiral scalar field $\phi = \phi_L + \phi_R$ with

$$\phi_L = \chi, \qquad \phi_R = U. \tag{3.7}$$

Since $\phi_R = U$ is periodic with the free-fermion radius, it is natural to suspect that the axion $\phi_L = \chi$ is periodic with the free-fermion radius as well.[30] Note that $(-1)^{F_L}$ as defined above acts by $(-1)^{F_L} : \phi \mapsto -\phi$.

The fact that the D-instanton has only one fermion zero mode means the operator that creates it carries fermion number 1. This indicates that $\oint_\gamma \overline{\partial} U = 1$, where γ is a contour containing the instanton. Further, from the coupling of the D-instanton to the RR axion, we expect that in the presence of a D-instanton $\oint_\gamma \partial\chi = 1$. A natural candidate for the effective operator with the right properties to create a D-instanton at the space-time location x is then

$$e^{i\phi(x)} = e^{i(\chi(x)+U(x))}. \tag{3.8}$$

Instantons are tunnelling events that interpolate between perturbative sectors. These sectors are characterized by an integer flux (here Σ denotes a space-filling contour):

$$\int_\Sigma \partial_0 \phi = k \qquad\qquad k \in \mathbf{Z}, \tag{3.9}$$

[30] This claim can be verified (or refuted) by computing the axion charge carried by a D-instanton [110], or the axion flux produced by a decaying D-brane, along the lines of [87,91].

which is the quantized momentum dual to the constant zero mode ϕ_0 of ϕ (ϕ_0 is periodic with period 2π). The integer k can be thought of as a slight generalization of the "s-charge" of [111].[31]

4. Dual Correspondence

We will now try match the physics of the Marinari-Parisi matrix model with that of the two-dimensional type IIB string theory. Following the logic of [86,87] we start by examining the open string spectrum of the unstable D-particles.

D-particles

In type II string theory, the boundary state for an unstable Dp-brane has the form [113,114,115,116]

$$|\hat{Dp}\rangle = |B, NSNS; +\rangle - |B, NSNS; -\rangle. \tag{4.1}$$

Here $|B, NSNS; \eta\rangle$ denotes a boundary state in the NSNS sector, satisfying $(G_r - i\eta\tilde{G}_r)|B;\eta\rangle = 0$; $G = G^+ + G^-$ is the gauged worldsheet supercurrent. The boundary state describing an unstable brane with unperturbed tachyon contains no term built on Ramond primaries. Experience with less supersymmetric Liouville models suggests that branes localized in the Liouville direction correspond to boundary states associated with the Liouville vacuum state, which we will call $|B_0;\eta\rangle$. The defining property of these states is that the corresponding bosonic open string spectrum (of NS-sector open strings for which both end-points satisfy this specific boundary condition) have support only at Liouville momentum $P = -i$, corresponding to the identity Liouville state.

Details regarding these boundary states appear in appendix B. Using the general formula (4.1), the basic unstable D0-brane of type IIB is represented by the

[31] Since $\partial_+\phi = \partial_+\chi$, sectors with nonzero p_L are backgrounds in which flux quanta of the RR axion are turned on. Backgrounds of two-dimensional type IIA strings with RR flux were described in [112].

boundary state $|\hat{D0}\rangle = |B_0;+\rangle - |B_0;-\rangle$. Study of the annulus amplitude for this D-brane, detailed in appendix B, reveals that the open string spectrum on this brane is precisely that of the Marinari-Parisi model, expanded as in (2.4), including the gauge supermultiplet. This correspondence is the first strong indication that the MP model describes the type IIB non-critical string theory.

Symmetry considerations

There are two prominent continuous symmetries of the MP model: there is the conserved energy H, and there is the overall fermion number $\hat{F} \equiv \sum_i \psi_i^\dagger \psi_i$. In the Hilbert space of the MP model, the quantum number F takes N different values, for which we take the CP-invariant choice $-N/2, \ldots, N/2$. We would like to identify $(-1)^{F_s}$ of the target space theory with $(-1)^F$ of the MP model. Further, as in the bosonic and type 0 cases, we identify the Hamiltonians of the systems

$$H = P_\tau.$$

The matrix model can also have a $\mathbf{Z}_2$ R-symmetry. Its interpretation can be understood as follows. Due to the coupling between the D-brane worldvolume fields and the closed strings, the worldvolume fields transform under $(-1)^{F_L}$. $(-1)^{F_L}$ acts by [114,117]

$$Y \leftrightarrow -Y, \quad \psi \leftrightarrow \psi^\dagger. \tag{4.2}$$

We will see below that this is consistent with $\Upsilon \leftrightarrow \Upsilon^\dagger$. Therefore $(-1)^{F_L}$ acts as an R-symmetry in the matrix quantum mechanics: it acts on the superspace coordinates as $(-1)^{F_L} : \theta \leftrightarrow \overline{\theta}$. In order for this to be a symmetry of the worldline action, W_0 *must be an odd function of* Y. This implies that supersymmetry is broken, since then none of the standard [118] candidate supersymmetric ground states $e^{\pm W}|0\rangle$ is normalizible.

Spectrum and $c = 1$ *Scaling*

In [95], it was shown that the supercharges act within the space of super matrix eigenvalues as

$$Q = \sum_k \psi_k^\dagger \Big(\frac{1}{N}\frac{\partial}{\partial z_k} + \frac{\partial W_{\rm eff}(z)}{\partial z_k} \Big) , \qquad\qquad Q^\dagger = \sum_k \psi_k \Big(\frac{1}{N}\frac{\partial}{\partial z_k} - \frac{\partial W_{\rm eff}(z)}{\partial z_k} \Big), \quad (4.3)$$

where

$$W_{\rm eff}(z) = \sum_k W_0(z_k) - \frac{1}{N}\sum_{k<l} \log(z_k - z_l). \quad (4.4)$$

These supercharges exactly coincide [119] with those of the supersymmetric Calogero-Moser model [120][121]. The corresponding Hamiltonian reads

$$\mathcal{H} = \sum_{i=1}^{N} \left(\frac{1}{2}p_i^2 + V(z_i) + \frac{2}{N} W_0^{''}(z_i)\psi_i^\dagger \psi_i \right) + \frac{1}{N^2}\sum_{i<j} \frac{1-\kappa_{ij}}{(z_i - z_j)^2} \quad (4.5)$$

where with $p_i = \frac{-i}{N}\frac{\partial}{\partial z_i}$ and $\kappa_{ij} = 1 - (\psi_i - \psi_j)(\psi_i^\dagger - \psi_j^\dagger)$ is the fermionic exchange operator [121]; it assigns fermi-statistics to the spin-down eigenvalues, and bose-statistics to the spin-up ones (here we are using the terminology of appendix A). The potential reads [95]

$$V(z) = \frac{1}{2}\Big(W_0^{'}(z)\Big)^2 - W_0^{''}(z), \quad (4.6)$$

where we used that for a cubic superpotential W_0 one has $\sum_{i<j} \frac{W_0'(z_i) - W_0'(z_j)}{z_i - z_j} = (N-1)\sum_i W_0^{''}(z_i)$. We see that the Hamiltonian describes a system of interacting eigenvalues. The interaction is such that eigenvalues always repel each other: it represents a $2/r^2$ repulsion between the boson states with $\kappa_{ij} = -1$, and although it vanishes between two fermionic states with $\kappa_{ij} = 1$, such particles still avoid each other since their wavefunctions are antisymmetric.

In the case all the eigenvalues have spin down, so that all $\kappa_{ij} = 1$, the Hamiltonian reduces to a decoupled set of one-particle Hamiltonians. It is easy to write the ground state wave function in this case. Let us introduce the notation $\Delta(i_1, ..i_k) = \prod_{i<j\in(i_1,..i_k)}(z_{ij})$, the vandermonde of the k variables $\{z_{i_k}\}$ (here $z_{ij} = z_i - z_j$). In this notation:

$$|f_0\rangle \equiv f_0|\downarrow\downarrow\cdots\downarrow\rangle = e^{\rm TrW_0}\Delta(1,2..n)|\downarrow\downarrow\cdots\downarrow\rangle. \quad (4.7)$$

This vacuum state represents the filled Fermi sea of the first N energy levels. In the harmonic potential, this is a supersymmetric ground state. In the cubic potential, there is a corresponding ground state in each well, only one of which is perturbatively supersymmetric.

In the sectors with non-zero fermion number, there are no normalizable supersymmetric ground states, even for the harmonic well. As described in Appendix C, the groundstate eigenfunction in the fermion number k sector is:

$$|f_k\rangle = \sum_{i_1<i_2<..<i_k} \Delta(i_1, ..i_k) \prod_{m=1}^{k} \psi^\dagger_{i_m} |f_0\rangle. \tag{4.8}$$

This wavefunction obeys the right statistics imposed by gauge invariance. From the expression (4.8) we can read off that the energy levels of the spin up eigenvalues are double spaced relative to that between spin down states [122].

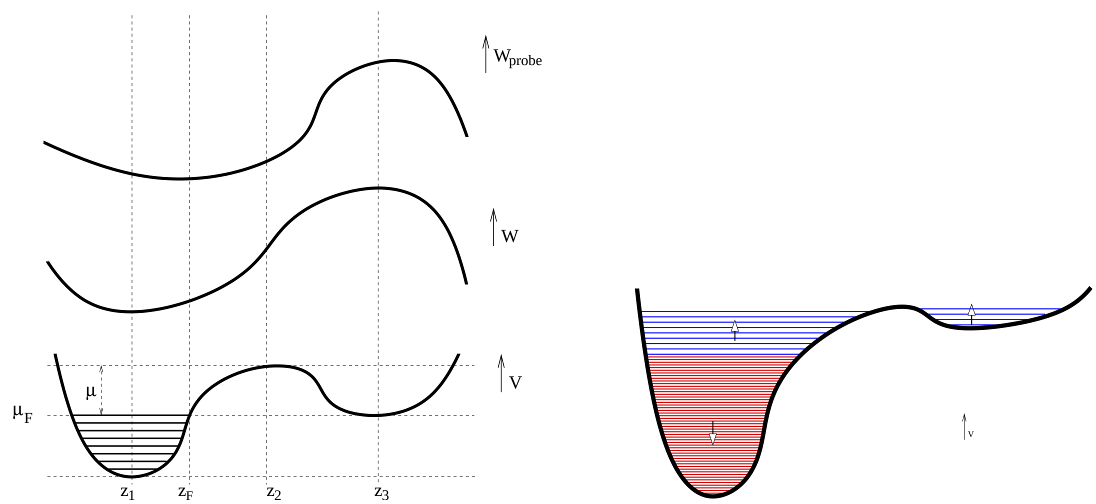

Fig. 5: The approximate probe superpotential, superpotential and bosonic potential of the f_0 state. The fermi level of the perturbatively supersymmetric ground state f_0 coincides with the minimum of the right well (*left*). The ground state in the sector with k spin up eigenvalues fills up to the $N + k$-th energy level, which in the double scaling limit approaches the unstable maximum (*right*).

Now let us discuss the system with a cubic superpotential. The state f_0 is a perturbatively supersymmetric state, with a filled fermi sea in the left well. It turns out that the fermi level exactly coincides with the bottom of the second well, as indicated on the left of Fig. 1. Although the super-Calogero-Moser model with a cubic potential has no known exact treatment, it seems reasonable to assume that the super-eigenvalues in each of the two potential wells behave qualitatively similar as for the single harmonic potential. This suggests that in the sector with k up-spins, the ground state is well-approximated by taking the single particle energy spectrum, and fill the first $N-k$ levels with the fermionic eigenvalues, and all levels $N-k+2m$ with $m=1,\ldots,k$ up to the $N+k$-th energy level. When k gets large enough, this $N+k$-th energy level starts approaching the unstable maximum of the potential, as indicated on the right of Fig. 1. Here we expect to find $c=1$ critical behavior. Our proposal is that the double scaled eigenvalue dynamics near this unstable maximum encodes the scattering non-perturbative physics of non-critical IIB strings.

Matrix model instantons

Single-eigenvalue tunneling events in the matrix model interpolate between the sectors with different number of up-spins: they relate the adjacent ground states $|f_k\rangle$ and $|f_{k+1}\rangle$. Note that these sectors have opposite parity of fermion number. The probe super-eigenvalue Z moves in the mean-field superpotential

$$W_{\text{probe}}(Z) = W_0(Z) - \frac{1}{2}\sum_i \ln(Z - X_i). \tag{4.9}$$

There exists [95] a BPS tunneling trajectory:

$$0 = \delta\psi = \dot{z}_{cl} - W'_{\text{probe}}(z_{cl}). \tag{4.10}$$

This trajectory breaks just one supersymmetry, and thus supports a single fermion zero mode.

We propose to identify the vacuum of the matrix model with fermion number k with the string theory vacuum with k units of flux, as defined in §3. This identification is directly supported by the interpretation of the MP matrix model as the worldline theory of the unstable branes and of the D-instantons with the tunnelling trajectories of the worldline tachyon field Y. Recall that an unstable type

II Dp-brane couples to the RR p-form potential $C_{(p)}$ via the gradient of its tachyon according to $\int C_{(p)} \wedge d\mathcal{W}(Y)$ [123,115,124,125] with $V(Y) = \frac{\partial}{\partial Y}\mathcal{W}(Y)$ [126]. For the unstable D-particle, this takes the form

$$S_{\hat{D}0} \supset \int \chi \, \frac{dY}{dt} \, V(Y) \, dt. \tag{4.11}$$

Combined with our proposal, this coupling implies that the tunnelling trajectory sources the RR axion. It can therefore be identified with a D-instanton, further vindicating the prescient analysis of [127].

5. Concluding remarks

We have collected evidence supporting the conjecture that the supersymmetric matrix model of Marinari and Parisi can be identified with the matrix mechanics of N unstable D-particles in two-dimensional IIB string theory. This suggests that in a suitable double scaling limit, the MP model, when viewed from this perspective, provides a non-perturbative definition of the string theory. The two systems on both sides of the conjectured duality, however, clearly need further study. We end with some concluding comments.

Space-time fields

An important open problem is the proper identification of the space-time fields in the MP model. It is reasonable to expect that, as for the $c = 1$ [128] and $\hat{c} = 1$ cases, the spacetime fields arise from the matrix model via the collective fields for the eigenvalue density. Possibly the supersymmetric collective field theory of [129,96,97,98] is the correct framework, though it seems that some Z_2 projection may be needed, since the target space boson χ and fermions Υ are chiral with opposite chirality. If Υ is linear in the fermionic component of the eigenvalue density (*e.g.* a Laplace transform of it), then the matrix model action of $(-1)^{F_L} : \psi \leftrightarrow \overline{\psi}$, is consistent with the action on closed-string fields $\Upsilon \leftrightarrow \overline{\Upsilon}$.

Space-time supersymmetry

Space-time supersymmetry should provide a helpful guideline in finding a precise dictionary. Expanded around the quadratic maximum of its bosonic potential, the model (2.4) has many symmetries; indeed the small bosonic and fermionic fluctuations are decoupled. It is not difficult to find fermionic operators which behave as in (3.5). The more mysterious question is how to describe the matrix model supersymmetry in the target space of the string theory.

D-brane decay

It should be possible to generalize the analysis of [86,87,88] to study the decay of a single unstable D-brane. This will presumably involve a superfield version of the fermion operator that creates and destroys the super-eigenvalues, and a superfield bosonization formula along the lines of [130]. This analysis would allow an independent determination of the compactification radius of the axion, along the lines of [87,91], confirming that χ is periodic at the free fermion radius.

Appendix A. The gauged Marinari-Parisi model

In this appendix, we describe two ways of introducing an auxiliary gauge field for the supersymmetric matrix model. We show that the second method is equivalent to the eigenvalue reduction of the MP model given in [95].

Gauged Model, Version I

The conventional method of gauging a supersymmetric action is to introduce a real matrix superfield $\mathcal{V}$, and replace the superderivatives in (2.2) with gauge-covariant superderivatives of the form

$$D_{\mathcal{V}}\Phi = e^{\mathrm{ad}\mathcal{V}} D(e^{-\mathrm{ad}\mathcal{V}}\Phi)\,, \qquad D = \partial_\theta + \overline{\theta}\partial_\tau \tag{A.1}$$

$$\overline{D}_{\mathcal{V}}\Phi = e^{\mathrm{ad}\mathcal{V}} \overline{D}_\theta(e^{-\mathrm{ad}\mathcal{V}}\Phi) \qquad \overline{D} = \partial_{\overline{\theta}} + \theta\partial_\tau \tag{A.2}$$

These derivatives are designed to be covariant under local gauge transformations

$$\Phi \mapsto e^{\mathrm{ad}\Lambda}\,\Phi\,, \qquad \mathcal{V} \mapsto \mathcal{V} + \Lambda \tag{A.3}$$

where Λ is an arbitrary real superfield. We can thus choose the gauge $\mathcal{V} = 0$. The presence of the gauge field still manifests itself, however, by means of the requirement that physical states must be annihilated by the generator of infinitesimal *bosonic* gauge rotations $\Phi \mapsto U^\dagger \Phi U$. This invariance can be used to diagonalize, say, the bosonic component ϕ of the matrix superfield. The off-diagonal fermionic matrix elements, however, remain as physical degrees of freedom.

Gauged Model, Version II

A second possible procedure is to introduce a complex superfield $\mathcal{A}$ and define superderivatives

$$D_{\mathcal{A}} \Phi = D\Phi - [\mathcal{A}, \Phi] \,, \qquad \overline{D}_{\mathcal{A}} \Phi = \overline{D}\Phi - [\overline{\mathcal{A}}, \Phi] \tag{A.4}$$

covariant under local gauge transformations

$$\Phi \mapsto e^{\mathrm{ad}\Lambda} \, \Phi \,, \qquad \mathcal{A} \mapsto \mathcal{A} + D\Lambda \,, \qquad \overline{\mathcal{A}} \mapsto \overline{\mathcal{A}} + \overline{D}\Lambda \tag{A.5}$$

with Λ an arbitrary real matrix superfield. In this case, the gauge invariance is not sufficient to choose a gauge in which $\mathcal{A}$ and $\overline{\mathcal{A}}$ are set equal to zero. However, since $\mathcal{A}$ and $\overline{\mathcal{A}}$ appear as non-dynamical fields, we can eliminate them via their equations of motion

$$\mathcal{G} \equiv [\, \Phi, \Pi \,] = 0 \,, \qquad \Pi \equiv D_{\mathcal{A}} \Phi \,. \tag{A.6}$$

We will impose the physical state conditions in the weak form

$$\mathcal{G} \, | \Psi_{\mathrm{phys}} \rangle \; = \; 0 \,. \tag{A.7}$$

The space of solutions to this constraint is characterized as follows. Let U be the bosonic unitary matrix that diagonalizes the bosonic component ϕ of the matrix superfield. We can then define

$$(U \Phi U^\dagger)_{kk} = z_k + \overline{\theta} \, \psi_k + \psi_k^\dagger \, \theta + \overline{\theta} \theta f_k \,. \tag{A.8}$$

Here the z_k are the eigenvalues of ϕ. A straightforward calculations shows that the gauge invariance conditions is solved by physical states that depend on z_k and ψ_k

only. Since the physical state constraint (A.6) is a supermultiplet of constraints, this guarantees that this subspace forms a consistent supersymmetric truncation of the full matrix model. One can also verify directly that it is invariant under supersymmetry transformations. After absorbing a factor of $\Delta = \prod_{i<j}(z_i - z_j)$ into our wave functions

$$\Psi(z,\psi) = \Delta(z)\tilde{\Psi}(z,\psi) \tag{A.9}$$

the supersymmetry generators take the form (4.3).

It is convenient to think of the system of eigenvalues as N particles moving in one dimension, each with an internal spin $\frac{1}{2}$ degree of freedom, a spin "up" or "down." Accordingly, we can define the Hilbert space on which the fermionic eigenvalues act by

$$\psi_i|\downarrow\downarrow\cdots\downarrow\rangle = 0, \quad \forall i; \quad \psi_i^\dagger|\downarrow\downarrow\cdots\downarrow\rangle \equiv |\underbrace{\downarrow\cdots\downarrow}_{i-1}\uparrow\downarrow\cdots\downarrow\rangle; \quad etc... \tag{A.10}$$

A general state in the physical Hilbert space is then

$$|f_\eta\rangle = \sum_\eta f_\eta(\mathbf{z})|\vec{\eta}\rangle \tag{A.11}$$

where η is a vector of N up or down arrows, and we have arranged the eigenvalues in a vector $\mathbf{z}$. Since it is possible via $U(N)$-rotations to interchange any eigenvalue superfield (z_i, ψ_i) with any other eigenvalue superfield (z_j, ψ_j), the matrix wavefunctions should be symmetric under this exchange operation.

However, since our wave-functions depend on anti-commuting variables, the model will inevitably contain bosonic as well as fermionic sectors. Let us now define a a fermionic interchange operation κ_{ij} with the property that it interchanges the i and j spin state, and also multiplies the overall wavefunction by a minus sign in case both spins point in the up-direction. This minus sign reflects the Fermi statistics of ψ_i and ψ_j. Define the total exchange operation as the product of the bosonic and fermionic one $\mathcal{K}_{ij} = K_{ij}\,\kappa_{ij}$ where K_{ij} interchanges z_i and z_j. We now specify the overall statistics of the physical wavefunctions by means of the requirement that

$$\mathcal{K}_{ij}\,|\tilde{\Psi}_{\rm phys}\rangle = -|\tilde{\Psi}_{\rm phys}\rangle \qquad \text{for all } i,j \tag{A.12}$$

The minus sign on the right-hand side ensures that the original wavefunction, before splitting off the Vandermonde determinant (see eqn (A.9)), is completely symmetric. The condition (A.12) implies that particles with spin "up" are fermions, while particles with spin "down" are bosons. We can call this the spin-statistics theorem for our model.

Appendix B. Boundary states for $\mathcal{N} = 2$ Liouville

In this appendix, we will attempt to write down the boundary state for the unstable D-particle of type IIB in the $\mathcal{N} = 2$ Liouville background. In doing this, we will take advantage of the worldsheet $\mathcal{N} = 2$ supersymmetry by expanding the boundary state in Ishibashi states which respect the $\mathcal{N} = 2$. In order to write the Ishibashi states, we will need to recall some facts about the primaries on which they are built, and their characters.

Characters of the $\mathcal{N} = 2$ superconformal algebra

The chiral $\mathcal{N} = 2$ characters are defined by

$$\chi_V(q, y) = \mathrm{Tr}_V q^{L_0 - c/24} y^{J_0}. \tag{B.1}$$

The trace is over an $\mathcal{N} = 2$ module V. These representations are built on primary states labelled by the eigenvalues h, ω of the central zeromodes L_0, J_0. It will be convenient to label our primary states by P and ω, related to the conformal dimension by $h = (Q^2/4 + P^2 + \omega^2)/2 = (1 + P^2 + \omega^2)/2$. The Liouville momentum P[32] is determined by this equation up to choice of branch, both of which have the same character.

These characters were written down in [131]. For the module associated with a generic NS primary, labelled by $[P, \omega]$, the character is

$$\chi^{NS}_{[P,\omega]}(q, y) = q^{P^2/2+\omega^2/2}\; y^{\omega} \frac{\vartheta_{00}(q, y)}{\eta^3(q)} \tag{B.2}$$

[32] Here we are defining Liouville momentum as P appearing in the wavefunction $e^{-(Q/2+iP)\rho}$.

In the R-sector, a primary is also annihilated by G_0^+ or G_0^-, and this results in an extra label $\sigma = \pm$ on the character. For the primary with labels P, ω, σ, the character is (let $y \equiv e^{2\pi i\nu}$)

$$\chi^R_{[P,\omega,\sigma]}(q,y) = 2\cos\pi\nu \; q^{P^2/2+\omega^2/2} \quad y^{\omega+\frac{\sigma}{2}} \frac{\vartheta_{1,0}(q,y)}{\eta^3(q)} \tag{B.3}$$

We will also need the character for the identity representation

$$\chi^{NS}_{\mathbf{1}}(q,y) = q^{-1} \frac{1-q}{(1+yq^{1/2})(1+y^{-1}q^{1/2})} \frac{\vartheta_{0,0}(q,y)}{\eta^3(q)}. \tag{B.4}$$

Modular properties of the characters

The modular transformation properties of the chiral characters of the $\mathcal{N}=2$ algebra will be crucial for our study of D-branes in $\mathcal{N}=2$ superLiouville. We use the notation $\tilde{q} = e^{2\pi i\tau}$, $\tilde{y} = e^{2\pi i\nu}$ for closed string modular variables, and $q = e^{-2\pi i/\tau}$, $y = e^{\pi i\nu/\tau}$ for their open string transforms. The characters participate in the following formulas [132][133].

$$\int_{-\infty}^{\infty} d\tilde{P} d\tilde{\omega} \;\; S(\tilde{p},\tilde{\omega}) \; \chi^{NS}_{[\tilde{p},\tilde{\omega}]}(\tilde{q},\tilde{y}) = \chi^{NS}_{\mathbf{1}}(q,y) \tag{B.5}$$

$$\int_{-\infty}^{\infty} d\tilde{P} d\tilde{\omega} \;\; S(\tilde{p},\tilde{\omega}) \; \chi^{NS}_{[\tilde{p},\tilde{\omega}]}(\tilde{q},-\tilde{y}) \equiv \; \chi^{R}_{\mathbf{1}}(q,y) \tag{B.6}$$

$$S(\tilde{p},\tilde{\omega}) = \frac{\sinh^2 \pi\tilde{p}}{2\cosh(\pi p/2 + i\pi\omega/2)\cosh(\pi p/2 - i\pi\omega/2)}. \tag{B.7}$$

Note that these formulas (B.5)–(B.7) are relevant for the case that R-charge is not quantized, on which we focus for simplicity in our study of boundary states. The refinement of these formulas to the case of compact euclidean time follows from (B.5)–(B.6) by Fourier decomposition. Further, these formulas arise by performing a formal sum; a careful treatment of convergence issues reveals additional contributions from discrete states [133].

Ishibashi states for $\mathcal{N}=2$

Using this notation for representations of the $\mathcal{N}=2$ algebra, let us now study Ishibashi states based on these representations[33]. Such states [137] provide a basis for D-brane states which respect the $\mathcal{N}=2$ algebra. They carry two kinds of labels: those which specify the primary of the chiral algebra on which the state is built; and those which specify the automorphism of the chiral algebra which was used to glue the left and right chiral algebras. In our notation, we will separate these labels by a semicolon. Only automorphisms of the $\mathcal{N}=2$ which preserve the gauged $\mathcal{N}=1$ subalgebra are allowed.

There is a $\mathbf{Z}_2$ automorphism group of the $\mathcal{N}=1$ subalgebra ($G=G^{+}+G^{-}$)

$$G \to \eta G \tag{B.8}$$

with $\eta=\pm 1$. There is an additional $\mathbf{Z}_2$ automorphism of the $\mathcal{N}=2$ algebra, which commutes with (B.8), and which is generated by

$$G^{\pm} \to G^{\pm\xi}, J \to -\xi J$$

with $\xi=\pm$ [134] (the trivial map, $\xi=+1$ is B-type, the nontrivial map $\xi=-1$ is A-type).

Let j label the $\mathcal{N}=2$ primaries; it is a multi-index with three components:

$$j = \left[h, n, \left\{ \begin{array}{c} NS \\ R+ \\ R- \end{array} \right\} \right].$$

(recall that an R primary is further specified by whether it is annihilated by G_0^{+} or G_0^{-}.) An A-type Ishibashi state satisfies

$$(L_n - \tilde{L}_n)|j;A,\eta\rangle\rangle, \quad (G_r^{\pm} - i\eta\tilde{G}_r^{\mp})|j;A,\eta\rangle\rangle = 0, \quad (J_n - \tilde{J}_n)|j;A,\eta\rangle\rangle \tag{B.9}$$

where r is half-integer moded if $|j\rangle$ is an NS primary, and integer moded if $|j\rangle$ is from an R sector.

[33] Useful references include [134,116,135,136].

In order to make type II D-branes from these states, we will need to know the action of the fermion number operators on them. In the NS sector [116],

$$(-1)^F|j,NS;\xi,\eta\rangle\rangle = -|j,NS;\xi,-\eta\rangle\rangle = (-1)^{\tilde{F}}|j,NS;\xi,\eta\rangle\rangle. \tag{B.10}$$

Note that the existence of the state with one value of η plus the conserved chiral fermion number implies the existence of the other. In the R sector, the action is more subtle because of the fermion zero modes. Since unstable branes may be built without using RR Ishibashi states, we will not discuss them further.

Next we need to know the matrix of inner products between these states. The inner product is defined by closed-string propagation between the two ends of a cylinder:

$$\langle\langle j_1;\xi_1\eta_1|\ D(\tilde{q},\tilde{y})\ |j_2;\xi_2\eta_2\rangle\rangle = \delta(j_1,j_2)\ \delta_{\xi_1,\xi_2}\ \chi_{j_1}(\tilde{q},\eta_1\eta_2\tilde{y}) \tag{B.11}$$

where $D(\tilde{q},\tilde{y})$ is the closed-string propagator, twisted by the R-current. The delta-function on primaries is obtained from the overlap of closed-string states:

$$\delta(j_1,j_2) \equiv \left(\langle j_1|\otimes\langle\tilde{j}_1|\right)|j_2\rangle\otimes|\tilde{j}_2\rangle \tag{B.12}$$

The D0 boundary state

We would now like to describe the boundary state for the unstable D-particle. Since it is extended in the R-symmetry direction, it should be a B-type brane. To construct consistent boundary states for $\mathcal{N}=2$ Liouville, we will follow the strategy which was successful for bosonic Liouville [138,139,140], and for $\mathcal{N}=1$ Liouville [141,142]. Basically, the consistent boundary states are Cardy states [140]; their wavefunctions can be written in terms of the modular matrix $U_j(i) = \frac{S^i_j}{\sqrt{S^i_0}}$ [143]. To be more precise, suppose, as in [138], that we can expand the desired boundary state in Ishibashi states for the non-degenerate (NS) representations of the $\mathcal{N}=2$ algebra:

$$|B_0,\eta\rangle = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dP\ U^\eta(P,\omega)\ |P,\omega,NS;B,\eta\rangle\rangle. \tag{B.13}$$

In writing an integral over ω in (B.13), we are focusing on the case when the time direction is infinite in extent. To build non-BPS type II branes from these boundary states, GSO-invariance requires $U(P,\omega) = U^{+}(P,\omega) = -U^{-}(P,\omega)$ according to (B.10). We will therefore write

$$|\hat{D0}\rangle = |B_0, \eta = +1\rangle\rangle - |B_0, \eta = -1\rangle\rangle. \tag{B.14}$$

Next, we make the self-consistent assumption that the bosonic open-string spectrum should contain only states with Liouville momentum $P = -i$ corresponding to the identity state. Given (B.5) a solution to this requirement is [132][133]

$$U(P,\omega) = \sqrt{2} \cdot e^{i\delta(P,\omega)} \cdot \frac{\sinh \pi P}{\cosh(\pi p/2 + i\pi\omega/2)}. \tag{B.15}$$

The phase $e^{i\delta(P,\omega)}$ is not actually determined by the modular hypothesis. As we will verify next, the D-brane with this wavefunction (B.15) indeed has only the identity representation in its bosonic open-string spectrum.

Open string spectrum

The vacuum annulus amplitude between the boundary state and itself,

$$\mathcal{A}(q) = \langle \hat{D0} | D(\tilde{q}) | \hat{D0} \rangle, \tag{B.16}$$

determines the open string spectrum by channel duality. In this expresseion, $D(\tilde{q})$ is the closed-string propagator, on a tube of length $\tau = \ln \tilde{q}/2\pi i$. Given the expression (B.13), it takes the form

$$\mathcal{A}(q) = \int_{-\infty}^{\infty} dP\, d\omega\, \hat{U}(P,r)\, \hat{U}^{\dagger}(P,\omega) \sum_{\eta_1\eta_2} \eta_1\eta_2\, \chi^{NS}_{[P,\omega]}(\tilde{q}, \eta_1\eta_2)\, \chi^{NS}_{gh}(q, \eta_1, \eta_2). \tag{B.17}$$

Note that there are two sets of terms which are identical. This reflects the fact that unstable branes are $\sqrt{2}$ times heavier than BPS D-branes.

Using the modular transformation formulas above, and the wavefunction (B.15), in the open-string channel this reads

$$\mathcal{A}(q) = 2\left(\chi^{NS}_{1}(q,1)\; \chi^{NS}_{gh}(q,1) + \chi^{R}_{1}(q,1)\; \chi^{R}_{gh}(q,1)\right). \tag{B.18}$$

As usual for two-dimensional strings, the contribution from the $\mathcal{N}=2$ $bc\beta\gamma$ ghost system cancels all of the modular functions from each term, and we find

$$Z \equiv \int \frac{dt}{2t} \mathrm{tr}\ q^{H_{\mathrm{open}}} = \int \frac{dt}{2t} \left[\left(q^{-1/2} - 2 + \ldots \right) - (1 + \ldots) \right] \tag{B.19}$$

with $q \equiv e^{-\pi t}$. Deducing the mass of the states propagating in the open string loop is subtle because the time direction participates in the $\mathcal{N}=2$ algebra, and the analytic continuation needs to be understood better. We can however understand the spectrum by looking at the large-t behaviour of the amplitude. The first term in brackets is the contribution of the NS sector. The two contributions indicated give rise to divergences and represent a tachyon Y, and a complex massless discrete state, respectively. This is similar to the case of the unstable D-particle in the bosonic $c=1$ string[34] (*q.v.* the lovely appendix B of [87]) and in the $\hat{c}=1$ type 0B string [90,91].

The second term in brackets is the contribution of the R sector, which produces the fermions which were absent in the above examples. The term indicated represents a massless complex fermion ψ,

This is the spectrum of the gauged Marinari-Parisi model (2.2) on a circle. The appearance of non-quantized energies at intermediate stages can be remedied [133] by employing characters of the $\mathcal{N}=2$ algebra extended by the spectral flow generators.

For the instanton-anti-instanton pair, the only difference is that we use A-type Ishibashi states, which describe branes which are localized in the R-symmetry direction. The spectrum of the BPS D-instanton is obtained from this by an open-string GSO projection, and therefore has half as many fermion zeromodes, namely one.

Clearly we have merely begun to study the interesting zoology of D-branes in this system. Further development appears in [132][133].

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[34] upon multiplying α' by 2

Appendix C. Ground states of the harmonic superpotential for arbitrary fermion number

For a quadratic superpotential $W_0(z) = \frac{1}{2}\omega z^2$, the Hamiltonian we are interested in is:

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N}(p_i^2 + \omega z_i^2) + \sum_{i<j}\frac{(1-\kappa_{ij})}{(z_i - z_j)^2} + \sum_{i=1}^{N}\omega\psi_i^\dagger\psi_i - E_0. \tag{C.1}$$

where $p_i = -i\frac{\partial}{\partial z_i}$ and κ_{ij} is the fermionic exchange operator, and $E_0 = \frac{1}{2}N^2$.
We introduce the notation $\Delta(i_1, ..i_k) = \prod_{i<j\in(i_1,..i_k)}(z_i - z_j)$, the vandermonde of the k variables $\{z_{i_k}\}$.
We saw earlier that the supersymmetric ground state of this hamiltonian is

$$|f_0\rangle \equiv f_0|\downarrow\downarrow\cdots\downarrow\rangle = e^{-\mathrm{TrW_0}}\Delta(1,2..n)|\downarrow\downarrow\cdots\downarrow\rangle. \tag{C.2}$$

We will show now that the groundstate eigenfunction in the fermion number k sector is:

$$|f_k\rangle = \sum_{i_1<i_2<..<i_k}\Delta(i_1,..i_k)\prod_{m=1}^{k}\psi_{i_m}^\dagger f_0|\downarrow\downarrow\cdots\downarrow\rangle. \tag{C.3}$$

This wavefunction obeys the right statistics imposed by gauge invariance:

$$K_{ij}|f_k\rangle = -\kappa_{ij}|f_k\rangle. \tag{C.4}$$

Conjugation by f_0 transformes the Hamiltonian to

$$\begin{aligned}\widetilde{\mathcal{H}} &= \sum_{i=1}^{N}(\frac{1}{2}p_i^2 + \omega z_i\partial_i + \omega\psi_i^\dagger\psi_i) + \sum_{i<j}\frac{1}{z_{ij}^2}(1-\kappa_{ij}) - \sum_{i<j}\frac{1}{z_{ij}}(\partial_i - \partial_j)\\ &\equiv \sum_{i=1}^{N}\widetilde{\mathcal{H}}_i + \sum_{i<j}\widetilde{\mathcal{H}}_{ij}\end{aligned} \tag{C.5}$$

and the wavefunctions (C.3) transform to

$$|\widetilde{f}_k\rangle = \sum_{i_1<i_2<..<i_k}\Delta(i_1,..i_k)\prod_{m=1}^{k}\psi_{i_m}^\dagger|\downarrow\downarrow\cdots\downarrow\rangle. \tag{C.6}$$

with the statistics

$$K_{ij}|f_k\rangle = \kappa_{ij}|f_k\rangle. \tag{C.7}$$

It is clear that $|\widetilde{f}_k\rangle$ are eigenfunctions of $\widetilde{\mathcal{H}}_i$, we will show below that they are also eigenstates of $\widetilde{\mathcal{H}}_{ij}$. To do so, we will show that

$$\sum_{i\neq j} \frac{1}{z_{ij}}(\partial_i - \partial_j)\Delta(i_1, ..i_k) = \sum_{i\neq j} \frac{1}{z_{ij}^2}(1 - K_{ij})\Delta(i_1, ..i_k). \tag{C.8}$$

Using (C.7), it follows that $\sum_{i<j} \widetilde{\mathcal{H}}_{ij}|\widetilde{f}_k\rangle = 0$.

For instance, it is easy to check that for $k = 0, 1, 2$, the action of $z_{ij}(\partial_i - \partial_j)$ on $\tilde{f}_k$ is identical to the action of $(1 - K_{ij})$ thus verifying (C.8) term by term. For a general value of k, we proceed as follows:

For a given k-tuple $(i_1...i_k)$, let us denote by roman alphabets the variables which are part of the k-tuple $a, b... \in (i_1...i_k)$ and by greek alphabets the variables which are not $\alpha, \beta... \in (i_1...i_k)^c$. We shall for now call $\Delta(i_1, ..i_k) = \Delta_k$[35]. Let us now analyze the terms in (C.8) according to whether $(i, j) = (\alpha, \beta)$, (a, b) or (a, α).

Two identities we shall use are:

1.

$$\partial_a^m \Delta_k = \sum_{a_1 \neq a_2 .. \neq a_m \neq a} \prod_{j=1}^{m} \frac{1}{z_{a a_j}} \Delta_k \tag{C.9}$$

2. If $P(z_1, ..z_n)$ is a completely antisymmetric polynomial in n variables, then it vanishes identically unless:

$$\deg(P) \geq \frac{1}{2}n(n-1). \tag{C.10}$$

Now let us analyse the three cases:

1. $(i, j) = (\alpha, \beta)$, $K_{\alpha\beta} = 1$. It is then trivially true that

$$\frac{1}{z_{\alpha\beta}}(\partial_\alpha - \partial_\beta)\Delta_k = \frac{1}{z_{\alpha\beta}^2}(1 - K_{\alpha\beta})\Delta_k = 0. \tag{C.11}$$

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[35] This is not the best notation because the function is not specified by k, but there is no ambiguity in this context

2. $(i,j) = (a,b)$, $K_{ab} = -1$.

$$\begin{aligned}
\sum_{a<b} \frac{1}{z_{ab}}(\partial_a - \partial_b)\Delta_k &= \sum_{a<b} \frac{2}{z_{ab}} \sum_{c\neq a} \frac{1}{z_{ac}} \Delta_k \\
&= \sum_{a<b} \frac{2}{z_{ab}^2} + \sum_{b\neq c\neq a} \frac{1}{z_{ab}} \frac{1}{z_{ac}} \Delta_k \\
&= \sum_{a<b} \frac{2}{z_{ab}^2} + \sum_{a} \partial_a^2 \Delta_k \qquad \text{using (C.9)} \\
&= \sum_{a<b} \frac{2}{z_{ab}^2} \Delta_k = \sum_{a<b} \frac{(1-K_{ab})}{z_{ab}^2} \Delta_k.
\end{aligned} \tag{C.12}$$

where in going to the last line, we have used (C.10).

3. $(i,j) = (a,\alpha)$. In this case, for each α,

$$\begin{aligned}
\sum_{a} \frac{1}{z_{a\alpha}^2}(1 - K_{a\alpha})\Delta_k &= \sum_{a} \frac{1}{z_{a\alpha}^2}(1 - \prod_{b\neq a} \frac{z_{\alpha b}}{z_{ab}})\Delta_k = \sum_{a,\alpha} \frac{1}{z_{a\alpha}^2} \left(1 - \prod_{b\neq a}(1 - \frac{z_{a\alpha}}{z_{ab}})\right) \Delta_k \\
&= \sum_{a} \frac{1}{z_{a\alpha}^2} \left(1 - \sum_{m=0}^{k} \sum_{a_1\neq a_2 .. \neq a_m \neq a} (-z_{a\alpha})^m \prod_{j=1}^{m} \frac{1}{z_{aa_j}}\right) \Delta_k \\
&= -\sum_{a} \sum_{m=1}^{k} \sum_{a_1\neq a_2 .. \neq a_m \neq a} (-z_{a\alpha})^{m-2} \prod_{j=1}^{m} \frac{1}{z_{aa_j}} \Delta_k \\
&= -\sum_{m=1}^{k} \sum_{a} (-z_{a\alpha})^{m-2} \partial_a^m \Delta_k, \qquad \text{using (C.9).} \\
&= \sum_{a} \frac{1}{z_{a\alpha}} \partial_a \Delta_k = \sum_{a} \frac{1}{z_{a\alpha}} (\partial_a - \partial_\alpha)\Delta_k
\end{aligned} \tag{C.13}$$

In going to the last line, we note that all the terms except for $m = 1$ are antisymmetric polynomials and hence (C.10) applies.

The hamiltonian $\mathcal{H}$ by virtue of its being supersymmetric, is positive semi-definite for any superpotential (in particular, for $\omega = 0$). Since conjugation does not change the spectrum of an operator, $\widetilde{\mathcal{H}}(\omega = 0)$ is also positive definite, which means

$$\widetilde{\mathcal{H}} \geq \sum_{i=1}^{N} (\omega z_i \partial_i + \omega \psi_i^\dagger \psi_i). \tag{C.14}$$

In the sector with fermion number k, the antisymmetry property[36] (C.10) tells us that $E \geq (\frac{1}{2}k(k-1)+k)\omega = \frac{1}{2}k(k+1)\omega$. This tells us that $|f_k\rangle$'s are indeed the ground states at fermion number k with energy $\frac{1}{2}k(k+1)\omega$.

[36] The bosonic part of the right hand side (the Euler operator) has homogeneous polynomials as its eigenfunctions.

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