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Spin-Gravity Coupling of Neutrinos in Long-Baseline Experiments

Masters thesis
Carried out for the purpose of obtaining the academic degree
of Diplom-Ingenieur (Dipl.-Ing or DI)

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submitted at
Technischen Universität Wien
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Vienna, 25.08.2024

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Abstract

In this study, we investigate the non-geodesic behaviour of neutrinos propagating in curved space-time. The non-geodesic motion arises as a consequence of spin-gravity coupling. We begin by reviewing the essential mathematical prerequisites, which subsequently allow us to derive the corresponding correction to the geodesic motion up to the order of $\hbar$. Specifically, we calculate this deviation using Fermi-normal coordinates to obtain an analytic estimation. Finally, leveraging this deviation, we have the potential to establish a lower bound for the neutrino mass, if the DUNE experiment is in operation.

Declaration of Authenticity

I declare that the present work was independently created by me in accordance with recognized principles for scientific papers. All tools used, especially the underlying literature, are cited and listed in this work. Literal excerpts from sources are clearly marked as such.

The topic of this work has not been previously submitted for assessment as an examination paper by any evaluator, either in Austria or abroad. This work aligns with the assessment made by the evaluators.

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Acknowlagement

I would like to express my sincere gratitude to my supervisor, Benjamin Koch, for his excellent support throughout this project. Our constructive conversations about the ongoing status of this Master's thesis were invaluable, and I appreciate the opportunity to engage with him on an equal footing.

Additionally, I am deeply thankful to my parents for their unwavering support during my years of study in Vienna. Their encouragement and assistance were far from self-evident, and I recognize the privilege of having such dedicated parents.

My time in Vienna was truly transformative, both intellectually and personally. I had the privilege of meeting fascinating and brilliant individuals during my (under)graduate years. I extend my heartfelt thanks to Victor Fitzek, Fabian Helmberger and Bruno Mittnik for our endless discussions on topics ranging from Physics to Philosophy. Similarly, I appreciate Lorenz Fischer, who helped me to create the figures in this work, and not to forget, whose insights enriched the final stages of my studies.

Science is anything but linear; it resembles a chaotic rollercoaster with unexpected twists and turns. While working on this Master's thesis, I experienced first-hand the challenges in research. There were moments when I came perilously close to giving up, but perseverance prevailed.

Thank you to everyone who contributed to this journey, and I am proud to stand by the integrity of my work.

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List of Abbreviations

1. WKB . . . Wentzel–Kramers–Brillouin
2. MPD . . . Mathisson–Papapetrou–Dixon
3. MP . . . Mathisson-Pirani
4. TD . . . Tulczykew-Dixon
5. CP . . . Corinaldesi-Papapetrou
6. NW . . . Newton-Wigner
7. OKS . . . Ohashi-Kyrian-Semerák
8. DUNE . . . Deep underground neutrino experiment

Contents

1 Introduction

The Glashow-Weinberg-Salam model describes the unification of electromagnetic and weak interaction through a local $SU(2)_L \times U(1)$ gauge theory. This theory is only exact for vanishing fermion masses, therefore the leptons need to lend their mass through the well known Higgs-mechanism [1, 2, 3], while the neutrinos stay massless. The discrepancy of expectation and measurements of solar neutrinos, is explainable by neutrino oscillation. Unlike being an eigenstate of the weak Hamiltonian, a neutrino is described as a superposition of at least two distinct mass states. This superposition results in the neutrino having a non-zero mass. For further details, refer to the excellent work by Bellini et al. [4]. By measuring this phase-shift, it is only possible to gain information about the squared mass-differences, but no information about the net neutrino masses [5].

Rather than relying solely on neutrino oscillations to infer mass, an alternative approach involves studying the motion of a spin-$1/2$ particle within a curved gravitational field. While spinless particles follow geodesic paths, those with spin deviate from geodesics. This deviation also applies to particles possessing classical spinning angular momentum, such as the Earth [6]. With this deviation, it is possible to extract information about the particle's mass. Although this description is not exclusive to neutrinos, they are particularly suitable due to their weak interaction with matter, making them ideal candidates for Long-baseline experiments. Notably, the deviation d scales inversely with the particle mass $d \sim 1/m$. Given the finite yet minuscule mass of neutrinos, we anticipate a non-vanishing deflections when considering them in this context. See figure (1) for more understanding. In this work, we will not distinguish between any flavour differences, in principle flavour differences could easily be described by including the spin-gravity deviation as a perturbation to neutrino oscillation and measure the different maximas on the detector. Further, we will only describe neutrinos as Dirac-fermions and not as Majorana-fermions, by which we construct the classical equation of motion up to the order of $\hbar$, without solving the Dirac equation exactly[1]. This work is strutted as the following. In section (2), we review the mathematical prerequisites necessary for understanding the derivation of the deflection, which is discussed in section (3). Subsequently, in section (4), we delve into an explicit calculation concerning geodesics within a Schwarzschild background. Finally, in section (5), we provide a concise summary of the underlying results and offer an outlook on potential research topics.

[1] In some sense this is like the derivation of the Ehrenfest theorem [7], which shows that in some circumstances a quantum system behaves similar to its classical counter part.

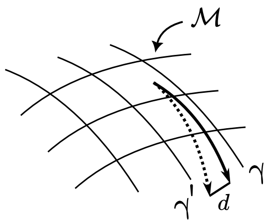

Figure 1: γ is the geodesic and γ' is the non-geodesic trajectory which deviates by a deflection d.

2 Mathematical Prerequisites

In this section, the most relevant mathematical tools to study the motion of fermions in curved space will be reviewed. We assume the reader is familiar with the concept of quantum mechanics and special relativity, but is not familiar of quantum mechanics in a curved background. In the realm of adequate mathematical rigour, we will provide proofs whenever appropriate, in order to enhance the reader's understanding of certain concepts. The following sections, on differential geometry, are mostly based on [8, 9, 10, 11]. In section (2.2) we review the definition of a manifold and their corresponding tangent spaces. We define the covariant derivative in section (2.3), followed by a brief introduction to curvature in section (2.4). In section (2.7) we generalize the Dirac equation to curved space, where we need to study the vierbein formalism more closely, elaborated in section (2.5). Followed by a quick derivation of Fermi-normal coordinates (2.6).

2.1 Convention

This work uses Planck units in where $\hbar = c = G = 1$. $\hbar$ will be later restored to make an adequate Wentzel–Kramers–Brillouin (WKB) expansion in powers of $\hbar$ see section (3). Additionally, we use the mostly plus sign convention, where the Minkowski metric takes the form $\eta = \mathrm{diag}\,(-1, +1, +1, +1)$, we follow the signature convention used in [12] and in [13].

2.2 Manifolds

Manifolds are the natural generalization of flat space, where the signature does not necessary need to be Riemannian[2]. Without mathematical rigour, a manifold is a space which looks locally flat, but has curvature if one looks

[2]Riemannian signature means the metric is positive definite, otherwise it is called pseudo-Riemannian or in the concept of general relativity Lorentzian.

from far apart. Imagine a sphere S^2 which is the easiest concept of a curved Manifold[3]. More mathematically speaking:

Definition 2.1. *An n-dimensional manifold is a Hausdorff topological space. which is connected and has the property that each point has a neighbourhood homeomorphic to some open set in Cartesian n-space. [11, p. 193]*

The above definition carries a lot of information, which is analysed in the following piece by piece. A Topological space is a mathematical space in which the set X are elements of a topology $T \subseteq \mathcal{P}(X)$ [4], with the following axioms;

$$\begin{aligned} &\emptyset, X \in T, \\ &\forall \mathcal{O} \subseteq T \mid \bigcup \mathcal{O} \in T, \\ &\forall U, V \in T \mid U \cap V \in T. \end{aligned} \tag{2.1}$$

The first axiom means, the whole set and the empty set are elements of the topology. The second means, the union of any amount of open sets is element of the topology and the third means, the intersection of finite amount of open sets is element of the topology. A topological space provides a natural notion of continuity, see [10] for some intuitive proofs. A Hausdorff topological space is roughly speaking a topological space, where for any two distinct points, there exists a disjoint neighbourhood. Nearly all interesting spaces in physics are Hausdorff. If this would not be the case, the underlining structure of the describing physics would be very weird. Connected means, that it is possible to construct a continuous path everywhere on the manifold connecting two arbitrary points on $\mathcal{M}$. And finally homeomorphic means, topological isomorphic, which is essentially what we meant previously of "looks locally flat" in the beginning of this section. More mathematically, there exists a map $\Phi \in C^q$ [5] such that $\Phi : O_p \mapsto \mathbf{R}^n$, where $\mathbf{R}^n$ is an open subset of $\mathbb{R}^n$ and there exists a corresponding inverse function $\Phi^{-1} : \Phi(O_p) \mapsto O_p$, where O_p is the open neighbourhood at a point $p \in \mathcal{M}$.

To work more efficiently with manifolds, it is mostly necessary to attach a coordinate system to the manifold. In rare cases it is enough to work with just one coordinate system, in general one uses a set of coordinate systems. Those coordinate system are often called *charts* and the set of all charts is called an *atlas*. Let A be an atlas such that $A = \{(\mathcal{O}_\alpha, \Phi_\alpha)\}$, where the $\mathcal{O}_\alpha$ is called a covering of $\mathcal{M}$, it is intuitively clear that $\bigcap_\alpha \mathcal{O}_\alpha = \mathcal{M}$ otherwise the there is a region which the atlas can not describe.

[3] S^n is the set of all point laying on the boundary of the n-dimensional sphere

[4] $\mathcal{P}(X)$ is the power set, which contains every possible **open** subset of X

[5] q-times differential function

Let $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$ for two open subsets, then the requirement per definitionem states that $\chi_1 := \Phi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \subseteq \mathbb{R}^n$ and $\chi_2 := \Phi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \subseteq \mathbb{R}^n$ with the map $\Psi_{\alpha\beta} := \Phi_\alpha \circ \Phi_\beta^{-1} : \chi_1 \mapsto \chi_2$ being differentiable.

Trivia 2.1. *It is clear that for a compact manifold, there needs to exist at least two charts. Otherwise, the chart would be closed and not part of the topology.*

Vectors and Tangent Space

One might be inclined to define a manifold of n dimensions within a higher-dimensional space, such as $\mathbb{R}^{n+1}$. However, it is generally not feasible to do so. Hassler Whitney [14] demonstrated that an n-dimensional manifold can be represented in $\mathbb{R}^{2n}$, but this approach is cumbersome and impractical. Therefore, it is more common to define manifolds intrinsically, as we have done above. This formalism offers the advantage of providing a natural description of a tangent space without the requirement of an embedding space.

In flat space $\mathbb{R}^n = \mathcal{M}$ the directional derivative of a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ in the direction of a vector $\mathbf{v}$ at a point $p \in \mathbb{R}^n$ is defined as,

$$D_{\mathbf{v}}(f) = \frac{d}{dt} f(p + t\mathbf{v})\bigg|_{t=0}, \tag{2.2}$$

which is in Cartesian coordinates identical to,

$$D_{\mathbf{v}}(f) = v^\mu \frac{\partial f}{\partial x^\mu}\bigg|_p, \tag{2.3}$$

here v^μ are the components of $\mathbf{v}$ in Cartesian coordinates, where we have adopted Einstein sum convention (summation over repeated indices is implied).

And therefore a generalization of $f : \mathcal{M} \mapsto \mathbb{R}$ requires the need of a chart $(\mathcal{O}, \Phi)$ with $\Phi = (x(p)^0, x(p)^1, \ldots x(p)^{n-1})$. Consequently, the derivative of a function f in the direction of a vector $\mathbf{v} \in T_pM$ at a point $p \in \mathcal{M}$ is defined as,

$$D_{\mathbf{v}}(f) = v^\mu \frac{\partial \left[f \circ \Phi^{-1}\right]}{\partial x^\mu} := \mathbf{v}(f). \tag{2.4}$$

In most cases it is not necessary to explicitly chose a mapping Φ so one defines,

$$\frac{\partial f}{\partial x^\mu} := \frac{\partial \left[f \circ \Phi^{-1}\right]}{\partial x^\mu}. \tag{2.5}$$

Since f is arbitrary, the vector $\mathbf{v}$ in the equation (2.4) is defined as a differential operator, at a point $p \in \mathcal{M}$. It is clear, with the above definition the differential $\partial/\partial x^\mu \mid_p =: \partial_\mu \mid_p$ span a vector space T_pM. Given a curve $\gamma : \mathbb{R} \mapsto \mathcal{M}$ it is also possible to define a tangent vector along γ as the directional derivative.

$$\mathbf{v}(f) = \frac{\partial \left[f \circ \Phi^{-1} \circ \Phi \circ \gamma \right]}{\partial \tau} = \frac{\partial \left[f \circ \gamma \right]}{\partial \tau}. \tag{2.6}$$

This needs to hold for arbitrary functions f and so $\mathbf{v} = \dot{\gamma}$.

Note: Defining $\mathbf{v}$ through equation (2.6) does not produce a vector field, $\mathbf{v}$ is only defined along the curve γ.

Definition 2.2. *The vector space T_pM at a point p of a manifold is called a real tangent space, if one has chosen a coordinate system, such that it gets spanned by basis elements ∂_μ.*

Theorem 2.1. *The transformation rule for vector components is,*

$$v^{\mu'} = v^\mu \frac{\partial x^{\mu'}}{\partial x^\mu}.$$

Proof. Let $p = \mathcal{O}_A \cap \mathcal{O}_B$ with the two charts $(\mathcal{O}_A, \Phi_A)$ and $(\mathcal{O}_B, \Phi_B)$ then,

$$f \circ \Phi_A^{-1} = f \circ \Phi_B^{-1} \circ \underbrace{\Phi_B \circ \Phi_A^{-1}}_{:=\Psi} = f \circ \Phi_B^{-1} \circ \Psi.$$

Since Ψ is per definitionem differentiable, the notation of equation. (2.5) is justified. Therefore, we write

$$\mathbf{v}(f) = v^\mu \frac{\partial f}{\partial x^\mu} = v^\mu \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial f}{\partial x^{\mu'}}, \tag{2.7}$$

where the chain rule of differentiation was used. The choice of a basis needs to be arbitrary, so one can also write

$$\mathbf{v}(f) = v^{\mu'} \frac{\partial f}{\partial x^{\mu'}} \tag{2.8}$$

leads to,

$$v^{\mu'} = v^\mu \frac{\partial x^{\mu'}}{\partial x^\mu}. \tag{2.9}$$

□

Note: There exist different conventions on indicating components in different coordinate systems like x'^{μ} and $x^{\mu'}$, where in the first notation the prime is on the tensor component and on the second on the index. Both notations are equivalent, and it is clear from the context how to interpret the notation. In this work we chose the second one, since it is the preferred way for objects which carry indices in different coordinate systems, see section (2.5).

The tangent space T_pM at a point $p \in \mathcal{M}$ is often visualized as a tangent plane at a point p but this image needs to be taken with care, since T_pM is not reaching into space where the manifold is not defined. It should be perceived as an abstract concept that is locally defined at p. The notion of a vector being a differential operator may initially seem peculiar, but this definition does not require a basis to be established in the first place. Although, when doing calculations, we need to define a curtain basis to extract the necessary information. That is why physicist often speak from a vector when they actually mean its components. In occasions, where it is clear from the context, we will adapt this loose vocabulary of calling v^i a vector.

Definition 2.3. *The dual space of T_pM is T_pM^* called cotangent space. [9, p. 40]*

The elements of T_pM^* are differentials, often called covectors which map elements from T_pM to the real numbers. For a function $f : \mathcal{M} \mapsto \mathbb{R}$ the differentials have the following property,

$$df(v)|_p = v(f)|_p \in \mathbf{R} \subset \mathbb{R}, \tag{2.10}$$

with $df \in T_pM^*$ and $v \in T_pM$.

Theorem 2.2. *The transformation rule for covector components are,*

$$v_{\mu'} = v_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}.$$

Proof.

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu = \frac{\partial f}{\partial x^{\mu'}} \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu = \frac{\partial f}{\partial x^{\mu'}} dx^{\mu'}$$

follows,

$$v_{\mu'} = v_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}.$$

□

With the above results, it is possible to construct tensors of higher rank, in particular a tensor of rank (p, q) is a element of $\underbrace{TM \otimes \ldots \otimes TM}_{p\in\mathbb{N}}$ $\underbrace{\otimes TM^* \otimes \ldots \otimes TM^*}_{q\in\mathbb{N}}$, where TM and TM^* are vector bundles.

$$T = T^{\mu^1 \ldots \mu^p}{}_{\nu^1 \ldots \nu^q} dx^{\nu^1} \otimes \ldots \otimes dx^{\nu^q} \otimes \partial_{\mu_1} \otimes \ldots \otimes \partial_{\mu_p}, \tag{2.11}$$

with the following transformation rule.

$$T^{\mu'^1 \ldots \mu'^p}{}_{\nu'^1 \ldots \nu'^q} = \frac{\partial x^{\mu'^1}}{\partial x^{\mu^1}} \cdots \frac{\partial x^{\mu'^p}}{\partial x^{\mu^p}} \frac{\partial x^{\nu^1}}{\partial x^{\nu'^1}} \cdots \frac{\partial x^{\nu^q}}{\partial x^{\nu'^q}} T^{\mu^1 \ldots \mu^p}{}_{\nu^1 \ldots \nu^q}. \tag{2.12}$$

The most important tensor is the metric tensor g of rank $(0, 2)$ which maps two vectors into $\mathbb{R}$

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{2.13}$$

In many cases the symbols $\otimes$ is dropped, and the metric is called ds^2 instead of g.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.14}$$

Since space-time does not differ in the sequence of how two vectors are compared [6], the metric is taken to be symmetric, such that $g_{\mu\nu} = g_{\nu\mu}$. Let the metric act on just one vector $\mathbf{u} = u^\mu \partial_\mu$ gives an element of T_pM^*.

Proof.

$$g_{\mu\nu} dx^\mu \otimes dx^\nu \left(u^\alpha \partial_\alpha\right) = g_{\mu\nu} dx^\mu u^\alpha \delta^\nu_\alpha = g_{\mu\alpha} u^\alpha dx^\mu := u_\mu dx^\mu \in T_pM^*.$$

□

So the metric can be used to go from T_pM to T_pM^* or vice versa, or expressed in components, $g_{\mu\nu}$ is used to lower and raise indices.

2.3 Covariant Derivative

A derivative acting on vectors, or more generally on tensors, is a mathematical operation that measures how a vector changes in spacetime regarding a specific direction. The covariant derivative maps vectors to vectors, or more precisely let $\mathbf{v} \in T_P\mathcal{M}$ and $\mathbf{u} : \mathcal{M} \mapsto T\mathcal{M}$ then $\nabla_\mathbf{v}\mathbf{u}|_P \in T_P\mathcal{M}$ is the directional derivative of $\mathbf{u}$ in the direction of $\mathbf{v}$ at a point $p \in \mathcal{M}$. The covariant derivative $\nabla_\mathbf{v}\mathbf{u}$ fulfils the following axioms;

[6] In example this is clearly not the case for a complex Hilbert-space.

- Linearity in $\mathbf{v} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$
  $\nabla_{\mathbf{v}}\mathbf{u} = \alpha\nabla_{\mathbf{v}_1}\mathbf{u} + \beta\nabla_{\mathbf{v}_2}\mathbf{u}$
- Additivity in $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$
  $\nabla_{\mathbf{v}}\mathbf{u} = \nabla_{\mathbf{v}}\mathbf{u}_1 + \nabla_{\mathbf{v}}\mathbf{u}_2$
- Leibniz-rule
  $\nabla_{\mathbf{v}}\mathbf{u}\otimes\mathbf{w} = (\nabla_{\mathbf{v}}\mathbf{u})\otimes\mathbf{w} + \mathbf{u}\otimes(\nabla_{\mathbf{v}}\mathbf{w})$
- (Metric compatibility)
  $\nabla_{\mathbf{v}}\mathbf{g} = 0\ \forall\mathbf{v}$

The metric compatibility is an additional condition, which has practical usage, which we will see later in following sections. To write the covariant derivative in index notation, we can not write $\nabla_\mu = \partial_\mu$, because the expression $\partial_\mu V^\nu$ has in general no tensorial behaviour. If a general coordinate transformation $\partial x^\alpha/\partial x'^\nu$ acts on $\partial_\mu V^\nu$, terms of the second order derivative arise, which destroy the tensorial behaviour, which means the result would not be an element of any tangent space. Per definitionem $\nabla_{\mathbf{v}}\mathbf{u}|_P \in T_P\mathcal{M}$, ∇ requires a more complicated structure, which contains a connection Γ. It is common to apply one additional constrain to make the connection Γ unique. This constrain is essential if we enforce a torsion free manifold, this means mathematically $T(\mathbf{u},\mathbf{v}) := \nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} - [\mathbf{u},\mathbf{v}] = 0$, the latter is the so-called Lie bracket [7]. This connection is known as the Levi-Civita connection. The covariant derivative can easily be rewritten in index notation by using $\mathbf{u} = u^\alpha(x)\partial_\alpha$ and $\mathbf{v} = v^\beta(x)\partial_\beta$. For the derivative on basis vectors it makes sense to define a connection in components in the following way $\nabla_{(\partial_\beta)}(\partial_\alpha) = \Gamma^\mu{}_{\alpha\beta}\partial_\mu$ Inserting of $\mathbf{u}$ and $\mathbf{v}$ into $\nabla_{\mathbf{v}}\mathbf{u}|_P$ gives,

$$\begin{aligned}\nabla_{v^\beta\partial_\beta}(u^\alpha(x)\partial_\alpha) = \overbrace{v^\beta\nabla_{(\partial_\beta)}(u^\alpha(x)\partial_\alpha)}^{\mathbf{I}} &= \overbrace{v^\beta\left(\nabla_{(\partial_\beta)}(u^\alpha(x))\partial_\alpha + u^\alpha(x)\nabla_{(\partial_\beta)}\partial_\alpha\right)}^{\mathbf{II}}\\ &= \underbrace{v^\beta\left(\partial_\beta u^\alpha(x)\partial_\alpha + u^\alpha(x)\Gamma^\mu{}_{\alpha\beta}\partial_\mu\right)}_{\mathbf{III}}\\ &= v^\beta\left(\partial_\beta u^\alpha(x) + u^\mu(x)\Gamma^\alpha{}_{\mu\beta}\right)\partial_\alpha. \qquad (2.15)\end{aligned}$$

I is obtained, by using linearity of $T_P\mathcal{M}$, **II** by the Leibniz rule of the derivation. **III** by substituting Γ and using the fact, that a covariant derivation of scalars reduces to conventional derivations and finally, the last expression is

[7]The Lie derivative of a vector field $\mathbf{u}$ with respect to another vector field $\mathbf{v}$ is referred to as the "Lie bracket" of $\mathbf{u}$ and $\mathbf{v}$. This mathematical concept is often denoted as $[\mathbf{u},\mathbf{v}] = \frac{\partial}{\partial\mathbf{u}}\frac{\partial}{\partial\mathbf{v}} - \frac{\partial}{\partial\mathbf{v}}\frac{\partial}{\partial\mathbf{u}}$ is only 0 if $\mathbf{u}$ and $\mathbf{v}$ form a coordinate basis.

obtained by renaming the indices. This expression is valid for all v^β, therefore it follows for the covariant derivation in index notation

$$\nabla_\beta u^\alpha = \partial_\beta u^\alpha + \Gamma^\alpha{}_{\mu\beta} u^\mu. \tag{2.16}$$

Because of $\nabla_\beta (u^\alpha u_\alpha) = \partial_\beta (u^\alpha u_\alpha)$ follows immediately

$$\nabla_\beta u_\alpha = \partial_\beta u^\alpha - \Gamma^\mu{}_{\alpha\beta} u_\mu. \tag{2.17}$$

Repeating this for $\nabla_\beta u^\mu u^\nu \ldots$ one gets the behaviour of the covariant derivative acting on a general tensor $T^{\mu_1\mu_2\cdots\mu_l}{}_{\nu_1\nu_2\ldots\nu_p}$. With the above defined axioms, the Levi-Civita connection can be written in components as (see Caroll [8] for a detailed proof),

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\kappa} \left(\frac{\partial g_{\nu\kappa}}{\partial x^\mu} + \frac{\partial g_{\mu\kappa}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa} \right). \tag{2.18}$$

This object is also well known under the name Christoffel-symbol. Based on the definitions provided earlier, we can now establish a formal definition for a geodesic. A geodesic represents a generalized straight line within a curved background. This concept arises from relaxing the third Euclid axiom [15], which asserts that parallel lines remain parallel. However, in curved space, this axiom no longer holds true. Consequently, a more refined approach to defining parallel lines becomes necessary. The physical interpretation of a geodesic is that it represents the path followed by a freely falling test particle. This holds true when the acceleration along this path becomes zero.

Definition 2.4. *Let $\boldsymbol{u} : \mathcal{M} \mapsto TM$ be a vector field, then $\boldsymbol{u}$ is a tangent vector field to a geodesic if $\nabla_{\boldsymbol{u}} \boldsymbol{u} = 0$.*

Solving $\dot{\gamma} = \mathbf{u}$ gives the geodesic γ. In index notation, this can be rewritten as,

$$u^\mu \nabla_\mu u^\nu = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta = 0. \tag{2.19}$$

The same result could be achieved by extremising[8] the length L of trajectory such that $\delta L = 0$ where L is

$$L[x] = \int ds = \int d\tau \sqrt{u^\mu u^\nu g_{\mu\nu}(x)}. \tag{2.20}$$

[8]Timelike geodesics are curves of maximum proper time τ since a timelike geodesic can be approximated picewise by null geodesics, therefore it is clear that a timelike geodesic can not be a curve of minimum proper time [8, p. 72].

Trivia 2.2. *This action is a one dimensional version of the Nambu-Goto action[16] a two-dimensional version with easier mathematical handling is the Polyakov action[17] which is mostly used in string theory to describe bosonic strings.*

Theorem 2.3. *Solutions of the geodesic equation are unique.*

Proof. for a detailed proof, see [18] □

2.4 Curvature and the Schwarzschild Solution

So far, a rigorous metamathematical treatment of curvature remains pending. However, we can gain insight by considering the behaviour of vectors on a curved surface. Imagine parallel-transporting a vector along different paths on such a surface (for instance, a sphere). As we vary the path, the vector's orientation changes. The discrepancy between these parallel-transported vectors along distinct paths defines what we refer to as curvature. When we take infinitesimal paths, we arrive at the Riemann curvature tensor $R(\mathbf{x}, \mathbf{y}) : T_pM \mapsto T_pM$, where $\mathbf{x}$ and $\mathbf{y}$ are vector fields [9, p. 244].

$$R(\mathbf{x}, \mathbf{y}) = [\nabla_{\mathbf{x}}, \nabla_{\mathbf{y}}] - \nabla_{[\mathbf{x},\, \mathbf{y}]}. \tag{2.21}$$

Choosing a coordinate system and switching to index notation the Riemann-tensor written in components is then [9],

$$[\nabla_\mu, \nabla_\nu]v^\rho = R^\rho{}_{\mu\nu\gamma}v^\gamma. \tag{2.22}$$

Expending the commutator, leads to the expression

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho{}_{\nu\sigma} - \partial_\nu\Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}. \tag{2.23}$$

Another interesting object is the Ricci-tensor defined by,

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}. \tag{2.24}$$

Additionally the Ricci scalar is defined by,

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{2.25}$$

Note: When the Ricci scalar R or the Ricci tensor $R_{\mu\nu}$ vanish, it does not necessarily imply that the underlying space is flat. A notable example illustrating this concept is the Schwarzschild spacetime.

[9] When $\mathbf{x}$ and $\mathbf{y}$ span a coordinate system the Lie bracket vanishes.

To derive the Einstein equations, we construct the simplest possible action which includes at least second order derivatives of the metric [10].

$$S_{EH} = \frac{-1}{2\kappa} \int d^4x\sqrt{-g}R + S_{matter}, \tag{2.26}$$

also known as the Einstein-Hilbert action, where κ is a prefactor defined by the Newtonian limit, S_{matter} is an additional matter term, $d^4x\sqrt{-g}$ is the invariant volume form. For simplicity, we have also neglected, the cosmological constant Λ. Varying the action $\delta S_{EH}/\delta g_{\mu\nu} = 0$ produces the Einstein equations. (see Appendix (A.1) for a detailed derivation) The Einstein equations are then,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.27}$$

A spherical symmetric vacuum solution of (2.27) which solves $R_{\mu\nu} = 0$ known as the Schwarzschild solution is given by [19],

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2, \tag{2.28}$$

with the substitution $f(r) = 1 - 2M/r$. $d\Omega^2$ is the S^2 line element. The quantity $2M$ is also called the Schwarzschild radius, which is the event horizon of a Schwarzschild black hole.

Note: At $r = 2M$, there exists no true singularity; rather, it is a pure coordinate singularity. A physical singularity arises at $r = 0$.

2.5 Vierbein Formalism

Since general relativity is defined on a manifold $\mathcal{M}$, it is possible to choose a coordinate system with Lorentzian signature, that is locally flat. In order to manifest these circumstances in a mathematical formalism, we define the *vierbein* or *tetrad* [20, 8] by, [11]

$$g_{\mu\nu}(x)e_a{}^{\mu}(x)e_b{}^{\nu}(x) = \eta_{ab}. \tag{2.29}$$

[10] It is not enough to construct an action, which only includes first order derivatives of the metric, since those can be set equal to zero, see section (2.6)

[11] Alsing et al [21] tried to calculate the deviation of a Dirac particle in curved space, but made a major mathematical mistake. Although, their introduction to the vierbein formalism is well-defined. It is worth noting, the mistake which is explained in Appendix (A.4), such that ongoing researches will not make the same mistake.

Equation (2.29) is essentially a coordinate transformation of the metric $g_{\mu\nu}(x)$. Here η_{ab} is the Minkowski metric. ${e_a}^{\mu}(x)$ are the so-called tetrads. To distinguish flat from general coordinates, Greek indices $\mu, \nu, \ldots$ are used for general coordinates and Latin indices $a, b, c, \ldots$ for tetrad coordinates. In order to not overload the notation, x dependencies are suppressed, and we write $g_{\mu\nu} = g_{\mu\nu}(x)$ and ${e_a}^{\mu} = {e_a}^{\mu}(x)$. A relation for the inverse metric can be defined analogously,

$$g^{\mu\nu} {\tilde{e}_{\mu}}{}^{a} {\tilde{e}_{\nu}}{}^{b} = \eta^{ab}. \tag{2.30}$$

e and $\tilde{e}$ are generally different objects, but since the index position of the Greek or Latin index clearly defines whether it is either e or $\tilde{e}$ the " ˜ " is omitted, but care must be taken, when expressions are explicitly written in components; since e.g. ${e_2}^{2} \neq {\tilde{e}_2}{}^{2}$. Further, due to

$$g^{\mu\nu} g_{\nu\alpha} = \delta^{\mu}_{\alpha} \tag{2.31}$$

it follows immediately,

$${e_a}^{\mu} {e_{\mu}}^{b} = \delta^{b}_{a} \ , \quad {e_a}^{\nu} {e_{\mu}}^{a} = \delta^{\nu}_{\mu}. \tag{2.32}$$

A general tensor ${T^{\mu_1\mu_2\ldots\mu_l}}_{\nu_1\nu_2\ldots\nu_p}$ can be written in flat coordinates by the following,

$${T^{a_1a_2\ldots a_l}}_{b_1b_2\ldots b_p} = {e_{\mu_1}}^{a_1} {e_{\mu_2}}^{a_2} \ldots {e_{\mu_l}}^{a_l} {e_{b_1}}^{\nu_1} {e_{b_2}}^{\nu_2} \ldots {e_{b_p}}^{\nu_p} {T^{\mu_1\mu_2\ldots\mu_l}}_{\nu_1\nu_2\ldots\nu_p} \tag{2.33}$$

and vice versa.

Since the Minkowski metric is Lorentz invariant, there are infinite tetrads distinguishable only by Lorentz transformations, it is therefore possible to transform a tensor like,

$${T^{a'_1a'_2\ldots a'_l}}_{b'_1b'_2\ldots b'_p} = {\Lambda_{a_1}}^{a'_1} {\Lambda_{a_2}}^{a'_2} {\Lambda_{a_l}}^{a'_l} {\Lambda_{b'_1}}^{b_1} {\Lambda_{b'_2}}^{b_2} \ldots {\Lambda_{b'_p}}^{b_p} {T^{a_1a_2\ldots a_l}}_{b_1b_2\ldots b_p}, \tag{2.34}$$

where Λ is a Lorentz-transformation.

A very interesting tetrad is the Fermi-tetrad, defined by setting ${e_0}^{\mu} = u^{\mu}$ where u^{μ} is a tangent vector along a geodesic. Additionally, we require that all tetrads are parallel transported along u^{μ}, which means, $u^{\mu} \nabla_{\mu} {e_a}^{\nu} = 0$. We will follow along Marck's construction [22], by using the simpler Schwarzschild geometry (2.28) instead of a Kerr geometry used in the original paper. Although, this construction is only valid along the geodesic, it will still play a crucial role by constructing Fermi-normal coordinates, see section (2.6). But first we need a timelike geodesic solution for the Schwarzschild geometry. This is achieved by using (2.28) deriving the Christoffel-Symbols with (2.18) and solving equation (2.19) with the constraint $u^\mu u_\mu = -1$ gives then for the equatorial plane at $\theta = \pi/2$

$$u^t := \frac{r\sqrt{2E+1}}{-2M+r}, \tag{2.35}$$

$$u^r := \frac{dr}{d\tau} = \pm\sqrt{2E + \frac{2M}{r} - \frac{L^2}{r^2} + \frac{2ML^2}{r^3}}, \tag{2.36}$$

$$u^\phi := \frac{d\phi}{d\tau} = \pm\frac{L}{r^2}, \tag{2.37}$$

$$u^\theta := \frac{d\theta}{d\tau} = 0. \tag{2.38}$$

For a detailed derivation, see [23], here M is the constant black hole mass in our case the Earth mass, E is the constant particle energy and L is the constant scalar valued angular momentum.

Definition 2.5. *The antisymmetric tensor $f_{\mu\nu} = -f_{\nu\mu}$ which fulfils $\nabla_\rho f_{\mu\nu} + \nabla_\nu f_{\mu\rho} = 0$ is called Killing-Yano tensor.*

Theorem 2.4. *The vector defined by $\lambda^\mu = f^\mu{}_\nu u^\nu$ is orthogonal to the tangent vector u^μ defined along a geodesic, and parallel transported along u^μ where $f_{\mu\nu}$ is a Killing-Yano tensor [24].*

Proof. $u_\mu f^\mu{}_\nu u^\nu = u^\mu u^\nu f_{\mu\nu}$, $u^\mu u^\nu$ is symmetric and $f_{\mu\nu}$ per derfinitionem antisymmetric, follows $u^\mu u^\nu f_{\mu\nu} = 0$. $u^\mu \nabla_\mu \lambda^\nu = u^\mu \nabla_\mu \left(f^\nu{}_\alpha u^\alpha\right) \Rightarrow$
$u^\mu \nabla_\mu \left(f_{\nu\alpha} u^\alpha\right) = u^\alpha u^\mu \nabla_\mu \left(f_{\nu\alpha}\right) + f_{\nu\alpha} \underbrace{u^\mu \nabla_\mu u^\alpha}_{=0} = -u^\mu u^\alpha \nabla_\alpha f_{\nu\mu}$ follows immediately $u^\mu \nabla_\mu \lambda^\nu = 0$. □

One possible solution for the Killing-Yano tensor for the Schwarzschild geometry is

$$f = r^3 \sin(\theta) d\theta \wedge d\phi. \tag{2.39}$$

Here, the wedge product is defined as $dx \wedge dy = -dy \wedge dx$. This gives for the equatorial plane $\theta = \pi/2$

$$e_2{}^\mu = \frac{1}{r}\delta_2^\mu, \tag{2.40}$$

where $e_2{}^\mu$ is already normalized to unity. This solution could be found by simply guessing, since we are interested in equatorial solutions, but it is always worth noting a more rigorous derivation.

A third vector can be found by swapping the first two components of u^μ and adding a proper factor such that $u^\mu e_{1\mu}$ vanishes.

$$\hat{e_1}^{\,\mu} = \frac{1}{\sqrt{\underbrace{u_\phi^2 r^2}_{=L^2/r^2} + 1}} \left[\frac{u_r}{f(r)} \delta_0^\mu + u_t f(r) \delta_1^\mu \right], \tag{2.41}$$

where we used $u_\phi = L/r^2$ and finally for $\hat{e_3}^{\,\mu}$,

$$\hat{e_3}^{\,\mu} = \frac{1}{\sqrt{1 + r^2/L^2}} \left[u_t \delta_0^\mu + u_r \delta_1^\mu + \frac{r^2 + L^2}{r^2 L} \delta_3^\mu \right]. \tag{2.42}$$

$\hat{e_2}$ and $\hat{e_3}$ are per constructionem orthogonal to each other, but not properly parallel transported along $\mathbf{u}$. Therefore, Marck suggested, creating a superposition, by the following

$$\underbrace{\begin{pmatrix} e_1^\mu \\ e_3^\mu \end{pmatrix}}_{:=e} = \underbrace{\begin{pmatrix} \cos \text{ש}(x) & -\sin \text{ש}(x) \\ \sin \text{ש}(x) & \cos \text{ש}(x) \end{pmatrix}}_{:=A} \underbrace{\begin{pmatrix} \hat{e_1^\mu} \\ \hat{e_3^\mu} \end{pmatrix}}_{:=\hat{e}}. \tag{2.43}$$

Where we have introduced a spacetime dependent rotation angle ש. For a visualisation of ש , see figure (2).

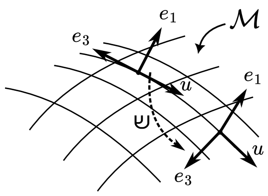

Figure 2: ש describes the rotation of e_1 and e_3 along u.

We introduce the operator $\mathcal{L}$ so we can write the expression, $u^\mu \nabla_\mu e_a{}^\nu = 0$ as $\mathcal{L}e = 0$.

$$0 = \mathcal{L}e = \mathcal{L}\left(A\hat{e}\right) = \left(\mathcal{L}A\right)\hat{e} + A\left(\mathcal{L}\hat{e}\right), \tag{2.44}$$

$$\Rightarrow \left(\mathcal{L}A\right)\hat{e} = -A\left(\mathcal{L}\hat{e}\right), \tag{2.45}$$

$$\mathcal{L}A = u^\mu \nabla_\mu A = u^\mu \partial_\mu A := B u^\mu \partial_\mu \text{ש} = B\dot{\text{ש}}. \tag{2.46}$$

Which leads to,

$$\dot{\text{ש}}\hat{e} = -\underbrace{B^{-1}A}_{=\aleph}\mathcal{L}\hat{e}. \tag{2.47}$$

Where we have defined,

$$\aleph = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.48}$$

This gives,

$$\dot{\boldsymbol{\varpi}} = \frac{-L\sqrt{2E+1}}{L^2+r^2}. \tag{2.49}$$

Note: Only $\dot{\boldsymbol{\varpi}}$ respectively $d\boldsymbol{\varpi}/dr = \dot{\boldsymbol{\varpi}}/u_r$ is defined by equation (2.49). Without any starting conditions, a solution $\boldsymbol{\varpi}$ is not unique, since it is always possible to add a constant and still arrive at (2.49). In order to determine the value of the integration constant, we consider the radial case where the angular momentum, denoted by L, is zero. In this specific scenario, the derivative of the rotation angel $\boldsymbol{\varpi}$ vanishes.

Radial motion corresponds to a scenario where the particle remains stationary with respect to the coordinates ϕ and θ. Therefore, we know that ${e_3}^{\mu} = 1/r\delta_3^{\mu}$ this comes from taking the square of the metric. But this is only satisfied if $\boldsymbol{\varpi} = 0$. By continuous variation of a radial geodesic, $\boldsymbol{\varpi}$ needs to stay continuous. Therefore, we claim that the integration constant is fixed to 0 for arbitrary geodesics.

2.6 Fermi-Normal Coordinates

In Riemann-normal coordinates, the Levi-Civita connection Γ vanishes at a specific point, $p \in \mathcal{M}$. Fermi generalized this concept in his dissertation, where he showed that is in principle possible to make a coordinate transformation along a curve, $\gamma : \mathbb{R} \mapsto \mathcal{M}$ such that the connection vanishes. O'Raifeartaigh [25] proved that this concept does not generalize to surfaces or even higher dimensional surfaces. Therefore, we keep an eye on curves. Fermi coordinates have a compelling physical interpretation, which describe a free-falling observer, where in a sufficient small region space looks flat, this aligns seamlessly with the definition of a manifold. Manasse et al. [26] derived the metric in Fermi coordinates in the neighbourhood of a geodesic up to the order of x^2. This expansion is valid if $||x^i|| \ll r$. In our context, if the deviation $d \ll R_0$, where R_0 is the Earth's radius. Fermi coordinates expressed by an expansion in the neighbourhood around a geodesic are often called Fermi-normal coordinates. In this section, we will give a brief summary of those results.

Let $\gamma : \mathbb{R} \mapsto \mathcal{M}$ be a geodesic such that $\gamma(0) := P_0$ is the starting point. Further let

$$\dot{\gamma}|_{\tau=0} := \mathbf{e}_0(\tau = 0), \tag{2.50}$$

be a unit tangent vector along the geodesic. On the geodesic, at $\tau = x^0$ we define a new orthogonal unit vector to $\mathbf{e}_0(x^0)$ as,

$$\mathbf{s} = \alpha^i \mathbf{e}_i(x^0), \qquad a^i a_i = 1. \tag{2.51}$$

This gives a new space like geodesic in the parameter λ. For a visualisation, see figure (3).

$$\gamma(x^0, \alpha^i, \lambda) \in \mathcal{M} \quad s.t \ \frac{\partial \gamma}{\partial \lambda} = \mathbf{s}. \tag{2.52}$$

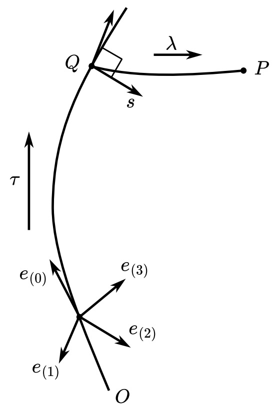

Figure 3: O is defined as $\gamma(0,0,0)$, $Q = \gamma(\tau, 0, 0)$ and $P = \gamma(x_0, \alpha^i, \lambda)$.

Theorem 2.5. $\gamma(\tau, c\alpha^i, \lambda) = \gamma(\tau, \alpha^i, c\lambda)$ *for* $c \in \mathbb{R}$

Proof. Since γ is a space like geodesic in the parameter λ starting at the central geodesic, it must satisfy the equation $\nabla_{\gamma'}\gamma' = 0$ with $\gamma' = \partial\gamma(\tau, \alpha^i, \lambda)/\partial\lambda$. Rescaling $\lambda \to c\lambda$ gives, $\nabla_{1/c\ \gamma'} 1/c\ \gamma' = 1/c^2 \nabla_{\gamma'}\gamma' = 0$ therefore the rescaled trajectory is also a geodesic. The starting condition satisfies $\gamma'(\tau, c\alpha^i, \lambda)|_{\lambda=0} = c\mathbf{s}(\tau) = c\gamma'(\tau, \alpha^i, \lambda)|_{\lambda=0} = \gamma'(\tau, \alpha^i, c\lambda)|_{\lambda=0}$. Where on the last step we used the inverse chain rule. Finally, with the uniqueness theorem for geodesics (T.2.3) this proof is complete. □

With the above results, and given a coordinate system ϕ, it is now possible to construct a coordinate transformation from Fermi-coordinates to non-Fermi-coordinates $y^{\mu'}(x^\alpha) = \Phi(\gamma(x^0, x^i, 1))^{\mu'}$. It needs to be shown that the inverse transformation exists. This is easily done by showing that the Jacobian is regular. Which essentially means, that we need to show, that

$$J = \det \frac{\partial y^{\mu'}}{\partial x^\alpha} \tag{2.53}$$

is different from 0. Since per constructionem the vectors $\mathbf{e}_\mu(\tau)$ form an orthogonal vierbein, they are linearly independent and so $J \neq 0$. By continuity around the neighbourhood the inverse transformation exists. The orthogonality of the vierbein essentially says, that along the geodesic G the metric reduces to the Minkowski metric.

$$g_{\mu\nu}|_G = g_{\mu'\nu'}\frac{\partial y^{\mu'}}{\partial x^\mu}\frac{\partial y^{\nu'}}{\partial x^\nu}\bigg|_G = <\mathbf{e}_\mu, \mathbf{e}_\nu>|_G = \eta_{\mu\nu}, \tag{2.54}$$

where $<,>$is the inner product of two vectors. By writing $\gamma(\tau, \alpha^i, \lambda)$ in Fermi coordinates as,

$$x^\mu = \Phi_F(\gamma(\tau, \alpha^i, \lambda)). \tag{2.55}$$

The geodesic equation (2.19) needs to hold.

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{2.56}$$

gives with equation (2.51),

$$\Gamma^\mu{}_{ij}\alpha^i\alpha^j = 0. \tag{2.57}$$

This needs to hold for every λ so

$$\Gamma^\mu{}_{ij}|_\gamma = 0. \tag{2.58}$$

Additionally the vector $\mathbf{v}$ respectively vectors $\mathbf{e}_i$ are parallel transported along $\dot\gamma^\mu = u^\mu$ and using $u^\mu = \delta^\mu_0$ in Fermi coordinates

$$u^\mu \nabla_\mu e_i{}^\nu = 0. \tag{2.59}$$

$$\underbrace{u^\mu\partial_\mu e_i{}^\alpha|_G}_{\partial_0 e_i{}^\alpha = 0} + u^\mu\Gamma^\alpha{}_{\mu\beta}e_i{}^\beta|_G = 0, \tag{2.60}$$

$$\Gamma^\nu{}_{0\alpha}e_i{}^\alpha|_G = 0, \tag{2.61}$$

again this need to hold for $\lambda = 0$ and due to symmetries follows,

$$\Gamma^\nu{}_{\alpha\beta}|_G = 0, \tag{2.62}$$

which implies $\partial g_{\mu\nu}/\partial x^\alpha|_G = 0$. To derive a Taylor expansion for the metric, $g_{\mu\nu}$ around G we need to evaluate second order derivatives. But instead of calculating $\partial_\beta\partial_\alpha g_{\mu\nu}$ we evaluate $\partial_\sigma\Gamma^\mu{}_{\alpha\beta}$. By using (2.23) we get,

$$R^\rho{}_{\sigma\mu\nu}|_G = \partial_\mu\Gamma^\rho{}_{\nu\sigma}|_G - \partial_\nu\Gamma^\rho{}_{\mu\sigma}|_G\,. \tag{2.63}$$

It is clear that $\partial_0\Gamma^\nu{}_{\alpha\beta}|_G = 0$ by construction, because the vierbein is defined along the geodesic and should not change, follows immediately

$$R^\rho{}_{\sigma\mu 0}|_G = \partial_\mu\Gamma^\rho{}_{0\sigma}|_G\,. \tag{2.64}$$

$\gamma(\tau, \alpha^i, 1)$ describes a family of geodesics. For $\gamma(\tau, 0)$ we arrive at the central timelike geodesic, a nearby geodesic could be described by $\gamma(\tau, \beta^i)$ where β^i is fixed. Likewise, $\gamma(x_0, \alpha^i)$ for fixed x_0, describing spacelike geodesics. We want a measure of how a nearby geodesic diverges from the central geodesic, which indirectly is a measure of curvature. An interesting observation is done in flat space, where geodesics may diverge from another (imagine two non-parallel straight lines). Therefore, we are only interested of the acceleration of this divergence.

$$\frac{D^2 \mathbf{s}_{(i)}}{d\lambda^2} = \nabla_{\mathbf{u}} \nabla_{\mathbf{u}} \mathbf{s}_{(i)}, \tag{2.65}$$

where $\mathbf{u} = \partial\gamma(x_0, \alpha^i, \lambda)/\partial\lambda$ with fixed α^i and $\mathbf{s}_{(i)} = \partial\gamma(x_0, \alpha^i, \lambda)/\partial\alpha^i$ with fixed λ. Since $\mathbf{u}$ and $\mathbf{s}$ are both generated from the same function γ, their Lie bracket vanishes. With equation (2.21) we can rewrite equation (2.65) as,

$$\frac{D^2 \mathbf{s}_{(i)}}{d\lambda^2} = R(\mathbf{u}, \mathbf{s}_{(i)})\mathbf{u}. \tag{2.66}$$

In index notation, the geodesic deviation equation then reads,

$$\frac{D^2 s_i^\mu}{d\lambda^2} = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho s_i^\sigma, \tag{2.67}$$

expanding the left term gives,

$$\begin{aligned} \frac{d^2 s_i^\mu}{d\tau^2} + 2\frac{ds_i^\sigma}{d\tau} u^\alpha \Gamma^\mu{}_{\sigma\alpha} + u^\alpha u^\beta s_i^\sigma \left(\partial_\beta \Gamma^\mu{}_{\sigma\alpha} + \Gamma^\tau{}_{\sigma\alpha}\Gamma^\mu{}_{\tau\beta} - \Gamma^\mu{}_{\sigma\tau}\Gamma^\tau{}_{\alpha\beta}\right) \\ = R^\mu{}_{\alpha\sigma\beta} s_i^\sigma u^\alpha u^\beta. \end{aligned} \tag{2.68}$$

In Fermi coordinates, $u^\mu = \delta_i^\mu \alpha^i$ and $s^\mu = \lambda\delta_i^\mu$. Evaluating equation (2.68) along the geodesic and Taylor expending $\Gamma^\mu{}_{\sigma\alpha}$ up to linear order gives,

$$\left(3\partial_k \Gamma^\mu{}_{ij} + R^\mu{}_{jik}\right) \alpha^i \alpha^k = 0. \tag{2.69}$$

This needs to hold even for $\alpha^i = 0$ and by exploiting the symmetries of the Reimann tensor one gets,

$$\partial_k \Gamma^\mu{}_{ij} = -\frac{1}{3}\left(R^\mu{}_{ijk} - R^\mu{}_{jik}\right)\Big|_G. \tag{2.70}$$

By using the definition of the Christoffel connection (2.18) we obtain,

$$\partial_\alpha g_{\mu\nu} = g_{\mu\sigma}\Gamma^\sigma{}_{\nu\alpha} + g_{\sigma\nu}\Gamma^\sigma{}_{\mu\alpha}, \tag{2.71}$$

taking one more derivative and insert the above results gives finally,

$$g_{00} = -1 + R_{0i0j}|_G \, x^i x^j, \tag{2.72a}$$
$$g_{0i} = -\tfrac{2}{3} \, R_{0jik}|_G \, x^j x^k, \tag{2.72b}$$
$$g_{ij} = \delta_{ij} - \tfrac{1}{3} \, R_{ikkl}|_G \, x^k x^l. \tag{2.72c}$$

To evaluate the Riemann tensor along the geodesic G in Fermi coordinates, we use the tetrads defined in section (2.5) and write,

$$R_{\alpha\beta\gamma\delta}|_G = e_\alpha{}^{\alpha'} e_\beta{}^{\beta'} e_\gamma{}^{\gamma'} e_\delta{}^{\delta'} \, R_{\alpha'\beta'\gamma'\delta'}|_G \, . \tag{2.73}$$

This expression is most easily evaluated with the help of a computer algebra system.

2.7 Dirac Equation in Curved Space

In 1928 Dirac [27] derived a relativistic wave equation which not only solved the negative probability problem, but also naturally embeds the concept of spin, by considering $\psi(x)$ not as a scalar valued field, but rather as a complex vector valued field. Which is nowadays called spinor valued field, to highlight the different transformation properties of a vector used in special relativity. To generalize the Dirac equation to curved space, we can not naively replace partial derivatives by covariant derivatives ∇. Since the action of ∇ is only well-define on vectors and not on spinors. To study the concept of covariant derivatives acting on spinors, we need to develop the vierbein formalism from section (2.5) more closely[20].

Since the Levi-Civita connection is not a tensor, we can not simply transform it by just transforming every index by going into a specific coordinate system, therefore we make the ansatz,

$$\nabla_\beta u^b = \partial_\beta u^b + \omega_{\beta a}{}^b u^a, \tag{2.74}$$

where $\mathbf{u}$ is an arbitrary vector field and ω the so called spin-connection. We know by construction that (2.74) transforms like a tensor, so we can rewrite it as,

$$\nabla_\beta u^a = e_\alpha{}^a \nabla_\beta u^\alpha \tag{2.75}$$

expanding the covariant derivative on both sides gives,

$$\partial_\beta u^a + \omega_{\beta b}{}^a u^b = e_\alpha{}^a \left(\partial_\beta u^\alpha + \Gamma^\alpha{}_{\mu\beta} u^\mu \right). \tag{2.76}$$

The right side becomes,

$$= \partial_\beta(\underbrace{u^\alpha e_\alpha{}^a}_{u^a}) - u^\alpha \partial_\beta(e_\alpha{}^a) + e_\alpha{}^a \Gamma^\alpha{}_{\mu\beta} u^\mu. \tag{2.77}$$

And therefore,

$$-u^b e_b{}^\alpha \partial_\beta(e_\alpha{}^a) + e_\alpha{}^a \Gamma^\alpha{}_{\mu\beta} u^b e_b{}^\mu = \omega_{\beta b}{}^a u^b. \tag{2.78}$$

This needs to hold for all u^b, so the spin-connection reads

$$\omega_{\mu ab} = \eta_{bc} \Gamma^\alpha{}_{\nu\mu} e_a{}^\nu e_\alpha{}^c - \eta_{bc} e_a{}^\alpha \partial_\mu(e_\alpha{}^c), \tag{2.79}$$

or more compact,

$$\omega_{\mu ab} = e_{a\beta} \nabla_\mu e_b{}^\beta. \tag{2.80}$$

Theorem 2.6. *ω_μ transforms like a two-form.*

Proof. Using the metric compatibility and the definition of the tetrads, we write $0 = \nabla_\mu \eta_{ab} = g_{\mu\nu} \nabla_\mu e_a{}^\mu e_b{}^\nu$, and use the Leibniz rule of derivates, follows $0 = g_{\mu\nu} e_b^\nu \nabla_\mu e_a{}^\mu + g_{\mu\nu} e_a{}^\mu \nabla_\mu e_b^\nu$ and therefore $\omega_{\mu ab} = -\omega_{\mu ba}$. □

Finally, the covariant derivative on a spinor $\Psi(x)$ is as follows,

$$D_\beta \Psi = \partial_\beta \Psi - \frac{1}{2} \omega_\mu{}^{ab} \Gamma_{(j)}(M_{ab}) \Psi, \tag{2.81}$$

where $\Gamma_{(j)}$ is the spin-j representation of the Lorentz algebra $\mathfrak{su}(3,1)$ with $M_{ab} \in \mathfrak{su}(3,1)$ satisfying the commutator relation[12],

$$[M^{ab}, M^{cd}] = -M^{ad}\eta^{bc} + M^{bd}\eta^{ac} - M^{bc}\eta^{ad} + M^{ac}\eta^{bd}. \tag{2.82}$$

If the spinor Ψ would have spin-1, it would transform like a Lorentz vector v^a and the corresponding representation would look like,

$$\Gamma_{(1)}(M^{ab}) = (M^{ab})_{cd} = \delta^b{}_c \delta^a{}_d - \delta^a{}_c \delta^b{}_d. \tag{2.83}$$

By plugging equation (2.83) into to the definition (2.81) one would get back (2.74).

If Ψ is a spin-1/2 spinor, the corresponding Lorentz representation is

$$\Gamma_{(1/2)}(M^{ab}) = \frac{1}{4}[\gamma^a, \gamma^b] := \frac{1}{2}\sigma^{ab}. \tag{2.84}$$

[12]The overall minus sign comes from the +2 signature.

It is worth introducing the so called spinor-connection, which reads

$$\Gamma_\mu = \frac{1}{8}\omega_{\mu ab}[\gamma^a, \gamma^b], \tag{2.85}$$

or due to the antisymmetry of ω, we can write

$$\Gamma_\mu = \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b. \tag{2.86}$$

With those definitions, the Dirac equation in curved space finally reads

$$\boxed{(\gamma^\alpha D_\alpha + m)\,\Psi = 0}. \tag{2.87}$$

To act on objects which carry Lorentz and two Dirac indices, let say $M \in \mathcal{V}\otimes\mathbb{C}^4\otimes\mathbb{C}^4$ [13] the covariant derivate generalizes to,

$$D_\mu M = \nabla_\mu M - [\Gamma_\mu, M]\,. \tag{2.88}$$

Using the metric compatibility and the Clifford algebra $\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}$ follows, $D_\mu\left(\{\gamma^\alpha, \gamma^\beta\}\right) = 0$. A sufficient condition is $D_\mu\gamma^\alpha = 0$ and since the commutator vanishes, follows $\nabla_\mu\gamma^\alpha = 0$. This result is later used in section (3).

By using the Fermi-normal coordinates from section (2.6) and using the shorthand notation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ we can derive tetrads in Fermi-normal coordinates by inserting the ansatz ${e_\mu}^\alpha = \delta^\alpha_\mu + \kappa {h_\mu}^\alpha$ in

$$\eta_{\mu\nu} = {e_\mu}^\alpha {e_\nu}^\beta\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right) \tag{2.89}$$

gives,

$$\eta_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + 2\kappa h_{\mu\nu} + \mathcal{O}(|h_{\mu\nu}|^2) \Rightarrow \kappa = -\frac{1}{2}. \tag{2.90}$$

So, we get for the tetrads,

$$ {e_0}^\alpha = \delta^\alpha_0 - \tfrac{1}{2}{R^\alpha}_{l0m}\big|_G\, x^l x^m, \tag{2.91a}$$
$$ {e_i}^\alpha = \delta^\alpha_i - \tfrac{1}{6}{R^\alpha}_{ilm}\big|_G\, x^l x^m. \tag{2.91b}$$

This was first derived by Parker [13]. Inserting the results (2.72) into the definition of the Christoffel connection (2.18) and using this result together with (2.91) and the definition of the spin-connection (2.80) and the spinor-connection (2.86) one finds by neglecting terms higher than x^2

[13] $\mathbb{C}^4$ is a complex fiber-bundle

$$\Gamma_0 = \tfrac{1}{2}\gamma_0\gamma_j\, R^{j0}{}_{0m}\big|_G\, x^m + \tfrac{1}{4}\gamma_k\gamma_j\, R^{kj}{}_{0m}\big|_G\, x^m, \tag{2.92a}$$
$$\Gamma_i = \tfrac{1}{4}\gamma_0\gamma_j\, R^{0j}{}_{im}\big|_G\, x^m + \tfrac{1}{8}\gamma_k\gamma_j\, R^{kj}{}_{im}\big|_G\, x^m. \tag{2.92b}$$

Like the identity (2.22), there is an equivalent relation with covariant derivatives acting on a spinor $\psi(x)$

$$[D_\mu, D_\nu]\,\psi(x) = \frac{i}{4} R_{\mu\nu\alpha\beta}\sigma^{\alpha\beta}\psi(x). \tag{2.93}$$

3 From Dirac to Mathisson–Papapetrou–Dixon

In this section, we will derive the classical Mathisson–Papapetrou–Dixon (MPD) [6, 28, 29] equations by doing a Wentzel–Kramers–Brillouin (WKB) [30, 31, 32] expansion of the Dirac equation followed by a Gordon decomposition [33]. The WKB expansion systematically expands ψ in powers of $\hbar$ up to the desired order. However, caution is necessary when applying MPD equations to calculate observables for spin-1/2 particles. Spin, being a pure quantum effect, vanishes entirely in the classical limit. Consequently, the use of MPD equations requires justification a priori. This justification holds only when we can assume the particle to be point-like and quantum effects treated as small corrections [34]. This derivation goes back to Lawrence [35], but he made the assumption $\overline{\psi}\psi = 1$, which is in general not true, as we will show. Audretsch [36] dropped this assumption, and derived the MPD equations from a WKB expansion by doing this in an unusual coordinate frame. He later generalized this derivation to gravitational fields with torsion [37], which do not come into play for our studies. At the same time, Rüdiger [38] derived the same results, in a slightly more comprehensive way. A modern derivation is given by Marius et al. [39], they generalized their derivation for charged fermions and fermions with no mass, which behave in particular differently. It still should be said, there exist multiple different ways to derive the MPD equations, for example Catenacci [40] and Cianfrani et al. [41] derived the MPD equations by iterating the Dirac equation twice. Restoring $\hbar$, the Dirac equation in curved space reads

$$(\hbar\gamma^\mu D_\mu + m)\,\psi(x) = 0. \tag{3.1}$$

Inserting the WKB-ansatz

$$\psi(x) = \exp\left(iS(x)/\hbar\right)\sum_{n=0}^{N}\left(i\hbar\right)^n \psi_n(x) \tag{3.2}$$

into (3.1) gives,

$$\sum_{n=0}^{N} i\partial\!\!\!/ S\,(i\hbar)^n\,\psi_n + \hbar\,(i\hbar)^n\,D\!\!\!\!/\,\psi_n + (i\hbar)^n\,m\psi_n = 0, \tag{3.3}$$

where we have used $D\!\!\!\!/\,S = \partial\!\!\!/ S$. Expanding up to $\mathcal{O}(\hbar^2)$ leads to,

$$\left(i\partial\!\!\!/ S\psi_0 + \hbar D\!\!\!\!/\,\psi_0 + m\psi_0\right) + \left(-\partial\!\!\!/ S\hbar\psi_1 + i\hbar m\psi_1\right) + \mathcal{O}(\hbar^2) = 0 \tag{3.4}$$

and therefore,

$$\hbar^0: \quad \left(i\partial\!\!\!/ S + m\right)\psi_0 = 0, \tag{3.5a}$$

$$\hbar^1: \quad \left(i\partial\!\!\!/ S + m\right)\psi_1 = iD\!\!\!\!/\,\psi_0. \tag{3.5b}$$

It should be said, that the function $S(x)$ has dimensions of an action, but it is not the action, since the action is a functional $C^p \mapsto \mathbb{R}$ and $S(x)$ is just an arbitrary real function which needs to be defined later on.

Using the conserved current $j^\mu = i\overline{\psi}\gamma^\mu\psi$ (for a proof, see Appendix (A.2)) and inserting the WKB-ansatz from equation (3.2) with $\overline{\psi} = \psi^\dagger i\gamma^0$. We obtain

$$j^\mu = i\overline{\psi_0}\gamma^\mu\psi_0 - \hbar\left[\overline{\psi_0}\gamma^\mu\psi_1 - \overline{\psi_1}\gamma^\mu\psi_0\right] + \mathcal{O}(\hbar^2), \tag{3.6}$$

where we define $i\overline{\psi_0}\gamma^\mu\psi_0 = j_0^\mu$, multiplying j_0^μ with m once from the left and once from the right and using (3.5a) gives the equations,

$$\begin{aligned} j_0^\mu m &= i\overline{\psi_0}\gamma^\mu\left(m\psi_0\right) = \overline{\psi_0}\gamma^\mu\gamma^\nu\partial_\nu S\psi_0, \\ j_0^\mu m &= i\left(m\overline{\psi_0}\right)\gamma^\mu\psi_0 = \partial_\nu S\overline{\psi_0}\gamma^\nu\gamma^\mu\psi_0. \end{aligned} \tag{3.7}$$

By using the Clifford-Algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ we obtain,

$$j_{0\mu} = \frac{\overline{\psi_0}\psi_0}{m}\partial_\mu S. \tag{3.8}$$

We define the quantities $\overline{\psi_0}\psi_0 = \mathcal{I}_0$ and $\partial_\mu S = k_\mu$, so we can rewrite equation (3.8) as,

$$j_0^\mu = \frac{\mathcal{I}_0 k^\mu}{m}. \tag{3.9}$$

Because of equation (3.5a), we are allowed to write

$$0 = \overline{\psi_0}\left(i\partial\!\!\!/ S + m\right)\psi = k_\mu \underbrace{i\overline{\psi_0}\gamma^\mu\psi_0}_{j_0^\mu} + m\overline{\psi_0}\psi, \tag{3.10}$$

with equation (3.9) one gets,

$$\mathcal{I}_0\left(\frac{1}{m}k^\mu k_\mu + m\right) = 0 \tag{3.11}$$

and therefore

$$k^\mu k_\mu = -m^2. \tag{3.12}$$

To get a non-trivial solution for equation (3.5a), we demand

$$\det\left(i\partial\!\!\!/ S + m\right) = 0. \tag{3.13}$$

This condition is equal with the already derived condition $k^\mu k_\mu = -m^2$. The matrix $i\partial\!\!\!/ S + m$ has rank 2, so we are only interested in two eigenvectors $\{\Sigma_0, \Sigma_1\}$, both with eigenvalue 0. Due to linearity ψ_0 is an arbitrary superposition of those two vectors,

$$\psi_0(x) = \sqrt{\mathcal{I}_0}\left(z_0(x)\Sigma_0 + z_1(x)\Sigma_1\right), \tag{3.14}$$

where $z_{1,0}$ are arbitrary complex functions, fulfilling

$$|z_0|^2 + |z_1|^2 = 1. \tag{3.15}$$

Theorem 3.1. *$i\partial\!\!\!/ S + m$ has rank 2.*

Proof. $\partial\!\!\!/ S = \gamma^a k_a = -\gamma^0 k^0 + \gamma^i k^i$ written in Dirac representation

$$\partial\!\!\!/ S = \begin{pmatrix} -ik^0 & 0 \\ 0 & +ik^0 \end{pmatrix} + \begin{pmatrix} 0 & i\sigma^i k^i \\ -i\sigma^i k^i & 0 \end{pmatrix},$$

since $\partial\!\!\!/ S$ is a Lorentz scalar, we switch to a rest frame, where $k^\mu = (m, 0)$ and so,

$$i\partial\!\!\!/ S + m = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}.$$

□

The eigenvectors are normalized and orthogonal. By defining $\{\Sigma_0, \Sigma_1\} = \{\Sigma_A, \Sigma_B\}$ such that $A, B = 0, 1$ the orthogonality condition reads $\overline{\Sigma_A}\Sigma_B = \delta_{AB}$ implying Einstein sum convention, we can write

$$\psi_0(x) = \sqrt{\mathcal{I}_0} z_A(x)\Sigma_A, \tag{3.16}$$

and

$$z_A^* z_A = 1. \tag{3.17}$$

Theorem 3.2. *Equation (3.5a) and (3.5b) form a linear system. The solvability condition implies that every solution of (3.5a) needs to be orthogonal to the inhomogeneous part of equation (3.5b). This is known as the Fredholm alternative [42].*

Proof. By defining $L = i\partial\!\!\!/ S + m$, equation (3.5a) can be written as, $L\Sigma_{0,1} = 0$, adjugate this equation gives, $\Sigma_{0,1}^\dagger L^\dagger = -\Sigma_{0,1}^\dagger \gamma^0 L \gamma^0 = i\overline{\Sigma_{0,1}} L\gamma^0 = 0$. Multiplying the last expression with $i\gamma^0$ to the right leads to $\overline{\Sigma_{0,1}} L = 0$, where we have used$(\gamma^0)^2 = -1$. Multiplying $\overline{\Sigma_{0,1}}$ from the left on equation (3.5b) leads to, $\overline{\Sigma_{0,1}} L\psi_1 = i\overline{\Sigma_{0,1}} D\!\!\!\!/\,\psi_0 = 0$. Therefore $\overline{\Sigma_{0,1}} D\!\!\!\!/\,\psi_0 = 0$. □

Additionally, since Σ_A and Σ_B both are solutions of (3.5a), we obtain the following identity

Theorem 3.3. $i\overline{\Sigma_A}\gamma^\mu\Sigma_B = \frac{1}{m}\delta_{AB}k^\mu$.

Proof. Defining $\tilde{j}_0^\mu = i\overline{\Sigma_0}\gamma^\mu\Sigma_0$ and likewise $\tilde{j}_1^\mu = i\overline{\Sigma_1}\gamma^\mu\Sigma_1$. Following along the procedure in equation (3.7) gives, $\tilde{j}_0^\mu = \frac{k^\mu}{m} = \tilde{j}_1^\mu$. With the orthogonality condition $\overline{\Sigma_A}\Sigma_B = \delta_{AB}$ this proof is complete. □

Using $D_\mu\gamma^\nu = 0$ and inserting equation (3.14) into the solvability condition $\overline{\Sigma_C} D\!\!\!\!/\,\psi_0 = 0$ gives,

$$\partial_\mu\left(\sqrt{\mathcal{I}_0}z_A\right)\overline{\Sigma_C}\gamma^\mu\Sigma_A + \sqrt{\mathcal{I}_0}z_A\overline{\Sigma_C}\gamma^\mu D_\mu\Sigma_A = 0, \tag{3.18}$$

and with,

$$\frac{\delta_{CA}}{m}\nabla_\mu k^\mu = D_\mu\left(i\overline{\Sigma_C}\gamma^\mu\Sigma_A\right) = i2\overline{\Sigma_C}\gamma^\mu D_\mu\Sigma_A. \tag{3.19}$$

Therefore equation (3.18) becomes

$$k^\mu\partial_\mu\left(\sqrt{\mathcal{I}_0}z_C\right) + \sqrt{\mathcal{I}_0}z_C\frac{\nabla_\mu k^\mu}{2} = 0. \tag{3.20}$$

With the above equation, we obtain

$$k^\alpha D_\alpha\psi_0 = -\frac{\nabla_\mu k^\mu}{2}\psi_0 + f_A k^\alpha D_\alpha\Sigma_A, \tag{3.21}$$

where we defined $f_A = \sqrt{\mathcal{I}_0}z_A$.

Theorem 3.4. $k^\alpha D_\alpha\Sigma_A = 0$.

Proof. This is best shown in a coordinate system where ${e_0}^\mu = k^\mu$ and $k^\mu \nabla_\mu {e_a}^\nu = 0$ also known as a Fermi-frame, see section (2.5).
$k^\alpha D_\alpha \Sigma_A = k^\alpha \partial_\alpha \Sigma_A - k^\alpha \Gamma_\alpha \Sigma_A$. $\Gamma_\alpha \propto \omega_{\alpha ab} \sigma^{ab}$. Where $\omega_{\alpha ab} = {e_a}^\beta \nabla_\alpha e_{b\beta}$ multiplying with k^α gives, ${e_a}^\beta k^\alpha \nabla_\alpha e_{b\beta} = 0$. Therefore $k^\alpha \Gamma_\alpha = 0$. Using equation (3.5a) follows with the Fermi tetrad $(i\gamma^0 + m)\Sigma_A = 0$, therefore Σ_A is constant in this frame. Since $k^\alpha D_\alpha \Sigma_A$ is a tensor expression, follows its validity is true in every coordinate system. □

With the above theorem, equation (3.21) can be rewritten as,

$$k^\alpha D_\alpha \psi_0 = -\frac{\nabla_\mu k^\mu}{2} \psi_0. \tag{3.22}$$

Now we define a new function f such that $|f|^2 = f_A^* f_A$. Now it is possible to rewrite the spinor $\psi_0 = f\phi$ with $\bar{\phi}\phi = 1$, which leads to the more compact equations

$$k^\alpha \partial_\alpha f = -\tfrac{\nabla_\mu k^\mu}{2} f, \tag{3.23a}$$

$$k^\alpha D_\alpha \phi = 0. \tag{3.23b}$$

To finally obtain a correction to the geodesic, we need the Gordon-decomposition [43], which states

Theorem 3.5. $\bar{\psi}\gamma^\mu\psi = \frac{g^{\mu\nu}\hbar}{2m}\left[\left(D_\nu\bar{\psi}\right)\psi - \bar{\psi}\left(D_\nu\psi\right)\right] + \frac{\hbar}{m} D_\nu\left(\bar{\psi}\sigma^{\mu\nu}\psi\right)$.

Proof. see appendix (A.3). □

So the Noether current j^μ can be written as,

$$j^\mu = j^\mu_c + j^\mu_m, \tag{3.24}$$

where j^μ_c is the convection current and j^μ_m is the magnetization current, in the following we are only interested in the convection current, since only the first term corresponds to the propagation of a Dirac particle. Plunging the WKB ansatz (3.2) with the solution from equation (3.23) into the convection current

$$j_{c\mu} = \frac{i\hbar}{2m}\left[\left(D_\mu\bar{\psi}\right)\psi - \bar{\psi}\left(D_\mu\psi\right)\right] \tag{3.25}$$

gives,

$$j_{c\mu} = \frac{i\hbar}{2m}\left(-2i\partial_\mu S\bar{\psi}\psi + \sum_{n,m}(i\hbar)^n(-i\hbar)^m\left[D_\mu\overline{\psi_n}\psi_m - \overline{\psi_n}D_\mu\psi_m\right]\right). \tag{3.26}$$

Normalising the convection current by dividing it by $\bar{\psi}\psi$ and neglecting terms of $\hbar^2$ gives,

$$\frac{j_{c\mu}}{\bar{\psi}\psi} = \frac{\partial_\mu S}{m} + \underbrace{\frac{i\hbar}{2m}\left[D_\mu\bar{\phi}\phi - \bar{\phi}D_\mu\phi\right]}_{:=\delta v_\mu}. \tag{3.27}$$

Defining $j_{c\mu}/\bar{\psi}\psi = v^\mu$ being the tangent vector field of the trajectory and $\partial_\mu S/m = u^\mu$ being the tangent vector field of the geodesic. We can write

$$\boxed{v^\mu = u^\mu + \delta v^\mu.} \tag{3.28}$$

This remarkable result demonstrates, that we have achieved a quantum correction of the order of $\hbar$ to the classical geodesic. This observation implies that a fermion moving through curved spacetime does not strictly follow a geodesic.

To finally derive the MPD-equations, we parallel transport the vector field v^μ along itself.

$$a^\nu := v^\mu\nabla_\mu v^\nu = (u^\mu + \delta v^\mu)\nabla_\mu(u^\nu + \delta v^\nu) \tag{3.29}$$

and neglecting terms higher than $\hbar$, we get

$$a^\nu = u^\mu\nabla_\mu\delta v^\nu + \delta v^\mu\nabla_\mu u^\nu + \mathcal{O}(\hbar^2), \tag{3.30}$$

where we have used that u^μ solves the geodesic equation (2.19).

Since $u^\mu \sim k^\mu$, we can use the following theorem,

Theorem 3.6. $\nabla_\mu k_\nu = \nabla_\nu k_\mu$.

Proof. Since $k_\mu = \partial_\mu S$, $\nabla_\mu k_\nu = \nabla_\mu\partial_\nu S = \partial_\mu\partial_\nu S - \Gamma^\alpha{}_{\mu\nu}\partial_\alpha S$ likewise, $\nabla_\nu k_\mu = \partial_\nu\partial_\mu S - \Gamma^\alpha{}_{\nu\mu}\partial_\alpha S$, by using the theorem of Schwarz, the partial derivatives could be interchanged and since the Christoffel symbol is symmetric in the lower indices, follows $\nabla_\mu k_\nu = \nabla_\nu k_\mu$. □

And therefore, we rewrite equation (3.30) as,

$$a_\nu = u^\mu\nabla_\mu\delta v_\nu + \delta v^\mu\nabla_\nu u^\mu. \tag{3.31}$$

Next, we use the fact that the correction δv^μ is orthogonal to the geodesic u^μ this can be easily seen by using equation (3.23b). Therefore $\nabla_\nu(\delta v^\mu u_\mu) = 0$, using the Leibniz rule of derivatives equation (3.31) can be written as,

$$a_\nu = u^\mu\nabla_\mu\delta v_\nu - u^\mu\nabla_\nu\delta v_\mu = 2u^\mu\nabla_{[\mu}\delta v_{\nu]}. \tag{3.32}$$

Inserting the second term of (3.27) gives,

$$a_\nu = \frac{i\hbar u^\mu}{m}\left[D_{[\mu}\left(D_{\nu]}\bar{\phi}\right)\phi - D_{[\mu}\left(\bar{\phi}\left(D_{\nu]}\phi\right)\right)\right]. \tag{3.33}$$

With identity (2.93) and equation (3.23b) follows finally

$$\dot{p}_\nu := m a_\nu = -\frac{\hbar u^\mu}{4} R_{\mu\nu\gamma\delta}\bar{\phi}\sigma^{\delta\gamma}\phi, \tag{3.34}$$

where we have used $D_{[\mu}D_{\nu]} = 2[D_\mu, D_\nu]$. To derive an equation for the motion of the intrinsic angular momentum we define the spin tensor as,

$$S^{\alpha\beta} = \frac{\hbar}{2}\frac{\bar{\psi}\sigma^{\alpha\beta}\psi}{\bar{\psi}\psi} = \frac{\hbar}{2}\bar{\phi}\sigma^{\alpha\beta}\phi + \mathcal{O}(\hbar^2). \tag{3.35}$$

The change of the spin tensor $S^{\alpha\beta}$ is given by,

$$\dot{S}^{\alpha\beta} = v^\mu \nabla_\mu S^{\alpha\beta} = 0 + \mathcal{O}(\hbar^2), \tag{3.36}$$

where again equation (3.23b) was used. Finally, we arrive at the MPD equations

$$\dot{p}_\nu = -\tfrac{1}{2} R_{\mu\nu\gamma\delta} u^\mu S^{\gamma\delta}, \tag{3.37a}$$
$$\dot{S}^{\alpha\beta} = 0. \tag{3.37b}$$

It is possible that the reader has encountered the MPD equations expressed in an alternative notation

$$\dot{p}_\nu = -\tfrac{1}{2} R_{\mu\nu\gamma\delta} v^\mu S^{\gamma\delta}, \tag{3.38a}$$
$$\dot{S}^{\alpha\beta} = p^\alpha v^\beta - p^\beta v^\alpha. \tag{3.38b}$$

Where $v^\mu = dx^\mu/d\tau$ is the velocity and p^μ is the momentum of the particle. The equations (3.38a) and (3.38b) are 10 equations, but we have 13 free parameters, so the set of equations (3.38a) and (3.38b) are undetermined, this is why it is often useful to require additional conditions, called spin-supplementary conditions. In the literature, there exists five different conditions [44].

1. Mathisson-Pirani (MP) condition : $v_\mu S^{\mu\nu} = 0$
2. Tulczykew-Dixon (TD) condition : $p_\mu S^{\mu\nu} = 0$
3. Corinaldesi-Papapetrou (CP) condition : $v_\mu^{\mathrm{lab}} S^{\mu\nu} = 0$, this condition sets the reference worldline as being the center of mass measured in the "laboratory" frame, which is chosen as the congruence of static observers at rest with respect to the source.

4. Newton-Wigner (NW) condition $V^\mu \propto v_\mu^{\text{lab}} + \frac{p^\mu}{m}, V_\mu S^{\mu\nu} = 0$
5. Ohashi-Kyrian-Semerák (OKS) condition which choses p^μ parallel to v^μ

With the WKB method we are able to derive the spin supplementary condition

$$v_\beta S^{\alpha\beta} = v_\beta \frac{\hbar}{2}\bar{\phi}\sigma^{\alpha\beta}\phi = u_\beta S^{\alpha\beta} + \mathcal{O}(\hbar^2). \tag{3.39}$$

With equation (3.5a) follows

$$v_\beta S^{\alpha\beta} \propto u_\beta u_\mu \bar{\phi} \underbrace{\left(\gamma^\mu \sigma^{\alpha\beta} + \sigma^{\alpha\beta}\gamma^\mu\right)}_{\propto g^{\mu\beta}\gamma^\alpha - g^{\alpha\beta}\gamma^\mu} \phi, \tag{3.40}$$

and therefore

$$v_\beta S^{\alpha\beta} = u_\beta S^{\alpha\beta} = 0. \tag{3.41}$$

By using TD-condition and linearizing (3.38a) and (3.38b) in the sense of neglecting quadratic terms of the spin tensor one arrives at (3.37a) and (3.37b), therefore (3.37a) and (3.37b) are only MPD-like equations, because we only consider terms of order $\hbar$ and neglected all higher order terms.

4 Analytic Estimation in Long-Baseline Experiments

In this section, we present an analytical estimation for the deviation. We achieve this by employing Fermi-normal coordinates. Initially, we apply this method to general geodesics in proximity. Subsequently, we explicitly address its application to Long-baseline experiments. Finally, we summarize relevant findings from existing literature.

An explicit solution of equation (3.5a) is

$$\Sigma_0(x) = \sqrt{\frac{k^0+m}{2m}} \begin{bmatrix} \frac{k^3}{k^0+m} \\ \frac{k^1+ik^2}{k^0+m} \\ 1 \\ 0 \end{bmatrix}, \tag{4.1}$$ $$\Sigma_1(x) = \sqrt{\frac{k^0+m}{2m}} \begin{bmatrix} \frac{k^1-ik^2}{k^0+m} \\ -\frac{k^3}{k^0+m} \\ 0 \\ 1 \end{bmatrix}. \tag{4.2}$$

Note, we are only interested in the deviation of real particles, so we neglect the negative energy states, which describe anti particles. By using the vierbein (2.91) from section (2.5), it is now possible to extend the definition of vectors, which would only be defined along a worldline G, which is in our case the geodesic, in the neighbourhood of G.

$$-k^0(x) = k_0 = k_\mu e^\mu{}_0 = k_a \left(\delta^a{}_0 - \frac{1}{2} R^a{}_{l0m} x^l x^m \right), \tag{4.3}$$

and

$$k_i(x) = k^i = k_\mu e^\mu{}_i = k_a \left(\delta^a{}_i - \frac{1}{6} R^a{}_{lim} x^l x^m \right). \tag{4.4}$$

We know that k^a along the geodesic G is $(m, 0, 0, 0)$, since in Fermi-normal coordinates a particle is at rest

$$k^0 = -m \left(1 - \frac{1}{2} R^a{}_{l0m} x^l x^m \right), \tag{4.5}$$

$$k^i = -\frac{m}{6} R^a{}_{lim} x^l x^m. \tag{4.6}$$

Expanding the spinors around the geodesic G gives,

$$\Sigma_A(x) = \Sigma_A(x)|_G + \partial_a \Sigma_A(x)|_G \, x^a + \mathcal{O}\left(\frac{x^2}{R_0^2} \right). \tag{4.7}$$

With equations (4.3) and (4.4), one finds that $\partial_a k^b(x)\big|_G = 0$. Therefore, we are allowed to write

$$\Sigma_0(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{4.8}$$

$$\Sigma_1(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.9}$$

From section (3) we know, we may write $\psi = f\Phi$ such that $\bar{\phi}\phi = 1$. We claim now, that it is possible for arbitrary spin polarization to rewrite ϕ as,

$$\phi(x) = \begin{bmatrix} 0 \\ 0 \\ \cos(\xi/2) \\ \exp(i\chi)\sin(\xi/2) \end{bmatrix}. \tag{4.10}$$

Where ξ and χ are the azimuthal and polar angels of the Bloch sphere. With the right-hand side of (3.27) we find for the deviation term

$$\delta v_a = \frac{i\hbar}{m} \bar{\phi} \Gamma_a \phi. \tag{4.11}$$

Inserting the spinor-connection (2.92) gives,

$$\delta v_a = \frac{i\hbar}{m} \begin{bmatrix} \frac{1}{2}\bar{\phi}\gamma_0\gamma_j\phi \; R^{0j}{}_{0m}\big|_G \, x^m + \frac{1}{4}\bar{\phi}\gamma_k\gamma_j\phi \; R^{kj}{}_{0m}\big|_G \, x^m \\ \frac{1}{4}\bar{\phi}\gamma_0\gamma_j\phi \; R^{0j}{}_{im}\big|_G \, x^m + \frac{1}{8}\bar{\phi}\gamma_k\gamma_j\phi \; R^{kj}{}_{im}\big|_G \, x^m \end{bmatrix}. \tag{4.12}$$

The expression in square brackets is purely imaginary, this is most easily shown by using the Clifford-Algebra and exploiting the symmetries of the

Riemann tensor. Equation (4.12) is a very interesting result, since it immediately tells us that along the geodesic G the correction velocity due to spin-gravity coupling vanishes. Since $\gamma(0) = \gamma'(0)$ the only way to get a non-vanishing deviation is by a non-zero acceleration at $\gamma(0)$. Additionally, equation (4.12) tells us, that there is no deviation to be expected in flat space, where the Riemann tensor would be identically 0.

4.1 General Case

To derive a symbolic expression, we need to evaluate the acceleration $a_a|_G = -1/2m\, R_{0abc}S^{bc}|_G$ in Fermi-normal coordinates along the geodesic G. With the help of a computer algebra system, we obtain

$$a_a|_G = \frac{3LM\sqrt{L^2+r^2}}{2mr^5}\begin{bmatrix} 0 \\ \sin(\chi)\sin(\xi)\cos(\mathfrak{w}(r)) \\ \sin(\xi)\cos(\chi)\cos(\mathfrak{w}(r)) + \sin(\mathfrak{w}(r))\cos(\xi) \\ \sin(\chi)\sin(\xi)\sin(\mathfrak{w}(r)) \end{bmatrix}. \tag{4.13}$$

The variable $\mathfrak{w}$ (see equation (2.49)) represents a highly intricate quantity within the context of Earth's mass, denoted by M. The dependence of $r(t)$ on M arises from the underlying curvature of spacetime. Consequently, obtaining a closed-form expression for $\mathfrak{w}$ becomes a formidable task. To address this, we employ an analytic approach by expanding $a_a|_G$ in terms of the mass M, while neglecting quadratic terms that would only contribute to $\mathfrak{w}$. This expansion remains valid when $2M \ll r$. We introduce the expansion parameter $\epsilon = 2M/r$ and proceed to expand the product,

$$f(\epsilon)g(\mathfrak{w}(\epsilon)) = f(0)g(\mathfrak{w}(0)) + \left[\frac{df}{d\epsilon}g(\mathfrak{w}(\epsilon)) + f(\epsilon)\frac{dg}{d\epsilon}\right]_{\epsilon=0}\xi + \mathcal{O}(\epsilon^2) \tag{4.14}$$

gives,

$$f(\epsilon)g(\mathfrak{w}(\epsilon)) = f(\epsilon)g(\mathfrak{w}(0)) + \mathcal{O}(\epsilon), \tag{4.15}$$

because $f(0) = 0$. By using the above result, the expansion of $a_a|_G$ up to ϵ leads to,

$$a_a|_G = \frac{3LM\sqrt{L^2+r^2}}{2mr^5}\begin{bmatrix} 0 \\ \sin(\chi)\sin(\xi)\cos(\mathfrak{w}(r,0)) \\ \sin(\xi)\cos(\chi)\cos(\mathfrak{w}(r,0)) + \sin(\mathfrak{w}(r,0))\cos(\xi) \\ \sin(\chi)\sin(\xi)\sin(\mathfrak{w}(r,0)) \end{bmatrix}, \tag{4.16}$$

where $\mathfrak{w}(r,0) = \mathfrak{w}(r)|_{\epsilon=0}$.

Given that $\mathfrak{w}$ is solely defined by its derivative, it remains possible to introduce an arbitrary constant. However, based on our justification in section (2.5), we set this constant to zero. Considering the relatively small change in r, we can further simplify by allowing $\mathfrak{w}$ to take any constant value, which turns out to be zero, even in the non-radial case where $L = 0$ and thus $d\mathfrak{w}/dr = 0$.

Proof. To show that $\mathfrak{w}$ can be considered to be constant, we have a look at tangential motion such that, $L \sim R_0$, where R_0 is in our case the Earth's radius. Since $\mathfrak{w}$ is evaluated at $\epsilon = 0$ which essentially means evaluated at flat space, we are allowed to write $r^2(t) \approx R_0^2 + t^2$, follows with equation (2.49) $\dot{\mathfrak{w}} = \frac{-\sqrt{2E-1}}{2R_0}\frac{1}{1+t^2/2R_0^2} \Rightarrow \mathfrak{w} = \frac{-\sqrt{2E-1}}{2R_0} \arctan \frac{t^2}{2R_0^2}$, since $E \sim 10^9$ and $R_0 \sim 3.712 * 10^{41}$ the prefactor is almost 0 and by justification of section (2.5) $\mathfrak{w} \sim 0$. $\square$

Having a deeper look at equation (4.16) it becomes evident that a specific polarization exists where the deviation vanishes. This particular scenario arises when the geodesic motion is in alignment with the direction of the polarization. To illustrate this concept, consider a metaphorical analogy where the particle is described by a screw "drilled" into in spacetime. It becomes apparent that no deviation can occur if the screw's direction coincides with the direction of motion. This metaphor serves to clarify the dynamics in this context.

Consider, for this justification, a circular geodesic where $du^r/d\tau = 0$, and in Schwarzschild coordinates, $\mathbf{u}$ has only a time and ϕ component. Likewise, ${e_z}^{\mu}$ has only temporal and ϕ components, meaning by considering only the three-dimensional part, the $z-$direction aligns with the motion of the particle. Choosing a z-polarization meaning $\xi = 0$ or π equation (4.16) vanishes.

From equation (4.16) we see, that the acceleration vanishes for $L = 0$, this means there is no deviation to be expected for the radial case.

4.2 An Explicit Formula for the Deviation

Since our spinor $\phi(x)$ from equation (4.10) is only determined up to the linear term by its spacial expansion, and so is the spin tensor $S^{\alpha\beta}$. In this case, we do not need to expand a_a in the neighbourhood of the geodesic, and so only the prefactors of (4.16) change, additionally the term Γvv occurring by writing out the covariant derivative, is already of order x^3 and of order $\hbar^2$ see equation (4.12). So we obtain

$$\frac{d\delta v^a}{d\tau} = a^a. \tag{4.17}$$

Because $\mathfrak{w}$ is approximately a constant, the direction of the deviation will not change in a sufficient small neighbourhood around the geodesic. Therefore, the vectorial differential equation reduces to a scalar valued differential equation, by defining $\tilde{a} = \sqrt{L^2 + r^2}/r^5$. For the radial coordinate, we make the approximation $r(t) = \sqrt{t^2 + R_0^2}$ since $t < R_0 \Rightarrow t^3 \ll R_0^3$ and get,

$$\tfrac{d\tilde{\delta v}}{d\tau} = \tfrac{\sqrt{2}}{R_0^4} - \tfrac{9\sqrt{2}t^2}{4R^6} + \mathcal{O}(\tfrac{t^3}{R_0^3}), \tag{4.18a}$$

$$\Rightarrow \tilde{x} = \tfrac{1}{2}\tfrac{\sqrt{2}}{R_0^4}t^2 + \mathcal{O}(\tfrac{t^3}{R_0^3}), \tag{4.18b}$$

where we have used the starting conditions $\tilde{\delta v} = 0$ and $\tilde{x} = 0$. The constant angular momentum was chosen to be $L = R_0$. Since we can not control the spin polarization, because it is not measurable, we average over all directions.

$$\tilde{d} = \frac{1}{4\pi}\int_{2\pi}^{0} d\chi \int_{\pi}^{0} d\xi ||\delta x^a||. \tag{4.19}$$

The deviation for a three-dimensional detector now reads,

$$\tilde{d} = \frac{3M}{4m}\frac{\sqrt{2}}{R_0^3}t^2, \tag{4.20}$$

but since the detector in most Long-baseline experiments is considered as a two-dimensional surface, we need to project the S^2 to D^2 [14] this gives a correction of a $4/\pi$ and so the deviation is,

$$\boxed{d = \frac{3M}{m\pi}\frac{\sqrt{2}}{R_0^3}t^2}. \tag{4.21}$$

Restoring SI units, this gives finally

$$d = \frac{3M\hbar G}{\pi mc}\frac{\sqrt{2}t^2}{R_0^3} + \mathcal{O}(\frac{t^3c^3}{R_0^3}, \frac{M^2G^2}{R_0^2c^4}, \hbar^2). \tag{4.22}$$

The expression (4.21) or in SI units (4.22) is a remarkable result, it shows perfectly that the deviation vanishes for zero curvature, this is essentially the case for $M \to 0$ or $R_0 \to \infty$. Additionally, it shows the inverse neutrino mass behaviour as expected. Finally, it explains that deviation increases approximately parabolically, which otherwise would lack of physical interpretation.

[14] Disk of dimension two.

(a) no gravity or no spin

(b) with gravity and spin

Figure 4: The grey area visualises the probability of residence. In a) there is no deviation from the centre of the beam to be expected. b) shows the probability of residence like a Stern-Gerlach experiment [45].

4.3 Applying Results to the DUNE Experiment

To accurately measure the deviation angle, it is insufficient to simply measure the deviation of the maxima observed at the Far-detector. Given that the opening angle of the neutrino beam is approximately $1°$, it significantly encompasses the detector area. By calculating the expected number of impact events in relation to the measured number, one can determine the deviation angle, which then allows estimating the mass of the lightest neutrinos. Due to the relatively big opening angle, the minimal measurable angle needs to be greater or equal to $1/2°$.

Considering the distance between the neutrino source and the detector (approximately $1300km$), the tangent of $1°$ is approximately $0,017$. Thus, we can reasonably consider $1/2°$ as a small neighbourhood along the geodesic. See figure (4) for a visualisation of the neutrino beam and its probability of residence.

Furthermore, let's examine the expansion parameters: $(t^3c^3/R_0^3 \approx 0,0018)$ and $(M^2G^2/R_0^2c^4 \approx 1,43 \times 10^{-9})$. Given these values, we can conclude that the formula (4.22) should hold for the DUNE experiment.

This result is significant because it provides the first opportunity to measure spin-gravity coupling. Additionally, if the DUNE experiment is in operation, it could be possible to establish a lower bound for the net mass of the lightest neutrinos. In neutrino oscillation experiments, only mass differences are typically measurable. Further, there is no experimental data, only en estimation by Stöcker et al. [5] based on cosmological and terrestrial data who determined an upper bound for the lightest neutrinos. For a visual representation of the mass versus the angle of deviation, refer to figure (5).

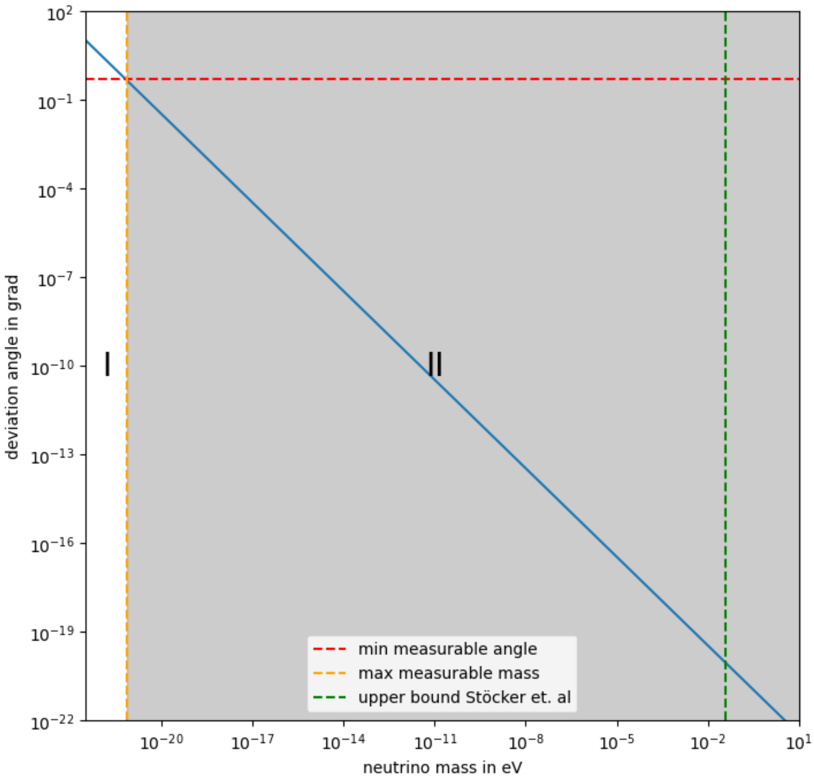

Figure 5: The blue curve illustrates the relationship between neutrino mass and the expected deviation angle for the DUNE experiment. The red line represents the minimal measurable angle $= 1/2°$. Area II is not detectable by the Far-detector, while area I represents the region where the lightest neutrinos are most likely to be found.

4.4 Comparison to Other Results

Koch et al. [46] studied the behaviour of a hypothetical massive photon in gravitational lensing, by using the Schwarzschild solution of the MPD equation. They get for the lensing angle

$$\Delta\Phi = 2\frac{r_0}{r_m}\left(1 + \frac{\hbar}{r_m c m}\right), \tag{4.23}$$

where r_m is the minimal radius, r_0 the starting radius and m the hypothetical mass of the photon. The second term of equation (4.23) corresponds to the correction due to photon spin. Notably, the second term in equation (4.23) accounts for the correction due to photon spin. Interestingly, measurements revealed a disagreement, indicating that the mass of photons is precisely zero. Besides, they studied a different problem, it shows the $\hbar/m$ correction to the geodesic. Rietdijk et al.[47] calculated a correction to the perihelion shift of a spinning test particle within a Schwarzschild geometry. The rotation in Planck units reads

$$\Delta\Phi = 2\pi \left(1 + \frac{3M}{k}\left(1 + \frac{\Sigma}{Lm}\right)\dots\right), \tag{4.24}$$

where $k \in \mathbb{N}$ and Σ is the scalar valued intrinsic angular momentum. Where the Σ/Lm term occoured due to the spinning of the test particle.

5 Summary and Conclusion

In section (2), we provided an introductory overview of the essential mathematical prerequisites for studying spin-gravity coupling in Long-baseline experiments. Moving forward, in section (3), we derived the MPD-"like" equations using a WKB expansion. We expressed the velocity and acceleration of the deviation in Fermi-normal coordinates. We then used the acceleration to finally obtain a formula for the deviation's length. This remarkable result not only may predict the lower net mass of neutrinos, it also describes the first ever measurable spin-gravity coupling.

However, it is crucial to recognize that this result holds only in the neighbourhood of the central geodesic. Extrapolating the deviation to arbitrary length scales would be inappropriate. In section (4.1), we demonstrated that there is no deviation to be expected for radial motion. This conclusion remains valid up to order $\hbar$, although higher-order corrections may yield non-vanishing results.

6 Outlook

While extending the calculations to order $\hbar^2$ and considering additional expansion terms of the metric in Fermi-normal coordinates could potentially enhance the result, deriving the necessary quantities in closed form remains a challenge. An intriguing approach involves utilizing spacetimes where exact solutions in Fermi coordinates exist (see [48]). These spacetimes could serve as approximations of Earth's gravitational field.

A very interesting idea for open problems would be to consider Einstein-Cartan theory, which reproduces the Einstein equations but with torsion, this torsion could be induced by spin-gravity coupling [49].

A Appendix

A.1 Einstein equations

The action reads

$$S[g_{\mu\nu}] = -\frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}R.$$

Varying this action with respect to the metric gives,

$$\delta S = \int_{\mathcal{M}} d^4x \underbrace{\delta\sqrt{-g}}_{-\frac{1}{2\sqrt{-g}}\delta g} R + \sqrt{-g} \underbrace{\underbrace{\delta R}_{\delta\left(g^{\mu\nu}R_{\mu\nu}\right)}}_{\delta g^{\mu\nu}R_{\mu\nu}+g^{\mu\nu}\delta R_{\mu\nu}} = 0.$$

We can write g as $\exp Tr(\ ln g_{\mu\nu})$ and therefore $\delta g = gg^{\mu\nu}\delta g_{\mu\nu}$. We obtain,

$$\delta S = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}.$$

$\delta R_{\mu\nu}$ is a total derivative term, most easily shown by using the definition of Riemann tensor (2.23). So we get,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0.$$

If we add a matter field S_{mat} and vary it with respect to the metric, we get the stress energy tensor $T^{\mu\nu}$. Then the Einstein equations are,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}.$$

A.2 Current Density

The Dirac action in curved spacetime is,

$$S\left[\overline{\psi},\psi\right] = -\int_{\mathcal{M}} d^4x\sqrt{-g}\overline{\psi}\left(\gamma^\mu D_\mu + m\right)\psi.$$

This action has a global $U(1)$ symmetry.

$$\delta\mathcal{L} = \mathcal{L}(\overline{\psi},\psi,D_\mu\overline{\psi},D_\mu\psi) - \mathcal{L}'(\overline{\psi}',\psi',D_\mu\overline{\psi}',D_\mu\psi') = 0.$$

Expanding $\mathcal{L}$ and by demanding, $\delta\mathcal{L} = D_\mu j^\mu$ we get for j^μ,

$$j^\mu = \frac{\partial\mathcal{L}}{\partial D_\mu\psi}\delta\psi - \delta\overline{\psi}\frac{\partial\mathcal{L}}{\partial D_\mu\overline{\psi}}.$$

Inserting $\delta\psi = i\epsilon\psi$ and $\delta\overline{\psi} = -i\epsilon\overline{\psi}$ and since ϵ is arbitrary, we get $j^\mu = \overline{\psi}\gamma^\mu\psi$. Taking the covariant derivative, we see that this quantity is indeed conserved $D_\mu j^\mu = 0$.

A.3 Gordon Decomposition

Lemma $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \sigma^{\mu\nu}$.

Proof.

$$\begin{aligned} \gamma^\mu\gamma^\nu - \sigma^{\mu\nu} &= g^{\mu\nu} \\ \Rightarrow 2\left(\gamma^\mu\gamma^\nu - \sigma^{\mu\nu}\right) &= \{\gamma^\mu, \gamma^\nu\} \\ \Rightarrow -2\sigma^{\mu\nu} &= -\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \\ \Rightarrow \sigma^{\mu\nu} &= \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right] \end{aligned}$$

□

Theorem A.1. $\bar{\psi}\gamma^\mu\psi = \frac{g^{\mu\nu}\hbar}{2m}\left[\left(D_\nu\bar{\psi}\right)\psi - \bar{\psi}\left(D_\nu\psi\right)\right] + \frac{\hbar}{m}D_\nu\left(\bar{\psi}\sigma^{\mu\nu}\psi\right)$.

Proof.

$$\bar{\psi}\gamma^\mu\left(m\psi\right) = -\hbar\bar{\psi}\gamma^\mu\gamma^\nu D_\nu\psi.$$

Where we have used the Dirac equation.

$$\left(\bar{\psi}m\right)\gamma^\mu m\psi = \hbar\left(D_\nu\bar{\psi}\right)\gamma^\nu\gamma^\mu\psi,$$

where we have used the adjugate Dirac equation. Adding those equations, we end up with,

$$\bar{\psi}\gamma^\mu\psi = \frac{\hbar}{2m}\left[\left(D_\nu\bar{\psi}\gamma^\nu\gamma^\mu - \bar{\psi}\gamma^\mu\gamma^\nu D_\nu\psi\right)\right].$$

Inserting $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \sigma^{\mu\nu}$ this proof is complete. □

A.4 A Mathematical Mistake

It is worth noting that Alsing et al. [21] demonstrated an incorrect approach, highlighting a significant mathematical error. Specifically, they computed the spin connection using the tetrads, which are only defined along the geodesic, as we discussed in section (2.5). The primary issue with this approach lies in the fact that $\nabla_\mu u^\nu$ is not well-defined; instead, only the directional derivative $u^\mu\nabla_\mu u^\nu$ holds mathematical validity. To be precise, the covariant derivative is defined only within an open subset around a point $p \in \mathcal{M}$. This is exactly the reason why the calculated correction terms lack of physical interpretation. As a well known fact it, is clear that in flat space the correction term should vanish, this is especially true when the Earth mass M goes to zero. Therefore, we need to expend the definition of u^μ to a vector field. This is most easily done in Fermi-normal coordinates. With this procedure, the covariant derivative is well-defined. [15]

[15] Also see equation (2.6) for the definition of a vector along a curve.

References

[1] P.W. Higgs. “Broken symmetries, massless particles and gauge fields”. In: *Physics Letters* (1964).

[2] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble. “Global Conservation Laws and Massless Particles”. In: *Physical Review Letters* (Nov. 1964).

[3] F. Englert and R. Brout. “Broken Symmetry and the Mass of Gauge Vector Mesons”. In: *Physical Review Letters* (Aug. 1964).

[4] G. Bellini et al. “Neutrino Oscillations”. In: *Advances in High Energy Physics* (2014).

[5] Patrick Stöcker et al. “Strengthening the bound on the mass of the lightest neutrino with terrestrial and cosmological experiments”. In: *Physical Review D* (2021).

[6] Myron Mathisson. *Neue mechanik materieller systeme*. 1937.

[7] Paul Ehrenfest. “Bemerkung über die angenäherte Gültigkeit der klassischen Mechanik innerhalb der Quantenmechanik”. In: *Zeitschrift für Physik* 45.7 (1927).

[8] Sean M Carroll. *Spacetime and geometry*. Cambridge University Press, 2019.

[9] Theodore Frankel. *The geometry of physics: an introduction*. Cambridge university press, 2011.

[10] Palash B Pal. *A Physicist’s Introduction to Algebraic Structures: Vector Spaces, Groups, Topological Spaces and More*. Cambridge University Press, 2019.

[11] Thomas James Willmore. *An introduction to differential geometry*. Courier Corporation, 2013.

[12] MN Hounkonnou and JEB Mendy. “Exact solutions of the Dirac equation in a nonfactorizable metric”. In: *Journal of Mathematical Physics* (1999).

[13] Leonard Parker. “One-electron atom as a probe of spacetime curvature”. In: *Physical Review D* (1980).

[14] Hassler Whitney. “The self-intersections of a smooth n-manifold in 2n-space”. In: *Annals of Mathematics* (1944).

[15] Euclid. *Euclid’s Elements*. Santa Fe, NM: Green Lion Press, Jan. 2002.

[16] Barton Zwiebach. *A First Course in String Theory*. en. 2nd ed. Cambridge, England: Cambridge University Press, Jan. 2009.

[17] Alexander M. Polyakov. "Quantum geometry of bosonic strings". In: *Physics Letters B* (1981).

[18] Herbert Seifert and William Threlfall. *Variationsrechnung im grossen:(theorie von Marston Morse)*. Vol. 49. American Mathematical Soc., 1971.

[19] Karl Schwarzschild. "On the gravitational field of a mass point according to Einstein's theory". In: *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* (1916).

[20] Peter Collas and David Klein. *The Dirac Equation in Curved Spacetime: A Guide for Calculations*. Springer, 2019.

[21] P. M. Alsing, G. J. Stephenson Jr. au2, and P. Kilian. *Spin-induced non-geodesic motion, gyroscopic precession, Wigner rotation and EPR correlations of massive spin 1/2 particles in a gravitational field*. 2009. arXiv: `0902.1396 [quant-ph]`.

[22] J-A Marck. "Solution to the equations of parallel transport in Kerr geometry; tidal tensor". In: *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* (1983).

[23] Yusuke Hagihara. "Theory of the relativistic trajeetories in a gravitational field of schwarzschild". In: *Japanese journal of astronomy and geophysics* (1930).

[24] Dennis Hansen. "Killing–Yano tensors". In: *Niels Bohr Institute, Independent Project Report* (2014).

[25] Lochlainn Ó Raifeartaigh. "Fermi coordinates". In: *Proceedings of the Royal Irish academy. Section A: Mathematical and physical sciences*. 1957.

[26] F. K. Manasse and Charles W. Misner. "Fermi normal coordinates and some basic concepts in differential geometry". In: *Journal of Mathematical Physics* (1963).

[27] P. A. M. Dirac. "The quantum theory of the electron". In: *Proceedings of the Royal Society of London* (1928).

[28] William G Dixon. "Dynamics of extended bodies in general relativity. I. Momentum and angular momentum". In: *Proceedings of the Royal Society of London* (1970).

[29] E. Corinaldesi and A. Papapetrou. "Spinning test-particles in general relativity. I". In: *Proceedings of the Royal Society of London* (1951).

[30] Gregor Wentzel. “Eine Verallgemeinerung der Quantenbedingungen fr die Zwecke der Wellenmechanik”. In: *Zeitschrift für Physik* (1926).

[31] Hendrik Anthony Kramers. “Wellenmechanik und halbzahlige Quantisierung”. In: *Zeitschrift für Physik* (1926).

[32] Léon Brillouin. “La mécanique ondulatoire de Schrödinger; une méthode générale de resolution par approximations successives”. In: *Compt. Rend. Hebd. Seances Acad. Sci.* (1926).

[33] William Francis Robert Gordon. “Der Strom der Diracschen Elektronentheorie”. In: *The European Physical Journal A* (1928).

[34] Francesco Cianfrani and Giovanni Montani. “Dirac equations in curved space-time vs. Papapetrou spinning particles”. In: *Europhysics Letters* (2008).

[35] John K Lawrence. “Fermions in general relativity”. In: *Annals of Physics* (1970).

[36] Jürgen Audretsch. “Trajectories and spin motion of massive spin-1/2 particles in gravitational fields”. In: *Journal of physics* (1981).

[37] Jürgen Audretsch. “Dirac electron in space-times with torsion: Spinor propagation, spin precession, and nongeodesic orbits”. In: *Physical Review D* (1981).

[38] R. Rüdiger. “The Dirac equation and spinning particles in general relativity”. In: *Proceedings of the Royal Society of London. Series A, Mathematical and physical sciences* (1981).

[39] Marius A Oancea and Achal Kumar. “Semiclassical analysis of Dirac fields on curved spacetime”. In: *Physical Review D* (2023).

[40] R Catenacci, M Martellini, et al. “Connection between the Dirac and the Dixon equation in a weak gravitational field”. In: *Lett. Nuovo Cimento;(Italy)* (1977).

[41] Francesco Cianfrani and Giovanni Montani. “Curvature-spin coupling from the semi-classical limit of the Dirac equation”. In: *International Journal of Modern Physics A* (2008).

[42] Ivar Fredholm. “Sur une classe d’équations fonctionnelles”. In: (1903).

[43] W. Gordon. “Der Strom der Diracschen Elektronentheorie”. In: *Zeitschrift für Physik* (1928).

[44] Georgios Lukes-Gerakopoulos. “Time parameterizations and spin supplementary conditions of the Mathisson-Papapetrou-Dixon equations”. In: *Physical Review D* (2017).

[45] Walther Gerlach and Otto Stern. “Der experimentelle Nachweis der Richtungsquantelung im Magnetfeld”. In: *Zeitschrift für Physik* (Dec. 1922).

[46] Sergio A Hojman and Benjamin Koch. “Closing a window for massive photons”. In: *Advances in High Energy Physics* 2013.1 (2013), p. 967805.

[47] Rachel Heloise Rietdijk and JW Van Holten. “Spinning particles in Schwarzschild spacetime”. In: *Classical and Quantum Gravity* (1993).

[48] David Klein and Peter Collas. “Exact Fermi coordinates for a class of space-times”. In: *Journal of Mathematical Physics* (2010).

[49] Andrzej Trautman. *Einstein-Cartan Theory*. 2006. arXiv: `gr-qc/0606062 [gr-qc]`.