COHOMOLOGY AND SPECTRAL SEQUENCES IN GAUGE THEORY

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Abstract. We define a double Chevalley-Eilenberg complex associated with the classical Becchi-Rouet-Stora-Tyutin symmetry of a gauge theory in the context of differential geometry, and describe the use of spectral sequences for the calculation of the corresponding cohomology groups.

Key words: gauge theory, spectral sequences, Lie algebra cohomology

1. Introduction

As is well know several years ago Bonora and Cotta-Ramusino [1] introduced a geometrical picture of the BRST [2] symmetry of a gauge theory, based on the concept of Lie algebra cohomology first introduced in the finite dimensional case by Chevalley and Eilenberg (CE) [3]. They considered the principal fiber bundle ξ where the gauge theory in question is defined and they generalized to the infinite dimensional case the CE theory. The generator of the symmetry could be identified with the coboundary operator associated with the representation of the Lie algebra of the gauge group with coefficients in the zero-forms on the space of connections; thus the symmetry (at least its generator) is not a property of a particular gauge theory but of the principal fiber bundle (which is a global geometrical concept) where the fields are defined. The local version of this cohomology was also shown in reference [1] to be related to the quantum anomalies of the theory; this point was studied in more detail in reference [4] and more recently by Dixon [5] and Dubois-Viollete *et al* [6].

In this note we extend the original BRST complex of a principal fiber bundle to a doble complex which in turn induces a total cochain complex: the *total BRST complex of a principal fiber bundle.* Double complexes appear when one considers representations with coefficients in forms of arbitrary order and uses the commutativity of the exterior and Lie derivatives. Though we have not made an explicit calculation of the relevant cohomology groups for any particular case, we present general arguments that show how spectral sequences [9] can be used to compute these groups. We think that the investigation of the meaning of the additional cohomology groups has both physical and geometrical interest, and leave it as a subject for further research.

2. Connections on principal bundles

2.1. Principal bundles and connections

The starting point is a smooth (C^∞) principal fiber bundle principal fiber bundle $\xi : G \to P \xrightarrow{\pi} B$ and the space of connections on it, $\mathcal{C}(\xi)$.

The *gauge group of* ξ, $\mathcal{G}(\xi)$, is the group of vertical automorphisms of ξ *i.e.* the set of C^∞ functions $P \xrightarrow{f} P$ such that the following diagram commutes:

$$\begin{array}{ccccc} P \times G & & \xrightarrow{f \times id} & & P \times G \\ \psi \downarrow & & & & \downarrow \psi \\ P & & \xrightarrow{f} & & P \\ & \pi \searrow & & \swarrow \pi & \\ & & B & & \end{array}$$

where ψ is the right free action of G on P. $\mathcal{G}(\xi)$ acts on $\mathcal{C}(\xi)$ through pull-backs *i.e.* if $\omega \in \Gamma(T^*P \otimes \mathbf{g})$ ($\mathbf{g} = LieG$) is a connection on ξ then $\omega' = f^*\omega$ gives the *gauge transformed* connection. This is the action that induces the BRST cohomology as we shall see below. We emphasize that gauge transformations are global and that it is only when one restricts to local trivializations and considers the pull-backs $A_U = \sigma_U^* \omega_U$ with local sections σ_U on open subsets U of the base space that one gets the familiar local gauge transformations used in physics. One has the quotient (moduli) space $\eta := \mathcal{C}/\mathcal{G}$, however the projection $\mathcal{C} \to \eta$ is not a principal fiber bundle and typically one restricts $\mathcal{G}$ to $\bar{\mathcal{G}} := \mathcal{G}/z$, where z is the center of $\mathcal{G}$ and $\mathcal{C}$ to $\mathcal{C}'$, the space of irreducible connections [10].

$\mathcal{C}(\xi)$ is an affine space modelled on $A^1(\xi) = \Gamma(T^*B \otimes E)$ where E is the bundle of Lie algebras of ξ (see below); once an arbitrary but fixed connection ω_0 is chosen in $\mathcal{C}(\xi)$ (base point) $\mathcal{C}(\xi) = \mathcal{C}^0(\xi)$ becomes an infinite dimensional real vector space (with origin in ω_0) and then an infinite dimensional differentiable manifold which is contractible to a point and therefore has vanishing cohomology groups except in dimension zero. We nevertheless expect that the *double complex* to be defined later will contain non-trivial information concerning any quantum gauge theory defined on ξ.

2.2. Associated bundles

Associated with ξ, there exist two *canonically* associated bundles: $\xi_G : G \to P \times_G G = F \stackrel{\pi_G}{\to} B$: the bundle of Lie groups of ξ and $\xi_{\mathbf{g}} : \mathbf{g} \to P \times_G \mathbf{g} = E \stackrel{\pi_{\mathbf{g}}}{\to} B$: the bundle of Lie algebras of ξ, where one has the left *adjoint action* of G on G and on $\mathbf{g}$ respectively. It can be shown that $\mathcal{G}(\xi) \cong \Gamma(F)$ and then $Lie\mathcal{G}(\xi) \cong \Gamma(E)$. We are interested in the Chevalley- Eilenberg [3] cohomology of $Lie\mathcal{G}(\xi)$ with coefficients in the differential forms on $\mathcal{C}(\xi)$, however it should be remarked here that the original definition of Lie algebra cohomology in reference [3] was for finite dimensional Lie algebras, while $Lie\mathcal{G}(\xi)$ is infinite dimensional.

2.3. Vector spaces of sections,covariant derivative and exterior covariant derivative

For $p = 0, 1, \ldots, dimB$ one defines the vector spaces $A^p = \Omega^p(E) := \Gamma(\Lambda^p T^* B \otimes E)$ of p-differential forms on B with values in E *i.e.* if $X_1, \ldots, X_p \in Vect(B)$ and $s \in A^p$, then $s(X_1, \ldots, X_p) \in \Gamma(E)$ (so $\Omega^p(E) \cong \Omega^p(B) \otimes_{C^\infty(B,R)} \Gamma(E)$); in particular $A^0 = \Gamma(E)$. There is a bijection between $\Gamma(E)$ and the set of equivariant maps $\gamma : P \to \mathbf{g}$, *i.e.*, smooth maps such that $\gamma(pg) = g^{-1}\gamma(p)$, defined as follows. If $s \in \Gamma(E)$ then $\gamma_s : P \to \mathbf{g}$ is given by $\gamma_s(p) = v$, where $s(b) = [p, v]$; if $\gamma : P \to \mathbf{g}$, then $s_\gamma \in \Gamma(E)$ is given by $s_\gamma(b) = [p, \gamma(p)]$, where $p \in \pi^{-1}(b)$. If $s \in \Gamma(E)$, $X \in Vect(B)$, and $\tilde{X} \in Vect(P)$ is the lifting of X by a connection ω, then the *covariant derivative, of s with respect to ω in the direction of* X, is defined by $\nabla^\omega_X s := s_{\tilde{X}(\gamma_s)}$, where $\tilde{X}(\gamma_s) := d\gamma_s(\tilde{X})$. In general if $f : P \to \mathbf{g}$ is equivariant, $Y \in Vect(P)$ and $\omega \in \mathcal{C}(\xi)$ then the covariant derivative of f with respect to ω in the direction Y is defined as $Df(Y) := df(horY)$, hence $\tilde{X}(\gamma_s) = D\gamma_s(\tilde{X})$. We define $d^\omega : A^0 \to A^1$ by $d^\omega s(X) := \nabla^\omega_X s$ and $\nabla^\omega : Vect(B) \times \Gamma(E) \to \Gamma(E)$ by $\nabla^\omega(X, s) := \nabla^\omega_X s$. The operator ∇^ω is a linear connection on E, *i.e.*, ∇^ω is $C^\infty(B, \mathbf{R})$- linear with respect to X but satisfies the Leibnitz rule $\nabla^\omega(X, fs) = f\nabla^\omega(X, s) + X(f)s$ with respect to s. The *covariant differential (linear) operator* $d^\omega : A^0 \to A^1$, extends to the *exterior covariant derivative (linear) operator* $\{\mathcal{D}^\omega_p\}_{p=0}^{dimB}$ in the same way as the De Rham exterior derivative extends the ordinary differential, namely $\mathcal{D}^\omega_0 = d^\omega$ and for $1 \le p \le dimB$, $\mathcal{D}^\omega_p(\alpha)(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}\nabla^\omega_{X_i}(\alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}))+$
$\sum_{1 \le i < j \le p+1}(-1)^{i+j}\alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1})$. In general

$A^0 \xrightarrow{d^\omega} A^1 \xrightarrow{\mathcal{D}^\omega_1} A^2 \xrightarrow{\mathcal{D}^\omega_2} \dots \xrightarrow{\mathcal{D}^\omega_{n-1}} A^n \to 0$ fails to be a complex *i.e.* $\mathcal{D}^\omega_{p+1} \circ \mathcal{D}^\omega_p \neq 0$, unless ω is *flat i.e.* $\mathcal{R}^\omega = 0$ where $\mathcal{R}^\omega : Vect(B) \times Vect(B) \to End(\Gamma(E))$, $\mathcal{R}^\omega(X,Y)(s) := (\nabla^\omega_X \circ \nabla^\omega_Y - \nabla^\omega_Y \circ \nabla^\omega_X - \nabla^\omega_{[X,Y]})(s)$ is the *curvature* of ∇^ω; so $\mathcal{R}^\omega \in \Gamma(\Lambda^2 T^*B \otimes Hom(E,E)) = \Omega^2(Hom(E,E))$ is an obstruction to have such a complex.

Assuming a compact, connected and 1-connected Lie group G and a compact and orientable base space B, we can define a positive definite non degenerate inner product on each of the A^p's, namely $< \alpha, \beta >_p := - \int_B tr(\alpha \wedge * \beta)$ [11], thus the A^p's become pre-Hilbert spaces; however they are not in general complete and to have Hilbert spaces one needs to specify a completion. Notice that the inner products induce norms $\| \alpha \|_p^2 := < \alpha, \alpha >_p$, so that the A^p's are normed vector spaces in a natural way.

2.4. Sobolev completions

Given $\omega \in \mathcal{C}(\xi)$ and ∇^ω the associated linear connection on $\xi_{\mathfrak{g}}$, the *Sobolev k-norm* on $\Omega^p(E)$ is defined by [8] $\| \phi \|^2_{p,k} := \| \phi \|^2_p + \| \mathcal{D}^\omega_p \phi \|^2_{p+1} + \dots + \| \mathcal{D}^\omega_{p+k-1} \circ \dots \circ \mathcal{D}^\omega_p \phi \|^2_{p+k}$. Different connections are equivalent in the sense that they lead to equivalent norms *i.e.* to topologically isomorphic vector spaces. The completion of $\Omega^p(E)$ with respect to this norm *i.e.* the set of formal limits of all Cauchy sequences in $\Omega^p(E)$ is called the *Sobolev k-norm completion* of $\Omega^p(E)$ and is denoted by $\Omega^p_k(E)$. So in particular $(Lie\mathcal{G}(\xi))_k = \Omega^0_k(E)$ and $\mathcal{C}^0_k(\xi) \cong \Omega^1_k(E)$. The Sobolev completion of $\mathcal{G}(\xi)$ is made through the following considerations. Let E_1, E_2 be smooth K-vector bundles over X ($K = \mathbf{R}$ or $\mathbf{C}$), then we have an isomorphism μ: $E_1^* \otimes E_2 \to Hom_K(E_1, E_2)$, given by $\mu(\alpha \otimes \omega)(v) = \alpha(v)\omega$. In particular for $E_1 = E_2 = V$, one has an isomorphism of vector bundles $V^* \otimes_K V \xrightarrow{\cong} Hom_K(V,V) \equiv End_K(V)$. A smooth map $h : V \to V$ is called *bundle map* if: *i*)$\pi \circ h = \pi$, where $\pi : V \to X$ is the projection; *ii*) $h|V_x : V_x \to V_x$ is linear, where $V_x = \pi^{-1}(x)$. We denote by $Map_X(V)$ the set of bundle maps. There is a bijection $\Psi : Map_X(V) \to \Gamma(End_K(V))$, given by $\Psi(h)(x) = h|V_x$, and with inverse $\Psi^{-1}(s)(v) = s(\pi(v))(v)$. There is a monoid structure on $Map_X(V)$ given by composition of bundle maps and a monoid structure on $\Gamma(End_K(V))$ given by $(s_1 \cdot s_2)(x) = s_1(x) \circ s_2(x)$. Clearly Ψ is a monoid isomorphism.

Let $K^n \to V = P \times_G K^n \to X$ be a vector bundle over X associated with the principal fiber bundle ξ: $G \to P \xrightarrow{\pi} X$, and a linear representation of G, $G \times K^n \to K^n$, then it can be easily verified that $\rho : \mathcal{G}(\xi) \to Map_X(V)$, given by $\rho(f)([p, \vec{x}]) := [f(p), \vec{x}]$ is a monoid monomorphism; therefore one has the following composition

$$\mathcal{G}(\xi) \xrightarrow{\rho} Map_X(V) \xrightarrow{\Psi} \Gamma(End_K(V)).$$

Clearly if $s = \Psi \circ \rho(f)$ *i.e.* if $s \in im(\Psi \circ \rho)$ then s has an inverse $s^{-1} = \Psi \circ \rho(f^{-1})$, so $\mathcal{G}(\xi)$ is isomorphic (as a group) to its image $\Psi \circ \rho(\mathcal{G}(\xi)) \hookrightarrow \Gamma(End_K(V))$. Let $< , >_x$ be an inner product in V_x, then the associated inner product in V_x^*, $< \alpha, \beta >_{*x} = < \lambda^{-1}(\alpha), \lambda^{-1}(\beta) >_x$, where $\lambda : V_x \to V_x^*$ is the vector space isomorphism $\lambda(v)(v') = < v, v' >_x$, induces the inner product $< , >_{\otimes x}$ in $V_x^* \otimes_K V_x$ as the linear extension of $< \alpha \otimes_K v, \alpha' \otimes_K v' >_{\otimes x} = < \alpha, \alpha' >_{*x} < v, v' >_x$. Thus one has the inner product $< , >_{End_x}$ in $End_K(V_x)$ given by $< \phi_{1x}, \phi_{2x} >_{End_x} = < \mu_x^{-1}(\phi_{1x}), \mu_x^{-1}(\phi_{2x}) >_{\otimes x}$.
Let $K = \mathbf{R}$, $V = E$ and $X = B$; for each $p = 0, 1, ..., dimB$ the inner product in $\Omega^p(E)$ defines an inner product in $\Omega^p(End_{\mathbf{R}}(E)) = \Gamma(\Lambda^p T^* B \otimes End_{\mathbf{R}}(E))$ for which one defines a Sobolev k-norm analogous to that defined for $\Omega^p(E)$; in particular *the Sobolev k-norm completion of* $\mathcal{G}(\xi)$, denoted by $\mathcal{G}(\xi)_k$, is the closure of $\mathcal{G}(\xi)$ it with respect to the Sobolev k-norm completion of $\Gamma(End_{\mathbf{R}}(P \times_G \mathbf{g}))$[8].

3. Cohomology of Lie algebras

3.1. Chevalley-Eilenberg cohomology

Let $\mathbf{g}$ and V be a finite dimensional Lie algebra and a finite dimensional vector space respectively, both over the field K ($K = \mathbf{R}$ or $\mathbf{C}$), and let $\phi : \mathbf{g} \to End_K(V)$ be a representation of $\mathbf{g}$ on V. Define $C^0 := V$ and $C^p := \{\alpha : \mathbf{g} \times \ldots \times \mathbf{g}(p\ times) \to V,\ \alpha\ multilinear\ alternating\}$ for $p = 1, 2, \ldots$. Define K-linear operators $\delta^p : C^p \to C^{p+1}$, by $\delta^p(\alpha)(a_1, \ldots, a_{p+1}) := \sum_{i=1}^{p+1} (-1)^{i+1} \phi(a_i)(\alpha(a_1, \ldots, \hat{a}_i, \ldots, a_{p+1})) +$
$\sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \alpha([a_i, a_j], \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{p+1})$. One can show [3] that $\delta^{p+1} \circ \delta^p = 0$, therefore one has a *cochain complex* $0 \to C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \ldots$ with p-cocycles $Z^p = ker\delta^p$ and p-coboundaries $B^p = im\delta^{p-1} \subset Z^p$. One defines the *Chevalley-Eilenberg cohomology of* $\mathbf{g}$ *with respect to the representation* ϕ *of* $\mathbf{g}$ *on* V ("with coefficients in V") as the graded group $H^*_{CE}(\mathbf{g}, \phi, V; K) := \{H^p_{CE}(\mathbf{g}, \phi, V; K)\}_{p=0}^{\infty}$, where $H^p_{CE}(\mathbf{g}, \phi, V; K) := Z^p / B^p$.

3.2. Double complexes and total cohomology

A *double cochain complex* is a triple (C, ∂, d), where $C = \{C^{p,q}\}, p, q = 0, 1, 2, \ldots$ is a set of abelian groups, and $d = \{d^{p,q} : C^{p,q} \to C^{p,q+1}\}$ and $\partial = \{\partial^{p,q} : C^{p,q} \to C^{p+1,q}\}$ are *differentials* *i.e.* group homomorphisms satisfying $d^2 = \partial^2 = 0$, such that the following diagrams are commutative:

$$\begin{array}{ccc} C^{p,q} & \xrightarrow{d^{p,q}} & C^{p,q+1} \\ \partial^{p,q} \downarrow & & \downarrow \partial^{p,q+1} \\ C^{p+1,q} & \xrightarrow{d^{p+1,q}} & C^{p+1,q+1} \end{array}$$

A double cochain complex (C,∂,d) naturally induces a (simple) cochain complex (K^*,D) as follows: for $n=0,1,2,\dots$ one defines the abelian groups $K^n := \oplus_{p+q=n}C^{p,q}$ and the operators $D^n := \oplus_{p+q=n}(\partial^{p,q}\oplus(-1)^p$ $d^{p,q})$. (K^*,D) is usually called the *total (cochain) complex* associated with the double (cochain) complex (C,∂,d) [7]. The cohomology of (K^*,D), namely $H^*(K^*,D) := \{H^n(K^*,D)\}_{p=0}^{\infty}$ with $H^n(K^*,D) := kerD^n/imD^{n-1}$ is called the *total cohomology of the double (cochain) complex* (C,∂,d).

3.3. Group actions and double complexes

Proposition: Let G be a Lie group, M a C^∞ manifold and $M\times G \xrightarrow{\psi} M$, $\psi(x,g)=xg$ a smooth action of G on M. Then there exists a double cochain complex involving $\mathbf{g} = LieG$ and $\Omega(M)$, the differential forms on M.

Proof:

i) $A\in\mathbf{g}$ induces the fundamental field of A in M, $A^*\in Vect(M)$ defined by $A^*_x(f) := d/dt(f(xexptA))\,|_{t=0}$.

ii) The Lie derivative of a tensor τ on M with respect to A^* is given by $\mathcal{L}_{A^*}\tau := d/dt(\Phi_t^*\tau)\,|_{t=0}$ where Φ_t is the flow of A^* and $\Phi_t^*\tau$ is the pull-back of τ by Φ_t, in particular this is valid for forms of arbitrary order and $\mathcal{L}_{A^*}\alpha\in\Omega^pM$ if $\alpha\in\Omega^pM$.

iii) Fix p in the set $\{0,1,\dots,n=dimM\}$ and define the infinite set of vector spaces $C_p^0 := \Omega^pM$ and $C_p^\nu := \{\mathbf{g}\times\dots\times\mathbf{g}(\nu\ times) \xrightarrow{\alpha} \Omega^pM, \alpha$ $alternating\}$, for $\nu=1,2,\dots$.

iv) Since $\mathcal{L}_{[A,B]^*} = [\mathcal{L}_{A^*},\mathcal{L}_{B^*}], \phi_p : \mathbf{g}\to End_{\mathbf{R}}(\Omega^pM)$, given by $\phi_p(A) := \mathcal{L}_{A^*}$ is a representation of $\mathbf{g}$ on Ω^pM, and then $C_p^0 \xrightarrow{\delta_p^0} C_p^1 \xrightarrow{\delta_p^1} C_p^2 \xrightarrow{\delta_p^2} \dots$ is a cochain complex with coboundary $\{\delta_p^\nu\}_{\nu=0}^{\infty}$, defined by $\delta_p^\nu(\alpha)(A_0,\dots,A_\nu) =$
$\sum_{i=0}^{\nu}(-1)^i\mathcal{L}_{A_i^*}(\alpha(A_0,\dots,\hat{A}_i,\dots,A_\nu$
$))+\sum_{0\le i<j\le\nu}(-1)^{i+j}\alpha([A_i,A_j],A_0,\dots,\hat{A}_i,\dots,\hat{A}_j,\dots,A_\nu)$. Therefore one has the Chevalley- Eilenberg cohomology of $\mathbf{g}$ with coefficients in Ω^pM, $H^*_{CE}(\mathbf{g},\phi_p,\Omega^pM;\mathbf{R}) = \{H^\nu_{CE}(\mathbf{g},\phi_p,\Omega^pM;\mathbf{R})\}_{\nu=0}^{\infty}$.

v) Let $d_p^\nu : C_p^\nu \to C_{p+1}^\nu$ be defined by $d_p^\nu(\alpha)(A_1,\dots,A_\nu) = d(\alpha$ $(A_1,\dots,A_\nu))$, where d is the exterior derivative operator on M. Since $d\mathcal{L} = \mathcal{L}d$, we have a double cochain complex (C,δ,d) given by the following lattice of commuting diagrams:

$$\begin{array}{ccccccccccc}
C_0^0 & \xrightarrow{d_0^0} & C_1^0 & \xrightarrow{d_1^0} & C_2^0 & \xrightarrow{d_2^0} & \dots & \xrightarrow{d_{n-2}^0} & C_{n-1}^0 & \xrightarrow{d_{n-1}^0} & C_n^0 \\
\delta_0^0\downarrow & & \delta_1^0\downarrow & & \delta_2^0\downarrow & & & & \delta_{n-1}^0\downarrow & & \delta_n^0\downarrow \\
C_0^1 & \xrightarrow{d_0^1} & C_1^1 & \xrightarrow{d_1^1} & C_2^1 & \xrightarrow{d_2^1} & \dots & \xrightarrow{d_{n-2}^1} & C_{n-1}^1 & \xrightarrow{d_{n-1}^1} & C_n^1 \\
\delta_0^1\downarrow & & \delta_1^1\downarrow & & \delta_2^1\downarrow & & & & \delta_{n-1}^1\downarrow & & \delta_n^1\downarrow \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \vdots
\end{array}$$

Therefore, the action $M \times G \xrightarrow{\psi} M$ has a naturally associated cohomology $H^*(K^*, D)$, namely that of the total (cochain) complex (K^*, D) associated with the double cochain complex $(\mathcal{C}, \delta, d)$ as specified in the previous subsection. *q.e.d.*

4. BRST cohomology

4.1. Total BRST cohomology of a principal fiber bundle

Using the results of the previous section and taking into account suitable Sobolev completions in the infinite dimensional case, one considers the case where $G = \mathcal{G}(\xi)$ and $M = \mathcal{C}(\xi)$; so for each $p = 0, 1, 2, \ldots$one has the representation of the Lie algebra of the gauge group on the differential p-forms on $\mathcal{C}(\xi)$, $\phi_p : A^0 \to End_{\mathbf{R}}(\Omega^p \mathrm{C}(\xi))$, $\phi_p(s) := \mathcal{L}_{s^\bullet}$. It is usual to give to $\mathcal{C}(\xi)$ and A^0 respectively Sobolev k- and $(k+1)$- norm completions with $k \geq (dimB/2) + 1$ which guarantee that the action $\mathcal{C} \times \mathcal{G} \to \mathcal{C}$ extends to a smooth action $\mathcal{C}_k \times \mathcal{G}_k \to \mathcal{C}_k$ [8]. Notice however that this condition does not fix the value of k and so the results of the BRST cohomology might depend on it. For $\nu = 1, 2, \ldots$ one defines the spaces $C_p^\nu(\xi) := \{$*alternating continuous functions* $A^0 \times \ldots \times A^0(\nu$ *times*$) \xrightarrow{\alpha} \Omega^p \mathcal{C}(\xi)\}$, and $C_p^0(\xi) := \Omega^p \mathcal{C}(\xi)$; the continuity of α is defined as follows: if $\alpha \in C_p^\nu(\xi)$ then for all $\omega \in \mathcal{C}_k$ and all $\xi_1, \ldots, \xi_p \in Vect(\mathcal{C}_k)$ the maps $\alpha_{\omega, \xi_1, \ldots, \xi_p} : A^0 \times \ldots \times A^0(\nu$ *times*$) \to \mathbf{R}$, where $\alpha_{\omega, \xi_1, \ldots, \xi_p}(s_1, \ldots, s_\nu) := \alpha(s_1, \ldots, s_\nu)(\xi_1, \ldots, \xi_p)(\omega)$, are continuous.

The first column in the double cochain complex

$$
\begin{array}{ccccc}
C_0^0(\xi) & \xrightarrow{d_0^0} & C_1^0(\xi) & \xrightarrow{d_1^0} & \cdots \\
\delta_0^0 \downarrow & & \delta_1^0 \downarrow & & \\
C_0^1(\xi) & \xrightarrow{d_0^1} & C_1^1(\xi) & \xrightarrow{d_1^1} & \cdots \\
\delta_0^1 \downarrow & & \delta_1^1 \downarrow & & \\
\vdots & & \vdots & &
\end{array}
$$

i.e. the one corresponding to 0-forms on $\mathcal{C}(\xi)$ leads to the usual BRST cohomology of the principal fiber bundle ξ [1] $H^*_{BRST}(\xi) = \{H^\nu_{BRST}(\xi)\}_{\nu=0}^\infty$ where $H^\nu_{BRST}(\xi) = ker\delta_0^\nu / im\delta_0^{\nu-1}$ $(\delta_0^{-1} = 0)$. ν is identified with the ghost number and the coboundary $\{\delta_0^\nu\}_{\nu=0}^\infty$ with the BRST nilpotent operator of any quantum field theory defined on ξ. The remaining columns for p=1,2,... have been here formally defined and lead to a total complex (K^*, D) with cohomology $H^*(K^*, D)$ which we call the *total BRST cohomology of the principal fiber bundle* ξ, and denote by $\mathcal{H}^*_{BRST}(\xi)$. We do not yet know the possible physical meaning of these additional cohomology groups; however the fact that for the case $p = 0$ the local version of this cohomology has an interpretation in terms of anomalies and since these groups are in principle computable with the technique of spectral sequences, their definition is of interest for further investigation.

5. Spectral sequences

5.1. Spectral sequence of a filtered complex

We give the basic definitions and results, for more details the reader may consult reference [9].

a) Let (K^*, d) be a *cochain complex*, $(K^*, d) = \{K^n, d^n : K^n \to K^{n+1}\}_{n \geq 0}$. Let $B^n = imd^{n-1} = n$-coboundaries $\subset Z^n = kerd^n = n$-cocycles, then the *Cohomology* $H^*(K^*)$ of (K^*, d) is given by $H^n(K^*) := Z^n/B^n$.

b) A *Filtration* F^*K^* of (K^*, d) is a sequence of subcomplexes $K^* = F^0K^* \supset F^1K^* \supset F^2K^* \supset \ldots \supset F^pK^* \supset \ldots$ so that $d : F^pK^* \to F^pK^*$. At each level in K^* one has $K^n = F^0K^n \supset F^1K^n \supset F^2K^n \supset \ldots \supset F^pK^n \supset \ldots$ and $d^n : F^pK^n \to F^pK^{n+1}$. (K^*, d) is said to be *finitely filtered* if for each n there exists an m such that $F^mK^n = 0$. We have short exact sequences of complexes $0 \to F^{p+1}K^* \to F^pK^* \to F^pK^*/F^{p+1}K^* \to 0$ which means that for each $p = 0, 1, 2, \ldots$ one has the array

$$\begin{array}{ccccccccc} 0 & \to & F^{p+1}K^0 & \to & F^pK^0 & \to & F^pK^0/F^{p+1}K^0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \Delta_p^0 & & \\ 0 & \to & F^{p+1}K^1 & \to & F^pK^1 & \to & F^pK^1/F^{p+1}K^1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \Delta_p^1 & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

where the coboundary $\Delta_p^n : F^pK^n/F^{p+1}K^n \to F^pK^{n+1}/F^{p+1}K^{n+1}$ for $n = 0, 1, 2, 3, \ldots$ is induced by d^n. Each sequence gives a long exact cohomology sequence

$$\begin{array}{ccccc} & & H^*(F^pK^*) & & \\ & i \nearrow & & \searrow \pi & \\ H^*(F^{p+1}K^*) & & \overset{\delta}{\leftarrow} & & H^*(F^pK^*/F^{p+1}K^*) \end{array}$$

where δ is a morphism of degree one.

Given a subcomplex $L^* \subset K^*$ we define a filtration F^*K^* by $F^0K^* = K^*$, $F^1K^* = L^*$ and $F^2K^* = 0$. The idea of the spectral sequences is that they play with respect to filtrations a róle analogous to that played by the long exact cohomology sequence with respect to subcomplexes.

c) The filtration of K^*, F^*K^*, gives a filtration of $H^*(K^*)$ as follows. Since F^pK^* is a subcomplex of K^*, the inclusion induces, for each $n \geq 0$, a homomorphism $i_p^n : H^n(F^pK^*) \to H^n(K^*)$, then the filtration is defined by $F^pH^n(K^*) = im\ i_p^n$.

d) A Spectral sequence consists of the following:

i) An infinite sequence of *bigraded abelian groups* $E_r = \{E_r^{p,q}\}_{p,q\geq 0}$, with $r \geq 0$,

ii) *differentials* of degree $(r, 1-r)$ *i.e.* group endomorphisms $d_r : E_r \to E_r$, $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ with $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$,

iii) *Cohomology relations* $E_{r+1} \cong H^*(E_r)$ *i.e.* $E_{r+1}^{p,q} \cong H^{p,q}(E_r)$ where $H^{p,q}(E_r) = ker d_r^{p,q} / im d_r^{p-r,q-1+r}$.

e) One says that a spectral sequence $(E_r, d_r)_{r=0}^{\infty}$ *converges finitely* to a filtered graded group H^*, provided: *i*) for each p and q, there exists an integer r_0 such that $E_r^{p,q} \cong E_{r_0}^{p,q}$ for $r \geq r_0$; *ii*) $E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$, where $E_\infty^{p,q} \equiv E_{r_0}^{p,q}$.

f) *Proposition*: If K^* is a complex finitely filtered by F^*K^*, then there exists a spectral sequence $(E_r, d_r)_{r=0}^{\infty}$ converging finitely to $H^*(K^*)$ with:

i) $E_0^{p,q} = F^pK^{p+q}/F^{p+1}K^{p+q}$,

ii) $E_1^{p,q} = H^{p+q}(F^pK^*/F^{p+1}K^*)$.

5.2. Application to double (cochain) complexes

For the total complex (K^*, D) associated with a double cochain complex (C, ∂, d) as defined in subsection 3.2 it can be verified the following:

i) K^* has *two* canonical filtrations $F_1^*K^*$ and $F_2^*K^*$ with $F_1^0K^* = K^*$ ($F_2^0K^* = K^*$), ..., $F_1^lK^*$ ($F_2^mK^*$): the standard complex associated with the doble complex obtained from C after making zero the first l (m) rows (columns) of C, ($l, m = 1, 2, ...$)

ii) The corresponding spectral sequences both converge finitely to $H^*(K^*)$.

In particular this result holds for the BRST double complex of subsection 4.1 and hence $\mathcal{H}_{BRST}^*(\xi)$ can be computed.

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